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# Invariance properties of Sliced Inverse Regression

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## Abstract

In this paper we consider a semiparametric regression model involving a  $p$ -dimensional explanatory variable  $\mathbf{x}$  and including a dimension reduction of  $\mathbf{x}$  via an index  $B'\mathbf{x}$ . In this model, the main goal is to estimate  $B$  and to predict the real response variable  $Y$  conditionally to  $\mathbf{x}$ . A standard approach is based on sliced inverse regression (SIR). We exhibit some invariance properties of this method.

The purpose of the regression of a univariate response  $y$  on a  $p$ -dimensional predictor vector  $\mathbf{x}$  is to make inference on the conditional distribution of  $y|\mathbf{x}$ . Following [17],  $\mathbf{x}$  can be replaced by its standardized version

$$\mathbf{z} = [\Sigma_{\mathbf{x}}]^{-1/2}(\mathbf{x} - \mu_{\mathbf{x}}) , \quad (1)$$

where  $\mu_{\mathbf{x}}$  and  $\Sigma_{\mathbf{x}}$  denote the mean and covariance matrix of  $\mathbf{x}$  respectively assuming non-singularity of  $\Sigma_{\mathbf{x}}$ .

The goal of dimension reduction in regression is to find out a  $p \times d$  matrix  $B$  such that

$$y \perp\!\!\!\perp \mathbf{z} | B'\mathbf{z} , \quad (2)$$

where “ $\perp\!\!\!\perp$ ” indicates independence. Then the  $p$ -dimensional  $\mathbf{z}$  can be replaced by the  $d$ -dimensional vector  $B'\mathbf{z}$  without specifying any parametric model and without losing any information on predicting  $y$ . The column space  $\text{Span}\{B\}$  is called a dimension reduction subspace. The smallest applicable  $d$  is called the dimension of the regression.

Based on the inverse mean  $E(\mathbf{z}|y)$ , [35] proposed Sliced Inverse Regression (SIR) for dimension reduction in regression. It is realized that SIR can not recover the symmetric dependency [35, 13]. After SIR, many dimension reduction methods have been introduced. Sliced Average Variance Estimate (SAVE) proposed by [13] and Principle Hessian Directions (pHd) proposed by [36] are another two popular ones. Both pHd and SAVE refer to the second inverse moment, centered or non-centered. Compared with SAVE, pHd can not detect certain dependency hidden in the second moment [67, 66] and the linear dependency [36, 16]. Among those dimension reduction methods using only the first two inverse moments, SAVE seems to be the preferred one. Nevertheless, SAVE is not always the winner. For example, [66] implied that a linear combination of

SIR and pHd may perform better than SAVE in some cases. It is not surprising since [35] already suggested that a suitable combination of two different methods might sharpen the dimension reduction results. [66] further proposed that a bootstrap method could be used to pick up the “best” linear combination of two known methods, as well as the dimension of the regression, in the sense of the variability of the estimators, although lower variability under the bootstrap procedure does not necessarily lead to a better estimator. [40] pointed out that linear combinations of two known methods selected by the bootstrap criterion may not perform as well as a single new method, their Directional Regression method (DR), even though the bootstrap one is computationally intensive.

Sliced Inverse Regression benefits from nice invariance properties. The following result can be established:

**Proposition.** *Let  $A$  be a  $p \times p$  orthogonal matrix, and let  $b$  a  $p$ -dimensional vector. Then, the SIR estimator computed on  $A\mathbf{x} + b$  is the  $A$  times the SIR estimator computed on  $\mathbf{x}$ .*

The proof of this result is based on the remark that, following (1), the standardized predictor associated to  $A\mathbf{x} + b$  is  $A$  times the standardized predictor associated to  $\mathbf{x}$ . The dimension reduction subspace associated to  $A\mathbf{x} + b$  is thus  $A$  times the dimension reduction subspace associated to  $\mathbf{x}$ .

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