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**CONSISTENT SEMI-IMPLICIT STAGGERED SCHEMES  
FOR COMPRESSIBLE FLOWS  
PART II: EULER EQUATIONS.**

R. HERBIN<sup>1</sup>, W. KHERIJI<sup>2</sup> AND J.-C. LATCHÉ<sup>3</sup>

**Abstract.** In this paper, we propose implicit and pressure correction schemes for the Euler equations, based on staggered space discretizations, namely the MAC finite volume scheme or the low-order (Rannacher-Turek or Crouzeix-Raviart) finite elements. Both schemes rely on the discretization of the internal energy balance equation, which offers two main advantages: first, we avoid the space discretization of the total energy, which involves cell-centered and face-centered variables; second, we obtain algorithms which boil down to usual schemes in the incompressible limit. However, since these schemes do not use the original total energy conservative equation, in order to obtain correct weak solutions (in particular, with shocks satisfying the Rankine-Hugoniot conditions), we need to introduce corrective terms in the internal energy balance. These corrective terms are found by deriving a discrete kinetic energy balance, observing that this relation contains residual terms which do not tend to zero (at least, under reasonable stability assumptions) and, finally, compensating them in the discrete internal energy balance. This latter step is performed under the guideline that, when adding the discrete kinetic equation to the discrete internal energy equation, the sum of the residual and corrective terms tends to zero with the mesh size and time step. It is then shown in the 1D case, that, if the scheme converges, the limit is indeed a weak solution. Finally, we present numerical results which confort this theory.

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1. INTRODUCTION

Let us consider the compressible Navier-Stokes equations, which read:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1a}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \operatorname{div}(\boldsymbol{\tau}(\mathbf{u})) = 0, \tag{1b}$$

$$\partial_t(\rho E) + \operatorname{div}(\rho E \mathbf{u}) + \operatorname{div}(p \mathbf{u}) = \operatorname{div}(\boldsymbol{\tau}(\mathbf{u}) \cdot \mathbf{u}), \tag{1c}$$

$$p = (\gamma - 1) \rho e, \quad E = \frac{1}{2} |\mathbf{u}|^2 + e, \tag{1d}$$

where  $t$  stands for the time,  $\rho$ ,  $\mathbf{u}$ ,  $p$ ,  $E$  and  $e$  are the density, velocity, pressure, total energy and internal energy in the flow,  $\boldsymbol{\tau}(\mathbf{u})$  stands for the shear stress tensor, and  $\gamma > 1$  is a coefficient specific to the considered fluid. The problem is supposed to be posed over  $\Omega \times (0, T)$ , where  $\Omega$  is an open bounded connected subset of  $\mathbb{R}^d$ ,  $1 \leq d \leq 3$ , and  $(0, T)$  is a finite time interval.

System (1) is complemented by initial conditions for  $\rho$ ,  $e$  and  $\mathbf{u}$ , denoted by  $\rho_0$ ,  $e_0$  and  $\mathbf{u}_0$  respectively, with  $\rho_0 > 0$  and  $e_0 > 0$ , and by a boundary condition which we suppose to be  $\mathbf{u} \cdot \mathbf{n} = 0$  at any time and *a.e.* on  $\partial\Omega$ , where  $\mathbf{n}$  stands for the normal vector to the boundary.

Let us suppose that the solution is regular. Taking the inner product of the momentum balance equation (1b) by  $\mathbf{u}$  and using the mass balance equation, we obtain the so-called kinetic energy balance equation:

$$\frac{1}{2} \partial_t(\rho |\mathbf{u}|^2) + \frac{1}{2} \operatorname{div}(\rho |\mathbf{u}|^2 \mathbf{u}) + \nabla p \cdot \mathbf{u} = \operatorname{div}(\boldsymbol{\tau}(\mathbf{u})) \cdot \mathbf{u}. \tag{2}$$

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Subtracting this relation from the total energy balance, we obtain the internal energy balance equation:

$$\partial_t(\rho e) + \operatorname{div}(\rho e \mathbf{u}) + p \operatorname{div}(\mathbf{u}) = \boldsymbol{\tau}(\mathbf{u}) : \boldsymbol{\nabla} \mathbf{u}. \quad (3)$$

Since,

- from thermodynamical arguments,  $\boldsymbol{\tau}(\mathbf{u}) : \boldsymbol{\nabla} \mathbf{u} \geq 0$ ,
- thanks to the mass balance equation, the first two terms in the left-hand side of (3) may be recast as a transport operator:  $\partial_t(\rho e) + \operatorname{div}(\rho e \mathbf{u}) = \rho [\partial_t e + \mathbf{u} \cdot \boldsymbol{\nabla} e]$ ,
- and, finally, from the equation of state, the pressure vanishes when  $e = 0$ ,

this equation implies that if  $e \geq 0$  at  $t = 0$  and with suitable boundary conditions, then  $e$  remains non-negative at all times.

Our aim in this paper is to build numerical schemes for the Euler equations, *i.e.* System (1) with  $\boldsymbol{\tau} = 0$ , which are stable and accurate at all Mach numbers, and, in particular, which boil down to usual schemes for incompressible flows (or, more generally speaking, for the asymptotic model of vanishing Mach number flows [19]) when the Mach number tends to zero; we thus choose to use staggered space discretizations. In incompressible models, the natural energy balance equation is the internal energy equation (3). In addition, discretizing (3) instead of the total energy balance (1c) presents two advantages:

- first, it avoids the space discretization of the total energy, which is rather unnatural for staggered schemes since the degrees of freedom for the velocity and the scalar variables are not collocated,
- second, a suitable discretization of (3) may yield, "by construction" of the scheme, the positivity of the internal energy [13].

However, in the inviscid case and for solutions with shocks, Equation (3) (with  $\boldsymbol{\tau} = 0$ ) is not equivalent to (1c) (with  $\boldsymbol{\tau} = 0$ ); more precisely speaking, at the locations of shocks, positive measures should replace, at the right-hand side of Equation (3), the term  $\boldsymbol{\tau}(\mathbf{u}) : \boldsymbol{\nabla} \mathbf{u}$  which is formally the product of vanishing quantities (for a Newtonian fluid, the viscosity) and infinite derivatives of the velocity. Discretizing (3) instead of (1c) may thus yield a scheme which does not compute the correct weak discontinuous solutions; in particular, the numerical solutions may present (smeared) shocks which do not satisfy the Rankine-Hugoniot conditions associated to (1c). The essential result of this paper is to provide solutions to circumvent this problem. To this purpose, we closely mimic the above performed formal computation:

- Starting from the discrete momentum balance equation, with an *ad hoc* discretization of the convection operator, we derive a discrete kinetic energy balance in which some residual terms do not tend to zero with space and time step (they are the discrete manifestations of the above mentioned measures).
- These residual terms are then compensated by corrective terms in the internal energy balance.

We provide a theoretical justification of this process by showing that, in the 1D case, if the scheme is stable and converges to a limit (in a sense to be defined), this limit satisfies a weak form of (1c) which implies the correct Rankine-Hugoniot conditions. Then, we perform numerical tests which substantiate this analysis. Different time discretizations are proposed: first, a fully implicit scheme (a solution to which may be rather difficult to obtain in practice) and, second, two pressure correction schemes; these latter algorithms are used in the tests presented here, and in the industrial open-source code ISIS [17], developed at IRSN on the basis of the software components library PELICANS [20]. All of these schemes conserve the integral of (a discrete equivalent of) the total energy over the domain. The first pressure correction scheme is appealing for its (relative) simplicity, but does not seem to warrant the sign of the internal energy (so that the unconditional stability induced by the above mentioned conservation of the total energy property is lost); the second scheme cures this problem, at the price of the introduction of an additional elliptic problem which must be solved at the beginning of each time step to determine a tentative pressure. Let us mention also that fully explicit versions may be built, and are now under study [16].

This work follows a companion paper [14], in which the same schemes are developed for the barotropic Euler equations. The present paper is self-consistent and in particular describes the definition of the space discretization, which is the same as that of [14]; we refer to [14] essentially for two issues: the derivation of the discrete kinetic energy balance, and the passage to the limit, in 1D, in the mass and momentum balance equations which are the same in both the barotropic and non-barotropic models.

Let us mention that a wide literature is related to the matter at hand. Indeed, schemes involving an elliptic pressure correction step for compressible flows appeared as soon as the late sixties [9, 10], with the so-called ICE method, and were followed by numerous works (see, among others, [24] for an introduction). In parallel, pressure correction schemes gained a wide popularity and were extensively studied for incompressible flows

(see [2,22] for seminal works and [8] for a review of most of the variants). We refer to [14] for a (more) complete bibliography.

This paper is organized as follows. We begin by describing the space discretizations (Section 2). We then study the implicit scheme (Section 3): we first give the general form of the algorithm (Section 3.1), then derive the kinetic energy balance and deduce the source terms to be included in the internal energy balance (Section 3.2), and, finally, we pass to the limit in the scheme to prove (in 1D) the consistency of the scheme (Section 3.3). Sections 4 and 5 follow the same lines for the two pressure correction algorithms. Finally, we present some numerical tests in Section 6.

## 2. MESHES AND UNKNOWNNS

Let  $\mathcal{M}$  be a decomposition of the domain  $\Omega$ , supposed to be regular in the usual sense of the finite element literature (eg. [3]). The cells may be:

- for a general domain  $\Omega$ , either convex quadrilaterals ( $d = 2$ ) or hexahedra ( $d = 3$ ) or simplices, both type of cells being possibly combined in a same mesh,
- for a domain the boundaries of which are hyperplanes normal to a coordinate axis, rectangles ( $d = 2$ ) or rectangular parallelepipeds ( $d = 3$ ) (the faces of which, of course, are then also necessarily normal to a coordinate axis).

By  $\mathcal{E}$  and  $\mathcal{E}(K)$  we denote the set of all  $(d-1)$ -faces  $\sigma$  of the mesh and of the element  $K \in \mathcal{M}$  respectively. The set of faces included in  $\Omega$  (resp. in the boundary  $\partial\Omega$ ) is denoted by  $\mathcal{E}_{\text{int}}$  (resp.  $\mathcal{E}_{\text{ext}}$ ), so that (*i.e.*  $\mathcal{E}_{\text{int}} = \mathcal{E} \setminus \mathcal{E}_{\text{ext}}$ ) is denoted by  $\mathcal{E}_{\text{int}}$ ; a face  $\sigma \in \mathcal{E}_{\text{int}}$  separating the cells  $K$  and  $L$  is denoted by  $\sigma = K|L$ . The outward normal vector to a face  $\sigma$  of  $K$  is denoted by  $\mathbf{n}_{K,\sigma}$ . For  $K \in \mathcal{M}$  and  $\sigma \in \mathcal{E}$ , we denote by  $|K|$  the measure of  $K$  and by  $|\sigma|$  the  $(d-1)$ -measure of the face  $\sigma$ . For  $1 \leq i \leq d$ , we denote by  $\mathcal{E}^{(i)} \subset \mathcal{E}$  and  $\mathcal{E}_{\text{ext}}^{(i)} \subset \mathcal{E}_{\text{ext}}$  the subset of the faces of  $\mathcal{E}$  and  $\mathcal{E}_{\text{ext}}$  respectively which are perpendicular to the  $i^{\text{th}}$  unit vector of the canonical basis of  $\mathbb{R}^d$ .

The space discretization is staggered, using either the Marker-And Cell (MAC) scheme [10,11], or non-conforming low-order finite element approximations, namely the Rannacher and Turek element (RT) [21] for quadrilateral or hexahedric meshes, or the lowest degree Crouzeix-Raviart (CR) element [4] for simplicial meshes.

For all these space discretizations, the degrees of freedom for the pressure, the density and the internal energy (*i.e.* the discrete pressure, density and internal energy unknowns) are associated to the cells of the mesh  $\mathcal{M}$ , and are denoted by:

$$\{p_K, \rho_K, e_K, K \in \mathcal{M}\}.$$

Let us then turn to the degrees of freedom for the velocity (*i.e.* the discrete velocity unknowns).

- **Rannacher-Turek** or **Crouzeix-Raviart** discretizations – The discrete velocity unknowns are located at the center of the faces of the mesh, and we choose the version of the element where they represent the average of the velocity through a face. The set of discrete velocity unknowns reads:

$$\{u_{\sigma,i}, \sigma \in \mathcal{E}, 1 \leq i \leq d\}.$$

- **MAC** discretization – The degrees of freedom for the  $i^{\text{th}}$  component of the velocity are located at the centre of the faces  $\sigma \in \mathcal{E}^{(i)}$ , so that the whole set of discrete velocity unknowns reads:

$$\{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}.$$

We now introduce a dual mesh, which will be used for the finite volume approximation of the time derivative and convection terms in the momentum balance equation.

- **Rannacher-Turek** or **Crouzeix-Raviart** discretizations – For the RT or CR discretizations, the dual mesh is the same for all the velocity components. When  $K \in \mathcal{M}$  is a simplex, a rectangle or a cuboid, for  $\sigma \in \mathcal{E}(K)$ , we define  $D_{K,\sigma}$  as the cone with basis  $\sigma$  and with vertex the mass center of  $K$  (see Figure 1). We thus obtain a partition of  $K$  in  $m$  sub-volumes, where  $m$  is the number of faces of the mesh, each sub-volume having the same measure  $|D_{K,\sigma}| = |K|/m$ . We extend this definition to general quadrangles and hexahedra, by supposing that we have built a partition still of equal-volume sub-cells, and with the same connectivities; note that this is of course always possible, but that such a volume  $D_{K,\sigma}$  may be no longer a cone; indeed, if  $K$  is far from a parallelogram, it may not be possible to build a cone having  $\sigma$  as basis, the opposite vertex lying in  $K$  and a volume equal to  $|K|/m$ . The volume  $D_{K,\sigma}$  is referred to

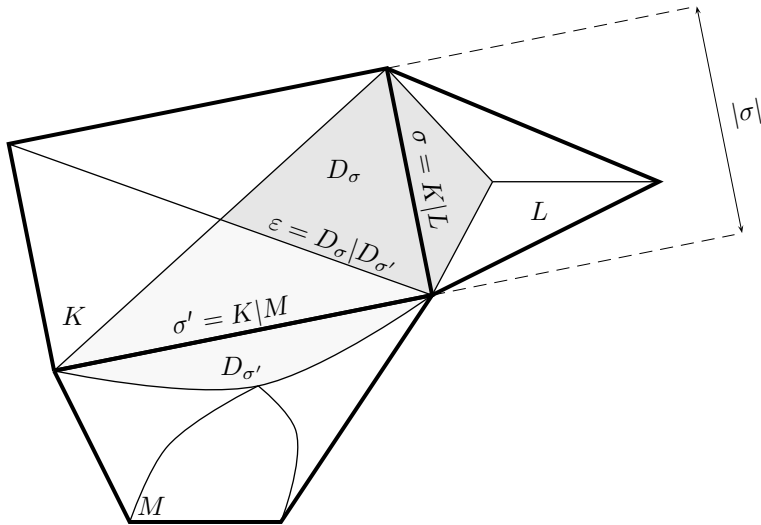


FIGURE 1. Primal and dual meshes for the Rannacher-Turek and Crouzeix-Raviart elements.

as the half-diamond cell associated to  $K$  and  $\sigma$ .

For  $\sigma \in \mathcal{E}_{\text{int}}$ ,  $\sigma = K|L$ , we now define the diamond cell  $D_\sigma$  associated to  $\sigma$  by  $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$ ; for an external face  $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K)$ ,  $D_\sigma$  is just the same volume as  $D_{K,\sigma}$ .

- **MAC** discretization – For the MAC scheme, the dual mesh depends on the component of the velocity. For each component, the MAC dual mesh only differs from the RT or CR dual mesh by the choice of the half-diamond cell, which, for  $K \in \mathcal{M}$  and  $\sigma \in \mathcal{E}(K)$ , is now the rectangle or rectangular parallelepiped of basis  $\sigma$  and of measure  $|D_{K,\sigma}| = |K|/2$ .

We denote by  $|D_\sigma|$  the measure of the dual cell  $D_\sigma$ , and by  $\varepsilon = D_\sigma|D_{\sigma'}$  the face separating two diamond cells  $D_\sigma$  and  $D_{\sigma'}$ .

Finally, we need to deal with the impermeability (*i.e.*  $\mathbf{u} \cdot \mathbf{n} = 0$ ) boundary condition. Since the velocity unknowns lie on the boundary (and not inside the cells), these conditions are taken into account in the definition of the discrete spaces. To avoid technicalities in the expression of the schemes, we suppose throughout this paper that the boundary is *a.e.* normal to a coordinate axis, (even in the case of the RT or CR discretizations), which allows to simply set to zero the corresponding velocity unknowns:

$$\text{for } i = 1, \dots, d, \forall \sigma \in \mathcal{E}_{\text{ext}}^{(i)}, \quad u_{\sigma,i} = 0. \quad (4)$$

Therefore, there are no discrete velocity unknowns on the boundary for the MAC scheme, and there are only  $d - 1$  discrete velocity unknowns on each boundary face for the CR and RT discretizations, which depend on the orientation of the face. In order to be able to write a unique expression of the discrete equations for both MAC and CR/RT schemes, we introduce the set of faces  $\mathcal{E}_S^{(i)}$  associated to the degrees of freedom of each component of the velocity ( $S$  stands for “scheme”):

$$\mathcal{E}_S^{(i)} = \begin{cases} \mathcal{E}^{(i)} \setminus \mathcal{E}_{\text{ext}}^{(i)} & \text{for the MAC scheme,} \\ \mathcal{E} \setminus \mathcal{E}_{\text{ext}}^{(i)} & \text{for the CR or RT schemes.} \end{cases}$$

Similarly, we unify the notation for the set of dual faces for both schemes by defining:

$$\bar{\mathcal{E}}_S^{(i)} = \begin{cases} \bar{\mathcal{E}}^{(i)} \setminus \bar{\mathcal{E}}_{\text{ext}}^{(i)} & \text{for the MAC scheme,} \\ \bar{\mathcal{E}} \setminus \bar{\mathcal{E}}_{\text{ext}}^{(i)} & \text{for the CR or RT schemes,} \end{cases}$$

where the symbol  $\bar{\cdot}$  refers to the dual mesh; for instance,  $\bar{\mathcal{E}}^{(i)}$  is thus the set of faces of the dual mesh associated to the  $i^{\text{th}}$  component of the velocity, and  $\bar{\mathcal{E}}_{\text{ext}}^{(i)}$  stands for the subset of these dual faces included in the boundary. Note that, for the MAC scheme, the faces of  $\bar{\mathcal{E}}^{(i)}$  are perpendicular to a unit vector of the canonical basis of  $\mathbb{R}^d$ , but not necessarily to the  $i^{\text{th}}$  one.

Note that general domains can easily be addressed (of course, with the CR or RT discretizations) by redefining, through linear combinations, the degrees of freedom at the external faces, so as to introduce the normal velocity as a new degree of freedom.

### 3. AN IMPLICIT SCHEME

#### 3.1. The scheme

Let us consider a uniform partition  $0 = t_0 < t_1 < \dots < t_N = T$  of the time interval  $(0, T)$ , and let  $\delta t = t_{n+1} - t_n$  for  $n = 0, 1, \dots, N-1$  be the constant time step. We consider an implicit-in-time numerical scheme for the discretization of the Euler equations, i.e. System (1) with  $\tau = 0$ . In its fully discrete form, this scheme reads, for  $0 \leq n \leq N-1$ :

$$\forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} = 0, \quad (5a)$$

$$\text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_S^{(i)},$$

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^n u_{\sigma,i}^n) + \sum_{\varepsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\varepsilon}^{n+1} u_{\varepsilon,i}^{n+1} + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} = 0, \quad (5b)$$

$$\forall K \in \mathcal{M},$$

$$\begin{aligned} \frac{|K|}{\delta t} (\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} e_\sigma^{n+1} \\ + |K| p_K^{n+1} (\operatorname{div} \mathbf{u})_K^{n+1} = S_K^{n+1}, \end{aligned} \quad (5c)$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\gamma - 1) \rho_K^{n+1} e_K^{n+1}. \quad (5d)$$

Equation (5a) is obtained by the discretization of the mass balance over the primal mesh, and  $F_{K,\sigma}^{n+1}$  stands for the mass flux across  $\sigma$  outward  $K$  which, by the impermeability boundary conditions, vanishes on external faces and is given on internal faces by:

$$\forall \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, \quad F_{K,\sigma}^{n+1} = |\sigma| \rho_\sigma^{n+1} u_{K,\sigma}^{n+1},$$

where  $u_{K,\sigma}^{n+1}$  is an approximation of the normal velocity to the face  $\sigma$  outward  $K$ . This latter quantity is defined by:

$$u_{K,\sigma}^{n+1} = \begin{cases} u_{\sigma,i}^{n+1} \mathbf{n}_{K,\sigma} \cdot \mathbf{e}^{(i)} & \text{for } \sigma \in \mathcal{E}^{(i)} \text{ in the MAC case,} \\ \mathbf{u}_\sigma^{n+1} \cdot \mathbf{n}_{K,\sigma} & \text{in the CR and RT cases,} \end{cases} \quad (6)$$

where  $\mathbf{e}^{(i)}$  denotes the  $i$ -th vector of the orthonormal basis of  $\mathbb{R}^d$ . The density at the face  $\sigma = K|L$  is approximated by the upwind technique:

$$\rho_\sigma^{n+1} = \begin{cases} \rho_K^{n+1} & \text{if } u_{K,\sigma}^{n+1} \geq 0, \\ \rho_L^{n+1} & \text{otherwise.} \end{cases} \quad (7)$$

We now turn to the discrete momentum balance (5b). For the MAC discretization, but also for the RT and CR discretizations, the time derivative and convection terms are approximated in (5b) by a finite volume technique over a dual mesh. For the discretization of the time derivative term, we must provide a definition for the quantities  $\rho_{D_\sigma}^{n+1}$  and  $\rho_{D_\sigma}^n$ , which approximate the density on the face  $\sigma$  at time  $t^{n+1}$  and  $t^n$  respectively. They are given by the following weighted average:

$$\forall \sigma = K|L \in \mathcal{E}_{\text{int}}, \text{ for } k = n \text{ or } k = n+1, \quad |D_\sigma| \rho_{D_\sigma}^k = |D_{K,\sigma}| \rho_K^k + |D_{L,\sigma}| \rho_L^k. \quad (8)$$

Let us now describe the discretization of the convection term. The first task is to define the discrete mass flux through the dual face  $\varepsilon$  outward  $D_\sigma$ , denoted by  $F_{\sigma,\varepsilon}^{n+1}$ ; the guideline for its construction is that we need a finite

volume discretization of the mass balance equation over the diamond cells of the form

$$\forall \sigma \in \mathcal{E}, \quad \frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + \sum_{\varepsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\varepsilon}^{n+1} = 0 \quad (9)$$

to hold in order to be able to derive a discrete kinetic energy balance (see Section 3.2 below). For the MAC scheme, the flux on a dual face which is located on two primal faces is the mean value of the sum of fluxes on the two primal faces, and the flux of a dual face located between two primal faces is again the mean value of the sum of fluxes on the two primal faces [15]. In the case of the CR and RT schemes, for a dual face  $\varepsilon$  included in the primal cell  $K$ , this flux is computed as a linear combination (with constant coefficients, *i.e.* independent of the cell) of the mass fluxes through the faces of  $K$ , *i.e.* the quantities  $(F_{K,\sigma}^{n+1})_{\sigma \in \mathcal{E}(K)}$  appearing in the discrete mass balance (5a). We refer to [1, 6] for a detailed construction of this approximation. Let us remark that a dual face lying on the boundary is then also a primal face, and the flux across that face is zero. Therefore, the values  $u_{\varepsilon,i}^{n+1}$  are only needed at the internal dual faces; we choose them to be centered or upwind, so, for  $\varepsilon = D_\sigma|D_{\sigma'}$ ,  $u_{\varepsilon,i}^{n+1}$  reads:

$$\text{Centered case: } u_{\varepsilon,i}^{n+1} = (u_{\sigma,i}^{n+1} + u_{\sigma',i}^{n+1})/2. \quad \text{Upwind case: } u_{\varepsilon,i}^{n+1} = \begin{cases} u_{\sigma,i}^{n+1} & \text{if } F_{\sigma,\varepsilon}^{n+1} \geq 0, \\ u_{\sigma',i}^{n+1} & \text{otherwise.} \end{cases}$$

The last term  $(\nabla p)_{\sigma,i}^{n+1}$  stands for the  $i$ -th component of the discrete pressure gradient at the face  $\sigma$ , which reads:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad (\nabla p)_{\sigma,i}^{n+1} = \frac{|\sigma|}{|D_\sigma|} (p_L^{n+1} - p_K^{n+1}) \mathbf{n}_{K,\sigma} \cdot \mathbf{e}^{(i)}.$$

Note that the pressure gradient only needs to be defined on internal faces since, thanks to the impermeability boundary conditions, no momentum balance equation is written at the external faces.

Equation (5c) is an approximation of the internal energy balance over the primal cell  $K$ . To ensure the positivity of the convection operator [18], we use an upwinding technique for this term:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad e_\sigma^{n+1} = \begin{cases} e_K^{n+1} & \text{if } F_{K,\sigma}^{n+1} \geq 0, \\ e_L^{n+1} & \text{otherwise.} \end{cases}$$

The divergence of the velocity,  $(\text{div} \mathbf{u})_K^{n+1}$ , is discretized as follows:

$$\text{for } K \in \mathcal{M}, \quad (\text{div} \mathbf{u})_K^{n+1} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{K,\sigma}^{n+1}.$$

Note that this definition implies that the discrete gradient and divergence operators are dual with respect to the  $L^2$  inner product:

$$\sum_{K \in \mathcal{M}} |K| p_K (\text{div} \mathbf{u})_K + \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} |D_\sigma| u_{\sigma,i}^{n+1} (\nabla p)_{\sigma,i}^{n+1} = 0.$$

The right-hand side,  $S_K^{n+1}$ , is derived using consistency arguments in the next section.

Finally, the initial approximations for  $\rho$ ,  $e$  and  $\mathbf{u}$  are given by the average of the initial conditions  $\rho_0$ ,  $e_0$  and  $\mathbf{u}_0$  on the primal and dual cells respectively:

$$\begin{aligned} \forall K \in \mathcal{M}, \quad \rho_K^0 &= \frac{1}{|K|} \int_K \rho_0(\mathbf{x}) \, d\mathbf{x}, & e_K^0 &= \frac{1}{|K|} \int_K e_0(\mathbf{x}) \, d\mathbf{x}, \\ \text{for } 1 \leq i \leq d, \quad \forall \sigma \in \mathcal{E}_S^{(i)}, \quad u_{\sigma,i}^0 &= \frac{1}{|D_\sigma|} \int_{D_\sigma} (\mathbf{u}_0(\mathbf{x}))_i \, d\mathbf{x}. \end{aligned} \quad (10)$$

### 3.2. The discrete kinetic energy balance equation and the corrective source terms

The following discrete kinetic energy balance is derived in [14] (Lemma 3.1).

**Lemma 3.1** (Discrete kinetic energy balance, implicit scheme).

A solution to the system (5) satisfies the following equality, for  $1 \leq i \leq d$ ,  $\sigma \in \mathcal{E}_S^{(i)}$  and  $0 \leq n \leq N-1$ :

$$\frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[ \rho_{D_\sigma}^{n+1} (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^n (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\varepsilon=D_\sigma|D_{\sigma'}} F_{\sigma,\varepsilon}^{n+1} u_{\sigma,i}^{n+1} u_{\sigma',i}^{n+1} + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1} = -R_{\sigma,i}^{n+1}, \quad (11)$$

where the remainder term  $R_{\sigma,i}^{n+1}$  reads:

$$R_{\sigma,i}^{n+1} = \frac{1}{2} \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 + \delta^{\text{up}} \left[ \sum_{\varepsilon=D_\sigma|D_{\sigma'}} \frac{1}{2} |F_{\sigma,\varepsilon}^{n+1}| (u_{\sigma,i}^{n+1} - u_{\sigma',i}^{n+1}) \right] u_{\sigma,i}^{n+1}, \quad (12)$$

with  $\delta^{\text{up}} = 1$  if an upwind discretization is used for the convection term in the momentum balance equation and  $\delta^{\text{up}} = 0$  in the centered case.

We recognize at the left-hand side a discrete conservative form of the kinetic energy balance, and the right-hand side gathers the residual terms. The next step is now to define corrective terms in the internal energy balance, with the aim to recover a consistent discretization of the total energy balance. The first idea to do this could be just to sum the (discrete) kinetic energy balance with the internal energy balance: it is indeed possible for a collocated discretization. But here, we face the fact that the kinetic energy balance is associated to the dual mesh, while the internal energy balance is discretized on the primal one. The way to circumvent this difficulty is to remark that we do not really need a discrete total energy balance; in fact, we only need to recover (a weak form of) this equation when the mesh and time steps tend to zero. To this purpose, we choose the quantities  $(S_K^{n+1})$  in such a way as to somewhat compensate the terms  $(R_{\sigma,i}^{n+1})$ :

$$\begin{aligned} \forall K \in \mathcal{M}, S_K^{n+1} &= \sum_{i=1}^d S_{K,i}^{n+1} \text{ with} \\ S_{K,i}^{n+1} &= \frac{1}{2} \rho_K^n \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_S^{(i)}} \frac{|D_{K,\sigma}|}{\delta t} (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 + \delta^{\text{up}} \sum_{\substack{\varepsilon \in \bar{\mathcal{E}}_S^{(i)}, \varepsilon \cap K \neq \emptyset, \\ \varepsilon = D_\sigma|D_{\sigma'}}} \alpha_{K,\varepsilon} \frac{|F_{\sigma,\varepsilon}^{n+1}|}{2} (u_{\sigma,i}^{n+1} - u_{\sigma',i}^{n+1})^2. \end{aligned}$$

The coefficient  $\alpha_{K,\varepsilon}$  is fixed to 1 if the face  $\varepsilon$  is included in  $K$ , and this is the only situation to consider for the RT and CR discretizations. For the MAC scheme, some dual faces are included in the primal cells, but some lie on their boundary; for such a dual face  $\varepsilon$ , we denote by  $\mathcal{N}_\varepsilon$  the set of cells  $M$  such that  $\bar{M} \cap \varepsilon \neq \emptyset$  (the cardinal of this set being always 4), and compute  $\alpha_{K,\varepsilon}$  by:

$$\alpha_{K,\varepsilon} = \frac{|K|}{\sum_{M \in \mathcal{N}_\varepsilon} |M|}. \quad (13)$$

For a uniform grid, this formula yields  $\alpha_{K,\varepsilon} = 1/4$ .

The expression of the  $(S_K^{n+1})_{K \in \mathcal{M}}$  is justified by the passage to the limit in the scheme (for a one-dimensional problem) performed in Section 3.3. However, its expression may be anticipated, thanks to the following remarks. First, we note that:

$$\sum_{K \in \mathcal{M}} S_K^{n+1} - \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} R_{\sigma,i}^{n+1} = 0. \quad (14)$$

Indeed, the first part of  $S_K^{n+1}$ , thanks to the expression (8) of the density at the face  $\rho_{D_\sigma}^n$ , results from a dispatching of the first part of the residual over the two adjacent cells:

$$\frac{1}{2} \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 = \underbrace{\frac{1}{2} \frac{|D_{K,\sigma}|}{\delta t} \rho_K^n (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2}_{\text{affected to K}} + \underbrace{\frac{1}{2} \frac{|D_{L,\sigma}|}{\delta t} \rho_L^n (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2}_{\text{affected to L}}.$$



For the second part of the remainder, a standard reordering of the sum yields:

$$\sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} \sum_{\varepsilon=D_\sigma|D_{\sigma'}} \frac{1}{2} |F_{\sigma,\varepsilon}^{n+1}| (u_{\sigma,i}^{n+1} - u_{\sigma',i}^{n+1}) u_{\sigma,i}^{n+1} = \sum_{i=1}^d \sum_{\varepsilon=D_\sigma|D_{\sigma'} \in \bar{\mathcal{E}}_S^{(i)}} \frac{1}{2} |F_{\sigma,\varepsilon}^{n+1}| (u_{\sigma,i}^{n+1} - u_{\sigma',i}^{n+1})^2.$$

However, we may wonder why we do not use in  $S_K^{n+1}$  the expression of this term as it is written in the remainder  $R_{\sigma,i}^{n+1}$ , *i.e.*, in other words, use the numerical diffusion multiplied by  $\mathbf{u}$  instead of the dissipation. A first answer is that we mimic what happens at the continuous level: the term which appears in the kinetic energy balance is  $\operatorname{div}(\boldsymbol{\tau}(\mathbf{u})) \cdot \mathbf{u}$  and the corresponding term in the internal energy balance is the dissipation  $\boldsymbol{\tau}(\mathbf{u}) : \nabla \mathbf{u}$ . A more involved argument is that the expression in  $S_K^{n+1}$  provides a positive source term to the internal energy balance, and we may hope that the difference between the numerical diffusion multiplied by  $\mathbf{u}$  and the associated dissipation tends to zero (because the numerical diffusion tends to zero) in the sense of distributions. To have an intuition of this fact, let us consider the toy elliptic problem, posed over  $\Omega$ :

$$v - \mu \Delta v = f,$$

where  $\mu$  is a positive parameter and  $f \in L^2(\Omega)$ . Assuming homogeneous Dirichlet boundary conditions, we obtain by standard variational arguments  $\|v\|_{L^2(\Omega)} + \mu^{1/2} \|\nabla v\| \leq C$ , with  $C$  only depending on  $\Omega$  and  $f$ . We thus get, with  $\varphi \in C_c^\infty(\Omega)$ :

$$\int_{\Omega} [\mu(\Delta v)v + \mu|\nabla v|^2] \varphi \, d\mathbf{x} = \mu \int_{\Omega} \operatorname{div}(v \nabla v) \varphi \, d\mathbf{x} = -\mu \int_{\Omega} v \nabla v \cdot \nabla \varphi \, d\mathbf{x},$$

and so, finally, by the Cauchy-Schwarz inequality:

$$\left| \int_{\Omega} [\mu(\Delta v)v + \mu|\nabla v|^2] \varphi \, d\mathbf{x} \right| \leq C \|\nabla \varphi\|_{L^\infty(\Omega)} \mu^{1/2}.$$

A discrete analogue of this simple computation is used to pass to the limit in the scheme in the next section (with a control on the unknown assumed and not proven).

Since, in the equation of state, the pressure vanishes for  $e = 0$ , and that  $S_K^{n+1}$  is a non-negative continuous function of the unknowns  $\rho$ ,  $\mathbf{u}$  and  $p$ , adapting the proof of [13, Theorem 4.1] to cope with this additional term, we obtain that the scheme admits at least one solution, which satisfies  $p \geq 0$ ,  $\rho \geq 0$  and  $e \geq 0$ . In addition, Equation (14) shows that the scheme conserves the integral of the total energy over the computational domain.

### 3.3. Passing to the limit in the scheme (1D case)

The objective of this section is to show, in the one dimensional case, that, if a sequence of solutions is controlled in suitable norms and converges to a limit, this latter necessarily satisfies a weak formulation of the continuous problem.

Let us recall the notations that we use in [14] for one dimensional meshes. These notations follow from an adaptation of the notations used for the multidimensional case in the previous sections, by taking the same index to refer to the dual faces and the primal cells and replacing the orientation defined by the outward normal vector to faces by the orientation of the coordinate axis. For any  $K \in \mathcal{M}$ , we denote by  $h_K$  its length (so  $|K| = h_K$ ); we define the mesh size  $h$  of  $\mathcal{M}$  by  $h = \sup_{K \in \mathcal{M}} h_K$ . When we write  $K = [\sigma\sigma']$ , this means that either  $K = (x_\sigma, x_{\sigma'})$  or  $K = (x_{\sigma'}, x_\sigma)$ ; if we need to specify the order, *i.e.*  $K = (x_\sigma, x_{\sigma'})$  with  $x_\sigma < x_{\sigma'}$ , then we write  $K = [\overrightarrow{\sigma\sigma'}]$ . For an interface  $\sigma = K|L$  between two cells  $K$  and  $L$ , we define  $h_\sigma = (h_K + h_L)/2$ , so, by definition of the dual mesh,  $h_\sigma = |D_\sigma|$ . If we need to specify the order of the cells  $K$  and  $L$ , say  $K$  is left of  $L$ , then we write  $\sigma = \overrightarrow{K|L}$ .

With these notations, the implicit scheme (5) reads:

$$\begin{aligned} \forall K \in \mathcal{M}, \quad \rho_K^0 &= \frac{1}{|K|} \int_K \rho_0(x) \, dx, & e_K^0 &= \frac{1}{|K|} \int_K e_0(x) \, dx, \\ \forall \sigma \in \mathcal{E}_{\text{int}}, \quad u_\sigma^0 &= \frac{1}{|D_\sigma|} \int_{D_\sigma} u_0(x) \, dx, \end{aligned} \tag{15a}$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^{n+1} - F_{\sigma}^{n+1} = 0, \quad (15b)$$

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \quad \frac{|D_{\sigma}|}{\delta t} (\rho_{D_{\sigma}}^{n+1} u_{\sigma}^{n+1} - \rho_{D_{\sigma}}^n u_{\sigma}^n) + F_L^{n+1} u_L^{n+1} - F_K^{n+1} u_K^{n+1} + p_L^{n+1} - p_K^{n+1} = 0, \quad (15c)$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + F_{\sigma'}^{n+1} e_{\sigma'}^{n+1} - F_{\sigma}^{n+1} e_{\sigma}^{n+1} + p_K^{n+1} (u_{\sigma'}^{n+1} - u_{\sigma}^{n+1}) = S_K^{n+1}, \quad (15d)$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\gamma - 1) \rho_K^{n+1} e_K^{n+1}. \quad (15e)$$

The mass flux in the discrete mass balance equation is given, for  $\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}$ , by:

$$F_{\sigma}^{n+1} = \rho_{\sigma}^{n+1} u_{\sigma}^{n+1}, \quad (16)$$

where the upwind approximation for the density at the face,  $\rho_{\sigma}^{n+1}$ , is given by Equation (7), *i.e.*  $\rho_{\sigma}^{n+1} = \rho_K^{n+1}$  if  $u_{\sigma}^{n+1} \geq 0$  and  $\rho_{\sigma}^{n+1} = \rho_L^{n+1}$  otherwise. In the momentum balance equation, the application of the procedure described in Section 3.1 yields for the density associated to the dual cell  $D_{\sigma}$  with  $\sigma = \overrightarrow{K|L}$  and for the mass fluxes at the dual face located at the center of the mesh  $K = [\overrightarrow{\sigma\sigma'}]$ :

$$\rho_{D_{\sigma}}^{n+1} = \frac{1}{2|D_{\sigma}|} (|K| \rho_K^{n+1} + |L| \rho_L^{n+1}), \quad F_K^{n+1} = \frac{1}{2} (F_{\sigma}^{n+1} + F_{\sigma'}^{n+1}), \quad (17)$$

and the approximation of the velocity at this face is either centered ( $\delta^{\text{up}} = 0$ ,  $u_K^{n+1} = (u_{\sigma}^{n+1} + u_{\sigma'}^{n+1})/2$ ) or upwind ( $\delta^{\text{up}} = 1$ ,  $u_K^{n+1} = u_{\sigma}^{n+1}$  if  $F_K^{n+1} \geq 0$  and  $u_K^{n+1} = u_{\sigma'}^{n+1}$  otherwise). In the convection terms of the internal energy balance, the approximation for  $e_{\sigma}^{n+1}$  is upwind with respect to  $F_{\sigma}^{n+1}$  (*i.e.*, for  $\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}$ ,  $e_{\sigma}^{n+1} = e_K^{n+1}$  if  $F_{\sigma}^{n+1} \geq 0$  and  $e_{\sigma}^{n+1} = e_L^{n+1}$  otherwise). The corrective term  $S_K^{n+1}$  reads:

$$\forall K = [\overrightarrow{\sigma\sigma'}], \quad S_K^{n+1} = \frac{|K|}{4\delta t} \rho_K^n [(u_{\sigma}^{n+1} - u_{\sigma}^n)^2 + (u_{\sigma'}^{n+1} - u_{\sigma'}^n)^2] + \delta^{\text{up}} \frac{|F_K^{n+1}|}{2} (u_{\sigma}^{n+1} - u_{\sigma'}^{n+1})^2. \quad (18)$$

To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual cells, so the density  $\rho$ , the pressure  $p$ , the internal energy  $e$  and the velocity  $u$  are defined almost everywhere on  $\Omega \times (0, T)$  by:

$$\begin{aligned} \rho(x, t) &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \rho_K^{n+1} \mathcal{X}_K(x) \mathcal{X}_{(n, n+1]}(t), & p(x, t) &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} p_K^{n+1} \mathcal{X}_K(x) \mathcal{X}_{(n, n+1]}(t), \\ e(x, t) &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} e_K^{n+1} \mathcal{X}_K(x) \mathcal{X}_{(n, n+1]}(t), & u(x, t) &= \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} u_{\sigma}^{n+1} \mathcal{X}_{D_{\sigma}}(x) \mathcal{X}_{(n, n+1]}(t), \end{aligned}$$

where  $\mathcal{X}_K$ ,  $\mathcal{X}_{D_{\sigma}}$  and  $\mathcal{X}_{(n, n+1]}$  stand for the characteristic function of  $K$ ,  $D_{\sigma}$  and the interval  $(t^n, t^{n+1}]$  respectively.

We recall the definition given in [14, Definition 3.6] of the interpolates and their discrete time and space derivatives that we use in the proof of Theorem 3.3.

**Definition 3.2** (Interpolates on one-dimensional meshes). Let  $\Omega$  be an open bounded interval of  $\mathbb{R}$ , let  $\varphi \in C_c^{\infty}(\Omega \times [0, T))$ , and let  $\mathcal{M}$  be a mesh over  $\Omega$ . The interpolate  $\varphi_{\mathcal{M}}$  of  $\varphi$  on the primal mesh  $\mathcal{M}$  is defined by:

$$\varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \varphi_K^n \mathcal{X}_K \mathcal{X}_{[t^n, t^{n+1})}, \quad (19)$$

where, for  $0 \leq n \leq N$ ,  $K \in \mathcal{M}$  and  $\sigma \in \mathcal{E}$ , we set  $\varphi_K^n = \varphi(x_K, t^n)$ , with  $x_K$  the mass center of  $K$ . The time discrete derivative of the discrete function  $\varphi_{\mathcal{M}}$  is defined by:

$$\partial_t \varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} \mathcal{X}_K \mathcal{X}_{[t^n, t^{n+1})}, \quad (20)$$

and its discrete space derivative by:

$$\tilde{\partial}_x \varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{\sigma=K|\vec{L} \in \mathcal{E}_{\text{int}}} \frac{\varphi_L^n - \varphi_K^n}{h_\sigma} \mathcal{X}_{D_\sigma} \mathcal{X}_{[t^n, t^{n+1})}, \quad (21)$$

Let  $\varphi_{\mathcal{E}}$  be an interpolate of  $\varphi$  on the dual mesh, defined by:

$$\varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} \varphi_\sigma^n \mathcal{X}_{D_\sigma} \mathcal{X}_{[t^n, t^{n+1})}, \quad (22)$$

where, for  $1 \leq n \leq N$ ,  $K \in \mathcal{M}$  and  $\sigma \in \mathcal{E}$ , we set  $\varphi_\sigma^n = \varphi(x_\sigma, t^n)$ , with  $x_\sigma$  the abscissa of the interface  $\sigma$ . We also define the time and space discrete derivatives of this discrete function by:

$$\begin{aligned} \tilde{\partial}_t \varphi_{\mathcal{E}} &= \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} \frac{\varphi_\sigma^{n+1} - \varphi_\sigma^n}{\delta t} \mathcal{X}_{D_\sigma} \mathcal{X}_{[t^n, t^{n+1})}, \\ \tilde{\partial}_x \varphi_{\mathcal{E}} &= \sum_{n=0}^{N-1} \sum_{K=[\vec{\sigma\sigma'}] \in \mathcal{M}} \frac{\varphi_{\sigma'}^n - \varphi_\sigma^n}{h_K} \mathcal{X}_K \mathcal{X}_{[t^n, t^{n+1})}. \end{aligned} \quad (23)$$

Finally, we define  $\tilde{\partial} \varphi_{\mathcal{M}, \mathcal{E}}$  by:

$$\tilde{\partial}_x \varphi_{\mathcal{M}, \mathcal{E}} = \sum_{n=0}^{N-1} \sum_{K=[\vec{\sigma\sigma'}] \in \mathcal{M}} \frac{\varphi_K^n - \varphi_\sigma^n}{h_K/2} \mathcal{X}_{D_{K,\sigma}} \mathcal{X}_{[t^n, t^{n+1})} + \frac{\varphi_{\sigma'}^n - \varphi_K^n}{h_K/2} \mathcal{X}_{D_{K,\sigma'}} \mathcal{X}_{[t^n, t^{n+1})}. \quad (24)$$

For the consistency result that we are seeking (Theorem 3.3 below), we have to assume that a sequence of discrete solutions  $(\rho^{(m)}, p^{(m)}, e^{(m)}, u^{(m)})_{m \in \mathbb{N}}$  is uniformly bounded in  $L^\infty((0, T) \times \Omega)^4$ , *i.e.*:

$$|(\rho^{(m)})_K^n| + |(p^{(m)})_K^n| + |(e^{(m)})_K^n| \leq C, \quad \forall K \in \mathcal{M}^{(m)}, \text{ for } 0 \leq n \leq N^{(m)}, \forall m \in \mathbb{N}, \quad (25)$$

and:

$$|(u^{(m)})_\sigma^n| \leq C, \quad \forall \sigma \in \mathcal{E}^{(m)}, \text{ for } 0 \leq n \leq N^{(m)}, \forall m \in \mathbb{N}. \quad (26)$$

Note that, by definition of the initial conditions of the scheme, these inequalities imply that the functions  $\rho_0$ ,  $e_0$  and  $u_0$  belong to  $L^\infty(\Omega)$ .

For discrete functions  $q$  and  $v$  defined on the primal and dual mesh, respectively, we define a discrete  $L^1((0, T); \text{BV}(\Omega))$  norm by:

$$\|q\|_{\mathcal{T}, x, \text{BV}} = \sum_{n=0}^N \delta t \sum_{\sigma=K|\vec{L} \in \mathcal{E}_{\text{int}}} |q_L^n - q_K^n|, \quad \|v\|_{\mathcal{T}, x, \text{BV}} = \sum_{n=0}^N \delta t \sum_{\sigma=D_\sigma|\vec{D}_{\sigma'} \in \bar{\mathcal{E}}_{\text{int}}} |v_{\sigma'}^n - v_\sigma^n|,$$

and a discrete  $L^1(\Omega; \text{BV}((0, T)))$  norm by:

$$\|q\|_{\mathcal{T}, t, \text{BV}} = \sum_{K \in \mathcal{M}} |K| \sum_{n=0}^{N-1} |q_K^{n+1} - q_K^n|, \quad \|v\|_{\mathcal{T}, t, \text{BV}} = \sum_{\sigma \in \mathcal{E}} |D_\sigma| \sum_{n=0}^{N-1} |v_\sigma^{n+1} - v_\sigma^n|.$$

For the consistency result below, we have to also assume that a sequence of discrete solutions satisfies the following uniform bounds with respect to these two norms:

$$\|\rho^{(m)}\|_{\mathcal{T}, x, \text{BV}} + \|e^{(m)}\|_{\mathcal{T}, x, \text{BV}} + \|u^{(m)}\|_{\mathcal{T}, x, \text{BV}} \leq C, \quad \forall m \in \mathbb{N}. \quad (27)$$

and:

$$\|u^{(m)}\|_{\mathcal{T}, t, \text{BV}} \leq C, \quad \forall m \in \mathbb{N}. \quad (28)$$

We are not able to prove the estimates (25)–(27) for the solutions of the scheme; however, such inequalities are satisfied by the "interpolation" (for instance, by taking the cell average) of the solution to a Riemann problem, and are also observed in computations.

A weak solution to the continuous problem satisfies, for any  $\varphi \in C_c^\infty([0, T] \times \Omega)$ :

$$- \int_{\Omega \times (0, T)} \left[ \rho \partial_t \varphi + \rho u \partial_x \varphi \right] dx dt - \int_{\Omega} \rho_0(x) \varphi(x, 0) dx = 0, \quad (29a)$$

$$- \int_{\Omega \times (0, T)} \left[ \rho u \partial_t \varphi + (\rho u^2 + p) \partial_x \varphi \right] dx dt - \int_{\Omega} \rho_0(x) u_0(x) \varphi(x, 0) dx = 0, \quad (29b)$$

$$- \int_{\Omega \times (0, T)} \left[ \rho E \partial_t \varphi + (\rho E + p) u \partial_x \varphi \right] dx dt - \int_{\Omega} \rho_0(x) E_0(x) \varphi(x, 0) dx = 0, \quad (29c)$$

$$p = (\gamma - 1)\rho e, \quad E = \frac{1}{2}u^2 + e, \quad E_0 = \frac{1}{2}u_0^2 + e_0. \quad (29d)$$

Note that these relations are not sufficient to define a weak solution to the problem, since they do not imply anything about the boundary conditions. However, they allow to derive the Rankine-Hugoniot conditions ; so, if we show that they are satisfied by the limit of a sequence of solutions to the discrete problem, this implies, loosely speaking, that *the scheme computes correct shocks*, which is the result we seek and state in Theorem 3.3.

**Theorem 3.3** (Consistency of the one-dimensional implicit scheme).

Let  $\Omega$  be an open bounded interval of  $\mathbb{R}$ . We suppose that  $\rho_0$ ,  $u_0$  and  $e_0$  are functions of  $L^\infty(\Omega)$ . Let  $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m \in \mathbb{N}}$  be a sequence of discretizations such that both the time step  $\delta t^{(m)}$  and the space step  $h^{(m)}$  tend to zero as  $m \rightarrow \infty$ . Let  $(\rho^{(m)}, p^{(m)}, e^{(m)}, u^{(m)})_{m \in \mathbb{N}}$  be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates (25)–(27) and converges in  $L^r((0, T) \times \Omega)^4$ , for  $1 \leq r < \infty$ , to  $(\bar{\rho}, \bar{p}, \bar{e}, \bar{u}) \in L^\infty((0, T) \times \Omega)^4$ .

Then the limit  $(\bar{\rho}, \bar{p}, \bar{e}, \bar{u})$  satisfies the system (29).

*Proof.* The fact that the limit  $(\bar{\rho}, \bar{p}, \bar{u})$  satisfies (29a) and (29b) is proven in the first part of this work [14, Proof of Theorem 3.7 and Remark 3.9], using milder estimates and the same convergence assumptions. On the other hand, the fact that  $(\bar{\rho}, \bar{p}, \bar{e})$  satisfies the equation of state is straightforward, in view of the supposed convergence. We thus only need to prove that  $(\bar{\rho}, \bar{p}, \bar{e}, \bar{u})$  satisfies (29c).

Let  $\varphi \in C_c^\infty(\Omega \times [0, T])$ . Let  $m \in \mathbb{N}$ ,  $\mathcal{M}^{(m)}$  and  $\delta t^{(m)}$  be given. Dropping for short the superscript  $(m)$ , let  $\varphi_{\mathcal{M}}$  and  $\varphi_{\mathcal{E}}$ ,  $\bar{\partial}_t \varphi_{\mathcal{M}}$ ,  $\bar{\partial}_x \varphi_{\mathcal{M}}$ ,  $\bar{\partial}_t \varphi_{\mathcal{E}}$ ,  $\bar{\partial}_x \varphi_{\mathcal{E}}$  and  $\bar{\partial}_x \varphi_{\mathcal{M}, \mathcal{E}}$  be the interpolates of  $\varphi$  on the primal and dual mesh and their discrete time and space derivatives, in the sense of Definition 3.2; thanks to the regularity of  $\varphi$ , these functions respectively converge in  $L^r(\Omega \times (0, T))$ , for  $r \geq 1$  (including  $r = +\infty$ ), to  $\varphi$ ,  $\varphi$ ,  $\partial_t \varphi$ ,  $\partial_x \varphi$ ,  $\partial_t \varphi$ ,  $\partial_x \varphi$  and  $\partial_x \varphi$  respectively.

On one hand, let us multiply the discrete kinetic energy equation (11) by  $\delta t \varphi_\sigma^n$  and sum over the faces and the time steps. On the other hand, let us multiply the discrete internal energy equation (5c) by  $\delta t \varphi_K^n$ , and sum over the primal cells and the time steps. Finally, let us sum the two obtained relations. Since the support of  $\varphi$  is compact in  $\Omega \times (0, T)$ , for space and time steps small enough (or, equivalently,  $m$  large enough), the interpolates of  $\varphi$  vanish for  $n = N$ , and, at any time, on the cells and faces located in a neighbourhood of the boundaries; we suppose that it is the case for the element  $m$  of the sequence under consideration. We thus get  $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} + T_5^{(m)} = R^{(m)}$  with:

$$T_1^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}} \frac{|D_\sigma|}{\delta t} [\rho_{D_\sigma}^{n+1} (u_\sigma^{n+1})^2 - \rho_{D_\sigma}^n (u_\sigma^n)^2] \varphi_\sigma^n, \quad (30a)$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} \frac{|K|}{\delta t} [\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n] \varphi_K^n, \quad (30b)$$

$$T_3^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma=K \xrightarrow{L} \in \mathcal{E}_{\text{int}}, \\ K=[\sigma' \sigma], L=[\sigma \sigma']}} [F_L^{n+1} u_\sigma^{n+1} u_{\sigma'}^{n+1} - F_K^{n+1} u_\sigma^{n+1} u_{\sigma'}^{n+1}] \varphi_\sigma^n, \quad (30c)$$

$$T_4^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} [F_{\sigma'}^{n+1} e_{\sigma'}^{n+1} - F_{\sigma}^{n+1} e_{\sigma}^{n+1}] \varphi_K^n, \quad (30d)$$

$$T_5^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} (p_L^{n+1} - p_K^{n+1}) u_{\sigma}^{n+1} \varphi_{\sigma}^n \\ + \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} p_K^{n+1} (u_{\sigma'}^{n+1} - u_{\sigma}^{n+1}) \varphi_K^n, \quad (30e)$$

$$R^{(m)} = - \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\text{int}}} R_{\sigma}^{n+1} \varphi_{\sigma}^n + \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} S_K^{n+1} \varphi_K^n, \quad (30f)$$

where, in the latter relation, the remainder term reads, for  $\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}$  with  $K = [\overrightarrow{\sigma'}\sigma]$  and  $L = [\overrightarrow{\sigma\sigma''}]$ :

$$R_{\sigma}^{n+1} = \frac{1}{2} \frac{|D_{\sigma}|}{\delta t} \rho_{D_{\sigma}}^n [u_{\sigma}^{n+1} - u_{\sigma}^n]^2 + \delta^{\text{up}} u_{\sigma}^{n+1} \left[ \frac{|F_K^{n+1}|}{2} (u_{\sigma}^{n+1} - u_{\sigma'}^{n+1}) + \frac{|F_L^{n+1}|}{2} (u_{\sigma}^{n+1} - u_{\sigma''}^{n+1}) \right].$$

We first study  $T_1^{(m)}$ . Reordering of the sums and then using the definition (8) of the density at the faces, we thus get:

$$T_1^{(m)} = -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}} |D_{\sigma}| \rho_{D_{\sigma}}^{n+1} (u_{\sigma}^{n+1})^2 \frac{\varphi_{\sigma}^{n+1} - \varphi_{\sigma}^n}{\delta t} - \frac{1}{2} \sum_{\sigma \in \mathcal{E}} |D_{\sigma}| \rho_{D_{\sigma}}^0 (u_{\sigma}^0)^2 \varphi_{\sigma}^0 \\ = -\frac{1}{2} \int_0^T \int_{\Omega} \rho^{(m)} (u^{(m)})^2 \partial_t \varphi_{\mathcal{E}} \, dx \, dt - \frac{1}{2} \int_{\Omega} (\rho^{(m)})^0(x) [(u^{(m)})^0(x)]^2 \varphi_{\mathcal{E}}(x, 0) \, dx.$$

By the definition (15a) of the initial conditions of the scheme, since both  $\rho_0$  and  $u_0$  are supposed to belong to  $L^{\infty}(\Omega)$ ,  $(\rho^{(m)})^0$  and  $(u^{(m)})^0$  converge to  $\rho_0$  and  $u_0$  respectively in  $L^r(\Omega)$ , for  $r \geq 1$ . Since, by assumption, the sequence of discrete solutions converges in  $L^r(\Omega \times (0, T))$  for  $r \geq 1$ , we can pass to the limit in the previous relation, to get:

$$\lim_{m \rightarrow +\infty} T_1^{(m)} = -\frac{1}{2} \int_0^T \int_{\Omega} \bar{\rho}(\bar{u})^2 \partial_t \varphi \, dx \, dt - \frac{1}{2} \int_{\Omega} \rho_0(x) u_0(x)^2 \varphi(x, 0) \, dx.$$

By a similar computation, we get for  $T_2^{(m)}$ :

$$T_2^{(m)} = - \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}} |K| \rho_K^{n+1} e_K^{n+1} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} - \sum_{\sigma \in \mathcal{E}} |K| \rho_K^0 e_K^0 \varphi_K^0 \\ = - \int_0^T \int_{\Omega} \rho^{(m)} e^{(m)} \partial_t \varphi_{\mathcal{M}} \, dx \, dt - \int_{\Omega} (\rho^{(m)})^0(x) (e^{(m)})^0(x) \varphi_{\mathcal{M}}(x, 0) \, dx,$$

and therefore:

$$\lim_{m \rightarrow +\infty} T_2^{(m)} = - \int_0^T \int_{\Omega} \bar{\rho} \bar{e} \partial_t \varphi \, dx \, dt - \int_{\Omega} \rho_0(x) e_0(x) \varphi(x, 0) \, dx.$$

Let us now turn to  $T_3^{(m)}$ . Reordering the sums, we get:

$$T_3^{(m)} = -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} F_K^{n+1} u_{\sigma}^{n+1} u_{\sigma'}^{n+1} (\varphi_{\sigma}^n - \varphi_{\sigma'}^n).$$

Using now the definition (17) of the mass fluxes at the dual edges, we have:

$$T_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} (\rho_{\sigma}^{n+1} u_{\sigma}^{n+1} + \rho_{\sigma'}^{n+1} u_{\sigma'}^{n+1}) u_{\sigma'}^{n+1} u_{\sigma}^{n+1} (\varphi_{\sigma}^n - \varphi_{\sigma'}^n).$$

We now split  $T_3^{(m)} = \mathcal{T}_3^{(m)} + \mathcal{R}_3^{(m)}$ , where

$$\mathcal{T}_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \rho_K^{n+1} [(u_\sigma^{n+1})^3 + (u_{\sigma'}^{n+1})^3] (\varphi_\sigma^n - \varphi_{\sigma'}^n) = -\frac{1}{2} \int_0^T \int_\Omega \rho^{(m)} (u^{(m)})^3 \bar{\partial}_x \varphi_\mathcal{E} \, dx \, dt,$$

so that

$$\lim_{m \rightarrow +\infty} \mathcal{T}_3^{(m)} = -\frac{1}{2} \int_0^T \int_\Omega \bar{\rho} \bar{u}^3 \partial_x \varphi \, dx \, dt,$$

and

$$\mathcal{R}_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \left[ (\rho_\sigma^{n+1} u_\sigma^{n+1} + \rho_{\sigma'}^{n+1} u_{\sigma'}^{n+1}) u_\sigma^{n+1} u_{\sigma'}^{n+1} - \rho_K^{n+1} \left( (u_\sigma^{n+1})^3 + (u_{\sigma'}^{n+1})^3 \right) \right] (\varphi_\sigma^n - \varphi_{\sigma'}^n).$$

Expanding the quantity  $(u_\sigma^{n+1})^3 + (u_{\sigma'}^{n+1})^3$  thanks to the identity  $a^3 + b^3 = (a+b)(ab + (a-b)^2)$ , and then reordering the sums, we obtain  $\mathcal{R}_3^{(m)} = \mathcal{R}_{3,1}^{(m)} + \mathcal{R}_{3,2}^{(m)}$  with:

$$\begin{aligned} \mathcal{R}_{3,1}^{(m)} &= -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \left[ (\rho_\sigma^{n+1} - \rho_K^{n+1}) u_\sigma^{n+1} + (\rho_{\sigma'}^{n+1} - \rho_K^{n+1}) u_{\sigma'}^{n+1} \right] u_\sigma^{n+1} u_{\sigma'}^{n+1} (\varphi_\sigma^n - \varphi_{\sigma'}^n), \\ \mathcal{R}_{3,2}^{(m)} &= \frac{1}{4} \sum_{n=0}^N \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \rho_K^{n+1} (u_\sigma^{n+1} + u_{\sigma'}^{n+1}) (u_\sigma^{n+1} - u_{\sigma'}^{n+1})^2 (\varphi_\sigma^n - \varphi_{\sigma'}^n). \end{aligned}$$

In the term  $\mathcal{R}_{3,1}^{(m)}$ , the differences  $\rho_\sigma^{n+1} - \rho_K^{n+1}$  and  $\rho_{\sigma'}^{n+1} - \rho_K^{n+1}$  either vanish or involve two adjacent cells densities. We thus get:

$$|\mathcal{R}_{3,1}^{(m)}| \leq h C_\varphi \|u^{(m)}\|_{L^\infty(\Omega \times (0,T))}^3 \|\rho^{(m)}\|_{\mathcal{T},x,BV},$$

and  $\mathcal{R}_{3,1}^{(m)}$  tends to zero when  $m$  tends to  $+\infty$ . By similar arguments:

$$|\mathcal{R}_{3,2}^{(m)}| \leq h C_\varphi \|\rho^{(m)}\|_{L^\infty(\Omega \times (0,T))} \|u^{(m)}\|_{L^\infty(\Omega \times (0,T))}^2 \|u^{(m)}\|_{\mathcal{T},x,BV},$$

and thus  $\mathcal{R}_{3,2}^{(m)}$  also tends to zero when  $m$  tends to  $+\infty$ .

Expressing the mass fluxes as a function of the unknowns in  $T_4^{(m)}$  and reordering the sums, we get:

$$T_4^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|\tilde{L}} \in \mathcal{E}_{\text{int}}} \rho_\sigma^{n+1} e_\sigma^{n+1} u_\sigma^{n+1} (\varphi_K^n - \varphi_{\tilde{L}}^n).$$

Let us write  $T_4^{(m)} = \mathcal{T}_4^{(m)} + \mathcal{R}_4^{(m)}$ , with:

$$\begin{aligned} \mathcal{T}_4^{(m)} &= -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|\tilde{L}} \in \mathcal{E}_{\text{int}}} \left[ |D_{K,\sigma}| \rho_K^{n+1} e_K^{n+1} + |D_{L,\sigma}| \rho_L^{n+1} e_L^{n+1} \right] u_\sigma^{n+1} \frac{\varphi_K^n - \varphi_{\tilde{L}}^n}{h_\sigma}, \\ \mathcal{R}_4^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|\tilde{L}} \in \mathcal{E}_{\text{int}}} \left[ |D_{K,\sigma}| (\rho_K^{n+1} e_K^{n+1} - \rho_\sigma^{n+1} e_\sigma^{n+1}) + |D_{L,\sigma}| (\rho_L^{n+1} e_L^{n+1} - \rho_\sigma^{n+1} e_\sigma^{n+1}) \right] \\ &\quad u_\sigma^{n+1} \frac{\varphi_K^n - \varphi_{\tilde{L}}^n}{h_\sigma}. \end{aligned}$$

We have:

$$\mathcal{T}_4^{(m)} = -\int_0^T \int_\Omega \rho^{(m)} e^{(m)} u^{(m)} \bar{\partial}_x \varphi_\mathcal{M} \, dx \, dt, \quad \text{so} \quad \lim_{m \rightarrow +\infty} \mathcal{T}_4^{(m)} = -\int_0^T \int_\Omega \bar{\rho} \bar{e} \bar{u} \partial_x \varphi \, dx \, dt.$$

Expanding the quantities  $(\rho_K^{n+1} e_K^{n+1} - \rho_\sigma^{n+1} e_\sigma^{n+1})$  and  $(\rho_L^{n+1} e_L^{n+1} - \rho_\sigma^{n+1} e_\sigma^{n+1})$  in the residual term  $\mathcal{R}_4^{(m)}$  thanks to the identity  $2(ab - cd) = (a + c)(b - d) + (b + d)(a - c)$ , we get:

$$|\mathcal{R}_4^{(m)}| \leq C_\varphi h \|u^{(m)}\|_{L^\infty(\Omega \times (0, T))} \left[ \|\rho^{(m)}\|_{L^\infty(\Omega \times (0, T))} \|e^{(m)}\|_{\mathcal{T}, x, \text{BV}} + \|e^{(m)}\|_{L^\infty(\Omega \times (0, T))} \|\rho^{(m)}\|_{\mathcal{T}, x, \text{BV}} \right],$$

so that  $\mathcal{R}_4^{(m)}$  tends to zero when  $m$  tends to  $+\infty$ .

Reordering the sums in the term  $T_5^{(m)}$ , we obtain:

$$T_5^{(m)} = \sum_{n=0}^{N-1} -\delta t \sum_{K=[\vec{\sigma\sigma'}] \in \mathcal{M}} p_K^{n+1} (u_{\sigma'}^{n+1} \varphi_{\sigma'}^n - u_\sigma^{n+1} \varphi_\sigma^n) + p_K^{n+1} (u_\sigma^{n+1} - u_{\sigma'}^{n+1}) \varphi_K^n,$$

hence:

$$\begin{aligned} T_5^{(m)} &= - \sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma\sigma'}] \in \mathcal{M}} \frac{|K|}{2} p_K^{n+1} u_\sigma^{n+1} \frac{\varphi_K^n - \varphi_\sigma^n}{h_K/2} + \frac{|K|}{2} p_K^{n+1} u_{\sigma'}^{n+1} \frac{\varphi_{\sigma'}^n - \varphi_K^n}{h_K/2} \\ &= - \int_0^T \int_\Omega p^{(m)} u^{(m)} \bar{\partial}_x \varphi_{\mathcal{M}, \varepsilon} dx dt. \end{aligned}$$

and so:

$$\lim_{m \rightarrow +\infty} T_5^{(m)} = - \int_0^T \int_\Omega \bar{p} \bar{u} \partial_x \varphi dx dt.$$

Finally we study  $R^{(m)}$ , which we decompose in  $R^{(m)} = R_c^{(m)} + R_{up}^{(m)}$ , the first part gathering the terms which are not linked to a possible upwinding. We have for this residual:

$$\begin{aligned} R_c^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \left[ \sum_{\sigma \in \mathcal{E}} -|D_\sigma| \rho_{D_\sigma}^n (u_\sigma^{n+1} - u_\sigma^n)^2 \varphi_\sigma^n + \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}(K)} |D_{K, \sigma}| \rho_K^n (u_\sigma^{n+1} - u_\sigma^n)^2 \varphi_K^n \right] \\ &= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}(K)} |D_{K, \sigma}| \rho_K^n (u_\sigma^{n+1} - u_\sigma^n)^2 (\varphi_K^n - \varphi_\sigma^n). \end{aligned}$$

We thus obtain:

$$R_c^{(m)} \leq h C_\varphi \|\rho^{(m)}\|_{L^\infty(\Omega \times (0, T))} \|u^{(m)}\|_{L^\infty(\Omega \times (0, T))} \|u^{(m)}\|_{\mathcal{T}, t, \text{BV}},$$

and therefore  $R_c^{(m)}$  tends to zero when  $m \rightarrow \infty$ . We now turn to the remainder term associated to the upwinding, which reads:

$$\begin{aligned} R_{up}^{(m)} &= - \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma=K | \vec{L} \in \mathcal{E}_{\text{int}}, \\ K=[\vec{\sigma'\sigma}], L=[\vec{\sigma\sigma'}]}} u_\sigma^{n+1} \left[ \frac{|F_K^{n+1}|}{2} (u_\sigma^{n+1} - u_{\sigma'}^{n+1}) + \frac{|F_L^{n+1}|}{2} (u_\sigma^{n+1} - u_{\sigma''}^{n+1}) \right] \varphi_\sigma^n \\ &\quad + \sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma\sigma'}] \in \mathcal{M}} \frac{|F_K^{n+1}|}{2} (u_\sigma^{n+1} - u_{\sigma'}^{n+1})^2 \varphi_K^n. \end{aligned}$$

As explained at the end of Section 3.2, the general idea is now to recast this term as a discrete version of the integral over space and time of a quantity of the form  $-u \partial_x u \partial_x \varphi$  scaled by a numerical viscosity vanishing with the space step; then, the supposed controls on the solution imply that the term tends to zero. We thus reorder the sums in  $R_{up}^{(m)}$ , which yields:

$$R_{up}^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma\sigma'}] \in \mathcal{M}} |F_K^{n+1}| (u_\sigma^{n+1} - u_{\sigma'}^{n+1}) (u_{\sigma'}^{n+1} \varphi_{\sigma'}^n - u_\sigma^{n+1} \varphi_\sigma^n) + |F_K^{n+1}| (u_{\sigma'}^{n+1} - u_\sigma^{n+1})^2 \varphi_K^n,$$

and thus:

$$R_{up}^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} |F_K^{n+1}| (u_\sigma^{n+1} - u_{\sigma'}^{n+1}) [u_\sigma^{n+1}(\varphi_K^n - \varphi_\sigma^n) + u_{\sigma'}^{n+1}(\varphi_{\sigma'}^n - \varphi_K^n)].$$

We thus get, using the definition (17) of the mass fluxes at the dual faces:

$$|R_{up}^{(m)}| \leq h C_\varphi \|\rho^{(m)}\|_{L^\infty(\Omega \times (0, T))} \|u^{(m)}\|_{L^\infty(\Omega \times (0, T))}^2 \|u^{(m)}\|_{\mathcal{T}, x, \text{BV}},$$

which yields the desired estimate.

Gathering the expression of the limits of each of the terms  $T_1^{(m)}$  to  $T_5^{(m)}$  and  $R^{(m)}$  concludes the proof.  $\square$

#### 4. A FIRST PRESSURE CORRECTION SCHEME

##### 4.1. The scheme

We derive in this section a pressure correction scheme from the implicit scheme (5). The first step, as usual, is to compute a tentative velocity by solving the momentum balance equation with the beginning-of-step pressure. Then, the velocity is corrected and the other variables are advanced in time; the correction step not standard, since it is performed here by a single coupled step for stability reasons detailed in [13]. Still for stability reasons, or, in other words, to be able to derive a kinetic energy balance, we need that a mass balance over the dual cells (9) holds; since the mass balance is not yet solved when performing the prediction step, this leads us to perform a time shift of the density at this step.

With the notations introduced in Section 3.1, the algorithm reads, for  $0 \leq n \leq N - 1$ :

**Prediction step** – Solve for  $\tilde{\mathbf{u}}^{n+1}$ :

$$\text{For } 1 \leq i \leq d, \quad \forall \sigma \in \mathcal{E}_S^{(i)},$$

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^n \tilde{u}_{\sigma,i}^{n+1} - \rho_{D_\sigma}^{n-1} u_{\sigma,i}^n) + \sum_{\varepsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\varepsilon}^n \tilde{u}_{\varepsilon,i}^{n+1} + |D_\sigma| (\nabla p)_\sigma^n = 0. \quad (31a)$$

**Correction step** – Solve for  $\rho^{n+1}$ ,  $p^{n+1}$ ,  $e^{n+1}$  and  $\mathbf{u}^{n+1}$ :

$$\text{For } 1 \leq i \leq d, \quad \forall \sigma \in \mathcal{E}_S^{(i)},$$

$$\frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}) + |D_\sigma| [(\nabla p)_{\sigma,i}^{n+1} - (\nabla p)_{\sigma,i}^n] = 0, \quad (31b)$$

$$\forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} = 0, \quad (31c)$$

$$\forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} e_\sigma^{n+1} + |K| p_K^{n+1} (\text{div} \tilde{\mathbf{u}})_K^{n+1} = S_K^{n+1}, \quad (31d)$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\gamma - 1) \rho_K^{n+1} e_K^{n+1}. \quad (31e)$$

The initialization of the scheme is performed as follows. First,  $\rho^{-1}$ ,  $e^0$  and  $\mathbf{u}^0$  are given by the average of the initial conditions  $\rho_0$ ,  $e_0$  and  $\mathbf{u}_0$  on the primal and dual cells:

$$\forall K \in \mathcal{M}, \quad \rho_K^{-1} = \frac{1}{|K|} \int_K \rho_0(\mathbf{x}) \, d\mathbf{x}, \quad e_K^0 = \frac{1}{|K|} \int_K e_0(\mathbf{x}) \, d\mathbf{x}, \quad (32)$$

$$\text{for } 1 \leq i \leq d, \quad \forall \sigma \in \mathcal{E}_S^{(i)}, \quad u_{\sigma,i}^0 = \frac{1}{|D_\sigma|} \int_{D_\sigma} (u_0(\mathbf{x}))_i \, d\mathbf{x}.$$



Then we compute  $\rho^0$  by solving the mass balance equation (31c), which allows to perform the first prediction step with  $(\rho_{D_\sigma}^{-1})_{\sigma \in \mathcal{E}}$ ,  $(\rho_{D_\sigma}^0)_{\sigma \in \mathcal{E}}$  and the dual mass fluxes satisfying the mass balance. Finally, the initial pressure  $p^0$  is computed from  $\rho^0$  and  $e^0$  by the equation of state:  $\forall K \in \mathcal{M}$ ,  $p_K^0 = (\gamma - 1) \rho_K^0 e_K^0$ .

#### 4.2. The discrete kinetic energy balance equation and the corrective source terms

As in the case of the implicit scheme, we now determine the corrective terms  $(S_K^n)_{K \in \mathcal{M}}$  which need to be added in the internal energy equation in order to recover a weak formulation of the total energy equation when passing to the limit. We thus begin by the derivation of a discrete kinetic energy equation, which is stated in the following lemma. Note that we cannot use directly here [14, Lemma 4.1] as we shall do in Section 5.2, because of the different order of the equations in the prediction-correction algorithm.

**Lemma 4.1** (Discrete kinetic energy balance, pressure correction scheme 1).

A solution to the system (31a)-(31b) satisfies the following equality, for  $1 \leq i \leq d$ ,  $\sigma \in \mathcal{E}_S^{(i)}$  and  $0 \leq n \leq N - 1$ :

$$\frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[ \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^{n-1} (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\varepsilon = D_\sigma | D_{\sigma'}} F_{\sigma,\varepsilon}^n \tilde{u}_{\sigma,i}^{n+1} \tilde{u}_{\sigma',i}^{n+1} + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} \tilde{u}_{\sigma,i}^{n+1} = -R_{\sigma,i}^n, \quad (33)$$

where the remainder term  $R_{\sigma,i}^{n+1}$  reads:

$$R_{\sigma,i}^{n+1} = \frac{|D_\sigma|}{2\delta t} \rho_{D_\sigma}^{n-1} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 - \frac{|D_\sigma|}{2\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1})^2 + \delta^{\text{up}} \left[ \sum_{\varepsilon = D_\sigma | D_{\sigma'}} \frac{1}{2} |F_{\sigma,\varepsilon}^n| (\tilde{u}_{\sigma,i}^{n+1} - \tilde{u}_{\sigma',i}^{n+1}) \right] \tilde{u}_{\sigma,i}^{n+1}, \quad (34)$$

with  $\delta^{\text{up}} = 1$  if an upwind discretization is used for the convection term in the momentum balance equation and  $\delta^{\text{up}} = 0$  in the centered case.

*Proof.* Let us sum the momentum balance equation (31a) with the velocity correction equation (31b), which yields:

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^n u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^{n-1} u_{\sigma,i}^n) + \sum_{\varepsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\varepsilon}^n \tilde{u}_{\varepsilon,i}^{n+1} + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} = 0.$$

Multiplying this equation by the corresponding degree of freedom of the predicted velocity  $\tilde{u}_{\sigma,i}^{n+1}$ , we obtain:

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^n u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^{n-1} u_{\sigma,i}^n) \tilde{u}_{\sigma,i}^{n+1} + \sum_{\varepsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\varepsilon}^n \tilde{u}_{\varepsilon,i}^{n+1} \tilde{u}_{\sigma,i}^{n+1} + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} \tilde{u}_{\sigma,i}^{n+1} = 0.$$

Let us recast the first two terms of this equation as  $T_{\sigma,i}^{(1)} + T_{\sigma,i}^{(2)}$ , with:

$$T_{\sigma,i}^{(1)} = \frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^n \tilde{u}_{\sigma,i}^{n+1} - \rho_{D_\sigma}^{n-1} u_{\sigma,i}^n) \tilde{u}_{\sigma,i}^{n+1} + \sum_{\varepsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\varepsilon}^n \tilde{u}_{\varepsilon,i}^{n+1} \tilde{u}_{\sigma,i}^{n+1},$$

$$T_{\sigma,i}^{(2)} = \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}) \tilde{u}_{\sigma,i}^{n+1}.$$

The term  $T_{\sigma,i}^{(1)}$  has the structure which allows to apply [14, Lemma A2], and we get:

$$T_{\sigma,i}^{(1)} = \frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[ \rho_{D_\sigma}^n (\tilde{u}_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^{n-1} (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\varepsilon = D_\sigma | D_{\sigma'}} F_{\sigma,\varepsilon}^n \tilde{u}_{\sigma,i}^{n+1} \tilde{u}_{\sigma',i}^{n+1} + R_{\sigma,i}^{(1)},$$

with:

$$R_{\sigma,i}^{(1)} = \frac{|D_\sigma|}{2\delta t} \rho_{D_\sigma}^{n-1} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 + \delta^{\text{up}} \left[ \sum_{\varepsilon = D_\sigma | D_{\sigma'}} \frac{1}{2} |F_{\sigma,\varepsilon}^n| (\tilde{u}_{\sigma,i}^{n+1} - \tilde{u}_{\sigma',i}^{n+1}) \right] \tilde{u}_{\sigma,i}^{n+1}.$$

Using the identity  $2(a - b)a = a^2 - b^2 + (a - b)^2$ , valid for any real numbers  $a$  and  $b$ , we get for  $T_2$ :

$$T_{\sigma,i}^{(2)} = \frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[ \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^n (\tilde{u}_{\sigma,i}^{n+1})^2 \right] + R_{\sigma,i}^{(2)},$$

with:

$$R_{\sigma,i}^{(2)} = -\frac{|D_\sigma|}{2\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1})^2.$$

Summing, we get the discrete kinetic energy balance equation:

$$\frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[ \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^{n-1} (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\varepsilon=D_\sigma|D_{\sigma'}} F_{\sigma,\varepsilon}^n \tilde{u}_{\sigma,i}^{n+1} \tilde{u}_{\sigma',i}^{n+1} + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} \tilde{u}_{\sigma,i}^{n+1} = R_{\sigma,i}^{(1)} + R_{\sigma,i}^{(2)} = R_{\sigma,i}^n,$$

which concludes the proof.  $\square$

By the same arguments as in the implicit case, we now get from the discrete kinetic energy balance associated to this time discretization:

$$\forall K \in \mathcal{M}, S_K^{n+1} = \sum_{i=1}^d S_{K,i}^{n+1} \quad \text{with}$$

$$S_{K,i}^{n+1} = \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_S^{(i)}} \frac{|D_{K,\sigma}|}{\delta t} \left[ \rho_K^{n-1} |\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n|^2 - \rho_K^n |u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}|^2 \right] + \delta^{\text{up}} \sum_{\substack{\varepsilon \in \mathcal{E}_S^{(i)}, \varepsilon \cap K \neq \emptyset, \\ \varepsilon = D_\sigma | D_{\sigma'}}} \alpha_{K,\varepsilon} \frac{|F_{\sigma,\varepsilon}^n|}{2} |\tilde{u}_{\sigma,i}^{n+1} - \tilde{u}_{\sigma',i}^{n+1}|^2.$$

Note that the term  $S_K^{n+1}$  may be negative, contrary to implicit case; this has indeed been observed in computations, and impedes the theoretical proof of the positivity of the internal energy for this prediction–correction scheme. However, even in very severe cases (as, for instance, Test 3 of [23, chapter 4]), at least with a reasonable time step, we still obtained a positive internal energy.

### 4.3. Passing to the limit in the scheme (1D case)

As for the implicit scheme, we show here that in the one dimensional case, if a sequence of solutions is controlled in suitable norms and converges to a limit, this limit necessarily satisfies a weak form of the Euler equations, namely the system (29).

Let us first write the pressure correction scheme (31a)–(31e) in the one-dimensional case, using the notations (16)–(17):

**Initialization** – Compute  $\rho^{-1}$ ,  $e^0$ ,  $u^0$ , solve for  $\rho^0$  and compute  $p^0$ :

$$\begin{aligned} \forall K \in \mathcal{M}, \quad \rho_K^{-1} &= \frac{1}{|K|} \int_K \rho_0(x) \, dx, & e_K^0 &= \frac{1}{|K|} \int_K e_0(x) \, dx, \\ \forall \sigma \in \mathcal{E}_{\text{int}}, \quad u_\sigma^0 &= \frac{1}{|D_\sigma|} \int_{D_\sigma} u_0(x) \, dx, \\ \forall K = \overrightarrow{[\sigma\sigma']} \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^0 - \rho_K^{-1}) + F_{\sigma'}^0 - F_\sigma^0 &= 0, \\ \forall K \in \mathcal{M}, \quad p_K^0 &= (\gamma - 1) \rho_K^0 e_K^0. \end{aligned} \tag{35a}$$

**Prediction step** – Solve for  $\tilde{u}^{n+1}$ :

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \quad \frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^n \tilde{u}_\sigma^{n+1} - \rho_{D_\sigma}^{n-1} u_\sigma^n) + F_L^n \tilde{u}_L^{n+1} - F_K^n \tilde{u}_K^{n+1} + p_L^n - p_K^n = 0. \tag{35b}$$

**Correction step** – Solve for  $\rho^{n+1}$ ,  $p^{n+1}$ ,  $e^{n+1}$  and  $u^{n+1}$ :

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \quad \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_\sigma^{n+1} - \tilde{u}_\sigma^{n+1}) + (p_L^{n+1} - p_K^{n+1}) - (p_L^n - p_K^n) = 0, \quad (35c)$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^{n+1} - F_\sigma^{n+1} = 0, \quad (35d)$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + F_{\sigma'}^{n+1} e_{\sigma'}^{n+1} - F_\sigma^{n+1} e_\sigma^{n+1} + p_K^{n+1} (\tilde{u}_{\sigma'}^{n+1} - \tilde{u}_\sigma^{n+1}) = S_K^{n+1}, \quad (35e)$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\gamma - 1) \rho_K^{n+1} e_K^{n+1}. \quad (35f)$$

The corrective term  $S_K^{n+1}$  reads:

$$\begin{aligned} \forall K = [\overrightarrow{\sigma\sigma'}], \quad S_K^{n+1} &= \frac{|K|}{4\delta t} \rho_K^{n-1} [(\tilde{u}_\sigma^{n+1} - u_\sigma^n)^2 + (\tilde{u}_{\sigma'}^{n+1} - u_{\sigma'}^n)^2] \\ &\quad - \frac{|K|}{4\delta t} \rho_K^n [(u_\sigma^{n+1} - \tilde{u}_\sigma^{n+1})^2 + (u_{\sigma'}^{n+1} - \tilde{u}_{\sigma'}^{n+1})^2] + \delta^{\text{up}} \frac{|F_K^n|}{2} (\tilde{u}_\sigma^{n+1} - \tilde{u}_{\sigma'}^{n+1})^2. \end{aligned}$$

For the proof of the consistency of the scheme (*i.e.* the proof of the theorem 4.2 below), we need to introduce the following stability assumptions on a sequence  $(\rho^{(m)}, p^{(m)}, e^{(m)}, \tilde{u}^{(m)}, u^{(m)})_{m \in \mathbb{N}}$  of discrete solutions:

$$|(\rho^{(m)})_K^n| + |(p^{(m)})_K^n| + |(e^{(m)})_K^n| \leq C, \quad \forall K \in \mathcal{M}^{(m)}, \text{ for } 0 \leq n \leq N^{(m)}, \forall m \in \mathbb{N}, \quad (36a)$$

$$\frac{1}{|(\rho^{(m)})_K^n|} \leq C, \quad \forall K \in \mathcal{M}^{(m)}, \text{ for } 0 \leq n \leq N^{(m)}, \forall m \in \mathbb{N}, \quad (36b)$$

$$|(u^{(m)})_\sigma^n| + |(\tilde{u}^{(m)})_\sigma^n| \leq C, \quad \forall \sigma \in \mathcal{E}^{(m)}, \text{ for } 0 \leq n \leq N^{(m)}, \forall m \in \mathbb{N}, \quad (36c)$$

$$\|\rho^{(m)}\|_{\mathcal{T},x,\text{BV}} + \|e^{(m)}\|_{\mathcal{T},x,\text{BV}} + \|p^{(m)}\|_{\mathcal{T},x,\text{BV}} + \|\tilde{u}^{(m)}\|_{\mathcal{T},x,\text{BV}} \leq C, \quad \forall m \in \mathbb{N}, \quad (36d)$$

$$\|\rho^{(m)}\|_{\mathcal{T},t,\text{BV}} + \|u^{(m)}\|_{\mathcal{T},t,\text{BV}} \leq C, \quad \forall m \in \mathbb{N}. \quad (36e)$$

Note that we do not suppose any control on time discrete derivatives of  $\tilde{u}^{(m)}$  (*i.e.* on  $\|\tilde{u}^{(m)}\|_{\mathcal{T},t,\text{BV}}$ ) and on the discrete space derivatives of  $u^{(m)}$  (*i.e.* on  $\|u^{(m)}\|_{\mathcal{T},x,\text{BV}}$ ).

**Theorem 4.2** (Consistency, first pressure correction scheme).

Let  $\Omega$  be an open bounded interval of  $\mathbb{R}$ . We suppose that  $\rho_0$ ,  $u_0$  and  $e_0$  are functions of  $L^\infty(\Omega)$ . Let  $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m \in \mathbb{N}}$  be a sequence of discretizations such that both the time step  $\delta t^{(m)}$  and the space step  $h^{(m)}$  tend to zero as  $m \rightarrow \infty$ . Let  $(\rho^{(m)}, p^{(m)}, e^{(m)}, \tilde{u}^{(m)}, u^{(m)})_{m \in \mathbb{N}}$  be the corresponding sequence of solutions. We suppose that this sequence satisfies (36) and converges in  $L^p(\Omega \times (0, T))^5$ , for  $1 \leq p < \infty$ , to  $(\bar{\rho}, \bar{p}, \bar{e}, \bar{\tilde{u}}, \bar{u}) \in L^\infty(\Omega \times (0, T))^5$ .

Then  $\bar{\tilde{u}} = \bar{u}$  and  $(\bar{\rho}, \bar{p}, \bar{e}, \bar{u})$  satisfies the system (29).

*Proof.* Let us first check that  $\bar{\tilde{u}} = \bar{u}$ . To this purpose, it is sufficient to note that the correction step (35c) yields:

$$|D_\sigma| |u_\sigma^{n+1} - \tilde{u}_\sigma^{n+1}| \leq \frac{\delta t}{\rho_{D_\sigma}^n} [|p_L^{n+1} - p_K^{n+1}| + |p_L^n - p_K^n|], \quad \forall \sigma = K|L \in \mathcal{E}_{\text{int}}, \text{ and for } 0 \leq n \leq N - 1,$$

so that:

$$\|u^{(m)} - \tilde{u}^{(m)}\|_{L^1(\Omega \times (0, T))} \leq C \delta t \|p^{(m)}\|_{\mathcal{T},x,\text{BV}}$$

which, passing to the limit when  $m \rightarrow +\infty$ , yields the result thanks to the assumption (36d).

We now turn to the proof that the limit satisfies (29). The passage to the limit on the mass equation (35d) and the momentum equation (35b) were already addressed in [14, Proof of Theorem 3.7] and [14, Proof of Theorem 4.6] respectively, so that we already know that equations (29a) and (29b) are satisfied by  $\bar{u}$ ,  $\bar{\rho}$  and  $\bar{p}$ . Therefore, there only remains to prove that (29c) is satisfied. We proceed as in the case of the implicit scheme. Let  $\varphi \in C_c^\infty(\Omega \times (0, T))$ . Let  $m \in \mathbb{N}$ ,  $\mathcal{M}^{(m)}$  and  $\delta t^{(m)}$  be given, and consider the primal and dual mesh

interpolates of  $\varphi$ , along with their discrete time and space discrete derivatives in the sense of Definition 3.2. On one hand, we multiply the discrete kinetic energy equation (33) by  $\delta t \varphi_\sigma^n$  and sum over the faces and the time steps. On the other hand, we multiply the discrete internal energy equation (31d) by  $\delta t \varphi_K^n$ , and sum over the primal cells and the time steps. Finally, we sum the two obtained relations. Supposing that  $m$  is large enough for  $\varphi$  to vanish on the boundary cells and at the last time steps, we get  $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} + T_5^{(m)} = R^{(m)}$ , with:

$$\begin{aligned} T_1^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{|D_\sigma|}{\delta t} [\rho_{D_\sigma}^n (u_\sigma^{n+1})^2 - \rho_{D_\sigma}^{n-1} (u_\sigma^n)^2] \varphi_\sigma^n, \\ T_3^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma=\overrightarrow{K} | \overleftarrow{L} \in \mathcal{E}_{\text{int}}, \\ K=[\sigma'\overrightarrow{\sigma}], L=[\overleftarrow{\sigma}\sigma']}} [F_L^n \tilde{u}_\sigma^{n+1} \tilde{u}_{\sigma'}^{n+1} - F_K^n \tilde{u}_\sigma^{n+1} \tilde{u}_{\sigma'}^{n+1}] \varphi_\sigma^n, \\ R^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\text{int}}} R_\sigma^{n+1} \varphi_\sigma^n + \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} S_K^{n+1} \varphi_K^n, \end{aligned}$$

and with  $T_2^{(m)}$ ,  $T_4^{(m)}$  given by (30b) and (30d) and  $T_5^{(m)}$  by (30e) with  $\tilde{u}$  instead of  $u$ .

The passage to the limit in the term  $T_1^{(m)}$  is done as in the implicit case; the only slight difference is the time shift in  $\rho$ , which is handled by remarking that  $\rho^{(m)}(\cdot, \cdot - \delta t)$  strongly converges to  $\bar{\rho}$ . For the term  $T_3^{(m)}$ , still by a computation similar to the implicit case, we get:  $T_3^{(m)} = \mathcal{T}_3^{(m)} + \mathcal{R}_3^{(m)}$ ,

$$\begin{aligned} \text{where} \quad \mathcal{T}_3^{(m)} &= -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\overrightarrow{\sigma'}] \in \mathcal{M}} \frac{|K|}{2} \rho_K^n [u_\sigma^n (\tilde{u}_\sigma^{n+1})^2 + u_{\sigma'}^n (\tilde{u}_{\sigma'}^{n+1})^2] \frac{\varphi_\sigma^n - \varphi_{\sigma'}^n}{h_K} \\ &= -\frac{1}{2} \int_0^T \int_\Omega \rho^{(m)}(x, t - \delta t) u^{(m)}(x, t - \delta t) (\tilde{u}^{(m)}(x, t))^2 \partial_x \varphi_\mathcal{E} \, dx \, dt, \end{aligned}$$

$$\text{so that} \quad \lim_{m \rightarrow +\infty} \mathcal{T}_3^{(m)} = -\frac{1}{2} \int_0^T \int_\Omega \bar{\rho} \bar{u}^3 \partial_x \varphi \, dx \, dt,$$

$$\text{and } \mathcal{R}_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\overrightarrow{\sigma'}] \in \mathcal{M}} \left[ (\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n) \tilde{u}_{\sigma'}^{n+1} \tilde{u}_\sigma^{n+1} - \rho_K^n \left( u_\sigma^n (\tilde{u}_\sigma^{n+1})^2 + u_{\sigma'}^n (\tilde{u}_{\sigma'}^{n+1})^2 \right) \right] (\varphi_\sigma^n - \varphi_{\sigma'}^n).$$

Reordering the terms in the sum, we get:

$$\mathcal{R}_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\overrightarrow{\sigma'}] \in \mathcal{M}} \left[ \underbrace{(\rho_\sigma^n \tilde{u}_{\sigma'}^{n+1} - \rho_K^n \tilde{u}_\sigma^{n+1}) u_\sigma^n \tilde{u}_\sigma^{n+1}}_{\mathcal{D}_1} + \underbrace{(\rho_{\sigma'}^n \tilde{u}_\sigma^{n+1} - \rho_K^n \tilde{u}_{\sigma'}^{n+1}) u_{\sigma'}^n \tilde{u}_{\sigma'}^{n+1}}_{\mathcal{D}_2} \right] (\varphi_\sigma^n - \varphi_{\sigma'}^n).$$

Using the identity  $2(ab - cd) = (a - c)(b + d) + (a + c)(b - d)$  for  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , we conclude that:

$$\begin{aligned} |\mathcal{R}_3^{(m)}| &\leq h C_\varphi \|u^{(m)}\|_{L^\infty(\Omega \times (0, T))} \|\tilde{u}^{(m)}\|_{L^\infty(\Omega \times (0, T))} \\ &\quad \left[ \|\rho^{(m)}\|_{\mathcal{T}, x, \text{BV}} \|\tilde{u}^{(m)}\|_{L^\infty(\Omega \times (0, T))} + \|\rho^{(m)}\|_{L^\infty(\Omega \times (0, T))} \|\tilde{u}^{(m)}\|_{\mathcal{T}, x, \text{BV}} \right], \end{aligned}$$

and thus  $\mathcal{R}_3^{(m)}$  tends to zero when  $m$  tends to  $+\infty$ .

Finally we study  $R^{(m)}$ , which we split into  $R^{(m)} = R_c^{(m)} + R_{up}^{(m)}$ , the first part, namely  $R_c^{(m)}$ , gathering the terms which are not associated to the upwinding:

$$\begin{aligned} R_c^{(m)} &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} |D_\sigma| \left[ \rho_{D_\sigma}^{n-1} (\tilde{u}_\sigma^{n+1} - u_\sigma^n)^2 - \rho_{D_\sigma}^n (\tilde{u}_\sigma^{n+1} - u_\sigma^{n+1})^2 \right] \varphi_\sigma^n \\ &\quad + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}(K)} |D_{K, \sigma}| \left[ \rho_K^{n-1} (\tilde{u}_\sigma^{n+1} - u_\sigma^n)^2 - \rho_K^n (\tilde{u}_\sigma^{n+1} - u_\sigma^{n+1})^2 \right] \varphi_K^n. \end{aligned}$$

Thanks to the definition of the density on the faces, we get:

$$R_c^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}(K)} |D_{K,\sigma}| \left[ \rho_K^{n-1} (\tilde{u}_\sigma^{n+1} - u_\sigma^n)^2 - \rho_K^n (\tilde{u}_\sigma^{n+1} - u_\sigma^{n+1})^2 \right] (\varphi_K^n - \varphi_\sigma^n),$$

so:

$$|R_c^{(m)}| \leq C_\varphi h \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}(K)} |D_{K,\sigma}| \left| \rho_K^{n-1} (\tilde{u}_\sigma^{n+1} - u_\sigma^n)^2 - \rho_K^n (\tilde{u}_\sigma^{n+1} - u_\sigma^{n+1})^2 \right|.$$

Developing

$$\begin{aligned} \rho_K^{n-1} (\tilde{u}_\sigma^{n+1} - u_\sigma^n)^2 - \rho_K^n (\tilde{u}_\sigma^{n+1} - u_\sigma^{n+1})^2 &= \\ &= (\rho_K^{n-1} - \rho_K^n) (\tilde{u}_\sigma^{n+1} - u_\sigma^n)^2 + \rho_K^n (u_\sigma^{n+1} - u_\sigma^n) (2\tilde{u}_\sigma^{n+1} - u_\sigma^n - u_\sigma^{n+1}) \end{aligned}$$

yields:

$$\begin{aligned} |R_c^{(m)}| &\leq h C_\varphi \|\rho^{(m)}\|_{\mathcal{T},t,BV} \left( \|u^{(m)}\|_{L^\infty(\Omega \times (0,T))}^2 + \|\tilde{u}^{(m)}\|_{L^\infty(\Omega \times (0,T))}^2 \right) \\ &\quad + \|\rho^{(m)}\|_{L^\infty(\Omega \times (0,T))} \|u^{(m)}\|_{\mathcal{T},t,BV} \left( \|u^{(m)}\|_{L^\infty(\Omega \times (0,T))} + \|\tilde{u}^{(m)}\|_{L^\infty(\Omega \times (0,T))} \right), \end{aligned}$$

and so  $R_c^{(m)}$  tends to zero as  $m$  tends to  $+\infty$ . Replacing  $u^{(m)}$  by  $\tilde{u}^{(m)}$ , the term  $R_{up}^{(m)}$  takes the same expression as in the implicit case, and so also tends to zero. Gathering all the limits yields the result we are seeking.  $\square$

## 5. A SECOND PRESSURE CORRECTION SCHEME

### 5.1. The scheme

We propose in this section a second pressure correction scheme, which differs from the first one by an additional pressure prediction step, as already proposed in [7] in the context of variable density incompressible flows (see also [5] for an implementation of this strategy for compressible barotropic flows). This step is an elliptic problem for the pressure, the discrete operator of which is obtained by combining the discrete gradient and the discrete divergence operators already used, for the pressure and velocity respectively, in the scheme. The advantage of the new algorithm is that the corrective term  $S_K^{n+1}$  in the internal energy equation is now non negative, which ensures the positivity of the internal energy and the existence of a solution to the scheme.

With the notations introduced in Section 3.1, the algorithm reads, for  $0 \leq n \leq N-1$ :

**Pressure prediction step** – Solve for  $\tilde{p}^{n+1}$ :

$$\forall K \in \mathcal{M}, \quad \sum_{\sigma=K|L} \frac{1}{\rho_{D_\sigma}^n} \frac{|\sigma|^2}{|D_\sigma|} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) = \sum_{\sigma=K|L} \frac{1}{(\rho_{D_\sigma}^n \rho_{D_\sigma}^{n-1})^{1/2}} \frac{|\sigma|^2}{|D_\sigma|} (p_K^n - p_L^n). \quad (37a)$$

**Prediction step** – Solve for  $\tilde{\mathbf{u}}^{n+1}$ :

For  $1 \leq i \leq d$ ,  $\forall \sigma \in \mathcal{E}_S^{(i)}$ ,

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^n \tilde{u}_{\sigma,i}^{n+1} - \rho_{D_\sigma}^{n-1} u_{\sigma,i}^n) + \sum_{\varepsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\varepsilon}^n \tilde{u}_{\varepsilon,i}^{n+1} + |D_\sigma| (\nabla \tilde{p})_{\sigma,i}^n = 0. \quad (37b)$$

**Correction step** – Solve for  $\rho^{n+1}$ ,  $p^{n+1}$ ,  $e^{n+1}$  and  $\mathbf{u}^{n+1}$ :

For  $1 \leq i \leq d$ ,  $\forall \sigma \in \mathcal{E}_S^{(i)}$ ,

$$\frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}) + |D_\sigma| [(\nabla p)_{\sigma,i}^{n+1} - (\nabla \tilde{p})_{\sigma,i}^n] = 0, \quad (37c)$$

$$\forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} = 0, \quad (37d)$$

$$\forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} e_\sigma^{n+1} + |K| p_K^{n+1} (\operatorname{div} \mathbf{u})_K^{n+1} = S_K^{n+1}, \quad (37e)$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\gamma - 1) \rho_K^{n+1} e_K^{n+1}. \quad (37f)$$

The initialization of the scheme is performed as for the previous pressure correction scheme.

## 5.2. The discrete kinetic energy balance equation and the corrective source terms

The following discrete kinetic energy balance is derived in [14] (Lemma 4.1).

**Lemma 5.1** (Discrete kinetic energy balance, pressure correction scheme 2).

A solution to the system (37) satisfies the following equality, for  $1 \leq i \leq d$ ,  $\sigma \in \mathcal{E}_S^{(i)}$  and  $0 \leq n \leq N - 1$ :

$$\begin{aligned} \frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[ \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^{n-1} (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\varepsilon = D_\sigma | D_{\sigma'}} F_{\sigma,\varepsilon}^n \tilde{u}_{\sigma,i}^{n+1} \tilde{u}_{\sigma',i}^{n+1} \\ + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1} = -R_{\sigma,i}^{n+1} - P_{\sigma,i}^{n+1}, \end{aligned} \quad (38)$$

where the remainder terms  $R_{\sigma,i}^{n+1}$  and  $P_{\sigma,i}^{n+1}$  read:

$$\begin{aligned} R_{\sigma,i}^{n+1} &= \frac{1}{2} \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^{n-1} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 + \delta^{\text{up}} \left[ \sum_{\varepsilon = D_\sigma | D_{\sigma'}} \frac{1}{2} |F_{\sigma,\varepsilon}^n| (\tilde{u}_{\sigma,i}^{n+1} - \tilde{u}_{\sigma',i}^{n+1}) \right] \tilde{u}_{\sigma,i}^{n+1}, \\ P_{\sigma,i}^{n+1} &= \frac{|D_\sigma| \delta t}{\rho_{D_\sigma}^n} \left[ |(\nabla p)_{\sigma,i}^{n+1}|^2 - |(\nabla \tilde{p})_{\sigma,i}^{n+1}|^2 \right]. \end{aligned}$$

For further use, we define the quantity  $P_\sigma^{n+1}$ , for  $0 \leq n \leq N - 1$  and  $\sigma \in \mathcal{E}_{\text{int}}$ , as:

$$P_\sigma^{n+1} = \frac{|D_\sigma| \delta t}{\rho_{D_\sigma}^n} \left[ |(\nabla p)_\sigma^{n+1}|^2 - |(\nabla \tilde{p})_\sigma^{n+1}|^2 \right] = \sum_{i=1}^d P_{\sigma,i}^{n+1}. \quad (39)$$

In order to compensate the residual terms  $R_{\sigma,i}^{n+1}$  of the discrete kinetic energy balance, we now choose the corrective term  $S_K^{n+1}$  in the internal energy balance (31d) as:

$$\begin{aligned} \forall K \in \mathcal{M}, \quad S_K^{n+1} &= \sum_{i=1}^d S_{K,i}^{n+1} + \sum_{\sigma \in \mathcal{E}(K)} Q_\sigma^{n+1} \text{ with} \\ S_{K,i}^{n+1} &= \frac{1}{2} \rho_K^{n-1} \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_S^{(i)}} \frac{|D_{K,\sigma}|}{\delta t} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 + \delta^{\text{up}} \sum_{\substack{\varepsilon \in \bar{\mathcal{E}}_S^{(i)}, \varepsilon \cap \bar{K} \neq \emptyset, \\ \varepsilon = D_\sigma | D_{\sigma'}}} \alpha_{K,\varepsilon} \frac{|F_{\sigma,\varepsilon}^n|}{2} (\tilde{u}_{\sigma,i}^{n+1} - \tilde{u}_{\sigma',i}^{n+1})^2, \quad (40) \\ Q_\sigma^{n+1} &= \delta t |D_\sigma| \left| \frac{1}{(\rho_{D_\sigma}^n)^{1/2}} (\nabla \tilde{p})_\sigma^{n+1} - \frac{1}{(\rho_{D_\sigma}^{n-1})^{1/2}} (\nabla p)_\sigma^n \right|^2. \end{aligned}$$

The terms  $(S_{K,i}^{n+1})_{1 \leq i \leq d}$  are obtained by the same construction as in Sections 3.2 and 4.2, with the same definition of the coefficients  $\alpha_{K,\varepsilon}$ , i.e.  $\alpha_{K,\varepsilon} = 1$  for the Rannacher-Turek or Crouzeix-Raviart element and  $\alpha_{K,\varepsilon}$  given by (13) for the MAC scheme. The last one compensates a remainder term due to the particular time discretization of the pressure used in this variant of the scheme, as stated in Lemma 5.2 below.

We emphasize that, contrary to the first prediction–correction algorithm, the term  $S_K^{n+1}$  is now non-negative, which implies that the internal energy remains non-negative at all time. As a consequence, we readily obtain

the same existence and stability results as for the implicit scheme, again adapting the proof of [13, Theorem 4.1].

**Lemma 5.2.** *Let  $\Omega$  be an open bounded interval of  $\mathbb{R}$ . Let  $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m \in \mathbb{N}}$  be a sequence of meshes and time steps, such that  $h^{(m)}$  and  $\delta t^{(m)}$  tend to zero as  $m$  tends to infinity, and satisfying the CFL-like condition:*

$$\forall m \in \mathbb{N}, \quad \frac{\delta t^{(m)}}{h^{(m)}} \leq C, \quad \text{with } \underline{h}^{(m)} = \min_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} \frac{|D_\sigma|}{|\sigma|},$$

and where  $C$  is a positive real number possibly greater than 1. Let  $(\rho^{(m)})_{m \in \mathbb{N}}$  and  $(p^{(m)})_{m \in \mathbb{N}}$  be (part of) the associated sequence of discrete solutions. We assume that the sequence  $(p^{(m)})_{m \in \mathbb{N}}$  is uniformly bounded in  $L^\infty(\Omega \times (0, T))$  and in the discrete  $L^1(0, T; \text{BV}(\Omega))$  norm:

$$\forall m \in \mathbb{N}, \quad \|p^{(m)}\|_{\mathcal{T}, x, \text{BV}} = \sum_{n=0}^{N^{(m)}} \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}^{(m)}} |\sigma| |(p^{(m)})_L^n - (p^{(m)})_K^n| \leq C. \quad (41)$$

We furthermore suppose that  $(\rho^{(m)})_{m \in \mathbb{N}}$  and  $(1/\rho^{(m)})_{m \in \mathbb{N}}$  are bounded in  $L^\infty(\Omega \times (0, T))$ . Let  $\varphi \in C_c^\infty(\Omega \times [0, T])$ , and, for  $m \in \mathbb{N}$ ,  $0 \leq n \leq N^{(m)}$  and  $\sigma \in \mathcal{E}_{\text{int}}^{(m)}$ , let  $\varphi_\sigma^n = \varphi(\mathbf{x}_\sigma, t^n)$ , with  $\mathbf{x}_\sigma$  the mass center of  $\sigma$ . We then have:

$$\lim_{m \rightarrow +\infty} \left[ \sum_{n=0}^{n-1} \delta t \sum_{\sigma \in \mathcal{E}_{\text{int}}} (P_\sigma^{n+1} - Q_\sigma^{n+1}) \varphi_\sigma^n \right] = 0,$$

where  $P_\sigma^{n+1}$  and  $Q_\sigma^{n+1}$  are defined by Equations (39) and (40) respectively.

*Proof.* We first give the main steps of this bound in the semi-discrete setting. In this formalism, the piecewise constant functions  $P^{n+1}$  and  $Q^{n+1}$ , defined by (39) and (40) respectively, read:

$$P^{n+1} = \frac{\delta t}{\rho^n} [|\nabla p^{n+1}|^2 - |\nabla \tilde{p}^{n+1}|^2], \quad Q^{n+1} = \delta t \left| \frac{1}{\rho^n} \nabla \tilde{p}^{n+1} - \frac{1}{\rho^{n-1}} \nabla p^n \right|^2.$$

The pressure prediction step (37c) reads, at step  $n+1$ :

$$\text{div} \left( \frac{1}{\rho^n} \nabla \tilde{p}^{n+1} \right) = \text{div} \left( \frac{1}{(\rho^{n-1} \rho^n)^{1/2}} \nabla p^n \right).$$

Let  $\varphi^n$  stand for an interpolate of  $\varphi(\cdot, t^n)$ . Multiplying by  $2\delta t \tilde{p}^{n+1} \varphi^n$  and integrating over space and time, we get:

$$2 \sum_{n=0}^{N-1} \delta t^2 \int_{\Omega} \left[ \frac{1}{(\rho^n)^{1/2}} \nabla \tilde{p}^{n+1} - \frac{1}{(\rho^{n-1})^{1/2}} \nabla p^n \right] \cdot \frac{1}{(\rho^n)^{1/2}} \nabla (\tilde{p}^{n+1} \varphi^n) = 0.$$

Developing the last gradient term and using the identity  $2(a-b, a) = (a, a) + (a-b, a-b) - (b, b)$ , valid for any inner product  $(\cdot, \cdot)$ , we get  $\mathcal{P}_{\text{ren}} + \mathcal{R} + \mathcal{Q} = 0$  with:

$$\begin{aligned} \mathcal{P}_{\text{ren}} &= \sum_{n=0}^{N-1} \delta t^2 \int_{\Omega} \left[ \frac{1}{\rho^n} |\nabla \tilde{p}^{n+1}|^2 - \frac{1}{\rho^{n-1}} |\nabla p^n|^2 \right] \varphi^n \, d\mathbf{x}, \\ \mathcal{R} &= 2 \sum_{n=0}^{N-1} \delta t^2 \int_{\Omega} \left[ \frac{1}{\rho^n} \nabla \tilde{p}^{n+1} - \frac{1}{(\rho^{n-1} \rho^n)^{1/2}} \nabla p^n \right] \cdot \nabla \varphi^n \tilde{p}^{n+1}, \\ \mathcal{Q} &= \sum_{n=0}^{N-1} \delta t^2 \int_{\Omega} \left| \frac{1}{\rho^n} \nabla \tilde{p}^{n+1} - \frac{1}{\rho^{n-1}} \nabla p^n \right|^2 \varphi^n \, d\mathbf{x} = \sum_{n=0}^{N-1} \delta t \int_{\Omega} Q^{n+1} \varphi^n \, d\mathbf{x}. \end{aligned}$$

The term  $\mathcal{R}$  is bounded as follows:

$$|\mathcal{R}| \leq C_\varphi \left\| \frac{1}{\rho} \right\|_{L^\infty(\Omega \times (0, T))} \|p\|_{L^\infty(\Omega \times (0, T))} \|\nabla p\|_{L^1(\Omega \times (0, T))} \delta t,$$

and thus tends to zero as  $\delta t$  (with, at the discrete level, the  $L^1$  norm of the pressure gradient replaced by its BV norm). Thus we also have:

$$\mathcal{P}_{\text{ren}} + \mathcal{Q} \rightarrow 0 \text{ as } \delta t \rightarrow 0. \quad (42)$$

On the other hand, multiplying  $P^{n+1}$  by  $\varphi^n$ , and integrating with respect to space and time, we get:

$$\mathcal{P} = \sum_{n=0}^{N-1} \delta t \int_{\Omega} P^{n+1} \varphi^n \, d\mathbf{x} = \sum_{n=0}^{N-1} \delta t^2 \int_{\Omega} \frac{1}{\rho^n} \left[ |\nabla p^{n+1}|^2 - |\nabla \tilde{p}^{n+1}|^2 \right] \varphi^n \, d\mathbf{x}.$$

From Equation (42), the proof will be finished if we show that  $\mathcal{P} + \mathcal{P}_{\text{ren}}$  tends to zero when  $m$  tends to infinity. We have:

$$\mathcal{P} + \mathcal{P}_{\text{ren}} = \sum_{n=0}^{N-1} \delta t^2 \int_{\Omega} \left[ \frac{1}{\rho^n} |\nabla p^{n+1}|^2 - \frac{1}{\rho^{n-1}} |\nabla p^n|^2 \right] \varphi^n \, d\mathbf{x}.$$

A discrete integration by parts with respect to the time yields:

$$\mathcal{P} + \mathcal{P}_{\text{ren}} = \delta t^2 \int_{\Omega} \frac{1}{\rho^{-1}} |\nabla p^0|^2 \varphi^0 \, d\mathbf{x} - \sum_{n=0}^{N-1} \delta t^2 \int_{\Omega} \frac{1}{\rho^n} |\nabla p^{n+1}|^2 (\varphi^{n+1} - \varphi^n) \, d\mathbf{x}.$$

Thanks to the regularity of  $\varphi$ , we get:

$$|\mathcal{P} + \mathcal{P}_{\text{ren}}| \leq C_{\varphi} \left\| \frac{1}{\rho} \right\|_{L^{\infty}(\Omega \times (0, T))} \delta t^2 \left[ \left| \int_{\Omega} |\nabla p^0|^2 \, d\mathbf{x} \right| + \sum_{n=0}^{N-1} \delta t \left| \int_{\Omega} |\nabla p^{n+1}|^2 \, d\mathbf{x} \right| \right],$$

and, in the fully discrete setting, we shall conclude that this term tends to zero by invoking the inverse inequality  $\|\nabla p\|_{L^{\infty}(\Omega \times (0, T))} \leq C \|p\|_{L^{\infty}(\Omega \times (0, T))} / \underline{h}$ , which yields:

$$|\mathcal{P} + \mathcal{P}_{\text{ren}}| \leq C_{\varphi} \left\| \frac{1}{\rho} \right\|_{L^{\infty}(\Omega \times (0, T))} \|p\|_{L^{\infty}(\Omega \times (0, T))} \left[ \|p_0\|_{\text{BV}(\Omega)} + \|p\|_{\mathcal{T}, x, \text{BV}} \right] \frac{\delta t^2}{\underline{h}}.$$

This is the estimate we are seeking.

Let us now undertake this program in the discrete setting. We multiply the pressure renormalization relation (31a) by  $\tilde{p}_K^{n+1} \varphi_K^n$ , where  $\varphi_K^n = \varphi(\mathbf{x}_K, t^n)$ , with  $\mathbf{x}_K$  the mass center of the cell  $K$ . We obtain:

$$\forall K \in \mathcal{M}, \quad \tilde{p}_K^{n+1} \varphi_K^n \sum_{\sigma=K|L} \frac{|\sigma|^2}{|D_{\sigma}|} \left[ \frac{1}{\rho_{D_{\sigma}}^n} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) - \frac{1}{(\rho_{D_{\sigma}}^{n-1} \rho_{D_{\sigma}}^n)^{1/2}} (p_K^n - p_L^n) \right] = 0.$$

Summing over the cells and reordering the sums, we get:

$$\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{|\sigma|^2}{|D_{\sigma}|} \left[ \frac{1}{\rho_{D_{\sigma}}^n} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) - \frac{1}{(\rho_{D_{\sigma}}^{n-1} \rho_{D_{\sigma}}^n)^{1/2}} (p_K^n - p_L^n) \right] [\tilde{p}_K^{n+1} \varphi_K^n - \tilde{p}_L^{n+1} \varphi_L^n] = 0.$$

Let us now split the difference  $\tilde{p}_K^{n+1} \varphi_K^n - \tilde{p}_L^{n+1} \varphi_L^n$  using the identity  $2(ab - cd) = (a - c)(b + d) + (a + c)(b - d)$ . Multiplying by  $\delta t$ , we get  $\mathcal{D}^{n+1} + \mathcal{R}_1^{n+1} + \mathcal{R}_2^{n+1} = 0$  with:

$$\begin{aligned} \mathcal{D}^{n+1} &= \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{|\sigma|^2}{|D_{\sigma}|} \left[ \frac{1}{(\rho_{D_{\sigma}}^n)^{1/2}} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) - \frac{1}{(\rho_{D_{\sigma}}^{n-1})^{1/2}} (p_K^n - p_L^n) \right] \\ &\quad \frac{1}{(\rho_{D_{\sigma}}^n)^{1/2}} [\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}] \varphi_{\sigma}^n, \\ \mathcal{R}_1^{n+1} &= \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{|\sigma|^2}{|D_{\sigma}|} \left[ \frac{1}{\rho_{D_{\sigma}}^n} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) - \frac{1}{(\rho_{D_{\sigma}}^{n-1} \rho_{D_{\sigma}}^n)^{1/2}} (p_K^n - p_L^n) \right] \tilde{p}_{\sigma}^{n+1} [\varphi_K^n - \varphi_L^n], \\ \mathcal{R}_2^{n+1} &= \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{|\sigma|^2}{|D_{\sigma}|} \left[ \frac{1}{\rho_{D_{\sigma}}^n} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) - \frac{1}{(\rho_{D_{\sigma}}^{n-1} \rho_{D_{\sigma}}^n)^{1/2}} (p_K^n - p_L^n) \right] \tilde{p}_{\sigma}^{n+1} [\tilde{\varphi}_{\sigma}^n - \varphi_{\sigma}^n]. \end{aligned}$$



with, for  $\sigma = K|L$ ,  $\tilde{\varphi}_\sigma^n = (\varphi_K^n + \varphi_L^n)/2$  and  $\tilde{p}_\sigma^{n+1} = (\tilde{p}_K^{n+1} + \tilde{p}_L^{n+1})/2$ . Thanks to the regularity of  $\varphi$ , we obtain:

$$\sum_{n=0}^{N-1} \delta t \left[ |\mathcal{R}_1^{n+1}| + |\mathcal{R}_2^{n+1}| \right] \leq C_\varphi \delta t \left\| \frac{1}{\rho} \right\|_{L^\infty(\Omega \times (0, T))} \|p\|_{L^\infty(\Omega \times (0, T))} \|p\|_{\mathcal{T}, x, \text{BV}},$$

and thus this quantity tends to zero when  $m$  tends to  $+\infty$ , which implies that so does  $\sum_{n=0}^{N-1} \delta t \mathcal{D}^{n+1}$ .

Using the identity  $2(a-b)a = a^2 + (a-b)^2 - b^2$ , we get  $\mathcal{D}^{n+1} = \mathcal{P}_{\text{ren}}^{n+1} + \mathcal{Q}^{n+1}$ , where

$$P_{\text{ren}}^{n+1} = \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{|\sigma|^2}{|D_\sigma|} \left[ \frac{1}{\rho_{D_\sigma}^n} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1})^2 - \frac{1}{\rho_{D_\sigma}^{n-1}} (p_K^n - p_L^n)^2 \right] \varphi_\sigma^n,$$

and

$$Q^{n+1} = \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{|\sigma|^2}{|D_\sigma|} \left[ \frac{1}{\rho_{D_\sigma}^n} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) - \frac{1}{\rho_{D_\sigma}^{n-1}} (p_K^n - p_L^n) \right]^2 \varphi_\sigma^n = \sum_{\sigma \in \mathcal{E}_{\text{int}}} Q_\sigma^{n+1} \varphi_\sigma^n.$$

To conclude the proof, it thus remains to show that, summing  $\mathcal{P}_{\text{ren}}^{n+1}$  and  $\mathcal{P}^{n+1} = \sum_{\sigma \in \mathcal{E}_{\text{int}}} P_\sigma^{n+1} \varphi_\sigma^n$ , we obtain a discrete time function the integral of which over  $(0, T)$  tends to zero. Let us recall the expression of  $P_\sigma^{n+1}$ :

$$P_\sigma^{n+1} = \frac{\delta t D_\sigma}{\rho_{D_\sigma}^n} \left[ ((\nabla p)_\sigma^{n+1})^2 - ((\nabla \tilde{p})_\sigma^{n+1})^2 \right] = \frac{\delta t}{\rho_{D_\sigma}^n} \frac{|\sigma|^2}{|D_\sigma|} \left[ (p_K^{n+1} - p_L^{n+1})^2 - (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1})^2 \right].$$

Summing  $\mathcal{P}^{n+1}$  with  $\mathcal{P}_{\text{ren}}^{n+1}$  and integrating with respect to the time, we obtain:

$$\sum_{n=0}^{N-1} \delta t [\mathcal{P}^{n+1} + \mathcal{P}_{\text{ren}}^{n+1}] = \sum_{n=0}^{N-1} \delta t^2 \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{|\sigma|^2}{|D_\sigma|} \left[ \frac{1}{\rho_{D_\sigma}^n} (p_K^{n+1} - p_L^{n+1})^2 - \frac{1}{\rho_{D_\sigma}^{n-1}} (p_K^n - p_L^n)^2 \right] \varphi_\sigma^n.$$

By a discrete integration by parts, we get:

$$\left| \sum_{n=0}^{N-1} \delta t [\mathcal{P}^{n+1} + \mathcal{P}_{\text{ren}}^{n+1}] \right| \leq \delta t^2 \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{1}{\rho_{D_\sigma}^{n-1}} \frac{|\sigma|^2}{|D_\sigma|} |p_K^0 - p_L^0|^2 \varphi_\sigma^0 + \sum_{n=0}^{N-1} \delta t^2 \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{1}{\rho_{D_\sigma}^n} \frac{|\sigma|^2}{|D_\sigma|} (p_K^{n+1} - p_L^{n+1})^2 |\varphi_\sigma^{n+1} - \varphi_\sigma^n|.$$

Using the fact that, for  $0 \leq n \leq N$  and  $\sigma = K|L \in \mathcal{E}_{\text{int}}$ ,  $(p_K^n - p_L^n)^2 \leq \|p\|_{L^\infty(\Omega \times (0, T))} |p_K^n - p_L^n|$ , we get, thanks to the regularity of  $\varphi$ , the desired estimate:

$$\left| \sum_{n=0}^{N-1} \delta t [\mathcal{P}^{n+1} + \mathcal{P}_{\text{ren}}^{n+1}] \right| \leq C_\varphi \frac{\delta t^2}{\underline{h}} \|p\|_{L^\infty(\Omega \times (0, T))} \left[ \|p^0\|_{\text{BV}(\Omega)} + \|p\|_{\mathcal{T}, x, \text{BV}} \right].$$

□

### 5.3. Passing to the limit in the scheme (1D case)

As for the other schemes, we now undertake the passage to the limit in the scheme (37) in the one-dimensional case. Using the notations (16)–(17), the scheme may be rewritten as follows:

**Initialization** – Compute  $\rho^{-1}$ ,  $e^0$ ,  $u^0$ ,  $\rho^0$  and  $p^0$  (35a).

**Pressure prediction step** – Solve for  $\tilde{p}^{n+1}$ :

$$\forall \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad \frac{1}{\rho_{D_\sigma}^n} \frac{|\sigma|^2}{|D_\sigma|} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) = \frac{1}{(\rho_{D_\sigma}^n \rho_{D_\sigma}^{n-1})^{1/2}} \frac{|\sigma|^2}{|D_\sigma|} (p_K^n - p_L^n). \quad (43a)$$

**Prediction step** – Solve for  $\tilde{u}^{n+1}$ :

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \quad \frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^n \tilde{u}_\sigma^{n+1} - \rho_{D_\sigma}^{n-1} u_\sigma^n) + F_L^n \tilde{u}_L^{n+1} - F_K^n \tilde{u}_K^{n+1} + \tilde{p}_L^{n+1} - \tilde{p}_K^{n+1} = 0. \quad (43b)$$

**Correction step** – Solve for  $\rho^{n+1}$ ,  $p^{n+1}$ ,  $e^{n+1}$  and  $u^{n+1}$ :

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \quad \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_\sigma^{n+1} - \tilde{u}_\sigma^{n+1}) + (p_L^{n+1} - p_K^{n+1}) - (\tilde{p}_L^{n+1} - \tilde{p}_K^{n+1}) = 0, \quad (43c)$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^{n+1} - F_\sigma^{n+1} = 0, \quad (43d)$$

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + F_{\sigma'}^{n+1} e_{\sigma'}^{n+1} - F_\sigma^{n+1} e_\sigma^{n+1} + p_K^{n+1} (u_{\sigma'}^{n+1} - u_\sigma^{n+1}) = S_K^{n+1}, \quad (43e)$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\gamma - 1) \rho_K^{n+1} e_K^{n+1}. \quad (43f)$$

Note that, in 1D, the pressure prediction step takes a simpler form than in the multi-dimensional case. To obtain (43a), we first remark that the (implicit) boundary conditions associated to the pressure elliptic operator are homogeneous Neumann boundary conditions; thus, equation (37b) written at one of the two boundary cells yields the relation (43a) for the inner face of this cell. We then proceed by induction on the faces of the mesh, which, roughly speaking, amounts to use the fact that, in one dimension, a divergence-free field is constant.

Consequently, the terms  $Q_\sigma^{n+1}$  vanish for  $\sigma \in \mathcal{E}_{\text{int}}$  and at all time, and the corrective term  $S_K^{n+1}$  reads:

$$\forall K = [\overrightarrow{\sigma\sigma'}], \quad S_K^{n+1} = \frac{|K|}{4\delta t} \rho_K^{n-1} [(\tilde{u}_\sigma^{n+1} - u_\sigma^n)^2 + (\tilde{u}_{\sigma'}^{n+1} - u_{\sigma'}^n)^2] + \delta^{\text{up}} \frac{|F_K^n|}{2} (\tilde{u}_\sigma^{n+1} - \tilde{u}_{\sigma'}^{n+1})^2.$$

For sequences of discretizations  $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m \in \mathbb{N}}$ , in the one-dimensional case, we introduce a CFL-like condition defined as follows:

$$\forall m \in \mathbb{N}, \quad \frac{\delta t^{(m)}}{\underline{h}^{(m)}} \leq C, \quad \text{with } \underline{h}^{(m)} = \min_{\sigma=K|L \in \mathcal{E}_{\text{int}}^{(m)}} \frac{1}{2} (h_K + h_L), \quad (44)$$

and  $C$  is a real number (which may take any positive value, and, in particular, is not necessarily lower than 1). With this definition, we are in position to state the following consistency result.

**Theorem 5.3.** *Let  $\Omega$  be an open bounded interval of  $\mathbb{R}$ . Let  $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m \in \mathbb{N}}$  be a sequence of discretizations such that both the time step  $\delta t^{(m)}$  and the space step  $h^{(m)}$  tend to zero as  $m \rightarrow \infty$  and satisfying the CFL-like condition (44). Let  $(\rho^{(m)}, p^{(m)}, e^{(m)}, \tilde{u}^{(m)}, u^{(m)})_{m \in \mathbb{N}}$  be the corresponding sequence of solutions. We suppose that this sequence satisfies the bounds (36) and converges in  $L^p(\Omega \times (0, T))^5$ , for  $1 \leq p < \infty$ , to  $(\bar{\rho}, \bar{p}, \bar{e}, \bar{\tilde{u}}, \bar{u}) \in L^\infty(\Omega \times (0, T))^5$ .*

*Then  $\bar{\tilde{u}} = \bar{u}$  and  $(\bar{\rho}, \bar{p}, \bar{e}, \bar{u})$  satisfies the system (29).*

*Proof.* Let us first check that  $\bar{\tilde{u}} = \bar{u}$ . We first note that thanks to assumptions (25) and (36b), we have  $|\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}| \leq C |p_K^{n+1} - p_L^{n+1}|$  and therefore the correction step on the velocity (43c) yields, again using assumption (36b):

$$\|u^{(m)} - \tilde{u}^{(m)}\|_{L^1(\Omega \times (0, T))} \leq C \delta t \|p^{(m)}\|_{\mathcal{T}, x, \text{BV}}$$

which, passing to the limit when  $m \rightarrow +\infty$ , yields the result.

Similarly to the proof of Theorem 4.2, we thus only need to prove that  $(\bar{\rho}, \bar{p}, \bar{e}, \bar{u})$  satisfies (29c). With the same notations as in the proof of Theorem 4.2, we first multiply the one dimensional version of the discrete kinetic energy equation (38) by  $\delta t \varphi_\sigma^n$  and sum over the faces and the time steps. Similarly, we multiply the discrete internal energy equation (37e) by  $\delta t \varphi_K^n$ , and sum over the primal cells and the time steps. Summing the two obtained relations, we get  $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} + T_5^{(m)} = R^{(m)}$ , where the terms  $T_i^{(m)}$  are identical to the terms  $T_i^{(m)}$  of the proof of Theorem 4.2, and

$$R^{(m)} = - \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}} (R_\sigma^{n+1} - P_\sigma^{n+1}) \varphi_\sigma^n + \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} S_K^{n+1} \varphi_K^n.$$

We already know that the terms  $T_i^{(m)}$  converge to what they should. The convergence of the term  $R^{(m)}$  to 0 is a straightforward consequence of the choice (40) of the term  $S_K^{m+1}$  and of Lemma 5.2.  $\square$

## 6. NUMERICAL TESTS

In this section, we assess the behaviour of the scheme on a one dimensional Riemann problem. We choose initial conditions such that the structure of the solution consists in two shock waves, separated by the contact discontinuity, with sufficiently strong shocks to allow an easy discrimination of correct numerical solutions. These initial conditions are those proposed in [23, chapter 4], for the test referred to as Test 5:

$$\text{left state: } \begin{bmatrix} \rho_{\text{left}} \\ u_{\text{left}} \\ p_{\text{left}} \end{bmatrix} = \begin{bmatrix} 5.99924 \\ 19.5975 \\ 460.894 \end{bmatrix} \quad \text{right state: } \begin{bmatrix} \rho_{\text{right}} \\ u_{\text{right}} \\ p_{\text{right}} \end{bmatrix} = \begin{bmatrix} 5.99242 \\ -6.19633 \\ 46.0950 \end{bmatrix}$$

The problem is posed over  $\Omega = (-0.5, 0.5)$ , and the discontinuity is initially located at  $x = 0$ .

At the boundaries, since, in this test, the flow is entering the domain, the solution is prescribed (which, in fact, is unimportant, the solution being constant at any time in a sufficiently large neighbourhood of these boundaries). Previous numerical experiments addressing barotropic flows [12] showed that, at least for one dimensional computations with schemes similar to the one under study here, it was not necessary to use upwinding in the momentum balance equation; consequently, we only employ a centered approximation of the velocity at the dual faces.

The computations are performed with the open-source software ISIS [17], developed at IRSN on the basis of the software component library and programming environment PELICANS [20].

On this test, results obtained with the two variants of the pressure correction scheme are almost the same, and we only report here the results associated to the first one. The density fields obtained with  $h = 1/2000$  (or a number of cells  $n = 2000$ ) at  $t = 0.035$ , with and without assembling the corrective source term in the internal energy balance  $(S_K)_{K \in \mathcal{M}}$ , together with the analytical solution, are shown on Figure 2. The density and the pressure obtained, still with and without corrective terms, for various meshes, are plotted on Figures 3 and 4 respectively. For these computations, we take  $\delta t = h/20$ , which yields a cfl number, with respect to the material velocity only, close to one. The first conclusion is that both schemes seem to converge, but the corrective term is necessary to obtain the correct solution. In this case, for instance, we obtain the correct intermediate state for the pressure and velocity up to four digits in the essential part of the corresponding zone:

$$\begin{aligned} \text{(analytical) intermediate state:} & \quad \begin{bmatrix} p^* \\ u^* \end{bmatrix} = \begin{bmatrix} 1691.65 \\ 8.68977 \end{bmatrix} \quad \text{for } x \in (0.028, 0.428) \\ \text{numerical results:} & \quad \begin{cases} p \in (1691.6, 1691.8) \\ u \in (8.689, 8.690) \end{cases} \quad \text{for } x \in (0.032, 0.417) \end{aligned}$$

Without a corrective term, one can check that the obtained solution is not a weak solution to the Euler system: indeed, the Rankine-Hugoniot condition applied to the total energy balance, with the states obtained numerically, yields a right shock velocity slightly greater than the analytical solution one, while the same shock velocity obtained numerically is clearly lower.

We also observe that the scheme is rather diffusive especially for contact discontinuities for which the beneficial compressive effect of the shocks does not apply. More accurate variants may certainly be derived, using for instance MUSCL-like techniques; this work is underway.

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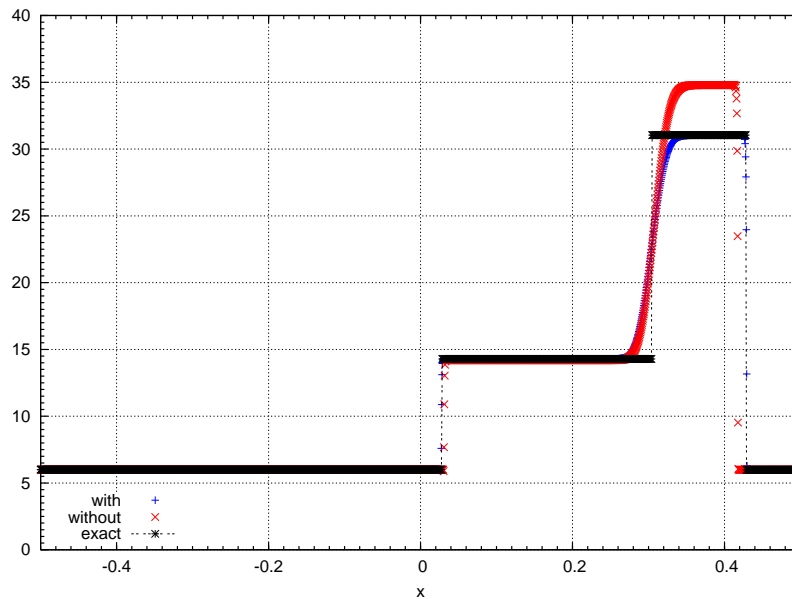


FIGURE 2. Test 5 of [23, chapter 4] - Density obtained with  $n = 2000$  cells, with and without corrective source terms, and analytical solution.

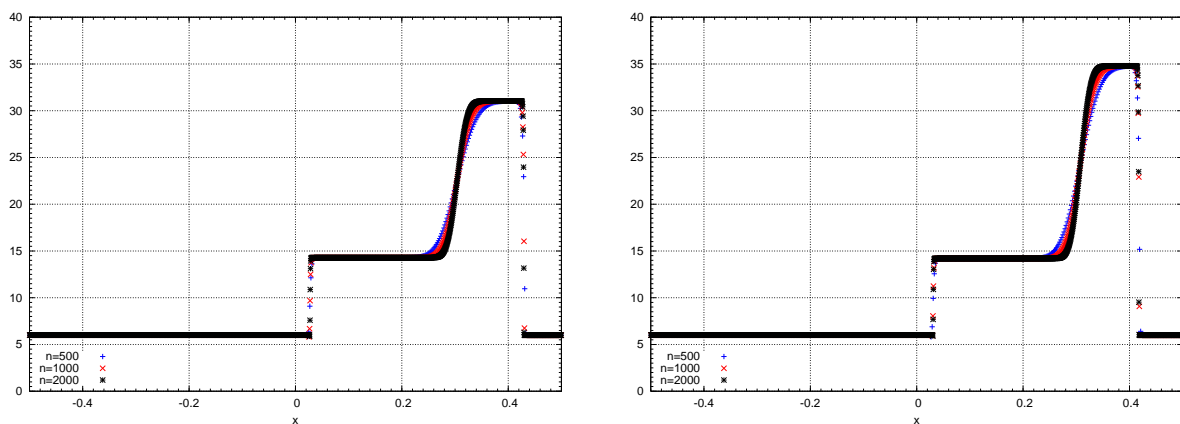


FIGURE 3. Test 5 of [23, chapter 4] - Density obtained with various meshes, with (left) and without (right) corrective source terms.

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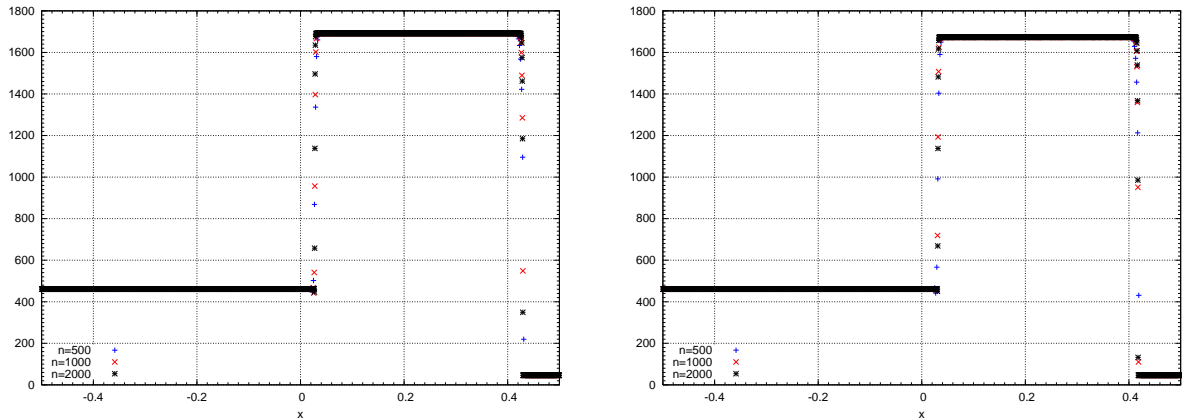


FIGURE 4. Test 5 of [23, chapter 4] - pressure obtained with various meshes, with (left) and without (right) corrective source terms.

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