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► **To cite this version:**

Raphaele Herbin, Walid Kheriji, Jean-Claude Latché. On some implicit and semi-implicit staggered schemes for the shallow water and Euler equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 2014, 48 (6), pp.1807-1857. hal-00805510v1

HAL Id: hal-00805510

<https://hal.science/hal-00805510v1>

Submitted on 28 Mar 2013 (v1), last revised 5 Mar 2016 (v2)

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CONSISTENT SEMI-IMPLICIT STAGGERED SCHEMES FOR COMPRESSIBLE FLOWS

PART I: THE BAROTROPIC EULER EQUATIONS.

R. HERBIN¹, W. KHERIJI² AND J.-C. LATCHÉ³

Abstract. In this paper, we analyze the stability and consistency of a time-implicit scheme and a pressure correction scheme, based on staggered space discretizations, for the compressible barotropic Euler equations. We first show that the solutions to these schemes satisfy a discrete kinetic energy and a discrete elastic potential balance equations. Integrating these equations over the domain readily yields discrete counterparts of the stability estimates known for the continuous problem. Then, in one space dimension, we prove that if the solutions to these schemes converge to some limit, then this limit is an entropy weak solution of the continuous problem.

2010 AMS Subject Classification. 35Q31,65N12,76M10,76M12.

September 2012.

1. INTRODUCTION

The objective pursued in this work is to develop and analyze a class of efficient numerical schemes for the simulation of compressible flows at all Mach number regimes. To this purpose, our basic choice is to extend algorithms that are classical in the incompressible framework, namely pressure correction schemes based on (*inf-sup* stable) staggered discretizations.

The possibility to build, thanks to a fractional step strategy involving an elliptic pressure correction step, algorithms which are not limited by stringent stability conditions (such as CFL conditions based on the celerity of the fastest waves) has been recognized since the first attempts to build "all flow velocity" schemes in the late sixties [17] or in the early seventies [18]; these algorithms may be seen as an extension to the compressible case of the celebrated MAC scheme, introduced some years before [19]. These seminal papers have been the starting point for the development of numerous schemes, using staggered finite volume space discretizations [4, 5, 25, 26, 29, 32, 35, 47–52, 54], collocated finite volumes [2, 9, 23, 24, 27, 28, 31, 33, 36–39, 41, 42, 44, 46, 53] or finite elements [3, 34, 40, 55]. Algorithms proposed in these works may be essentially implicit-in-time, and the pressure correction step is then an ingredient of a SIMPLE-like iterative procedure, or only semi-implicit, with a single (or a limited number of) prediction and correction step(s), as in projection methods for incompressible flows (see [6, 45] for seminal works and [14] for a review of most of the variants).

For practical applications, the schemes that we develop and study here are of non-iterative pressure correction type; however, we also deal with the fully implicit scheme for its (relative) simplicity of analysis. The exposition is split in two companion papers: in a first step (the present part), we deal with the barotropic Euler equations; we further extend the study in [20] to the Euler equations. This splitting is however somewhat artificial, since, in particular, the entropy balance inequality for barotropic flows coincides with the total energy equation of the non-barotropic model; hence, proving the consistency in both cases is more or less the same task. Consequently, we proceed as follows: results concerning the entropy balance are stated here with a brief sketch of proof, and we refer to [20] for complete proofs. Note also that an extension of the proposed algorithms to the compressible Navier-Stokes equations is described in [21].

Keywords and phrases: finite volumes, finite elements, staggered, pressure correction, Euler equations, compressible flows, analysis.

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We address in this paper the following system (the so-called barotropic Euler equations):

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \quad (1a)$$

$$\partial_t (\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \quad (1b)$$

$$p = \wp(\rho) = \rho^\gamma. \quad (1c)$$

This problem is posed over an open bounded connected subset Ω of \mathbb{R}^d , $1 \leq d \leq 3$, with boundary $\partial\Omega$, and a finite time interval $(0, T)$. The variable t stands for the time, ρ , $\mathbf{u} = (u_1, \dots, u_d)$ and p are the density, velocity and pressure in the flow. The three above equations are respectively the mass balance, the momentum balance and the equation of state of the fluid, which is supposed to take the form $\wp(s) = s^\gamma$, where $\gamma \geq 1$ is a coefficient which is specific to the considered fluid. This system must be supplemented by initial conditions ρ_0 and \mathbf{u}_0 , and we assume $\rho_0 > 0$. It must also be supplemented by a suitable boundary condition, which we suppose to be:

$$\mathbf{u} \cdot \mathbf{n} = 0,$$

at any time and *a.e.* on $\partial\Omega$, where \mathbf{n} stands for the normal vector to the boundary.

We study two schemes for the numerical solution of System (1) which differ by the time discretization: the first one is implicit, and the second one is a non-iterative pressure-correction scheme introduced in [11]. This latter algorithm (and, by an easy extension, also the first one) was shown in [11] to have at least one solution, to provide solutions satisfying $\rho > 0$ (and therefore also $p > 0$) and to be unconditionally stable, in the sense that its (their) solution(s) satisfies an inequality corresponding to the control in $L^\infty(0, T)$ of the integral of the discrete entropy over the domain. In this paper, we complement this work in several directions. For the implicit scheme, we obtain the following results.

- First we pass from a (discrete) global (*i.e.* integrated over Ω) entropy balance to (discrete) local balance equations. Precisely speaking, a discrete kinetic energy balance is established on dual cells, while a discrete potential elastic balance is established on primal cells. These equations yield the stability of the scheme (*i.e.* the same global entropy conservation as in [11]) by a simple integration in space (*i.e.* summation over the primal and dual control volumes).
- Second, in one space dimension, the limit of any convergent sequence of solutions to the scheme is shown to be a weak solution to the continuous problem, and thus to satisfy the Rankine-Hugoniot conditions.
- Finally, passing to the limit in the discrete kinetic energy and elastic potential balances, such a limit is also shown to satisfy the usual entropy inequality.

The numerical study of this scheme (which is the only one implemented in practice) is performed in [30]. It confornts the present theoretical study: in particular, the scheme is observed to converge to weak entropy solutions of Riemann problems, with an approximately first order rate; in addition, it yields qualitatively correct solutions for CFL numbers much larger than one.

This paper is organized as follows. We first introduce the considered space discretizations (Section 2). We then study the implicit and pressure correction schemes (Sections 3 and 4 respectively). In several theoretical developments, we are lead to use a derived form of a discrete finite volume convection operator (for instance, typically, a convection operator for the kinetic energy, possibly with residual terms, obtained from the finite volume discretization of the convection of the velocity components); the proofs of various related discrete identities are given in the Appendix.

2. MESHES AND UNKNOWNNS

Let \mathcal{M} be a decomposition of the domain Ω , supposed to be regular in the usual sense of the finite element literature (*e.g.* [7]). The cells may be:

- for a general domain Ω , either convex quadrilaterals ($d = 2$) or hexahedra ($d = 3$) or simplices, both types of cells being possibly combined in a same mesh,
- for a domain with boundaries that are normal hyperplanes to a coordinate axis, rectangles ($d = 2$) or rectangular parallelepipeds ($d = 3$) (the faces of which, of course, are then also necessarily normal to a coordinate axis).

By \mathcal{E} and $\mathcal{E}(K)$ we denote the set of all $(d-1)$ -faces σ of the mesh and of the element $K \in \mathcal{M}$ respectively. The set of faces included in Ω (resp. in the boundary $\partial\Omega$) is denoted by \mathcal{E}_{int} (resp. \mathcal{E}_{ext}), so that (*i.e.* $\mathcal{E}_{\text{int}} = \mathcal{E} \setminus \mathcal{E}_{\text{ext}}$) is denoted by \mathcal{E}_{int} ; a face $\sigma \in \mathcal{E}_{\text{int}}$ separating the cells K and L is denoted by $\sigma = K|L$. The outward normal

vector to a face σ of K is denoted by $\mathbf{n}_{K,\sigma}$. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by $|K|$ the measure of K and by $|\sigma|$ the $(d-1)$ -measure of the face σ . For $1 \leq i \leq d$, we denote by $\mathcal{E}^{(i)} \subset \mathcal{E}$ and $\mathcal{E}_{\text{ext}}^{(i)} \subset \mathcal{E}_{\text{ext}}$ the subset of the faces of \mathcal{E} and \mathcal{E}_{ext} respectively which are perpendicular to the i^{th} unit vector of the canonical basis of \mathbb{R}^d .

The space discretization is staggered, using either the Marker-And Cell (MAC) scheme [18, 19], or non-conforming low-order finite element approximations, namely the Rannacher and Turek element (RT) [43] for quadrilateral or hexahedric meshes, or the lowest degree Crouzeix-Raviart (CR) element [8] for simplicial meshes.

For all these space discretizations, the degrees of freedom for the pressure and the density (*i.e.* the discrete pressure and density unknowns) are associated to the cells of the mesh \mathcal{M} , and are denoted by:

$$\{p_K, \rho_K, K \in \mathcal{M}\}.$$

Let us then turn to the degrees of freedom for the velocity (*i.e.* the discrete velocity unknowns).

- **Rannacher-Turek** or **Crouzeix-Raviart** discretizations – The degrees of freedom for the velocity components are located at the center of the faces of the mesh, and we choose the version of the element where they represent the average of the velocity through a face. The set of degrees of freedom reads:

$$\{u_{\sigma,i}, \sigma \in \mathcal{E}, 1 \leq i \leq d\}.$$

- **MAC** discretization – The degrees of freedom for the i^{th} component of the velocity are defined at the centre of the faces $\sigma \in \mathcal{E}^{(i)}$, so that the whole set of discrete velocity unknowns reads:

$$\{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}.$$

We now introduce a dual mesh, which is used for the finite volume approximation of the time derivative and convection terms in the momentum balance equation.

- **Rannacher-Turek** or **Crouzeix-Raviart** discretizations – For the RT or CR discretizations, the dual mesh is the same for all the velocity components. When $K \in \mathcal{M}$ is a simplex, a rectangle or a cuboid, for $\sigma \in \mathcal{E}(K)$, we define $D_{K,\sigma}$ as the cone with basis σ and with vertex the mass center of K (see Figure 1). We thus obtain a partition of K in m sub-volumes, where m is the number of faces of the mesh, each sub-volume having the same measure $|D_{K,\sigma}| = |K|/m$. We extend this definition to general quadrangles and hexahedra, by supposing that we have built a partition still of equal-volume sub-cells, and with the same connectivities; note that this is of course always possible, but that such a volume $D_{K,\sigma}$ may be no longer a cone; indeed, if K is far from a parallelogram, it may not be possible to build a cone having σ as basis, the opposite vertex lying in K and a volume equal to $|K|/m$. The volume $D_{K,\sigma}$ is referred to as the half-diamond cell associated to K and σ .

For $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, we now define the diamond cell D_σ associated to σ by $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$; for an external face $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K)$, D_σ is just the same volume as $D_{K,\sigma}$.

- **MAC** discretization – For the MAC scheme, the dual mesh depends on the component of the velocity. For each component, the MAC dual mesh only differs from the RT or CR dual mesh by the choice of the half-diamond cell, which, for $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, is now the rectangle or rectangular parallelepiped of basis σ and of measure $|D_{K,\sigma}| = |K|/2$.

We denote by $|D_\sigma|$ the measure of the dual cell D_σ , and by $\epsilon = D_\sigma|D_{\sigma'}$ the face separating two diamond cells D_σ and $D_{\sigma'}$.

Finally, we need to deal with the impermeability (*i.e.* $\mathbf{u} \cdot \mathbf{n} = 0$) boundary condition. Since the velocity unknowns lie on the boundary (and not inside the cells), these conditions are taken into account in the definition of the discrete spaces. To avoid technicalities in the expression of the schemes, we suppose throughout this paper that the boundary is *a.e.* normal to a coordinate axis, (even in the case of the RT or CR discretizations), which allows to simply set to zero the corresponding velocity unknowns:

$$\text{for } i = 1, \dots, d, \forall \sigma \in \mathcal{E}_{\text{ext}}^{(i)}, \quad u_{\sigma,i} = 0. \quad (2)$$

Therefore, there are no degrees of freedom for the velocity on the boundary for the MAC scheme, and there are only $d-1$ degrees of freedom on each boundary face for the CR and RT discretizations, which depend on the orientation of the face. In order to be able to write a unique expression of the discrete equations for both MAC

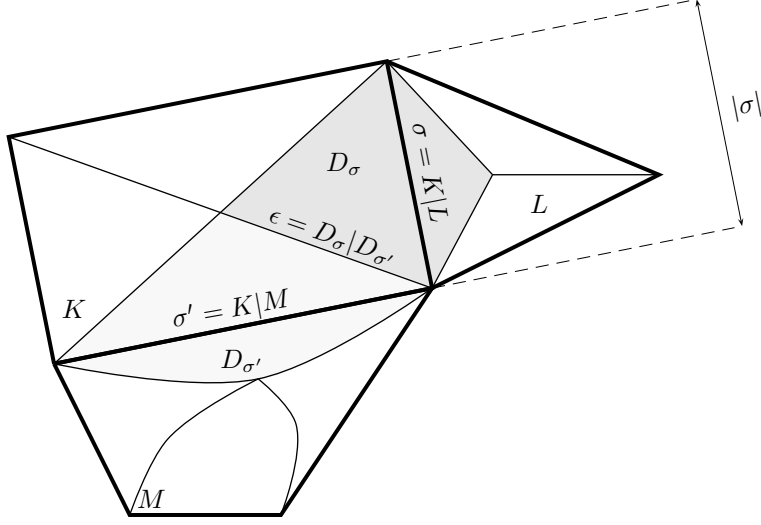


FIGURE 1. Primal and dual meshes for the Rannacher-Turek and Crouzeix-Raviart elements.

and CR/RT schemes, we introduce the set of faces $\mathcal{E}_S^{(i)}$ associated to the degrees of freedom of each component of the velocity (\mathcal{S} stands for “scheme”):

$$\mathcal{E}_S^{(i)} = \begin{cases} \mathcal{E}^{(i)} \setminus \mathcal{E}_{\text{ext}}^{(i)} & \text{for the MAC scheme,} \\ \mathcal{E} \setminus \mathcal{E}_{\text{ext}}^{(i)} & \text{for the CR or RT schemes.} \end{cases}$$

Similarly, we unify the notation for the set of dual faces for both schemes by defining:

$$\tilde{\mathcal{E}}_S^{(i)} = \begin{cases} \tilde{\mathcal{E}}^{(i)} \setminus \tilde{\mathcal{E}}_{\text{ext}}^{(i)} & \text{for the MAC scheme,} \\ \tilde{\mathcal{E}} \setminus \tilde{\mathcal{E}}_{\text{ext}}^{(i)} & \text{for the CR or RT schemes,} \end{cases}$$

where the symbol $\tilde{\cdot}$ refers to the dual mesh; for instance, $\tilde{\mathcal{E}}^{(i)}$ is thus the set of faces of the dual mesh associated to the i^{th} component of the velocity, and $\tilde{\mathcal{E}}_{\text{ext}}^{(i)}$ stands for the subset of these dual faces included in the boundary. Note that, for the MAC scheme, the faces of $\tilde{\mathcal{E}}^{(i)}$ are perpendicular to a unit vector of the canonical basis of \mathbb{R}^d , but not necessarily to the i^{th} one.

Note that general domains can easily be addressed (of course, with the CR or RT discretizations) by redefining, through linear combinations, the degrees of freedom at the external faces, so as to introduce the normal velocity as a new degree of freedom.

3. AN IMPLICIT SCHEME

3.1. The scheme

Let us consider a uniform partition $0 = t_0 < t_1 < \dots < t_N = T$ of the time interval $(0, T)$, and let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \dots, N-1$ be the constant time step. We consider an implicit-in-time scheme, which reads in its fully discrete form, for $0 \leq n \leq N-1$:

$$\forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} = 0, \quad (3a)$$

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}_S^{(i)}$,

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^n u_{\sigma,i}^n) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^{n+1} u_{\epsilon,i}^{n+1} - |D_\sigma| (\Delta^{\mathcal{M}} \mathbf{u})_{\sigma,i}^{n+1} + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} = 0, \quad (3b)$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma, \quad (3c)$$

where the terms introduced for each discrete equation are defined hereafter.

Equation (3a) is obtained by discretization of the mass balance over the primal mesh, and $F_{K,\sigma}^{n+1}$ stands for the mass flux across σ outward K , which, because of the impermeability condition, vanishes on external faces and is given on the internal faces by:

$$\forall \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad F_{K,\sigma}^{n+1} = |\sigma| \rho_\sigma^{n+1} u_{K,\sigma}^{n+1},$$

where $u_{K,\sigma}^{n+1}$ is an approximation of the normal velocity to the face σ outward K . This latter quantity is defined by:

$$u_{K,\sigma}^{n+1} = \begin{cases} u_{\sigma,i}^{n+1} \mathbf{e}^{(i)} \cdot \mathbf{n}_{K,\sigma} & \text{for } \sigma \in \mathcal{E}^{(i)} \text{ in the MAC case,} \\ \mathbf{u}_\sigma^{n+1} \cdot \mathbf{n}_{K,\sigma} & \text{in the CR and RT cases,} \end{cases} \quad (4)$$

where $\mathbf{e}^{(i)}$ denotes the i -th vector of the orthonormal basis of \mathbb{R}^d . The density at the face $\sigma = K|L$ is approximated by the upwind technique:

$$\rho_\sigma^{n+1} = \begin{cases} \rho_K^{n+1} & \text{if } u_{K,\sigma}^{n+1} \geq 0, \\ \rho_L^{n+1} & \text{otherwise.} \end{cases} \quad (5)$$

We now turn to the discrete momentum balance (3b). For the discretization of the time derivative, we must provide a definition for the values $\rho_{D_\sigma}^{n+1}$ and $\rho_{D_\sigma}^n$, which approximate the density on the face σ at time t^{n+1} and t^n respectively. They are given by the following weighted average:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\text{int}}, \text{ for } k = n \text{ and } k = n + 1, \quad |D_\sigma| \rho_{D_\sigma}^k = |D_{K,\sigma}| \rho_K^k + |D_{L,\sigma}| \rho_L^k. \quad (6)$$

Let us then detail the discretization of the convection term. The first task is to define the discrete mass flux through the dual face ϵ outward D_σ , denoted by $F_{\sigma,\epsilon}^{n+1}$; the guideline for its construction is that a finite volume discretization of the mass balance equation over the diamond cells, of the form

$$\forall \sigma \in \mathcal{E}, \quad \frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^{n+1} = 0 \quad (7)$$

must hold in order to be able to derive a discrete kinetic energy balance (see Section 3.1 below). For the MAC scheme, the flux on a dual face which is located on two primal faces is the mean value of the sum of fluxes on the two primal faces, and the flux of a dual face located between two primal faces is again the mean value of the sum of fluxes on the two primal faces [22]. In the case of the CR and RT schemes, for a dual face ϵ included in the primal cell K , this flux is computed as a linear combination (with constant coefficients, *i.e.* independent of the cell) of the mass fluxes through the faces of K , *i.e.* the quantities $(F_{K,\sigma}^{n+1})_{\sigma \in \mathcal{E}(K)}$ appearing in the discrete mass balance (3a). We refer to [1, 12] for a detailed construction of this approximation. Let us remark that a dual face lying on the boundary is then also a primal face, and the flux across that face is zero. Therefore, the values $u_{\epsilon,i}^{n+1}$ are only needed at the internal dual faces; we choose them to be centered:

$$\text{for } \epsilon = D_\sigma|D'_\sigma, \quad u_{\epsilon,i}^{n+1} = \frac{1}{2} (u_{\sigma,i}^{n+1} + u_{\sigma',i}^{n+1}). \quad (8)$$

The quantity $(\Delta^{\mathcal{M}} \mathbf{u})_{\sigma,i}^{n+1}$ stands for a possible stabilizing diffusion term which may be written under a finite volume form over any diamond cell D_σ associated to the i^{th} component of the velocity:

$$-|D_\sigma| (\Delta^{\mathcal{M}} \mathbf{u})_{\sigma,i}^{n+1} = \sum_{\epsilon = D_\sigma|D'_\sigma} \nu h_\epsilon^{d-2} (u_{\sigma,i}^{n+1} - u_{\sigma',i}^{n+1}), \quad (9)$$

where h_ϵ is a characteristic dimension of the face ϵ , and ν stands for a non-negative coefficient, possibly depending on a power of h_ϵ . Note that this term is usually (*i.e.* for general meshes) not consistent with a Laplace operator. The usual upwind scheme (*i.e.* an upwind choice of the quantity $u_{\epsilon,i}^{n+1}$ with respect to the flux $F_{\sigma,\epsilon}^{n+1}$) leads to the discrete equation (3b), with the velocity at the face defined by (8) and the numerical diffusion term defined by (9), with:

$$\nu h_\epsilon^{d-2} = \frac{1}{2} |F_{\sigma,\epsilon}^{n+1}|, \quad (10)$$

and ν behaves as h_ϵ in this case. Writing the numerical diffusion of the scheme as the separate term (9) presents two advantages. First, with such a diffusion term, assuming that the coefficient ν behaves as h_ϵ^α with $0 < \alpha < 2$, we obtain a weak $L^2(\mathbf{H}^1)$ control of the velocity which is sufficient, at least in one space dimension, to pass to the limit in the scheme (see Section 3.3). Unfortunately, this assumption is not satisfied by the upwind discretization of the nonlinear convection term (10); note however that the convergence analysis of Section 3.3 would still hold in this case under the additional assumption that the approximate solution satisfies a BV estimate. Second, this formalism may prepare for a stabilization strategy which could be less diffusive than the upwind choice, for instance choosing ν on the basis of an *a posteriori* analysis of the local regularity of the solution [15, 16].

The last term $(\nabla p)_{\sigma,i}^{n+1}$ stands for the i -th component of the discrete pressure gradient at the face σ . The gradient operator is built as the transpose of the natural discrete divergence operator defined by

$$|K| (\operatorname{div} \mathbf{u})_K = \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{K,\sigma}. \quad (11)$$

In the CR and RT case, the duality between the divergence and gradient operators simply reads:

$$\sum_{K \in \mathcal{M}} |K| p_K (\operatorname{div} \mathbf{u})_K + \sum_{\sigma \in \mathcal{E}} |D_\sigma| \mathbf{u}_\sigma \cdot (\nabla p)_\sigma = 0.$$

This duality relation may be rewritten so as to fit both the CR/RT scheme and the MAC scheme as follows:

$$\sum_{K \in \mathcal{M}} |K| p_K (\operatorname{div} \mathbf{u})_K + \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} |D_\sigma| u_{\sigma,i} (\nabla p)_{\sigma,i} = 0. \quad (12)$$

Therefore, on any internal face, the components of the gradients are given by:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad (\nabla p)_{\sigma,i}^{n+1} = \frac{|\sigma|}{|D_\sigma|} (p_L^{n+1} - p_K^{n+1}) \mathbf{n}_{K,\sigma} \cdot \mathbf{e}^{(i)}.$$

Note that because of the impermeability boundary conditions, the discrete gradient is not defined at the external faces.

Finally, the initial approximations for ρ and \mathbf{u} are given by the average of the initial conditions ρ_0 and \mathbf{u}_0 on the primal and dual cells respectively:

$$\begin{aligned} \forall K \in \mathcal{M}, \quad \rho_K^0 &= \frac{1}{|K|} \int_K \rho_0(\mathbf{x}) \, d\mathbf{x}, \\ \text{for } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_S^{(i)}, \quad u_{\sigma,i}^0 &= \frac{1}{|D_\sigma|} \int_{D_\sigma} (\mathbf{u}_0(\mathbf{x}))_i \, d\mathbf{x}. \end{aligned} \quad (13)$$

3.2. Estimates

We begin with an estimate on the velocity which is a discrete equivalent of the kinetic energy balance. Recall that in the continuous setting, the kinetic energy balance is formally obtained by multiplying the i^{th} component of the momentum balance equation (1b) by the i^{th} component u_i of \mathbf{u} ; this yields for $1 \leq i \leq d$, using the mass balance equation (1a) twice:

$$\partial_t \left(\frac{1}{2} \rho u_i^2 \right) + \operatorname{div} \left(\left(\frac{1}{2} \rho u_i^2 \right) \mathbf{u} \right) + (\partial_{x_i} p) u_i = 0,$$

and thus, summing over the components:

$$\partial_t (\rho E_k) + \operatorname{div} (\rho E_k \mathbf{u}) + \nabla p \cdot \mathbf{u} = 0, \quad \text{with } E_k = \frac{1}{2} |\mathbf{u}|^2. \quad (14)$$

In the discrete setting, this multiplication must be performed on the dual mesh, since the velocity unknowns are defined on the faces. This is the reason why we chose the fluxes on the faces of the dual mesh in such a way that a discrete mass balance equation holds on the dual grid cells, thus allowing us to use Lemma A.2 (which performs the discrete equivalent of the above formal computations) on the dual mesh.

Lemma 3.1 (Discrete kinetic energy balance, implicit scheme).

A solution to the system (3) satisfies the following equality, for $1 \leq i \leq d$, $\sigma \in \mathcal{E}_S^{(i)}$ and $0 \leq n \leq N-1$:

$$\frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[\rho_{D_\sigma}^{n+1} (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^n (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\epsilon=D_\sigma|D_{\sigma'}} F_{\sigma,\epsilon}^{n+1} u_{\sigma,i}^{n+1} u_{\sigma',i}^{n+1} + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1} = -R_{\sigma,i}^{n+1}, \quad (15)$$

where

$$R_{\sigma,i}^{n+1} = \frac{1}{2} \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 + \left[\sum_{\epsilon=D_\sigma|D_{\sigma'}} \nu h_\epsilon^{d-2} (u_{\sigma,i}^{n+1} - u_{\sigma',i}^{n+1}) \right] u_{\sigma,i}^{n+1}. \quad (16)$$

Proof. Let us multiply equation (3b) by the corresponding velocity unknown $u_{\sigma,i}^{n+1}$; this yields $T_{\sigma,i}^{\text{conv}} + T_{\sigma,i}^\Delta + T_{\sigma,i}^\nabla = 0$, with:

$$\begin{aligned} T_{\sigma,i}^{\text{conv}} &= \left[\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^n u_{\sigma,i}^n) + \sum_{\epsilon=D_\sigma|D_{\sigma'}} \frac{1}{2} F_{\sigma,\epsilon}^{n+1} (u_{\sigma,i}^{n+1} + u_{\sigma',i}^{n+1}) \right] u_{\sigma,i}^{n+1}, \\ T_{\sigma,i}^\Delta &= \left[\sum_{\epsilon=D_\sigma|D_{\sigma'}} \nu h_\epsilon^{d-2} (u_{\sigma,i}^{n+1} - u_{\sigma',i}^{n+1}) \right] u_{\sigma,i}^{n+1}, \\ T_{\sigma,i}^\nabla &= |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1}. \end{aligned}$$

Applying Lemma A.2 with $P = D_\sigma$, we get from the identity (56) that

$$T_{\sigma,i}^{\text{conv}} = \frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[\rho_{D_\sigma}^{n+1} (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^n (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\epsilon=D_\sigma|D_{\sigma'}} F_{\sigma,\epsilon}^{n+1} u_{\sigma,i}^{n+1} u_{\sigma',i}^{n+1} + \frac{|D_\sigma|}{2\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2.$$

Noting that $R_{\sigma,i}^{n+1}$ defined by (16) is the sum of $T_{\sigma,i}^\Delta$ and of the last term of $T_{\sigma,i}^{\text{conv}}$, this concludes the proof of (15). \square

Let us now define the elastic potential \mathcal{P} :

$$\mathcal{P}(z) = \int_0^z \frac{\wp(s)}{s^2} ds \text{ i.e. } \mathcal{P}(z) = \begin{cases} \frac{z^{\gamma-1}}{\gamma-1} & \text{if } \gamma > 1, \\ \ln(z) & \text{if } \gamma = 1, \end{cases} \quad (17)$$

and let \mathcal{H} be the function defined over $(0, +\infty)$ by

$$\mathcal{H}(z) = z \mathcal{P}(z) = \begin{cases} \frac{z^\gamma}{\gamma-1} & \text{if } \gamma > 1, \\ z \ln(z) & \text{if } \gamma = 1. \end{cases} \quad (18)$$

It may easily be checked that $s\mathcal{H}'(s) - \mathcal{H}(s) = \wp(s)$; therefore, by a formal computation detailed in the appendix (see 51), multiplying (1a) by $\mathcal{H}'(\rho)$ yields:

$$\partial_t(\mathcal{H}(\rho)) + \text{div}(\mathcal{H}(\rho) \mathbf{u}) + p \text{div}(\mathbf{u}) = 0. \quad (19)$$

We now derive a discrete analogue of this relation.

Lemma 3.2 (Discrete potential balance). *Let \mathcal{H} be defined by (18). A solution to the system (3) satisfies the following equality, for $K \in \mathcal{M}$ and $0 \leq n \leq N-1$:*

$$\frac{|K|}{\delta t} \left[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n) \right] + \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \mathcal{H}(\rho_\sigma^{n+1}) u_{K,\sigma}^{n+1} + |K| p_K^{n+1} (\text{div} \mathbf{u}^{n+1})_K = -R_K^{n+1}, \quad (20)$$

with:

$$R_K^{n+1} = \frac{1}{2} \frac{|K|}{\delta t} \mathcal{H}''(\bar{\rho}_K^{n+1}) (\rho_K^{n+1} - \rho_K^n)^2 + \frac{1}{2} \sum_{\sigma=K|L} |\sigma| (u_{K,\sigma}^{n+1})^- \mathcal{H}''(\bar{\rho}_\sigma^{n+1}) (\rho_L^{n+1} - \rho_K^{n+1})^2, \quad (21)$$

where $\bar{\rho}_K^{n+1} \in [\min(\rho_K^{n+1}, \rho_K^n), \max(\rho_K^{n+1}, \rho_K^n)]$, $\bar{\rho}_\sigma^{n+1} \in [\min(\rho_\sigma^{n+1}, \rho_K^{n+1}), \max(\rho_\sigma^{n+1}, \rho_K^{n+1})]$ for all $\sigma \in \mathcal{E}(K)$, and, for $a \in \mathbb{R}$, $a^- \geq 0$ is defined by $a^- = -\min(a, 0)$. Note that, since the function \mathcal{H} is convex, R_K^{n+1} is non-negative.

Proof. Let us multiply the discrete mass balance (3a) by $\mathcal{H}'(\rho_K^{n+1})$. The result is then a consequence of Lemma A.1, using the fact that $z\mathcal{H}'(z) - \mathcal{H}(z) = \wp(z)$ and that ρ_σ^{n+1} is the upwind choice between ρ_K and ρ_L in the remainder term $R_{K,\delta t}$. \square

Summing (14) and (19) yields:

$$\partial_t \mathcal{S} + \operatorname{div}((\mathcal{S} + p) \mathbf{u}) = 0,$$

with $\mathcal{S} = \rho E_k + \mathcal{H}(\rho)$ the entropy of the system. This relation is only valid for regular solutions, and should be replaced by an inequality to take into account the presence of shocks (see Relations (34)-(35)). Integrating over Ω and using the boundary conditions yields:

$$\frac{d}{dt} \int_{\Omega} \mathcal{S}(\mathbf{x}, t) \, d\mathbf{x} \leq 0 \quad (\text{for regular solutions, } \frac{d}{dt} \int_{\Omega} \mathcal{S}(\mathbf{x}, t) \, d\mathbf{x} = 0),$$

and, for $t \in (0, T)$,

$$\int_{\Omega} \mathcal{S}(\mathbf{x}, t) \, d\mathbf{x} \leq \int_{\Omega} \mathcal{S}(\mathbf{x}, 0) \, d\mathbf{x}.$$

The following proposition states a discrete analogue to this relation.

Proposition 3.3 (Global discrete entropy inequality, existence of a solution).

Assume that the initial density ρ_0 is positive. Then there exists a solution $(u^n)_{0 \leq n \leq N}$ and $(\rho^n)_{0 \leq n \leq N}$ to the scheme, and, for $1 \leq n \leq N$, $\rho^n > 0$ and the following inequality holds:

$$\frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} |D_\sigma| \rho_{D_\sigma}^n (u_{\sigma,i}^n)^2 + \sum_{K \in \mathcal{M}} |K| \mathcal{H}(\rho_K^n) + \mathcal{R}^n \leq C, \quad (22)$$

where the real number $C \in \mathbb{R}_+$ only depends on the initial conditions, and \mathcal{R}^n is the following non-negative remainder which depends on the space and time translates of the unknowns:

$$\begin{aligned} \mathcal{R}^n &= \sum_{i=1}^d \sum_{k=1}^n \left[\frac{1}{2} \sum_{\sigma \in \mathcal{E}_S^{(i)}} |D_\sigma| \rho_{D_\sigma}^k (u_{\sigma,i}^k - u_{\sigma,i}^{k-1})^2 + \delta t \sum_{\epsilon = D_\sigma | D_{\sigma'} \in \mathcal{E}_S^{(i)}} \nu h_{\sigma'}^{d-2} (u_{\sigma,i}^k - u_{\sigma',i}^k)^2 \right] \\ &+ \frac{\gamma}{2} \sum_{k=1}^n \delta t \sum_{\sigma = K | L \in \mathcal{E}_{\text{int}}} |\sigma| (\rho_{\sigma,\gamma}^k)^{\gamma-2} |u_{K,\sigma}| (\rho_K^k - \rho_L^k)^2, \end{aligned} \quad (23)$$

with $\rho_{\sigma,\gamma}^k$ equal to either ρ_K^k or ρ_L^k and such that $(\rho_{\sigma,\gamma}^k)^{\gamma-2} = \min((\rho_K^k)^{\gamma-2}, (\rho_L^k)^{\gamma-2})$.

Remark 3.4. For $\gamma > 1$, the function \mathcal{H} is positive and increasing over $(0, +\infty)$. The inequality (22) thus readily provides an estimate on the unknowns.

This is still true also for $\gamma = 1$, since in this case $\mathcal{H}(s) = s \ln s$ and therefore $\mathcal{H}(s) \geq -1/e$, $\forall s \in (0, +\infty)$, and \mathcal{H} is increasing over $(1/e, +\infty)$. In fact, in order to get the usual formulation of an estimate, we may rephrase the inequality (22) by changing the expression of \mathcal{H} to $\mathcal{H}(s) = \max(s \log(s), 0)$ and adding $|\Omega|/e$ to the constant C at the right-hand side.

Proof. Let us give the proof of Proposition 3.3. The positivity of the density is a consequence of the properties of the upwind choice (5) for ρ [13, Lemma 2.1]; note that it may also be proved applying Lemma A.1 with $\psi(s) = \frac{1}{2}(s^-)^2$ and $P = K$.

Let us then sum Equation (15) over the components i and the faces $\sigma \in \mathcal{E}_S^{(i)}$, Equation (20) over $K \in \mathcal{M}$, and, finally, the two obtained relations.

Since the discrete gradient and divergence operators are dual with respect to the L^2 inner product (see (12)), noting that the conservative fluxes vanish in the summation, we get, for $1 \leq k \leq N$:

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} \frac{|D_\sigma|}{\delta t} \left[\rho_{D_\sigma}^k (u_{\sigma,i}^k)^2 - \rho_{D_\sigma}^{k-1} (u_{\sigma,i}^{k-1})^2 \right] + \sum_{K \in \mathcal{M}} \frac{|K|}{\delta t} (\mathcal{H}(\rho_K^k) - \mathcal{H}(\rho_K^{k-1})) \\ = - \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} R_{\sigma,i}^k - \sum_{K \in \mathcal{M}} R_K^k. \end{aligned} \quad (24)$$

Summing (24) for $k = 1$ to n , and using the fact that $\mathcal{H}''(s) = \gamma s^{\gamma-2}$ for any $\gamma \geq 1$ yields (22), with \mathcal{R}^n given by (23) and

$$C = \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} |D_\sigma| \rho_\sigma^0 |u_{\sigma,i}^0|^2 + \sum_{K \in \mathcal{M}} |K| \mathcal{H}(\rho_K^0).$$

Finally, the existence of a solution may be inferred by the Brouwer fixed point theorem, by an easy adaptation of the proof of [10, Proposition 5.2]. This proof relies on the following set of mesh-dependent estimates: the conservativity of the mass balance discretization, together with the fact that the density is positive, yields an estimate for ρ in the L^1 -norm, and so, by a norm equivalence argument, of the pressure in any norm; the discrete momentum balance equation then provides a control on the velocity. Therefore, computing

- (i) ρ from the mass balance for fixed \mathbf{u} ,
- (ii) p from ρ by the equation of state,
- (iii) and finally \mathbf{u} from the momentum balance equation with fixed ρ and p ,

yields an iteration in a bounded convex subset of a finite dimensional space. \square

3.3. Passing to the limit in the scheme

The objective of this section is to show, in the one dimensional case, that if a sequence of solutions is controlled in suitable norms and converges to a limit, this latter necessarily satisfies a (part of the) weak formulation of the continuous problem.

The 1D version of the scheme which is studied in this section may be obtained from Scheme (3) by taking the MAC variant, only one horizontal stripe of grid cells, supposing that the vertical component of the velocity (the degrees of freedom of which are located on the top and bottom boundaries) vanishes, and that the measure of the vertical faces is equal to 1. For the sake of readability, however, we completely rewrite this 1D scheme, and, to this purpose, we first introduce some adaptations of the notations to the one dimensional case. For any $K \in \mathcal{M}$, we denote by h_K its length (so $h_K = |K|$); when we write $K = [\sigma\sigma']$, this means that either $K = (x_\sigma, x_{\sigma'})$ or $K = (x_{\sigma'}, x_\sigma)$; if we need to specify the order, *i.e.* $K = (x_\sigma, x_{\sigma'})$ with $x_\sigma < x_{\sigma'}$, then we write $K = \overrightarrow{[\sigma\sigma']}$. For an interface $\sigma = K|L$ between two cells K and L , we define $h_\sigma = (h_K + h_L)/2$, so, by definition of the dual mesh, $h_\sigma = |D_\sigma|$. If we need to specify the order of the cells K and L , say K is left of L , then we write $\sigma = \overrightarrow{K|L}$. With these notations, the implicit scheme (3) may be written as follows in the one dimensional setting:

$$\begin{aligned} \forall K \in \mathcal{M}, \quad \rho_K^0 &= \frac{1}{|K|} \int_K \rho_0(x) dx, \\ \forall \sigma \in \mathcal{E}_{\text{int}}, \quad u_\sigma^0 &= \frac{1}{|D_\sigma|} \int_{D_\sigma} u_0(x) dx, \end{aligned} \quad (25a)$$

$$\begin{aligned} \forall K = \overrightarrow{[\sigma\sigma']} \in \mathcal{M}, \\ \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^{n+1} - F_\sigma^{n+1} &= 0, \end{aligned} \quad (25b)$$

$$\begin{aligned} \forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \\ \frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^{n+1} u_L^{n+1} - F_K^{n+1} u_K^{n+1} \\ - |D_\sigma| (\Delta_{\mathcal{M}u})_\sigma^{n+1} + p_L^{n+1} - p_K^{n+1} &= 0, \end{aligned} \quad (25c)$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma. \quad (25d)$$

The mass flux in the discrete mass balance equation is given, for $\sigma \in \mathcal{E}_{\text{int}}$, by:

$$F_\sigma^{n+1} = \rho_\sigma^{n+1} u_\sigma^{n+1}, \quad (26)$$

where the upwind approximation for the density at the face, ρ_σ^{n+1} , is defined by (5). In the momentum balance equation, the application of the procedure described in Section 3.1 yields for the density associated to the dual cell D_σ with $\sigma = K|L$ and for the mass fluxes at the dual face located at the center of the cell $K = [\sigma\sigma']$:

$$\rho_{D_\sigma}^{n+1} = \frac{1}{2|D_\sigma|} (|K|\rho_K^{n+1} + |L|\rho_L^{n+1}), \quad F_K^{n+1} = \frac{1}{2} (F_\sigma^{n+1} + F_{\sigma'}^{n+1}), \quad (27)$$

and the approximation of the velocity at this face is centered: $u_K^{n+1} = (u_\sigma^{n+1} + u_{\sigma'}^{n+1})/2$. Finally, for a face $\sigma = \overrightarrow{K|L}$ with $K = [\overrightarrow{\sigma'}\sigma]$ and $L = [\sigma\overrightarrow{\sigma'}]$, the stabilization diffusion term reads:

$$-|D_\sigma| (\Delta_{\mathcal{M}} u)_\sigma^{n+1} = \nu \left[\frac{1}{h_K} (u_\sigma^{n+1} - u_{\sigma'}^{n+1}) + \frac{1}{h_L} (u_\sigma^{n+1} - u_{\sigma'}^{n+1}) \right]. \quad (28)$$

Definition 3.5 (Regular sequence of discretizations, implicit case).

We define a regular sequence of discretizations $(\mathcal{M}^{(m)}, \delta t^{(m)}, \nu^{(m)})_{m \in \mathbb{N}}$ as a sequence of meshes, time steps and numerical diffusion coefficients satisfying:

- (i) both the time step $\delta t^{(m)}$ and the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$, defined by $h^{(m)} = \sup_{K \in \mathcal{M}^{(m)}} h_K$, tend to zero as $m \rightarrow \infty$,
- (ii) there exists $\theta > 0$ such that:

$$\theta \leq \frac{h_K}{h_L} \leq \frac{1}{\theta}, \quad \forall m \in \mathbb{N} \text{ and } K, L \in \mathcal{M}^{(m)} \text{ sharing a face,}$$

- (iii) the sequence of numerical diffusion coefficients $(\nu^{(m)})_{m \in \mathbb{N}}$ satisfies:

$$\lim_{m \rightarrow +\infty} \nu^{(m)} = 0, \quad \lim_{m \rightarrow +\infty} \frac{(h^{(m)})^2}{\nu^{(m)}} = 0.$$

Let such a regular sequence of discretizations be given, and $\rho^{(m)}$, $p^{(m)}$ and $u^{(m)}$ be the solution given by the scheme (25) with the mesh $\mathcal{M}^{(m)}$, the time step $\delta t^{(m)}$ and the numerical diffusion coefficient $\nu^{(m)}$. To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, so that the density $\rho^{(m)}$, the pressure $p^{(m)}$ and the velocity $u^{(m)}$ are defined almost everywhere on $\Omega \times (0, T)$ by:

$$\begin{aligned} \rho^{(m)}(x, t) &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} (\rho^{(m)})_K^{n+1} \mathcal{X}_K(x) \mathcal{X}_{(n, n+1]}(t), & p^{(m)}(x, t) &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} (p^{(m)})_K^{n+1} \mathcal{X}_K(x) \mathcal{X}_{(n, n+1]}(t), \\ u^{(m)}(x, t) &= \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} (u^{(m)})_\sigma^{n+1} \mathcal{X}_{D_\sigma}(x) \mathcal{X}_{(n, n+1]}(t), \end{aligned}$$

where \mathcal{X}_K , \mathcal{X}_{D_σ} and $\mathcal{X}_{(n, n+1]}$ stand for the characteristic functions of the intervals K , D_σ and $(t^n, t^{n+1}]$ respectively.

A weak solution to the continuous problem satisfies, for any $\varphi \in C_c^\infty(\Omega \times [0, T])$:

$$- \int_0^T \int_\Omega \left[\rho \partial_t \varphi + \rho u \partial_x \varphi \right] dx dt - \int_\Omega \rho_0(x) \varphi(x, 0) dx = 0, \quad (29a)$$

$$- \int_0^T \int_\Omega \left[\rho u \partial_t \varphi + (\rho u^2 + p) \partial_x \varphi \right] dx dt - \int_\Omega \rho_0(x) u_0(x) \varphi(x, 0) dx = 0, \quad (29b)$$

$$p = \rho^\gamma. \quad (29c)$$

Note that these relations are not sufficient to define a weak solution to the problem, since they do not imply anything about the boundary conditions. However, they allow to derive the Rankine-Hugoniot conditions; hence if we show that they are satisfied by the limit of a sequence of solutions to the discrete problem, this implies, loosely speaking, that *the scheme computes correct shocks* (i.e. shocks where the jumps of the unknowns and of the fluxes are linked to the shock speed by Rankine-Hugoniot conditions). This is the result we are seeking and which we state in Theorem 3.7. In order to prove this theorem, we need some definitions of interpolates of regular test functions on the primal and dual mesh.

Definition 3.6 (Interpolates on one-dimensional meshes). Let Ω be an open bounded interval of \mathbb{R} , let $\varphi \in C_c^\infty(\Omega \times [0, T])$, and let \mathcal{M} be a mesh over Ω . The interpolate $\varphi_{\mathcal{M}}$ of φ on the primal mesh \mathcal{M} is defined by:

$$\varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \varphi_K^n \mathcal{X}_K \mathcal{X}_{[t^n, t^{n+1})},$$

where, for $0 \leq n \leq N$ and $K \in \mathcal{M}$, we set $\varphi_K^n = \varphi(x_K, t^n)$, with x_K the mass center of K . The time and space discrete derivatives of the discrete function $\varphi_{\mathcal{M}}$ are defined by:

$$\bar{\partial}_t \varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} \mathcal{X}_K \mathcal{X}_{[t^n, t^{n+1})}, \text{ and } \bar{\partial}_x \varphi_{\mathcal{M}} = \sum_{n=0}^{N-1} \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} \frac{\varphi_L^n - \varphi_K^n}{h_\sigma} \mathcal{X}_{D_\sigma} \mathcal{X}_{[t^n, t^{n+1})}.$$

Let $\varphi_{\mathcal{E}}$ be an interpolate of φ on the dual mesh, defined by:

$$\varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} \varphi_\sigma^n \mathcal{X}_{D_\sigma} \mathcal{X}_{[t^n, t^{n+1})},$$

where, for $1 \leq n \leq N$ and $\sigma \in \mathcal{E}$, we set $\varphi_\sigma^n = \varphi(x_\sigma, t^n)$, with x_σ the abscissa of the interface σ . We also define the time and space discrete derivatives of this discrete function by:

$$\bar{\partial}_t \varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} \frac{\varphi_\sigma^{n+1} - \varphi_\sigma^n}{\delta t} \mathcal{X}_{D_\sigma} \mathcal{X}_{[t^n, t^{n+1})}, \text{ and } \bar{\partial}_x \varphi_{\mathcal{E}} = \sum_{n=0}^{N-1} \sum_{K = [\overleftarrow{\sigma\sigma'}] \in \mathcal{M}} \frac{\varphi_{\sigma'}^n - \varphi_\sigma^n}{h_K} \mathcal{X}_K \mathcal{X}_{[t^n, t^{n+1})}.$$

Theorem 3.7 (Consistency of the one-dimensional implicit scheme).

Let Ω be an open bounded interval of \mathbb{R} . We suppose that the initial data satisfies $\rho_0 \in L^\infty(\Omega)$ and $u_0 \in L^\infty(\Omega)$. Let $(\mathcal{M}^{(m)}, \delta t^{(m)}, \nu^{(m)})_{m \in \mathbb{N}}$ be a regular sequence of discretizations in the sense of Definition 3.5, and $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m \in \mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence converges in $L^p(\Omega \times (0, T))^3$, for $1 \leq p < \infty$, to $(\bar{\rho}, \bar{p}, \bar{u}) \in L^\infty(\Omega \times (0, T))^3$. We suppose in addition that both sequences $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(1/\rho^{(m)})_{m \in \mathbb{N}}$ are uniformly bounded in $L^\infty(\Omega \times (0, T))$.

Then the limit $(\bar{\rho}, \bar{p}, \bar{u})$ satisfies the system (29).

Proof. With the assumed convergence for the sequence of solutions, the limit clearly satisfies the equation of state. The proof of this theorem is thus obtained by passing to the limit in the scheme, first for the mass balance equation and then for the momentum balance equation. Thanks to the assumption on the initial condition, the stability estimate of Proposition 3.3 is uniform with respect to m , and thus provides uniform bounds for some space translates of the solution (see the expression (23) of the remainder term), which are used all along the proof. In particular, using in addition the assumption that both sequences $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(1/\rho^{(m)})_{m \in \mathbb{N}}$ are uniformly bounded in $L^\infty(\Omega \times (0, T))$, exploiting the last part of the remainder term, we get the following weak BV estimate for ρ :

$$\forall m \in \mathbb{N}, \quad \sum_{n=0}^{N-1} \delta t \sum_{\sigma = K|L} |(u^{(m)})_\sigma^{n+1}| \left[(\rho^{(m)})_K^{n+1} - (\rho^{(m)})_L^{n+1} \right]^2 \leq C, \quad (30)$$

where C stands for a real number which is independent of m .

Mass balance equation – Let $\varphi \in C_c^\infty(\Omega \times [0, T])$. Let $m \in \mathbb{N}$, $\mathcal{M}^{(m)}$, $\delta t^{(m)}$ and $\nu^{(m)}$ be given. Dropping for short the superscript $^{(m)}$, let $\varphi_{\mathcal{M}}$ be an interpolate of φ on the primal mesh and let $\bar{\partial}_t \varphi_{\mathcal{M}}$ and $\bar{\partial}_x \varphi_{\mathcal{M}}$ be its

time and space discrete derivatives in the sense of Definition 3.6. Thanks to the regularity of φ , these functions respectively converge in $L^r(\Omega \times (0, T))$, for $r \geq 1$ (including $r = +\infty$), to φ , $\partial_t \varphi$ and $\partial_x \varphi$ respectively. In addition, $\varphi_{\mathcal{M}}(\cdot, 0)$ converges to $\varphi(\cdot, 0)$ in $L^r(\Omega)$ for $r \geq 1$. Since the support of φ is compact in $\Omega \times [0, T]$, for m large enough, the interpolate of φ vanishes at the boundary cells and at the final time; hereafter, we systematically assume that we are in this case.

Let us multiply the discrete mass balance equation (25b) by $\delta t \varphi_K^n$, and sum the result on $n \in \{0, \dots, N-1\}$ and $K \in \mathcal{M}$, to obtain $T_1^{(m)} + T_2^{(m)} = 0$ with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} |K| (\rho_K^{n+1} - \rho_K^n) \varphi_K^n, \quad T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} [F_{\sigma'}^{n+1} - F_{\sigma}^{n+1}] \varphi_K^n.$$

Reordering the sums in $T_1^{(m)}$ yields:

$$T_1^{(m)} = - \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} |K| \rho_K^{n+1} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} - \sum_{K \in \mathcal{M}} |K| \rho_K^0 \varphi_K^0,$$

so that:

$$T_1^{(m)} = - \int_0^T \int_{\Omega} \rho^{(m)} \partial_t \varphi_{\mathcal{M}} dx dt - \int_{\Omega} (\rho^0)^{(m)}(x) \varphi_{\mathcal{M}}(x, 0) dx.$$

Since, by assumption, the sequence of discrete solutions converges in $L^r(\Omega \times (0, T))$ for $r \geq 1$, and by definition of the discrete initial conditions (25a), the sequence $((\rho^{(m)})^0)_{m \in \mathbb{N}}$ converges to ρ_0 in $L^r(\Omega)$ for $r \geq 1$, we get:

$$\lim_{m \rightarrow +\infty} T_1^{(m)} = - \int_0^T \int_{\Omega} \bar{\rho} \partial_t \varphi dx dt - \int_{\Omega} \rho_0(x) \varphi(x, 0) dx.$$

Reordering the sums in $T_2^{(m)}$, we get:

$$T_2^{(m)} = - \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|\vec{L} \in \mathcal{E}_{\text{int}}} |D_{\sigma}| \rho_{\sigma}^{n+1} u_{\sigma}^{n+1} \frac{\varphi_L^n - \varphi_K^n}{h_{\sigma}},$$

where h_{σ} (which is equal to $|D_{\sigma}|$) is by definition equal to $|x_L - x_K|$ and we recall that ρ_{σ}^{n+1} is the upwind approximation of ρ^{n+1} at the face σ . Using the fact that $|D_{\sigma}| = (|K| + |L|)/2$, we may write $T_2^{(m)} = \mathcal{T}_2^{(m)} + \mathcal{R}_2^{(m)}$ with:

$$\begin{aligned} \mathcal{T}_2^{(m)} &= - \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|\vec{L} \in \mathcal{E}_{\text{int}}} \left[\frac{|K|}{2} \rho_K^{n+1} + \frac{|L|}{2} \rho_L^{n+1} \right] u_{\sigma}^{n+1} \frac{\varphi_L^n - \varphi_K^n}{h_{\sigma}}, \\ \mathcal{R}_2^{(m)} &= - \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|\vec{L} \in \mathcal{E}_{\text{int}}} \left[\frac{|K|}{2} (\rho_K^{n+1} - \rho_L^{n+1})(u_{\sigma}^{n+1})^- + \frac{|L|}{2} (\rho_K^{n+1} - \rho_L^{n+1})(u_{\sigma}^{n+1})^+ \right] \frac{\varphi_L^n - \varphi_K^n}{h_{\sigma}}. \end{aligned}$$

Therefore we get

$$\mathcal{T}_2^{(m)} = - \int_0^T \int_{\Omega} \rho^{(m)} u^{(m)} \partial_x \varphi_{\mathcal{M}} dx dt, \quad \text{and} \quad \lim_{m \rightarrow +\infty} \mathcal{T}_2^{(m)} = - \int_0^T \int_{\Omega} \bar{\rho} \bar{u} \partial_x \varphi dx dt,$$

and the remainder term $\mathcal{R}_2^{(m)}$ is bounded as follows:

$$\begin{aligned} |\mathcal{R}_2^{(m)}| &\leq \|\varphi'\|_{\infty} \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |D_{\sigma}| |\rho_K^{n+1} - \rho_L^{n+1}| |u_{\sigma}^{n+1}| \\ &\leq \|\varphi'\|_{\infty} h^{1/2} \sum_{n=0}^{N-1} \delta t \left[\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |u_{\sigma}^{n+1}| |\rho_K^{n+1} - \rho_L^{n+1}|^2 \right]^{1/2} \left[\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |D_{\sigma}| |u_{\sigma}^{n+1}| \right]^{1/2}. \end{aligned}$$

Thanks to the stability estimate (30), this term tends to zero when m tends to $+\infty$.

Momentum balance equation – Let $\varphi_{\mathcal{E}}$, $\bar{\partial}_t \varphi_{\mathcal{E}}$ and $\bar{\partial}_x \varphi_{\mathcal{E}}$ be the interpolate of φ on the dual mesh and its discrete time and space derivatives, in the sense of Definition 3.6, which converge in $L^r(\Omega \times (0, T))$, for $r \geq 1$ (including $r = +\infty$), to φ , $\partial_t \varphi$ and $\partial_x \varphi$ respectively.

Let us multiply the discrete momentum balance equation (25c) by $\delta t \varphi_{\sigma}^n$, and sum the result over $n \in \{0, \dots, N-1\}$ and $\sigma \in \mathcal{E}_{\text{int}}$. We obtain $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} = 0$ with:

$$\begin{aligned} T_1^{(m)} &= \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}_{\text{int}}} |D_{\sigma}| (\rho_{D_{\sigma}}^{n+1} u_{\sigma}^{n+1} - \rho_{D_{\sigma}}^n u_{\sigma}^n) \varphi_{\sigma}^n, \\ T_2^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} \left[F_L^{n+1} u_L^{n+1} - F_K^{n+1} u_K^{n+1} \right] \varphi_{\sigma}^n, \\ T_3^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} (p_L^{n+1} - p_K^{n+1}) \varphi_{\sigma}^n, \\ T_4^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\text{int}}} \left[\sum_{K=[\sigma\sigma']} \frac{\nu}{h_K} (u_{\sigma}^{n+1} - u_{\sigma'}^{n+1}) \right] \varphi_{\sigma}^n. \end{aligned}$$

Thanks to the definition (27) of the density on the dual mesh $\rho_{D_{\sigma}}$, reordering the sums, we get for $T_1^{(m)}$:

$$T_1^{(m)} = - \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} \left[\frac{|K|}{2} \rho_K^{n+1} + \frac{|L|}{2} \rho_L^{n+1} \right] u_{\sigma}^{n+1} \frac{\varphi_{\sigma}^{n+1} - \varphi_{\sigma}^n}{\delta t} - \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} \left[\frac{|K|}{2} \rho_K^0 + \frac{|L|}{2} \rho_L^0 \right] u_{\sigma}^0 \varphi_{\sigma}^0.$$

Therefore:

$$T_1^{(m)} = - \int_0^T \int_{\Omega} \rho^{(m)} u^{(m)} \bar{\partial}_t \varphi_{\mathcal{M}} dx dt - \int_{\Omega} (\rho^0)^{(m)}(x) (u^0)^{(m)}(x) \varphi_{\mathcal{M}}(x, 0) dx.$$

Since, from the definition (25a) of the initial conditions, the sequences $((\rho^{(m)})^0)$ and $((u^{(m)})^0)$ converge in $L^r(\Omega)$, for $r \geq 1$, to ρ_0 and u_0 respectively, thanks to the convergence assumption of the sequence of discrete solutions, we get:

$$\lim_{m \rightarrow +\infty} T_1^{(m)} = - \int_0^T \int_{\Omega} \bar{\rho} \bar{u} \partial_t \varphi dx dt - \int_{\Omega} \rho_0(x) u_0(x) \varphi(x, 0) dx.$$

Let us now turn to $T_2^{(m)}$. From the expression (27) of the fluxes F_K and the values u_K , reordering the sums, we get:

$$T_2^{(m)} = - \frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} (\rho_{\sigma}^{n+1} u_{\sigma}^{n+1} + \rho_{\sigma'}^{n+1} u_{\sigma'}^{n+1}) (u_{\sigma}^{n+1} + u_{\sigma'}^{n+1}) (\varphi_{\sigma'}^n - \varphi_{\sigma}^n),$$

which we write $T_2^{(m)} = \mathcal{T}_2^{(m)} + \mathcal{R}_2^{(m)}$ with:

$$\mathcal{T}_2^{(m)} = - \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \rho_K^{n+1} \left[(u_{\sigma}^{n+1})^2 + (u_{\sigma'}^{n+1})^2 \right] (\varphi_{\sigma'}^n - \varphi_{\sigma}^n).$$

This term reads:

$$\mathcal{T}_2^{(m)} = - \int_0^T \int_{\Omega} \rho^{(m)} (u^{(m)})^2 \bar{\partial}_x \varphi_{\mathcal{E}} dx dt, \text{ and so } \lim_{m \rightarrow +\infty} \mathcal{T}_2^{(m)} = - \int_0^T \int_{\Omega} \bar{\rho} \bar{u}^2 \partial_x \varphi dx dt.$$

The remainder term $\mathcal{R}_2^{(m)}$ reads:

$$\mathcal{R}_2^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \left[(\rho_\sigma^{n+1} u_\sigma^{n+1} + \rho_{\sigma'}^{n+1} u_{\sigma'}^{n+1})(u_\sigma^{n+1} + u_{\sigma'}^{n+1}) - 2\rho_K^{n+1} \left((u_\sigma^{n+1})^2 + (u_{\sigma'}^{n+1})^2 \right) \right] (\varphi_{\sigma'}^n - \varphi_\sigma^n).$$

Expanding the quantity $2\rho_K^{n+1}((u_\sigma^{n+1})^2 + (u_{\sigma'}^{n+1})^2)$ thanks to the identity $2(a^2 + b^2) = (a+b)^2 + (a-b)^2$, we get $\mathcal{R}_2^{(m)} = \mathcal{R}_{2,1}^{(m)} + \mathcal{R}_{2,2}^{(m)}$:

$$\begin{aligned} \mathcal{R}_{2,1}^{(m)} &= -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \left[\left((\rho_\sigma^{n+1} - \rho_K^{n+1}) u_\sigma^{n+1} + (\rho_{\sigma'}^{n+1} - \rho_K^{n+1}) u_{\sigma'}^{n+1} \right) (u_\sigma^{n+1} + u_{\sigma'}^{n+1}) \right] (\varphi_{\sigma'}^n - \varphi_\sigma^n), \\ \mathcal{R}_{2,2}^{(m)} &= \frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \rho_K^{n+1} (u_\sigma^{n+1} - u_{\sigma'}^{n+1})^2 (\varphi_{\sigma'}^n - \varphi_\sigma^n). \end{aligned}$$

First we study $\mathcal{R}_{2,1}^{(m)}$. Thanks to the definition (5) of the upwind value ρ_σ^{n+1} , reordering the sum by faces, we get that:

$$|\mathcal{R}_{2,1}^{(m)}| = \frac{1}{4} \left| \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=L \rightarrow K, K=[\overrightarrow{\sigma\sigma'}]}} (\rho_L^{n+1} - \rho_K^{n+1}) u_\sigma^{n+1} (u_\sigma^{n+1} + u_{\sigma'}^{n+1}) (\varphi_\sigma^n - \varphi_{\sigma'}^n) \right|,$$

where the notation $L \rightarrow K$ means that the flow is going from L to K , or, in other words, that if $u_\sigma^{n+1} \geq 0$ (resp. $u_\sigma^{n+1} \leq 0$), the cells K and L are chosen such that $\sigma = \overrightarrow{L|K}$ (resp. $\sigma = \overrightarrow{K|L}$). Since $|\varphi_\sigma^n - \varphi_{\sigma'}^n| \leq C_\varphi |K| \leq C_\varphi (|D_\sigma| + |D_{\sigma'}|)$, we get:

$$|\mathcal{R}_{2,1}^{(m)}| \leq \frac{C_\varphi}{4} \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=L \rightarrow K, K=[\overrightarrow{\sigma\sigma'}]}} (|D_\sigma| + |D_{\sigma'}|) |\rho_L^{n+1} - \rho_K^{n+1}| |u_\sigma^{n+1}| |u_\sigma^{n+1} + u_{\sigma'}^{n+1}|.$$

Therefore, by the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} |\mathcal{R}_{2,1}^{(m)}| &\leq \frac{C_\varphi}{4} h^{1/2} \sum_{n=0}^{N-1} \delta t \left[\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |u_\sigma^{n+1}| (\rho_L^{n+1} - \rho_K^{n+1})^2 \right]^{1/2} \\ &\quad \left[\sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=L \rightarrow K, K=[\overrightarrow{\sigma\sigma'}]}} (|D_\sigma| + |D_{\sigma'}|) |u_\sigma^{n+1}| (u_\sigma^{n+1} + u_{\sigma'}^{n+1})^2 \right]^{1/2}. \end{aligned} \quad (31)$$

Since the ratio of the size of two neighbouring meshes is bounded by the regularity assumption on the mesh (Item (ii) of Definition 3.5), we get from the estimate (30) on the solution:

$$|\mathcal{R}_{2,1}^{(m)}| \leq C h^{1/2} \|u^{(m)}\|_{L^3(\Omega \times (0,T))}^{3/2}, \quad (32)$$

where the real number C is independent of m and therefore $\mathcal{R}_{2,1}^{(m)}$ tends to zero when m tends to $+\infty$. For $\mathcal{R}_{2,2}^{(m)}$, we have, thanks to the estimate (22):

$$|\mathcal{R}_{2,2}^{(m)}| \leq C_\varphi h^2 \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} \rho_K^{n+1} \frac{1}{h_K} (u_\sigma^{n+1} - u_{\sigma'}^{n+1})^2 \leq C \frac{h^2}{\nu^{(m)}} \|\rho^{(m)}\|_{L^\infty(\Omega \times (0,T))},$$

where C does not depend on m ; therefore, this term also tends to zero when m tends to $+\infty$, since, by assumption, $h^2/\nu^{(m)}$ tends to zero.

We turn to the term $T_3^{(m)}$:

$$T_3^{(m)} = - \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} |K| p_K^{n+1} \frac{\varphi_{\sigma'}^n - \varphi_{\sigma}^n}{h_K} = - \int_0^T \int_{\Omega} p^{(m)} \bar{\partial}_x \varphi_{\mathcal{E}} \, dx \, dt,$$

and therefore,

$$\lim_{m \rightarrow +\infty} T_3^{(m)} = - \int_0^T \int_{\Omega} \bar{p} \bar{u} \, \partial_x \varphi \, dx \, dt.$$

Let us finally study $T_4^{(m)}$. Reordering the sums, we get:

$$T_4^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \frac{\nu^{(m)}}{h_K} (u_{\sigma}^{n+1} - u_{\sigma'}^{n+1}) (\varphi_{\sigma}^n - \varphi_{\sigma'}^n).$$

The Cauchy-Schwarz inequality yields:

$$|T_4^{(m)}| \leq \left[\sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \frac{\nu^{(m)}}{h_K} (u_{\sigma}^{n+1} - u_{\sigma'}^{n+1})^2 \right]^{1/2} \left[\sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \frac{\nu^{(m)}}{h_K} (\varphi_{\sigma}^n - \varphi_{\sigma'}^n)^2 \right]^{1/2},$$

and thus, in view of the estimate (22), this term tends to zero when $\nu^{(m)}$ tends to zero.

Conclusion – Gathering the limits of all the terms of the mass and momentum balance equation concludes the proof. \square

Remark 3.8 (Sharper bounds and convergence assumptions).

The convergence properties and bounds assumed for the solution have been chosen so as to match what may be observed in practice. However, examining the proof of this theorem, we observe that what we really need is that the sequences $\rho^{(m)} u^{(m)}$, $\rho^{(m)} (u^{(m)})^2$, $p^{(m)} u^{(m)}$ converge in the distribution sense to $\bar{\rho} \bar{u}$, $\bar{\rho} \bar{u}^2$ and $\bar{p} \bar{u}$ respectively, that $(\rho^{(m)})^{\gamma}$ converge a.e. to $\bar{\rho}^{\gamma}$, and that the sequence $(u^{(m)})_{m \in \mathbb{N}}$ be bounded in $L^3(\Omega \times (0, T))$. The required second assumption for $(\nu^{(m)})_{m \in \mathbb{N}}$ is in fact:

$$\lim_{m \rightarrow +\infty} \frac{(h^{(m)})^2}{\nu^{(m)}} \|\rho^{(m)}\|_{L^{\infty}(\Omega \times (0, T))} = 0,$$

and may be verified, for instance supposing a relation between $\delta t^{(m)}$ and $h^{(m)}$ and invoking inverse inequalities, with milder estimates on $(\rho^{(m)})_{m \in \mathbb{N}}$. Finally, the bound of $(1/\rho^{(m)})_{m \in \mathbb{N}}$ in $L^{\infty}(\Omega \times (0, T))$ (which, loosely speaking, means that the appearance of void is excluded) is needed to obtain the weak-BV estimate:

$$\lim_{m \rightarrow +\infty} h^{(m)} \sum_{n=0}^N \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |u_{\sigma}^n| (\rho_K^n - \rho_L^n)^2 = 0 \quad (33)$$

from the "weighted weak-BV estimate" (22):

$$\lim_{m \rightarrow +\infty} h^{(m)} \sum_{n=0}^N \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} (\rho_{\sigma, \gamma}^n)^{\gamma-2} |u_{\sigma}^n| (\rho_K^n - \rho_L^n)^2 = 0,$$

where we recall that $\rho_{\sigma, \gamma}^n$ is equal to either ρ_K^n or ρ_L^n . This assumption is thus useless for $\gamma \leq 2$. For $\gamma > 2$, in the case of a non-vanishing viscosity, Equation (33) may perhaps be derived by using the density itself as test function in the discrete mass balance equation, and invoking a control of the divergence of the velocity (from the momentum balance diffusion term), see [10, Proposition 5.5] for such a computation in the steady case.

Remark 3.9 (Less sharp bounds and more general meshes).

The assumption that the ratio of the size of two neighbouring meshes is bounded, *i.e.* Assumption (ii) of

Definition 3.5, is only used to derive (32) from (31). If we now suppose that the sequence of approximate solutions is uniformly bounded, we may replace (32) by

$$|\mathcal{R}_{2,1}^{(m)}| \leq C h^{1/2} \|u^{(m)}\|_{L^\infty(\Omega \times (0,T))}^{3/2},$$

and Assumption (ii) is useless.

We now turn to the entropy condition. Let us first recall that $\mathcal{S} = \rho E_k + \mathcal{H}(\rho)$ is an entropy of the continuous problem (1), in the sense that if we sum the formal kinetic energy (14) and elastic potential balance (19), we get:

$$\partial_t \mathcal{S} + \operatorname{div}((\mathcal{S} + p) \mathbf{u}) = 0. \quad (34)$$

In fact, in order to avoid to invoke unrealistic regularity assumption, such a computation should be done on regularized equations (obtained by adding diffusion perturbation terms), and, when making these regularization terms tend to zero, positive measures appear at the left-hand-side of (34), so that we get in the distribution sense:

$$\partial_t \mathcal{S} + \operatorname{div}((\mathcal{S} + p) \mathbf{u}) \leq 0. \quad (35)$$

An entropy solution to (1) is thus required to satisfy, for any $\varphi \in C_c^\infty(\Omega \times [0, T])$, $\varphi \geq 0$:

$$\int_0^T \int_\Omega [\mathcal{S} \partial_t \varphi + (\mathcal{S} + p) \mathbf{u} \cdot \nabla \varphi] \, d\mathbf{x} \, dt + \int_\Omega \mathcal{S}_0(\mathbf{x}) \varphi(\mathbf{x}, 0) \, d\mathbf{x} \geq 0, \quad (36)$$

where $\mathcal{S}_0 = \frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \mathcal{H}(\rho_0)$.

Theorem 3.10 (Entropy consistency). *Under the assumptions of Theorem 3.7, $(\bar{\rho}, \bar{p}, \bar{u})$ satisfies the entropy condition (36).*

Proof. Let $\varphi \in C_c^\infty(\Omega \times [0, T])$, $\varphi \geq 0$. Using the notations introduced in Definition 3.6, we multiply the kinetic balance equation (15) by φ_σ^n , and the elastic potential balance (20) by φ_K^n , sum over the faces and cells respectively and over the time steps, to get:

$$\sum_{\mathcal{E} \in \mathcal{E}_{\text{int}}} T_\sigma^{n+1} \varphi_\sigma^n + \sum_{K \in \mathcal{M}} T_K^{n+1} \varphi_K^n = - \sum_{\mathcal{E} \in \mathcal{E}_{\text{int}}} R_\sigma^{n+1} \varphi_\sigma^n - \sum_{K \in \mathcal{M}} R_K^{n+1} \varphi_K^n, \quad (37)$$

where, for $\sigma = \overrightarrow{K|L}$, $K = [\overrightarrow{\sigma'}\sigma]$ and $L = [\overrightarrow{\sigma\sigma'}]$:

$$T_\sigma^{n+1} = \frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[\rho_{D_\sigma}^{n+1} (u_\sigma^{n+1})^2 - \rho_{D_\sigma}^n (u_\sigma^n)^2 \right] + \frac{1}{2} F_L^{n+1} u_\sigma^{n+1} u_{\sigma'}^{n+1} - \frac{1}{2} F_K^{n+1} u_\sigma^{n+1} u_{\sigma'}^{n+1} + (p_L^{n+1} - p_K^{n+1}) u_\sigma^{n+1},$$

for $K = [\overrightarrow{\sigma\sigma'}]$:

$$T_K^{n+1} = \frac{|K|}{\delta t} \left[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n) \right] + F_\sigma^{n+1} \mathcal{H}(\rho_\sigma^{n+1}) - F_{\sigma'}^{n+1} \mathcal{H}(\rho_{\sigma'}^{n+1}) + p_K^{n+1} (u_\sigma^{n+1} - u_{\sigma'}^{n+1}),$$

and the quantities R_σ^{n+1} and R_K^{n+1} are given by (the one-dimensional version of) Equation (16) and (21) respectively.

For the passage to the limit in this equation, we essentially refer to the study performed in Part II of the present paper [20, Proof of Theorem 3.4]. Indeed, the entropy inequality for the barotropic model is the same as the total energy balance for non-barotropic flows (up to the change from an inequality to an equality), and the passage to the limit in this latter equation, with the same discretization as here, is detailed in [20].

The treatment of the terms at the left-hand side of (37) follows similar arguments as in [20], and we thus skip it here. Since we only seek an inequality, the non-negative part of the remainder terms, *i.e.* the first part in R_σ^{n+1} and the whole term R_K^{n+1} , poses no problem, and we only have to study the second part of R_σ^{n+1} , which reads:

$$(R_{\text{diff}})_{\sigma}^{n+1} = \left[\sum_{K=[\sigma\sigma']} \frac{\nu}{h_K} (u_\sigma^{n+1} - u_{\sigma'}^{n+1}) \right] u_\sigma^{n+1}.$$

For $0 \leq n \leq N - 1$ and $K \in \mathcal{M}$, $K = [\sigma\sigma']$, let us define the quantity S_K^{n+1} by:

$$S_K^{n+1} = \frac{\nu}{h_K} (u_\sigma^{n+1} - u_{\sigma'}^{n+1})^2.$$

We have $S_K \geq 0$, and we prove in [20, proof of Theorem 3.4] that the difference $\mathcal{R}^{(m)}$ between the discrete functions associated to $((R_{\text{diff}})_\sigma^{n+1})_{\sigma \in \mathcal{E}_{\text{int}}, 0 \leq n \leq N-1}$ and $(S_K^{n+1})_{K \in \mathcal{M}, 0 \leq n \leq N-1}$ tends to zero in the distribution sense as soon as the diffusion coefficient tends to zero, *i.e.*:

$$\left| \sum_{n=0}^{N-1} \delta t \left[\sum_{\sigma \in \mathcal{E}_{\text{int}}} \varphi_\sigma^n (R_{\text{diff}})_\sigma^{n+1} - \sum_{K \in \mathcal{M}} \varphi_K^n S_K^{n+1} \right] \right| \leq C (\nu^{(m)})^{1/2}, \quad (38)$$

where C only depends on φ and on bounds on the solution either assumed or given by (22). This concludes the proof. \square

4. A PRESSURE CORRECTION SCHEME

4.1. The scheme

The implicit scheme which we studied in the previous section is easy to write, but difficult to implement in practice, because it yields at the algebraic level a large nonlinear system. Pressure correction methods are based on the idea that one may compute the velocity and the pressure in a sequential way, thus yielding a more practical scheme. More precisely, as shown in the algorithm given below, the velocity is predicted by solving the momentum balance equation with a known pressure. This latter is obtained from the beginning-of-step pressure through “renormalization” step, in order to be able to perform the stability analysis (stability of the scheme and satisfaction of the entropy condition). Then, the velocity is corrected and the other variables are advanced in time. As in the case of the implicit scheme, we can derive a discrete kinetic energy balance provided that the mass balance over the dual cells (7) holds; since the mass balance is not yet solved when performing the prediction step, this leads us to perform a time shift of the density at this stage.

The algorithm reads, for $0 \leq n \leq N - 1$:

Renormalization step – Solve for \tilde{p}^{n+1} :

$$\forall K \in \mathcal{M}, \quad \sum_{\sigma=K|L} \frac{1}{\rho_{D_\sigma}^n} \frac{|\sigma|^2}{|D_\sigma|} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) = \sum_{\sigma=K|L} \frac{1}{(\rho_{D_\sigma}^n \rho_{D_\sigma}^{n-1})^{1/2}} \frac{|\sigma|^2}{|D_\sigma|} (p_K^n - p_L^n). \quad (39a)$$

Prediction step – Solve for $\tilde{\mathbf{u}}^{n+1}$:

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}_S^{(i)}$,

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^n \tilde{u}_{\sigma,i}^{n+1} - \rho_{D_\sigma}^{n-1} u_{\sigma,i}^n) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^n \tilde{u}_{\epsilon,i}^{n+1} - |D_\sigma| (\Delta^M \tilde{\mathbf{u}})_{\sigma,i}^{n+1} + |D_\sigma| (\nabla \tilde{p})_{\sigma,i}^{n+1} = 0. \quad (39b)$$

Correction step – Solve for ρ^{n+1} , p^{n+1} and \mathbf{u}^{n+1} :

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}_S^{(i)}$,

$$\frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}) + |D_\sigma| [(\nabla p)_{\sigma,i}^{n+1} - (\nabla \tilde{p})_{\sigma,i}^{n+1}] = 0, \quad (39c)$$

$$\forall K \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} = 0 \text{ with } F_{K,\sigma}^{n+1} = |\sigma| \rho_\sigma^{n+1} u_{K,\sigma}^{n+1}, \quad (39d)$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\rho_K^{n+1})^\gamma. \quad (39e)$$

Recall that the notation ρ_σ^{n+1} in (39d) stands for the upwind choice of ρ defined by (5), while $\rho_{D_\sigma}^n$ in (39c) is the convex combination of ρ_K^n and ρ_L^n defined by (6).

The initialization of the scheme is performed as follows. First, ρ^{-1} and \mathbf{u}^0 are given by the average of the initial conditions ρ_0 and \mathbf{u}_0 on the primal and dual cells respectively:

$$\begin{aligned} \forall K \in \mathcal{M}, \quad \rho_K^{-1} &= \frac{1}{|K|} \int_K \rho_0(\mathbf{x}) \, d\mathbf{x}, \\ \text{for } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_S^{(i)}, \quad u_{\sigma,i}^0 &= \frac{1}{|D_\sigma|} \int_{D_\sigma} (\mathbf{u}_0(\mathbf{x}))_i \, d\mathbf{x}. \end{aligned} \quad (40)$$

Then, we compute ρ^0 by solving the mass balance equation (39d). Finally, the initial pressure p^0 is computed from the initial density ρ^0 by the equation of state: $\forall K \in \mathcal{M}$, $p_K^0 = (\rho_K^0)^\gamma$. This procedure allows to perform the first prediction step with $(\rho_{D_\sigma}^{-1})_{\sigma \in \mathcal{E}}$, $(\rho_{D_\sigma}^0)_{\sigma \in \mathcal{E}}$ and the dual mass fluxes satisfying the mass balance.

4.2. Estimates

Lemma 4.1 (Discrete kinetic energy balance, pressure correction scheme).

A solution to the system (39) satisfies the following equality, for $1 \leq i \leq d$, $\sigma \in \mathcal{E}_S^{(i)}$ and $0 \leq n \leq N-1$:

$$\begin{aligned} \frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[\rho_{D_\sigma}^n (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^{n-1} (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\epsilon = D_\sigma | D_{\sigma'}} F_{\sigma,\epsilon}^n \tilde{u}_{\sigma,i}^{n+1} \tilde{u}_{\sigma',i}^{n+1} \\ + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1} = -R_{\sigma,i}^{n+1} - P_{\sigma,i}^{n+1}, \end{aligned} \quad (41)$$

where

$$\begin{aligned} R_{\sigma,i}^{n+1} &= \frac{1}{2} \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^{n-1} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 + \left[\sum_{\epsilon = D_\sigma | D_{\sigma'}} \nu h_\epsilon^{d-2} (\tilde{u}_{\sigma,i}^{n+1} - \tilde{u}_{\sigma',i}^{n+1}) \right] \tilde{u}_{\sigma,i}^{n+1}, \\ P_{\sigma,i}^{n+1} &= \frac{|D_\sigma| \delta t}{\rho_{D_\sigma}^n} \left[((\nabla p)_{\sigma,i}^{n+1})^2 - ((\nabla \tilde{p})_{\sigma,i}^{n+1})^2 \right]. \end{aligned} \quad (42)$$

Proof. Let us multiply the velocity prediction equation (39b) by the corresponding velocity unknown $\tilde{u}_{\sigma,i}^{n+1}$, and use the equality (56) of Lemma A.2, on the dual mesh and with $P = D_\sigma$. We obtain:

$$\begin{aligned} \frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[\rho_{D_\sigma}^n (\tilde{u}_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^{n-1} (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\epsilon = D_\sigma | D_{\sigma'}} F_{\sigma,\epsilon}^n \tilde{u}_{\sigma,i}^{n+1} \tilde{u}_{\sigma',i}^{n+1} + \frac{1}{2} \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^{n-1} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 \\ - |D_\sigma| (\Delta^{\mathcal{M}} \tilde{\mathbf{u}})_{\sigma,i}^{n+1} \tilde{u}_{\sigma,i}^{n+1} + |D_\sigma| (\nabla \tilde{p})_{\sigma,i}^{n+1} \tilde{u}_{\sigma,i}^{n+1} = 0. \end{aligned} \quad (43)$$

Dividing the velocity correction equation (39c) by $(\frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n)^{\frac{1}{2}}$, we obtain:

$$\left[\frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n \right]^{1/2} u_{\sigma,i}^{n+1} + \left[\frac{|D_\sigma| \delta t}{\rho_{D_\sigma}^n} \right]^{1/2} (\nabla p)_{\sigma,i}^{n+1} = \left[\frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n \right]^{1/2} \tilde{u}_{\sigma,i}^{n+1} + \left[\frac{|D_\sigma| \delta t}{\rho_{D_\sigma}^n} \right]^{1/2} (\nabla \tilde{p})_{\sigma,i}^{n+1}.$$

Squaring this relation and summing it with (43) yields the result, using the definition (9) of $(\Delta^{\mathcal{M}} \tilde{\mathbf{u}})^{n+1}$. \square

The discrete potential balance is again valid for the pressure correction algorithm, thanks to the fact that the mass balance (39d) is satisfied. The proof is identical to that of Lemma 3.2 given for the implicit scheme.

Lemma 4.2 (Discrete potential balance). A solution to the system (39) satisfies the discrete potential balance (20), with R_K^{n+1} defined by (21).

Proposition 4.3 (Global discrete entropy inequality, existence of a solution).

There exists a solution $(u^n)_{0 \leq n \leq N}$ and $(\rho^n)_{0 \leq n \leq N}$ to the scheme, the density satisfies $\rho > 0$ and, for $1 \leq n \leq N$, the following inequality holds:

$$\frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} |D_\sigma| \rho_{D_\sigma}^{n-1} (u_{\sigma,i}^n)^2 + \sum_{K \in \mathcal{M}} |K| \mathcal{H}(\rho_K^n) + \mathcal{R}^n \leq C, \quad (44)$$

where C only depends on the initial conditions and on the density field ρ^0 computed at the initialization of the algorithm. The remainder term \mathcal{R} is non-negative, and gathers some estimates of the space and time translates of the unknowns:

$$\begin{aligned} \mathcal{R}^n = & \sum_{i=1}^d \sum_{k=1}^n \left[\frac{1}{2} \sum_{\sigma \in \mathcal{E}_S^{(i)}} |D_\sigma| \rho_{D_\sigma}^{k-2} (\tilde{u}_{\sigma,i}^k - u_{\sigma,i}^{k-1})^2 + \delta t \sum_{\epsilon = D_\sigma | D_{\sigma'} \in \tilde{\mathcal{E}}_S^{(i)}} \nu h_\sigma^{d-2} (\tilde{u}_{\sigma,i}^k - \tilde{u}_{\sigma',i}^k)^2 \right] \\ & + \frac{\gamma}{2} \sum_{k=1}^n \delta t \sum_{\sigma = K | L \in \mathcal{E}_{\text{int}}} |\sigma| (\rho_{\sigma,\gamma}^k)^{\gamma-2} |u_{K,\sigma}^k| |\rho_K^k - \rho_L^k|^2 + \delta t^2 \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{|D_\sigma|}{\rho_{D_\sigma}^{n-1}} |(\nabla p)_\sigma^n|^2, \end{aligned}$$

with $\rho_{\sigma,\gamma}^k$ equal to either ρ_K^k or ρ_L^k and such that $(\rho_{\sigma,\gamma}^k)^{\gamma-2} = \min((\rho_K^k)^{\gamma-2}, (\rho_L^k)^{\gamma-2})$.

Proof. The essential arguments of the proof of this proposition are given in [11, Theorem 3.8], and we only briefly recall here how to obtain this estimate, for the sake of completeness. As in the implicit case, we sum the kinetic energy balance equation (41) over the faces, and the elastic potential balance (20) (which is the same, and obtained by the same computation as in the implicit case) over the cells, and finally sum the two obtained relations. We obtain a "local in time" version of Equation (44), which reads:

$$\mathcal{T}^{n+1} - \mathcal{T}^n + \mathcal{R}^{n+1} + \mathcal{P}^{n+1} = 0, \quad (45)$$

where:

$$\mathcal{T}^{n+1} = \sum_{K \in \mathcal{M}} |K| \mathcal{H}(\rho_K^{n+1}) + \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_S^{(i)}} |D_\sigma| \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1})^2,$$

and:

$$\mathcal{R}^{n+1} = \sum_{1 \leq i \leq d} \sum_{\sigma \in \mathcal{E}_S^{(i)}} R_{\sigma,i}^{n+1}, \quad \mathcal{P}^{n+1} = \sum_{1 \leq i \leq d} \sum_{\sigma \in \mathcal{E}_S^{(i)} \cap \mathcal{E}_{\text{int}}} P_{\sigma,i}^{n+1},$$

with $R_{\sigma,i}^{n+1}$ and $P_{\sigma,i}^{n+1}$ given by Equation (42). The term \mathcal{P}^{n+1} thus reads:

$$\mathcal{P}^{n+1} = \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{|D_\sigma| \delta t^2}{\rho_{D_\sigma}^n} \left[|(\nabla p)_\sigma^{n+1}|^2 - |(\nabla \tilde{p})_\sigma^{n+1}|^2 \right]$$

Before summing over the time steps, we need to transform \mathcal{P}^{n+1} to get a difference between a same expression written at two consecutive time levels, which is possible thanks to the renormalization step. Indeed, multiplying (39a) by \tilde{p}_K^{n+1} and summing over the cells yields, after a discrete integration by parts and use of the identity $2(a-b)a = a^2 + (a-b)^2 - b^2$:

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{|D_\sigma| \delta t^2}{\rho_{D_\sigma}^n} |(\nabla \tilde{p})_\sigma^{n+1}|^2 \leq \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{|D_\sigma| \delta t^2}{\rho_{D_\sigma}^{n-1}} |(\nabla p)_\sigma^n|^2.$$

Summing this relation with (45) and summing over the time steps yields the estimate (44) with:

$$C = \sum_{K \in \mathcal{M}} |K| \mathcal{H}(\rho_K^0) + \frac{1}{2} \sum_{1 \leq i \leq d} \sum_{\sigma \in \mathcal{E}_S^{(i)}} |D_\sigma| \rho_{D_\sigma}^{-1} (\mathbf{u}_{\sigma,i}^0)^2 + \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{|D_\sigma| \delta t^2}{\rho_{D_\sigma}^{-1}} |(\nabla p)_\sigma^0|^2.$$

□

Remark 4.4 (Regularity assumptions for the initial conditions). For a given mesh, the quantity denoted above by C is bounded whenever ρ_0 is positive and belongs to $L^1(\Omega)$ and \mathbf{u}_0 belongs to $L^1(\Omega)^d$. When dealing with a sequence of discretizations to pass to the limit in the scheme, we need to assume that C is controlled independently of the mesh and time step, which necessitates (i) that the initial kinetic energy is bounded, (ii) that $\mathcal{H}(\rho_K^0)$ is bounded in $L^1(\Omega)$, and (iii) that the last term involving the discrete pressure gradient does not blow-up.

Assumption (ii) (and, of course, (i)) may be obtained by supposing that both \mathbf{u}_0 and ρ_0 belongs to $L^\infty(\Omega)$ and $L^\infty(\Omega)^d$ respectively and that $\delta t/h$ is bounded (possibly by a number much larger than 1); indeed, ρ^0 is then

obtained in this case by a single time step of a (discrete) transport equation with a velocity field the divergence of which is controlled by $1/h$, and so ρ^0 is controlled in $L^\infty(\Omega)$. Assumption (iii) may then be inferred by the same assumption on the ratio $\delta t/h$, together with the hypothesis that the data ρ_0 (and so ρ^{-1}) is bounded away from zero. Indeed, since ρ^0 is bounded, so is p^0 and we get:

$$\sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{|D_\sigma| \delta t^2}{\rho_{D_\sigma}^{-1}} |(\nabla p)_\sigma^0|^2 = \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{\delta t^2 |\sigma|^2}{\rho_{D_\sigma}^{-1} |D_\sigma|} (p_K^0 - p_L^0)^2 \leq C \left\| \frac{1}{\rho^{-1}} \right\|_{L^\infty(\Omega)} \|p^0\|_{L^\infty(\Omega)}^2 \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{h^2 |\sigma|^2}{|D_\sigma|}$$

and the last sum is bounded. We shall work under these assumptions for the passage to the limit in the scheme.

4.3. Passing to the limit in the scheme

As for the implicit scheme, we show in this section, in the one dimensional case, that if a sequence of solutions is controlled in suitable norms and converges to a limit, this latter necessarily satisfies a (part of the) weak formulation of the continuous problem.

Using the notations (26)-(28) already introduced for the implicit scheme, the pressure correction scheme reads, in one space dimension:

Initialization – Compute ρ^{-1} , u^0 , solve for ρ^0 and compute p^0 :

$$\begin{aligned} \forall K \in \mathcal{M}, \quad \rho_K^{-1} &= \frac{1}{|K|} \int_K \rho_0(x) \, dx, \\ \forall \sigma \in \mathcal{E}_{\text{int}}, \quad u_\sigma^0 &= \frac{1}{|D_\sigma|} \int_{D_\sigma} u_0(x) \, dx, \\ \forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \quad \frac{|K|}{\delta t} (\rho_K^0 - \rho_K^{-1}) + F_{\sigma'}^0 - F_\sigma^0 &= 0, \\ \forall K \in \mathcal{M}, \quad p_K^0 &= (\rho_K^0)^\gamma. \end{aligned} \tag{46a}$$

Pressure renormalization step – Solve for \tilde{p}^{n+1} :

$$\forall K \in \mathcal{M}, \quad \sum_{\sigma=K|L} \frac{1}{\rho_{D_\sigma}^n} \frac{|\sigma|^2}{|D_\sigma|} (\tilde{p}_K^{n+1} - \tilde{p}_L^{n+1}) = \sum_{\sigma=K|L} \frac{1}{(\rho_{D_\sigma}^n \rho_{D_\sigma}^{n-1})^{1/2}} \frac{|\sigma|^2}{|D_\sigma|} (p_K^n - p_L^n). \tag{46b}$$

Prediction step – Solve for \tilde{u}^{n+1} :

$$\begin{aligned} \forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \\ \frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^n u_\sigma^{n+1} - \rho_{D_\sigma}^{n-1} u_\sigma^n) + F_L^n \tilde{u}_L^{n+1} - F_K^n \tilde{u}_K^{n+1} \\ - |D_\sigma| (\Delta_{\mathcal{M}} \tilde{u})_\sigma^{n+1} + \tilde{p}_L^{n+1} - \tilde{p}_K^{n+1} = 0, \end{aligned} \tag{46c}$$

Correction step – Solve for ρ^{n+1} , p^{n+1} and u^{n+1} :

$$\begin{aligned} \forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}, \\ \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^n (u_\sigma^{n+1} - \tilde{u}_\sigma^{n+1}) + (p_L^{n+1} - p_K^{n+1}) - (\tilde{p}_L^{n+1} - \tilde{p}_K^{n+1}) = 0, \end{aligned} \tag{46d}$$

$$\begin{aligned} \forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \\ \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^{n+1} - F_\sigma^{n+1} = 0, \end{aligned} \tag{46e}$$

$$\forall K \in \mathcal{M}, \quad p_K^{n+1} = (\rho_K^{n+1})^\gamma. \tag{46f}$$

Definition 4.5 (Regular sequence of discretizations, pressure correction case).

We define a regular sequence of discretizations $(\mathcal{M}^{(m)}, \delta t^{(m)}, \nu^{(m)})_{m \in \mathbb{N}}$ as a sequence of meshes, time steps and

numerical diffusion coefficients satisfying the assumptions (i)-(iii) of Definition 3.5 and the additional following condition:

(iv) there exists $C > 0$ such that:

$$\forall m \in \mathbb{N}, \quad \frac{\delta t^{(m)}}{\underline{h}^{(m)}} \leq C,$$

$$\text{where } \underline{h}^{(m)} = \min_{\sigma \in \mathcal{E}_{\text{int}}^{(m)}} \frac{|D_\sigma|}{|\sigma|}.$$

Theorem 4.6 (Consistency of the pressure correction scheme).

Let Ω be an open bounded interval of \mathbb{R} . We suppose that $\rho_0 \in L^\infty(\Omega)$, $1/\rho_0 \in L^\infty(\Omega)$ and $u_0 \in L^\infty(\Omega)$. Let $(\mathcal{M}^{(m)}, \delta t^{(m)}, \nu^{(m)})_{m \in \mathbb{N}}$ be a regular sequence of discretizations in the sense of Definition 4.5, and let $(\rho^{(m)}, p^{(m)}, u^{(m)}, \tilde{u}^{(m)})_{m \in \mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence converges in $L^p(\Omega \times (0, T))^4$, for $1 \leq p < \infty$, to $(\bar{\rho}, \bar{p}, \bar{u}, \bar{\tilde{u}}) \in L^\infty(\Omega \times (0, T))^4$. We suppose in addition that both sequences $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(1/\rho^{(m)})_{m \in \mathbb{N}}$ are uniformly bounded in $L^\infty(\Omega \times (0, T))$.

Then $\bar{u} = \bar{\tilde{u}}$ and the triplet $(\bar{\rho}, \bar{p}, \bar{u})$ satisfies the system (29).

Remark 4.7 (On the "non appearance of void assumption").

The assumption that $(1/\rho^{(m)})_{m \in \mathbb{N}}$ is bounded in $L^\infty(\Omega \times (0, T))$ is used twice in the proof of this theorem. We use it for the first time to obtain $\bar{u} = \bar{\tilde{u}}$. Here, the hypothesis may be circumvented by replacing this conclusion by $\bar{\rho}\bar{u} = \bar{\rho}\bar{\tilde{u}}$ (or, in other words, $\bar{u} = \bar{\tilde{u}}$ wherever $\bar{\rho} \neq 0$), which is easily obtained from Inequality (47) below. The second time is, as for the implicit case, to obtain an "non-weighted" estimate of the density space translates for $\gamma \geq 2$, and we do not repeat here the above discussion on this issue.

Proof. Let $m \in \mathbb{N}$ be given. Dropping for short the superscript (m) , the estimate of Proposition 4.3 yields:

$$\sum_{k=1}^n \delta t \sum_{\sigma \in \mathcal{E}_{\text{int}}} |D_\sigma| \rho_{D_\sigma}^{k-1} (\tilde{u}_\sigma^k - u_\sigma^{k-1})^2 \leq C \delta t, \quad (47)$$

where, by the assumption on the initial data, the real number C is independent of m (see Remark 4.4). We thus get:

$$\|\tilde{u}^{(m)} - u^{(m)}(\cdot, \cdot - \delta t)\|_{L^2(\Omega \times (0, T))}^2 \leq C \delta t^{(m)} \left\| \frac{1}{\rho^{(m)}} \right\|_{L^\infty(\Omega \times (0, T))}.$$

Letting m tend to $+\infty$ in this equation yields $\bar{u} = \bar{\tilde{u}}$.

The passage to the limit in the mass balance equation is the same as in the implicit case, and we only need to address here the momentum balance equation. Let $\varphi \in C_c^\infty(\Omega \times [0, T])$, and let its interpolate $\varphi_\mathcal{E}$ and its discrete derivatives be defined by Definition 3.6. Summing the velocity prediction and correction equations, multiplying the result by $\delta t \varphi_\sigma^n$ and then summing over the faces and time steps, we get $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} = 0$, with:

$$\begin{aligned} T_1^{(m)} &= \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}_{\text{int}}} |D_\sigma| [\rho_{D_\sigma}^n u_\sigma^{n+1} - \rho_{D_\sigma}^{n-1} u_\sigma^n] \varphi_\sigma^n, \\ T_2^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} [F_L^n \tilde{u}_L^{n+1} - F_K^n \tilde{u}_K^{n+1}] \varphi_\sigma^n, \\ T_3^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}} (p_L^{n+1} - p_K^{n+1}) \varphi_\sigma^n, \\ T_4^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\text{int}}} \left[\sum_{K=[\sigma\sigma']} \frac{\nu}{h_K} (\tilde{u}_\sigma^{n+1} - \tilde{u}_{\sigma'}^{n+1}) \right] \varphi_\sigma^n. \end{aligned}$$

The passage to the limit in $T_1^{(m)}$ is the same as for the implicit scheme, just noting that the sequence $(\rho^{(m)}(\cdot, \cdot - \delta t))_{m \in \mathbb{N}}$ converges to $\bar{\rho}$ as $(\rho^{(m)})_{m \in \mathbb{N}}$.

Let us now turn to $T_2^{(m)}$. By a computation similar to the implicit case, we get:

$$T_2^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} (\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n) (\tilde{u}_\sigma^{n+1} + \tilde{u}_{\sigma'}^{n+1}) (\varphi_{\sigma'}^n - \varphi_\sigma^n),$$

which we write $T_2^{(m)} = \mathcal{T}_2^{(m)} + \mathcal{R}_2^{(m)}$ with:

$$\mathcal{T}_2^{(m)} = -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \rho_K^n \left[u_\sigma^n \tilde{u}_\sigma^{n+1} + u_{\sigma'}^n \tilde{u}_{\sigma'}^{n+1} \right] (\varphi_{\sigma'}^n - \varphi_\sigma^n).$$

This term reads:

$$\mathcal{T}_2^{(m)} = - \int_0^T \int_\Omega \rho^{(m)}(\cdot, \cdot - \delta t) u^{(m)}(\cdot, \cdot - \delta t) \tilde{u}^{(m)} \bar{\partial}_x \varphi_\varepsilon \, dx \, dt,$$

and therefore,

$$\lim_{m \rightarrow +\infty} T_2^{(m)} = - \int_0^T \int_\Omega \bar{\rho} \bar{u}^2 \partial_x \varphi \, dx \, dt.$$

The remainder term $\mathcal{R}_2^{(m)}$ reads:

$$\mathcal{R}_2^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \left[(\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n) (\tilde{u}_\sigma^{n+1} + \tilde{u}_{\sigma'}^{n+1}) - 2\rho_K^n (u_\sigma^n \tilde{u}_\sigma^{n+1} + u_{\sigma'}^n \tilde{u}_{\sigma'}^{n+1}) \right] (\varphi_{\sigma'}^n - \varphi_\sigma^n).$$

Expanding the quantity $2\rho_K^n (u_\sigma^n \tilde{u}_\sigma^{n+1} + u_{\sigma'}^n \tilde{u}_{\sigma'}^{n+1})$ thanks to the identity $2(ab+cd) = (a+c)(b+d) + (a-c)(b-d)$, we get $\mathcal{R}_2^{(m)} = \mathcal{R}_{2,1}^{(m)} + \mathcal{R}_{2,2}^{(m)}$:

$$\begin{aligned} \mathcal{R}_{2,1}^{(m)} &= -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \left[\left((\rho_\sigma^n - \rho_K^n) u_\sigma^n + (\rho_{\sigma'}^n - \rho_K^n) u_{\sigma'}^n \right) \right. \\ &\quad \left. (\tilde{u}_\sigma^{n+1} + \tilde{u}_{\sigma'}^{n+1}) \right] (\varphi_{\sigma'}^n - \varphi_\sigma^n), \\ \mathcal{R}_{2,2}^{(m)} &= \frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \rho_K^n (u_\sigma^n - u_{\sigma'}^n) (\tilde{u}_\sigma^{n+1} - \tilde{u}_{\sigma'}^{n+1}) (\varphi_{\sigma'}^n - \varphi_\sigma^n). \end{aligned}$$

First we study $\mathcal{R}_{2,1}^{(m)}$. Thanks to the definition of the upwind approximation, reordering the sum by faces, we get:

$$\mathcal{R}_{2,1}^{(m)} = \frac{\varepsilon}{4} \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=L \rightarrow K, K=[\sigma\sigma']}} (\rho_L^n - \rho_K^n) u_\sigma^n (\tilde{u}_\sigma^{n+1} + \tilde{u}_{\sigma'}^{n+1}) (\varphi_\sigma^n - \varphi_{\sigma'}^n),$$

where we recall that the notation $\sigma = L \rightarrow K$ means that the face σ separates K and L and the flow goes from L to K , and where $\varepsilon = \pm 1$. Since $|\varphi_\sigma^n - \varphi_{\sigma'}^n| \leq C_\varphi |K| \leq C_\varphi (|D_\sigma| + |D_{\sigma'}|)$, we get:

$$|\mathcal{R}_{2,1}^{(m)}| \leq \frac{C_\varphi}{4} \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=L \rightarrow K, K=[\sigma\sigma']}} (|D_\sigma| + |D_{\sigma'}|) |\rho_L^n - \rho_K^n| |u_\sigma^n| |\tilde{u}_\sigma^{n+1} + \tilde{u}_{\sigma'}^{n+1}|.$$

Therefore, by the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} |\mathcal{R}_{2,1}^{(m)}| &\leq \frac{C_\varphi}{4} (h^{(m)})^{1/2} \left[\sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |u_\sigma^n| (\rho_L^n - \rho_K^n)^2 \right]^{1/2} \\ &\quad \left[\sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=L \rightarrow K, K=[\sigma\sigma']}} (|D_\sigma| + |D_{\sigma'}|) |u_\sigma^n| (\tilde{u}_\sigma^{n+1} + \tilde{u}_{\sigma'}^{n+1})^2 \right]^{1/2}. \end{aligned}$$

Since the ratio of the size of two neighbouring meshes is bounded by the regularity assumption on the mesh, we get from the estimate (44) on the solution:

$$|\mathcal{R}_{2,1}^{(m)}| \leq C (h^{(m)})^{1/2} \left[\|u^{(m)}\|_{L^2(\Omega \times (0,T))} + \|\tilde{u}^{(m)}\|_{L^4(\Omega \times (0,T))}^2 \right], \quad (48)$$

where C does not depend on m , and so $\mathcal{R}_{2,1}^{(m)}$ tends to zero when m tends to $+\infty$. For $\mathcal{R}_{2,2}^{(m)}$, we have, by the Cauchy-Schwarz inequality:

$$\begin{aligned} |\mathcal{R}_{2,2}^{(m)}| &\leq C_\varphi \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} |K| \rho_K^{n+1} |u_\sigma^n + u_{\sigma'}^n| (\tilde{u}_\sigma^{n+1} - \tilde{u}_{\sigma'}^{n+1}) \\ &\leq C_\varphi \frac{h^{(m)}}{(\nu^{(m)})^{1/2}} \|\rho^{(m)}\|_{L^\infty(\Omega \times (0,T))} \|u^{(m)}\|_{L^2(\Omega \times (0,T))} \left[\sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \frac{\nu^{(m)}}{h_K} (\tilde{u}_\sigma^{n+1} - \tilde{u}_{\sigma'}^{n+1})^2 \right]^{1/2}, \end{aligned}$$

and thus, thanks to the estimate (44):

$$|\mathcal{R}_{2,2}^{(m)}| \leq C \frac{h^{(m)}}{(\nu^{(m)})^{1/2}} \|\rho^{(m)}\|_{L^\infty(\Omega \times (0,T))} \|u^{(m)}\|_{L^2(\Omega \times (0,T))},$$

where C does not depend on m . Therefore, this term also tends to zero when m tends to $+\infty$.

Finally, the terms $T_3^{(m)}$ and $T_4^{(m)}$ are dealt with as in the implicit case. \square

Remark 4.8 (Less sharp bounds and more general meshes).

As in the implicit case, the assumption that the ratio of the size of two neighbouring meshes is bounded, *i.e.* Assumption (ii) of Definition 4.5, is only used for the remainder associated to the convection term in the momentum balance. It may be avoided if we suppose that the sequence of solution is uniformly bounded, replacing (48) by

$$|\mathcal{R}_{2,1}^{(m)}| \leq C (h^{(m)})^{1/2} \|u^{(m)}\|_{L^\infty(\Omega \times (0,T))}^{1/2} \|\tilde{u}^{(m)}\|_{L^\infty(\Omega \times (0,T))}.$$

For any piecewise constant function q on primal cells, we define its $L^1(0, T; \text{BV}(\Omega))$ norm by:

$$\|q\|_{\mathcal{T}, x, \text{BV}} = \sum_{n=0}^N \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |q_L^n - q_K^n|. \quad (49)$$

With this notation, we are now in position to state the following result.

Theorem 4.9 (Entropy consistency). *Under the assumptions of Theorem 4.6, we furthermore assume that the sequence $(p^{(m)})_{m \in \mathbb{N}}$ is uniformly bounded in the discrete $L^1(0, T; \text{BV}(\Omega))$ norm defined by (49). Then the limit $(\bar{\rho}, \bar{p}, \bar{u})$ satisfies the entropy condition (36).*

Proof. Let $\varphi \in C_c^\infty(\Omega \times [0, T])$, $\varphi \geq 0$. Again using the notations of Definition 3.6, we multiply the kinetic balance equation (41) by φ_σ^n , and the elastic potential balance (20) by φ_K^n , sum over the faces and cells respectively and over the time steps, to get:

$$\sum_{\mathcal{E} \in \mathcal{E}_{\text{int}}} T_\sigma^{n+1} \varphi_\sigma^n + \sum_{K \in \mathcal{M}} T_K^{n+1} \varphi_K^n = - \sum_{\mathcal{E} \in \mathcal{E}_{\text{int}}} R_\sigma^{n+1} \varphi_\sigma^n - \sum_{K \in \mathcal{M}} R_K^{n+1} \varphi_K^n - \sum_{\mathcal{E} \in \mathcal{E}_{\text{int}}} P_\sigma^{n+1} \varphi_\sigma^n, \quad (50)$$

where, for $\sigma = \overrightarrow{K|L}$, $K = [\overrightarrow{\sigma'}]$ and $L = [\overrightarrow{\sigma''}]$:

$$T_\sigma^{n+1} = \frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[\rho_{D_\sigma}^n (u_\sigma^{n+1})^2 - \rho_{D_\sigma}^{n-1} (u_\sigma^n)^2 \right] + \frac{1}{2} F_L^{n+1} \tilde{u}_\sigma^{n+1} \tilde{u}_{\sigma''}^{n+1} - \frac{1}{2} F_K^{n+1} \tilde{u}_\sigma^{n+1} \tilde{u}_{\sigma'}^{n+1} + (p_L^{n+1} - p_K^{n+1}) u_\sigma^{n+1},$$

for $K = [\overrightarrow{\sigma\sigma'}]$:

$$T_K^{n+1} = \frac{|K|}{\delta t} \left[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n) \right] + F_\sigma^{n+1} \mathcal{H}(\rho_\sigma^{n+1}) - F_{\sigma'}^{n+1} \mathcal{H}(\rho_{\sigma'}^{n+1}) + p_K^{n+1} (u_\sigma^{n+1} - u_{\sigma'}^{n+1}),$$

the quantities R_σ^{n+1} and P_σ^{n+1} are given by (the one-dimensional version of) Equation (42), and R_K^{n+1} is given by (the one-dimensional version of) Equation (21).

As in the implicit case, we refer to [20, Theorem 5.3] for the passage to the limit in the terms at the left-hand side of (50). The term associated to R_σ^{n+1} is dealt with as in the implicit case, and R_K^{n+1} is non-negative. The last difficulty lies in the control of the additional remainder term involving the pressure gradients, *i.e.* the last term of (50); we show in [20, Lemma 5.2] that this term may be written as the sum of a positive part and a quantity tending to zero. \square

APPENDIX A. SOME RESULTS ASSOCIATED TO FINITE VOLUME CONVECTION OPERATORS

We gather in this section some results concerning the finite volume discretization of the two convection operators which appear in the Navier–Stokes equations:

- the convection operator appearing in the mass balance, which reads, at the continuous level, $\rho \rightarrow \mathcal{C}(\rho) = \partial_t \rho + \operatorname{div}(\rho \mathbf{u})$, where \mathbf{u} stands for a given velocity field, which is not assumed to satisfy any divergence constraint,
- the convection operator appearing in the momentum and energy balances, which reads, in the continuous setting, $z \rightarrow \mathcal{C}_\rho(z) = \partial_t(\rho z) + \operatorname{div}(\rho z \mathbf{u})$, where ρ (resp. \mathbf{u}) stands for a given scalar (resp. vector) field; we wish to obtain some property of \mathcal{C}_ρ under the assumption that ρ and \mathbf{u} satisfy a mass balance equation, *i.e.* $\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0$.

Multiplying these operators by functions depending on the unknown is a often used technique to obtain convection operators acting over different variables, possibly with residual terms: one may think, for instance, to the theory of renormalized solutions or entropy solutions for the operator \mathcal{C} , or, in mechanics, to the derivation of the so-called kinetic energy transport identity for the operator \mathcal{C}_ρ , with z standing for a component of the velocity. The results provided in this section are the discrete analogs of these properties.

We begin with a property of \mathcal{C} , which, at the continuous level, may be formally obtained as follows. Let ψ be a regular function from $(0, +\infty)$ to \mathbb{R} ; then:

$$\psi'(\rho) \mathcal{C}(\rho) = \psi'(\rho) \partial_t(\rho) + \psi'(\rho) \mathbf{u} \cdot \nabla \rho + \psi'(\rho) \rho \operatorname{div} \mathbf{u} = \partial_t(\psi(\rho)) + \mathbf{u} \cdot \nabla \psi(\rho) + \rho \psi'(\rho) \operatorname{div} \mathbf{u},$$

so adding and subtracting $\psi(\rho) \operatorname{div} \mathbf{u}$ yields:

$$\psi'(\rho) \mathcal{C}(\rho) = \partial_t(\psi(\rho)) + \operatorname{div}(\psi(\rho) \mathbf{u}) + (\rho \psi'(\rho) - \psi(\rho)) \operatorname{div} \mathbf{u}. \quad (51)$$

This computation is of course completely formal and only valid for regular functions ρ and \mathbf{u} . The following lemma states a discrete analogue to (51), and its proof follows the formal computation which we just described.

Lemma A.1. *Let P be a polygonal (resp. polyhedral) bounded set of \mathbb{R}^2 (resp. \mathbb{R}^3), and let $\mathcal{E}(P)$ be the set of its edges (resp. faces). Let ψ be a continuously differentiable function defined over $(0, +\infty)$. Let $\rho_P^* > 0$, $\rho_P > 0$, $\delta t > 0$; consider three families $(\rho_\eta)_{\eta \in \mathcal{E}(P)} \subset \mathbb{R}_+ \setminus \{0\}$, $(V_\eta)_{\eta \in \mathcal{E}(P)} \subset \mathbb{R}$ and $(F_\eta)_{\eta \in \mathcal{E}(P)} \subset \mathbb{R}$ such that*

$$\forall \eta \in \mathcal{E}(P), \quad F_\eta = \rho_\eta V_\eta,$$

and define:

$$\begin{aligned} R_{P,\delta t} = & \left[\frac{|P|}{\delta t} (\rho_P - \rho_P^*) + \sum_{\eta \in \mathcal{E}(P)} F_\eta \right] \psi'(\rho_P) \\ & - \left[\frac{|P|}{\delta t} (\psi(\rho_P) - \psi(\rho_P^*)) + \sum_{\eta \in \mathcal{E}(P)} \psi(\rho_\eta) V_\eta + (\rho_P \psi'(\rho_P) - \psi(\rho_P)) \sum_{\eta \in \mathcal{E}(P)} V_\eta \right] \quad (52) \end{aligned}$$

Then

- (i) *If ψ is convex and $\rho_\eta = \rho_P$ whenever $V_\eta > 0$, then $R_{P,\delta t} \geq 0$.*
- (ii) *If ψ is twice continuously differentiable then*

$$R_{P,\delta t} = \frac{1}{2} \frac{|P|}{\delta t} \psi''(\bar{\rho}_P) (\rho_P - \rho_P^*)^2 - \frac{1}{2} \sum_{\eta \in \mathcal{E}(P)} V_\eta \psi''(\bar{\rho}_\eta) (\rho_\eta - \rho_P)^2,$$

with $\bar{\rho}_P \in [\min(\rho_P, \rho_P^*), \max(\rho_P, \rho_P^*)]$ and $\forall \eta \in \mathcal{E}(P)$, $\bar{\rho}_\eta \in [\min(\rho_\eta, \rho_P), \max(\rho_\eta, \rho_P)]$.

Proof. We have:

$$\begin{aligned} \left[\frac{|P|}{\delta t} (\rho_P - \rho_P^*) + \sum_{\eta \in \mathcal{E}(P)} F_\eta \right] \psi'(\rho_P) &= \frac{|P|}{\delta t} (\rho_P - \rho_P^*) \psi'(\rho_P) + \sum_{\eta \in \mathcal{E}(P)} \psi(\rho_\eta) V_\eta \\ &\quad + \sum_{\eta \in \mathcal{E}(P)} [\rho_\eta \psi'(\rho_P) - \psi(\rho_\eta)] V_\eta, \end{aligned}$$

so the remainder term $R_{P,\delta t}$ reads $R_{P,\delta t} = \frac{|P|}{\delta t} r_P + \sum_{\eta \in \mathcal{E}(P)} V_\eta r_\eta$, with:

$$r_P = (\rho_P - \rho_P^*) \psi'(\rho_P) - [\psi(\rho_P) - \psi(\rho_P^*)], \quad r_\eta = \rho_\eta \psi'(\rho_P) - \psi(\rho_\eta) - [\rho_P \psi'(\rho_P) - \psi(\rho_P)].$$

If the function ψ is convex, r_P is non-negative while r_η is non-positive (and vanishes if $\rho_\eta = \rho_P$). If ψ is twice continuously differentiable, a Taylor expansion gives that:

$$\begin{aligned} (\rho_P - \rho_P^*) \psi'(\rho_P) &= \psi(\rho_P) - \psi(\rho_P^*) + \frac{1}{2} \psi''(\bar{\rho}_P) (\rho_P - \rho_P^*)^2, \\ \rho_\eta \psi'(\rho_P) - \psi(\rho_\eta) &= \rho_P \psi'(\rho_P) - \psi(\rho_P) - \frac{1}{2} \psi''(\bar{\rho}_\eta) (\rho_\eta - \rho_P)^2, \end{aligned}$$

with $\bar{\rho}_P \in [\min(\rho_P, \rho_P^*), \max(\rho_P, \rho_P^*)]$ and for any $\eta \in \mathcal{E}(P)$, $\bar{\rho}_\eta \in [\min(\rho_\eta, \rho_P), \max(\rho_\eta, \rho_P)]$; hence the result. \square

We now turn to the second operator; formally, at the continuous level, using twice the assumption $\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0$ yields:

$$\begin{aligned} \psi'(z) \mathcal{C}_\rho(z) &= \psi'(z) [\partial_t(\rho z) + \operatorname{div}(\rho z \mathbf{u})] = \psi'(z) \rho [\partial_t z + \mathbf{u} \cdot \nabla z] \\ &= \rho [\partial_t \psi(z) + \mathbf{u} \cdot \nabla \psi(z)] = \partial_t(\rho \psi(z)) + \operatorname{div}(\rho \psi(z) \mathbf{u}). \end{aligned}$$

Taking for z a component of the velocity field, this relation is the central argument used to derive the kinetic energy balance. The following lemma states a discrete counterpart of this identity.

Lemma A.2. *Let P be a polygonal (resp. polyhedral) bounded set of \mathbb{R}^2 (resp. \mathbb{R}^3) and let $\mathcal{E}(P)$ be the set of its edges (resp. faces). Let $\rho_P^* > 0$, $\rho_P > 0$, $\delta t > 0$, and $(F_\eta)_{\eta \in \mathcal{E}(P)} \subset \mathbb{R}$ be such that*

$$\frac{|P|}{\delta t} (\rho_P - \rho_P^*) + \sum_{\eta \in \mathcal{E}(P)} F_\eta = 0. \quad (53)$$

Let ψ be a continuously differentiable function defined over $(0, +\infty)$. For $u_P^ \in \mathbb{R}$, $u_P \in \mathbb{R}$ and $(u_\eta)_{\eta \in \mathcal{E}(P)} \subset \mathbb{R}$ let us define:*

$$\begin{aligned} R_{P,\delta t} &= \left[\frac{|P|}{\delta t} (\rho_P u_P - \rho_P^* u_P^*) + \sum_{\eta \in \mathcal{E}(P)} F_\eta u_\eta \right] \psi'(u_P) \\ &\quad - \left[\frac{|P|}{\delta t} [\rho_P \psi(u_P) - \rho_P^* \psi(u_P^*)] + \sum_{\eta \in \mathcal{E}(P)} F_\eta \psi(u_\eta) \right]. \end{aligned} \quad (54)$$

Then:

- (i) *If ψ is convex and $u_\eta = u_P$ whenever $F_\eta > 0$, then $R_{P,\delta t} \geq 0$.*
- (ii) *If ψ is twice continuously differentiable, then*

$$R_{P,\delta t} = \frac{1}{2} \frac{|P|}{\delta t} \rho_P^* \psi''(\bar{u}_P) (u_P - u_P^*)^2 - \frac{1}{2} \sum_{\eta \in \mathcal{E}(P)} F_\eta \psi''(\bar{u}_\eta) (u_\eta - u_P)^2, \quad (55)$$

with, $\bar{u}_P \in [\min(u_P, u_P^), \max(u_P, u_P^*)]$ and, $\forall \eta \in \mathcal{E}(P)$, $\bar{u}_\eta \in [\min(u_\eta, u_P), \max(u_\eta, u_P)]$.*

(iii) As a consequence of (ii), for ψ defined by $\psi(s) = s^2/2$ and $\forall \eta \in \mathcal{E}(P)$, u_η such that $u_\eta = (u_P + u_{P_\eta})/2$ (this is simply obtained by defining $u_{P_\eta} = 2u_\eta - u_P$), we get the following identity:

$$\left[\frac{|P|}{\delta t} (\rho_P u_P - \rho_P^* u_P^*) + \sum_{\eta \in \mathcal{E}(P)} F_\eta u_\eta \right] u_P = \frac{1}{2} \frac{|P|}{\delta t} [\rho_P u_P^2 - \rho_P^* (u_P^*)^2] + \sum_{\eta \in \mathcal{E}(P)} F_\eta u_P u_{P_\eta} + R_{P,\delta t}, \quad (56)$$

$$\text{with } R_{P,\delta t} = \frac{1}{2} \frac{|P|}{\delta t} \rho_P^* (u_P - u_P^*)^2.$$

Proof. Let T_P be defined by:

$$T_P = \left[\frac{|P|}{\delta t} (\rho_P u_P - \rho_P^* u_P^*) + \sum_{\eta \in \mathcal{E}(P)} F_\eta u_\eta \right] \psi'(u_P).$$

Using Equation (53), we obtain:

$$T_P = \left[\frac{|P|}{\delta t} \rho_P^* (u_P - u_P^*) + \sum_{\eta \in \mathcal{E}(P)} F_\eta (u_\eta - u_P) \right] \psi'(u_P).$$

We now define the remainder terms r_P and $(r_\eta)_{\eta \in \mathcal{E}(P)}$ by:

$$r_P = (u_P - u_P^*) \psi'(u_P) - [\psi(u_P) - \psi(u_P^*)], \quad r_\eta = (u_P - u_\eta) \psi'(u_P) - [\psi(u_P) - \psi(u_\eta)].$$

With these notations, we get:

$$T_P = \frac{|P|}{\delta t} \rho_P^* [\psi(u_P) - \psi(u_P^*)] + \sum_{\eta \in \mathcal{E}(P)} F_\eta [\psi(u_\eta) - \psi(u_P)] + \frac{|P|}{\delta t} \rho_P^* r_P - \sum_{\eta \in \mathcal{E}(P)} F_\eta r_\eta.$$

Using Equation (53) once again, we have:

$$T_P = \frac{|P|}{\delta t} [\rho_P \psi(u_P) - \rho_P^* \psi(u_P^*)] + \sum_{\eta \in \mathcal{E}(P)} F_\eta \psi(u_\eta) + \frac{|P|}{\delta t} \rho_P^* r_P - \sum_{\eta \in \mathcal{E}(P)} F_\eta r_\eta,$$

and thus:

$$R_{P,\delta t} = \frac{|P|}{\delta t} \rho_P^* r_P - \sum_{\eta \in \mathcal{E}(P)} F_\eta r_\eta.$$

If ψ is convex, the remainder terms r_P and $(r_\eta)_{\eta \in \mathcal{E}(P)}$ are non-negative, and if $u_\eta = u_P$, $r_\eta = 0$; hence, if we suppose that $u_\eta = u_P$ when $F_\eta \geq 0$, then $R_{P,\delta t} \geq 0$. If ψ is twice continuously differentiable, a Taylor expansion yields:

$$r_P = \frac{1}{2} \psi''(\bar{u}_P) (u_P - u_P^*)^2, \quad r_\eta = \frac{1}{2} \psi''(\bar{u}_\eta) (u_\eta - u_P)^2$$

with $\bar{u}_P \in [\min(u_P, u_P^*), \max(u_P, u_P^*)]$ and, $\forall \eta \in \mathcal{E}(P)$, $\bar{u}_\eta \in [\min(u_\eta, u_P), \max(u_\eta, u_P)]$. Thus (ii) holds, and, as a direct consequence, so does (iii). \square

APPENDIX B. PROOFS OF THE ENTROPY CONSISTENCY THEOREMS

Proof of Theorem 3.10 – The discrete weak form of the entropy balance is obtained by integrating in time (*i.e.* summing over the time steps) Equation (37). We obtain $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} + T_5^{(m)} = R$ with:

$$\begin{aligned} T_1^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}_{\text{int}}} |D_\sigma| \left[\rho_{D_\sigma}^{n+1} (u_\sigma^{n+1})^2 - \rho_{D_\sigma}^n (u_\sigma^n)^2 \right] \varphi_\sigma^n, \\ T_2^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} |K| \left[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n) \right] \varphi_K^n, \\ T_3^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma = \overrightarrow{K} | \overrightarrow{L} \in \mathcal{E}_{\text{int}}, \\ K = [\overrightarrow{\sigma'}], L = [\overrightarrow{\sigma''}]} } \left[F_L^{n+1} u_\sigma^{n+1} u_{\sigma''}^{n+1} - F_K^{n+1} u_\sigma^{n+1} u_{\sigma'}^{n+1} \right] \varphi_\sigma^n, \\ T_4^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma'}] \in \mathcal{M}} \left[F_\sigma^{n+1} \mathcal{H}(\rho_\sigma^{n+1}) - F_{\sigma'}^{n+1} \mathcal{H}(\rho_{\sigma'}^{n+1}) \right] \varphi_K^n, \\ T_5^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \left[\sum_{\sigma = \overrightarrow{K} | \overrightarrow{L} \in \mathcal{E}_{\text{int}}} (p_L^{n+1} - p_K^{n+1}) u_\sigma^{n+1} \varphi_\sigma^n + \sum_{K = [\overrightarrow{\sigma'}] \in \mathcal{M}} p_K^{n+1} (u_\sigma^{n+1} - u_{\sigma'}^{n+1}) \varphi_K^n \right]. \end{aligned}$$

We detail here the treatment of the terms which significantly differ from [20, Proof of Theorem 3.4], *i.e.* the terms $T_3^{(m)}$ and $T_4^{(m)}$ and the residual term.

Reordering the sums in $T_3^{(m)}$, we get:

$$T_3^{(m)} = -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma'}] \in \mathcal{M}} F_K^{n+1} u_\sigma^{n+1} u_{\sigma'}^{n+1} (\varphi_\sigma^n - \varphi_{\sigma'}^n).$$

Using now the definition of the mass fluxes at the dual edges, we have:

$$T_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma'}] \in \mathcal{M}} (\rho_\sigma^{n+1} u_\sigma^{n+1} + \rho_{\sigma'}^{n+1} u_{\sigma'}^{n+1}) u_\sigma^{n+1} u_{\sigma'}^{n+1} (\varphi_\sigma^n - \varphi_{\sigma'}^n).$$

We now split $T_3^{(m)} = \mathcal{T}_3^{(m)} + \mathcal{R}_3^{(m)}$, where

$$\mathcal{T}_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma'}] \in \mathcal{M}} \rho_K^{n+1} [(u_\sigma^{n+1})^3 + (u_{\sigma'}^{n+1})^3] (\varphi_\sigma^n - \varphi_{\sigma'}^n) = -\frac{1}{2} \int_0^T \int_\Omega \rho^{(m)} (u^{(m)})^3 \partial_x \varphi \, dx \, dt,$$

so that

$$\lim_{m \rightarrow +\infty} \mathcal{T}_3^{(m)} = -\frac{1}{2} \int_0^T \int_\Omega \bar{\rho} \bar{u}^3 \partial_x \varphi \, dx \, dt,$$

and

$$\mathcal{R}_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma'}] \in \mathcal{M}} \left[(\rho_\sigma^{n+1} u_\sigma^{n+1} + \rho_{\sigma'}^{n+1} u_{\sigma'}^{n+1}) u_\sigma^{n+1} u_{\sigma'}^{n+1} - \rho_K^{n+1} \left((u_\sigma^{n+1})^3 + (u_{\sigma'}^{n+1})^3 \right) \right] (\varphi_\sigma^n - \varphi_{\sigma'}^n).$$

Expanding the quantity $(u_\sigma^{n+1})^3 + (u_{\sigma'}^{n+1})^3$ thanks to the identity $a^3 + b^3 = (a+b)(ab + (a-b)^2)$, and then reordering the sums, we obtain $\mathcal{R}_3^{(m)} = \mathcal{R}_{3,1}^{(m)} + \mathcal{R}_{3,2}^{(m)}$ with:

$$\begin{aligned}\mathcal{R}_{3,1}^{(m)} &= -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \left[(\rho_\sigma^{n+1} - \rho_K^{n+1}) u_\sigma^{n+1} + (\rho_{\sigma'}^{n+1} - \rho_K^{n+1}) u_{\sigma'}^{n+1} \right] u_\sigma^{n+1} u_{\sigma'}^{n+1} (\varphi_\sigma^n - \varphi_{\sigma'}^n), \\ \mathcal{R}_{3,2}^{(m)} &= \frac{1}{4} \sum_{n=0}^N \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \rho_K^{n+1} (u_\sigma^{n+1} + u_{\sigma'}^{n+1}) (u_\sigma^{n+1} - u_{\sigma'}^{n+1})^2 (\varphi_\sigma^n - \varphi_{\sigma'}^n).\end{aligned}$$

Reordering the sums, the term $\mathcal{R}_{3,1}^{(m)}$ reads:

$$\mathcal{R}_{3,1}^{(m)} = \frac{\varepsilon}{4} \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}, \\ \sigma=L \rightarrow K, K=[\sigma\sigma']}} (\rho_L^{n+1} - \rho_K^{n+1}) u_\sigma^{n+1} u_\sigma^{n+1} u_{\sigma'}^{n+1} (\varphi_\sigma^n - \varphi_{\sigma'}^n),$$

where $\varepsilon = \pm 1$ and the notation $L \rightarrow K$ means that the flow is going from L to K . Thanks to the Cauchy-Schwarz inequality, we get, by the regularity of φ :

$$|\mathcal{R}_{3,1}^{(m)}| \leq C_\varphi h^{1/2} \left[\sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |u_\sigma^{n+1}| (\rho_L^{n+1} - \rho_K^{n+1})^2 \right]^{1/2} \left[\sum_{n=0}^{N-1} \delta t \sum_{\sigma=K \in \mathcal{M}} |K| |u_\sigma^{n+1}| (u_\sigma^{n+1} u_{\sigma'}^{n+1})^2 \right]^{1/2},$$

and thus:

$$|\mathcal{R}_{3,1}^{(m)}| \leq C_\varphi h^{1/2} \|u^{(m)}\|_{L^5(\Omega \times (0,T))}^{5/2}.$$

We now turn to $\mathcal{R}_{3,2}^{(m)}$. Thanks to the regularity of φ , we get:

$$|\mathcal{R}_{3,2}^{(m)}| \leq C_\varphi \frac{(h^{(m)})^2}{\nu^{(m)}} \|\rho^{(m)}\|_{L^\infty(\Omega \times (0,T))} \|u^{(m)}\|_{L^\infty(\Omega \times (0,T))}^2 \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \frac{\nu^{(m)}}{h_K} (u_\sigma^{n+1} - u_{\sigma'}^{n+1})^2,$$

and thus $\mathcal{R}_{3,2}^{(m)}$ also tends to zero when m tends to $+\infty$ as soon as the ratio $(h^{(m)})^2/\nu^{(m)}$ tends to zero.

Expressing the mass fluxes as a function of the unknowns in $T_4^{(m)}$ and reordering the sums, we get:

$$T_4^{(m)} = - \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overline{K|L} \in \mathcal{E}_{\text{int}}} \mathcal{H}(\rho_\sigma^{n+1}) u_\sigma^{n+1} (\varphi_K^n - \varphi_L^n).$$

Let us write $T_4^{(m)} = \mathcal{T}_4^{(m)} + \mathcal{R}_4^{(m)}$, with:

$$\begin{aligned}\mathcal{T}_4^{(m)} &= - \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overline{K|L} \in \mathcal{E}_{\text{int}}} \left[|D_{K,\sigma}| \mathcal{H}(\rho_K^{n+1}) + |D_{L,\sigma}| \mathcal{H}(\rho_L^{n+1}) \right] u_\sigma^{n+1} \frac{\varphi_K^n - \varphi_L^n}{h_\sigma}, \\ \mathcal{R}_4^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overline{K|L} \in \mathcal{E}_{\text{int}}} |D_{L,\sigma}| \left[\mathcal{H}(\rho_K^{n+1}) + \mathcal{H}(\rho_L^{n+1}) - 2\mathcal{H}(\rho_\sigma^{n+1}) \right] u_\sigma^{n+1} \frac{\varphi_K^n - \varphi_L^n}{h_\sigma}.\end{aligned}$$

We have:

$$\mathcal{T}_4^{(m)} = - \int_0^T \int_\Omega \mathcal{H}(\rho^{(m)}) u^{(m)} \bar{\partial}_x \varphi_{\mathcal{M}} \, dx \, dt, \quad \text{so} \quad \lim_{m \rightarrow +\infty} \mathcal{T}_4^{(m)} = - \int_0^T \int_\Omega \mathcal{H}(\bar{\rho}) \bar{u} \partial_x \varphi \, dx \, dt.$$

Thanks to the definition of the upwind density at the face, we get:

$$|\mathcal{R}_4^{(m)}| \leq C_\varphi h^{(m)} \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} |\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_L^{n+1})| |u_\sigma^{n+1}|.$$

Since both sequences $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(1/\rho^{(m)})_{m \in \mathbb{N}}$ are supposed to be uniformly bounded, we have $|\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_L^{n+1})| \leq C |\rho_K^{n+1} - \rho_L^{n+1}|$ with a constant real number C , and therefore $\mathcal{R}_4^{(m)}$ tends to zero as $h^{(m)}$.

Finally, let us turn to the residual terms, *i.e.* to the proof of the relation (38). Using the expression of the residual terms and reordering the sums, we have:

$$\begin{aligned} \mathcal{R}_{\text{diff}}^{(m)} &= \sum_{n=0}^{N-1} \delta t \left[\sum_{\sigma \in \mathcal{E}_{\text{int}}} \varphi_{\sigma}^n (R^{\text{diff}})_{\sigma}^{n+1} - \sum_{K \in \mathcal{M}} \varphi_K^n S_K^{n+1} \right] = \\ &= \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \frac{\nu}{h_K} (u_{\sigma}^{n+1} - u_{\sigma'}^{n+1})(u_{\sigma}^{n+1} \varphi_{\sigma}^n - u_{\sigma'}^{n+1} \varphi_{\sigma'}^n) - \frac{\nu}{h_K} (u_{\sigma}^{n+1} - u_{\sigma'}^{n+1})^2 \varphi_K^n. \end{aligned}$$

Hence:

$$\mathcal{R}_{\text{diff}}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \frac{\nu}{h_K} (u_{\sigma}^{n+1} - u_{\sigma'}^{n+1}) [u_{\sigma}^{n+1} (\varphi_{\sigma}^n - \varphi_K^n) - u_{\sigma'}^{n+1} (\varphi_{\sigma'}^n - \varphi_K^n)].$$

The Cauchy-Schwarz inequality yields, using the regularity of the function φ :

$$|\mathcal{R}_{\text{diff}}^{(m)}| \leq C_{\varphi} \nu^{1/2} \left[\sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \frac{\nu}{h_K} (u_{\sigma}^{n+1} - u_{\sigma'}^{n+1})^2 \right]^{1/2} \left[\sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} |K| ((u_{\sigma}^{n+1})^2 + (u_{\sigma'}^{n+1})^2) \right]^{1/2}.$$

Thus, thanks to the estimate (22), we get:

$$|\mathcal{R}_{\text{diff}}^{(m)}| \leq C (\nu^{(m)})^{1/2} \|u^{(m)}\|_{L^2(\Omega \times (0,T))},$$

with C independent of m and $|\mathcal{R}_{\text{diff}}^{(m)}|$ tends to zero. \square

Proof of Theorem 4.9 – The only term which significantly differs from its counterpart in [20, Proof of Theorems 3.4, 4.2 and 5.3] and Theorem 3.10 is the kinetic energy convection term which reads:

$$T_3^{(m)} = \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma=K \rightarrow L \in \mathcal{E}_{\text{int}}, \\ K=[\sigma'\sigma], L=[\sigma\sigma']}} \left[F_L^n \tilde{u}_{\sigma}^{n+1} \tilde{u}_{\sigma'}^{n+1} - F_K^n \tilde{u}_{\sigma}^{n+1} \tilde{u}_{\sigma'}^{n+1} \right] \varphi_{\sigma}^n.$$

Reordering the sums, we get:

$$T_3^{(m)} = -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} F_K^n \tilde{u}_{\sigma}^{n+1} \tilde{u}_{\sigma'}^{n+1} (\varphi_{\sigma'}^n - \varphi_{\sigma}^n)$$

We write $T_3^{(m)} = \mathcal{T}_3^{(m)} + \mathcal{R}_3^{(m)}$, where

$$\begin{aligned} \mathcal{T}_3^{(m)} &= -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \frac{|K|}{2} \rho_K^n [u_{\sigma}^n (\tilde{u}_{\sigma}^{n+1})^2 + u_{\sigma'}^n (\tilde{u}_{\sigma'}^{n+1})^2] \frac{\varphi_{\sigma}^n - \varphi_{\sigma'}^n}{h_K} \\ &= -\frac{1}{2} \int_0^T \int_{\Omega} \rho^{(m)}(x, t - \delta t) u^{(m)}(x, t - \delta t) (\tilde{u}^{(m)}(x, t))^2 \partial_x \varphi \, dx \, dt, \end{aligned}$$

so that:

$$\lim_{m \rightarrow +\infty} \mathcal{T}_3^{(m)} = -\frac{1}{2} \int_0^T \int_{\Omega} \bar{\rho} \bar{u}^3 \partial_x \varphi \, dx \, dt,$$

and

$$\mathcal{R}_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\sigma\sigma'] \in \mathcal{M}} \left[(\rho_{\sigma}^n u_{\sigma}^n + \rho_{\sigma'}^n u_{\sigma'}^n) \tilde{u}_{\sigma'}^{n+1} \tilde{u}_{\sigma}^{n+1} - \rho_K^n \left(u_{\sigma}^n (\tilde{u}_{\sigma}^{n+1})^2 + u_{\sigma'}^n (\tilde{u}_{\sigma'}^{n+1})^2 \right) \right] (\varphi_{\sigma}^n - \varphi_{\sigma'}^n).$$

Reordering the terms in the sum, we get:

$$\mathcal{R}_3^{(m)} = -\frac{1}{4} \sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma}\vec{\sigma}'] \in \mathcal{M}} \left[\underbrace{(\rho_\sigma^n \tilde{u}_{\sigma'}^{n+1} - \rho_K^n \tilde{u}_\sigma^{n+1}) u_\sigma^n \tilde{u}_\sigma^{n+1}}_{\mathcal{D}_1} + \underbrace{(\rho_{\sigma'}^n \tilde{u}_\sigma^{n+1} - \rho_K^n \tilde{u}_{\sigma'}^{n+1}) u_{\sigma'}^n \tilde{u}_{\sigma'}^{n+1}}_{\mathcal{D}_2} \right] (\varphi_\sigma^n - \varphi_{\sigma'}^n).$$

Let us consider the term involving \mathcal{D}_1 , and skip the exposition of the treatment of the term with \mathcal{D}_2 , which is totally similar. Using the identity $2(ab - cd) = (a - c)(b + d) + (a + c)(b - d)$, we split this first part of $\mathcal{R}_3^{(m)}$ into $\mathcal{R}_{31}^{(m)} + \mathcal{R}_{32}^{(m)}$, with:

$$\begin{aligned} \mathcal{R}_{31}^{(m)} &= -\frac{1}{8} \sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma}\vec{\sigma}'] \in \mathcal{M}} u_\sigma^n \tilde{u}_\sigma^{n+1} (\rho_\sigma^n - \rho_K^n) (\tilde{u}_{\sigma'}^{n+1} + \tilde{u}_\sigma^{n+1}) (\varphi_\sigma^n - \varphi_{\sigma'}^n), \\ \mathcal{R}_{32}^{(m)} &= -\frac{1}{8} \sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma}\vec{\sigma}'] \in \mathcal{M}} u_\sigma^n \tilde{u}_\sigma^{n+1} (\rho_\sigma^n + \rho_K^n) (\tilde{u}_{\sigma'}^{n+1} - \tilde{u}_\sigma^{n+1}) (\varphi_\sigma^n - \varphi_{\sigma'}^n). \end{aligned}$$

Thanks to the regularity of φ , the Cauchy-Schwarz inequality yields:

$$|\mathcal{R}_{31}^{(m)}| \leq C_\varphi h^{1/2} \left[\sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma}\vec{\sigma}'] \in \mathcal{M}} |u_\sigma^n| (\rho_\sigma^n - \rho_K^n)^2 \right]^{1/2} \left[\sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma}\vec{\sigma}'] \in \mathcal{M}} |K| |u_\sigma^n| \left(\tilde{u}_\sigma^{n+1} (\tilde{u}_{\sigma'}^{n+1} + \tilde{u}_\sigma^{n+1}) \right)^2 \right]^{1/2},$$

and thus, invoking the estimate (44),

$$|\mathcal{R}_{31}^{(m)}| \leq C (h^{(m)})^{1/2} \left[\|u^{(m)}\|_{L^2(\Omega \times (0, T))}^{1/2} + \|\tilde{u}^{(m)}\|_{L^8(\Omega \times (0, T))}^2 \right].$$

Similarly, we get:

$$|\mathcal{R}_{32}^{(m)}| \leq C_\varphi \frac{h^2}{\nu} \left[\sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma}\vec{\sigma}'] \in \mathcal{M}} \frac{\nu}{h_K} (\tilde{u}_{\sigma'}^{n+1} - \tilde{u}_\sigma^{n+1})^2 \right]^{1/2} \left[\sum_{n=0}^{N-1} \delta t \sum_{K=[\vec{\sigma}\vec{\sigma}'] \in \mathcal{M}} |K| \left(u_\sigma^n \tilde{u}_\sigma^{n+1} (\rho_\sigma^n + \rho_K^n) \right)^2 \right]^{1/2},$$

and thus:

$$|\mathcal{R}_{32}^{(m)}| \leq C \frac{(h^{(m)})^2}{\nu^{(m)}} \left[\|u^{(m)}\|_{L^6(\Omega \times (0, T))}^3 + \|\tilde{u}^{(m)}\|_{L^6(\Omega \times (0, T))}^3 + \|\rho^{(m)}\|_{L^6(\Omega \times (0, T))}^3 \right].$$

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