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ABSTRACT. We study the renormalized volume of asymptotically hyperbolic Einstein (AHE in short) manifolds (M, g) when the conformal boundary ∂M has dimension n even. Its definition depends on the choice of metric h_0 on ∂M in the conformal class at infinity determined by g, we denote it by $\mathrm{Vol}_R(M, g; h_0)$. We show that $\mathrm{Vol}_R(M, g; \cdot)$ is a functional admitting a "Polyakov type" formula in the conformal class $[h_0]$ and we describe the critical points as solutions of some non-linear equation $v_n(h_0) = \text{constant}$, satisfied in particular by Einstein metrics. In dimension $n = 2$, choosing extremizers in the conformal class amounts to uniformizing the surface, while in dimension $n = 4$ this amounts to solving the σ_2-Yamabe problem. Next, we consider the variation of $\mathrm{Vol}_R(M, \cdot; \cdot)$ along a curve of AHE metrics g^t with boundary metric h_0^t and we use this to show that, provided conformal classes can be (locally) parametrized by metrics h solving $v_n(h) = \int_{\partial M} v_n(h) \mathrm{dvol}_h$, the set of ends of AHE manifolds (up to diffeomorphisms isotopic to the identity) can be viewed as a Lagrangian submanifold in the cotangent space to the space $\mathcal{T}(\partial M)$ of conformal structures on ∂M. We obtain as a consequence a higher-dimensional version of McMullen's quasifuchsian reciprocity. We finally show that conformal classes admitting negatively curved Einstein metrics are local minima for the renormalized volume for a warped product type filling.

1. INTRODUCTION

By Mostow rigidity, the volume of complete oriented finite volume hyperbolic 3-manifolds is an important topological invariant, also related to the Jones polynomial of knots. For infinite volume hyperbolic 3-manifolds, one should still expect some invariant derived from the volume form as well. Following ideas coming from the physics literature [41, 63, 44], Takhtajan-Teo [64] and Krasnov-Schlenker [45] defined a regularized (or renormalized) version of the volume in the case of convex co-compact hyperbolic quotients $M = \Gamma \backslash \mathbb{H}^3$, and studied some of its properties. The renormalized volume is actually related to the uniformization theory of the boundary of the conformal compactification of M. Indeed, such hyperbolic manifolds can be compactified into smooth manifolds with boundary $\overline{M}$ by adding a compact surface N to M, and the metric on M is conformal to a smooth metric $\bar{g}$ on $\overline{M}$, inducing a conformal class $[\bar{g}|_{TN}]$ on N. The renormalized volume plays the role of an action on the conformal class $[h_0]$ with critical points at the constant curvature metrics, in a way similar

to the determinant of the Laplacian. It turns out that this action has interesting properties when we deform the hyperbolic metric in the bulk.

In this paper, we study the higher dimensional analog of this invariant and compute its variation on the so-called quantum conformal superspace, the higher-dimensional analog of the Teichmüller space.

1.1. **Dimension** $n+1 = 3$. Let us explain in more details the definition of the renormalized volume of hyperbolic convex co-compact 3 manifolds. Outside a compact set, the hyperbolic metric on $M = \Gamma \backslash \mathbb{H}^3$ is isometric to a *hyperbolic end*

$$((0, \varepsilon_0)_x \times N, g), \qquad g = \frac{dx^2 + h_x}{x^2}, \qquad h_x = h_0 + x^2 h_2 + \tfrac{1}{4} x^4 h_2^2 \qquad (1)$$

for some $\varepsilon_0 > 0$, where (N, h_0) is a compact Riemannian surface (possibly disconnected), and h_2 is a symmetric 2-tensor on N satisfying the constraints

$$\mathrm{Tr}_{h_0}(h_2) = -\tfrac{1}{2} \mathrm{Scal}_{h_0}, \qquad \qquad \delta_{h_0}(h_2) = \tfrac{1}{2} d \, \mathrm{Scal}_{h_0}. \qquad (2)$$

Here δ_{h_0} stands for divergence with respect to the background metric h_0. The remaining term h_2^2 is the square of h_2, identified to an endomorphism using the metric h_0. The manifold $\overline{M} := M \sqcup N$ becomes a smooth compactification of M by declaring that the function x is smooth on $\overline{M}$ and vanishes to order 1 on the boundary $\partial \overline{M} = N$. Near the boundary, x is the distance function to N with respect to the metric $\bar{g} := x^2 g$. In particular it defines a foliation with leaves $\{x = \text{constant}\}$ near the boundary. We call x a *geodesic boundary defining function* associated to h_0. The important observation explained in [27, Lemma 2.1] is that the function x and the metric h_0 above are not determined by a given g, but the conformal class of h_0 is. Moreover, for each representative $\hat{h}_0$ of the conformal class $[h_0]$, there is a unique smooth boundary defining function $\hat{x}$ on $\overline{M}$ near N such that the hyperbolic metric g near N has the form (1) with x, h_0, h_2 replaced by $\hat{x}, \hat{h}_0, \hat{h}_2$ for some tensor $\hat{h}_2$ satisfying the constraints (2) with respect to the metric $\hat{h}_0$. In other words, conformal representatives of the conformal infinity N of M correspond to certain geometric foliations in the end of M. For (M, g) fixed, the *renormalized volume* $\mathrm{Vol}_R(M, g; \cdot)$ is a function on the conformal class $[h_0]$ defined by the regular value at $z = 0$ of the meromorphic function $F(z) := \int_M x^z \mathrm{dvol}_g$ if x is the geodesic boundary defining function associated to h_0:

$$\mathrm{Vol}_R(M, g; h_0) = \mathrm{FP}_{z=0} \int_M x^z \mathrm{dvol}_g \qquad (3)$$

(here $\mathrm{FP}_{z=0}$ denotes finite part). The meromorphic function $F(z)$ has a pole at $z = 0$, and the residue is the topological invariant $-\frac{\pi}{2} \chi(N)$ computed from (2) using the Gauss-Bonnet formula. The functional $\mathrm{Vol}_R(M, g; \cdot)$, defined on the conformal class $[h_0]$, has the following remarkable properties:

(1) **"Polyakov Formula"**: If $\omega \in C^\infty(N)$, then

$$\mathrm{Vol}_R(M, g; e^{2\omega} h_0) - \mathrm{Vol}_R(M, g; h_0) = -\tfrac{1}{4} \int_{\partial M} (|\nabla \omega|_{h_0}^2 + \mathrm{Scal}_{h_0} \omega) \mathrm{dvol}_{h_0}.$$

(2) **Critical points**: For fixed g, $\mathrm{Vol}_R(M, g; \cdot)$ is critical, among metrics in the conformal class $[h_0]$ with fixed area, exactly at constant curvature metrics.

(3) **Extrema**: If $\chi(N) < 0$, the critical point is unique in a conformal class (with fixed area) and is a maximum.

The Polyakov-type formula is easily shown (see Proposition 3.11) and the extremum is obtained by the classical variational approach, see for example [66, Ch. 14.2]. When $n = 2$, the renormalized volume Vol_R is essentially the Liouville functional defined by Takhtajan and Zograf [65] for Schottky and quasifuchsian hyperbolic manifolds, see [64].

Recall that if N is a smooth compact surface with $\chi(N) < 0$, the Teichmüller space $\mathcal{T}(N)$ is the space of conformal classes of metrics on N quotient by the group $\mathcal{D}_0(N)$ of diffeomorphisms of N isotopic to the identity; equivalently it is given by the quotient $\mathcal{G}\backslash\mathcal{M}(N)$, where $\mathcal{M}(N)$ is the space of smooth metrics on N and $\mathcal{G} = C^\infty(N) \rtimes \mathcal{D}_0(N)$ is the semi-direct product of the conformal group with $\mathcal{D}_0(N)$. By choosing the unique hyperbolic metric in each conformal class, one identifies $\mathcal{T}(N)$ with the space of hyperbolic metrics up to $\mathcal{D}_0(N)$. This can be represented as a slice of hyperbolic metrics transverse to the action of $\mathcal{D}_0(N)$. As mentioned above, ends of convex co-compact hyperbolic 3 manifolds are hyperbolic ends in the sense defined by (1) with h_0, h_2 satisfying the constraint equations (2). Actually, more is true: for every surface N, a metric g of the form (1) on $(0, \varepsilon) \times N$ is hyperbolic if and only if h_0, h_2 satisfy (2). A hyperbolic end is thus determined by the pair (h_0, h_2) and we denote by $\mathcal{E}(N)$ the space of these ends. The gauge group for $\mathcal{E}(N)$ is $\mathcal{G}$, acting by:

$$\text{if } g = \frac{dx^2 + h_x}{x^2}, \text{ then } (f, \phi).g = \psi^* g, \text{ with } \psi(x, y) = (x, \psi_x(y))$$

where $\psi_x : N \to N$ is the diffeomorphism defined so that $\psi_0 = \phi^{-1}$ and $y \in N \mapsto (\hat{x}, \psi_x \circ \phi(y)) \in [0, \varepsilon) \times N$ is the flow at time x of the gradient $\nabla^{\hat{x}^2 g}\hat{x}$ (with respect to $\hat{x}^2 g$) of the function $\hat{x}$ defined to be the unique boundary defining function of $[0, \varepsilon) \times N$ satisfying $|d\hat{x}|_{\hat{x}^2 g} = 1$ and $(\hat{x}^2 g)|_{x=0} = e^{2f}(\phi^{-1})^* h_0$. This can be seen as an action on the pairs (h_0, h_2) which determine the end, and the action on the data h_0 gives rise to $\mathcal{T}(N)$ as the space of orbits. The action of the conformal part of $\mathcal{G}$ does not have so nice properties on h_2, for instance h_2 is not conformally covariant. We then represent each conformal class $[h_0]$ by its hyperbolic metric h_0, which is the maximizer of the renormalized volume in the conformal class of area $-2\pi\chi(N)$. This allows to identify the quotients

$$\mathcal{G}\backslash\mathcal{E}(N) \simeq \mathcal{D}_0(N)\backslash\{(h_0, h_2) \in \mathcal{E}(N); h_0 \text{ hyperbolic}\}$$

where $\phi \in \mathcal{D}_0(N)$ acts by $\phi.(h_0, h_2) = ((\phi^{-1})^* h_0, (\phi^{-1})^* h_2)$. If $h_2^\circ := h_2 - \frac{1}{2}\mathrm{Tr}_{h_0}(h_2)h_0$ is the trace-free part of h_2, we obtain that the pair (h_0, h_2°) satisfies

$$\mathrm{Scal}_{h_0} = -2, \qquad\qquad \mathrm{Tr}_{h_0}(h_2^\circ) = 0, \qquad\qquad \delta_{h_0}(h_2^\circ) = 0. \qquad (4)$$

The cotangent space $T_{h_0}^* \mathcal{T}(N)$ to $\mathcal{T}(N)$ at a hyperbolic metric h_0 is naturally represented by symmetric 2-tensors on M which are trace-free and divergence-free with respect to h_0 (such a tensor is the real part of a holomorphic quadratic differential on N). Consequently, since the action of $\mathcal{D}_0(N)$ on (h_0, h_2°) coincides with the action of $\mathcal{D}_0(N)$ on the cotangent space to the space of hyperbolic metrics, we have

(4) **Cotangent vectors as ends**: There is a natural isomorphism $\mathcal{G}\backslash\mathcal{E}(N) \to T^*\mathcal{T}(N)$, given by the correspondence $(h_0, h_2) \mapsto (h_0, h_2^\circ)$.

For a hyperbolic end, we call h_0 and h_2° the Dirichlet and Neumann boundary data at $x = 0$, by analogy with linear elliptic boundary value problems of order 2. Here the elliptic equation on the end $(0, \varepsilon) \times N$ is Einstein's equation $\mathrm{Ric}_g = -2g$ and it is solvable near $x = 0$ for any choice of boundary data (h_0, h_2°) satisfying the constraints equation (2) with h_0 hyperbolic.

From a global point of view, it is interesting to understand which pairs (h_0, h_2°) occur as boundary data of a convex co-compact hyperbolic manifold. A famous theorem of Ahlfors and Bers [1, 5] in the quasifuchsian setting, extended by Marden [47, 48], states that for a given convex co-compact hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^3$ with conformal boundary $N = \partial M$, there is a smooth map

$$\Phi : \mathcal{T}(N) \to \mathcal{M}_{-1}(M) \quad \text{with} \quad [x^2 \Phi(h_0)|_N] = [h_0] \tag{5}$$

where $[\cdot]$ denotes conformal class, x is any smooth boundary defining function of the compactification $\overline{M}$ and $\mathcal{M}_{-1}(M)$ is the space of hyperbolic metrics on M which are convex co-compact, considered up to isotopy. From this viewpoint, the relevant object is the Dirichlet-to-Neumann map $h_0 \mapsto h_2^\circ$, where h_2° is the Neumann data of the metric $\Phi(h_0)$. This map can be understood as a 1-form on $\mathcal{T}(N)$. The following facts were proved by Krasnov-Schlenker [45] (see also [53] for the Lagrangian property when M is quasifuchsian and [35] for an alternative proof using Chern-Simons invariants):

(5) **Lagrangian submanifold**: The space $\mathcal{L}$ of couples of boundary data (h_0, h_2°) corresponding to convex co-compact metrics on M is Lagrangian in $T^* \mathcal{T}(N)$, endowed with the natural symplectic structure.

(6) **Generating function**: $\mathcal{L}$ is the graph of the exact 1-form given by the differential of $h_0 \mapsto \mathrm{Vol}_R(M, \Phi(h_0); h_0)$ over $\mathcal{T}(N)$. More precisely, if $\dot{h}_0 \in T_{h_0} \mathcal{T}(N)$ is a variation of hyperbolic metrics at $h_0 \in \mathcal{T}(N)$, then

$$d\mathrm{Vol}_R(M, \Phi(h_0), h_0).\dot{h}_0 = -\tfrac{1}{4} \int_N \langle h_2^\circ, \dot{h}_0 \rangle \mathrm{dvol}_{h_0}. \tag{6}$$

Another interesting property of the function $h_0 \mapsto \mathrm{Vol}_R(M, \Phi(h_0); h_0)$ was discovered by Takhtajan-Teo [64] (using the correspondence with the Liouville action of [65]) and Krasnov-Schlenker [45] (when M is a Schottky or a quasifuchsian manifold): it generates the Weil-Petersson Kähler form ω_{WP} on $\mathcal{T}(N)$. This was extended by Guillarmou-Moroianu [35] in the general setting using Chern-Simons invariants:

(7) **Kähler potential**: The function $\mathrm{Vol}_R : h_0 \mapsto \mathrm{Vol}_R(M, \Phi(h_0); h_0)$ is a Kähler potential for the Weil-Petersson form:

$$\partial \bar{\partial} \mathrm{Vol}_R = \tfrac{i}{16} \omega_{WP}.$$

An important class of convex co-compact hyperbolic 3-manifolds are quasifuchsian manifolds, diffeomorphic to a topological cylinder $M = \mathbb{R}_t \times S$ with basis a surface S of genus $g \geq 2$. The conformal boundary has two connected components $S^+ \sqcup S^-$, both diffeomorphic to S, with conformal classes $[h_0^\pm]$ corresponding respectively to the limit $t \to \pm\infty$. Here we choose the hyperbolic representative $h_0^\pm$ in the conformal classes $[h_0^\pm]$ on $S^\pm$. For each pair $h_0 = (h_0^+, h_0^-)$ there exists a quasifuchsian hyperbolic metric $g = \Phi(h_0)$ on M and, as seen

before, each of the 2 ends $e^\pm$ has a metric of the form $x^{-2}(dx^2 + h_0^\pm + x^2 h_2^\pm + \frac{1}{4}x^4(h_2^\pm)^2)$, with the trace free part $h_2^{0\pm}$ of $h_2^\pm$ interpreted as a cotangent vector to $\mathcal{T}(S)$. The pair (h_0^+, h_0-) then defines two points $h_2^{0-} \in T_{h_0^-}\mathcal{T}(S)$ and $h_2^{0+} \in T_{h_0^+}\mathcal{T}(S)$. We can now fix h_0^+ and consider the linear maps

$$\phi_{h_0^+} : T_{h_0^-}\mathcal{T}(S) \to T_{h_0^+}^*\mathcal{T}(S), \qquad\qquad \phi_{h_0^-} : T_{h_0^+}\mathcal{T}(S) \to T_{h_0^-}^*\mathcal{T}(S)$$

sending a first-order variation of $h_0^\mp$ to the corresponding variation of $h_2^{0\pm}$.

(8) **Quasifuchsian reciprocity**: The maps $\phi_{h_0^+}$ and $\phi_{h_0^-}$ are adjoint.

This was discovered by McMullen [53] (see also Krasnov-Schlenker [45] for an alternative proof). Still in the setting of quasifuchsian manifolds, the renormalized volume behaves essentially like a squared distance for the Weil-Petersson metric near the Fuchsian locus (i.e., the diagonal in $\mathcal{T}(S) \times \mathcal{T}(S)$ corresponding to $h_0^+ = h_0^-$); this follows from

(9) **A local distance in Teichmüller**: for $h_0^+ \in \mathcal{T}(S)$ fixed, and denoting $h_0 = (h_0^+, h_0^-) \in \mathcal{T}(S) \times \mathcal{T}(S)$ the function $h_0^- \mapsto \mathrm{Vol}_R(M, \Phi(h_0); h_0)$ has a critical point at $h_0^- = h_0^+$, unique near h_0^+, which is a local minimum (the Hessian is positive definite).

(10) **A global distance in Teichmüller**: There are constants $C_0, C_1 > 0$ such that

$$\frac{d_{\mathrm{WP}}(h_0^+, h_0^-)}{C_0} - C_1 \leq \mathrm{Vol}_R(M, \Phi(h_0); h_0) \leq C_0 d_{\mathrm{WP}}(h_0^+, h_0^-) + C_1$$

where $d_{\mathrm{WP}}(\cdot, \cdot)$ denotes distance with respect to the Weil-Petersson metric.

We give a proof of the first property in Prop 7.1 but it can also be seen as an application of the computations of [45, Prop 8.10]; the second property follows by combining the result of Schlenker [61] and of Brock [11].

1.2. **Dimension $n + 1$ odd.** We aim to understand here to which extent the theory in dimension $2 + 1$ makes sense in higher odd dimensions. By analogy with $n = 2$, we are interested in the set $\mathcal{T}(N)$ of conformal classes of metrics on a compact manifold N of even dimension n, up to the group $\mathcal{D}_0(N)$ of diffeomorphisms isotopic to identity. This space can be defined as a quotient of the space of smooth metrics $\mathcal{M}(N)$ by the action of the semi-direct product $C^\infty(N) \rtimes \mathcal{D}_0(N)$. We assume that (N, h_0) does not admit nonzero conformal Killing vector fields, so that a neighbourhood of the image of h_0 in the quotient is an infinite dimensional Fréchet manifold (in contrast to $n = 2$, where $\dim \mathcal{T}(N) = -3\chi(N)$). Following Fefferman-Graham [24], we can view the conformal class $(N, [h_0])$ as the conformal boundary of a *Poincaré-Einstein end*, that is a cylinder $(0, \varepsilon)_x \times N$ equipped with a metric

$$g = \frac{dx^2 + h_x}{x^2}, \qquad\qquad h_x \sim_{x \to 0} \sum_{\ell=0}^{\infty} h_{x,\ell}(x^n \log x)^\ell \qquad (7)$$

where $h_{x,\ell}$ are one-parameter families of tensors on M depending smoothly on x, and satisfying the approximate Einstein equation as $x \to 0$

$$\mathrm{Ric}_g = -ng + \mathcal{O}(x^\infty).$$

The tensor $h_{x,0}$ has a Taylor expansion at $x = 0$ given by

$$h_{x,0} \sim_{x \to 0} \sum_{j=0}^{\infty} x^{2j} h_{2j}$$

where h_{2j} are formally determined by h_0 if $j < n/2$ and formally determined by the pair (h_0, h_n) for $j > n/2$; for $\ell \geq 1$, the tensors $h_{x,\ell}$ have a Taylor expansion at $x = 0$ formally determined by h_0, h_n. Like h_2 in (2), h_n is a formally undetermined tensor which satisfies some constraints equations: there exist a function T_n and a 1-form D_n, natural in terms of the tensor h_0 (see Definition 2.4), such that

$$\mathrm{Tr}_{h_0}(h_n) = T_n, \qquad\qquad \delta_{h_0}(h_n) = D_n. \qquad\qquad (8)$$

The formula for T_n, D_n is complicated and not known in general, but in principle it can be computed reccursively. An *Asymptotically Hyperbolic Einstein* (AHE) manifold is an Einstein manifold (M, g) with $\mathrm{Ric}_g = -ng$ which compactifies smoothly to some $\overline{M}$ so that there exists a smooth boundary defining function x with respect to which g has the form (7). The conformal boundary $N = \partial\overline{M}$ inherits naturally a conformal class $[x^2 g|_{TN}]$. Exactly like when $n = 2$, each conformal representative $h_0 \in [x^2 g|_{TN}]$ determines a unique geodesic boundary defining function x near N so that g has the form (7). The renormalized volume $\mathrm{Vol}_R(M, g; h_0)$ was apparently introduced by physicists [41], and appeared in [27] in the mathematics literature. We define it using a slightly different procedure from [41, 29], using the same approach as for $n = 2$ above, that is using the formula

$$\mathrm{Vol}_R(M, g; h_0) := \mathrm{FP}_{z=0} \int_M x^z \mathrm{dvol}_g; \qquad\qquad (9)$$

the function $F(z) = \int_M x^z \mathrm{dvol}_g$ has a pole at $z = 0$ with residue $\int_N v_n \mathrm{dvol}_{h_0}$, where v_n is the function appearing as the coefficient of x^n in the expansion of the volume form near N:

$$\mathrm{dvol}_g = (v_0 + v_2 x^2 + \cdots + v_n x^n + o(x^n)) dx\, \mathrm{dvol}_{h_0}, \qquad\qquad v_0 = 1. \qquad (10)$$

This method for renormalizing the volume was used for AHE manifolds e.g. in the work of Albin [2]. The quantities v_{2j} for $j \leq n/2$ are formally determined by h_0 (they are local expressions in terms of h_0), the term v_n is called a conformal anomaly in the physics literature and its integral $L := \int_N v_n \mathrm{dvol}_{h_0}$ is a conformal invariant [33]. For instance in dimension $n = 4$,

$$4v_4 = \sigma_2(\mathrm{Sch}_{h_0})$$

is the symmetric function of order 2 in the eigenvalues of the Schouten tensor $\mathrm{Sch}_{h_0} = \frac{1}{2}(\mathrm{Ric}_g - \frac{1}{6}\mathrm{Scal}_{h_0} h_0)$, see Lemma 3.9. We first show

Theorem 1.1. *Let (M, g) be an odd dimensional AHE manifold with conformal boundary N equipped with the conformal class $[h_0]$.*

(1) **Polyakov type formula**: *Under conformal change $e^{2\omega_0} h_0$, the renormalized volume $\mathrm{Vol}_R(M, g; h_0)$ satisfies*

$$\mathrm{Vol}_R(M, g; e^{2\omega_0} h_0) = \mathrm{Vol}_R(M, g; h_0) + \int_{\partial M} \sum_{j=0}^{n/2} v_{2j}(h_0)\omega_{n-2j}\, \mathrm{dvol}_{h_0}$$

where v_{2i} are the volume coefficients of (10) and ω_{2j} are polynomial expressions in ω_0 and its derivatives of order at most j.

(2) **Critical points**: The critical points of $\mathrm{Vol}_R(M, g; \cdot)$, among metrics of fixed volume in the conformal class $[h_0]$ are those metrics h_0 satisfying $v_n(h_0) = \text{constant}$.

(3) **Extrema**: Assume that $[h_0]$ contains an Einstein metric h_0 with non-zero Ricci curvature. Then h_0 is a local extremum for $\mathrm{Vol}_R(M, g; \cdot)$ in its conformal class with fixed volume: it is a maximum if $\mathrm{Ric}_{h_0} < 0$ or $n/2$ is odd, it is a minimum if $n/2$ is even. Moreover if $(N, [h_0])$ is not the sphere, then for each conformal classes $[h]$ close to $[h_0]$, there is a a metric $h \in [h]$ solving $v_n(h) = \text{constant}$ and $\mathrm{Vol}_R(M, g; h)$ is a local extremum in $[h]$ with fixed volume.

These properties are proved in Section 3. Property (2) follows directly from the discussion after [27, Th. 3.1] and is certainly known, but to be self-contained we give an elementary proof.

Following the theory in dimension $n = 2$, after choosing representatives in the conformal class satisfying the condition $v_n = \text{constant}$, it is natural to expect a correspondence between Poincaré-Einstein ends and cotangent vectors to the space $\mathcal{T}(N)$ of conformal structures (i.e. conformal classes modulo $\mathcal{D}_0(N)$). A Poincaré-Einstein end is determined by the pair (h_0, h_n). When $\mathcal{T}(N)$ (or an open subset) has a Fréchet manifold structure, we can use a symplectic reduction of the cotangent space $T^*\mathcal{M}(N)$ of the space of metrics $\mathcal{M}(N)$ by the semi-direct product $C^\infty(N) \rtimes \mathcal{D}_0(N)$, and we can identify $T^*_{[h_0]}\mathcal{T}(N)$ to the space of trace-free and divergence-free tensors on N (with respect to h_0). Unlike for $n = 2$, even after choosing a metric h_0 with $v_n(h_0) = \text{constant}$, the formally undetermined tensor h_n is neither divergence-free, nor is its trace constant. However, we show the following

Theorem 1.2. There exists a symmetric tensor F_n, formally determined by h_0, such that $G_n := -\frac{1}{4}(h_n + F_n)$ satisfies

$$\mathrm{Tr}_{h_0}(G_n) = \tfrac{1}{2}v_n, \qquad\qquad \delta_{h_0}(G_n) = 0. \qquad\qquad (11)$$

(4) **Cotangent vectors as ends**: Assume that there exists an open set $\mathcal{U} \subset \mathcal{T}(N)$ and a smooth Fréchet submanifold $\mathcal{S}_0 \subset \mathcal{M}(N)$ of metrics h_0 solution to $v_n(h_0) = \int_N v_n(h_0)\mathrm{dvol}_{h_0}$ so that the projection $\pi : \mathcal{M}(N) \to \mathcal{T}(N)$ is a homeomorphism from $\mathcal{S}$ to $\mathcal{U}$. Then there is a bijection between the space of Poincaré-Einstein ends with $h_0 \in \mathcal{S}$ and the space $T^*_{\mathcal{U}}\mathcal{T}(N)$ given by $(h_0, h_n) \mapsto (h_0, G_n^\circ)$, where G_n° is the trace-free part of G_n.

The existence of a slice $\mathcal{S}_0$ is proved for instance in Corollary 4.5 in a neighbourhood of a conformal class containing an Einstein metric which is not the sphere. We have learnt from Robin Graham that there is a result related to the first part of the Theorem about G_n in the physics literature [20], although the renormalization for the volume seems a bit different from ours.

As for $n = 2$, we can define the Cauchy data for the Einstein equation to be (h_0, G_n°) where h_0 solves $v_n(h_0) = \int_N v_n(h_0)\mathrm{dvol}_{h_0}$. We may ask if those Cauchy data which are ends of AHE manifolds span a Lagrangian submanifold of $T^*\mathcal{T}(N)$.

Theorem 1.3. *Assume that there is a smooth submanifold $\mathcal{S}_0 \subset \mathcal{M}(N)$ of metrics h_0 solving $v_n(h_0) = \int_N v_n(h_0)\mathrm{dvol}_{h_0}$ so that the projection $\pi : \mathcal{M}(N) \to \mathcal{T}(N)$ is a homeomorphism from $\mathcal{S}_0$ to $\mathcal{U}$. Let $\overline{M}$ be a smooth manifold with boundary N and assume that there is a smooth map $\Phi : \mathcal{S}_0 \to \mathcal{M}(M)$ such that $\mathrm{Ric}_{\Phi(h_0)} = -n\Phi(h_0)$ and $[x^2\Phi(h_0)|_N] = [h_0]$ for some boundary defining function x.*

(5) ***Lagrangian submanifold****: The set $\mathcal{L}$ of Cauchy data (h_0, G_n°) of the AHE metrics $\Phi(h_0)$ with $h_0 \in \mathcal{S}_0$ is a Lagrangian submanifold in $T^*\mathcal{T}(N)$ with respect to the canonical symplectic structure.*

(6) ***Generating function****: $\mathcal{L}$ is the graph of the exact 1-form given by the differential of $h_0 \mapsto \mathrm{Vol}_R(M, \Phi(h_0); h_0)$ over $\mathcal{S}_0$. More precisely, if $\dot{h}_0 \in T_{h_0}\mathcal{S}_0$ is a variation of metrics satisfying $v_n = \int_N v_n$ and $h_0 \in \mathcal{S}_0$, then*

$$d\mathrm{Vol}_R(M, \Phi(h_0), h_0).\dot{h}_0 = \int_N \langle G_n^\circ, \dot{h}_0\rangle\mathrm{dvol}_{h_0}. \tag{12}$$

Here, what we mean by Lagrangian is an isotropic submanifold such that the projection on the base is a diffeomorphism. In Section 6, we show that the assumptions of Theorem 1.3 are satisfied for instance in a neighbourhood of what we call a Fuchsian-Einstein manifold, in a way similar to the quasifuchsian metrics near a Fuchsian metric. A Fuchsian-Einstein metric is a product $M = \mathbb{R}_t \times N$ with a metric $g^0 := dt^2 + \cosh^2(t)\gamma$ where γ is a metric on N such that $\mathrm{Ric}_\gamma = -(n-1)\gamma$ and the sectional curvatures of γ are non-positive. By Corollary 4.5, near an Einstein metric γ on a compact manifold N with negative Ricci curvature, there is a smooth slice $\mathcal{S}_0 \subset \mathcal{M}(N)$ of metrics solution to $v_n = \int_N v_n$ and so that the projection $\pi : \mathcal{M}(N) \to \mathcal{T}(N)$ is a homeomorphism from $\mathcal{S}_0$ to a neighbourhood $\mathcal{U}$ of $[h_0]$. Using a result by Lee [46], and possibly after taking an open subset of $\mathcal{S}_0$ instead of $\mathcal{S}_0$, for each pair $(h_0^+, h_0^-) \in \mathcal{S}_0 \times \mathcal{S}_0$ there exists an AHE metric $g = \Phi(h_0^+, h_0^-)$ satisfying

$$\mathrm{Ric}_g = -ng \text{ on } M, \qquad \Phi(\gamma, \gamma) = g^0, \qquad [x^2g|_{t=\pm\infty}] = [h_0^\pm] \text{ for } x := e^{-|t|}.$$

For each of the two ends ($t \to \pm\infty$) we have a traceless symmetric 2-tensor $G_n^{\circ\pm}$. We denote $\mathcal{G}_n^{\circ\pm} := G_n^{\circ\pm} \otimes \mathrm{dvol}_{h_0^\pm}$, and consider $\mathcal{G}_n^{\circ\pm}$ as a vector in $T_{h_0^\pm}^*\mathcal{T}(N)$: if $\dot{h}_0^\pm \in T_{h_0^\pm}\mathcal{T}(N)$ are symmetric 2-tensors on N, then

$$\mathcal{G}_n^{\circ+}(\dot{h}_0^+) = \int_N \langle \mathcal{G}_n^{\circ+}, \dot{h}_0^+\rangle_{h_0^+} := \int_N \langle G_n^{\circ+}, \dot{h}_0^+\rangle_{h_0^+}\mathrm{dvol}_{h_0^+}$$

and similarly for $\dot{h}_0^-$.

Theorem 1.4. *Fix $h_0^+, h_0^- \in \mathcal{S}_0$ and consider the linear maps*

$$\phi_{h_0^+} : T_{h_0^-}\mathcal{S}_0 \to T_{h_0^+}^*\mathcal{T}(N), \qquad\qquad \phi_{h_0^-} : T_{h_0^+}\mathcal{S}_0 \to T_{h_0^-}^*\mathcal{T}(N)$$

defined as

$$\phi_{h_0^+} : \dot{h}_0^- \mapsto (d\mathcal{G}_n^{\circ+})_{(h_0^-, h_0^+)}(\dot{h}_0^-, 0)$$
$$\phi_{h_0^-} : \dot{h}_0^+ \mapsto (d\mathcal{G}_n^{\circ-})_{(h_0^-, h_0^+)}(0, \dot{h}_0^+)$$

where $\mathcal{G}_n^{\circ\pm}$ and its variation are obtained using the AHE metrics $g = \Phi(h_0^+, h_0^-)$. Then

(7) **Quasifuchsian reciprocity**: *The linear maps $\phi_{h_0^+}$ and $\phi_{h_0^-}$ are adjoint.*

A more general statement holds, which states that $(d\mathcal{G}_n^\circ)_{h_0}$ is self-adjoint if $\mathcal{G}_n^\circ$ is the cotangent data coming from an Einstein filling in the bulk M and $d\mathcal{G}_n^\circ$ is its linearisation (see Corollary 6.12).

Finally, we study the second variation of $h_0 = (h_0^+, h_0^-) \mapsto \mathrm{Vol}_R(M, \Phi(h_0), h_0)$ at the Fuchsian-Einstein metric, i.e., when $\Phi(h_0) = g^0$.

Theorem 1.5. *In the setting of Theorem 1.4, consider the function $\mathrm{Vol}_R : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ defined by $\mathrm{Vol}_R(h_0) := \mathrm{Vol}_R(M, \Phi(h_0), h_0)$ for $h_0 = (h_0^+, h_0^-) \in \mathcal{S} \times \mathcal{S}$, and set $n = 4$. Assume that $L_\gamma - 2 > 0$ where $L_\gamma := \Delta_\gamma - 2\mathring{R}_\gamma \geq 0$ is the linearized Einstein operator at γ acting on divergence-free, trace-free tensors (see Section 7 for precise definition).*

(8) **Hessian at the Fuchsian-Einstein locus**: *The point $h_0 = (\gamma, \gamma)$ is a critical point for Vol_R, i.e., $d\mathrm{Vol}_R(h_0) = 0$ on $T_\gamma\mathcal{S} \times T_\gamma\mathcal{S}$, the Hessian at (γ, γ) is positive in the sense that there exists $c_0 > 0$ such that for all $\dot{h}_0 = (\dot{h}_0^+, \dot{h}_0^-) \in T_\gamma\mathcal{S} \times T_\gamma\mathcal{S}$ with $\delta_\gamma(\dot{h}_0^\pm) = 0$*
$$\mathrm{Hess}_{h_0}(\mathrm{Vol}_R)(\dot{h}_0, \dot{h}_0) \geq c_0 ||\dot{h}_0||^2_{H^2(N)}$$
where $H^2(N)$ is the L^2-based Sobolev space of order 2.

The lower bound $L_\gamma - 2 > 0$ is for instance satisfied if γ has constant sectional curvature -1 and $\ker d_D = \ker d_D^* = 0$ where d_D is the exterior derivative on T^*N-valued 1-forms and d_D^* its adjoint. In Proposition 7.6, we compute the Hessian explicitly: the quadratic form acting on divergence-free tensors tangent to $\mathcal{S} \times \mathcal{S}$ is given by a self-adjoint linear elliptic pseudo-differential operator $\mathcal{H}$, $\mathrm{Hess}_{(\gamma,\gamma)}(\mathrm{Vol}_R)(\dot{h}_0, \dot{h}_0) = \langle \mathcal{H}\dot{h}_0, \dot{h}_0 \rangle_{L^2}$, and $\mathcal{H}$ is in fact a function of L_γ (the condition $\dot{h}_0^\pm \in T_\gamma\mathcal{S}$ and $\delta_\gamma(\dot{h}_0^\pm) = 0$ actually implies that $\mathrm{Tr}_\gamma(\dot{h}_0^\pm) = 0$). If $L_\gamma - 2$ has non-positive eigenvalues, the same result remains true along deformations orthogonal to (the finite dimensional) range of $\mathbb{1}_{\mathbb{R}_-}(L_\gamma - 2)$.

The equivalent of the Kähler potential property valid in dimension $n = 2$ does not seem to extend to general dimensions without additional geometric assumptions.

1.3. **Dimension $n+1$ even.** When $n+1$ is even, the renormalized volume defined by (9) has been more extensively studied. It is also an interesting quantity, more tractable than in the case $n+1$ odd, but has quite different properties. For instance it is related to the Chern-Gauss-Bonnet formula and does not depend on the choice of conformal representative h_0 (i.e., it is independent of the geodesic boundary-defining function x). Anderson [3] gave a formula when $n+1 = 4$ for $\mathrm{Vol}_R(M, g)$ in terms of the L^2 norm of the Weyl tensor and the Euler characteristic $\chi(M)$ if g is AHE. This was extended by Chang-Qing-Yang [18] in higher dimensions (see also Albin [2] for Chern-Gauss-Bonnet formula), while Epstein [57, Appendix A] proved that for convex co-compact hyperbolic manifolds it equals a constant times $\chi(M)$. When $n = 4$, Chang-Qing-Yang [17] also proved a rigidity theorem if the renormalized volume is pinched enough near that of hyperbolic space $\mathbb{H}^4$. As for variations, Anderson [3] and Albin [2] proved that the derivative of the renormalized volume for AHE metrics is given by the formally undetermined tensor $-\frac{1}{4}h_n$, see Theorem 5.2. A byproduct

of our computation in Section 7 is a formula for the Hessian of the renormalized volume when $n + 1$ is even, at a Fuchsian-Einstein metric, see expresion (87) in Proposition 7.6.

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2. MODULI SPACE OF CONFORMAL STRUCTURES AND POINCARÉ-EINSTEIN MANIFOLDS

2.1. **Spaces of metrics and conformal structures.** We use the notions of tame Fréchet manifold and Fréchet Lie groups as in Hamilton [40]. Let N be a compact smooth manifold of dimension n, and $\mathcal{M}(N)$ the set of Riemannian metrics on N. This set is an open convex subset in the Fréchet space $C^\infty(N, S^2 N)$ of symmetric smooth 2-tensors on N. It has a tautological non-complete Riemannian metric given on $T_h \mathcal{M}(N) = C^\infty(N, S^2 N)$ by the L^2 product with respect to $h \in \mathcal{M}(N)$:

$$\langle k_1, k_2 \rangle := \int_N \langle k_1, k_2 \rangle_h \mathrm{dvol}_h, \qquad k_1, k_2 \in T_h \mathcal{M}(N)$$

where $\langle k_1, k_2 \rangle_h = \mathrm{Tr}(h^{-1} k_1 h^{-1} k_2)$ is the scalar product on $S^2 N$ induced naturally by h (here $K = h^{-1} k$ means the symmetric endomorphism defined by $h(K\cdot, \cdot) = k$). Let $\mathcal{D}(N)$ be the group of smooth diffeomorphisms of N and $\mathcal{D}_0(N)$ the connected component of the identity. The groups $\mathcal{D}_0(N)$ and $C^\infty(N)$ are Fréchet Lie groups, the latter being in fact a Fréchet vector space. Consider the map

$$\Phi : C^\infty(N) \times \mathcal{D}_0(N) \times \mathcal{M}(N) \to \mathcal{M}(N), \qquad (f, \phi, h) \mapsto e^{2f} (\phi^{-1})^* h.$$

This map defines an action of the semi-direct product $\mathcal{G} := C^\infty(N) \rtimes \mathcal{D}_0(N)$ on $\mathcal{M}(N)$, and this action is smooth and proper if N is not the sphere S^n (see Ebin [22], Fischer-Moncrief [25]). The isotropy group at a metric h for the action Φ is the group of conformal diffeomorphism of (N, h) isotopic to the Identity; by Obata [55] it is compact if N is not the sphere.

Definitions. The object studied in this paper is the moduli space of conformal structures on N (called *quantum conformal superspace* in physics), denoted by

$$\mathcal{T}(N) := \mathcal{G}\backslash\mathcal{M}(N). \tag{13}$$

This space is the Teichmüller space when $n = 2$ and N has negative Euler characteristic. In higher dimension, it is infinite dimensional and has a complicated structure near general metrics. In [25], Fischer-Moncrief describe the structure of $\mathcal{T}(N)$: they show for instance that it is a smooth Inverse Limit Hilbert orbifold if the degree of symmetry of N is 0 (the isotropy group is then finite). Moreover, if the action is proper and the isotropy group at a metric h is trivial, then a neighbourhood of $[h]$ in $\mathcal{T}(N)$ is a Fréchet manifold. By a result of Frenkel [26], the isotropy group is trivial if $h \in \mathcal{M}(N)$ is a metric of negative Ricci

curvature and non-positive sectional curvatures. An equivalent way to define $\mathcal{T}(N)$ is to consider $\mathcal{D}_0(N)\backslash\mathcal{C}(N)$, where

$$\mathcal{C}(N) := C^\infty(N)\backslash\mathcal{M}(N) \tag{14}$$

is the space of conformal classes of metrics on N.

Slices. Since we will use this later, let us describe the notion of *slice* introduced by Ebin [22] in these settings. We will say that $\mathcal{S} \subset \mathcal{M}(N)$ is a slice at $h_0 \in \mathcal{M}(N)$ for the conformal action of $C^\infty(N)$ if it is a tame Fréchet submanifold such that there is a neighbourhood U of 0 in $C^\infty(N)$ and a neighbourhood $V \subset \mathcal{M}(N)$ of h_0 such that

$$\Psi : U \times \mathcal{S} \to V, \qquad\qquad (f,h) \mapsto e^{2f}h \tag{15}$$

is a diffeomorphism of Fréchet manifolds. Since the action of $C^\infty(N)$ on $\mathcal{M}(N)$ is free and proper, it is easy to see that Ψ extends to $C^\infty(N) \times \mathcal{S} \to \mathcal{M}(N)$ and is injective. In other words, $\mathcal{S}$ defines a tame Fréchet structure on $\mathcal{C}(N)$ near the conformal class $[h_0]$. Similarly, if $\mathcal{S} \subset \mathcal{M}(N)$ is a Fréchet submanifold containing h_0, on which a neighbourhood $U \subset \mathcal{D}_0(N)$ of Id acts smoothly, then a Fréchet submanifold $\mathcal{S}_0$ of $\mathcal{S}$ is a slice at h_0 for the action of $\mathcal{D}_0(N)$ if there exists a neighbourhood $V \subset \mathcal{S}$ of h_0 such that

$$\Phi : U \times \mathcal{S}_0 \to V, \qquad\qquad (\phi,h) \mapsto (\phi^{-1})^*h \tag{16}$$

is a diffeomorphism of Fréchet manifolds. Extending Φ to $\mathcal{D}_0(N) \times \mathcal{S}_0 \to \mathcal{M}(N)$, and assuming that the action of $\mathcal{D}_0(N)$ on $\Phi(\mathcal{D}_0(N) \times \mathcal{S}_0)$ is free and proper, the extension of Φ is injective in a small neighbourhood of h_0 in $\mathcal{S}_0$. If $\mathcal{S}$ was a slice for the conformal action, then $\mathcal{S}_0$ is a slice at h_0 for the action of $\mathcal{G}$ on $\mathcal{M}(N)$, thus giving a tame Fréchet structure on $\mathcal{T}(N)$ near the class of h_0 in $\mathcal{T}(N)$.

Cotangent bundles. The tangent bundle $T\mathcal{M}(N)$ over $\mathcal{M}(N)$ is the trivial Fréchet bundle $\mathcal{M}(N) \times C^\infty(N, S^2N)$. For each base point h, which by definition is a Riemannian metric on N, we can identify symmetric 2-vectors with symmetric bilinear forms, so that elements of the topological dual $T^*\mathcal{M}_h(N)$ can be described as distributional sections of $S^2N \otimes \Omega^n N$. Such spaces of distributions are not Fréchet manifolds.

In this work, we are interested in C^∞ objects and Fréchet manifolds, we thus define the *smooth cotangent space* $T^*_h\mathcal{M}(N)$ to be the vector space of continuous linear forms on $T_h\mathcal{M}(N)$ which are represented by smooth tensors through the h pairing followed by integration on N:

$$k^* \in T^*_h\mathcal{M}(N) \text{ if } \exists k \in C^\infty(N, S^2N \otimes \Omega^n N),\ \forall v \in T_h\mathcal{M}(N),\ k^*(v) = \int_N \langle k, v \rangle_h.$$

This identifies the smooth cotangent bundle $T^*\mathcal{M}(N)$ with $T\mathcal{M}(N) = \mathcal{M}(N) \times C^\infty(N, S^2N \otimes \Omega^n N)$, making it a Fréchet bundle.

There exists a symplectic form Ω on $T^*\mathcal{M}(N)$, derived from the Liouville canonical 1-form:

$$\Omega_{(h,k)}((\dot{h}_1, \dot{k}_1), (\dot{h}_2, \dot{k}_2)) = \int_N \langle \dot{k}_1, \dot{h}_2 \rangle_h - \langle \dot{k}_2, \dot{h}_1 \rangle_h . \tag{17}$$

The group $\mathcal{G}$ acts on $T^*\mathcal{M}(N)$, with a symplectic action induced from the base and using the Riemannian metric on $\mathcal{M}(N)$:

$$(f, \phi) : (h, k) \mapsto \left(e^{2f} (\phi^{-1})^* h, e^{2f} (\phi^{-1})^* k \right). \tag{18}$$

We then define (locally) the cotangent bundle to $\mathcal{T}(N)$. We will always assume that there is a slice $\mathcal{S}_0$ at h_0 representing a neighbourhood $U \subset \mathcal{T}(N)$ of the class $[h_0]$, as we just explained. The tangent space $T_{[h]}\mathcal{T}(N)$ at a point $[h] \in \mathcal{T}(N)$ near $[h_0]$ is then identified with $T_h \mathcal{S}_0$ where h is the representative of $[h]$ in $\mathcal{S}_0$, and $T\mathcal{T}(N)$ is then locally represented near $[h_0]$ as a Fréchet subbundle of $T_{\mathcal{S}_0}\mathcal{M}(N)$. We define the *smooth cotangent space* $T^*_{[h]}\mathcal{T}(N)$ to be the vector space of continuous linear forms on $T_h \mathcal{S}_0 \simeq T_{[h]}\mathcal{T}(N)$ which are represented by smooth sections of $S^2 N \otimes \Omega^n N$ through the L^2 pairing and vanish on the tangent space of the orbit $\mathcal{G}_h$ of h by the group $\mathcal{G}$:

$$k^* \in T^*_{[h]}\mathcal{T}(N) \text{ if } \exists k \in C^\infty(S^2 N \otimes \Omega^n N), \, \forall v \in T_h \mathcal{S}_0, \ k^*(v + T_h \mathcal{G}_h) = \int_N \langle k, v \rangle_h \ .$$

Since $T_h \mathcal{G}_h = \{ L_X h + fh; X \in C^\infty(N, TN), f \in C^\infty(N) \}$ (where $L_X h$ is the Lie derivative), k must satisfy

$$\int_N \langle k, L_X h + fh \rangle_h = 0, \qquad\qquad \forall X \in C^\infty(N, TN), f \in C^\infty(N),$$

which is equivalent to asking that $k = k' \otimes \mathrm{dvol}_h$, with $\delta_h(k') = 0$ and $\mathrm{Tr}_h(k') = 0$. The smooth cotangent bundle $T^*\mathcal{T}(N)$ over a neighbourhood $U \subset \mathcal{T}(N)$ of $[h_0]$ represented by a slice $\mathcal{S}_0$ is then

$$T^*_U \mathcal{T}(N) = \{ (h, k \otimes \mathrm{dvol}_h) \in \mathcal{S}_0 \times C^\infty(N, S^2 N \otimes \Omega^n N); \delta_h(k) = 0, \mathrm{Tr}_h(k) = 0 \}. \tag{19}$$

Lemma 2.1. *Assume that h has no conformal Killing vector fields for all $h \in \mathcal{S}_0$. The space $T^*_U \mathcal{T}(N)$ is a Fréchet subbundle of $T_{\mathcal{S}_0}\mathcal{M}(N)$, therefore a Fréchet bundle over $\mathcal{S}_0$.*

Proof. We are going to exhibit a trivialisation of the fiber bundle defined by (19). Define

$$\Phi_h : C^\infty(N, S^2 N) \to C^\infty(N, TN \oplus \mathbb{R}), \qquad \Phi_h(k) = (\delta_h k + \tfrac{1}{n} d \, \mathrm{Tr}_h(k), \mathrm{Tr}_h(k)).$$

Evidently, $\ker \Phi_h = T^*_h \mathcal{T}(N)$. The formal adjoint of the differential operator Φ_h is

$$\Phi^*_h(\sigma, f) = \delta^*_h \sigma + (\tfrac{1}{n} d^* \sigma + f) h.$$

Since h is a metric without conformal Killing vector fields, Φ^*_h is injective. The projector on the kernel of Φ_h is $P_h := 1 - \Phi^*_h (\Phi_h \Phi^*_h)^{-1} \Phi_h$. We claim that $P_{h_0} : T^*_h \mathcal{T}(N) \to T^*_{h_0} \mathcal{T}(N)$ is a tame isomorphism. Let us check that it is indeed a tame family of 0-th order pseudo-differential operators. In matrix form, the operator $\Phi_h \Phi^*_h$ is

$$\Phi_h \Phi^*_h = \begin{bmatrix} \delta_h \delta^*_h - \frac{1}{n} dd^* & 0 \\ 0 & n \end{bmatrix}$$

where $n = \mathrm{Tr}_h(h)$ is the dimension of N. This operator acts on mixed Sobolev spaces as follows: $\Phi_h \Phi_h^* : H^s(N, TN) \times H^s(N) \to H^{s-2}(N, TN) \times H^s(N)$ for every $s \in \mathbb{R}$. The self-adjoint operator $A_h := \delta_h \delta_h^* - \frac{1}{n} dd^*$ is elliptic and invertible and thus has a tame family of pseudo-differential inverses of order -2 (see [40, Section II.3.3]). Then the inverse of $\Phi_h \Phi_h^*$ is also invertible and tame. In particular, we see that P_h is smooth tame family of pseudodifferential operators of order 0. We have that $P_h P_{h_0} : T_h^* \mathcal{T}(N) \to T_h^* \mathcal{T}(N)$ and $P_{h_0} P_h : T_{h_0}^* \mathcal{T}(N) \to T_{h_0}^* \mathcal{T}(N)$ are invertible for h close to h_0 in some Sobolev norm, since they are the Identity when $h = h_0$ and by the Calderón-Vaillancourt theorem; moreover the inverse are tame, by the results of [58, Th. 4.5]. This gives the desired trivialisation. $\qquad\square$

To obtain a local description of $T^* \mathcal{T}(N)$ which is independent of the choice of slice, it is necesary and sufficient that for another choice of metric $\hat{h} = (f, \phi).h$ in the orbit $\mathcal{G}_h$, the new representative for k becomes $e^{(2-n)f}(\phi^{-1})^* k$, which is indeed a divergence-free/trace-free tensor with respect to $\hat{h}$ (note that $(h, k \otimes \mathrm{dvol}_h)$ transforms as $e^{2f}((\phi^{-1})^* h, (\phi^{-1})^*(k \otimes \mathrm{dvol}_h))$, but this means that k transforms as $e^{(2-n)f}(\phi^{-1})^* k$). In a small neighbourhood of $[h_0] \in \mathcal{T}(N)$, we can therefore identify $T^* \mathcal{T}(N)$ with the quotient

$$\mathcal{G} \backslash \{(h, k) \in \mathcal{M}(N) \times C^\infty(N, S^2 N); \mathrm{Tr}_h(k) = 0, \delta_h(k) = 0\} \qquad (20)$$

where the group action of $\mathcal{G}$ is (18). The action of $\mathcal{G}$ is Hamiltonian, and $T^* \mathcal{T}(N)$ is the symplectic reduction of $T^* \mathcal{M}(N)$, where the moment map is given at $(f, v) \in C^\infty(N) \times C^\infty(N, TN) = \mathrm{lie}(\mathcal{G})$ in terms of the L^2 inner product with respect to h by

$$\mu_{(f,v)}(k) = \langle -2\mathrm{tr}_h(k), f \rangle + \langle \delta_h(k), v \rangle, \qquad\qquad k \in T_h^* \mathcal{M}(N).$$

Therefore the zero set of the moment map is exactly the space appearing in (20) before quotienting. Finally, the symplectic form Ω descends to $T^* \mathcal{T}(N)$.

2.2. Asymptotically hyperbolic Einstein manifolds.

The reader can find more details about the theory of this section in the books [23, 43, 24].

Definition 2.2. Let $\overline{M}^{n+1}$ be a compact smooth manifold with boundary, and $M \subset \overline{M}$ its interior. A metric g on M is called *asymptotically hyperbolic Einstein* (or AHE) if $\mathrm{Ric}_g = -ng$ and if there exists a smooth boundary defining function $x : \overline{M} \to [0, \infty)$ such that, in a collar neighbourhood of $\partial \overline{M}$ induced by x, g is of the form

$$g = x^{-2}(dx^2 + h_x) \qquad (21)$$

where h_x is a continuous family of smooth metrics on $N := \partial M$, depending smoothly on the variable x when n is odd, and on the variables $x, x^n \log x$ when n is even. The conformal class $[h_0]$ of h_0 on $\partial \overline{M}$ (which is independent of the choice of x) is called the *conformal infinity of* (M, g).

By a collar neighbourhood induced by x we mean a diffeomorphism $\Phi : [0, \varepsilon)_t \times \partial \overline{M} \to \overline{M}$ onto its image, such that $\Phi^*(x) = t$, $\Phi(0, \cdot) = \mathrm{Id}_{\partial \overline{M}}$ and the meaning of (21) is $\Phi^* g = (dt^2 + h_t)/t^2$ on $(0, \varepsilon)_t \times \partial \overline{M}$.

In particular, AHE metrics are smooth on M and of class C^{n-1} on $\overline{M}$. In even dimension, the definition with the regularity statement is justified by the result of Chrusciel-Delay-Lee-Skinner [19], which states that an Einstein metric on a conformally compact C^2 manifold with smooth conformal infinity admits an expansion at the boundary in integral powers of x and $x^n \log x$. We notice that the sectional curvatures of a AHE metric are $-1 + \mathcal{O}(x)$ and that the metric g is complete.

In this paper we will be essentially interested in the more complicated case where n is even (so that the dimension of M is odd) but at the moment we do not fix the parity of n.

We say that a function f is *polyhomogeneous conormal* (with integral index set) on M if it is smooth in M and for all $J \in \mathbb{N}$, f has an expansion at $\partial \overline{M}$ of the form:

$$f = \sum_{j=0}^{J} \sum_{\ell=0}^{\ell_j} x^j \log(x)^\ell f_{j,\ell} + o(x^J)$$

where $f_{j,\ell} \in C^\infty(\partial \overline{M})$ and x is a smooth boundary defining function. The same definition applies to tensors on M. There are natural topologies of Fréchet space for polyhomogeneous conormal functions or tensors; we refer to [52, Chap 4] and [51] for details and properties of these conormal polyhomogeneous spaces.

2.3. **Poincaré-Einstein ends.** There is a weaker notion of metric that will prove useful, that of Poincaré-Einstein metrics, introduced by Fefferman-Graham [24]. Let (M, g) be an $(n + 1)$-dimensional asymptotically hyperbolic Einstein manifold. Since by [19], the metric g in a collar $(0, \varepsilon)_x \times \partial M$ induced by x near ∂M has an expansion of the form

$$g = \frac{dx^2 + h_x}{x^2}, \qquad\qquad h_x \sim_{x \to 0} \sum_{\ell=0}^{\infty} h_{x,\ell} (x^n \log x)^\ell \tag{22}$$

where $h_{x,\ell}$ are one-parameter families of tensors on M depending smoothly on x, we want to define the asymptotic version of AHE manifolds:

Definition 2.3. An *Poincaré-Einstein end* is a half-cylinder $\mathcal{Z} = [0, \varepsilon) \times N$ equipped with a smooth metric g on $(0, \varepsilon) \times N$ with an expansion of the form (22) near $x = 0$, such that $\mathrm{Ric}_g + ng = \mathcal{O}(x^\infty)$. If $(\mathcal{Z}, g)$ is Einstein, we call it an *exact Poincaré-Einstein end*.

In [24], Fefferman and Graham analyze the properties of Poincaré-Einstein ends. To explain their results we need the notion of formally determined tensors.

2.4. **Formally determined tensors.**

Definition 2.4. Let N be a compact manifold, and $m, \ell \in \mathbb{N}_0$. A map $F : \mathcal{M}(N) \to C^\infty(M, (T^*M)^\ell)$ from metrics on N to covariant ℓ-tensors is said to be natural of order m (and the tensor $F(h_0)$ is said to be formally determined by h_0 of order $m \in \mathbb{N}$) if there exists a tensor-valued polynomial P in the variables h_0, h_0^{-1}, $\sqrt{\det(h_0)}$, $\partial^\alpha h_0$ with $|\alpha| \le m$, so that in any local coordinates y

$$F(h_0) = P(h_0, h_0^{-1}, \sqrt{\det(h_0)}, \partial_y^\alpha h_0).$$

Remark 2.5. A formally determined tensor $F(h_0)$ is preserved by local isometries: if $\phi : U \to U'$ is a diffeomorphism where U, U' are open sets of Riemannian manifolds N, N' and h_0, h_0' are metrics on U, U' then if $h_0 = \phi^* h_0'$ on U, we get $F(h_0) = \phi^* F(h_0')$ on U. As a consequence, a formally determined tensor is 0 if it vanishes for all metrics on the sphere S^n.

Lemma 2.6. Let h_0^t be a smooth one-parameter family of metrics on N with $h_0^t = h_0 + t\dot{h}_0 + \mathcal{O}(t^2)$ at $t = 0$, and let $P(h_0^t), Q(h_0^t)$ be tensors formally determined by h_0^t of respective order p, q. There exists a formally determined tensor $R(h_0)$ in h_0 of order $r = p + q$ such that

$$\langle \partial_t P(h_0^t)|_{t=0}, Q(h_0) \rangle_{L^2(N,h_0)} = \langle \dot{h}_0, R(h_0) \rangle_{L^2(N,h_0)}.$$

Proof. By using a partition of unity we can assume that h_0^t has support in a coordinate domain. Then $\partial_t P(h_0^t)|_{t=0}$ is a polynomial in the variables $\partial_y^\beta \dot{h}_0$, h_0, h_0^{-1}, $\sqrt{\det(h_0)}$, $\partial_y^\alpha h_0$, linear in $\dot{h}_0$. Integrating by parts with respect to the coordinates y_j it is clear that there exists a polynomial R such that

$$\langle \partial_t P(h_0^t)|_{t=0}, Q(h_0) \rangle_{L^2(N,h_0)} = \langle \dot{h}_0, R(h_0(y)) \rangle_{L^2(N,h_0)}. \tag{23}$$

The polynomial R is the same for different coordinate systems. To see that it defines a formally determined tensor, we need to prove that the 2-tensor $R(h_0(y))$ is independent of the coordinate system y. This follows from the identity (23) since $\dot{h}_0$ is arbitrary, and all the terms except $R(h_0(y))$ are known to be intrinsically defined. $\square$

Proposition 2.7 (**Fefferman-Graham** [24]). *Let $(\mathcal{Z}, g)$ be a Poincaré-Einstein end. Using the expansion (22), define $h_j = \frac{1}{j!}\partial_x^j h_{x,0}|_{x=0}$ and $k := h_{x,1}|_{x=0}$. Then the following hold:*

(1) When n is odd, $h_{x,\ell} = 0$ when $\ell \geq 1$.
(2) The tensors h_{2j+1} are 0 for $2j + 1 < n$.
(3) The tensors h_{2j} for $j < n/2$ and k are formally determined by h_0, of order $2j$.
(4) The tensors h_{2j} for $j > n/2$ are formally determined by h_0 and h_n.
(5) The trace $\text{Tr}_{h_0}(h_n)$ depends only on h_0 and defines a formally determined function $T_n = T_n(h_0)$ of order n, which is zero for n odd.
(6) The divergence $\delta_{h_0}(h_n)$ depends only on h_0 and not on h_n, and defines a formally determined tensor $D_n = D_n(h_0)$ of order $n + 1$ which is zero for n odd.
(7) The tensor k, called *obstruction tensor*, is trace- and divergence-free with respect to h_0.
(8) All coefficients in the Taylor expansion at $x = 0$ of $h_{x,\ell}$ for $\ell \geq 1$ are formally determined by h_0 and h_n.

A consequence of this is the expansion for h_x

$$h_x = h_0 + h_2 x^2 + \cdots + k x^n \log(x) + h_n x^n + o(x^n). \tag{24}$$

This proposition follows (not directly though) from the decomposition of the Ricci tensor of g in terms of h_x in the collar neighbourhood $\mathcal{Z}$. Since we shall use it later, we recall some standard computations of Ricci curvatures on a generalized cylinder, see e.g. [4]. On $M := \mathbb{R} \times N$ consider a metric $g = dt^2 + g_t$ and let

$$\text{II} := -\tfrac{1}{2}\partial_t g_t = g_t(W\cdot, \cdot), \qquad\qquad W := g_t^{-1}\text{II}$$

be the second fundamental form, respectively the Weingarten operator. Set $\nu = \partial_t$ the unit normal vector field to the level hypersurfaces $\{t = \text{constant}\}$. Then, for U, V tangent vectors to N, the Ricci tensor of g is described by

$$\text{Ric}_g(\nu, \nu) = \text{tr}(W^2) - \tfrac{1}{2}\text{tr}(g_t^{-1}\partial_t^2 g_t),$$
$$\text{Ric}_g(\nu, V) = V(\text{tr}(W)) + \langle \delta_{g_t} W, V \rangle \tag{25}$$
$$\text{Ric}_g(U, V) = \text{Ric}_{g_t}(U, V) + 2\langle W(U), W(V) \rangle - \text{tr}(W)\langle W(U), V \rangle - \tfrac{1}{2}\partial_t^2 g_t(U, V).$$

Using these equations, the Einstein equation $\text{Ric}_g = -ng$ for $g = x^{-2}(dx^2 + h_x)$ can be restated using the variable $t = e^x$ in terms of the 1-parameter family of endomorphisms A_x defined by

$$A_x := h_x^{-1}\partial_x h_x = 2x^{-1}(1 - W) \tag{26}$$

as follows:

$$\partial_x \text{Tr}(A_x) + \tfrac{1}{2}|A_x|^2 = x^{-1}\text{Tr}(A_x), \tag{27}$$
$$\delta_{h_x}(\partial_x h_x) = -d\text{Tr}(A_x),$$
$$x\partial_x A_x + (1 - n + \tfrac{1}{2}x\text{Tr}(A_x))A_x = 2xh_x^{-1}\text{Ric}(h_x) + \text{Tr}(A_x)\text{Id}.$$

The same equations are valid modulo x^∞ on asymptotic Poincaré-Einstein ends.

The coefficients in the asymptotic expansion of h_x in (24) near $\{x = 0\}$ can be recursively computed from h_0 until the n-term, and the dependence is local: one has the following formulas

(1) In dimension $n = 2$, the obstruction tensor k is 0, and the coefficient h_2 can be any symmetric tensor satisfying (see [24, Th 7.4])

$$\text{Tr}_{h_0}(h_2) = -\tfrac{1}{2}\text{Scal}_{h_0}, \qquad\qquad \delta_{h_0}(h_2) = \tfrac{1}{2}d\,\text{Scal}_{h_0}. \tag{28}$$

(2) In dimension $n > 2$, the tensors h_2 is minus the Schouten tensor of h_0 and in dimension $n > 4$, h_4 is expressed in terms of Schouten and Bach tensors of h_0 (see [24, Eq (3.18)]):

$$-h_2 = \text{Sch}_{h_0} := \tfrac{1}{n-2}\left(\text{Ric}_{h_0} - \tfrac{1}{2(n-1)}\text{Scal}_{h_0}h_0\right) \qquad h_4 = \tfrac{1}{4}\left(h_2^2 - \tfrac{1}{n-4}B_{h_0}\right) \tag{29}$$

where B_{h_0} is the Bach tensor of h_0 if $n > 4$ and $h_2^2(\cdot, \cdot) := h_0(H_2^2\cdot, \cdot)$ if H_2 is the endomorphism of TN defined by $h_2 = h_0(H_2\cdot, \cdot)$.

(3) In dimension $n > 4$, when h_0 is locally conformally flat, one has $k = 0$ and

$$-h_2 = \text{Sch}_{h_0}, \qquad\qquad h_4 = \tfrac{1}{4}h_2^2, \qquad\qquad h_{2j} = 0 \ \text{ for } 2 < j < \tfrac{n}{2}. \tag{30}$$

See [24, Th 7.4] or [63] for a proof. When $h_n = 0$, then the metric $g = x^{-2}(dx^2 + h_x)$ has constant sectional curvature -1 in a small neighbourhood of $x = 0$ if $h_x = h_0 + x^2 h_2 + x^4 h_4$ with h_2, h_4 of (30). When $n = 4$, one still has $h_2 = -\text{Sch}_{h_0}$ but h_4 is not necessarily h_2^2.

(4) When h_0 is an Einstein metric with $\mathrm{Ric}_{h_0} = \lambda(n-1)h_0$, it is easily checked that $k = 0$ and

$$h_2 = -\tfrac{\lambda}{2}h_0, \qquad\qquad h_4 := \tfrac{\lambda^2}{16}h_0, \qquad\qquad h_{2j} = 0 \ \text{ for } 2 < j < \tfrac{n}{2}. \qquad (31)$$

When $h_n = 0$, the metric $g = (dx^2 + h_x)/x^2$ with $h_x := (1 - \tfrac{\lambda x^2}{4})^2 h_0$ is an exact Poincaré-Einstein end in $x < x_0$ for some small $x_0 > 0$, see Section 6.2.

2.5. The conformal class at infinity.

By [32, 19], the whole conformal class $[h_0]$ of the metric h_0 induced by g on the boundary at infinity (with respect to a given boundary defining function x) can be parametrized by a family of "geodesic" boundary defining functions:

Lemma 2.8. *Let (M, g) be an odd dimensional AHE manifold, of the form (21) near $\partial\overline{M}$ for some x. Let h_0 be the induced metric at infinity. For any $\hat{h}_0 \in [h_0]$, there is a neighborhood V of $\partial\overline{M}$ and a unique boundary defining function $\hat{x}$ such that $\hat{x}^2 g|_{T\partial M} = \hat{h}_0$ and $|d\hat{x}|_{\hat{x}^2 g} = 1$ in V. The function $\hat{x}$ has a polyhomogeneous expansion with respect to x and the metric g is of the form $(d\hat{x}^2 + \hat{h}_{\hat{x}})/\hat{x}^2$ in a collar near ∂M, where $\hat{h}_{\hat{x}}$ is a one-parameter family of tensors on $\partial\overline{M}$ which is smooth in $\hat{x}, \hat{x}^n \log(\hat{x})$.*

Proof. The existence and polyhomogeneity of $\hat{x}$ is shown in [19, Lemma 6.1]. The form of the metric in the collar neighborhood induced by $\hat{x}$ follows for instance from Theorem A in [19]. Since it will be used later, we recall that the proof amounts to seting $\hat{x} = e^\omega x$ for some unknown function ω defined on $\overline{M}$ near $N = \partial\overline{M}$ which solves near the boundary the equation $|d\hat{x}|^2_{\hat{x}^2 h} = 1$ with $\hat{h}_0 = e^{2\omega_0} h_0$. This leads to the following Hamilton-Jacobi equation in the collar neighbourhood $[0, \varepsilon) \times N$ of the boundary:

$$\partial_x \omega + \tfrac{x}{2}\left((\partial_x \omega)^2 + |d_N \omega|^2_{h_x}\right) = 0, \qquad\qquad \omega|_N = \omega_0. \qquad (32)$$

where d_N is the de Rham differential on N. $\qquad\square$

Geometrically, the function $\hat{x}$ corresponding to $\hat{h}_0$ yields a particular foliation by hypersurfaces $\{x = \text{const}\}$ diffeomorphic to N near infinity, induced by the choice of conformal representative at infinity.

2.6. Cauchy data for Einstein equation, non-linear Dirichlet-to-Neumann map.

By Proposition 2.7, a Poincaré-Einstein end is uniquely determined modulo $\mathcal{O}(x^\infty)$. There is in fact a stronger statement proved by Biquard [9], based on unique continuation for elliptic equations:

Proposition 2.9 (Biquard). *An exact Poincaré-Einstein end $([0, \varepsilon)_x \times N, g = \frac{dx^2 + h_x}{x^2})$ is uniquely determined by the data (h_0, h_n) where $h_x = \sum_{j=0}^{n/2} x^{2j} h_{2j} + kx^n \log x + o(x^n)$.*

On a manifold with boundary $\overline{M}$, the unique continuation of [9] also holds true: if two AHE metrics on M agree to infinite order at $\partial\overline{M}$, then, near the boundary, one is the pull back of the other by a diffeomorphism of $\overline{M}$ which is the identity on $\partial\overline{M}$.

We will then call (h_0, h_n) the *Cauchy data* for the Einstein equation,

$$h_0 \text{ is the } \textit{Dirichlet datum}, \qquad\qquad h_n \text{ is the } \textit{Neumann datum}. \qquad (33)$$

We emphasize that here the pair (h_0, h_n) is associated to the geodesic boundary function of Lemma 2.8 determined by h_0.

It is of interest to study those pairs (h_0, h_n) for which there does exist an AHE manifold (M, g) which can be written in a collar neighbourhood $[0, \varepsilon)_x \times \partial M$ under the form $g = \frac{dx^2 + h_x}{x^2}$ with $h_x = \sum_{j=0}^{n/2} x^{2j} h_{2j} + kx^n \log x + o(x^n)$.

We can define a Dirichlet-to-Neumann map under the assumption that a local existence result for the following Dirichlet problem on M holds: let g^0 be an AHE metric on M and $h_0 = (x^2 g^0)_{|TN}$ be a representative of the conformal infinity of g associated to a geodesic boundary defining function x, then there exists a smooth submanifold $\mathcal{S} \subset \mathcal{M}(N)$ containing h_0 (with $N = \partial \overline{M}$), transverse to the action of $C^\infty(N)$ on $\mathcal{M}(N)$, such that for any $h \in \mathcal{S}$, there is an AHE metric g near g^0 such that

$$\text{Ric}_g = -ng, \qquad\qquad (x^2 g)_{|\partial \overline{M}} = h \qquad\qquad (34)$$

and g depends smoothly on h. The topology here can be chosen to be some $C^{k,\alpha}(M)$ norms for some $k \in \mathbb{N}$ and $\alpha > 0$. Such an existence result has been proved by Graham-Lee [32] when $(M, g^0) = (\mathbb{H}^{n+1}, g_{\mathbb{H}^{n+1}})$ where $\mathbb{H}^{n+1}$ is viewed as the unit ball in $\mathbb{R}^{n+1}$, and has been extended by Lee [46] to the case where g^0 is AHE with negative sectional curvatures. We can then define a (local) *Dirichlet-to-Neumann map*[1] near h_0

$$\mathcal{N} : C^\infty(M, S_+^2 T^* \partial \overline{M}) \to C^\infty(M, S^2 T^* \partial \overline{M}), \qquad\qquad h \mapsto h_n. \qquad (35)$$

where h_n is the Neumann datum of the metric g satisfying (34). Graham [28] computes its linearization at the hyperbolic metric in the case n odd and when $(M, g^0) = (\mathbb{H}^{n+1}, g_{\mathbb{H}^{n+1}})$. For n odd, this was also studied by Wang [68] in a general setting: she proved that this linearized operator is a pseudo-differential operator on the boundary and she computed its principal symbol.

3. The renormalized volume in a fixed conformal class

3.1. **The renormalized volume.** We follow the method introduced by Henningson-Skenderis [41], Graham [27]. The volume form near the boundary is

$$\text{dvol}_g = v(x) \text{dvol}_{h_0} \frac{dx}{x^{n+1}} = \det(h_0^{-1} h_x)^{\frac{1}{2}} \text{dvol}_{h_0} \frac{dx}{x^{n+1}}.$$

Since $\text{Tr}(k) = 0$, the function $v(x)$ has an asymptotic expansion of the form

$$v(x) = 1 + v_2 x^2 + \cdots + v_n x^n + o(x^n). \qquad\qquad (36)$$

Definition 3.1. The renormalized volume of (M, g) with respect to a conformal representative h_0 of $[h_0]$ is the Hadamard regularized integral

$$\text{Vol}_R(M, g; h_0) = \text{FP}_{\varepsilon \to 0} \int_{x > \varepsilon} \text{dvol}_g. \qquad\qquad (37)$$

---

[1]In even dimension n, we will see later that it is more natural to modify h_n with a certain formally determined tensor in the definition of $\mathcal{N}$.

where, near ∂M, x is the geodesic boundary defining function such that $x^2 g|_{T\partial M} = h_0$. When g is fixed, we consider $\mathrm{Vol}_R(M, g; h_0)$ as a function of h_0, we shall write it $\mathrm{Vol}_R(M; h_0)$.

An equivalent definition for Vol_R was given by Albin [2] using Riesz regularization

$$\mathrm{Vol}_R(M, h_0) = \mathrm{FP}_{z=0} \int_M x^z \mathrm{dvol}_g, \qquad\qquad z \in \mathbb{C} \qquad (38)$$

where x is any positive function equal to the geodesic boundary-defining function associated to h_0 near ∂M. With this definition we can easily compute the variation of Vol_R inside the conformal class $[h_0]$.

Proposition 3.2. *Let (M, g) be an odd dimensional Einstein conformally compact manifold with conformal infinity $[h_0]$. The renormalized volume $\mathrm{Vol}_R(M, \cdot)$ of M, as a functional on $\mathcal{M}_{[h_0]} := \{h_0 \in [h_0]; \int_{\partial M} \mathrm{dvol}_{h_0} = 1\}$, admits a critical point at h_0 if and only if $v_n(h_0)$ is constant.*

Proof. We set $h_0^s := h_0 e^{2s\omega_0}$ for $s \geq 0$, then from Lemma 2.8 there exists a unique function ω^s such that the geodesic boundary defining function x_s associated to h_0^s is given by

$$x^s = e^{\omega^s} x, \qquad\qquad \omega^s = s\omega_0 + \mathcal{O}(xs^2). \qquad (39)$$

Indeed, for all s we have $|d\log(x^s)|_g^2 = 1$, thus ω^s must satisfy

$$2\partial_x \omega^s = -x((\partial_x \omega^s)^2 + |d_y \omega^s|_{h_x}^2), \qquad\qquad \omega^s|_{x=0} = s\omega_0.$$

This is a non-characteristic Hamilton-Jacobi equation which has a unique solution depending smoothly in s on the initial data with $\omega^0 = 0$. Then $\omega^s = \mathcal{O}(s)$ and thus $\partial_x \omega^s = \mathcal{O}(xs^2)$, which implies that (39) holds. Taking the derivative of (38) at $s = 0$, we obtain using the expansion (36)

$$\partial_s \mathrm{Vol}_R(M, h_0^s)|_{s=0} = \mathrm{FP}_{z=0} \int_M z\omega_0 x^z v(x) \mathrm{dvol}_{h_0} \frac{dx}{x^{n+1}} = \int_{\partial M} \omega_0 v_n \mathrm{dvol}_{h_0}. \qquad (40)$$

We now make a variation within constant volume metrics in $[h_0]$, thus $\int_{\partial M} \omega_0 \mathrm{dvol}_{h_0} = 0$. We thus conclude that

$$v_n = \mathrm{constant} \qquad (41)$$

is the equation describing a critical point of the renormalized volume functional in the conformal class with constant total volume. $\qquad\square$

Remark 3.3. From Graham-Zworski [33], the following identity holds

$$\int_{\partial M} v_n \mathrm{dvol}_{h_0} = C_n \int_{\partial M} Q_n \mathrm{dvol}_{h_0}, \qquad (42)$$

where C_n is an explicit constant and Q_n is Branson's Q-curvature. This integral depends only on the conformal class $[h_0]$ and not on h_0. For locally conformally flat metrics, this is a constant times the Euler characteristic, as proved by Graham-Juhl [31].

Remark 3.4. According to Graham-Hirachi [30], the infinitesimal variation of the integral of v_n along a 1-parameter family of Poincaré-Einstein metrics g_s inducing h_0^s on N with $\dot{h}_0 := \partial_s(h_0^s)_{|s=0}$ is determined by the obstruction tensor k of h_0:

$$\partial_s \left(\int_{\partial M} v_n \, \mathrm{dvol}_{h_0} \right)_{|s=0} = \tfrac{1}{4} \int_{\partial M} \langle k, \dot{h}_0 \rangle \mathrm{dvol}_{h_0}. \tag{43}$$

In fact, we can give a formula for the renormalized volume $\mathrm{Vol}_R(M, e^{2\omega_0}h_0)$ in terms of ω_0.

Lemma 3.5. Let $h_0 \in [h_0]$ be fixed, $\omega_0 \in C^\infty(\partial M)$, and let

$$\omega = \sum_{j=0}^{\frac{n}{2}} \omega_{2j} x^{2j} + \mathcal{O}(x^{n+1})$$

be the solution of the Hamilton-Jacobi equation $|dx/x + d\omega|_g^2 = 1$ near ∂M with boundary condition $\omega|_{\partial M} = \omega_0$. The renormalized volume $\mathcal{V}_n(\omega_0) := \mathrm{Vol}_R(M, e^{2\omega_0}h_0)$ as a function of ω_0 is given by

$$\mathcal{V}_n(\omega_0) = \mathcal{V}_n(0) + \int_{\partial M} \sum_{i=0}^{n/2} v_{2i}(h_0)\omega_{n-2i} \, \mathrm{dvol}_{h_0}$$

where $v_{2i}(h_0) \in C^\infty(\partial M)$ are the terms in the expansion of the volume element (36) at ∂M.

Proof. From the expansion $e^{z\omega} = 1 + z\omega + \mathcal{O}(z^2)$ near $z = 0$, we get

$$\mathrm{Vol}_R(M, e^{2\omega_0}h_0) = \mathrm{FP}_{z=0} \int_M x^{z-n} e^{z\omega} v(x) \frac{dx}{x} \mathrm{dvol}_{h_0}$$

$$= \mathrm{Vol}_R(M, h_0) + \mathrm{FP}_{z=0} \left(z \int_M x^{z-n} \omega(x) v(x) \frac{dx}{x} \mathrm{dvol}_{h_0} \right)$$

$$= \mathrm{Vol}_R(M, h_0) + \mathrm{Res}_{z=0} \int_M x^{z-n} \omega(x) v(x) \frac{dx}{x} \mathrm{dvol}_{h_0}$$

$$= \mathrm{Vol}_R(M, h_0) + \int_{\partial M} \sum_{i=0}^{n/2} v_{2i} \omega_{n-2i} \, \mathrm{dvol}_{h_0}$$

where in the last equality we have exhibited the residue as the coefficient of x^n in $\omega(x)v(x)$ $\square$

We mention a similar statement after Theorem 3.1 in [27].

Let us now give some properties of the ω_{2i} in the expansion of $\omega(x)$ at $x = 0$:

Lemma 3.6. The function ω solving the equation $|dx/x+d\omega|_g = 1$ near $x = 0$ and $\omega|_{x=0} = \omega_0$ satisfies $\omega(x) = \sum_{i=0}^{n/2} x^{2i}\omega_{2i} + o(x^n)$ for some $\omega_{2i} \in C^\infty(M)$ with

$$\omega_2 = -\tfrac{1}{4}|\nabla\omega_0|_{h_0}^2$$

$$\omega_4 = \tfrac{1}{8} \left(-\frac{1}{4}|\nabla\omega|^4 + h_2(\nabla\omega_0, \nabla\omega_0) - 2h_0(\nabla\omega_0, \nabla\omega_2) \right).$$

where ∇ denotes the gradient with respect to h_0. If we replace ω_0 by $s\omega_0$ for $s > 0$ small, for all $i > 0$ one has as $s \to 0$

$$\omega_{2i} = -\frac{s^2}{4i}h^{(2i-2)}(d\omega_0, d\omega_0) + \mathcal{O}(s^3). \tag{44}$$

where $h_x^{-1} = \sum_{i=0}^{n/2} x^{2i}h^{(2i)} + O(x^n \log x)$ is the metric induced by h_x on the cotangent bundle $T^*\partial M$.

Proof. The computation for ω_2 and ω_4 is simply obtained by expanding in powers of x the equation $2\partial_x \omega = -x((\partial_x \omega)^2 + |d_y \omega|^2_{h(x)})$ and identifying the terms:

$$\sum_{i=0}^{n/2} 4ix^{2i-1}\omega_{2i} = -x\Big(\sum_{i=0}^{n/2} 2ix^{2i-1}\omega_{2i}\Big)^2 - \sum_{i,j,k=0}^{n/2} x^{2(i+j+k)+1}h^{(2i)}(d\omega_{2j}, d\omega_{2k}) + o(x^{n-1})$$

where $h_x^{-1} = \sum_{i=0}^{n/2} x^{2i}h^{(2i)} + O(x^n \log x)$ if h_x^{-1} is the metric on the cotangent bundle. In particular, we have $h^{(2)}(d\omega_{2k}, d\omega_{2j}) = -h_2(\nabla \omega_{2k}, \nabla \omega_{2j})$. Now for (44), we observe that $\omega_{2i} = \mathcal{O}(s^2)$ for each $i \neq 0$, and so by looking at the terms modulo s^3 in the equation above, only the terms with $j = k = 0$ appear and we get

$$\sum_{i=0}^{n/2} 4ix^{2i-1}\omega_{2i} = -s^2 \sum_{i=0}^{n/2} x^{2i+1}h^{(2i)}(d\omega_0, d\omega_0) + \mathcal{O}(s^3)$$

which implies the desired identity. $\qquad \square$

From this, we can give an expression for the Hessian of $\omega_0 \mapsto \mathrm{Vol}_R(M, e^{2\omega_0}h_0)$ at a critical point h_0, as a quadratic form of ω_0.

Corollary 3.7. *Let (M, g) be an $(n+1)$-dimensional Poincaré-Einstein manifold with conformal infinity $(\partial M, [h_0])$. Then for $\omega_0 \in C^\infty(M)$ we have*

$$\mathrm{Hess}_{h_0}(\mathcal{V}_n)(\omega_0) := \partial_s^2 \mathrm{Vol}_R(M, e^{2s\omega_0}h_0)_{|s=0} = -\sum_{j=1}^{n/2} \int_{\partial M} \frac{v_{n-2j}(h_0)}{2j}h^{(2j-2)}(d\omega_0, d\omega_0)\mathrm{dvol}_{h_0}$$

where $h_x^{-1} = \sum_{i=0}^{n/2} x^{2j}h^{(2j)} + O(x^n \log x)$ is the metric induced by h_x on the cotangent bundle $T^*\partial M$, and $v_{2j}(h_0) \in C^\infty(\partial M)$ are the coefficients in the expansion (36) of the volume element at ∂M.

Remark that the Hessian of $\mathcal{V}_n$ depends only on the conformal infinity $(\partial M, [h_0])$ of M. Since the positive/negative definiteness of the Hessian of $\mathcal{V}_n = \mathrm{Vol}_R$ is entirely characterized by the tensor $-\sum_{j=1}^{n/2} \int_{\partial M} \frac{v_{n-2j}(h_0)}{2j}h^{(2j-2)}$ we shall call this tensor the Hessian of $\mathcal{V}_n$ at h_0 and denote it

$$\mathrm{Hess}_{h_0}(\mathcal{V}_n) = -\sum_{j=1}^{n/2} \frac{1}{2j}v_{n-2j}(h_0)h^{(2j-2)}. \tag{45}$$

Remark 3.8. We remark that the tensors $h^{(2j-2)}$ are symmetric tensors on $T^*\partial M$ and thus $\mathrm{Hess}_{h_0}(\mathcal{V}_n)$ is also symmetric. While we were finishing this work, we learnt that this computation also appears in the work of Chang-Fang-Graham [14, eq. (3.6)].

3.2. **Computations of** v_2, v_4, v_6**.** To express the renormalized volume functional in dimension $2, 4, 6$, we need to compute the volume coefficients v_2, v_4, v_6. This will serve also later for the variation formula for the renormalized volume of AHE metrics. The formulas are already known [29] (see also [43, Th 6.10.2] for a proof) but to be self-contained we give a couple of details of how the computations go. We recall first that for a symmetric endomorphism A on an n-dimensional vector space equipped with a scalar product, the elementary symmetric function of order k of A is defined by

$$\sigma_k(A) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \ldots \lambda_{i_k} \tag{46}$$

where $(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of A repeated with multiplicities.

Lemma 3.9. *Let* $((0, \varepsilon)_x \times N, g = \frac{dx^2 + h_x}{x^2})$ *be an asymptotic Poincaré-Einstein end, and* H_{2j}, K *the endomorphisms of* TN *defined by*

$$h_x(\cdot, \cdot) = h_0\Big(\Big(\sum_{j=0}^{n} H_{2j} x^{2j} + K x^n \log(x)\Big)\cdot, \cdot\Big) + o(x^n).$$

If v_{2j} *are the volume coefficients in* (36)*, one has*

$$v_2 = \tfrac{1}{2}\sigma_1(H_2) = \tfrac{1}{2}\mathrm{Tr}(H_2),$$
$$v_4 = \tfrac{1}{4}\sigma_2(H_2) = \tfrac{1}{8}(\mathrm{Tr}(H_2)^2 - \mathrm{Tr}(H_2^2))$$
$$v_6 = \tfrac{1}{8}\sigma_3(H_2) + \tfrac{1}{24(n-4)}\langle B_{h_0}, h_2\rangle,$$

where $h_0(H_2\cdot, \cdot) = h_2 = -\mathrm{Sch}_{h_0}$*. In addition, we have*

$$4\,Tr(H_4) - Tr(H_2^2) = 0, \qquad\qquad 6\,Tr(H_6) - 4\,Tr(H_2 H_4) + Tr(H_2^3) = 0. \tag{47}$$

Proof. From (26) we obtain modulo $\mathcal{O}(x^6)$

$$A_x = 2xH_2 + x^3(4H_4 - 2H_2^2) + x^5(6H_6 - 6H_2 H_4 + 2H_2^3) + x^{n-1}K(n\log(x) + 1)$$

Taking the trace and using that the obstruction tensor is trace-free (ie. $\mathrm{Tr}(K) = 0$), we get modulo $\mathcal{O}(x^6)$

$$\mathrm{Tr}(A_x) = 2x\mathrm{Tr}(H_2) + x^3(4\mathrm{Tr}(H_4) - 2\mathrm{Tr}(H_2^2)) + 6x^5(\mathrm{Tr}(H_6) - \mathrm{Tr}(H_2 H_4) + \tfrac{1}{3}\mathrm{Tr}(H_2^3))$$
$$\tfrac{1}{2}x|A_x|^2 = \tfrac{1}{2}x\mathrm{Tr}(A_x^2) = 2x^3\mathrm{Tr}(H_2^2) + 4x^5(2\mathrm{Tr}(H_2 H_4) - \mathrm{Tr}(H_2^3)) + \mathcal{O}(x^6).$$

Now from (27), we obtain (47). We can expand the volume form (using the expansion of determinant in traces) modulo $\mathcal{O}(x^7)$ and use (47)

$$\det(h_0^{-1}h_x) = 1 + x^2\mathrm{Tr}(H_2) + x^4\Big(-\tfrac{1}{4}\mathrm{Tr}(H_2^2) + \tfrac{1}{2}(\mathrm{Tr}(H_2))^2\Big)$$
$$+ x^6\Big(\tfrac{1}{6}\mathrm{Tr}(H_2^3) - \tfrac{1}{3}\mathrm{Tr}(H_2 H_4) + \tfrac{1}{6}(\mathrm{Tr}(H_2))^3 + \tfrac{1}{4}\mathrm{Tr}(H_2)\mathrm{Tr}(H_2^2)\Big)$$

thus taking the square root and using the expression of H_4 given by (29), we obtain the desired formula for v_2, v_4, v_6. $\qquad\square$

Remark 3.10. If h_0 is a locally conformally flat metric on N, the expression of $v_{2j}(h_0)$ has been computed by Graham-Juhl [31]: they obtain

$$v_{2j}(h_0) = 2^{-j}\sigma_j(H_2), \qquad h_2(\cdot,\cdot) = h_0(H_2\cdot,\cdot) = -\mathrm{Sch}_{h_0}(\cdot,\cdot). \qquad (48)$$

3.3. The renormalized volume in dimension $n = 2$. Combining Lemma 3.5 with Lemma 3.6 and Lemma 3.9, we obtain:

Proposition 3.11. *The renormalized volume functional $\mathcal{V}_2(\omega_0) = \mathrm{Vol}_R(M, e^{2\omega_0}h_0)$ on the conformal class $[h_0]$ in dimension 2 is given by the expression*

$$\mathcal{V}_2(\omega_0) = \mathcal{V}_2(0) - \tfrac{1}{4}\int_{\partial M}(|\nabla\omega_0|^2_{h_0} + \mathrm{Scal}_{h_0}\omega_0)\mathrm{dvol}_{h_0}.$$

Its Hessian at h_0 is $\mathrm{Hess}_{h_0}(\mathcal{V}_2) = -\tfrac{1}{2}h_0^{-1}$.

The critical points of the functional $\mathcal{V}_2$ restricted to the set

$$\{\omega_0 \in C^\infty(\partial M); \int_{\partial M} e^{2\omega_0}\mathrm{dvol}_{h_0} = 1\}$$

are the solutions of the equation $\mathrm{Scal}_{e^{2\omega_0}h_0} = 4\pi\chi(\partial M)$. We notice that this is the usual functional for uniformizing surfaces, that is, of finding the constant curvature metrics in the conformal class as critical points. When $\chi(\partial M) < 0$, there is existence and uniqueness of critical points by strict convexity of the functional (see e.g [66]). The renormalized volume is maximized at the hyperbolic metric in the conformal class.

It is instructive to recall here the Polyakov formula for the regularized determinant of the Laplacian (see e.g. [56, Eq (1.13)])

$$3\pi\log(\det'\Delta_{e^{2\omega_0}h_0}) - 3\pi\log(\det'\Delta_{h_0}) = -\tfrac{1}{4}\int_{\partial M}(|\nabla\omega_0|^2_{h_0} + \mathrm{Scal}_{h_0}\omega_0)\mathrm{dvol}_{h_0}.$$

As a consequence, we deduce

Lemma 3.12. *Let $(N, [h_0])$ be a closed compact Riemann surface, and let M be a Poincaré-Einstein manifold with conformal infinity $(N, [h_0])$. Then the functional*

$$F_M : [h_0] \to \mathbb{R}, \qquad h \mapsto \det'(\Delta_h)\exp\left(-\frac{\mathrm{Vol}_R(M, h)}{3\pi}\right)$$

is constant.

The constant $F_M([h_0])$, which depends on M and $[h_0]$, is computed by Zograf [70] for the case where M is a Schottky 3-manifold: M is a handlebody, its interior is equipped with a complete hyperbolic metric and the space of conformal classes $[h_0]$ on the conformal infinity ∂M is identified to the Teichmüller space $\mathcal{T}_{\partial M}$ of ∂M. The function $F_M : \mathcal{T}_{\partial M} \to \mathbb{R}^+$ can be expressed in terms of a period matrix determinant on ∂M and the modulus of a holomorphic function on $\mathcal{T}_{\partial M}$.

3.4. **The renormalized volume in dimension** $n = 4$. Combining Lemma 3.5, Lemma 3.6 and Lemma 3.9, we obtain an explicit formula for the functional

$$\mathcal{V}_4 : C^\infty(\partial M) \to \mathbb{R}, \qquad\qquad \mathcal{V}_4(\omega_0) := \text{Vol}_R(M, h_0 e^{2\omega_0}).$$

Proposition 3.13. *The renormalized volume functional $\mathcal{V}_4$ on the conformal class $[h_0]$ in dimension 4 is given by the expression*

$$\mathcal{V}_4(\omega_0) = \mathcal{V}_4(0) + \int_{\partial M} [\tfrac{1}{4}\sigma_2(H_2)\omega_0 + \tfrac{1}{8}(h_2 - \text{Tr}_{h_0}(h_2)h_0)(\nabla\omega_0, \nabla\omega_0)$$
$$+ \tfrac{1}{16}\Delta_{h_0}\omega_0.|\nabla\omega_0|^2_{h_0} - \tfrac{1}{32}|\nabla\omega_0|^4_{h_0}]\text{dvol}_{h_0}$$

where $h_2 = h_0(H_2\cdot, \cdot)$. Its Hessian at h_0 is given by

$$\text{Hess}_{h_0}(\mathcal{V}_4)(\omega_0) = \tfrac{1}{4}\int_{\partial M}(h_2(\nabla\omega_0, \nabla\omega_0) - \text{Tr}_{h_0}(h_2)|\nabla\omega_0|^2_{h_0})\text{dvol}_{h_0}$$
$$= -\tfrac{1}{8}\int_{\partial M}(\text{Ric}_{h_0} - \tfrac{1}{2}\text{Scal}_{h_0}h_0)(\nabla\omega_0, \nabla\omega_0)\text{dvol}_{h_0}.$$

The critical points of the functional $\mathcal{V}_4$ restricted to the set

$$\left\{\omega_0 \in C^\infty(\partial M); \int_{\partial M} e^{4\omega_0}\text{dvol}_{h_0} = 1\right\}$$

are, as we have seen, the solutions of the equation

$$\sigma_2(\text{Sch}_{e^{2\omega_0}h_0}) = 4\int_{\partial M} v_4 \, \text{dvol}_{h_0} = C_4\left(4\pi^2\chi(\partial M) - \tfrac{1}{2}\int_{\partial M}|W|^2_{h_0}\right)$$

with C_4 is an universal constant, $\chi(M)$ the Euler characteristic, W the Weyl tensor of h_0, and Sch_{h_0} the Schouten tensor.

4. METRICS WITH v_n CONSTANT

Equations of the type $v_k = $ constant appeared first in the work of Chang-Fang [13], who proved that for $k < n/2$, these equations are variational. We will exhibit some cases where the equation $v_n = $ constant has solutions. We shall consider either $n \le 4$ or perturbations of computable cases, typically conformal classes containing Einstein manifolds or locally conformally flat manifolds.

First let us give an expression for the linearisation of v_n in the conformal class.

Lemma 4.1. *Let h_0 be a smooth metric, then for any $\omega_0 \in C^\infty(M)$*

$$\partial_s(e^{ns\omega_0}v_n(e^{2s\omega_0}h_0))|_{s=0} = d^*_{h_0}(H_{h_0}(d\omega_0))$$

where $H_{h_0} \in C^\infty(N, \text{End}(T^*N))$ is defined by $h_0^{-1}(H_{h_0}\cdot, \cdot) = \text{Hess}_{h_0}(\mathcal{V}_n)(\cdot, \cdot)$, using the notation (45).

Proof. Let (M, g) is a Poincaré-Einstein manifold with conformal infinity $[h_0]$, then we have seen from (40) that $\partial_s(\text{Vol}_R(M, e^{2s\omega_0}h_0)) = \int_N v_n(e^{2s\omega_0}h_0)\omega_0\text{dvol}_{e^{2s\omega_0}h_0}$ thus

$$\partial_s^2(\text{Vol}_R(M, e^{2s\omega_0}h_0))|_{s=0} = \int_N \partial_s(v_n(e^{2s\omega_0}h_0))|_{s=0}\omega_0\text{dvol}_{e^{2\omega_0}h_0} + n\int_N v_n(h_0)\omega_0^2\text{dvol}_{h_0}.$$

We therefore have

$$\int_N \partial_s(v_n(e^{2s\omega_0}h_0))|_{s=0}\omega_0\mathrm{dvol}_{h_0} = \int_N \mathrm{Hess}_{h_0}(\mathcal{V}_n)(d\omega_0, d\omega_0) - nv_n(h_0)\omega_0^2\mathrm{dvol}_{h_0}. \qquad (49)$$

Using the symmetry of the tensor $\mathrm{Hess}_{h_0}(\mathcal{V}_n)$ as mentionned in Remark 3.8, this quadratic form can be polarized and this provides the desired expression for the linearisation of v_n. $\quad\square$

This Lemma suggests that $v_n(e^{\omega_0}h_0)$ depends only on derivatives of order 2 of ω_0. In fact Graham [29, Th. 1.4] proved a stronger statement, namely that $v_n(h_0)$ depends only on two derivatives of h_0.

Using the Nash-Moser implicit function theorem we can deal with perturbations of model cases for which we know that v_n is constant.

Proposition 4.2. *Let N be an n-dimensional compact manifold with a conformal class $[h_0]$ admitting a representative h_0 with $v_n(h_0) = \int_N v_n(h_0)\mathrm{dvol}_{h_0}$. Assume that $\mathrm{Hess}(\mathcal{V}_n)$ is a positive (resp. negative) definite tensor at h_0 and that the quadratic form*

$$f \mapsto \int_N \left(\mathrm{Hess}_{h_0}(\mathcal{V}_n)(df, df) - nv_n(h_0)f^2\right)\mathrm{dvol}_{h_0} \qquad (50)$$

is non-degenerate on $C^\infty(N)$. Then there is a neighbourhood $U_{h_0} \subset \mathcal{M}(N)$ of h_0 such that

$$\mathcal{S} := \{h \in U_{h_0}; v_n(h) = \int_N v_n(h)\mathrm{dvol}_h\}$$

is a slice at h_0 for the conformal action of $C^\infty(N)$ as defined in (15).

Proof. We shall use the Nash-Moser implicit function theorem. We first take a slice S_{h_0} at h_0 for the conformal action, in order to view a neighbourhood $U_{[h_0]} \subset \mathcal{C}(N)$ of $[h_0]$ as a Fréchet submanifold of $\mathcal{M}(N)$ and a neighbourhood U_{h_0} in $\mathcal{M}(N)$ as a product space $S_{h_0} \times C^\infty(N)$: for instance, take the open subset of Fréchet space

$$B_{h_0} = \{r \in C^\infty(N, S^2N); \mathrm{Tr}_{h_0}(r) = 0, \sup_{m\in N} |r(m)|_{h_0} < 1\};$$

then the map

$$\Psi : B_{h_0} \times C^\infty(N) \to \mathcal{M}(N), \qquad\qquad \Psi(r, \omega_0) = e^{2\omega_0}(h_0 + r)$$

is a tame Fréchet diffeomorphism onto its image and $S_{h_0} := \Psi(B_{h_0} \times \{0\})$ is a slice. Let Φ be the smooth map of Fréchet manifolds

$$\Phi : B_{h_0} \times C^\infty(N) \to C^\infty(N), \quad \Phi(r, \omega_0) := v_n(\Psi(r, \omega_0)) - \int_N v_n(\Psi(r, \omega_0))\mathrm{dvol}_{\Psi(r, \omega_0)}.$$

where we recall from Remark 3.3 that $\int_N v_n(h)\mathrm{dvol}_h$ is a conformal invariant. The map Φ is a non-linear differential operator and thus is tame in the sense of [40]. Notice that $\Phi(0, 0) = 0$. We compute its differential with respect to the coordinate ω_0:

$$D\Phi_{(r,\omega_0)}(0, f) = \partial_s(v_n(e^{2sf}\Psi(r, \omega_0)))|_{s=0}.$$

Using Lemma 4.1 and writing $h = \Psi(r, \omega_0)$, we therefore have

$$D\Phi_{(r,\omega_0)}(0, f) = d_h^*(H_h df) - nv_n(h)f$$

where d_h^* is the adjoint of d with respect to h. If H_h (or equivalently $\mathrm{Hess}_h(\mathcal{V}_n)$) is positive definite or negative definite, then $f \mapsto D\Phi_{(r,\omega_0)}(0,f)$ is an elliptic self-adjoint differential operator of order 2 acting on $C^\infty(N)$. If in addition the quadratic form (50) is non-degenerate, then by continuity of $h \mapsto H_h$ and $h \mapsto v_n(h)$ in $C^\infty(N, S^2N)$ and the theory of elliptic differential operators, we deduce that $f \mapsto D\Phi_{(r,\omega_0)}(0,f)$ is an isomorphism on $C^\infty(N)$ for (r,ω_0) in a small neighbourhood of $(0,0)$ in $B_{h_0} \times C^\infty(N)$. Moreover the inverse is a pseudo-differential operator of order -2, depending continuously on (r,ω_0), which is automatically tame (see for instance [40, Chap II.3]). Therefore we can apply the Nash-Moser theorem and we obtain that there exists a smooth tame map

$$r \mapsto \omega_0(r) \tag{51}$$

of Fréchet spaces such that $\Phi(r, \omega_0(r)) = 0$, if r is in a small open subset of B_{h_0}. The slice $\mathcal{S}$ is simply the image of $r \mapsto \Psi(r, \omega_0(r))$ for r near 0. $\qquad\square$

Remark 4.3. Notice that, if we were not interested in the Frechet structure of the slice, we could instead apply the implicit function theorem in some Hölder $C^{j,\alpha}(N)$ space with j large enough by using [29, Th. 1.4] which says that $\omega_0 \mapsto v_n(e^{2\omega_0}h_0)$ maps $C^{j,\alpha}(N)$ to $C^{j-2,\alpha}(N)$, and then use uniqueness of the solution near the model cases to show that the solution $e^{2\omega_0}h_0$ is indeed $C^\infty(N)$ if h_0 is smooth. The proof amounts essentially to the same argument as Proposition 4.2 except that only the isomorphism of $D\Phi(0, h_0)$ is needed.

We now apply the existence result of Proposition 4.2 to a couple of cases.

4.1. **Einstein manifolds.** We now consider the behavior of the renormalized volume in Poincaré-Einstein manifolds with a conformal infinity containing an Einstein metric. A prime example is given by the "Fuchsian" Poincaré-Einstein manifolds defined in the previous section.

Lemma 4.4. *Let N be an n-dimensional manifold with a conformal class $[h_0]$ that contains an Einstein metric. Then the Einstein representative $h_0 \in [h_0]$ with $Ric_{h_0} = \lambda(n-1)h_0$, satisfies*

$$v_n(h_0) = \frac{n!}{(n/2)!^2}(-\tfrac{\lambda}{4})^{\frac{n}{2}}.$$

The Hessian of the renormalized volume $\mathcal{V}_n$ at h_0, viewed as a symmetric tensor on T^*N, is given by

$$\mathrm{Hess}(\mathcal{V}_n) = -\tfrac{1}{4}\left(-\tfrac{\lambda}{4}\right)^{\frac{n}{2}-1}\frac{n!}{(n/2)!^2}h_0^{-1}. \tag{52}$$

The Einstein metric h_0 is a local maximum for Vol_R in $\{h_0 \in [h_0]; \int_N \mathrm{dvol}_{h_0} = 1\}$ if either $\lambda < 0$ or if $\lambda > 0$ and $\frac{n}{2}$ is odd. If $\lambda > 0$ and $\frac{n}{2}$ is even, it is a local minimum.

Proof. In all these cases, one has from the expression (31)

$$h_2 = -\tfrac{\lambda}{2}h_0, \quad h_4 = \tfrac{\lambda^2}{16}h_0, \quad h_{2j} = 0 \text{ for } j > 2, \quad v_{2j} = C_j^n(-1)^j(\tfrac{\lambda}{4})^j$$

$$h_x^{-1} = h_0^{-1}\sum_{j=0}^{\infty}(j+1)(\tfrac{\lambda}{4})^j x^{2j}, \quad h^{(2j)} = (j+1)(\tfrac{\lambda}{4})^j h_0^{-1}. \tag{53}$$

In particular the Einstein metric h_0 satisfies $v_n(h_0) = C^n_{n/2}(-\frac{1}{2})^{\frac{n}{2}}\lambda^{\frac{n}{2}}$, which is constant. Now Corollary 3.7 gives the expression for the Hessian of $\text{Vol}_R(M, e^{2\omega_0}h_0)$:

$$\text{Hess}_{h_0}(\mathcal{V}_n)(\omega_0) = -\frac{1}{2}(\frac{\lambda}{4})^{\frac{n}{2}-1}\sum_{k=0}^{\frac{n}{2}-1} C^n_k(-1)^k \int_{\partial M} |\nabla\omega_0|^2_{h_0}\text{dvol}_{h_0}.$$

Using the binomial formula we get $2\sum_{k=0}^{\frac{n}{2}-1} C^n_k(-1)^k = -C^n_{n/2}(-1)^{\frac{n}{2}}$, which achieves the computation.

Let us check this is a local maximum in the $\lambda < 0$ case, the other cases are similar. One has

$$\mathcal{V}_n(\omega_0) - \mathcal{V}_n(0) = \int_0^1 (1-s)\partial_s^2(\text{Vol}_R(M, e^{2s\omega_0}h_0))ds = \int_0^1 (1-s)\text{Hess}_{e^{2s\omega_0}h_0}(\mathcal{V}_n)(\omega_0)ds.$$

Now from the formula giving the hessian in Corollary 3.7 and the negativity of (52), we have by continuity that there exists $\varepsilon > 0$ small, $k \gg n$ large and $c_0 > 0$ such that for all $||\omega_0||_{C^k(N)} \leq \varepsilon$ and all $s \in [0,1]$

$$\text{Hess}_{e^{2s\omega_0}h_0}(\mathcal{V}_n)(\omega_0) \leq -c_0||d\omega_0||^2_{L^2}.$$

This implies that $\mathcal{V}_n(\omega_0) \leq \mathcal{V}_n(0)$ with equality if and only if ω_0 is constant, but since we restrict to $\int_N e^{n\omega_0}\text{dvol}_{h_0} = \int_N \text{dvol}_{h_0} = 1$, the equality happens only if $\omega_0 = 0$. $\square$

The extremals of v_n in conformal classes containing Einstein metrics is also considered independently by Chang-Fang-Graham [14].

Using this computation and applying Proposition 4.2, we obtain that in a neighbourhood of a conformal class admitting an Einstein metric, the equation $v_n = $ constant can be solved except for the case of the canonical sphere.

Corollary 4.5. *Let $[h_0]$ be a conformal class on N admitting a metric h_0 with $\text{Ric}_{h_0} = \lambda(n-1)h_0 \neq 0$, which is not conformal to the canonical sphere. Then, there is a neighbourhood $U_{h_0} \subset \mathcal{M}(N)$ of h_0 such that $\mathcal{S} := \{h \in U_{h_0}; v_n(h) = \int_N v_n(h)\text{dvol}_h\}$ is a slice at h_0 for the conformal action of $C^\infty(N)$.*

Proof. The quadratic form (50) is a non-zero constant times $\langle(\Delta_{h_0} - n\lambda)\omega_0, \omega_0\rangle_{L^2}$ and using the Lichnerowicz-Obata theorem [54], then $\Delta_{h_0} - n\lambda$ has trivial kernel except for the case of the sphere. The result follows from Proposition 4.2. $\square$

4.2. **Locally conformally flat metrics.** In this case, we can take the Poincaré-Einstein metric to be of the form (30), which can be rewritten

$$g = \frac{dx^2 + h_x}{x^2}, \qquad\qquad h_x(\cdot,\cdot) = h_0((1 + \tfrac{1}{2}x^2H_2)^2\cdot,\cdot)$$

with H_2 some endomorphism of TN (representing $-\text{Sch}_{h_0}$). The metric h_x^{-1} dual to h_x has expansion near $x = 0$ given by

$$h_x^{-1} = h_0^{-1}(\sum_{j=0}^{\frac{n}{2}} H^{2j}\cdot,\cdot) + \mathcal{O}(x^{n+2}), \qquad\qquad H^{2j} = 2^{-j}(j+1)(-H_2^*)^j$$

where H_2^* here denotes the endomorphism of T^*N dual of H_2. Recall by (48) that

$$v_n(h_0) = 2^{-\frac{n}{2}} \sigma_{\frac{n}{2}}(H_2).$$

Lemma 4.6. *The Hessian of $\mathcal{V}_n$ at a locally conformally flat metric h_0 is given by*

$$\mathrm{Hess}_{h_0}(\mathcal{V}_n) = 2^{-\frac{n}{2}} h_0^{-1} \Big(\sum_{j=0}^{\frac{n}{2}-1} \sigma_j(H_2^*)(-H_2^*)^{\frac{n}{2}-j-1} \cdot, \cdot \Big).$$

where H_2^* is the dual endomorphism to H_2 defined by $h_2(\cdot, \cdot) = h_0(H_2 \cdot, \cdot)$ and $\sigma_j(H_2^*)$ is the elementary symmetric function of order j of H_2^*, as defined in (46). If $e_1, \ldots, e_n$ is an orthogonal basis of eigenvectors of H_2^*, then

$$\mathrm{Hess}_{h_0}(\mathcal{V}_n)|_{\mathbb{R}e_j} = 2^{-\frac{n}{2}} \sigma_{\frac{n}{2}-1}(H_2^*|_{(\mathbb{R}e_j)^\perp}) h_0^{-1}. \tag{54}$$

Proof. The first formula for the Hessian is a direct application of (45) and (48), it remains to prove (54). Let λ_ℓ be the eigenvalue of H_2^* corresponding to e_ℓ. Then, denoting by $F(t)_{[j]}$ the coefficient of t^j in a power series $F(t)$, we compute

$$
\begin{aligned}
\sum_{j=0}^{\frac{n}{2}-1}(-\lambda_\ell)^{\frac{n}{2}-j-1}\sigma_j(H_2^*) &= \sum_{j=0}^{\frac{n}{2}-1}(-\lambda_\ell)^{\frac{n}{2}-j-1}\det(1+tH_2^*)_{[j]} \\
&= \sum_{j=0}^{\frac{n}{2}-1}[(-t\lambda_\ell)^{\frac{n}{2}-j-1}\det(1+tH_2^*)]_{[\frac{n}{2}-1]} \\
&= [(1+t\lambda_\ell)^{-1}\det(1+tH_2^*)_{[\frac{n}{2}-1]]} \\
&= \sum_{\substack{i_1<\cdots<i_{\frac{n}{2}-1} \\ i_\bullet \neq \ell}} \lambda_{i_1}\ldots\lambda_{i_{\frac{n}{2}-1}}
\end{aligned}
$$

which is the claimed formula. $\qquad\square$

We remark that $\sum_{j=0}^{\frac{n}{2}-1}\sigma_j(H_2^*)(-H_2^*)^{\frac{n}{2}-j-1}$ is the so called $(\frac{n}{2}-1)$-Newton transform $T_{\frac{n}{2}-1}(H_2^*)$ associated with H_2^*. The fact that $\partial_t\sigma_{\frac{n}{2}}(A(t)) = T_{\frac{n}{2}-1}(A(t)).\partial_t A(t)$ for a family of symmetric matrices is well-known, see [59]. When the eigenvalues of H_2^* are in the connected component containing $(\mathbb{R}_+)^n$ inside the positive cone

$$\Gamma_{\frac{n}{2}}^+ := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n; \sigma_j(\lambda) > 0, \forall j = 1, \ldots, \tfrac{n}{2}\}$$

then $T_{\frac{n}{2}-1}(H_2^*)$ is positive definite, while when they are in $-\Gamma_{\frac{n}{2}}^+$, it is negative definite, see e.g. [12]. In the first case, it is proved in [36] that if $\sigma_{\frac{n}{2}}(H_2) > 0$ in the locally conformally flat case, then the manifold has to be of constant positive sectional curvature. On the other hand, when the eigenvalues of H_2 are in $-\Gamma_{\frac{n}{2}}^+$, there seem to be no existence result for the equation $\sigma_{\frac{n}{2}}(H_2) = \mathrm{const}$ (although there are interesting partial results in Gursky-Viaclovsky [38]).

4.3. **Dimension** 4. By Lemma 3.9,the equation $v_4(e^{2\omega_0}h_0) = \text{const}$ is the σ_2-Yamabe equation, as introduced in the work of Viaclovsky [67]. It has solutions in dimension $n = 4$ under certain ellipticity condition: when the eigenvalues of the Schouten tensor (viewed as an endomorphism via h_0) are in the connected component containing $(\mathbb{R}_+)^4$ inside

$$\Gamma_2^+ := \{\lambda = (\lambda_1, \ldots, \lambda_4) \in \mathbb{R}^4; \sigma_2(\lambda) > 0, \sigma_1(\lambda) > 0\},$$

then Chang-Gursky-Yang [15, 16] proved that there is a solution ω_0 of $v_4(e^{2\omega_0}h_0) = \text{const}$. Another proof appears in the work of Gursky-Viaclovsky [39, Cor. 1.2] and in Sheng-Trudinger-Wang [62].

We now give a uniqueness result using maximum principle.

Lemma 4.7. *Let (N, h_0) be a compact manifold. Assume that $\int_N v_4(h_0)\text{dvol}_{h_0} > 0$ and that $\text{Sch}_{h_0} - \text{Tr}_{h_0}(\text{Sch}_{h_0})$ is positive definite. Then the equation $v_4(e^{2\omega_0}h_0) = \int_N v_4(e^{2\omega_0}h_0)\text{dvol}_{e^{2\omega_0}h_0}$ has at most one solution $\omega_0 \in C^\infty(N)$. These conditions are satisfied in a neighbourhood of an Einstein metric h_0 with negative Ricci curvature.*

Proof. Assume there are two solutions. Changing h_0 by a conformal factor we can assume that 0 is a solution and let ω_0 be the other solution, we then have $v_4(h_0) = v_4(e^{2\omega_0}h_0)$ as $\int_N v_4$ is a conformal invariant. At the minimum $p \in N$ of ω_0, one has $\nabla\omega_0(p) = 0$. Since

$$\text{Sch}_{e^{2\omega_0}h_0} = \text{Sch}_{h_0} - 2\nabla^2\omega_0 + 2d\omega_0 \otimes d\omega_0 - |d\omega_0|^2_{h_0}h_0$$

where $\nabla^2\omega_0$ is the Hessian with respect to h_0, we deduce by using the expresion of v_4 in Lemma 3.9 that

$$v_4(h_0) = v_4(e^{2\omega_0}h_0) = e^{-4\omega_0}\Big(v_4(h_0) + \sigma_2(B_{\omega_0}) + \tfrac{1}{2}\langle\text{Sch}_{h_0} - \text{Tr}_{h_0}(\text{Sch}_{h_0}), \nabla^2\omega_0\rangle_{h_0}\Big).$$

where $\sigma_2(B_{\omega_0})$ is the symmetric function of order 2 in the eigenvalues of the symmetric endomorphism B_{ω_0} defined by $\nabla^2\omega_0 = h_0(B\cdot, \cdot)$. At p, the eigenvalues of B_{ω_0} are non negative, thus $\sigma_2(B_{\omega_0}) \geq 0$ there. Moreover, if $v_4(h_0) \neq 0$, $\int_N(e^{4\omega_0} - 1)\text{dvol}_{h_0} = 0$ and thus $\omega_0(p) < 0$ if $\omega_0 \neq 0$, which gives $1 - e^{-4\omega_0(p)} < 0$. We then obtain, if $v_4(h_0) > 0$,

$$\langle\text{Sch}_{h_0} - \text{Tr}_{h_0}(\text{Sch}_{h_0}), \nabla^2\omega_0\rangle_{h_0}(p) < 0$$

thus if $\text{Sch}_{h_0} - \text{Tr}_{h_0}(\text{Sch}_{h_0})$ is positive definite, we obtain a contradiction with the maximum principle. $\square$

5. General variations of the renormalized volume

We shall now compute the variation of Vol_R for a family of Einstein metrics.

5.1. **The Schläfli formula.** We recall the Schläfli formula proved by in [60] for Einstein manifolds with boundary and non zero Einstein constant. For completeness, we give a short proof of this formula arising from the variation formula for scalar curvature, this is similar to [3, Lemma 2.1].

Lemma 5.1 (see [60]). *Let M be an $n+1$-dimensional manifold with boundary and g^t a family of Einstein metrics on M with H^t the mean curvature at ∂M, and II^t the second fundamental form at ∂M, computed with respect to the inward-pointing unit normal vector field to ∂M. Let $\mathrm{Ric}_{g^t} = n\lambda_t g^t$ and assume that $\lambda_0 \neq 0$. Then*

$$\partial_t \mathrm{Vol}(M, g^t)|_{t=0} = -\frac{(n+1)\dot\lambda}{2\lambda_0}\mathrm{Vol}(M, g) - \frac{1}{n\lambda_0}\int_{\partial M}(\dot H + \tfrac{1}{2}\langle \dot g, \mathrm{II}\rangle_g)\mathrm{dvol}_{\partial M} \qquad (55)$$

where dot denotes the time derivative at $t = 0$ and $\mathrm{dvol}_{\partial M}$ is the volume form induced by the restriction of g^0 on ∂M, and $g = g^0$.

Proof. We use the variation formula of the scalar curvature of a 1-parameter family of Riemannian metrics [6, Theorem 1.174]:

$$\partial_t \mathrm{Scal}_{g^t}|_{t=0} = \Delta_g \mathrm{Tr}_g(\dot g) + d^* \delta^g \dot g - \langle \mathrm{Ric}_g, \dot g\rangle. \qquad (56)$$

Since g^t is Einstein, we have $\mathrm{Ric}_{g^t} = \frac{\mathrm{Scal}_{g^t}}{n+1}g^t$ and hence

$$\langle \mathrm{Ric}_g, \dot g\rangle \mathrm{dvol}_g = \frac{\mathrm{Scal}_g}{n+1}\mathrm{Tr}_g(\dot g)\mathrm{dvol}_g = \frac{2\mathrm{Scal}_g}{n+1}\partial_t \mathrm{dvol}_{g^t}|_{t=0}.$$

Let ν be the inward-pointing unit vector field on ∂M. Integrating (56) times dvol_g over M and using Stokes we get

$$2n\lambda_0 \partial_t \mathrm{Vol}(M, g^t)|_{t=0} = \int_M (\Delta_g \mathrm{Tr}_g(\dot g) + d^* \delta^g \dot g - n(n+1)\dot\lambda)\mathrm{dvol}_g$$

$$= -n(n+1)\dot\lambda \mathrm{Vol}(M, g) + \int_{\partial M}(\nu(\mathrm{Tr}_g(\dot g)) + \delta^g(\dot g)(\nu))\mathrm{dvol}_{\partial M}.$$

To compute the right-hand side we reduce to the case where the metric is of the form $g^t = dx^2 + h_x^t$ near the boundary where x, the distance function to the boundary for g^t, is independent of t, and h_x^t are metrics on ∂M depending smoothly on x, t. One way to do that is to pull-back g^t by a diffeomorphism ψ^t which is the identity on ∂M and constructed as follows: let

$$\phi^t : \partial M \times [0, \varepsilon) \to M, \qquad\qquad \phi^t(p, s) := \exp_p^{g^t}(s\nu^t)$$

be the normal geodesic flow where ν^t is the inward-pointing unit normal to ∂M with respect to g^t, and then set ψ^t to be any diffeomorphism of M extending $\phi^0 \circ (\phi^t)^{-1}$ defined near ∂M. We replace g^t by $(\psi^t)^* g^t$ and remark that all the terms in (55) are invariant by this operation.

We have $\mathrm{II}^t = -\frac{1}{2}\partial_x h_x^t|_{x=0}$ and $H^t = \mathrm{Tr}_{h_0^t}(\mathrm{II}^t) = -\frac{1}{2}\mathrm{Tr}((h_0^t)^{-1}\partial_x h_x^t)|_{x=0}$. Since $\partial_x \mathrm{Tr}(A^{-1}\partial_t A) = \partial_t \mathrm{Tr}(A^{-1}\partial_x A)$ if $A = A(x, t)$ is a family of invertible matrices, we deduce

$$\nu(\mathrm{Tr}_g(\dot g)) = \partial_x \mathrm{Tr}_{h_x}(\dot h_x)|_{x=0} = \partial_x \mathrm{Tr}(h_x^{-1}\partial_t h_x^t)|_{t=0, x=0}$$

$$= \partial_t \mathrm{Tr}((h_0^t)^{-1}\partial_x h_x^t)|_{t=0, x=0} = -2\dot H.$$

Using that $\dot g = \dot h_x$, it is easy to see that

$$\delta_g(\dot g)(\nu) = -\langle \dot g, \mathrm{II}\rangle,$$

which concludes the proof. $\qquad\qquad\square$

5.2. **Variation of the renormalized volume in arbitrary dimensions.** Let g^t be a family of asymptotically hyperbolic Einstein metrics on M, and choose a family of boundary defining functions x^t. We can pull back g^t by a diffeomorphism ψ^t so that $x^t = (\psi^t)^* x$ is a fixed function on M and consider $(\psi^t)^* g^t$ instead of g^t. Clearly $\mathrm{Vol}_R(M, g^t; x^t) = \mathrm{Vol}_R(M, (\psi^t)^* g^t; x)$ therefore we can assume that x^t is independent of t and to simplify notation we will write $\mathrm{Vol}_R^t(M)$ for $\mathrm{Vol}_R(M, g^t; x)$.

We write $g^t = (dx^2 + h_x^t)/x^2$ and use the dot notation for $\partial_t|_{t=0}$ and $h_x := h_x^0$.

Regularity Assumption: we assume in this section that g^t is a C^1 function of t, near $t = 0$, with values in the space of smooth metrics on M, and that h_x^t is a C^1 function of t with values in the space of conormal polyhomogeneous tensors equipped with the natural topology (i.e., the asymptotic expansions of h_x^t at $x = 0$ are C^1 in t).

In dimension $n + 1$ even, Albin [2, Th. 1.3] and Anderson [3, Th. 0.2] proved

Theorem 5.2 (**Albin, Anderson**). *Let g^t be a family of AHE metrics on M with n odd and let $h_0^t = h_0 + t\dot{h}_0 + o(t)$ be a C^1 family of representatives of the conformal infinity $(\partial M, [h_0^t])$ of (M, g^t). Let h_n be the Neumann datum of g^0 in the sense of (33). Then*

$$\partial_t \mathrm{Vol}_R^t(M)|_{t=0} = -\tfrac{1}{4} \int_{\partial M} \langle h_n, \dot{h}_0 \rangle \mathrm{dvol}_{h_0}$$

Here, we study the more complicated case when n even and obtain:

Theorem 5.3. *Let g^t be a family of AHE metrics on M with n even, satisfying the regularity assumption described above when written under the form $g^t = (dx^2 + h_x^t)/x^2$ for some fixed x near ∂M. We write $h_0^t = h_0 + t\dot{h}_0 + o(t)$ and let h_n be the Neumann datum of g^0. There exists a symmetric covariant 2-tensor F_n formally determined by h_0, of order n, such that*

$$\partial_t \mathrm{Vol}_R^t(M)|_{t=0} = \int_{\partial M} \langle G_n, \dot{h}_0 \rangle \mathrm{dvol}_{h_0} \tag{57}$$

where $G_n := -\tfrac{1}{4}(h_n + F_n)$ satisfies $\delta_{h_0}(G_n) = 0$ and $\mathrm{Tr}_{h_0}(G_n) = \tfrac{1}{2} v_n$.

Proof. We will use the Schläfli formula for the compact manifold with boundary $\{x \le \varepsilon\}$. The second fundamental form, mean curvature and their variation on the hypersurface $\{x = \varepsilon\}$ are given by the value at $x = \varepsilon$ of

$$\begin{aligned}
\mathrm{I\!I} &= -\tfrac{1}{2} x \partial_x(h_x/x^2) = -x^{-2}(\tfrac{1}{2} x \partial_x h_x - h_x), \\
H &= \mathrm{Tr}_{h_x}(\mathrm{I\!I}) = -\tfrac{1}{2} \mathrm{Tr}_{h_x}(x \partial_x h_x) - n, \\
\dot{H} &= \tfrac{1}{2} \langle \dot{h}_x, x \partial_x h_x \rangle_{h_x} - \tfrac{1}{2} \mathrm{Tr}_{h_x}(x \partial_x \dot{h}_x).
\end{aligned}$$

Let us denote $\dot{\mathrm{Vol}}_R = \partial_t \mathrm{Vol}_R^t(M)|_{t=0}$. We are interested in computing

$$-n\dot{\mathrm{Vol}}_R = \tfrac{1}{2} \mathrm{FP}_{\varepsilon \to 0} \int_{x=\varepsilon} \left(\mathrm{Tr}_{h_x}((x\partial_x - 1)\dot{h}_x) - \tfrac{1}{2} \langle \dot{h}_x, x\partial_x h_x \rangle_{h_x} \right) \frac{v_x}{x^n} \mathrm{dvol}_{h_0} \tag{58}$$

where $v_x \mathrm{dvol}_{h_0} = \mathrm{dvol}_{h_x}$. Viewing symmetric tensors as matrices in local coordinates, we write modulo $o(x^n)$

$$v_x = \sum_{2q \leq n} x^{2q} v_{2q}, \qquad\qquad\qquad v_0 = 1$$

$$h_x = h_0 \Big(\sum_{2j \leq n} x^{2j} H_{2j} + x^n \log(x) K \Big), \qquad\qquad H_0 = 1$$

$$h_x^{-1} = \Big(\sum_{2j \leq n} x^{2j} H^{2j} - x^n \log(x) K \Big) h_0^{-1}, \qquad\qquad H^0 = 1$$

$$\dot{h}_x = \dot{h}_0 \Big(\sum_{2j \leq n} x^{2j} H_{2j} + x^n \log(x) K \Big) + h_0 \Big(\sum_{2j \leq n} x^{2j} \dot{H}_{2j} + x^n \log(x) \dot{K} \Big)$$

$$x \partial_x h_x = h_0 \Big(\sum_{2j \leq n} 2j x^{2j} H_{2j} + n x^n \log(x) K + x^n K \Big)$$

$$(x \partial_x - 1) \dot{h}_x = \dot{h}_0 \Big(\sum_{2j \leq n} (2j - 1) x^{2j} H_{2j} + (n-1) x^n \log(x) K + x^n K \Big)$$
$$+ h_0 \Big(\sum_{2j \leq n} (2j - 1) x^{2j} \dot{H}_{2j} + (n-1) x^n \log(x) \dot{K} + x^n \dot{K} \Big).$$

Taking the term of degree x^n and using $\mathrm{Tr}(K) = \mathrm{Tr}(\dot{K}) = 0$, we get

$$[v_x \mathrm{Tr}_{h_x}((x \partial_x - 1) \dot{h}_x)]_n = \langle \dot{h}_0, k \rangle + \sum_{i+j+q=\frac{n}{2}} (2j-1)[\mathrm{Tr}(H^{2i} \dot{H}_{2j}) \tag{59}$$
$$+ \mathrm{Tr}(h_0^{-1} \dot{h}_0 H_{2j} H^{2i})] v_{2q}$$

$$[v_x \langle \dot{h}_x, x \partial_x h_x \rangle]_n = \langle \dot{h}_0, k \rangle + \sum_{i+j+m+\ell+q=\frac{n}{2}} 2\ell (\mathrm{Tr}(H^{2i} \dot{H}_{2j} H^{2m} H_{2\ell})$$
$$+ \mathrm{Tr}(h_0^{-1} \dot{h}_0 H_{2j} H^{2m} H_{2\ell} H^{2i})) v_{2q}$$
$$= \langle \dot{h}_0, k \rangle + \sum_{i+j+m+\ell+q=\frac{n}{2}} 2\ell (\mathrm{Tr}(H^{2i} \dot{H}_{2j} H^{2m} H_{2\ell}) v_{2q} \tag{60}$$
$$+ \sum_{i+\ell+q=\frac{n}{2}} 2\ell \mathrm{Tr}(h_0^{-1} \dot{h}_0 H_{2\ell} H^{2i})) v_{2q}$$

where in the last line we used $\sum_{j+m=u} H_{2j} H^{2m} = 0$ for all $u > 0$. Let us single out the terms in $-n \dot{\mathrm{Vol}}_R$ which do not depend formally on h_0. Since the H_{2j}, H^{2j}, v_{2j} are formally determined by h_0 of order $2j$ when $j < n/2$, by Lemma 2.6 we know that there exist R_n formally determined by h_0 of order n such that

$$-n \dot{\mathrm{Vol}}_R = \tfrac{1}{2} \Big((n-1)(\mathrm{Tr}(\dot{H}_n) + \langle \dot{h}_0, h_n \rangle) - \mathrm{Tr}(h_0^{-1} \dot{h}_0 H^n) \Big) - \tfrac{n}{4} \langle \dot{h}_0, h_n \rangle + \langle \dot{h}_0, R_n \rangle.$$

But since $H^n + H_n$ depends formally on h_0, this reduces to considering terms containing $H_n, \dot{H}_n$ and we get that there exists R'_n formally determined by h_0 of order n such that

$$-n \dot{\mathrm{Vol}}_R = \tfrac{n-1}{2} \partial_t \mathrm{Tr}_{h_0^t}(h_n^t)|_{t=0} + \tfrac{n}{4} \langle \dot{h}_0, h_n \rangle + \langle \dot{h}_0, R'_n \rangle.$$

Now we know that $\mathrm{Tr}_{h_0^t}(h_n^t)$ is formally determined with respect to h_0^t of order n for each t, therefore we have established (57) with $G_n = -\frac{1}{4}(h_n + F_n)$ for some F_n formally determined by h_0 of order n.

Let us now show that $\mathrm{Tr}_{h_0}(G_n) = \frac{1}{2}v_n$ and $\delta_{h_0}(G_n) = 0$. Let $h_0^t = e^{2t\omega_0}h_0$ for some function $\omega_0 \in C^\infty(\partial M)$. We have $\dot{h}_0 = 2\omega_0 h_0$, and combining (57) with Lemma (3.5) and (3.6) we get

$$\dot{\mathrm{Vol}}_R = \int_{\partial M} v_n \omega_0 \mathrm{dvol}_{h_0} = 2 \int \mathrm{Tr}_{h_0}(G_n)\omega_0 \mathrm{dvol}_{h_0}$$

for all ω_0, and so $2\mathrm{Tr}_{h_0}(G_n) = v_n$. It remains to compute the divergence of G_n. Let $\phi^t = \exp(tV)$ be a one-parameter family of diffeomorphisms of M generated by a vector field V such that $dx(V) = 0$ near ∂M. Then $\mathrm{Vol}_R(M, (\phi^t)^*g; x)$ is independent of t because ϕ^t preserves the regions $\{x > \varepsilon\}$ for any small $\varepsilon > 0$. Therefore from (57) applied to $\dot{h}_0 = L_V h_0 = 2\delta_{h_0}^*V$ we get

$$0 = \dot{\mathrm{Vol}}_R = \langle \dot{h}_0, G_n \rangle = 2\langle V, \delta_{h_0}(G_n) \rangle.$$

Since $V|_{\partial M}$ can be chosen arbitrarily, we conclude that $\delta_{h_0}(G_n) = 0$. $\qquad\square$

Although F_n has been defined as a function of h_0 when h_0 is the conformal infinity of an Einstein metric, the fact that it is formally determined implies that we can consider $F_n(h_0)$ for any metric h_0.

Corollary 5.4. *Let (N, h_0) be a Riemannian manifold. There exists a tensor $F_n = F_n(h_0)$ formally determined by h_0, of order n, such that*

$$\mathrm{Tr}_{h_0}(F_n) = -T_n - 2v_n, \qquad\qquad \delta_{h_0}(F_n) = -D_n \qquad\qquad (61)$$

where D_n, T_n are the formally determined tensors of Proposition 2.7 and v_n is the formally determined function defined by the volume expansion in (36). If (h_0, h_n) is a Poincaré-Einstein end, then $\delta_{h_0}(h_n + F_n) = 0$ and $\mathrm{Tr}_{h_0}(h_n + F_n) = -2v_n$.

Proof. Since $F_n(h_0)$ is formally determined by h_0, we see by Remark 2.5 that it suffices to prove the result on metrics on the sphere S^n. For the round metric h_{S^n} on S^n, or any other metrics which is the conformal infinity of an AHE metric on the unit ball B^{n+1}, the conclusion (61) follows directly from Theorem 5.3, more precisely from the last part of its proof. If now h_0 is any metric on S^n, we define the metrics $h_0^t := th_0 + (1-t)h_{S^n}$ for $t \in [0,1]$. By Graham-Lee [32], for small $t \in [0, \varepsilon]$, the metric h_0^t is the conformal infinity of some AHE metric g^t on B^{n+1} and we have seen that this implies (61) for h_0^t with $t \in [0, \varepsilon]$. But $F_n(h_0^t)$, $T_n(h_0^t)$ and $v_n(h_0^t)$ are real analytic in t, therefore by unique continuation we deduce that (61) holds for $h_0 = h_0^1$. $\qquad\square$

5.3. Case $n = 2$.

We do not give full details of the computation, since this case has been analyzed in [45, 35]. With the notation of the proof of Theorem 5.3) we have

$$k = 0, \qquad\qquad v_2 = \frac{1}{2}\mathrm{Tr}_{h_0}(h_2) = \frac{1}{2}\mathrm{Tr}(H_2), \qquad\qquad H^2 = -H_2, \qquad (62)$$

and from (58), (59), (60), we obtain

$$\dot{\mathrm{Vol}}_R = -\tfrac{1}{4} \left(\int_{\partial M} 2\partial_t(\mathrm{Tr}_{h_0}(h_2))|_{t=0} \mathrm{dvol}_{h_0} + \left\langle \dot{h}_0, h_2 - v_2 h_0 \right\rangle \right).$$

By [24, Prop 7.2], $\mathrm{Tr}_{h_0}(h_2) = -\tfrac{1}{2}\mathrm{Scal}_{h_0}$, and thus, using the Gauss-Bonnet formula, we easily get $\int_{\partial M} \mathrm{Scal}_{h_0} \mathrm{dvol}_{h_0} = -\tfrac{1}{2}\langle \dot{h}_0, \mathrm{Scal}_{h_0} h_0 \rangle$. We conclude

$$\dot{\mathrm{Vol}}_R = -\tfrac{1}{4} \left\langle \dot{h}_0, h_2 + \tfrac{1}{2}\mathrm{Scal}_{h_0} h_0 \right\rangle, \qquad -4G_2 = h_2 + \tfrac{1}{2}\mathrm{Scal}_{h_0} h_0, \qquad F_2 = \tfrac{1}{2}\mathrm{Scal}_{h_0} h_0. \qquad (63)$$

5.4. **Case $n = 4$.** First, we have the relations (with the notation of the proof of Theorem 5.3)

$$H^2 = -H_2, \qquad\qquad H^4 = -H_4 - H_2 H^2 = -H_4 + (H_2)^2.$$

From (58), (59) and (60) we obtain

$$\begin{aligned}
-8\dot{\mathrm{Vol}}_R = {}& \left\langle \dot{h}_0, \tfrac{1}{2}k + 2h_4 - h_2^2 + v_2 h_2 - v_4 h_0 \right\rangle \\
& + \int_{\partial M} (v_2 \mathrm{Tr}(\dot{H}_2) - 2\mathrm{Tr}(H_2 \dot{H}_2) + 3\mathrm{Tr}(\dot{H}_4)) \mathrm{dvol}_{h_0}.
\end{aligned} \qquad (64)$$

where $h_2^2 := h_0^{-1}(H_2)^2$ is the tensor obtained by composing the endomorphism H_2 with itself. Now, recall Lemma 3.9 obtained from the constraint equation on the trace of the shape operator, which gives

$$\begin{aligned}
v_2 \mathrm{Tr}(\dot{H}_2) - 2\mathrm{Tr}(H_2 \dot{H}_2) + 3\mathrm{Tr}(\dot{H}_4) &= \partial_t\Big(\tfrac{1}{4}\mathrm{Tr}(H_2)^2 - \mathrm{Tr}(H_2^2) + 3\mathrm{Tr}(H_4) \Big)|_{t=0} \\
&= 2\dot{v}_4.
\end{aligned} \qquad (65)$$

But we also have from (43)

$$\int_{\partial M} \dot{v}_4 \, \mathrm{dvol}_{h_0} + \tfrac{1}{2}\langle \dot{h}_0, v_4 h_0 \rangle = \tfrac{1}{4}\langle \dot{h}_0, k \rangle$$

and by combining with (64) and (65), we obtain

$$\dot{\mathrm{Vol}}_R = \langle \dot{h}_0, G_4 \rangle, \qquad\qquad -4G_4 := h_4 - \tfrac{1}{2}h_2^2 + \tfrac{1}{2}v_2 h_2 - v_4 h_0 + \tfrac{1}{2}k$$

and by Lemma 3.9 this can be rewritten as

$$-4G_4 = h_4 - \tfrac{1}{2}h_2^2 + \tfrac{1}{4}\mathrm{Tr}_{h_0}(h_2)h_2 - \tfrac{1}{4}\sigma_2(h_2)h_0 + \tfrac{1}{2}k,$$

where $h_2 = -\mathrm{Sch}_{h_0} = -\tfrac{1}{2}(\mathrm{Ric}_{h_0} - \tfrac{1}{6}\mathrm{Scal}_{h_0} h_0)$.

5.5. **Einstein metric in the conformal infinity.** If $\mathrm{Ric}_{h_0} = \lambda(n-1)h_0$ for some $\lambda \in \mathbb{R}$, one can prove that the tensor F_n is a constant times h_0:

Lemma 5.5. *Let h_0 be Einstein, $\mathrm{Ric}_{h_0} = \lambda(n-1)h_0$. Then $F_n = -2\frac{(n-1)!}{(n/2)!^2}\left(-\frac{\lambda}{4}\right)^{\frac{n}{2}} h_0$ and $G_n = -\tfrac{1}{4}\big(h_n - 2\frac{(n-1)!}{(n/2)!^2}\left(-\frac{\lambda}{4}\right)^{\frac{n}{2}} h_0\big)$. In particular, $\partial_s\mathrm{Vol}_R(M, g^s; h_0^s)|_{s=0} = 0$ if g^s is a family of AHE metrics with $(g^s, h_0^s)|_{s=0} = (g, h_0)$ and $\mathrm{Vol}(N, h_0^s) = 1$, and if the trace-free part of the tensor h_n in the expansion of g is 0.*

Proof. First, notice that $T_n = 0$ in that case since $T_n = \mathrm{Tr}_{h_0}(h_n)$ depends only on h_{2j} for $j < n/2$ and the metric $g := x^{-2}(dx^2 + (1 - \lambda x^2/4)^2 h_0)$ is an exact Einstein metric near $x = 0$ which has $h_n = 0$ (see Section 6.2 below). Therefore it suffices to prove that F_n is proportional to h_0 and the multiplicative constant is deduced directly from (61) and the formula $v_n = C_{n/2}^n (-\frac{\lambda}{4})^{\frac{n}{2}}$ of Lemma 4.4. Let $A_x = h_x^{-1} \partial_x h_x = \frac{-\lambda x}{(1 - \lambda x^2/4)} \mathrm{Id}$ if $h_x = (1 - \lambda x^2/4)^2 h_0$. If $g^t = (dx^2 + h_x^t)/x^2$ is a family of Poincaré-Einstein metrics near $x = 0$, with $g^0 = g$, then differentiating the first constraint equation in (27) at $t = 0$ gives

$$\partial_x F(x) - \frac{\lambda x}{1 - \lambda x^2/4} F(x) = 0, \qquad F(x) = x^{-1}\mathrm{Tr}(\dot{A}_x) \in C^\infty([0, \varepsilon))$$

and $\dot{A}_x = \partial_t A_x|_{t=0}$. In particular

$$\mathrm{Tr}(\dot{A}_x) = a_0 x \exp\left(\int_0^x \frac{\lambda t}{1 - \lambda t^2/4} dt \right) \tag{66}$$

is determined by a constant $a_0 \in \mathbb{R}$.

Using the notations in the proof of Theorem 5.3, we claim that there exists $c_j, d_j \in \mathbb{R}$ such that for all $j \leq n/2$,

$$a_0 = 2\mathrm{Tr}(\dot{H}_2), \qquad \mathrm{Tr}(\dot{H}_{2j}) = c_j \mathrm{Tr}(\dot{H}_2), \qquad \mathrm{Tr}(\dot{H}^{2j}) = d_j \mathrm{Tr}(\dot{H}_2). \tag{67}$$

Since $\sum_{j,k=0}^{n/2} x^{2(j+k)} H_{2j} H^{2k} = \mathrm{Id} + \mathcal{O}(x^{n+1})$ and $H_{2j}|_{t=0}, H^{2k}|_{t=0}$ are multiples of Id, then $\mathrm{Tr}(\dot{H}^{2j}) = -\mathrm{Tr}(\dot{H}_{2j}) + \sum_{k=0}^{j-1} b_k \mathrm{Tr}(\dot{H}_{2k})$ for some constants $b_k \in \mathbb{R}$. But modulo $o(x^n)$, we have

$$x^{-1}\mathrm{Tr}(\dot{A}_x) = \sum_{k=1}^{n/2} \sum_{j=0}^{n/2} 2k(\alpha_k \mathrm{Tr}(\dot{H}^{2j}) + \beta_j \mathrm{Tr}(\dot{H}_{2k})) x^{2(j+k-1)}$$

for some $\alpha_k, \beta_j \in \mathbb{R}$ such that $\beta_0 = 1$, thus an easy induction and (66) prove (67).

Inserting (67) in (59) and (60), and using that v_{2q} are constant if h_0 is Einstein for $q \leq n/2$ by (53), we deduce directly that there exists $C \in \mathbb{R}$ such that

$$\dot{\mathrm{Vol}}_R = -\tfrac{1}{4}\langle h_n, \dot{h}_0 \rangle + C \int_{\partial M} \mathrm{Tr}(\dot{H}_2) \mathrm{dvol}_{h_0}.$$

Since $\mathrm{Tr}(\dot{H}_2) = \partial_t(\mathrm{Tr}_{h_0^t}(h_2^t))|_{t=0}$ and $\mathrm{Tr}_{h_0^t}(h_2^t) = C'\mathrm{Scal}_{h_0^t}$, we can use the variation formula (56) for the scalar curvature, integration by parts and the fact that $\mathrm{Ric}_{h_0^t} = \lambda(n-1)h_0^t$ when $t = 0$ to conclude that $\int_{\partial M} \mathrm{Tr}(\dot{H}_2) \mathrm{dvol}_{h_0} = C''\langle h_0, \dot{h}_0 \rangle$ for some $C'' \in \mathbb{R}$. If (M, g) is an AHE manifold with conformal infinity containing an Einstein representative h_0, then the traceless part of G_n is the traceless part of the formally undetermined term h_n (for the choice of x associated to the metric h_0). This achieves the proof. $\qquad \square$

6. Cotangent space of conformal structures and quasifuchsian reciprocity in higher dimension

We can now explain how the results of the previous section for hyperbolic manifolds in three dimensions can be used to identify Poincaré-Einstein ends modulo gauge with cotangent bundles to the space of conformal structures. This allows to extend McMullen's quasifuchsian

reciprocity, or more generally Kleinian reciprocity [53], in dimension $n + 1$. We will work in both even and odd dimensions, but for n even we shall need more hypotheses.

6.1. **Assumptions and the slice** $v_n = $ constant. To get a satisfactory picture where the analogs of the 3-dimensional phenomena can be stated and proved, two technical hypothesis will be necessary. We show below that those hypothesis are satisfied in non-trivial situations.

Like in Section 2.1, we denote by $\mathcal{M}(N)$ the space of smooth metrics on N and by $\mathcal{M}(M)$ the space of polyhomogeneous metrics on M in the sense of Section 2.2 together with its natural Fréchet structure.

We will consider in this section the situation where (some of) the following hypotheses hold. Let $h_0 \in \mathcal{M}(N)$ be a fixed metric.

Hypothesis 6.1. Frechet structure: The metric h_0 has no conformal Killing fields and the quotient space $\mathcal{T}(N) = \mathcal{G} \backslash \mathcal{M}(N)$ has a Fréchet manifold structure near $[h_0] \in \mathcal{T}(N)$.

Hypothesis 6.2. Slice $\mathbf{v_n} = \mathbf{const}$: There is a slice $\mathcal{S}_0$ at h_0 for the action of $\mathcal{G} = \mathcal{D}_0(N) \ltimes C^\infty(N)$ on $\mathcal{M}(N)$ as defined in (16), and $\mathcal{S}_0$ is included in the subset of metrics $\{h \in \mathcal{M}(N); v_n(h) = \int_N v_n(h) \mathrm{dvol}_h\}$.

Hypothesis 6.3. Einstein filling: If $\mathcal{S}_0$ is a slice at h_0 for the action of $\mathcal{G} = \mathcal{D}_0(N) \ltimes C^\infty(N)$ on $\mathcal{M}(N)$, then there is a C^1 map of Fréchet manifolds $\Xi : \mathcal{S}_0 \to \mathcal{M}(M)$ such that $\Xi(h)$ is asymptotically hyperbolic Einstein with conformal boundary $(N, [h])$.

Using the existence results for Einstein equation obtained by Biquard or Lee [7, 46] and the result of Corollary 4.5, we obtain

Proposition 6.4. *Let $h_0 \in \mathcal{M}(N)$ be an Einstein metric with negative sectional curvatures and let $g_0 \in \mathcal{M}(M)$ be an AHE metric with non-positive sectional curvatures on a manifold M with conformal boundary $(N, [h_0])$. Then Hypothesis 6.1 and 6.3 are satisfied. If n is even, Hypothesis 6.2 is also satisfied. Moreover $\mathcal{S}_0$ can be chosen so that $T_{h_0}\mathcal{S}_0 = \{r_0 \in C^\infty(N, S^2 N); \mathrm{Tr}_{h_0}(r_0) = 0, \delta_{h_0}(r_0) = 0\}$.*

Proof. Hypothesis 6.1 comes from the fact that $\mathcal{G}$ acts properly since N is not the sphere and there is no conformal Killing field for h_0 since the Ricci curvature is negative (by Yano [69]), ie. the isotropy group at h_0 is finite and in fact it is trivial, by a result of Frenkel [26], since we assumed the sectional curvatures to be non-positive. If $\mathcal{S}_0$ is any given slice at h_0 for the action of $\mathcal{G}$ and if $g_0 \in \mathcal{M}(M)$ is an AHE metric with non-positive sectional curvatures on M and with conformal boundary $[h_0]$, then Hypothesis 6.3 holds, after intersecting $\mathcal{S}_0$ with a small enough neighbourhood of h_0; this is proved by Biquard [7] and Lee [46, Theorem A]. In fact, technically speaking, [46] does not prove it with the topology we need, (i.e. that for which the whole expansion of the metric at the boundary depends in a C^1 fashion on h_0), but the arguments used by Biquard in the Kähler-Einstein setting [8] give the right property, in fact it is even simpler in our case. If n is even, we know by Corollary 4.5 that there is a slice $\mathcal{S} \subset \mathcal{M}(N)$ at h_0 for the conformal action with $\mathcal{S} = \{h \in U_{h_0}; v_n(h) = \int_N v_n(h) \mathrm{dvol}_h\}$

for some neighbourhood $U_{h_0} \subset \mathcal{M}(N)$ of h_0. There is an action by pull-back

$$\Theta : \mathcal{D}_0(N) \times \mathcal{S} \to \{h \in \mathcal{M}(N); v_n(h) = \int_N v_n(h)\mathrm{dvol}_h\}, \qquad \Theta(\phi, h) = \phi^* h.$$

The set on the right is a Fréchet submanifold when intersected with a small neighbourhood of h_0 in $\mathcal{M}(N)$. Let us first define a slice $\mathcal{S}_0 \subset \mathcal{S}$ at h_0 for the action Θ in the sense of (16). To that aim, we return to the proof of Proposition 4.2 and use the notations there. We define the smooth tame map

$$\Pi : B_{h_0} \to \mathcal{S}, \qquad\qquad r \mapsto e^{2\omega_0(r)}(h_0 + r) \qquad\qquad (68)$$

where $\omega_0(r)$ is obtained from (51) by solving $\Phi(r, \omega_0(r)) = 0$. This is a Fréchet chart for $\mathcal{S}$. The derivative is the tame family of isomorphisms defined on $\{\dot{r} \in C^\infty(N, S^2 N); \mathrm{Tr}_{h_0}(\dot{r}) = 0\}$

$$D\Pi_r(\dot{r}) = e^{2\omega_0(r)}\dot{r} + 2(D\omega_0)_r(\dot{r})\Pi(r) \qquad\qquad (69)$$

where, from the proof of Proposition 4.2 using the Nash-Moser implicit function theorem, we have that $r \mapsto (D\omega_0)_r$ is a tame map into pseudo-differential operator on N of order 0. We take the open neighbourhood $B'_{h_0} := \{r \in B_{h_0}; \delta_{h_0}(r) = 0\}$ of the Fréchet space of trace-free/divergence-free tensors with respect to h_0. We will call $\mathcal{S}_0$ the image by Π of a neighbourhood of h_0 contained in B'_{h_0}; this is a Fréchet submanifold of $\mathcal{S}$ and we are now going to show that it is a slice for the action of $\mathcal{D}_0(N)$. In that aim, we apply the Nash-Moser inverse function theorem to the restriction $\Theta_0 : \mathcal{D}_0(N) \times \mathcal{S}_0 \to \Theta(\mathcal{D}_0(N) \times \mathcal{S}_0)$ of Θ to $\mathcal{D}_0(N) \times \mathcal{S}_0$. The derivative at (ϕ, h) is

$$(D\Theta_0)_{(\phi,h)}(X, \dot{h}) = \phi^*(L_X h + \dot{h}) \in T_{\phi^* h}\mathcal{S}$$

where $X \in \mathrm{lie}(\mathcal{D}_0(N))$ is a smooth vector field and $\dot{h} \in T_h \mathcal{S}_0$. Here (ϕ, h) are in a small neighbourhood of (Id, h_0) so that $\phi^* h \in \mathcal{S}$. Now $T_{\phi^* h}\mathcal{S} = \mathrm{Im}(D\Pi_{\Pi^{-1}(\phi^* h)})$. Then we want to find a smooth tame map $((\phi, h), \dot{r}) \mapsto (X, \dot{h}) \in \mathrm{lie}(\mathcal{D}_0(N)) \times T_h \mathcal{S}_0$ so that

$$\phi^*(L_X h + \dot{h}) = D\Pi_r(\dot{r})$$

where $\mathrm{Tr}_{h_0}(\dot{r}) = 0$ and $r = \Pi^{-1}(\phi^* h) \in B_{h_0}$. Using the chart Π, we translate this into the problem of solving for $(X, \dot{r}_0)$ in

$$D\Pi_r^{-1}(L_X h) + \dot{r}_0 = D\Pi_r^{-1}\Big((\phi^{-1})^* D\Pi_r(\dot{r})\Big) \qquad\qquad (70)$$

with $h = \Pi(r)$ and $\mathrm{Tr}_{h_0}(\dot{r}_0) = 0, \delta_{h_0}(\dot{r}_0) = 0$. Applying δ_{h_0}, this leads to

$$\delta_{h_0} D\Pi_r^{-1}(L_X h) = \delta_{h_0} D\Pi_r^{-1}\Big((\phi^{-1})^* D\Pi_r(\dot{r})\Big) \qquad\qquad (71)$$

First, observe that the map $F_h : X \mapsto \delta_{h_0} D\Pi_r^{-1}(L_X h)$ is a pseudo-differential operator on N of order 2 acting on vector fields, depending smoothly in a tame way on h. We now state the following Lemma, the proof of which is defered below the proof of this Proposition.

Lemma 6.5. *Let $h_0^s \subset \mathcal{S}$ be a one-parameter smooth family of metrics on N, i.e. with $v_n(h_0^s) = \int_N v_n(h_0^s)\mathrm{dvol}_{h_0^s}$, such that $h_0^0 = h_0$. Let $\dot{h}_0 := \partial_s h_0^s|_{s=0}$ and assume that $\delta_{h_0}(\dot{h}_0) = 0$, then $\mathrm{Tr}_{h_0}(\dot{h}_0) = 0$. Moreover we have $D\Pi_0 = \mathrm{Id}$.*

We denote by $\Psi^m(N)$ the class of classical pseudo-differential operator of order m on N (acting on vector fields). The operator F_{h_0} being equal to the elliptic differential operator $F_{h_0}(X) = \delta_{h_0} L_X h_0$ of order 2, we deduce by smoothness of F_h with respect to h that $F_h \in \Psi^2(N)$ is elliptic when $||h - h_0||_{H^L}$ is small enough (for some L). The operator F_{h_0} is elliptic self-adjoint and invertible from H^2 to L^2 since there is no Killing field on (N, h_0) by [10], therefore F_h is also invertible from H^2 to L^2 with inverse an operator $F_h^{-1} \in \Psi^{-2}(N)$ and $(h, X) \to F_h^{-1}(X)$ is a tame map (by [58, Th 4.5]). This allows to solve for X in (71). Note that X is uniquely determined, according to the argument we used. Then $\dot{r}_0$ is obtained by (70), it has $\mathrm{Tr}_{h_0}(\dot{r}_0) = 0$ by the property of $D\Pi^{-1}$ and it satisfies $\delta_{h_0}(\dot{r}_0) = 0$ by construction of X solving (71). We can therefore apply the Nash-Moser inverse function theorem to deduce that $\mathcal{S}_0$ is a slice for the $\mathcal{D}_0(N)$ action on $\mathcal{S}$. $\qquad\square$

Proof of Lemma 6.5. Here we take a family of Poincaré Einstein metrics $g^s = (dx^2 + h_x^s)/x^2$ near the conformal infinity $x = 0$. We use the notation in the proof of Theorem 5.3 and remove the superscript s when s is set to be 0. We are going to show that $\dot{v}_n = c_n \dot{v}_2$ for some $c_n \neq 0$. To prove that, for the moment we do not assume that $v_n(h_0^s)$ is constant and we simply assume that g^s is Poincaré-Einstein for $s \neq 0$ with $g^0 = g$. Using $\partial_x v_x^s = \frac{1}{2} v_x^s \mathrm{Tr}_{h_x^s}(\partial_x h_x^s)$, differentiating this identity with respect to s at $s = 0$, one has modulo $o(x^n)$

$$\sum_{j,k\leq\frac{n}{2}} \dot{v}_{2k} j \gamma_j x^{2k+2j-1} + \sum_{i,j,k\leq n/2} x^{2i+2j+2k-1} v_{2k} j (\alpha_j \mathrm{Tr}(\dot{H}^{2i}) + \beta_i \mathrm{Tr}(\dot{H}_{2j})) = \sum_{j\leq\frac{n}{2}} 2j \dot{v}_{2j} x^{2j-1};$$

notice that we have used that $H^{2j} = \alpha_{2j}\mathrm{Id}$, $H_{2j} = \beta_{2j}\mathrm{Id}$ for some $\alpha_{2j}, \beta_{2j} \in \mathbb{R}$, and γ_i are some constants. Then by a straightforward induction and using (67), we deduce that $\dot{v}_{2j} = c_{2j}\dot{v}_2$ for some $c_{2j} \in \mathbb{R}$ if $1 \leq j \leq \frac{n}{2}$. To compute c_n, we notice that the obstruction tensor k vanishes for an Einstein metric, so

$$\partial_s \Big(\int_N v_n^s \mathrm{dvol}_{h_0^s} \Big) |_{s=0} = \tfrac{1}{4}\langle k, \dot{h}_0 \rangle = 0 = \int_N \dot{v}_n \mathrm{dvol}_{h_0} + \int_N \frac{v_n}{2} \mathrm{Tr}_{h_0}(\dot{h}_0) \mathrm{dvol}_{h_0}$$

and thus $c_n \int_N \dot{v}_2 = -\frac{1}{2} v_n \int_N \mathrm{Tr}_{h_0}(\dot{h}_0)$; but since $v_2^s = -\frac{1}{4(n-1)}\mathrm{Scal}_{h_0^s}$, we can use (56) to deduce that $\int_N \dot{v}_2 = -\frac{1}{4} \int_N \mathrm{Tr}_{h_0}(\dot{h}_0)$, and since $\mathrm{Tr}_{h_0}(\dot{h}_0)$ can be chosen so that its integral is not 0, we obtain that $c_n = 2v_n$. Now we come back to our setting where g^s is AHE with $\delta_{h_0}(\dot{h}_0) = 0$. Since $\dot{v}_n = 2v_n\dot{v}_2$ and $v_n \neq 0$ (by Lemma 4.4), we deduce from (56)

$$\dot{v}_n = 0 \iff (\Delta - \lambda(n-1))\mathrm{Tr}_{h_0}(\dot{h}_0) = 0 \iff \mathrm{Tr}_{h_0}(\dot{h}_0) = 0$$

if $\mathrm{Ric}_{h_0} = \lambda(n-1)h_0$. This concludes the first part of the proof since $\dot{v}_n = \frac{1}{4}\langle k, \dot{h}_0 \rangle = 0$ if $v_n(h_0^s) = \int_N v_n(h_0^s)\mathrm{dvol}_{h_0^s}$.

Let us finally show that $D\Pi_0 = \mathrm{Id}$ where Π is defined in (68). Let $\dot{r}_0$ be divergence-free and trace free with respect to h_0, then by the discussion above, we have $(Dv_n)_{h_0}(D\Pi_0(\dot{r}_0)) = 0 = (Dv_2)_{h_0}(D\Pi_0(\dot{r}_0))$ and by (69), we have also have $D\Pi_0(\dot{r}_0) = \dot{r}_0 + 2(D\omega_0)_0(\dot{r}_0)h_0$. By (56) we deduce that $(\Delta_{h_0} - \lambda n)(D\omega_0)_0(\dot{r}_0) = 0$ and thus $D\Pi_0(\dot{r}_0) = \dot{r}_0$. If now X is a vector field so that $\mathrm{Tr}_{h_0}(L_X h_0) = 0$, we set $\phi_t = e^{tX}$ and write $\phi_t^* h_0 = \Pi(r_t)$ for some r_t with $\mathrm{Tr}_{h_0}(r_t) = 0$. Then, differentiation gives $L_X h_0 = D\Pi_0(\dot{r})$ and since $\Pi(r_t) = e^{2\omega_0(r_t)}(h_0 + r_t)$, we also deduce $L_X h_0 = D\Pi_0(\dot{r}) = 2(D\omega_0)_0(\dot{r})h_0 + \dot{r}$. Taking the trace with respect to h_0,

we obtain $2(D\omega_0)_0(\dot{r}) = 0$ and $\dot{r} = L_X h_0 = D\Pi_0(L_X h_0)$. Since any trace free tensor $\dot{r}$ can be decomposed as a sum $L_X h_0 + \dot{r}_0$, this achieves the proof that $D\Pi_0 = \text{Id}$. $\square$

6.2. **Examples.** We give two examples where these hypotheses are satisfied.

The case $n = 2$. This is our archetypal motivation. We consider here a 3-manifold M which admits a convex co-compact hyperbolic metric — this is the same, in dimension 3, as an AHE metric. Then $N = \partial M$ is the disjoint union of a finite set of closed surfaces of genus at least 2. The classical Ahlfors-Bers theorem [1, 5], extended by Marden [47, 48], gives the map Φ of Hypothesis 6.3, for any choice of slice $\mathcal{S}_0$ (in fact the map Φ is well defined on Teichmüller space in this case). Moreover, we have seen that $v_2 = -\frac{1}{4}\text{Scal}_{h_0}$, so given a metric h_0 on $N = \partial M$, it has $v_2 = -\pi\chi(N)$ if and only it has constant curvature. Since there is a unique constant curvature metric with volume 1 on each connected component of ∂M, Hypothesis 6.2 is also satisfied.

Fuchsian-Einstein manifolds. We now recall a particularly simple type of AHE manifolds. Let (N, h_0) be a closed Einstein manifold with $\text{Ric}_{h_0} = -(n-1)h_0$. Its conformal class will be denoted $[h_0]$ as before.

We consider the product $M = \mathbb{R} \times N$, with the warped product metric:

$$g := dt^2 + \cosh^2(t)h_0. \tag{72}$$

We will call *Fuchsian* a Riemannian manifold of this type, the reason being that, for $n = 2$, we find precisely the Fuchsian hyperbolic 3-manifolds, that is, quotients of $\mathbb{H}^3$ by co-compact Fuchsian groups $\Gamma \subset \text{PSL}_2(\mathbb{R}) \hookrightarrow \text{PSL}_2(\mathbb{C})$, or equivalently hyperbolic 3-manifolds which are topologically the product of a surface of genus at least 2 by an interval, and which contain a closed totally geodesic surface.

It follows directly from (25) that $\text{Ric}_g = -ng$. To prove that (M, g) is actually AHE, set $x = 2e^{-|t|}$ away from $t = 0$. In this new variable,

$$g = \frac{dx^2}{x^2} + \left(\frac{1 + \frac{1}{4}x^2}{x}\right)^2 h_0 \ ,$$

so g is Poincaré-Einstein. The subset corresponding to $t = 0$ is a closed totally geodesic hypersurface since the warping function is even.

Write $g = dt^2 + f^2(t)h_0$ with $f(t) = \cosh(t)$. Let v, w be some (t-independent) vector fields on N and let $V := f^{-1}v$, $W := f^{-1}w$ and $T = \partial/\partial t$, then one has by a direct computation

$$\nabla_T T = 0, \quad \nabla_V T = f^{-1}f'V, \quad \nabla_T V = 0, \quad \nabla_V W = f^{-2}\nabla_v^N w - f^{-1}f'\langle v, w\rangle_{h_0} T. \tag{73}$$

This implies for X, Y tangent to N

$$R_{X,T}T = -X \ , \qquad\qquad R_{X,Y}T = 0. \tag{74}$$

Moreover, for X, Y, Z, W tangent to N,

$$\langle R(X,Y)Z, W\rangle_g = \langle R^{h_0}(X,Y)Z, W\rangle_g - \frac{(f')^2}{f^2}(\langle Y, Z\rangle_g\langle X, W\rangle_g - \langle X, Z\rangle_g\langle Y, W,\rangle_g) \tag{75}$$

where R^{h_0} is the Riemann tensor on (N, h_0), showing in particular that if h_0 has non-positive sectional curvature, then g also has non-positive sectional curvature.

The conformal boundary of M is the disjoint union of two copies of $(N, [h_0])$, one corresponding to $t = -\infty$ and the other to $t = \infty$. We call these two components of the conformal boundary $(N_\pm, [h_0])$.

We summarize the discussion in the

Lemma 6.6. *Let (N, h_0) be a closed Einstein manifold with non-positive sectional curvatures and negative Ricci curvature, and let $M = N \times \mathbb{R}$ be endowed with the warped product metric $g = dt^2 + \cosh^2(t)h_0$. Then g satisfies the assumptions of Proposition 6.4.*

6.3. **Poincaré-Einstein ends as cotangent vectors to conformal structures.** We then use the description in Section 2.1: $\mathcal{T}(N)$ correspond to the quotient of the space of metrics $\mathcal{M}(N)$ by the group $C^\infty(N) \rtimes \mathcal{D}_0(N)$ which is the semi-direct product of the group of conformal transformations by the group of diffeomorphisms of N isotopic to the Identity, or equivalently it is the space of conformal classes up to diffeomorphisms isotopic to the Identity. We will work near a metric $h_0 \in \mathcal{M}(N)$ where $\mathcal{T}(N)$ can be locally represented by a slice. By the discussion of Section 2.1, $T^*_{[h]}\mathcal{T}(N)$ can be identified with the space of trace-free and divergence-free (for h) symmetric 2-tensors on ∂M.

Let $\mathcal{E}$ be the space of Poincaré-Einstein ends (with conformal boundary N), i.e. the set of products $N \times (0, \varepsilon)_x$ equipped with a Poincaré-Einstein metric $g = (dx^2 + h_x)/x^2$; here $\varepsilon > 0$ is not relevant since a Poincaré-Einstein metric is defined only up to $\mathcal{O}(x^\infty)$. The group $\mathcal{D}_0(N)$ acts naturally on $\mathcal{E}$ by $\phi.g = (dx^2 + (\phi^{-1})^*h_x)/x^2$ where $(\phi^{-1})^*h_x$ is just the pull-back of h_x by ϕ^{-1}, viewed as a metric on N. The group $C^\infty(N)$ also acts on $\mathcal{E}$ as follows: $\omega_0.g := (d\hat{x}^2 + \hat{h}_{\hat{x}})/\hat{x}^2$ where $\hat{x}$ is the geodesic boundary defining function associated to the conformal representative $e^{2\omega_0}h_0$, in the sense of Lemma 2.8. This induces an action of $C^\infty(N) \rtimes \mathcal{D}_0(N)$ by $(\omega_0, \phi).g := \omega_0.(\phi.g)$. This group action corresponds to the action of the group of those diffeomorphisms which map a Poincaré-Einstein end to another one: this is the natural gauge group of $\mathcal{E}$.

Case n odd. We observe that the action of an element $(f, \phi) \in C^\infty(N) \rtimes \mathcal{D}_0(N)$ on a Poincaré-Einstein end g transforms the pair (h_0, h_n) in the expansion of g into the pair $(e^{2\omega_0}(\phi^{-1})^*h_0, e^{(2-n)\omega_0}(\phi^{-1})^*h_n)$ in the expansion of $(\omega_0, \phi).g$. This is easy to show: the $\mathcal{D}_0(N)$ action is clear, as for the conformal action, it comes from the fact that h_n is the coefficient of the first odd power of x in the expansion of g and that the geodesic boundary defining function $\hat{x}$ associated to $e^{2\omega_0}h$ is of the form $\hat{x} = xe^{\omega_x}$ with ω_x an even function of x up to $\mathcal{O}(x^{n+2})$ (see for e.g. Lemma 2.1 in [34] and its proof). Notice that the action $(f, \phi).g$ corresponds exactly to the action (18) of (ω_0, ϕ) on $T^*\mathcal{M}(N)$ if we view $(h_0, h_n \otimes \mathrm{dvol}_{h_0})$ as an element in $T^*\mathcal{T}(N)$ (here h_n is a divergence-free trace-free tensor). We therefore deduce

Proposition 6.7. *If n is odd, over the points where $\mathcal{T}(N)$ has a Fréchet manifold structure, the space $\mathcal{G}\backslash\mathcal{E}$ of Poincaré-Einstein ends, up to the gauge group $\mathcal{G} = C^\infty(N) \rtimes \mathcal{D}_0(N)$, identifies naturally to the cotangent space $T^*\mathcal{T}(N)$ of the set of conformal structures.*

Case n even. In even dimension, the pairs (h_0, h_n) representing a Poincaré-Einstein ends are not identified directly to an element in $T^*_{[h_0]}\mathcal{T}(N)$ as for n odd. Indeed, it is easy to verify that for a change of conformal representative $\hat{h}_0 = e^{2\omega_0}h_0 \in [h_0]$, the formally undetermined term $\hat{h}_n$ in the end is of the form $\hat{h}_n = e^{(2-n)\omega_0}h_n + P(\omega_0, h_0)$ where P is some non-linear differential operator. Moreover h_n is neither trace-free nor divergence-free with respect to h_0. However, Theorem 5.3 and Corollary 5.4 tell us that if h_0 satisfies $v_n(h_0) = \int_N v_n(h_0)\mathrm{dvol}_{h_0}$ then there is a formally determined tensor $F_n = F_n(h_0)$ such that the trace-free part $G_n^\circ = G_n - \frac{v_n}{2n}h_0$ of $G_n = -\frac{1}{4}(h_n + F_n)$ is divergence-free. By the description (19) of $T^*\mathcal{T}(N)$ in Section 2.1, we can thus see $\mathcal{G}_n^\circ = G_n^\circ \otimes \mathrm{dvol}_{h_0}$ as a cotangent vector at h_0. We then obtain

Proposition 6.8. *Let n be even and assume Hypothesis 6.3. Near the base point $[h_0] \in \mathcal{T}(N)$, we can identify the cotangent space $T^*\mathcal{T}(N)$ of the set of conformal structures to the space $\mathcal{G}\backslash\mathcal{E}$ of Poincaré-Einstein ends as follows: if $h \in \mathcal{S}_0$ and $r \in C^\infty(N, S^2 N)$ with $\mathrm{Tr}_h(r) = 0$, $\delta_h(r) = 0$, we assign to the cotangent data $(h, r\otimes\mathrm{dvol}_h) \in T^*_{[h]}\mathcal{T}(N)$ the Poincaré-Einstein end $(h, -4r - F_n(h) - \frac{2v_n(h)}{n}h)$.*

Example: $n = 2$. In this case, N is a closed surface of genus at least 2, and $\mathcal{E}$ is the space of hyperbolic ends on $N \times (0, \infty)$. Hyperbolic ends on $N \times (0, \infty)$ are in one-to-one correspondence to complex projective structures on N. Let $\mathcal{CP}$ the space of complex projective structures on N. Given $\sigma \in \mathcal{CP}$, one can consider the underlying complex structure c, and the Fuchsian complex projective structure σ_0 obtained by applying Riemann uniformization to c. Let ϕ be the holomorphic map isotopic to the identity between (N, σ_0) to (N, σ), and let $q = \mathcal{S}(\phi)$ be the Schwarzian derivative of ϕ.

Lemma 6.9. *Let h be the hyperbolic metric in the conformal class c, then, for all $g \in \mathcal{E}$, $(h, \frac{1}{2}\mathrm{Re}(q))$ is the associated cotangent data to c.*

Proof. It is proved in [45, Lemma 8.3] that $\mathrm{II}_0^* = -\mathrm{Re}(q)$, where II^* is the "second fundamental form at infinity" considered in [45] and II_0^* is its traceless part. However comparing the expressions of the hyperbolic metric at infinity in terms of h_2 used here, and in terms of II^* as in [45], shows that $h_2 = 2\mathrm{II}^*$. Finally we have seen in Section 5.3 that $h_2^\circ = -4G_2^\circ$. The result follows. $\qquad\square$

6.4. **Lagrangian submanifold in $T^*\mathcal{T}(N)$.** We now come back to the situation where Hypothesis 6.1, 6.2 and 6.3 apply (we use the same notations as there). Using again that $T^*\mathcal{T}(N)$ near $[h_0]$ is represented by (19), we define the modified Dirichlet-to-Neumann map

$$\mathcal{N}_\Xi : h \in \mathcal{S}_0 \mapsto \mathcal{G}_n^\circ(h) \in T^*_{[h]}\mathcal{T}(N) \tag{76}$$

where $\mathcal{G}_n^\circ(h) = G_n^\circ(h) \otimes \mathrm{dvol}_h$ with $G_n^\circ(h)$ the divergence-free/trace-free tensor G_n° associated to the Poincaré-Einstein end of the AHE metric $\Xi(h)$.

Proposition 6.10. *The section $\mathcal{N}_\Xi$ is an exact 1-form on the slice $\mathcal{S}_0$.*

Proof. By linearizing the identity $v_n(h) = \int_N v_n(h)\mathrm{dvol}_h$ valid for every $h \in \mathcal{S}_0$, we get $\int_N v_n(h)\,\mathrm{Tr}_h(\dot{h})\mathrm{dvol}_h = 0$ if $\dot{h} \in T_h\mathcal{S}_0$, and therefore $\langle G_n, \dot{h}\rangle = \langle G_n^\circ, \dot{h}\rangle$. By Theorem 5.3 we deduce that $\mathcal{N}_\Xi$ is the differential of the map $h \mapsto \mathrm{Vol}_R(M, \Xi(h); h)$. $\square$

Corollary 6.11. *The image of $\mathcal{N}_\Xi$ is a Lagrangian Fréchet submanifold in $T^*\mathcal{T}(N)$.*

Proof. The image is a submanifold since it is the image of a smooth section. It is isotropic for the symplectic form Ω of (17) since the section is an exact form and Ω is the exterior derivative of the Liouville 1-form. Moreover, it is maximal isotropic since it is diffeomorphic to the base by the projection. $\square$

The following corollary is the analog in our higher-dimensional setting of McMullen's Kleinian reciprocity, see [53, Theorem 9.1].

Corollary 6.12. *Let $h \in \mathcal{S}_0$, and let $u, v \in T_h\mathcal{S}_0$. Let u^*, v^* be the corresponding first-order variations of $\mathcal{G}_n^\circ$, so that $u^*, v^* \in T^*_{[h]}\mathcal{T}(N)$. Then*

$$\langle v, u^*\rangle = \langle u, v^*\rangle$$

where $\langle\,,\rangle$ is the bilinear pairing with respect to h followed by integration on N. Equivalently, the linearization $d\mathcal{N}_\Xi$ of $\mathcal{N}_\Xi$ is such that $(d\mathcal{N}_\Xi)_h$ is self-adjoint.

Proof. This is a direct translation of Corollary 6.11 using the definition of the cotangent symplectic structure induced by (17) on $T^*\mathcal{T}(N)$. $\square$

Quasifuchsian reciprocity for Poincaré-Einstein manifolds. We now consider a more specific setting, analogous to the situation occuring for the quasifuchsian reciprocity for 3-dimensional hyperbolic manifolds, see [53]. We consider a manifold M such that ∂M has two connected components, N_+ and N_-. We denote by $\mathcal{M}(N_\pm)$ and $\mathcal{T}(N_\pm)$ the space of Riemannian metrics and the space of conformal structures on $N_\pm$, and we assume that Hypothesis 6.1, 6.3 apply and Hypothesis 6.2 applies on N_+, N_- separately with $v_n \neq 0$, i.e. $\mathcal{S}_0 = \mathcal{S}_0^- \times \mathcal{S}_0^+$.

Given $h = (h_-, h_+) \in \mathcal{S}_0$, let $\mathcal{N}_\Xi^+(h) \in T^*_{h_+}\mathcal{T}(N_+)$ and $\mathcal{N}_\Xi^- \in T^*_{h_-}\mathcal{T}(N_-)$ be $N_\pm$ component of $\mathcal{N}_\Xi(h)$. For fixed h_- we have a section $\mathcal{N}_\Xi^+(h_-, \cdot)$ of $T^*\mathcal{T}(N_+)$, while for fixed h_+ we have a section $\mathcal{N}_\Xi(\cdot, h_+)$ of $T^*\mathcal{T}(N_-)$.

For fixed $h = (h_-, h_+)$, we now consider the linear maps

$$\phi_{h_+} : T_{h_-}\mathcal{S}_0^- \to T^*_{h_+}\mathcal{T}(N_+), \qquad\qquad v_- \mapsto (d\mathcal{N}_\Xi^+)_h(v_-, 0)$$
$$\phi_{h_-} : T_{h_+}\mathcal{S}_0^+ \to T^*_{h_-}\mathcal{T}(N_-), \qquad\qquad v_+ \mapsto (d\mathcal{N}_\Xi^-)_h(0, v_+)$$

Proposition 6.13. *ϕ_{h_-} and ϕ_{h_+} are adjoint.*

Proof. This is simply a particular case of Corollary 6.12. $\square$

7. The Dirichlet-to-Neumann map for the Fuchsian-Einstein case and Hessian of the renormalized volume

In this last section we compute the Hessian of the renormalized volume at a Fuchsian-Einstein metric ($M = \mathbb{R} \times N, g$), when n is odd, and when $n = 2, 4$. In the latter case we will consider the renormalized volume as a function on the slice $\mathcal{S} \subset \mathcal{M}(N)$ of metrics satisfying $v_n = \int_N v_n$ near h_0 Einstein with negative Ricci curvature.

7.1. Hessian of Vol_R at the Fuchsian locus when $n = 2$.

It is instructive to do first the computation for $n = 2$. Let $g = dt^2 + \cosh(t)^2 h_0$ be a Fuchsian metric on $M = N \times \mathbb{R}_t$ for a hyperbolic surface (N, h_0). The conformal boundary consists of (M, g) is $\partial M = N_+ \sqcup N_-$ (corresponding to $t \to \pm\infty$) where each $N_\pm$ is N equipped with the conformal class of h_0. The geodesic boundary defining function associated to h_0 is $x := 2e^{-|t|}$ near $t = \pm\infty$; the metric g takes the form near ∂M

$$g = x^{-2}(dx^2 + h_0 + \tfrac{1}{2}x^2 h_0 + \tfrac{1}{16}x^4 h_0) \text{ as } x \to 0.$$

We have the following result

Proposition 7.1. *Let h_0 be a hyperbolic metric on a Riemann surface N with genus ≥ 2. We identify Teichmüller space $\mathcal{T}(N)$ of N with a slice of hyperbolic metrics, with tangent spaces at each point the space of divergence-free/trace-free tensors. Let $\Phi : h_- \mapsto \Phi(h_-)$ be the Bers map sending a hyperbolic metric $h_- \in \mathcal{T}(N)$ on N to the quasifuchsian hyperbolic metric on $N \times \mathbb{R}_t$ with conformal boundary h_0 at N_+, h_- at N_-. Then the map $V_{h_0} : h_- \mapsto \mathrm{Vol}_R(M, \Phi(h_-); (h_0, h_-))$ has a unique critical point at $h_- = h_0$ on $\mathcal{T}(N)$ and the Hessian there is*

$$\mathrm{Hess}_{h_0}(V_{h_0})(k) = \frac{1}{8} \int_N |k|^2_{h_0} \mathrm{dvol}_{h_0}, \quad k \in T_{h_0}\mathcal{T}(N).$$

Proof. The fact that the Fuchsian metric g is a critical point is a consequence of the fact that the trace-free part G_2° of G_2 is 0 by (63). To see that it is the unique critical point, we claim that for a critical point quasifuchsian metric $g = \Phi(h_-)$, the trace-free part G_2° of G_2 at both conformal boundaries is 0, which means that the 2 hyperbolic ends are of the form $x^{-2}(dx^2 + (1 + \frac{x^2}{4})^2 h_\pm)$ for $h_+ = h_0$ and h_- some hyperbolic metric; thus the quasifuchsian metric g would have two embedded totally geodesic surfaces corresponding to $x = 2$ if $h_+ \neq h_0$, and this is not possible by topological reasons since by doubling the region bounded by theses surfaces, we would get a closed 3 dimensional hyperbolic manifold which is $S^1 \times N$ with 2 embedded totally geodesic surfaces, thus toroidal, contradicting Thurston hyperbolisation theorem.

Next we compute the Hessian. We deform g by a 1-parameter family of quasifuchsian metrics g^s by means of a divergence-free/trace-free tensor k as follows:

$$g^s := dt^2 + h_0 + e^{-2t} h_2^s + \tfrac{1}{4}e^{-4t}(h_2^s)^2, \qquad h_2^s = \tfrac{1}{2}h_0 + sk.$$

This amounts to changing the conformal class on N_- only. We denote conformal representatives in the conformal boundary by pairs (h_+^s, h_-^s) corresponding to the components $N_\pm$.

For small s, the expression for g^s makes sense for all $t \in \mathbb{R}$; at N_+, x induces the conformal representative $h_+^s = h_0$ and at N_- one has $h_-^s = 4(h_2^s)^2$ since the metric near $t = -\infty$ is

$$g^s = x^{-2}(dx^2 + h_-^s + x^2 h_2^s + x^4 h_4^s).$$

Notice that $h_-^0 = h_0$ is hyperbolic, but for other values of s it is not. The variation formula from Section 5.3 gives for s near 0 (with $\dot{h}_-^s = \partial_s h_-^s$)

$$-4\partial_s \mathrm{Vol}_R(M, g^s; (h_0, h_-^s)) = \int_N \langle \dot{h}_-^s, h_2^s - \mathrm{tr}_{h_-^s}(h_2^s)h_-^s \rangle_{h_-^s} \mathrm{dvol}_{h_-^s}.$$

We have

$$h_- := h_-^0 = h_0, \qquad \dot{h}_- := \dot{h}_-^0 = 4k, \qquad \partial_s^2 h_-^s|_{s=0} = 8k^2, \qquad \mathrm{tr}_{h_-^s}(h_2^s) = 1 + \mathcal{O}(s^2),$$

and we compute (with the dot notation for $\partial_s|_{s=0}$)

$$-4\partial_s^2 \mathrm{Vol}_R(M, g^s; (h_0, h_-^s))|_{s=0} = \langle \dot{h}_-, \dot{h}_2 \rangle - \tfrac{1}{2}\langle \partial_s^2 h_-^s|_{s=0}, h_0 \rangle = 0.$$

We are interested in the renormalized volume where the boundary families are uniformized to have scalar curvature -2. This means, we must consider $\mathrm{Vol}_R(M, g^s; (h_0, \hat{h}_-^s))$ where $\hat{h}_-^s := e^{2\omega_0^s}h_-^s$ is the unique hyperbolic metric in the conformal class of h_-^s. Then $\partial_s \hat{h}^s$ is the sum of a Lie derivative of $\hat{h}^s$ and a divergence free/trace free tensor, in particular $\int_N \mathrm{tr}_{\hat{h}_-^s}(\partial_s \hat{h}_-^s)\mathrm{dvol}_{\hat{h}_-^s} = 0$. From this we can derive the identity

$$\int_N \dot{\omega}_0^s \mathrm{dvol}_{h_-^s} = 4s \int_N |k|_{h_0}^2 \mathrm{dvol}_{h_0} + \mathcal{O}(s^2).$$

Since $\mathrm{Scal}_{h_-^s} = -2\mathrm{tr}_{h_-^s}(h_2^s) = -2 + \mathcal{O}(s^2)$, it follows that $\omega_0^s = \tfrac{1}{2}s^2\alpha + o(s^2)$ for some α with $\int_N \alpha = 4 \int_N |k|_{h_0}^2 \mathrm{dvol}_{h_0}$. Proposition 3.11 shows that

$$\mathrm{Vol}_R(M, g^s; (h_0, \hat{h}_-^s)) = -\frac{1}{4}\int_M (|\nabla \omega_0^s| + \mathrm{Scal}_{h_-^s}\,\omega_0^s)\mathrm{dvol}_{h_-^s}.$$

The only term of order 2 which survives is

$$\mathrm{Vol}_R(M, g^s; (h_0, \hat{h}_-^s)) = s^2 \int_N |k|_{h_0}^2 \mathrm{dvol}_{h_0} + o(s^3).$$

This computes the Hessian of the renormalized volume at (h_0, h_0) in the direction $(0, 4k)$. $\quad\square$

7.2. Higher dimensions. Second variation of the volume in terms of G_n and h_n

Let us consider a family of AHE metrics g^s (for s near 0) on $M = \mathbb{R}_t \times N$ with N compact and $g^0 = g$ with

$$g = dt^2 + \cosh^2(t)h_0$$

where $\mathrm{Ric}_{h_0} = -(n-1)h_0$. The conformal infinity of (M, g) is $\partial M = N_+ \sqcup N_-$ where each $N_\pm$ is N equipped with the conformal class of h_0. Notice that $x := 2e^{-|t|}$, defined outside $t = 0$, is the geodesic boundary defining function associated to the conformal representative h_0 on ∂M. When n is even, we choose (for s near 0) the smooth family h_0^s of metrics on ∂M so that $v_n(h_0^s) = \int_N v_n(h_0^s)\mathrm{dvol}_{h_0^s} \neq 0$ and $[h_0^s]$ is the conformal infinity of (M, g^s); this is possible by Corollary 4.5 and implies $\mathrm{Vol}(N, h_0^s) = 1$ and $\int_N \mathrm{Tr}_{h_0^s}(\partial_s h_0^s)\mathrm{dvol}_{h_0^s} = 0$.

Case n odd. By a result of Albin [2] (see Theorem 5.3), we have $\partial_s \text{Vol}_R(M, g^s; h_0^s) = -\frac{1}{4}\langle h_n^s, \partial_s h_0^s \rangle_{L^2}$ and since $h_n = 0$ at $s = 0$,

$$\partial_s^2 \text{Vol}_R(M, g^s)|_{s=0} = -\frac{1}{4}\langle \dot{h}_n, \dot{h}_0 \rangle_{L^2}.$$

Case n even. By Theorem 5.3, we have $\partial_s \text{Vol}_R(M, g^s; h_0^s) = \langle G_n^s, \partial_s h_0^s \rangle_{L^2}$ and $\text{Tr}_{h_0^s}(G_n^s) = \frac{1}{2}v_n(h_0^s)$ is constant, it follows that $\partial_s \text{Vol}_R(M, g^s; h_0^s) = \langle (G_n^s)^\circ, \partial_s h_0^s \rangle$ where $(G_n^s)^\circ$ is the trace-free part of G_n^s, which vanishes at $s = 0$ by Lemma 5.5. Hence

$$\partial_s^2 \text{Vol}_R(M, g^s; h_0^s)|_{s=0} = \langle \dot{G}_n^\circ, \dot{h}_0 \rangle_{L^2}$$

where $\dot{G}_n^\circ := \partial_s[(G_n^s)^\circ]|_{s=0}$. Moreover since $\text{Tr}_{h_0^s}(G_n^s) = \frac{1}{2}v_n(h_0^s)$,

$$\partial_s^2 \text{Vol}_R(M, g^s; h_0^s)|_{s=0} = \langle \dot{G}_n, \dot{h}_0 \rangle_{L^2} - \frac{v_n}{2n}|\dot{h}_0|_{L^2}^2 - \frac{1}{2n}\int_N \dot{v}_n \text{Tr}_{h_0}(\dot{h}_0)\text{dvol}_{h_0}. \tag{77}$$

Variation of the local term F_n when $n = 4$. Since $-4G_n = h_n + F_n$ where F_n is local in terms of h_0, we have to compute the variation $\dot{F}_n$. In general even dimension n, we do not have a formula for F_n, thus we will restrict to $n = 4$.

Will will now assume that $\dot{h}_0$ is divergence free, so that by Lemma 6.5,

$$\text{Tr}_{h_0}(\dot{h}_0) = 0, \quad \delta_{h_0}(\dot{h}_0) = 0. \tag{78}$$

Using this, we compute $\langle \dot{F}_n, \dot{h}_0 \rangle$ for $n = 4$.

Lemma 7.2. *In dimension $n = 4$, assuming (78), we have*

$$\langle \dot{F}_4, \dot{h}_0 \rangle_{L^2} = (\tfrac{1}{8} - v_4)|\dot{h}_0|_{L^2}^2 + \tfrac{1}{2}\langle \dot{k}, \dot{h}_0 \rangle_{L^2}, \qquad\qquad \text{Tr}_{h_0}(\dot{F}_4) = 0.$$

Proof. We recall that $F_4^s = -\frac{1}{2}(h_2^s)^2 + \frac{1}{4}\text{Tr}_{h_0^s}(h_2^s)h_2^s - v_4^s h_0^s + \frac{1}{2}k^s$. Using that $h_2 = \frac{1}{2}h_0$ and $\text{Scal}_{h_0} = -12$, we obtain

$$\langle \dot{F}_4, \dot{h}_0 \rangle = -\langle \frac{\dot{\text{Scal}}}{48}, \dot{h}_0 \rangle + (\tfrac{1}{8} - v_4)|\dot{h}_0|^2 + \tfrac{1}{2}\langle \dot{k}, \dot{h}_0 \rangle.$$

Moreover by (56) and the fact that $\text{Tr}_{h_0}(\dot{h}_0) = 0$ and $\delta_{h_0}(\dot{h}_0) = 0$, we have $\dot{\text{Scal}} = 0$. Similarly, using $\dot{v}_4 = 0$, $\text{Tr}_{h_0}(\dot{k}) = 0$, and that $\text{Tr}_{h_0}(\dot{h}_2) = 2\dot{v}_2$ is a multiple of $\dot{\text{Scal}} = 0$, we easily see that the trace of $\dot{F}_4$ is 0. $\square$

Bianchi gauge. Let us define $\dot{g} := \partial_s g^s|_{s=0}$, which solves the linearized Einstein equation. Since this equation is not elliptic due to gauge invariance (by diffeomorphism actions) we have to fix a gauge, as is well known in the study of Einstein equation. We shall use Bianchi gauge: using for instance Proposition 4.5 in [68], there exists a smooth vector field X on M so that

$$q := \dot{g} + L_X g \text{ solves } \delta_g(q) + \tfrac{1}{2}d\text{Tr}_g(q) = 0 \tag{79}$$

and q has an asymptotic expansion $q = x^{-2}(q_0 + \sum_{j \leq n} q_j x^j + x^n \log(x)q_{n,1}) + o(x^n)$ as $x \to 0$ for some x independent tensors $q_j, q_{n,1}$ on $[0, \varepsilon)_x \times N$ and

$$q_0 = \dot{h}_0, \qquad\qquad q_n = \dot{h}_n + T_n\dot{h}_0$$

with T_n a differential operator. We notice (see [68]) that X is the vector field dual to the form ω solving

$$(\Delta_g + n)\omega = -2\delta_g(\dot{g}) - d\text{Tr}_g(\dot{g}). \tag{80}$$

where $\Delta_g = \nabla^*\nabla$. In dimension $n = 4$, we compute q_4:

Lemma 7.3. *Let $n = 4$, then assuming* (78), *we have for q defined by* (79)

$$q = \dot{g} + o(x^4),$$

where the $o(x^4)$ is with respect to the norm induced by g.

Proof. In this proof, all error terms are measured with respect to the metric g. First, since $\text{Tr}_{h_0}(\dot{h}_0) = 0$, $\text{Tr}_{h_0}(\dot{h}_2) = \dot{\text{Scal}} = 0$ and $\text{Tr}_{h_0}(\dot{k}) = 0$ (since $k = 0$ for Einstein manifold and $\text{Tr}_{h_0^s}(k^s) = 0$ for all s), we have modulo $o(x^4)$

$$\text{Tr}_g(\dot{g}) = x^4\text{Tr}_{h_0}(\dot{h}_4), \qquad\qquad d\text{Tr}_g(\dot{g}) = 4x^4\text{Tr}_{h_0}(\dot{h}_4)\frac{dx}{x}.$$

For the divergence, we use formula (84) and $\delta_{h_0}(\dot{h}_0) = 0$, we get modulo $o(x^4)$ (we use $x = 2e^{-|t|}$)

$$\delta_g(\dot{g}) = x^4\text{Tr}_{h_0}(\dot{h}_4)\frac{dx}{x} + x^2\delta_{h_0}(\dot{h}_2).$$

But since $\dot{\text{Scal}} = 0$, $\delta_{h_0^s}(h_0^s) = 0$ and $\delta_{h_0^s}(\text{Ric}_{h_0^s} - \frac{1}{2}\text{Scal}_{h_0^s}) = 0$ we have

$$\delta_{h_0}(\dot{h}_2) = \frac{1}{2}\dot{\delta}(\text{Ric}_{h_0} - \frac{1}{2}\text{Scal}_{h_0}) = \frac{3}{2}\dot{\delta}(h_0) = -\frac{3}{2}\delta(\dot{h}_0) = 0.$$

where $\dot{\delta} = \partial_s\delta_{h_0^s}|_{s=0}$. Therefore modulo $o(x^4)$

$$-2\delta_g(\dot{g}) - d\text{Tr}_g(\dot{g}) = -6x^4\text{Tr}_{h_0}(\dot{h}_4)\frac{dx}{x}.$$

Now by Theorem 5.3, $\text{Tr}_{h_0}(\dot{h}_4) + \text{Tr}_{h_0}(\dot{F}_4) = -2\dot{v}_4 = -\frac{1}{2}\langle k, \dot{h}_0\rangle = 0$ thus $\text{Tr}_{h_0}(\dot{h}_4) = 0$ by Lemma 7.2. We now use Section 4 in [68] and refer the reader to that for details: the construction of [68] (based on an approximate solution using indicial equations and the correction using the Green's function of $\Delta_g + n$ on 1-forms on M) yields that there is a polyhomogeneous form $\omega = o(x^4)$, satisfying $(\Delta_g + n)\omega = -2\delta_g(\dot{g}) - d\text{Tr}_g(\dot{g})$. A straightforward computation gives that if X is the dual vector field defined by $g(X, \cdot) = \omega$, then $L_X g = o(x^4)$. $\square$

Linearized Einstein operator In this section, n can be either even or odd. Now that q is in the kernel of the Bianchi operator $\delta_g + \frac{1}{2}d\text{Tr}_g$, then we see by linearizing the Einstein equation that q solves

$$L_g q := (\nabla^*\nabla - 2\mathring{R})q = 0 \tag{81}$$

where $\mathring{R}$ is the operator acting on symmetric 2 tensors defined by

$$(\mathring{R}q)(Y, Z) = -\sum_{i,j}\langle R_{Y,E_i}Z, E_j\rangle q(E_i, E_j)$$

if $(E_j)_j$ is an orthonormal basis for g and R the Riemann tensor of g. Notice that if u is a function, then $L_g(ug) = ((\Delta_g + 2n)u)g$. Since moreover L_g maps trace-free tensors to trace-free tensors, we deduce that $(\Delta_g + 2n)\text{Tr}_g(q) = 0$. From the work of Mazzeo [50],

the solutions of this equation are polyhomogeneous, they are combinations of functions in $x^{\frac{n}{2}\pm s}C^\infty(\overline{M})$, where $s = \frac{1}{2}\sqrt{n(n+8)}$, and thus since $\mathrm{Tr}_g(q) \in C^\infty(\overline{M}) + x^n \log(x)C^\infty(\overline{M})$, then $\mathrm{Tr}_g(q) = 0$, and thus

$$\delta_g(q) = 0, \qquad\qquad\qquad \mathrm{Tr}_g(q) = 0. \qquad (82)$$

We want to express the operator L in the decomposition $\mathbb{R}_t \times N$ acting on divergence-free, trace-free tensors. We will decompose such a tensor q into

$$q = u dt^2 + \xi \overset{s}{\otimes} dt + r$$

where u is a function, $\xi \in \Lambda^1(N)$ is a one form on N and $r \in S^2(N)$ a symmetric tensor on N. Here $\overset{s}{\otimes}$ denote the symmetric tensor product. The following Lemma is proved by Delay [21][2], we give a couple of details of the computations for the reader's convenience.

Lemma 7.4. *Let $g := dt^2 + f^2 h_0$ on $M = \mathbb{R}_t \times N$ for some compact manifold (N, h_0) and $f \in C^2(\mathbb{R})$ some positive function. Then, if $q = u dt^2 + \xi \overset{s}{\otimes} dt + r$ with $\mathrm{Tr}_g(q) = 0$ and $\delta_g(q) = 0$, we have*

$$L_g q = \left(-u'' + f^{-2}\Delta_{h_0}u - (n+4)\frac{f'}{f}u' - 2[(n+1)\frac{(f')^2}{f^2} + \frac{f''}{f}]u\right)dt^2$$

$$+ \left(-\xi'' - (n+2)\frac{f'}{f}\xi' - [(n-1)\frac{(f')^2}{f^2} + 2\frac{f''}{f}]\xi + f^{-2}\Delta_{h_0}\xi - 2\frac{f'}{f}d^N u\right)\overset{s}{\otimes}dt$$

$$+ 2(ff'' - (f')^2)uh_0 - 4\frac{f'}{f}\delta_{h_0}^*\xi - r'' - n\frac{f'}{f}r' + 2\frac{(f')^2}{f^2}\mathrm{Tr}_{h_0}(r)h_0 + f^{-2}L_{h_0}r$$

where $\xi' := \nabla_{\partial_t}\xi$, $\xi'' = \nabla_{\partial_t}\nabla_{\partial_t}\xi$ with the same notation for r'', r'. Here L_{h_0} is the linearized Einstein operator defined like (81) but on N with the metric h_0.

Proof. First, since $\mathrm{Tr}_g(q) = 0$, we have

$$u = -f^{-2}\mathrm{Tr}_{h_0}(r). \qquad (83)$$

Let $T := \partial_t$, let v be some (t-independent) vector field on N, and set $V := f^{-1}v$. From (73) we deduce that if $A = fa$ with $a \in \Lambda_1(N)$ independent of t,

$$\nabla dt = f'fh_0, \qquad \nabla_T A = 0, \qquad \nabla_V A = \nabla_V^N A - f^{-1}f'A(V)dt.$$

We also have that for any $q \in S^2(M)$

$$\nabla^*(dt \otimes q) = -n\frac{f'}{f}q - \nabla_T q.$$

By direct computation we also obtain the formula for the divergence

$$\delta_g(q) = \left(-u' - n\frac{f'}{f}u + f^{-2}\delta_{h_0}(\xi) + \frac{f'}{f^3}\mathrm{Tr}_{h_0}(r)\right)dt$$

$$- (\nabla_T\xi + (n+1)\frac{f'}{f}\xi) + f^{-2}\delta_{h_0}(r). \qquad (84)$$

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[2]The t-derivative denoted by prime in our setting is with respect to the connection of g and is not exactly the same as Delay, which is why the coefficients are slightly different.

From (83) and (84), since q is divergence-free we obtain

$$u' = -(n+1)\frac{f'}{f}u + f^{-2}\delta_{h_0}(\xi), \qquad\qquad \nabla_T\xi + (n+1)\frac{f'}{f}\xi = f^{-2}\delta_{h_0}(r). \qquad (85)$$

Let y_j be Riemannian normal coordinates at $p \in N$. Then $e_j := \partial_{y_j}$ are parallel and orthonormal at p: $\nabla^N_{e_i}e_j = 0$ and $h_0(e_i, e_j) = \delta_{ij}$. Set $E_i = f^{-1}e_i$. At the point (t, p) for all $t \in \mathbb{R}$ we have

$$\nabla^*\nabla q = -\nabla_T\nabla_T q - n\frac{f'}{f}\nabla_T q - \sum_{j=1}^n \nabla_{E_i}\nabla_{E_i}q.$$

Using this we compute for $q_1 = u\,dt^2$

$$\nabla^*\nabla(u\,dt^2) = \left(-u'' + f^{-2}\Delta_N u - n\frac{f'}{f}u' + 2n\frac{(f')^2}{f^2}u\right)dt^2 - 2\frac{f'}{f}(d^N u \overset{s}{\otimes} dt) - 2(f')^2 u h_0.$$

For $q_2 = \xi \overset{s}{\otimes} dt$, we get

$$\nabla^*\nabla(\xi \overset{s}{\otimes} dt) = \left(-\xi'' - n\frac{f'}{f}\xi' + (n+3)\frac{(f')^2}{f^2}\xi + f^{-2}\Delta_{h_0}\xi\right) \overset{s}{\otimes} dt$$
$$- 4\frac{f'}{f^3}\delta_{h_0}(\xi)dt^2 - 4\frac{f'}{f}\delta^*_{h_0}\xi.$$

Finally for the tangential part r, we get

$$\nabla^*\nabla r = -2f^{-2}\mathrm{tr}_{h_0}(r)\frac{(f')^2}{f^2}dt^2 - 2\frac{f'}{f^3}\delta_{h_0}r \overset{s}{\otimes} dt - r'' - n\frac{f'}{f}r' + 2\frac{(f')^2}{f^2}r + f^{-2}\Delta_{h_0}r.$$

In conclusion, using (83) and (85) to substitute for $\mathrm{tr}_{h_0}(r), \delta_{h_0}\xi$ and $\delta_{h_0}r$ we get

$$\nabla^*\nabla q = \left(-u'' + f^{-2}\Delta_N u - (n+4)\frac{f'}{f}u' - 2(n+1)\frac{(f')^2}{f^2}u\right)dt^2$$
$$+ \left(-\xi'' - (n+2)\frac{f'}{f}\xi' - (n-1)\frac{(f')^2}{f^2}\xi + f^{-2}\Delta_{h_0}\xi - 2\frac{f'}{f}d^N u\right) \overset{s}{\otimes} dt$$
$$- 2(f')^2 u h_0 - 4\frac{f'}{f}\delta^*_{h_0}\xi - r'' - n\frac{f'}{f}r' + 2\frac{(f')^2}{f^2}r + f^{-2}\Delta_{h_0}r$$

On the other hand, from (75) and (83) we get

$$(\mathring{R}q) = \frac{f''}{f}(u\,dt^2 + \xi \overset{s}{\otimes} dt) + f^{-2}(\mathring{R}_{h_0}r) - ff''uh_0 + \frac{(f')^2}{f^2}(r - \mathrm{Tr}_{h_0}(r)h_0).$$

Combining this with the formula for $\nabla^*\nabla$, the Lemma is proved. $\qquad\square$

7.3. Computation of q for n odd or $n = 4$.

We start by showing that the dt^2 and $\xi \overset{s}{\otimes} dt$ components of q vanish identically. In the following Lemma, n can be either odd or even.

Lemma 7.5. *Assume that $f(t) = \cosh(t)$. Let $q = \dot{g} + L_X g = u\,dt^2 + \xi \overset{s}{\otimes} dt + r$ be the trace-free and divergence-free tensor in $\ker L_g$ defined in (79). Then $u = 0$ and $\xi = 0$.*

Proof. Let $(\varphi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of eigenvectors for the Laplacian Δ_{h_0} acting on functions on N, with eigenvalues λ_j. From Lemma 7.4, we see after writing $u = \sum_{j\in\mathbb{N}} u_j = \sum_{j\in\mathbb{N}} \varphi_j\langle u, \varphi_j\rangle\varphi_j$ that u_j satisfies the ODE

$$-u_j'' - (n+4)\tanh(t)u_j' - \Big(2(n+1)\tanh(t)^2 + 2 - \frac{\lambda_j}{\cosh(t)^2}\Big)u_j = 0.$$

Setting $u_j = f^{-\frac{n}{2}-2}v_j$, this equation can be rewritten

$$-v_j'' + \Big(\frac{n(n-2)}{4}\tanh(t)^2 + \frac{\lambda_j}{\cosh(t)^2} + \frac{n}{2}\Big)v_j = 0$$

and since $\lambda_j \geq 0$, this equation has no solution in $L^2(\mathbb{R}, dt)$ because the corresponding operator is strictly positive. By standard ODE theory (e.g this equation is also a hypergeometric equation after setting $\sinh(t)^2 = z$), the solutions are linear combinations of two independent functions F_1, F_2 such that $F_1(t) \sim_{t\to+\infty} e^{\alpha_+ t}$ and $F_2(t) \sim_{t\to+\infty} e^{\alpha_- t}$ with $\alpha_\pm = \pm\frac{n}{2}$ the roots of the polynomial $-\alpha^2 + \frac{n^2}{4}$. Since $\delta_{h_0}(\mathring{h}_0) = 0$, we have $|\delta_{h_0}(r)|_g = \mathcal{O}(1)$ and thus by the second equation of (85), we deduce that $|\xi|_g = \mathcal{O}(x^2)$, which implies that $\delta_{h_0}(\xi) = \mathcal{O}(x)$ and by the first equation of (85) we get $u = \mathcal{O}(x^3) = \mathcal{O}(e^{-3|t|})$ (here recall that $x = 2e^{-|t|}$ for large $|t|$). Therefore v_j is a constant times F_2 and thus of order $\mathcal{O}(e^{-\frac{n}{2}t})$ when $|t| \to +\infty$. This shows that $v_j = \mathcal{O}(e^{-\frac{n}{2}|t|}) \in L^2(\mathbb{R}, dt)$, thus $v_j = 0$ and hence $u = 0$.

Writing the mixed component of $L_g(q)$ to be 0, using $u = 0$ and decomposing $\xi = \sum_j \xi_j f\psi_j$ where $(\psi_j)_j$ is an $L^2(N, \mathrm{dvol}_{h_0})$ orthonormal basis of eigenvectors of Δ_{h_0} on 1-forms with eigenvalues α_j, we get from Lemma 7.4 and $\nabla_{\partial_t}(f\psi_j) = 0$

$$-\partial_t^2\xi_j - (n+2)\tanh(t)\partial_t\xi_j - \Big((n-1)\tanh(t)^2 + 2 - \frac{\alpha_j}{\cosh(t)^2}\Big)\xi_j = 0.$$

Setting $\xi_j = \zeta_j f^{-\frac{n}{2}-1}$, one has

$$-\partial_t^2\zeta_j + \Big(\big(\frac{n^2}{4} - \frac{n}{2} + 1\big)\tanh(t)^2 + \frac{\alpha_j}{\cosh(t)^2} + \frac{n}{2} - 1\Big)\zeta_j = 0.$$

One one hand, again by positivity, this equation has no solutions in $L^2(\mathbb{R}, dt)$. On the other hand, we have seen above that $|\xi|_g = \mathcal{O}(x^2) = \mathcal{O}(e^{-2|t|})$ hence $\zeta_j = \mathcal{O}(e^{|t|})$. Since the indicial roots in the above equation are ± 2, ζ_j must be of order $\mathcal{O}(e^{-2|t|})$ which is clearly in $L^2(\mathbb{R}, dt)$, so actually we deduce $\xi_j = 0$. $\square$

We are going now to compute the coefficient of x^{n-2} in the expansion of r. Recall that we chose $x = 2e^{-|t|}$ for $t \neq 0$.

Proposition 7.6. *Let $r = q$ be the TT tensor in $\ker L_g$ defined in (79) under the assumption (78). Let $r_0^\pm, r_n^\pm, r_{n,1}^\pm$ be the tensors on N so that as $t \to \pm\infty$*

$$r = x^{-2}\Big(r_0^\pm + \sum_{j=1}^{n} r_j^\pm x^j + r_{n,1}^\pm x^n \log(x) + o(x^n)\Big).$$

Let $L_{h_0} = \Delta_{h_0} - 2\mathring{R}_{h_0}$ be the linearized Einstein operator on (N, h_0). For every j denote by r_j^0, r_j^1 the even, respectively the odd component of the pair $r_j = (r_j^+, r_j^-)$ with respect to $t \mapsto -t$.

(1) *When n is even, $r_{n,1}^{\pm}$ is given by a differential (hence, local) operator of order n in terms of r_0:*

$$r_{n,1} = \frac{(-1)^{\frac{n}{2}+1} 2^{1-n}}{\frac{n}{2}!(\frac{n}{2}-1)!} \prod_{j=0}^{\frac{n}{2}-1} (L_{h_0} - j(n-1-j)) r_0. \tag{86}$$

(2) *If n is odd, then $r_{n,1}^{\pm} = 0$, and for $\varepsilon \in \{0,1\}$, r_n^{ε} are given by*

$$r_n^{\varepsilon} = \mathcal{N}_{\varepsilon}(r_0^{\varepsilon}) \tag{87}$$

where $\mathcal{N}_{\epsilon}$ is the pseudodifferential operator of order n

$$\mathcal{N}_{\varepsilon} = 2^{-n} \frac{\Gamma(-\frac{n}{2})}{\Gamma(\frac{n}{2})} \mathcal{F}_{\varepsilon}(\sqrt{L_{h_0} - (n-1)^2/4})$$

for $\sqrt{\cdot} : \mathbb{R} \to \mathbb{R}^+ \cup i\mathbb{R}^-$ being the square root function, and $\mathcal{F}_{\varepsilon}$ defined by

$$\mathcal{F}_{\varepsilon}(u) := u(\tanh(\tfrac{\pi}{2}u))^{(-1)^{\frac{n-1}{2}-\varepsilon}} \prod_{\ell=1}^{\frac{n-1}{2}} (u^2 + \ell^2).$$

(3) *For $n = 4$ and $\varepsilon \in \{0,1\}$, r_4^{ε} are given by $r_4^{\varepsilon} = \mathcal{G}_{\varepsilon}(\sqrt{L_{h_0} - \frac{9}{4}}) r_0^{\varepsilon}$ with*

$$\mathcal{G}_{\varepsilon}(u) := -\frac{1}{32} \Big[\Big(c_0 - (-1)^{\varepsilon} \pi \frac{1 - \text{Im}(\sinh(\pi u))}{\cosh(\pi u)} + 2\text{Re}\Psi(\tfrac{5}{2} - iu) \Big)(u^2 + \tfrac{1}{4})(u^2 + \tfrac{9}{4})$$
$$- 2\text{Im}(u)(2u^2 + \tfrac{5}{2}) + (u^2 + \tfrac{1}{4})^2 \Big]$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, and $c_0 := -\frac{5}{2} + 2\gamma - 2\ln(2)$.

Proof. Setting $r = sf^{-\frac{n}{2}+2}$ and $s = \sum_{j \in \mathbb{N}} s_j \phi_j$ where ϕ_j is an $L^2(N, \text{dvol}_{h_0})$ orthonormal basis of eigenvectors for L_{h_0} with eigenvalues γ_j, then since $\nabla_{\partial_t}(f^2 \phi_j) = 0$

$$r' = f^{-\frac{n}{2}+2} \sum_j (\partial_t s_j - \frac{n}{2}\frac{f'}{f} s_j)\phi_j, \qquad r'' = f^{-\frac{n}{2}+2} \sum_j ((\partial_t - \frac{n}{2}\frac{f'}{f})^2 s_j)\phi_j.$$

We then have from Lemma 7.4 and Lemma 7.5 that s_j satisfies the equation

$$-\partial_t^2 s_j + \Big(z^2 - \frac{\nu_j(\nu_j+1)}{\cosh^2(t)} \Big) s_j = 0, \tag{88}$$

with $\nu_j \in (-\frac{1}{2} + i\mathbb{R}_+) \cup [-\frac{1}{2}, \infty)$, $\nu_j(\nu_j+1) = \frac{n(n-2)}{4} - \gamma_j$, and $z = \frac{n}{2}$. Let us consider more generally this equation for $z \in \mathbb{R}$ near $n/2$. From [37, Appendix], it has two independent solutions on $\mathbb{R}$, one odd and one even in t:

$$E_1(t) = \sinh(t)\cosh(t)^{1+\nu_j} F_1(\tfrac{\nu_j+z+2}{2}, \tfrac{\nu_j-z+2}{2}, \tfrac{3}{2}; -\sinh(t)^2)$$
$$E_0(t) = \cosh(t)^{1+\nu_j} F_1(\tfrac{\nu_j+z+1}{2}, \tfrac{\nu_j-z+1}{2}, \tfrac{1}{2}; -\sinh(t)^2)$$

where $F_1(a, b, c; \tau)$ is the hypergeometric function. The solution E_1 corresponds to taking $r_0^- = -r_0^+$ while E_0 corresponds to $r_0^+ = r_0^-$. Using the identity

$$F_1(a, b, c; \tau) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-\tau)^{-a}F_1(a, a+1-c, a+1-b; \tfrac{1}{\tau})$$
$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-\tau)^{-b}F_1(b, b+1-c, b+1-a; \tfrac{1}{\tau})$$

and $F_1(a, b, c; 0) = 1$, we see that, for $x = 2e^{-|t|}$, there exist meromorphic coefficients $a_{2k}(z)$ such that

$$E_1(t)\frac{\Gamma(\frac{\nu_j+z+2}{2})\Gamma(\frac{-\nu_j+z+1}{2})}{\Gamma(\frac{3}{2})\Gamma(z)}$$
$$= \text{sign}(t)x^{-z}\left(1 + \sum_{k=1}^{\frac{n}{2}} x^{2k}a_{2k}(z, \nu_j) + x^{2z}\frac{\Gamma(-z)\Gamma(\frac{-\nu_j+z+1}{2})\Gamma(\frac{\nu_j+z+2}{2})}{\Gamma(z)\Gamma(\frac{-\nu_j-z+1}{2})\Gamma(\frac{\nu_j-z+2}{2})} + o(x^{2z})\right). \tag{89}$$

We shall use the same type of arguments as in the work of Graham-Zworski [33]; the coefficients $a_k(z, \nu_j)$ are regular near $z = n/2$ except for $a_n(z, \nu_j)$ when n is even, which has a first order pole at $z = n/2$. The coefficient

$$S_j^1(z) = \frac{\Gamma(-z)\Gamma(\frac{-\nu_j+z+1}{2})\Gamma(\frac{\nu_j+z+2}{2})}{\Gamma(z)\Gamma(\frac{-\nu_j-z+1}{2})\Gamma(\frac{\nu_j-z+2}{2})}. \tag{90}$$

of x^z in (89) also has a pole in that case, and its residue is $-\text{Res}_{\frac{n}{2}}a_n(z, \nu_j)$. Notice that S_j^1 is the action of the scattering operator of L_{h_0} at $z \in \mathbb{C}$ on the odd pair of tensors $(\phi_j, -\phi_j)$. Using the formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ and $\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\sqrt{\pi}\Gamma(2s)$ we rewrite

$$S_j^1(z) = 2^{-2z}\frac{\Gamma(-z)}{\Gamma(z)}\frac{\sin(\frac{\pi}{2}(\nu_j-z))}{\sin(\frac{\pi}{2}(\nu_j+z))}\frac{\Gamma(z-\nu_j)}{\Gamma(-z-\nu_j)}. \tag{91}$$

When n is even, the right-hand side of (89) at $z = n/2$ has the asymptotic expansion as $t \to \pm\infty$

$$\pm x^{-\frac{n}{2}}\left(1 + \sum_{k=1}^{\frac{n}{2}-1} x^{2k}a_{2k}(\tfrac{n}{2}, \nu_j) + 2(\text{Res}_{\frac{n}{2}}S_j^1(z))x^n\log(x)\right.$$
$$\left. + \text{FP}_{\frac{n}{2}}(a_n(z, \nu_j) + S_j^1(z))x^n + o(x^n)\right) \tag{92}$$

where FP denotes finite part. From (91), we deduce the formula (86) by taking the residue at $z = n/2$ and the fact that $L_{h_0}\phi_j = (\frac{n(n-2)}{4} - \nu_j(\nu_j+1))\phi_j$.

If now n is odd, we can take the limit $z \to \frac{n}{2}$ in (89) and each coefficient is smooth at $z = n/2$ (a_n does not exist in this parity) writing $\nu_j = -\frac{1}{2} + i\alpha_j$ with $\alpha_j = \sqrt{\gamma_j - \frac{(n-1)^2}{4}}$ (the convention is $i\alpha_j \in \mathbb{R}^+$ if $\gamma_j < \frac{(n-1)^2}{4}$), we obtain directly that the coefficient of $x^{\frac{n}{2}}$ in (89) is

$$2^{-n}\frac{\Gamma(-\frac{n}{2})}{\Gamma(\frac{n}{2})}\alpha_j\left(\frac{\cosh(\frac{\pi}{2}\alpha_j)}{\sinh(\frac{\pi}{2}\alpha_j)}\right)^{(-1)^{\frac{n-1}{2}}}\prod_{\ell=1}^{\frac{n-1}{2}}(\alpha_j^2 + \ell^2) \tag{93}$$

which implies formula (87) for the odd component r_n^1. Now we can do the same analysis with the even solution $E_0(t)$; we do not give details of the calculations which are very similar to the above, notice however that by locality, the formula (93) for the logarithmic term cannot change at all. We eventually obtain in this case

$$S_j^0(z) = \frac{\Gamma(-z)\Gamma(\frac{-\nu_j+z}{2})\Gamma(\frac{\nu_j+z+1}{2})}{\Gamma(z)\Gamma(\frac{-\nu_j-z}{2})\Gamma(\frac{\nu_j-z+1}{2})} = 2^{-2z}\frac{\Gamma(-z)}{\Gamma(z)}\frac{\Gamma(-\nu_j+z)}{\Gamma(-\nu_j-z)}\frac{\cos(\frac{\pi}{2}(\nu_j-z))}{\cos(\frac{\pi}{2}(\nu_j+z))} \tag{94}$$

which implies

$$r_n^0 = 2^{-n}\frac{\Gamma(-\frac{n}{2})}{\Gamma(\frac{n}{2})}\sum_j \Big(\alpha_j\Big(\frac{\sinh(\frac{\pi}{2}\alpha_j)}{\cosh(\frac{\pi}{2}\alpha_j)}\Big)^{(-1)^{\frac{n-1}{2}}}\prod_{\ell=1}^{\frac{n-1}{2}}(\alpha_j^2+\ell^2)\Big)\langle r_0^0,\phi_j\rangle\phi_j. \tag{95}$$

Let us now consider the case $n = 4$. Let us first compute $a_4(z,\nu_j)$ in (89). We rewrite equation (88) in terms of $x = 2e^{-|t|}$ for $|t| > 1$:

$$-(x\partial_x)^2 s_j + (z^2 - x^2\nu_j(\nu_j+1) + \tfrac{\nu_j(\nu_j+1)}{2}x^4 + \mathcal{O}(x^6))s_j = 0.$$

Solving this equation as a series in x, the coefficients $a_{2k}(z,\nu_j)$ are uniquely determined and we obtain for $t \to \infty$ the asymptotic expansion for the even, respectively odd solution

$$s_j^\varepsilon(z) = x^{-z}\Big(1 + \tfrac{\nu_j(\nu_j+1)}{4(z-1)}x^2 + \tfrac{1}{8(z-2)}(\tfrac{\nu_j^2(\nu_j+1)^2}{4(z-1)} - \tfrac{\nu_j(\nu_j+1)}{2})x^4\Big) + x^z S_j^\varepsilon(z) + o(x^4)$$

and thus $a_4(z,\nu_j) = \frac{1}{8(z-2)}(\frac{\nu_j^2(\nu_j+1)^2}{4(z-1)} - \frac{\nu_j(\nu_j+1)}{2})$. By (92),

$$r_n^\varepsilon = \sum_j \Big(\mathrm{FP}_2 S_j^\varepsilon(z) - \frac{\nu_j^2(\nu_j+1)^2}{32}\Big)\langle r_0^\varepsilon,\phi_j\rangle\phi_j. \tag{96}$$

We now compute $\mathrm{FP}_2 S_j^1(z)$. We assume that $L_{h_0} \geq 2$ so that we can write $\nu_j = -\frac{1}{2} + i\alpha_j$ with $\alpha_j = \sqrt{\gamma_j - \frac{9}{4}} \geq 0$ if $\gamma_j \geq \frac{9}{4}$ and $i\alpha_j \in [0,\frac{1}{2}]$ if $\gamma_j \leq 9/4$. We use formula (91) for $S_j^1(z)$, then for $\nu_j \in \mathbb{R}$ we see that $S_j^1(z) \in \mathbb{R}$, but we also notice that (90) implies that

$$S_j^1(z) = \frac{\Gamma(-z)|\Gamma(\frac{z+\frac{3}{2}+i\alpha_j}{2})|^2}{\Gamma(z)|\Gamma(\frac{-z+\frac{3}{2}+i\alpha_j}{2})|^2} \in \mathbb{R} \text{ if } z \in \mathbb{R}, \alpha_j \in \mathbb{R}^+.$$

Let $\gamma = -\Gamma'(1)$ be the Euler constant, then for z close to 2

$$2^{-2z}\frac{\Gamma(-z)}{\Gamma(z)} = -\tfrac{1}{32}[(z-2)^{-1} + 2\gamma - \tfrac{5}{2} - 2\ln(2)] + \mathcal{O}((z-2)). \tag{97}$$

We write

$$\frac{\Gamma(z-\nu_j)}{\Gamma(-z-\nu_j)} = \frac{\Gamma(z-\nu_j)}{\Gamma(-z+4-\nu_j)}(z+\nu_j)(z+\nu_j-1)(z+\nu_j-2)(z+\nu_j-3)$$

and consider its Taylor expansion at $z = 2$:

$$\frac{\Gamma(z-\nu_j)}{\Gamma(-z-\nu_j)} = (\alpha_j^2+\tfrac{1}{4})(\alpha_j^2+\tfrac{9}{4}) + (z-2)\Big[2\Psi(2-\nu_j)(\alpha_j^2+\tfrac{1}{4})(\alpha_j^2+\tfrac{9}{4})$$
$$- 2i\alpha_j(2\alpha_j^2+\tfrac{5}{2}))\Big] + \mathcal{O}((z-2)^2). \tag{98}$$

where $\Psi(z) = \partial_z \Gamma(z)/\Gamma(z)$ is the digamma function. Finally we expand

$$\frac{\sin(\frac{\pi}{2}(\nu_j - z))}{\sin(\frac{\pi}{2}(\nu_j + z))} = 1 - (z-2)\pi\frac{\cos(\frac{\pi}{2}\nu_j)}{\sin(\frac{\pi}{2}\nu_j)} + \mathcal{O}((z-2)^2). \tag{99}$$

Using (96) and combining (97), (98), (99) we get when $\nu_j = -\frac{1}{2} + \lambda_j$ with $\lambda_j := i\alpha_j \in \mathbb{R}^+$

$$\langle r_4^\pm, \phi_j \rangle = -\frac{1}{32}\left[\left(c_0 - \pi\frac{\cos(\frac{\pi}{2}(\lambda_j - \frac{1}{2}))}{\sin(\frac{\pi}{2}(\lambda_j - \frac{1}{2}))} + 2\Psi(\tfrac{5}{2} - \lambda_j)\right)(-\lambda_j^2 + \tfrac{1}{4})(-\lambda_j^2 + \tfrac{9}{4})\right.$$
$$\left. + 2\lambda_j(2\lambda_j^2 - \tfrac{5}{2}) + (-\lambda_j^2 + \tfrac{1}{4})^2\right]$$

where $c_0 := 2\gamma - \frac{5}{2} - 2\ln(2)$. When $\nu_j = -\frac{1}{2} + i\alpha_j$ with $\alpha_j \in \mathbb{R}$ we get (using that $S_j^1(z)$ is real)

$$\langle r_4^\pm, \phi_j \rangle = -\frac{1}{32}\left[\left(c_0 + \frac{\pi}{\cosh(\pi\alpha_j)} + 2\mathrm{Re}(\Psi(\tfrac{5}{2} - i\alpha_j))\right)(\alpha_j^2 + \tfrac{1}{4})(\alpha_j^2 + \tfrac{9}{4}) + (\alpha_j^2 + \tfrac{1}{4})^2\right]$$

This gives the desired result when $r_0^+ = -r_0^-$ by using (96). When $r_0^+ = r_0^-$, we consider the expansion of the even solution $E_0(t)$, this is a similar computation to what we did for E_1, but using formula (94) instead of (90)), and

$$\frac{\cos(\frac{\pi}{2}(\nu_j - z))}{\cos(\frac{\pi}{2}(\nu_j + z))} = 1 + (z-2)\pi\frac{\sin(\frac{\pi}{2}\nu_j)}{\cos(\frac{\pi}{2}\nu_j)} + \mathcal{O}((z-2)^2).$$

instead of (99). We find for $\alpha_j \geq 0$

$$\langle r_4^0, \phi_j \rangle = -\frac{1}{32}\left[\left(c_0 - \frac{\pi}{\cosh(\pi\alpha_j)} + 2\mathrm{Re}\Psi(\tfrac{5}{2} - i\alpha_j)\right)(\alpha_j^2 + \tfrac{1}{4})(\alpha_j^2 + \tfrac{9}{4}) + (\alpha_j^2 + \tfrac{1}{4})^2\right]$$

and for $\lambda_j = i\alpha_j \in \mathbb{R}^+$

$$\langle r_4^0, \phi_j \rangle = -\frac{1}{32}\left[\left(c_0 - \pi\tan(\tfrac{\pi}{2}(\tfrac{1}{2} - \lambda_j)) + 2\Psi(\tfrac{5}{2} - \lambda_j)\right)(-\lambda_j^2 + \tfrac{1}{4})(-\lambda_j^2 + \tfrac{9}{4})\right.$$
$$\left. + 2\lambda_j(2\lambda_j^2 - \tfrac{5}{2}) + (-\lambda_j^2 + \tfrac{1}{4})^2\right].$$

This finishes the proof. $\square$

As a first corollary, we recover a formula proved recently by Matsumoto [49] for the Hessian of the functional $h_0 \mapsto \int_N v_n(h_0)\mathrm{dvol}_{h_0}$ defined on the space $\mathcal{C}(N)$ of conformal structures.

Corollary 7.7. *Let n be even, let h_0 satisfies $\mathrm{Ric}_{h_0} = -(n-1)h_0$ on N, and let $L_{h_0} = \nabla^*\nabla - 2\mathring{R}_{h_0}$ be the linearized Einstein operator at h_0. Then the obstruction tensor k linearized at h_0 and acting on divergence-free/trace free tensors $\dot{h}_0$ is given by*

$$Dk_{h_0}.\dot{h}_0 = \frac{(-1)^{\frac{n}{2}+1}2^{1-n}}{\frac{n}{2}!(\frac{n}{2}-1)!}\prod_{j=0}^{\frac{n}{2}-1}(L_{h_0} - j(n-1-j))\dot{h}_0.$$

Proof. If g^s is a deformation of Einstein metrics as before and $\dot{g} = \partial_s g^s|_{s=0}$, then the first log term in the expansion of $\dot{g}$ is $\dot{k}x^{n-1}\log(x)$ where $\dot{k}$ is the variation of the obstruction tensor

k^s of g^s. We modify $\dot{g}$ by $L_X g$ as in (79), and we apply [68, Prop. 4.5][3] to deduce that $L_X g$ has no log term before $x^n \log(x)$, thus $q = L_X g + \dot{g}$ has first log term given by $\dot{k} x^{n-1} \log(x)$ in its expansion. Now it remains to use formula (86) and this gives $\dot{k}$ in terms of $\dot{h}_0$. $\square$

Our second corollary is

Corollary 7.8. *Let g^s be a family of AHE metrics such that $g^0 = g$ is the Fuchsian-Einstein metric, and $L_{h_0} = \Delta_{h_0} - 2\mathring{R}_{h_0}$ be the linearized Einstein operator.*

(1) *If n is odd, there exists $C > 0$ such that for all $\dot{h}_0$ satisfying $\delta_{h_0}(\dot{h}_0) = 0$, $\mathrm{Tr}_{h_0}(\dot{h}_0) = 0$ and $\mathbb{1}_{[0, \frac{(n-1)^2}{4}]}(L_{h_0})\dot{h}_0 = 0$*

$$(-1)^{\frac{n+1}{2}} \partial_s^2 \mathrm{Vol}_R(M, g^s)|_{s=0} \geq C|\dot{h}_0|_{H^{\frac{n}{2}}(N)}^2.$$

(2) *Assume $n = 4$ and let h_0^s be a smooth family of conformal representatives of the conformal infinity satisfying $v_n(h_0^s) = \int_N v_n(h_0^s)\mathrm{dvol}_{h_0^s}$. Assuming that $L_{h_0} - 2 > 0$ on the subspace of trace-free/divergence free tensors, there exists $C > 0$ such that for all $\dot{h}_0$ satisfying $\delta_{h_0}(\dot{h}_0) = 0$ and $\mathrm{Tr}_{h_0}(\dot{h}_0) = 0$*

$$\partial_s^2 \mathrm{Vol}_R(M, g^s; h_0^s)|_{s=0} \geq C|\dot{h}_0|_{H^2(N)}^2.$$

Remark. By using a Weitzenböck type formula, one obtains that the assumption $L_{h_0} - 2 \geq 0$ is satisfied for hyperbolic metrics h_0, see for example the proof of [6, Th 12.67]; and the strict inequality $L_{h_0} - 2 > 0$ holds if $\ker d_D = \ker d_D^* = 0$ where d_D is the exterior derivative on T^*N-valued 1-forms and d_D^* its adjoint.

Proof. The case n odd is a direct consequence of Proposition 7.6. For the case $n = 4$, we use (77), Lemma 6.5, Lemma 7.2 and Lemma 7.5 to deduce that

$$4\partial_s^2 \mathrm{Vol}_R(M, g^s; h_0^s)|_{s=0} = -\langle r_4, \dot{h}_0 \rangle_{L^2} + (\tfrac{1}{2}v_4 - \tfrac{1}{8})|\dot{h}_0|_{L^2}^2 - \tfrac{1}{2}\langle r_{4,1}, \dot{h}_0 \rangle_{L^2}$$

where r_4 and $r_{4,1}$ are given in Proposition 7.6. By Lemma 4.4, $v_4 = 3/8$ and so

$$2^7 \partial_s^2 \mathrm{Vol}_R(M, g^s; h_0^s)|_{s=0} = \langle \mathcal{H}_1(\sqrt{L_{h_0} - \tfrac{9}{4}})\dot{h}_0^1, \dot{h}_0^1 \rangle_{L^2} + \langle \mathcal{H}_0(\sqrt{L_{h_0} - \tfrac{9}{4}})\dot{h}_0^0, \dot{h}_0^0 \rangle_{L^2}$$

where $\mathcal{H}_\varepsilon$, $\varepsilon \in \{0, 1\}$ are the functions defined by

$$\mathcal{H}_\varepsilon(u) := \left(c_0 + \frac{-(-1)^\varepsilon \pi}{\cosh(\pi u)} + 2\mathrm{Re}(\Psi(\tfrac{5}{2} - iu)) \right)(u^2 + \tfrac{1}{4})(u^2 + \tfrac{9}{4}) + (u^2 + \tfrac{1}{4})^2 + 2 \quad (100)$$

for $u \geq 0$, and to

$$\mathcal{H}_\varepsilon(-iu) := \left(c_0 + 1 - (-1)^\varepsilon \pi(\tan(\tfrac{\pi}{2}(\tfrac{1}{2} - u)))^{(-1)^\varepsilon} + 2\Psi(\tfrac{5}{2} - u) \right)(-u^2 + \tfrac{1}{4})(-u^2 + \tfrac{9}{4})$$
$$+ 2u(2u^2 - \tfrac{5}{2}) + (-u^2 + \tfrac{1}{4})^2 + 2 \quad (101)$$

for $u \geq 0$. Let us show that the functions in (100) are positive for $u \geq 0$ (numerically this follows from Figure 1 but we give a formal proof). Since $\mathcal{H}_1(u) \geq \mathcal{H}_0(u)$ for $u \geq 0$, it suffices

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[3]The proof in [68] is technically for n odd, but the same arguments apply, once we have noticed that $\delta_g(\dot{g}) + \frac{1}{2}d\mathrm{Tr}(\dot{g})$ has no log coefficient before $x^{n+1}\log(x)$ when measured with respect to g, this is easy to check.
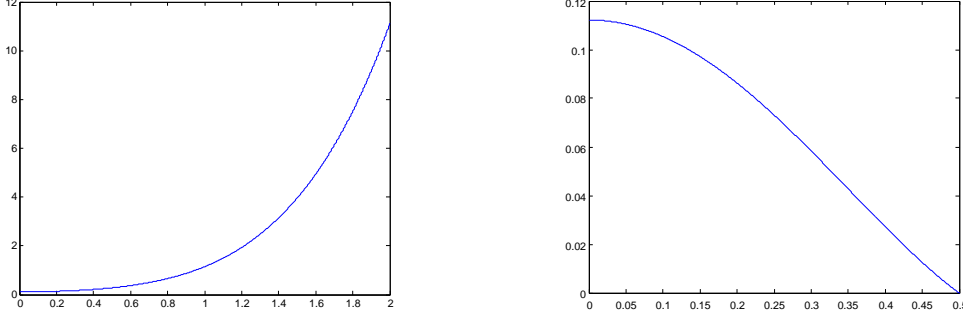
FIGURE 1. Left: the graph of the function $\tilde{\mathcal{H}}(u)$ bounding $\mathcal{H}_0(u)$ from below, where $\tilde{\mathcal{H}}(u) := \left(c_0 + 1 - \frac{\pi}{\cosh(\pi u)} + 2\Psi(\frac{5}{2})\right)(u^2 + \frac{1}{4})(u^2 + \frac{9}{4}) + (u^2 + \frac{1}{4})^2 + 2$. Right: the graph of the function $\mathcal{H}_0(-iu)$ for $u \in [0, 1/2]$.

to show that $\mathcal{H}_0(u) > 0$. We write

$$\mathcal{H}_0(u) = a(u)x(x+2) + x^2 + 2, \quad x := u^2 + \tfrac{1}{4}, \quad a(u) := c_0 + 1 - \frac{\pi}{\cosh(\pi u)} + 2\mathrm{Re}(\Psi(\tfrac{5}{2} - iu)).$$

The real part of the digamma function $\Re(\Psi(\frac{5}{2} + it))$ is increasing as a function of $|t|$:

$$\Re(\Psi(\sigma + it)) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{n+\sigma}{(n+\sigma)^2 + t^2} \right), \tag{102}$$

thus

$$\mathrm{Re}(\Psi(\tfrac{5}{2} - i\tfrac{u}{2})) \geq \Psi(\tfrac{5}{2}) = -\gamma - 2\ln 2 + \tfrac{8}{3}, \tag{103}$$

and so $a(u)$ is increasing and satisfies

$$a(u) \geq a(0) = \tfrac{23}{6} - 6\ln 2 - \pi. \tag{104}$$

For $a \in \mathbb{R}$ let $P_a(x) := ax(x+2) + x^2 + 2$. Since $a(u)$ is increasing, we have $\mathcal{H}_0(u) = P_{a(u)}(x) \geq P_{a(u_0)}(x)$ for all $u \geq u_0$. We use a bootstrap argument: by (104), $a(0) > -3.468 =: a_0$ and so $P_{a_0}(x) > 0$ for $x < x_1$ with some explicitly computable x_1. This means that $\mathcal{H}_0(u) > 0$ for $u < (x_1 - 0.25)^{1/2} =: u_1$. Using (103), we have a lower bound $a(u_1) > a_1$. The binomial $P_{a_1}(x)$ is positive for $x < x_2$, etc. This tedious algorithm stops after a finite number of steps. Hence $\mathcal{H}_0(u) > 0$ for all $u > 0$.

We now show that the function $\mathcal{H}_0(-iu)$ from (101) is positive for $0 \leq u < 1/2$. We introduce the notation

$$u := \tfrac{1}{2} - v, \qquad\qquad a(v) := c_0 + 1 - \pi(\tan(\tfrac{\pi}{2}v)) + 2\Psi(v+2)$$

and we compute $\mathcal{H}_0(-iu) = v(2-v)[a(v)(v^2 - 1) + v^2 - 4v - 1]$. Therefore for $0 < v \leq 1/2$, $\mathcal{H}_0(-iu) > 0$ if and only if $-(a(v) + 1) < \frac{4v}{1-v^2}$. At $v = 0$ this is verified since $c_0 > -4$. It is thus enough to show that for $0 < v \leq 1/2$ we have

$$-a'(v) < 2\left(\frac{1}{(1-v)^2} + \frac{1}{(1+v)^2} \right).$$

Using (102), we have

$$-a'(v) = \frac{\pi^2}{2\cos^2(\pi v/2)} - 2\Psi'(2+v) = 2\sum_{k\in\mathbb{Z}}\frac{1}{(v+2k+1)^2} - 2\sum_{k\geq 0}\frac{1}{(v+k+2)^2}$$

$$= 2\left(\frac{1}{(1-v)^2} + \frac{1}{(1+v)^2}\right) + 2\sum_{k=1}^{\infty}\frac{(2v-1)(4k+1)}{(v-2k-1)^2(v+2k)^2}$$

and this finishes the proof since $2v - 1 < 0$. $\qquad\square$

Remark 7.9. For n odd, we notice that from Proposition 7.6, the Fuchsian-Einstein metric is a saddle point for regularized volume if $L_{h_0} - \frac{(n-1)^2}{4}$ has negative eigenvalues.

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