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# Frobenius structure for rank one $p$ -adic differential equations.

Andrea Pulita

ABSTRACT. We generalize to all rank one  $p$ -adic differential equations over  $\mathcal{R}$  the theorem 2.3.1 of [Ch-Ch] which provides the existence of a Frobenius structure of order  $h$  for soluble rank one operators of the form  $\frac{d}{dx} + g(x)$ ,  $g(x) \in x^{-2}K[x^{-1}]$ . It follows a generalization of a theorem of Matsuda which asserts that the Robba's exponential  $\exp(\sum_{i=0}^m \pi_{m-i} x^{p^i} / p^i)$  has a Frobenius structure. Namely our theorem works in the case  $p = 2$ .

In the appendix we describe the variation of the radius of convergence of a differential module by pull-back by a Kummer ramification.

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## 1. Notations

Let  $K$  be a complete field with respect to an ultra-metric absolute value  $|\cdot|$ . Let  $\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$  be the ring of integers of  $K$ , and let  $D(0, 1^-) = \{x \in K \mid |x| < 1\}$  be its maximal ideal. Let  $k$  be its residue field which will be supposed to be a perfect field of characteristic  $p > 0$ . Let  $\bar{K}$  be an algebraic closure of  $K$ .

Let  $E$  (resp.  $E_\rho$ ) be the completion of  $K(x)$  for the Gauss norm (resp. the norm  $|\cdot|_\rho$  defined by  $|\sum_i a_i x^i|_\rho := \sup_i |a_i| \rho^i$ ).

Let  $\mathcal{E}$  be the Amice's ring. The elements of  $\mathcal{E}$  are bounded series  $f = \sum_{i \in \mathbb{Z}} a_i x^i$ ,  $a_i \in K$ ,  $|a_i| \rightarrow 0$  for  $i \rightarrow -\infty$ , for which there exists a constant  $M(f) \in \mathbb{R}$  such that  $|a_i| \leq M(f)$  for all  $i \in \mathbb{Z}$ . The topology of  $\mathcal{E}$  is defined by the Gauss norm and  $\mathcal{E}$  is a complete ring. We have a canonical isometric embedding  $K(x) \subset E \subset \mathcal{E}$ .

Let  $I \subseteq \mathbb{R}_{\geq 0}$  be an interval. Let  $\mathcal{C}(I) := \{x \in K \mid |x| \in I\}$ . Let  $\mathcal{A}(I)$  be the ring of analytic functions over  $\mathcal{C}(I)$ , the elements of  $\mathcal{A}(I)$  are power series  $f = \sum_{i \in \mathbb{Z}} a_i x^i$ ,  $a_i \in K$ , such that  $\lim_{i \rightarrow \pm\infty} |a_i| \rho^i = 0$ , for all  $\rho \in I$ .  $\mathcal{A}(I)$  is complete for the topology defined by the family of absolute values  $\{|\cdot|_\rho\}_{\rho \in I}$ , where

$$|f|_\rho := \sup_{i \in \mathbb{Z}} |a_i| \rho^i.$$

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Let  $\mathcal{R}$  be the Robba's Ring. The elements  $f$  of  $\mathcal{R}$  are germs of convergent analytic functions at the edge of  $D(0, 1^-)$ , namely

$$(1.0.1) \quad \mathcal{R} = \cup_{0 < \varepsilon < 1} \mathcal{A}(|1 - \varepsilon, 1|).$$

In other words  $\mathcal{R}$  is the inductive limit of the sequence  $\mathcal{A}(r_1, 1) \subset \mathcal{A}(r_2, 1)$ ,  $0 < r_1 < r_2 < 1$  and it is equipped with the limit topology.

All the rings  $E_\rho$ ,  $\mathcal{E}$ ,  $\mathcal{A}(I)$ ,  $\mathcal{R}$  are differential rings with respect to the continue derivation  $\frac{d}{dx}$ .

**1.1. Berkovich spaces and  $p$ -adic differential equations.** Let  $\mathcal{M}(\mathcal{A}(I))$  be the analytic space (in the sense of Berkovich [Ber] 1.2) attached to the affinoid algebra  $\mathcal{A}(I)$  ([Ber] 2.1.1). The Berkovich's point defined by the norm  $|\cdot|_\rho$  can be (and will be) identified with the Dwork's generic point  $t_\rho$  of radius  $\rho$  ([Ch-Ro], 9.1.2). Following this identification let  $M$  be the differential module defined by  $\frac{d}{dx} + G(x)$ ,  $G(x) \in M_n(\mathcal{A}(I))$ , then the radius of convergence of  $M$  at the point  $|\cdot|_\rho$  is defined as

$$(1.1.1) \quad \text{Ray}(M, |\cdot|_\rho) := \inf \left( \liminf_s \left( \frac{|G_s(x)|_\rho}{|s!|} \right)^{-1/s}, \rho \right)$$

where  $G_s \in M_n(\mathcal{A}(I))$  is defined by the recursion formula

$$(1.1.2) \quad G_{s+1} = \frac{d}{dx}(G_s) + G_s \cdot G, \quad G_1 := G.$$

The function  $\rho \mapsto \text{Ray}(M, |\cdot|_\rho)$  is continuous and there exists a partition  $I = \cup_{j \in \mathbb{Z}} I_j$ ,  $\sup I_j = \inf I_{j+1}$ , such that  $\text{Ray}(M, |\cdot|_\rho) = \alpha_j \rho^{\beta_j}$ , for all  $\rho \in I_j$ .

For simplicity we will write  $\text{Ray}(M, \rho)$  instead of  $\text{Ray}(M, |\cdot|_\rho)$ .

1.1.1. More generally let  $|\cdot|_t \in \mathcal{M}(\mathcal{A}(I))$  be a bounded multiplicative semi-norm ([Ber] 1.2). We define the *radius of the generic disk of center  $|\cdot|_t$*  as

$$(1.1.3) \quad \rho(|\cdot|_t) := \inf(|x - a|_t \mid a \in \overline{K})$$

and we put

$$(1.1.4) \quad \text{Ray}(M, |\cdot|_t) := \inf \left( \liminf_s \left( \frac{|G_s(x)|_t}{|s!|} \right)^{-1/s}, \rho(|\cdot|_t) \right).$$

REMARK 1.1. Observe that the function  $\rho : \mathcal{M}(\mathcal{A}(I)) \rightarrow [0, \sup I]$ ,  $|\cdot|_t \mapsto \rho(|\cdot|_t)$  is semi-continuous in the sense that the set  $\rho^{-1}([0, r])$  is open in  $\mathcal{M}(\mathcal{A}(I))$ <sup>1</sup>, but  $\rho$  is not continuous. Indeed for all  $a \in K$  such that  $|a| \in I$  we define  $|\cdot|_a$  as the semi-norm given by  $f \mapsto |f(a)|$ . It is clear that  $\rho(|\cdot|_a) = 0$ . Now let  $I := [0, 1]$  and let  $|\cdot|_1$  be the semi-norm attached to the unit disk  $|f|_1 := \sup_{|a| \leq 1, a \in K} (|f(a)|)$ .<sup>2</sup> Let  $K = \overline{K}$ . We choose a sequence  $\{a_n\}_n$ ,  $a_n \in K$ , such that  $\bar{a}_i \neq \bar{a}_j \in k$ ,  $\forall i \neq j$ , then we have  $\lim_n |\cdot|_{a_n} = |\cdot|_1$  in  $\mathcal{M}(\mathcal{A}(I))$ , but  $\rho(|\cdot|_1) = 1$ . This results from the fact that every function  $f \in \mathcal{A}(I)$  has only a finite number of zeros in  $\mathcal{C}([0, 1])$ .

However let  $\gamma : I \rightarrow \mathcal{M}(\mathcal{A}(I))$  be a continue section of  $\rho$ . Then the function  $r \mapsto \text{Ray}(M, \gamma(r))$  is a continuous function ([Ch-Dw]).

<sup>1</sup>Observe that the open set  $\rho^{-1}([0, r])$  is not an affinoid in the sense of [Ber] 2.2.1.

<sup>2</sup>This norm is the Gauss norm.

DEFINITION 1.2 ([**Astx**]). A differential module  $M$  over  $E$  (resp. over  $\mathcal{E}$ ) is called *soluble* if  $\text{Ray}(M, 1) = 1$ . A differential module  $M$  over  $\mathcal{A}(I)$  is called *soluble* at  $\rho \in I$  if  $\text{Ray}(M, \rho) = \rho$ . A differential module  $M$  over  $\mathcal{R}$  is called *soluble* if

$$(1.1.5) \quad \lim_{\rho \rightarrow 1^-} \text{Ray}(M, \rho) = 1.$$

1.1.2. The radius of convergence at  $|\cdot|_{c,r} \in \mathcal{M}(\mathcal{A}(I))$  of a differential module  $M$  can be viewed as the smallest radius of convergence of the solutions of  $M$  at some ‘‘incarnation’’  $t_{c,r}$  of  $|\cdot|_{c,r}$  (cf. 5.1.1). Observe that a function of  $\mathcal{A}(I)$  has no poles and no zeros in the generic disk  $D_\Omega(t_{c,r}, r^-)$ . Hence all points of  $D_\Omega(t_{c,r}, r^-)$  are non singular for all differential modules.

**1.2. Frobenius structure.** Let  $A$  be one of the rings  $E_\rho$ ,  $\mathcal{E}$ ,  $\mathcal{A}(I)$  or  $\mathcal{R}$  and let  $A^p$  be one of the rings  $E_{\rho^p}$ ,  $\mathcal{E}$ ,  $\mathcal{A}(I^p)$  or  $\mathcal{R}$  respectively. Let  $\sigma : K \rightarrow K$  be an automorphism of  $K$  such that  $|a^\sigma - a^p| < 1$ , for all  $a \in \mathcal{O}_K$ . For all functions  $f(x) = \sum_i a_i x^i \in A^p$  we set  $f^\sigma(x) := \sum_i a_i^\sigma x^i \in A$ . We define a Frobenius map  $\varphi : A^p \rightarrow A$  by

$$\varphi(f(x)) := f^\sigma(x^p).$$

This morphism defines an functor, called  $\varphi^*$  (cf. [**Astx**]), from the category of  $A^p$ -differential modules into the category of  $A$ -differential modules. Let  $M$  be the  $A^p$ -differential module, defined by  $\frac{d}{dx} + G(x)$ ,  $G(x) \in M_n(A^p)$ . The Frobenius functor sends  $M$  into the module  $\varphi^*(M)$  defined by  $\frac{d}{dx} + px^{p-1}G^\sigma(x^p)$ . In the appendix (cf. Corollary 5.7) we show that

$$\text{Ray}(\varphi^*(M), |\cdot|_{c,r}) = \min(\text{Ray}(M, |\cdot|_{c^p, r'})^{1/p}, |p|^{-1} \sup(|c|, r)^{1-p} \text{Ray}(M, |\cdot|_{c^p, r'}))$$

where  $r' = \max(r^p, |p||c|^{p-1}r)$  (cf. equation 5.2.3).

DEFINITION 1.3 (Frobenius structure). Let  $A$  be one of the rings  $E$ ,  $\mathcal{E}$  or  $\mathcal{R}$ . Let  $M$  be the differential module defined by  $\frac{d}{dx} + G(x)$ ,  $G(x) \in M_n(A)$  over  $A$ . We will say that  $M$  has a Frobenius structure of order  $h$  over  $A$  if there exists an  $A$ -isomorphism  $M \xrightarrow{\sim} \varphi_h^*(M)$ , where  $\varphi_h^*(M)$  is the differential module defined by  $\frac{d}{dx} + p^h x^{p^h-1} G^{\sigma^h}(x^{p^h})$ . In other words there exists an invertible matrix  $H(x) \in GL_n(A)$  such that

$$p^h x^{p^h-1} G^{\sigma^h}(x^{p^h}) = H(x)G(x)H^{-1}(x) + H(x)'H^{-1}(x).$$

THEOREM 1.4 ([**Ro**] 5.3). Let  $L := \frac{d}{dx} + g(x)$ ,  $g(x) \in K(x)$  be a soluble differential operator. By the Mittag-Leffler decomposition we may write  $g(x) = g^+(x) + \sum_{-n \leq i \leq -1} a_i x^i$ ,  $a_i \in K$ , where  $g^+(x) \in K(x)$  has no poles in  $D(0, 1^-)$ . Then  $L$  is isomorphic over the ring  $\mathcal{A}([0, 1])[1/x]$  to the operator  $\frac{d}{dx} + \sum_{-n \leq i \leq -1} a_i x^i$ .

THEOREM 1.5 ([**Ch-Ch**] 2.3.1). Let  $k$  be perfect. Let  $L = \frac{d}{dx} + \sum_{-n \leq i \leq -1} a_i x^i$ ,  $a_i \in K$ , be a soluble first order differential operator such that  $a_{-1} \in \mathbb{Z}_{(p)}$ . Then  $L$  has a Frobenius structure over  $\mathcal{R}$ . In other words there exist some  $h > 0$  and an invertible function  $f(x) \in \mathcal{R}^\times$  for which the following equality holds

$$\frac{f(x)'}{f(x)} = \left( \sum_{-n \leq i \leq -1} a_i x^i \right) - (p^h x^{p^h-1} \sum_{-n \leq i \leq -1} a_i^{\sigma^h} x^{ip^h})$$

## 2. Robba's exponentials

In this section  $z = x^{-1}$ . Let  $\{\xi_m\}_{m \geq 0}$  be a sequence of (primitive)  $p^{m+1}$ -roots of 1 such that  $\xi_m^{p^j} = \xi_{m-j}$ ,  $j \geq 0$  and such that  $\xi_0$  is a non trivial  $p$ -th root of 1. Let  $\pi_m := \xi_m - 1$ .

THEOREM 2.1. *For all  $m \geq 0$  the function*

$$(2.0.1) \quad E_m(z) := \exp(\pi_m z + \pi_{m-1} \frac{z^p}{p} + \cdots + \pi_0 \frac{z^{p^m}}{p^m}) \in \overline{K}[[z]]$$

*has radius of convergence equal to 1.*

*Proof:* Let  $E(z) = \exp(z + \frac{z^p}{p} + \frac{z^{p^2}}{p^2} + \cdots)$  be the Artin-Hasse exponential. If  $(\lambda_0, \lambda_1, \dots) \in W(\mathcal{O}_K)$ , then by a straightforward computation ([Bou] exercice 58-b) one shows that

$$\prod_{i \geq 0} E(\lambda_i z^{p^i}) = \exp(\phi_0 z + \phi_1 \frac{z^p}{p} + \phi_2 \frac{z^{p^2}}{p^2} \cdots)$$

where  $\phi_k = \lambda_0^{p^k} + p\lambda_1^{p^{k-1}} + \cdots + p^k \lambda_k$  are the phantom components of  $(\lambda_0, \lambda_1, \dots)$ . Since  $|\lambda_i| \leq 1$ , hence this infinite product defines a *bounded* analytic function on  $D(0, 1^-)$ . If  $(\lambda_0, \lambda_1, \dots) = (\xi_m, 0, \dots) - (1, 0, \dots)$ , then  $\phi_i = \pi_{m-i}$ . The fact that the radius of convergence of  $E_m(z)$  is exactly 1 will be a consequence of the fact that the operator

$$\frac{d}{dx} + E_m(z^{-1})'/E_m(z^{-1}) = \frac{d}{dx} - (\pi_m z^{-2} + \pi_{m-1} z^{-p-1} + \cdots + \pi_0 z^{-p^m-1})$$

is soluble<sup>3</sup> and its radius of convergence, for  $\rho$  close to 0, is  $\rho^{p^m+1}$  (cf. Corollary 3.1). Then, by the log-concavity property of the radius of convergence, we have that this operator has radius of convergence equal to  $\rho^{p^m+1}$ , for all  $\rho < 1$ .  $\square$

REMARK 2.2. Observe that for  $|z| < 1$  close to 1, we have  $|\pi_m z + \cdots + \pi_0 \frac{z^{p^m}}{p^m}| > |\pi_0|$ . On the other hand, the analytic function  $\exp(y)$  converges for  $|y| < |\pi_0|$ , so the convergent composition of  $\pi_m z + \cdots + \pi_0 \frac{z^{p^m}}{p^m}$  and  $\exp(y)$  does not exist. The precedent theorem asserts that the *formal composition*, after resummation, has radius of convergence equal to 1.

REMARK 2.3. Observe that in the formal case (cf. [Man]) a logarithmic derivative of a formal Laurent series has always an  $x$ -adic valuation  $\geq -1$ . But in the  $p$ -adic case the Robba-Matsuda's exponentials give an example of logarithmic derivatives of negative  $x$ -adic valuation. Then the definitions of  $p$ -adic irregularity and formal ( $x$ -adic) irregularity must be different (cf. [Ro]).

THEOREM 2.4 ([Ma]). *Let  $p \neq 2$ . Then the exponential  $E_m^\sigma(z^p)/E_m(z)$  is overconvergent. In other words, if  $p$  is different from 2, the differential operator*

$$\frac{d}{dx} + E_m^\sigma(x^{-1})'/E_m(x^{-1}) = \frac{d}{dx} - (\pi_m x^{-2} + \pi_{m-1} x^{-p-1} + \cdots + \pi_0 x^{-p^m-1})$$

*has a Frobenius structure of order 1.*

<sup>3</sup>The solubility of this operator is due to the fact that the convergent function  $E_m(z^{-1})$  is a solution of this operator at infinity and  $E_m(z^{-1})$  converges in the set  $\{x \in K \mid |x| > 1\}$ .

### 3. Formal slopes and $p$ -adic slopes

LEMMA 3.1 (Young, cf. [Astx]). *Let  $L := \sum_{s=0}^r g_s(x)(\frac{d}{dx})^s$  be a differential operator such that  $g_r(x) = 1$ ,  $g_s \in E_\rho$ ,  $s = 0, \dots, r-1$ . Let  $\rho \in I$ , then  $R(M, \rho) < |\pi_0|\rho$  if and only if  $|g_s|_\rho > \rho^{s-r}$  for some  $s < r$ , and in this case we have:*

$$R(M, |\cdot|_\rho) = |\pi_0| \min_{0 \leq s < r} (|g_s|_\rho^{-1/r-s})$$

Let  $M$  be a soluble  $p$ -adic differential module over  $\mathcal{R}$ . Then there exist  $0 < \varepsilon < 1$  and a rational number  $\beta \geq 0$  such that  $\text{Ray}(M, \rho) = \rho^{\beta+1}$  for all  $\rho \in ]1-\varepsilon, 1[$  (cf. [Astx]). If  $M$  is defined in some basis by the operator  $\frac{d}{dx} + G(x)$ , with  $G(x) \in M_n(\mathcal{A}([0, 1][1/x]))$ , then it is easy to show that there exist  $0 < \delta < 1$  and a rational number  $\alpha \geq 0$  such that  $\text{Ray}(M, \rho) = \rho^{\alpha+1}$  for  $\rho \in ]0, \delta[$ . By log-concavity we have  $\alpha \geq \beta$ .

DEFINITION 3.2. The number  $\beta$  is called the  $p$ -adic slope of  $M$ . We set  $pt(M) := \beta$ . If  $M$  is defined by the operator  $\frac{d}{dx} + G(x)$ ,  $G(x) \in \mathcal{A}([0, 1][1/x])$  we set  $pt_F(M) := \alpha$  and we will call  $pt_F(M)$  the formal slope. We have  $pt(M) \leq pt_F(M)$ .

REMARK 3.3. The precedent definition is justified by the fact that if  $M$  is defined by a linear differential operator  $L := \sum_{s=0}^r g_s(x)(\frac{d}{dx})^s$ ,  $g_r(x) = 1$ , such that  $g_s(x) \in \mathcal{A}([0, 1][1/x]) \subset \mathcal{R}$ , then  $pt_F(M)$  is the usual formal slope defined by

$$(3.0.2) \quad pt_F(L) = \max \left( 0, \max_s \left( \frac{s-r-v(g_s)}{r-s} \right) \right)$$

where  $v(g_s)$  is the  $x$ -adic valuation of  $g_s(x)$ . The formal slope is the largest slope of the Formal Newton polygon of  $L$ .<sup>4</sup> This follows from lemma 3.1 and some continuity and convexity arguments. Indeed, as  $M$  is soluble, observe that, for  $\rho$  close to 0, by continuity and log-concavity we have only two cases:  $\text{Ray}(M, \rho) = \rho$  or  $\text{Ray}(M, \rho) < |\pi_0|\rho$ .

LEMMA 3.4. *Let  $L = \frac{d}{dx} + \sum_{i \geq -d} a_i x^i$ ,  $a_{-d} \neq 0$ ,  $d \geq 1$  be a soluble rank one differential operator with  $\sum_{i \geq -d} a_i x^i \in \mathcal{R}$ . Then we have that  $\text{Ray}(L, \rho) = |\pi_0| |a_{-d}|^{-1} \rho^d$ .*

### 4. Frobenius structure over $\mathcal{R}$

4.1. **Reduction to a  $K[x, x^{-1}]$ -lattice.** Let  $L := \frac{d}{dx} + g(x)$ ,  $g(x) = \sum_i a_i x^i \in \mathcal{R}$ , be a rank one soluble differential operator. In this section we show that  $L$  is isomorphic over  $\mathcal{R}$  to the operator  $\tilde{L} = \frac{d}{dx} + \sum_{-d \leq i \leq -1} a_i x^i$  for a suitable  $d \geq 1$  (cf. Theorem 4.7). Moreover, if  $K = \overline{K}$ , then  $d$  can be choosed equal to  $pt(L) + 1$ .

LEMMA 4.1. *The operator  $L := \frac{d}{dx} + g(x)$ ,  $g(x) = \sum_i a_i x^i \in \mathcal{R}$  is isomorphic over  $\mathcal{R}$  to the operator  $\tilde{L} := \frac{d}{dx} + \sum_{-d \leq i \leq \infty} a_i x^i$  for a suitable  $d \geq 1$ .*

*Proof:* Let  $d \geq 1$  be an integer such that  $\sup(|\frac{a_i}{i+1}| \rho^{i+1}) < |\pi_0|$ , for all  $i < -d$ . Such an integer exists because  $g(x) \in \mathcal{R}$ . Then the series  $f(x) := \exp(-\sum_{i < -d} \frac{a_i}{i+1} x^{i+1})$  lies in  $\mathcal{R}$  and then  $L$  is isomorphic over  $\mathcal{R}$  to the operator  $\tilde{L} = \frac{d}{dx} + g(x) + f(x)'/f(x)$ .  $\square$

<sup>4</sup>We recall that the formal Newton polygon is the convex hull of the set formed by the points of the form  $(s, v(a_s) - s)$  and the two additional points  $(-\infty, 0)$ ,  $(0, +\infty)$ . In particular we observe that in our case the last point is  $(r, -r)$ .

LEMMA 4.2. Let  $L := \frac{d}{dx} + g(x)$ ,  $g(x) = \sum_{-d \leq i \leq \infty} a_i x^i$ ,  $d \geq 2$  be a soluble differential operator over  $\mathcal{R}$ . Then  $|a_{-d}| \leq |\pi_0|$ .

*Proof:* Let  $g_s(x) = g_{s-1}(x)' + g_s(x)g(x)$  be as in the equation 1.1.2. An explicit computation shows that  $g_s(x) = a_{-d}^s x^{-sd} + \{\text{terms of degree } \geq -sd + 1\}$ . So the equation 1.1.1 shows that  $\text{Ray}(L, \rho) \leq \frac{|\pi_0| \rho^d}{|a_{-d}|}$ . The solubility implies  $1 \leq |\pi_0|/|a_{-d}| \cdot \rho^d$ .  $\square$

LEMMA 4.3 ([Ch-Ro] 11.2.4). Let  $L = \frac{d}{dx} + a_{-1}x^{-1}$  be a differential equation. Then  $L$  is soluble if and only if  $a_{-1} \in \mathbb{Z}_p$ .

LEMMA 4.4 ([Ch-Ro] 18.4.4). Let  $L = \frac{d}{dx} + a_{-1}x^{-1}$  be a differential equation. Then  $L$  has a Frobenius structure if and only if  $a_{-1} \in \mathbb{Z}_{(p)}$ .

LEMMA 4.5. Let  $L = \frac{d}{dx} + a_{-1}x^{-1}$ . Let

$$\alpha(a_{-1}) := \limsup_n (|a_{-1}(a_{-1}-1)(a_{-1}-2) \cdots (a_{-1}-n+1)|^{\frac{1}{n}}).$$

Then for all  $\rho > 0$  we have  $\text{Ray}(L, \rho) = \frac{|\pi_0|}{\alpha(a_{-1})} \cdot \rho$ .

*Proof:* A computation shows that  $g_n(x) = \alpha_n(a_{-1})x^{-n}$ , where  $\alpha_n(a_{-1}) := a_{-1}(a_{-1}-1) \cdots (a_{-1}-n+1)$ . So  $\text{Ray}(L, \rho) = \liminf_n (|\alpha_n(a_{-1})|^{-1/n}) |\pi_0| \rho$ .  $\square$

LEMMA 4.6. Let  $L = \frac{d}{dx} + g(x)$ ,  $g(x) = \sum_{i \geq -1} a_i x^i \in \mathcal{R}$  be a soluble differential equation. Then  $a_{-1} \in \mathbb{Z}_p$ . Moreover there exists an analytic function  $f(x) \in \mathcal{A}([0, 1[)$  such that  $f'(x)/f(x) = \sum_{i \geq 0} a_i x^i$ . In other words  $L$  is isomorphic to  $\frac{d}{dx} + a_{-1}x^{-1}$ .

*Proof:* We proceed by absurd.  $L$  is the tensor product of  $L_1 := \frac{d}{dx} + a_{-1}x^{-1}$  and  $L_2 := \frac{d}{dx} + \sum_{i \geq 0} a_i x^i$ . As  $L_2$  has a convergent solution in 0, we have  $\text{Ray}(L_2, \rho) = \rho$ , for  $\rho$  sufficiently close to 0. On the other hand, by the lemma 4.5 we have  $\text{Ray}(L_1, \rho) = \frac{|\pi_0|}{\alpha(a_{-1})} \cdot \rho$  for all  $\rho > 0$ . The radius of the tensor product of two operators with different radius is the minimum of the radius.<sup>5</sup> So, if  $L_1$  is not soluble, then  $\text{Ray}(L_1, 1) = \text{Ray}(L_2, 1) = R < 1$ . Then  $\text{Ray}(L_1, \rho) = R$  for all  $\rho > 0$ . But in this case, by convexity,  $\text{Ray}(L_2, \rho) > R$  for  $\rho < 1$ . Then if  $\rho < 1$  we would get  $\text{Ray}(L, \rho) = \min(\text{Ray}(L_1, \rho), \text{Ray}(L_2, \rho)) = R$ . Since  $L$  is soluble, hence by continuity of  $\text{Ray}(L, \rho)$  we get a contraddiction.

We have shown that  $\text{Ray}(L_1, \rho) = \text{Ray}(L_2, \rho) = \text{Ray}(L, \rho) = 1$ , for all  $0 < \rho \leq 1$ . Now by transfert theorem ([Ch-Ro] 9.3.2) there exists a convergent solution  $f(x)$  of  $L_2$  in the disk  $D(0, 1^-)$ . In particular  $f \in \mathcal{R}$  and  $f'(x)/f(x) = \sum_{i \geq 0} a_i x^i$ .  $\square$

THEOREM 4.7. Let  $L := \frac{d}{dx} + g(x)$ ,  $g(x) = \sum_i a_i x^i \in \mathcal{R}$  be a rank one soluble differential operator. Then  $a_{-1} \in \mathbb{Z}_p$  and there exists a  $d \geq \max(1, \text{pt}(M) + 1)$  such that  $L$  is isomorphic, over  $\mathcal{R}$ , to  $\frac{d}{dx} + \sum_{-d_1 \leq i \leq -1} a_i x^i$ , for all  $d_1 \geq d$ . Moreover

<sup>5</sup>Observe that this fact depends on the definition given in the equation 1.1.1. For example  $(x-1)(x-1)^{-1} = 1$ , so in this case the radius of 1 is not equal to the minimum of the other two radius. This depends on the definition of "radius". In general let  $L_1 = \frac{d}{dx} + g(x)$ ,  $L_2 = \frac{d}{dx} + f(x)$  be two differential operators. Suppose that  $g, f \in \mathcal{A}(I)$ . Let  $s_i(x)$  be a power series solution of  $L_i$  at  $a \in \mathcal{C}(I)$ . Let  $r = \inf(|a-b|, b \in \overline{K} - C_{\overline{K}}(I))$ . Then for all  $\rho \leq r$  we have the equality

$$\min(\text{Ray}(s_1 \cdot s_2), \rho) = \min(\min(\text{Ray}(s_1), \rho), \min(\text{Ray}(s_2), \rho)).$$

for all  $-pt(M) - 1 \leq i \leq -2$ , there exist  $b_i \in \overline{K}$  such that  $L$  is isomorphic, over  $\mathcal{R}_{\overline{K}}$ , to  $\frac{d}{dx} + \sum_{-pt(M)-1 \leq i \leq -2} b_i x^i + a_{-1} x^{-1}$ , and  $|b_{-pt(M)-1}| = |\pi_0|$ .

*Proof:* By the lemma 4.1 we can suppose that  $L = \frac{d}{dx} + \sum_{-d \leq i \leq \infty} a_i x^i$ , with  $d \geq 1$ . By the lemma 4.6 we can suppose that  $d \geq 2$ . The theorem will be proved applying the lemma 4.6. To apply this lemma we need that the operator  $\frac{d}{dx} + \sum_{i \geq -1} a_i x^i$  is soluble. The solubility is invariant by extension of the field of constants and then we can actually suppose that  $K = \overline{K}$ .

Let  $d - 1 = n \cdot p^m$ ,  $(n, p) = 1$  and let  $(\frac{a-d}{n \cdot \pi_0})^{1/p^m}$  be a  $p^m$ -th root of  $\frac{a-d}{n \cdot \pi_0}$ . Let us introduce the following analytic function:

$$f_d(x) := E_m \left( \left( \frac{a-d}{n \cdot \pi_0} \right)^{1/p^m} x^{-n} \right) = \exp \left( \pi_m \left( \frac{a-d}{n \cdot \pi_0} \right)^{1/p^m} x^{-n} + \dots + a_{-d} \frac{x^{-(d-1)}}{d-1} \right).$$

By the lemma 4.2 we have  $|a_{-d}| \leq |\pi_0|$ . Then, as  $|n| = 1$ ,  $f_d$  is an analytic function in  $\mathcal{C}([1, \infty[)$ . The logarithmic derivative  $f'_d/f_d = -a_{-d} x^{-d} + \dots + (-n) \cdot \pi_m \left( \frac{a-d}{n \cdot \pi_0} \right)^{1/p^m} x^{-n-1}$  defines a differential operator

$$L_d := \frac{d}{dx} + f'_d/f_d$$

which is soluble because its solution at infinity is  $f_d(x)$ .

We proceed now by induction on  $d \geq 2$ . If  $|a_{-d}| < |\pi_0|$ , then  $f_d$  is an element of  $\mathcal{R}$  and then the operator  $\frac{d}{dx} + g(x)$  is isomorphic to  $\frac{d}{dx} + g(x) + f'_d/f_d$ . Now the  $x$ -adic valuation of the function  $g(x) + f'_d/f_d$  is strictly larger than  $-d$ .

Otherwise, if  $|a_{-d}| = |\pi_0|$ , then the operator  $L_d$  is soluble and not trivial by the transfert theorem at infinity ([**Ch-Ro**] 9.3.2).<sup>6</sup> So the tensor product operator  $L \otimes L_d$ , defined by  $\frac{d}{dx} + g(x) + f'_d/f_d$ , is still soluble, and  $g(x) + f'_d/f_d$  is of degree  $\geq -d+1$  and we can proceed by induction on  $d$ . Observe that  $f'_d/f_d$  is a polynomial in  $x^{-2}\overline{K}[x^{-1}]$ . Iterating, we get that there exist functions  $f_d, \dots, f_2 \in \mathcal{A}_{\overline{K}}([1, \infty[)$  which are analytic in  $\mathcal{C}([1, \infty[)$  and are such that

$$g(x) = f'_d/f_d + \dots + f'_2/f_2 + \left( \sum_{i \geq -1} a_i x^i \right).$$

Then  $\frac{d}{dx} + \sum_{i \geq -1} a_i x^i$  is soluble. Observe that, as in the proof of lemma 4.6, if  $|a_{-d}| = |\pi_0|$  we have  $\text{Ray}(L, \rho) = \min_i \text{Ray}(L_i, \rho) = \text{Ray}(L_d, \rho)$ , then

$$\text{Ray}(L, \rho) = \text{Ray} \left( \frac{d}{dx} + f'_d/f_d, \rho \right) = |\pi_0| |a_{-d}|^{-1} \rho^d = \rho^d, \quad \forall 0 < \rho < 1.$$

And in this case  $d = pt(M) + 1$ . Moreover if  $d > pt(M) + 1$ , then it must be  $|a_{-d}| < |\pi_0|$  and then  $f_d \in \mathcal{R}_{\overline{K}}$ . Iterating this process we can show that, over  $\overline{K}$ , we can obtain  $d = \max(1, pt(M) + 1)$ .  $\square$

**REMARK 4.8.** Observe that in the proof of the precedent lemma we show that, over  $\overline{K}$ ,  $L$  is isomorphic to the operator

$$(4.1.1) \quad \frac{d}{dx} + f'_d/f_d + \dots + f'_2/f_2 + a_{-1} x^{-1}.$$

In other words  $L$  is the tensor product of the operators  $L_d := \frac{d}{dx} + f'_d/f_d, \dots, L_2 := \frac{d}{dx} + f'_2/f_2$ , and  $L_{-1} := \frac{d}{dx} + a_{-1} x^{-1}$ , where  $f_i(x)$  are functions obtained from a

<sup>6</sup>Indeed if the operator  $L_d$  is trivial over  $\mathcal{R}$  then  $\text{Ray}(L_d, \rho) = \rho$  for all  $\rho > 1 - \varepsilon$ ,  $\exists \varepsilon$  and by the transfert theorem the solution at infinity converges in the disk  $\{|x| > 1 - \varepsilon\}$ .



Robba's exponential by a substitution of the variable. Moreover  $L_i$  is trivial or  $\text{Ray}(L_i, \rho) = \rho^i$  for all  $\rho < 1$ . Over  $\overline{K}$  we can choose  $d = pt(M) + 1$ . This is a kind of canonical form for rank one soluble differential equation over  $\mathcal{R}_{\overline{K}}$ .

#### 4.2. Frobenius structure over $\mathcal{R}$ .

**THEOREM 4.9.** *Let  $k$  be a perfect field of characteristic  $p > 0$ . Let  $L = \frac{d}{dx} + g(x)$ ,  $g(x) = \sum_i a_i x^i \in \mathcal{R}$  be a soluble differential equation. Then  $L$  has a Frobenius structure if and only if  $a_{-1} \in \mathbb{Z}_{(p)}$ .*

*Proof:* The operator  $L$  is isomorphic to the operator  $\frac{d}{dx} + \sum_{-d \leq i \leq -1} a_i x^i$ . We can now apply the theorem of Christol-Chiarellotto 1.5.  $\square$

**COROLLARY 4.10.** *The Robba-Matsuda operator  $\frac{d}{dx} + \frac{E'_m(x^{-1})}{E_m(x^{-1})}$  (cf. equation 2.0.1) has a Frobenius structure for all  $m \geq 0$ .*

*Proof:* We must show that the formal series  $E_m^{\sigma^h}(x^{-p^h})/E_m(x^{-1})$  defines a function of  $\mathcal{R}$  for some  $h \geq 1$ . Since the convergence does not change by extension of the field of constants, hence we can suppose  $K = \overline{K}$ . We can actually apply the theorem 4.9.  $\square$

**REMARK 4.11.** The corollary 4.10 works for all  $p > 0$ . On the other hand, the proof of Matsuda (cf. Theorem 2.4) is a very strong computation and is stronger than our result because it shows that, if  $p \neq 2$ , the operator  $\frac{d}{dx} + E_m(x^{-1})'/E_m(x^{-1})$  has a Frobenius structure of order 1.

### 5. Appendix: Variation of Radius of convergence by ramifications.

In this section we precise some known (but not published) facts about the variation of the radius of convergence of the pull-back of a module by a covering of the form  $x \mapsto x^n$ . We study the ramification  $\phi_n^* : f(x) \mapsto f(x^n)$  instead of  $\varphi$ , because the application  $f(x) \mapsto f^\sigma(x)$  (cf. 1.2) defines an auto-equivalence of the category of differential modules which preserves the radius of convergence.

**5.1. Scalar extension.** Let  $I$  be a (non empty) interval. Let  $L/K$  be an extension of valued fields. Let  $\mathcal{A}_L(I)$  denote the ring of analytic functions over  $I$ , with coefficients in  $L$ . Then we have the following diagram

$$\begin{array}{ccc} \mathcal{M}(\mathcal{A}_K(I)) & \xleftarrow{\psi_L} & \mathcal{M}(\mathcal{A}_L(I)) \\ \cup & & \cup \\ \mathcal{C}_K(I) & \subseteq & \mathcal{C}_L(I) \end{array}$$

where the vertical inclusions are the canonical inclusions  $a \mapsto | \cdot |_a$  (cf. remark 1.1) and the map  $\psi_L$  is the functorial morphism of analytic spaces corresponding to the inclusion  $\mathcal{A}_K(I) \subseteq \mathcal{A}_L(I)$ . This diagram is commutative in the sense that the inclusion  $\mathcal{C}_K(I) \subseteq \mathcal{C}_L(I)$  is a section of the map  $\psi_L$ .

5.1.1. By [Ch-Ro] 9.1 there exists a field  $\Omega$  such that  $\psi_\Omega(\mathcal{C}_\Omega(I)) = \mathcal{M}(\mathcal{A}_K(I))$ . In this sense all points  $| \cdot |_t$  of  $\mathcal{M}(\mathcal{A}_K(I))$  have an "incarnation" in a true point of  $\mathcal{C}_\Omega(I)$ . If  $t \in \mathcal{C}_\Omega(I)$  is an incarnation of the seminorm  $| \cdot |_t \in \mathcal{M}(\mathcal{A}_K(I))$  (i.e.  $\psi_\Omega(t) = | \cdot |_t$ ) we will say that  $D_\Omega(t, \rho(| \cdot |_t)^-)$  is a generic disk for  $| \cdot |_t$  (cf. 1.1.1).

**5.2. Ramifications and image of a point of Berkovich.** Let  $\phi_n^* : \mathcal{A}(I^n) \rightarrow \mathcal{A}(I)$  be the morphism  $\sum_i a_i x^i \mapsto \sum_i a_i x^{ni}$ . Let  $\phi_n : \mathcal{M}(\mathcal{A}(I)) \rightarrow \mathcal{M}(\mathcal{A}(I^n))$  be the corresponding morphism of analytic spaces. In this section we compute the image of a point of  $\mathcal{M}(\mathcal{A}(I))$  by the ramification  $\phi_n$ .

5.2.1. By a result of Berkovich ([Ber] 1.4.4) we know that every point of  $\mathcal{M}(\mathcal{A}(I))$  is a limit of points of the form  $|\cdot|_{c,r}$ ,  $c \in K$

$$|f(x)|_{c,r} := \sup_{x \in D(c,r^-)} |f(x)|$$

In other words if  $|\cdot|_t$  is a point of  $\mathcal{M}(\mathcal{A}(I))$ , then  $|\cdot|_t$  is the seminorm attached to some disk, or  $|\cdot|_t$  is the seminorm attached to a totally ordered<sup>7</sup> sequence of disks. If  $K$  is spherically complete then all points are of the form  $|\cdot|_{c,r}$  and, since  $K$  is contained in some spherically complete and algebraically closed field  $K'$ , hence we can suppose that all points of  $\mathcal{M}(\mathcal{A}(I))$  and  $\mathcal{M}(\mathcal{A}(I^n))$  are of the form  $|\cdot|_{c,r}$  for a suitable  $c \in K'$ . For simplicity we will suppose that  $K = K'$ .

REMARK 5.1. Let  $t_{c,r}$  be an incarnation of  $|\cdot|_{c,r}$ . It is clear that  $\rho(|\cdot|_{c,r}) = r$  (cf. 1.1), and that  $|t_{c,r}|_\Omega = \max(|c|, r)$ .

5.2.2. Rational fractions are dense in  $\mathcal{A}(I)$ , hence to compute the seminorm  $|\cdot|_{c',r'} := \phi_n(|\cdot|_{c,r})$  it is enough to know the value  $|x - a|_{c',r'}$  for all  $a \in K$ . On the other hand  $\phi_n(|\cdot|_{c,r}) = |\cdot|_{c,r} \circ \phi_n^*$ , then  $|x - a|_{c',r'} = |\phi_n^*(x - a)|_{c,r} = |x^n - a|_{c,r}$ . We write  $x^n - a = [(x - c) + c]^n - a = [\sum_{i=1}^n \binom{n}{i} c^{n-i} (x - c)^i] + (c^n - a)$ . Then we have that

$$(5.2.1) \quad |x^n - a|_{c,r} = \sup_{1 \leq i \leq n} (|\binom{n}{i}| |c|^{n-i} r^i, |c^n - a|)$$

On the other hand  $|x - a|_{c',r'} = \sup(r', |c' - a|)$ . Therefore  $c' = c^n$  and

$$(5.2.2) \quad r' = \sup_{1 \leq i \leq n} (|\binom{n}{i}| |c|^{n-i} r^i) = |c|^n \sup_{1 \leq i \leq n} (|\binom{n}{i}| (r/|c|)^i).$$

We can compute  $r'$  in some particular case:

$$(5.2.3) \quad r' = \begin{cases} \min(r^p, |p| |c|^{p-1} r) & \text{if } n = p \\ \min(r^n, |c|^{n-1} r) & \text{if } (n, p) = 1. \end{cases}$$

This process can be applied to compute the image of  $|\cdot|_{c,r}$  under the action of an arbitrary polynomial map instead of  $\phi_n$ . To recover the value of  $r'$  it is sufficient to look at the Taylor's development of this polynomial at  $c$ .

COROLLARY 5.2 (Deformation of the Generic Disk). *Let  $D_\Omega(t_{c,r}, r^-)$  be a generic disk for  $|\cdot|_{c,r}$  (cf. 5.1.1). Then  $t_{c,r}^n$  is an incarnation of  $|\cdot|_{c^n, r'} := \phi_n(|\cdot|_{c,r})$  and  $\phi_n(D_\Omega(t_{c,r}, r^-)) \subseteq D_\Omega(t_{c,r}^n, r'^-)$ . More precisely let  $y \in D_\Omega(t_{c,r}, r^-)$ . If  $(n, p) = 1$ , then*

$$(5.2.4) \quad |y^n - t_{c,r}^n| = |t_{c,r}|^{n-1} \cdot |y - t_{c,r}|.$$

If  $n = p$ , then

$$(5.2.5) \quad |y^p - t_{c,r}^p| \leq \max(|t_{c,r}|^{p-1} |p| |y - t_{c,r}|, |y - t_{c,r}|^p)$$

and the equality holds if  $|t_{c,r}|^{p-1} |p| |y - t_{c,r}| \neq |y - t_{c,r}|^p$ .

<sup>7</sup>Ordered by inclusion.

*Proof:* In all cases we have that  $|y - t_{c,r}| < |t_{c,r}|$  (cf. 5.1). If  $(n, p) = 1$  we have  $|y^n - t_{c,r}^n| = |(y - t_{c,r} + t_{c,r})^n - t_{c,r}^n| = |t_{c,r}|^n \left| \sum_{i=1}^n \binom{n}{i} ((y - t_{c,r})/t_{c,r})^i \right|$ . Observe that  $|\binom{n}{1}| = 1$ . If  $n = p$ , observe that  $|\binom{p}{i}| = |p|$  for all  $i = 1, \dots, p-1$ , then the same computation gives that  $|y^n - t_{c,r}^n| \leq |t_{c,r}|^{p-1} |y - t_{c,r}| \cdot \sup(|p|, |(y - t_{c,r})/t_{c,r}|^{p-1})$ .  $\square$

**5.3. Variation of the radius of convergence by ramification.** Let  $M$  be an  $\mathcal{A}(I^p)$ -differential module. In this section we compute the radius of convergence at  $|\cdot|_{c,r}$  of the pull-back  $\mathcal{A}(I^p)$ -differential module  $\phi_n^*(M)$ . Observe that the radius of  $\varphi^*(M)$  and  $\phi_p^*(M)$  are equal (cf. 5).

5.3.1. *Frobenius:* Let  $s(x) = \sum_{i=0}^{\infty} a_i(x - t_{c,r}^p)^i$  be a convergent analytic function at  $t_{c,r}^p$ . Let  $R = \liminf_i |a_i|^{-1/i}$  be the radius of convergence of  $s(x)$  at  $t_{c,r}^p$ . Let  $\phi_p^*(s)(y) := s(y^p) = \sum_i a_i(y^p - t_{c,r}^p)^i$  be its pull-back and  $R'$  the radius of  $\phi_p^*(s)(y)$  at  $t_{c,r}$ . By composition we have (cf. corollary 5.2)

$$(5.3.1) \quad R' \geq \min(|p|^{-1} |t_{c,r}|^{1-p} R, R^{1/p})$$

LEMMA 5.3. *Let  $x_l \in \mathbb{R}$  be a sequence. Then for all  $m \in \mathbb{Z}$  we have:*

$$\liminf_l (x_l) = \min_{0 \leq \varepsilon \leq m-1} (\liminf_{l \in \varepsilon + m\mathbb{Z}} (x_l)) \leq \liminf_{l \in m\mathbb{Z}} (x_l).$$

LEMMA 5.4. *Let  $i, n, l \in \mathbb{N}$ ,  $i, n \geq 1$ ,  $i \leq l \leq n \cdot i$ . Let*

$$B(i, l, n) := \sum_{\substack{j_1 + \dots + j_i = l \\ 1 \leq j_k \leq n}} \binom{n}{j_1} \cdots \binom{n}{j_i}.$$

*Then  $B(i, i, n) = n^i$  and  $B(i, in, n) = 1$ .*

LEMMA 5.5. *Let  $m \in \mathbb{N}$ ,  $m \geq 1$ . Let  $\{c_l\}_l$  be a sequence in some ultrametric ring. Let  $R := \liminf_l |c_l|^{-1/l}$ . If  $R > 0$ , then we have*

$$(5.3.2) \quad \liminf_{l \in m\mathbb{Z}} |c_{\frac{l}{m}}|^{-\frac{1}{l}} = R^{\frac{1}{m}}$$

*Proof:* This equation is equivalent to the equation  $\liminf_l |c_l|^{-\frac{1}{ml}} = R^{\frac{1}{m}}$ .  $\square$

THEOREM 5.6. *We have  $R' = \min(R|p|^{-1} |t_{c,r}|^{1-p}, R^{1/p})$ .*

*Proof:* We write  $y^p - t_{c,r}^p = (y - t_{c,r} + t_{c,r})^p - t_{c,r}^p$ . We have  $s(y^p) = \sum_i a_i \cdot (\sum_{j=1}^p \binom{p}{j} t_{c,r}^{p-j} y^j)^i$ . After a resommation we get

$$s(y^p) = \sum_{l=0}^{\infty} \left[ \sum_{i=\lceil \frac{l}{p} \rceil}^l a_i \cdot B(i, l, p) \cdot t_{c,r}^{ip-l} \right] (y - t_{c,r})^l$$

where  $\lceil l/p \rceil := \min(i \in \mathbb{N} \mid i \geq l/p)$ . Since the term  $\sum_{i=\lceil \frac{l}{p} \rceil}^l a_i \cdot B(i, l, p) \cdot t_{c,r}^{ip-l}$  is a polynomial in  $t_{c,r}$  with coefficients in  $K$ , hence its valuation is given by  $\sup_{i=\lceil \frac{l}{p} \rceil, \dots, l} (|a_i| \cdot |B(i, l, p)| \cdot |t_{c,r}|^{ip-l})$ . We have then

$$R' = \liminf_l \left( \sup_{i=\lceil \frac{l}{p} \rceil, \dots, l} (|a_i| \cdot |B(i, l, p)| \cdot |t_{c,r}|^{ip-l}) \right)^{-1/l}.$$

Applying the lemma 5.3 we have

$$(5.3.3) \quad R' \leq \liminf_{l \in p\mathbb{Z}} \left( \sup_{l/p \leq i \leq l} (|a_i| \cdot |B(i, l, p)| \cdot |t_{c,r}|^{ip-l}) \right)^{-1/l}.$$

By the lemma 5.4 we have  $B(l/p, l, p) = 1$  and  $B(l, l, p) = p^l$  and then clearly  $\sup_{l/p \leq i \leq l} (|a_i| \cdot |B(i, l, p)| \cdot |t_{c,r}|^{ip-l}) \geq \sup(|a_{l/p}|, |a_l| \cdot |p|^l \cdot |t_{c,r}|^{l(p-1)})$ . This fact and the lemma 5.5 show that

$$R' \leq \liminf_{l \in p\mathbb{Z}} \left( \sup(|a_{l/p}|, |a_l| \cdot |p|^l \cdot |t_{c,r}|^{l(p-1)}) \right)^{-\frac{1}{l}} = \inf(R^{1/p}, R|p|^{-1}|t_{c,r}|^{1-p}). \quad \square$$

Recalling that  $|t_{c,r}| = \sup(|c|, r)$  (cf. remark 5.1), we can state the following

**COROLLARY 5.7.** *Let  $A$  be one of the rings  $E_\rho$ ,  $\mathcal{A}(I)$ ,  $\mathcal{E}$  or  $\mathcal{R}$ . Let  $A^p$  be the ring  $E_\rho^p$ ,  $\mathcal{A}(I^p)$ ,  $\mathcal{E}$  or  $\mathcal{R}$ . respectively. Let  $M$  be a differential module over  $A^p$ , let  $\phi_p^*(M)$  be its pull-back over  $A$  by the morphism  $f(x) \rightarrow f(x^p) : A^p \rightarrow A$ . Then we have*

$$\text{Ray}(\phi_p^*(M), |\cdot|_{c,r}) = \min \left( \text{Ray}(M, |\cdot|_{c^p, r'})^{1/p}, |p|^{-1} \sup(|c|, r)^{1-p} \text{Ray}(M, |\cdot|_{c^p, r'}) \right)$$

where  $r' = \max(r^p, |p||c|^{p-1}r)$  (cf. equation 5.2.3).

**5.3.2. Ramification prime to  $p$ .** Let  $(n, p) = 1$ . Following the same method of the precedent discussion we get that  $R' = |t_{c,r}|^{1-n}R$ , and the following

**THEOREM 5.8.** *Let  $A$  be one of the rings  $E_\rho$ ,  $\mathcal{A}(I)$ ,  $\mathcal{E}$  or  $\mathcal{R}$ . Let  $A^n$  be one of the rings  $E_\rho^n$ ,  $\mathcal{A}(I^n)$ ,  $\mathcal{E}$  or  $\mathcal{R}$ . Let  $M$  be a differential module over  $A^n$ , let  $\phi_n^*(M)$  be its pull-back over  $A$ . Then we have*

$$\text{Ray}(\phi_n^*(M), |\cdot|_{c,r}) = \sup(|c|, r)^{1-n} \text{Ray}(M, |\cdot|_{c^n, r'})$$

where  $r' = \max(r^n, |c|^{n-1}r)$  (cf. equation 5.2.3).

## References

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