A simple introduction to water waves

Dimitrios Mitsotakis

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Abstract. The theory of the water waves is the main subject in coastal hydrodynamics and plays a significant role in applied mathematics and in physics. In these notes we present the basics of the water wave theory. Specifically, after introducing briefly the basic concepts of continuum mechanics, we derive the physical laws describing the physics of an inviscid, incompressible fluid, namely the Euler equations. Euler equations are the governing equations of water waves, but because of the great difficulties in the theoretical and numerical studies of these equations, simple, approximate mathematical models have been derived instead. The models are usually simplified so as to be valid for specific types of waves.Water waves are usually divided into categories depending on their amplitude and their wavelength compared to the water depth. Some waves such as the Tsunami waves are generated in the deep ocean as waves of small amplitude with large wavelength, but as they approach the shoreline they grow in amplitude while their wavelength is decreased. Here, we will derive models for long waves with or without using the small amplitude assumption.

Key words and phrases: Water waves; Boussinesq equations; Serre equations; KdV equation; BBM equation; dispersive waves; solitary waves

1. Introduction to Kinematics

Kinematics is the branch of classical mechanics that describes the motion of bodies and systems of bodies without consideration of the causes of the motion. In this section we present the basic tools for studying the motion of a continuous body moving in $\mathbb{R}^d$ with $d = 2$ or 3. For further reading we suggest the books [Hun06, Log06, Pat83, Whi99] on which these notes are based on.

Let $\vec{a}$ be (the label of) a particle of a continuous body that at $t = 0$ occupies the region $\mathcal{P}_0$. By a *fluid motion* we mean a mapping $\phi_t : \mathcal{P}_0 \to \mathcal{P}_t$, which maps the region $\mathcal{P}_0$ into the region $\mathcal{P}_t = \phi_t(\mathcal{P}_0)$ which is occupied by the same fluid at time t. We assume that ϕ_t is represented by the formula

$$\vec{x} = \vec{X}(\vec{a}, t) \tag{1}$$

with $X(\vec{a}, 0) = \vec{a}$. $\vec{a}$ is the Lagrangian coordinates or the particle's label at $t = 0$ while $\vec{x}$ is the Eulerian coordinate representing the position of the same particle $\vec{a}$ at time t. Usually, the mapping $\vec{X} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, is also referred to as the *particle path*. We assume that $\vec{X}$ is a diffeomorphism of $\mathbb{R}^d$, i.e. is smooth and invertible where the derivative $D_{\vec{a}}\vec{X} = \left(\partial_{a_j} X_i\right)_{ij}$ is a nonsingular matrix. Roughly speaking,

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the last condition means that the motion does not crush a nonzero material volume to zero volume.

The region $\mathcal{P}_t$ occupied by the material particles $\vec{a}$ at time t can be described also as the bounded set with smooth boundary such that

$$\mathcal{P}_t = \{\vec{X}(\vec{a},t) : \vec{A} \in \mathcal{P}_0\}.$$

For a given fluid motion the *velocity* is defined by $\vec{U}(\vec{a},t) = \vec{X}_t(\vec{a},t)$. Then the corresponding spatial velocity $\vec{u}(\vec{x},t)$ is defined by

$$\vec{u}\left(\vec{X}(\vec{a},t)\right) = \vec{U}(\vec{a},t).$$

Conversely, given a smooth spatial velocity $\vec{u}(\vec{x},t)$, we may reconstruct the motion of $\vec{X}(\vec{a},t)$ by solving the system of ODEs

$$\vec{X}_t(\vec{a},t) = \vec{u}\left(\vec{X}(\vec{a},t),t\right),$$

with $\vec{X}(\vec{a},0) = \vec{a}$ as initial condition.

Assume that we take measurements for the material volume at time t. Let a measurement f be a function of the spatial coordinates $(\vec{x},t)$, then the corresponding measurement F of the material coordinates is a function of $(\vec{a},t)$ and they are connected by the formula

$$F(\vec{a},t) = f\left(\vec{X}(\vec{a},t),t\right).$$

The rate of change of f at a given spatial point is given by the derivative f_t (or $\partial_t f$), while the rate of change of f following a particle path is given by F_t. The last derivative is called the *material derivative* of f and is usually denoted by Df/Dt. According to the chain rule,

$$\begin{aligned}\frac{Df}{Dt}(\vec{X}(\vec{a},t),t) &= f_t(\vec{X}(\vec{a},t),t) + \vec{X}_t(\vec{a},t)\cdot\nabla f(\vec{X}(\vec{a},t),t)\\ &= f_t(\vec{X}(\vec{a},t),t) + \vec{u}(\vec{X}_t(\vec{a},t),t)\cdot\nabla f(\vec{X}(\vec{a},t),t)\end{aligned}$$

The last relationship implies the compact form

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla.$$

We also define the *vorticity* as the curl of the velocity field, i.e. $\omega = \nabla\times\vec{u}$.

Here we present the *Reynolds' transport theorem* which may be thought of as a generalization of the Leibniz rule[1] for differentiating one dimensional integrals with variable endpoints:

$$\frac{d}{dt}\int_{\mathcal{P}_t} f\,d\vec{x} = \int_{\mathcal{P}_t}\{f_t + \nabla\cdot(f\vec{u})\}\,d\vec{x}, \tag{2}$$

or after using the divergence theorem it can be written as

$$\frac{d}{dt}\int_{\mathcal{P}_t} f\,d\vec{x} = \int_{\mathcal{P}_t} f_t\,d\vec{x} + \int_{\partial\mathcal{P}_t} f\vec{u}\cdot\vec{n}\,dS. \tag{3}$$

Now we have all the tools needed to derive the Euler's equations.

[1] $\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)} f_t(x,t)dx + f(b(t),t)\cdot b'(t) - f(a(t),t)\cdot a'(t)$.

2. Derivation of the Euler equations

The Euler equations consist of a set of physical *conservation laws* such as the conservation of mass and momentum, together with the assumption that the density of the fluid is constant, cf. [Whi99].

First we derive the *conservation of mass*. We consider a fluid of density $\rho(\vec{x},t)$. The mass contained in a material volume $\mathcal{P}_t$ is then given by

$$\int_{\mathcal{P}_t} \rho \, d\vec{x}.$$

If the mass of the material volume does not change with time, then

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho \, d\vec{x} = 0.$$

Hence, using (2), we get that

$$\int_{\mathcal{P}_t} \{\rho_t + \nabla\cdot(\rho\vec{u})\} \, d\vec{x} = 0.$$

Since this equality holds for all $t \geq 0$ and for an arbitrary smooth region $\mathcal{P}_t$, we conclude that

$$\rho_t + \nabla\cdot(\rho\vec{u}) = 0. \tag{4}$$

Assuming that the medium is homogenous, i.e. that the density of the fluid is constant, the conservation of mass reduces to the equation

$$\nabla\cdot(\vec{u}) = 0. \tag{5}$$

We proceed with the derivation of the equation of *conservation of momentum*. The total momentum of a material volume $\mathcal{P}_t$ is

$$\int_{\mathcal{P}_t} \rho\vec{u} \, d\vec{x}.$$

Newton's second law states that the rate of change of the momentum of a material volume is equal to the force acting on it. Taking into account that the only forces acting on the material volume is the surface pressure force p that acts in the inward normal direction and the gravity force $\vec{F} = -\rho g\vec{k}$, where $\vec{k}$ is the unit vector perpendicular to the horizontal plane. Thus

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\vec{u} \, d\vec{x} = -\int_{\partial\mathcal{P}_t} p\vec{n} \, dS + \int_{\mathcal{P}_t} \vec{F} \, d\vec{x}.$$

Using (2), and the divergence theorem, we find that

$$\int_{\mathcal{P}_t} \left\{(\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla p - \vec{F}\right\} d\vec{x} = 0,$$

where the tensor product $\vec{u}\otimes\vec{u} = (u_i u_j)_{ij}$ and therefore

$$\int_{\mathcal{P}_t} \left\{\frac{\partial}{\partial t}(\rho u_i) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial x_i} - F_i\right\} d\vec{x} = 0.$$

Since this equation holds for any $t \geq 0$ and for any arbitrary smooth region $\mathcal{P}_t$, we conclude that

$$(\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla p = \vec{F}. \tag{6}$$

Assuming that the density of the fluid is constant, i.e. $\rho_t = |\nabla\rho|^2 = 0$, and using (5), the conservation of momentum (6) reduces to

$$\vec{u}_t + (\vec{u}\cdot\nabla)\,\vec{u} + \frac{1}{\rho}\nabla p = -g\vec{k}. \tag{7}$$

We close the derivation of the Euler equations by studying the *vorticity* of the velocity field. The vorticity is defined as $\vec{\omega} = \nabla\times\vec{u}$. Taking curl on both sides of the conservation of momentum (7) we have

$$\vec{\omega}_t + \vec{u}\cdot\nabla\vec{\omega} = \vec{\omega}\cdot\nabla\vec{u}. \tag{8}$$

If $d = 2$ then $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, and $\vec{\omega}\cdot\nabla\vec{u} = 0$, so (8) reduces to the transport equation

$$\omega_t + \vec{u}\cdot\nabla\omega = 0. \tag{9}$$

From equations (8) and (9) we conclude that if the flow is irrotational initialy (i.e. $\nabla\times\vec{u} = 0$ for $t = 0$) then the flow remains irrotational for all time t. The irrotationality condition is formulated as

$$\nabla\times\vec{u} = 0 \tag{10}$$

One main consequence of the irrotationality condition is that the momentum equation (7) can be written as

$$\vec{u}_t + \frac{1}{2}\nabla|\vec{u}|^2 + \frac{1}{\rho}\nabla p = -g\vec{k}. \tag{11}$$

3. Boundary and initial conditions

The Euler equations consist of the conservation of mass (5), conservation of momentum (7) and the irrotationality condition (10) defined on a strip-like domain bounded above from the free surface $\eta(\vec{x},t)$ and bellow from the horizontal bottom h_0. To close the system of these conservation laws we need to impose two boundary conditions and two initial conditions. There are two kinds of boundary condition for the free surface; the kinematic and the dynamic boundary conditions. The kinematic boundary condition on a free surface states that the surface of the water is impermeable, and thus the fluid velocity $\vec{u}$ satisfies

$$\vec{u}\cdot\vec{n} = \vec{V}\cdot\vec{n},$$

on the free surface, where $\vec{n}$ is the unit normal to the surface and $\vec{V}$ is the velocity tangent to the surface. Assuming that the free surface has the equation $F(\vec{x},t) = 0$ and that it is smooth we have that the unit normal and the normal velocity are given in terms of F as

$$\vec{n} = \frac{\nabla F}{|\nabla F|}, \qquad \vec{V} = -\frac{F_t}{|\nabla F|}\vec{n}.$$

Using these expressions, the kinematic boundary condition becomes:

$$\frac{DF}{Dt} = 0, \qquad \text{on } F = 0,$$

meaning that the material particles on the surface remain on the surface. For a surface that is a graph with equation $z = \eta(x,t)$ then the kinematic condition can be written as

$$\eta_t + \vec{u}\cdot\nabla\eta = 0, \qquad \text{on } z = \eta(x,t), \tag{12}$$

where now x denotes only the horizontal independent variable and z the vertical.

The dynamic boundary condition on the free surface is that the stresses on either side of the surface are equal. In the case of an air-water interface, we neglect the motion of the air, because of its smaller density, and assume that the atmospheric pressure is constant $p = 0$.

The boundary condition on the bottom is the impermeability condition of the bottom, i.e.

$$\vec{u} \cdot \vec{n} = 0, \qquad \text{on } h_0,$$

where $\vec{n}$ is the unit outward normal to the boundary.

4. Non-dimensionalization and normalization

For simplicity, we will restrict the analysis only in the case $d = 2$. In this case, the Euler equations for inviscid, incompressible and irrotational flow with a free surface over a horizontal bottom at height $y = -h_0$ (the undisturbed level of fluid is at $y = 0$) may be written in dimensional and unscaled variables as follows: Let $\eta(x, t)$ be the deviation of the free surface of the fluid above its level of rest and consider the domain $\Omega_t = \{(x, y) : -\infty < x < \infty, -h_0 \le y \le \eta(x, t)\}$. The for $(x, y) \in \Omega_t$ and $t \ge 0$ we have

$$u_t + uu_x + vu_y + \frac{1}{\rho} p_x = 0, \tag{13}$$

$$v_t + uv_x + vv_y + \frac{1}{\rho} p_y = -g, \tag{14}$$

$$u_x + v_y = 0, \tag{15}$$

$$u_y = v_x, \tag{16}$$

where g is the acceleration of gravity, $u = u(x, y, t)$, respectively $v = v(x, y, t)$, denotes the horizontal, respectively the vertical, velocity component, ρ is the (constant) density of the fluid, and $p = p(x, y, t)$ the pressure. The system (13)–(16) is supplemented by the free surface kinematic and dynamic boundary conditions

$$\eta_t + u\eta_x = v, \qquad \text{at } y = \eta(x, t), \tag{17}$$

$$p = 0, \qquad \text{at } y = \eta(x, t). \tag{18}$$

At the bottom $y = -h_0$ we assume that the normal component of the velocity vanishes, i.e. that

$$v = 0, \qquad \text{at } y = -h_0, \tag{19}$$

We also assume that initial conditions for η and u have been specified and let

$$\eta(x, 0) = \eta_0(x), \qquad x \in \mathbb{R}, \tag{20}$$

$$u(x, y, 0) = u_0(x, y), \qquad (x, y) \in \Omega_0, \tag{21}$$

where η_0 and u_0 are given real functions. (We assume that u_0 satisfies the compatibility condition $\frac{\partial u_0}{\partial y}(x, y) = -\int_{h_0}^{y} \frac{\partial^2 u_0}{\partial x^2}(x, y')\, dy'$ in Ω_0, which follows assuming that (15), (16) and (19) hold at $t = 0$).

Water waves are usually characterized by their amplitude and their wavelength compared to the water depth. We assume that a characteristic wavelength is λ, and a characteristic amplitude A. The so called shallow water condition states that the water depth h_0 is much smaller compared to the wavelength i.e. $\sigma = h_0/\lambda \ll 1$. We define also another important parameter $\varepsilon = A/h_0$. As it will be more clear also later in these notes, ε measures the local size of nonlinear effects, while σ measures the local size of dispersive effects. Usually, waves in the deep water are characterized by the small amplitude assumption since $\varepsilon \ll 1$ but as the waves approach the shoreline they gain amplitude, i.e. $\varepsilon = O(1)$ and finally they break $\sigma^2 \ll \varepsilon$. The smallness of σ is used to derive simple model equations that do not depend on the vertical coordinate y. It is also common to divide water waves into three main categories depending on the value of the quotient $S = \varepsilon/\sigma^2$ known as the Stokes number (sometimes also referred to as Ursel number). Stokes number is a measure of the relative strength of nonlinear and dispersive effects. For example, nonlinear but non-dispersive waves are characterized by $\sigma^2 \ll \varepsilon$ or $S \gg 1$, while for weakly nonlinear and weakly dispersive $\sigma^2 \ll 1$ and $\sigma^2 = O(\varepsilon)$, i.e. $S = O(1)$, and finally we speak about strongly nonlinear but weakly dispersive waves when $\sigma^2 \ll 1$ but $\varepsilon = O(1)$ and $S > 1$.

To derive simple mathematical models from the Euler equations the first step is to introduce non-dimensional independent variables that are defined as follows:

$$x^* = \frac{x}{\lambda}, \quad y^* = \frac{y}{h_0}, \quad \text{and} \quad t^* = \frac{\sigma g}{c_0} t. \tag{22}$$

We know from the linear theory that the horizontal velocity $u(x, y, t)$ has the following order of magnitude: $u = O(\varepsilon c_0)$, where $c_0 = \sqrt{g h_0}$ is the linear wave phase velocity. Then the nondimensional horizontal velocity is:

$$u^* = \frac{u}{\varepsilon\, c_0}. \tag{23}$$

In the case of long waves the motion of the fluid is essentially horizontal, i.e. the vertical component of the velocity is usually very small. That is $v = O(\sigma^2 \varepsilon c_0)$, thus we define

$$v^* = \frac{v}{\sigma^2 \varepsilon c_0}, \tag{24}$$

also the pressure is scaled by the static pressure, $p = O(\rho g h_0)$, and thus

$$p^* = \frac{p}{\rho g h_0}. \tag{25}$$

Finally, since a typical amplitude of a wave is considered in this analysis to be A, i.e. $\eta = O(A)$, the nondimensional free surface elevation is given

$$\eta^* = \frac{1}{\varepsilon h_0} \eta. \tag{26}$$

Then, the problem (13)–(21) is transformed into the equivalent problem that follows, in which *the dependent variables and the initial conditions are of order one*, while powers of ε and σ signify the order of magnitude of terms they multiply. We seek $u^* = u^*(x^*, y^*, t^*)$, $v^* = v^*(x^*, y^*, t^*)$, $p^* = p^*(x^*, y^*, t^*)$, $\eta^* = \eta^*(x^*, t^*)$, defined for $-\infty < x^* < \infty$, $-1 \le y^* \le \varepsilon \eta^*(x^*, t^*)$, $t^* \ge 0$, such that,

$$\varepsilon u^*_{t^*} + \varepsilon^2 u^* u^*_{x^*} + \varepsilon^2 v^* u^*_{y^*} + p^*_{x^*} = 0, \tag{27}$$

$$\varepsilon \sigma^2 v^*_{t^*} + \varepsilon^2 \sigma^2 u^* v^*_{x^*} + \varepsilon^2 \sigma^2 v^* v^*_{y^*} + p^*_{y^*} = -1, \tag{28}$$

$$u^*_{x^*} + v^*_{y^*} = 0, \tag{29}$$

$$u^*_{y^*} - \sigma^2 v^*_{x^*} = 0. \tag{30}$$

This system is supplemented with the free surface and bottom boundary conditions, which now take the form:

$$\eta^*_{t^*} + \varepsilon u^* \eta^*_{x^*} = v^*, \ \text{ for } y^* = \varepsilon \eta^*(x^*, t^*), \tag{31}$$

$$p^* = 0, \ \text{ for } y^* = \varepsilon \eta^*(x^*, t^*), \tag{32}$$

$$v^* = 0, \text{ for } y^* = -1, \tag{33}$$

and the initial conditions

$$\eta^*(x^*, 0) = \eta_0^*(x^*), \quad u^*(x^*, y^*, 0) = u_0^*(x^*, y^*), \tag{34}$$

where, in terms of the functions η_0, u_0 in (20)–(21) we have

$$\eta_0^*(x^*) := \frac{1}{h_0\varepsilon}\eta_0\left(\frac{x^* h_0}{\sigma}\right), \quad u_0^*(x^*, y^*) := \frac{1}{\varepsilon c_0}u_0\left(\frac{h_0}{\sigma}x^*, h_0 y^*\right). \tag{35}$$

To derive model equations we will follow the great lines of [Bar04, BC98, BCS02, BS76, DM08, Whi99]. Moreover, in what follows we simplify the notation by dropping the $*$. We proceed first with the derivation of models that describe the bidirectional wave propagation.

5. Model equations

It has been observed that the horizontal velocity of the fluid is usually uniform across the fluid depth and thus it is convenient to study either the depth averaged velocity or the velocity of the fluid at a certain height above the bottom. It is expected that in both cases the values of the velocities will be close. Here, we will obtain approximate models to the Euler equations by using the mean velocity with respect to depth:

$$\bar{u} = \frac{1}{1+\varepsilon\eta}\int_{-1}^{\varepsilon\eta} u(x,y,t)dy. \tag{36}$$

Integrating the equation of the conservation of mass (continuity equation) (29) we have

$$\int_{-1}^{\varepsilon\eta} u_x\, dy + v(x, \varepsilon\eta, t) - v(x, -1, t) = 0. \tag{37}$$

Using Leibniz rule[2] we have

$$\int_{-1}^{\varepsilon\eta} u_x\, dy = \frac{\partial}{\partial x}\int_{-1}^{\varepsilon\eta} u\, dy - u(x, \varepsilon\eta, t)\cdot\varepsilon\eta_x = [h\bar{u}]_x - u(x, \varepsilon\eta, t)\cdot\varepsilon\eta_x. \tag{38}$$

Applying the boundary conditions (31) and (33) we obtain from (38) the (exact) equation

$$\eta_t + [h\bar{u}]_x = 0, \tag{39}$$

where h denotes the total depth $1+\varepsilon\eta$.

Similarelly, integrating the momentum equation (27) and using Leibniz rule, the continuity equation (39) and the boundary conditions (31)–(33) we have

$$\varepsilon h\bar{u}_t + \varepsilon^2 h\bar{u}\bar{u}_x + \varepsilon^2\frac{\partial}{\partial x}\int_{-1}^{\varepsilon\eta}(u^2 - \bar{u}^2)\, dy = -\int_{-1}^{\varepsilon\eta} p_x\, dy. \tag{40}$$

[2] $\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)} f_t(x,t)dx + f(b(t),t)\cdot b'(t) - f(a(t),t)\cdot a'(t)$.

5.1. **The Serre-Green-Naghdi equations.** In this subsection we will derive model equations for strongly nonlinear long waves known to as the Serre-Green-Naghdi equations (sometimes also referred to as also as the Su and Gardner equations), cf. [Ser53a, Ser53b, SG69, GLN74]. For the derivation of model equations, crucial role plays the assumptions on the pressure field. Using Leibniz rule and boundary condition (32) gives

$$\int_{-1}^{\varepsilon\eta} p_x\, dy = [h\bar{p}]_x - h_x p(h) = [h\bar{p}]_x. \tag{41}$$

To compute $\bar{p}$ we write the y momentum equation (28) as

$$-p_y = 1 + \varepsilon\sigma^2\Gamma(x,y,t), \tag{42}$$

where

$$\Gamma(x,y,t) = v_t + \varepsilon u v_x + \varepsilon v v_y. \tag{43}$$

Integrating (42) from y to $\varepsilon\eta$ we get

$$-p(x,y,t) = (y+1-h) - \varepsilon\sigma^2 \int_y^{\varepsilon\eta} \Gamma(x,z,t)\, dz, \tag{44}$$

and taking the mean value

$$-h\bar{p} = -\frac{1}{2}h^2 - \varepsilon\sigma^2 \int_{-1}^{\varepsilon\eta}\int_y^{\varepsilon\eta} \Gamma(x,z,t)\, dz\, dy, \tag{45}$$

and therefore, equation (40) is written

$$\bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x + \frac{\sigma^2}{h}\frac{\partial}{\partial x}\int_{-1}^{\varepsilon\eta}\int_y^{\varepsilon\eta} \Gamma(x,z,t)\, dz\, dy = -\frac{\varepsilon}{h}\frac{\partial}{\partial x}\int_{-1}^{\varepsilon\eta}(u^2-\bar{u}^2)\, dy \tag{46}$$

To compare u and $\bar{u}$ we compute the Taylor[3] polynomial of u around the bottom -1. Denoting by u_b the horizontal velocity at the bottom and using the boundary condition (33) and the irrotationality condition (30), the Taylor polynomial for u is

$$u(x,y,t) = u_b(x,t) - \frac{1}{2}\sigma^2(y+1)^2\frac{\partial^2 u_b}{\partial x^2} + O(\sigma^4) \tag{47}$$

and for v

$$v(x,y,t) = -(y+1)\frac{\partial u_b}{\partial x} + \frac{1}{3!}\sigma^2(y+1)^3\frac{\partial^3 u_b}{\partial x^3} + O(\sigma^4). \tag{48}$$

Integrating (47) it follows that

$$u_b = \bar{u} + \frac{1}{6}\sigma^2 h^2\frac{\partial^2\bar{u}}{\partial x^2} + O(\sigma^4, \varepsilon\sigma^4), \tag{49}$$

and consequently,

$$u(x,y,t) = \bar{u} + \frac{1}{6}\sigma^2 h^2\frac{\partial^2\bar{u}}{\partial x^2} - \frac{1}{2}\sigma^2(y+1)^2\frac{\partial^2\bar{u}}{\partial x^2} + O(\sigma^4, \varepsilon\sigma^4). \tag{50}$$

Taking squares in the previous equation we have:

$$u^2(x,y,t) = \bar{u}^2 + \frac{1}{3}\bar{u}\sigma^2 h^2\frac{\partial^2\bar{u}}{\partial x^2} - \bar{u}\sigma^2(y+1)^2\frac{\partial^2\bar{u}}{\partial x^2} + O(\sigma^4, \varepsilon\sigma^4). \tag{51}$$

Integrating (51) from -1 to $\varepsilon\eta$ and after simplifications follows that:

$$\int_{-1}^{\varepsilon\eta}(u^2-\bar{u}^2)dy = O(\sigma^4, \varepsilon\sigma^4). \tag{52}$$

[3] $f(x) = f(a) + (x-a)f_x(a) + \frac{1}{2!}(x-a)^2 f_{xx}(a) + O(x-a)^3$.

Moreover, the vertical velocity

$$v(x,y,t) = -(y+1)\frac{\partial \bar{u}}{\partial x} + O(\sigma^2). \tag{53}$$

Substituting the last relation into (43) gives

$$\Gamma(x,y,t) = -(y+1)[\bar{u}_{xt} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon(\bar{u}_x)^2] + O(\sigma^2, \varepsilon\sigma^2). \tag{54}$$

Combining (54), (52), (46) and taking into account that the quantity $\bar{u}_{xt} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon(\bar{u}_x)^2$ is independent of y we have

$$\bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x - \frac{\sigma^2}{3h}\frac{\partial}{\partial x}[h^3(\bar{u}_{xt} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon(\bar{u}_x)^2)] = O(\sigma^4, \varepsilon\sigma^4). \tag{55}$$

In this setting we assume that $\sigma \ll 1$ and $\varepsilon = O(1)$. Summarizing, we derived the Serre-Green-Naghdi system of equations given by

$$\eta_t + [(1+\varepsilon\eta)\bar{u}]_x = 0, \tag{56}$$

$$\bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x - \frac{\sigma^2}{3h}\frac{\partial}{\partial x}[h^3(\bar{u}_{xt} + \varepsilon\bar{u}\bar{u}_{xx} - \varepsilon(\bar{u}_x)^2)] = O(\sigma^4, \varepsilon\sigma^4). \tag{57}$$

Dropping the high-order terms and going back to dimensional variables is easy to see that system (56)–(57) writes

$$h_t + [h\bar{u}]_x = 0, \tag{58}$$

$$\bar{u}_t + gh_x + \bar{u}\bar{u}_x - \frac{1}{3h}\frac{\partial}{\partial x}[h^3(\bar{u}_{xt} + \bar{u}\bar{u}_{xx} - (\bar{u}_x)^2)] = 0, \tag{59}$$

with $h = h_0 + \eta$.

5.2. **The 'classical' Boussinesq system.** Considering long waves of small amplitude, i.e. when $\sigma \ll 1$ and $\varepsilon \ll 1$ Serre system could be simplified more. For example, keeping the terms of $O(\varepsilon, \sigma^2)$ in (57) we obtain the 'classical' Boussinesq system, [Bou72, Per80, BCS02]:

$$\eta_t + [(1+\varepsilon\eta)\bar{u}]_x = 0, \tag{60}$$

$$\bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x - \frac{\sigma^2}{3}\bar{u}_{xxt} = O(\sigma^4, \varepsilon\sigma^2), \tag{61}$$

and in dimensional variables, setting the right-hand side equal to zero, we have

$$h_t + [h\bar{u}]_x = 0, \tag{62}$$

$$\bar{u}_t + gh_x + \bar{u}\bar{u}_x - \frac{1}{3}\bar{u}_{xxt} = 0, \tag{63}$$

with $h = h_0 + \eta$.

5.3. **Unidirectional model equations: The KdV and the BBM equations.** The previous systems describe the two-way propagation of water waves. In this section we will derive equations that describe water waves traveling mainly in one direction, such as the KdV and the BBM equations.

From the equations (60) and (61) of the 'classical' Boussinesq system we observe that $\eta_t + \bar{u}_x = O(\varepsilon)$ and $\bar{u}_t + \eta_x = O(\varepsilon, \sigma^2)$ respectively. From these equations the wave equation follows $\eta_{tt} + \eta_{xx} = O(\varepsilon, \sigma^2)$, from which we choose the component traveling to the right, i.e. the solutions such that $\eta_t + \eta_x = O(\varepsilon, \sigma^2)$. If we choose such η and set $\bar{u} = \eta$ then we obtain an $O(1)$ solution, traveling mainly towards one direction. Moreover, since $\eta_t + \eta_x = O(\varepsilon, \sigma^2)$ we deduce the general low order approximation $\partial_t + \partial_x = O(\varepsilon, \sigma^2)$.

To improve the accuracy of the solution we assume that

$$\bar{u} = \eta + \varepsilon A + \sigma^2 B + O(\varepsilon^2, \sigma^4), \tag{64}$$

where A, B are functions of η and its derivatives with respect of x. Substitution into the system (60)–(61) and collecting the terms of the same order we have

$$\eta_t + \eta_x + \varepsilon(A_x + 2\eta\eta_x) + \sigma^2 B_x = O(\varepsilon^2, \sigma^4), \tag{65}$$

$$\eta_t + \eta_x + \varepsilon(A_t + \eta\eta_x) + \sigma^2(B_t - \frac{1}{3}\eta_{xxt}) = O(\varepsilon^2, \sigma^4). \tag{66}$$

Equating the terms of the same order so as equations (65)–(66) coincide and using the low order approximations $A_t + A_x = O(\varepsilon, \sigma^2)$ and $B_t + B_x = O(\varepsilon, \sigma^2)$ we get $A = -\frac{1}{4}\eta^2$ and $B = \frac{1}{6}\eta_{xx}$. Therefore, (64) reduces to

$$u = \eta - \frac{\varepsilon}{4}\eta^2 + \frac{\sigma^2}{6}\eta_{xx}$$

and so (60) reduces to the KdV equation:

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x + \frac{\sigma^2}{6}\eta_{xxx} = O(\varepsilon^2, \sigma^4). \tag{67}$$

Ignoring the high-order terms and turning back to the dimensional and unscaled variables, KdV takes the form

$$\eta_t + \sqrt{gh_0}\eta_x + \frac{3}{2}\sqrt{\frac{g}{h_0}}\eta\eta_x + \frac{h_0^3}{6}\sqrt{\frac{g}{h_0}}\eta_{xxx} = 0. \tag{68}$$

To derive the BBM equation we use the fact that $\eta_t + \eta_x = O(\varepsilon, \sigma^2)$ and thus $\eta_{xxx} = -\eta_{xxt} + O(\varepsilon, \sigma^2)$. Substituting the last identity into (67) we get the BBM equation

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{\sigma^2}{6}\eta_{xxt} = O(\varepsilon^2, \sigma^4), \tag{69}$$

which in dimensional and unscaled variables could be written in the form:

$$\eta_t + \sqrt{gh_0}\eta_x + \frac{3}{2}\sqrt{\frac{g}{h_0}}\eta\eta_x - \frac{h_0^2}{6}\eta_{xxt} = 0. \tag{70}$$

5.4. **The Shallow Water equations.** Assuming that the pressure is hydrostatic, i.e. the vertical accelerations are negligible, and $p_y = -1$, (or in dimensional variables $p_y = -\rho g$), equation (45) reduces to $h\bar{p} = \frac{1}{2}h^2$. Therefore, equation (46) with the help of (52) leads to

$$\bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x = O(\sigma^4, \varepsilon\sigma^4), \tag{71}$$

and the Shallow Water Wave system:

$$h_t + \varepsilon[h\bar{u}]_x = 0, \tag{72}$$

$$\bar{u}_t + \eta_x + \varepsilon\bar{u}\bar{u}_x = O(\sigma^4, \varepsilon\sigma^4). \tag{73}$$

In dimensional variables, dropping the high order terms the Shallow Water equations are written as

$$h_t + [h\bar{u}]_x = 0, \tag{74}$$

$$\bar{u}_t + gh_x + \bar{u}\bar{u}_x = 0, \tag{75}$$

which is a nonlinear system of conservation laws without dispersive terms. This system can also be derived using the shallow water assumption $\sigma^2 \ll 1$ and keeping only the terms of $O(\varepsilon)$ in (63).

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University of California, Merced, 5200 North Lake Road, Merced, CA 94353, USA
E-mail address: dmitsot@gmail.com
URL: http://dmitsot.googlepages.com/