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Frobenius structure for rank one p -adic differential equations.

Andrea Pulita

ABSTRACT. We generalize to all rank one p -adic differential equations over \mathcal{R} the theorem 2.3.1 of [Ch-Ch] which provides the existence of a Frobenius structure of order h for soluble rank one operators of the form $\frac{d}{dx} + g(x)$, $g(x) \in x^{-2}K[x^{-1}]$. It follows a generalization of a theorem of Matsuda which asserts that the Robba's exponential $\exp(\sum_{i=0}^m \pi_{m-i} x^{p^i} / p^i)$ has a Frobenius structure. Namely our theorem works in the case $p = 2$.

In the appendix we describe the variation of the radius of convergence of a differential module by pull-back by a Kummer ramification.

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1. Notations

Let K be a complete field with respect to an ultra-metric absolute value $|\cdot|$. Let $\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$ be the ring of integers of K , and let $D(0, 1^-) = \{x \in K \mid |x| < 1\}$ be its maximal ideal. Let k be its residue field which will be supposed to be a perfect field of characteristic $p > 0$. Let \bar{K} be an algebraic closure of K .

Let E (resp. E_ρ) be the completion of $K(x)$ for the Gauss norm (resp. the norm $|\cdot|_\rho$ defined by $|\sum_i a_i x^i|_\rho := \sup_i |a_i| \rho^i$).

Let \mathcal{E} be the Amice's ring. The elements of \mathcal{E} are bounded series $f = \sum_{i \in \mathbb{Z}} a_i x^i$, $a_i \in K$, $|a_i| \rightarrow 0$ for $i \rightarrow -\infty$, for which there exists a constant $M(f) \in \mathbb{R}$ such that $|a_i| \leq M(f)$ for all $i \in \mathbb{Z}$. The topology of \mathcal{E} is defined by the Gauss norm and \mathcal{E} is a complete ring. We have a canonical isometric embedding $K(x) \subset E \subset \mathcal{E}$.

Let $I \subseteq \mathbb{R}_{\geq 0}$ be an interval. Let $\mathcal{C}(I) := \{x \in K \mid |x| \in I\}$. Let $\mathcal{A}(I)$ be the ring of analytic functions over $\mathcal{C}(I)$, the elements of $\mathcal{A}(I)$ are power series $f = \sum_{i \in \mathbb{Z}} a_i x^i$, $a_i \in K$, such that $\lim_{i \rightarrow \pm\infty} |a_i| \rho^i = 0$, for all $\rho \in I$. $\mathcal{A}(I)$ is complete for the topology defined by the family of absolute values $\{|\cdot|_\rho\}_{\rho \in I}$, where

$$|f|_\rho := \sup_{i \in \mathbb{Z}} |a_i| \rho^i.$$

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Let \mathcal{R} be the Robba's Ring. The elements f of \mathcal{R} are germs of convergent analytic functions at the edge of $D(0, 1^-)$, namely

$$(1.0.1) \quad \mathcal{R} = \cup_{0 < \varepsilon < 1} \mathcal{A}(|1 - \varepsilon, 1|).$$

In other words \mathcal{R} is the inductive limit of the sequence $\mathcal{A}(r_1, 1) \subset \mathcal{A}(r_2, 1)$, $0 < r_1 < r_2 < 1$ and it is equipped with the limit topology.

All the rings E_ρ , \mathcal{E} , $\mathcal{A}(I)$, \mathcal{R} are differential rings with respect to the continue derivation $\frac{d}{dx}$.

1.1. Berkovich spaces and p -adic differential equations. Let $\mathcal{M}(\mathcal{A}(I))$ be the analytic space (in the sense of Berkovich [Ber] 1.2) attached to the affinoid algebra $\mathcal{A}(I)$ ([Ber] 2.1.1). The Berkovich's point defined by the norm $|\cdot|_\rho$ can be (and will be) identified with the Dwork's generic point t_ρ of radius ρ ([Ch-Ro], 9.1.2). Following this identification let M be the differential module defined by $\frac{d}{dx} + G(x)$, $G(x) \in M_n(\mathcal{A}(I))$, then the radius of convergence of M at the point $|\cdot|_\rho$ is defined as

$$(1.1.1) \quad \text{Ray}(M, |\cdot|_\rho) := \inf \left(\liminf_s \left(\frac{|G_s(x)|_\rho}{|s!|} \right)^{-1/s}, \rho \right)$$

where $G_s \in M_n(\mathcal{A}(I))$ is defined by the recursion formula

$$(1.1.2) \quad G_{s+1} = \frac{d}{dx}(G_s) + G_s \cdot G, \quad G_1 := G.$$

The function $\rho \mapsto \text{Ray}(M, |\cdot|_\rho)$ is continuous and there exists a partition $I = \cup_{j \in \mathbb{Z}} I_j$, $\sup I_j = \inf I_{j+1}$, such that $\text{Ray}(M, |\cdot|_\rho) = \alpha_j \rho^{\beta_j}$, for all $\rho \in I_j$.

For simplicity we will write $\text{Ray}(M, \rho)$ instead of $\text{Ray}(M, |\cdot|_\rho)$.

1.1.1. More generally let $|\cdot|_t \in \mathcal{M}(\mathcal{A}(I))$ be a bounded multiplicative semi-norm ([Ber] 1.2). We define the *radius of the generic disk of center $|\cdot|_t$* as

$$(1.1.3) \quad \rho(|\cdot|_t) := \inf(|x - a|_t \mid a \in \overline{K})$$

and we put

$$(1.1.4) \quad \text{Ray}(M, |\cdot|_t) := \inf \left(\liminf_s \left(\frac{|G_s(x)|_t}{|s!|} \right)^{-1/s}, \rho(|\cdot|_t) \right).$$

REMARK 1.1. Observe that the function $\rho : \mathcal{M}(\mathcal{A}(I)) \rightarrow [0, \sup I]$, $|\cdot|_t \mapsto \rho(|\cdot|_t)$ is semi-continuous in the sense that the set $\rho^{-1}([0, r])$ is open in $\mathcal{M}(\mathcal{A}(I))$ ¹, but ρ is not continuous. Indeed for all $a \in K$ such that $|a| \in I$ we define $|\cdot|_a$ as the semi-norm given by $f \mapsto |f(a)|$. It is clear that $\rho(|\cdot|_a) = 0$. Now let $I := [0, 1]$ and let $|\cdot|_1$ be the semi-norm attached to the unit disk $|f|_1 := \sup_{|a| \leq 1, a \in K} (|f(a)|)$ ². Let $K = \overline{K}$. We choose a sequence $\{a_n\}_n$, $a_n \in K$, such that $\bar{a}_i \neq \bar{a}_j \in k$, $\forall i \neq j$, then we have $\lim_n |\cdot|_{a_n} = |\cdot|_1$ in $\mathcal{M}(\mathcal{A}(I))$, but $\rho(|\cdot|_1) = 1$. This results from the fact that every function $f \in \mathcal{A}(I)$ has only a finite number of zeros in $\mathcal{C}([0, 1])$.

However let $\gamma : I \rightarrow \mathcal{M}(\mathcal{A}(I))$ be a continue section of ρ . Then the function $r \mapsto \text{Ray}(M, \gamma(r))$ is a continuous function ([Ch-Dw]).

¹Observe that the open set $\rho^{-1}([0, r])$ is not an affinoid in the sense of [Ber] 2.2.1.

²This norm is the Gauss norm.

DEFINITION 1.2 ([**Astx**]). A differential module M over E (resp. over \mathcal{E}) is called *soluble* if $\text{Ray}(M, 1) = 1$. A differential module M over $\mathcal{A}(I)$ is called *soluble* at $\rho \in I$ if $\text{Ray}(M, \rho) = \rho$. A differential module M over \mathcal{R} is called *soluble* if

$$(1.1.5) \quad \lim_{\rho \rightarrow 1^-} \text{Ray}(M, \rho) = 1.$$

1.1.2. The radius of convergence at $|\cdot|_{c,r} \in \mathcal{M}(\mathcal{A}(I))$ of a differential module M can be viewed as the smallest radius of convergence of the solutions of M at some ‘‘incarnation’’ $t_{c,r}$ of $|\cdot|_{c,r}$ (cf. 5.1.1). Observe that a function of $\mathcal{A}(I)$ has no poles and no zeros in the generic disk $D_\Omega(t_{c,r}, r^-)$. Hence all points of $D_\Omega(t_{c,r}, r^-)$ are non singular for all differential modules.

1.2. Frobenius structure. Let A be one of the rings E_ρ , \mathcal{E} , $\mathcal{A}(I)$ or \mathcal{R} and let A^p be one of the rings E_{ρ^p} , \mathcal{E} , $\mathcal{A}(I^p)$ or \mathcal{R} respectively. Let $\sigma : K \rightarrow K$ be an automorphism of K such that $|a^\sigma - a^p| < 1$, for all $a \in \mathcal{O}_K$. For all functions $f(x) = \sum_i a_i x^i \in A^p$ we set $f^\sigma(x) := \sum_i a_i^\sigma x^i \in A$. We define a Frobenius map $\varphi : A^p \rightarrow A$ by

$$\varphi(f(x)) := f^\sigma(x^p).$$

This morphism defines an functor, called φ^* (cf. [**Astx**]), from the category of A^p -differential modules into the category of A -differential modules. Let M be the A^p -differential module, defined by $\frac{d}{dx} + G(x)$, $G(x) \in M_n(A^p)$. The Frobenius functor sends M into the module $\varphi^*(M)$ defined by $\frac{d}{dx} + px^{p-1}G^\sigma(x^p)$. In the appendix (cf. Corollary 5.7) we show that

$$\text{Ray}(\varphi^*(M), |\cdot|_{c,r}) = \min(\text{Ray}(M, |\cdot|_{c^p, r'})^{1/p}, |p|^{-1} \sup(|c|, r)^{1-p} \text{Ray}(M, |\cdot|_{c^p, r'}))$$

where $r' = \max(r^p, |p||c|^{p-1}r)$ (cf. equation 5.2.3).

DEFINITION 1.3 (Frobenius structure). Let A be one of the rings E , \mathcal{E} or \mathcal{R} . Let M be the differential module defined by $\frac{d}{dx} + G(x)$, $G(x) \in M_n(A)$ over A . We will say that M has a Frobenius structure of order h over A if there exists an A -isomorphism $M \xrightarrow{\sim} \varphi_h^*(M)$, where $\varphi_h^*(M)$ is the differential module defined by $\frac{d}{dx} + p^h x^{p^h-1} G^{\sigma^h}(x^{p^h})$. In other words there exists an invertible matrix $H(x) \in GL_n(A)$ such that

$$p^h x^{p^h-1} G^{\sigma^h}(x^{p^h}) = H(x)G(x)H^{-1}(x) + H(x)'H^{-1}(x).$$

THEOREM 1.4 ([**Ro**] 5.3). Let $L := \frac{d}{dx} + g(x)$, $g(x) \in K(x)$ be a soluble differential operator. By the Mittag-Leffler decomposition we may write $g(x) = g^+(x) + \sum_{-n \leq i \leq -1} a_i x^i$, $a_i \in K$, where $g^+(x) \in K(x)$ has no poles in $D(0, 1^-)$. Then L is isomorphic over the ring $\mathcal{A}([0, 1])[1/x]$ to the operator $\frac{d}{dx} + \sum_{-n \leq i \leq -1} a_i x^i$.

THEOREM 1.5 ([**Ch-Ch**] 2.3.1). Let k be perfect. Let $L = \frac{d}{dx} + \sum_{-n \leq i \leq -1} a_i x^i$, $a_i \in K$, be a soluble first order differential operator such that $a_{-1} \in \mathbb{Z}_{(p)}$. Then L has a Frobenius structure over \mathcal{R} . In other words there exist some $h > 0$ and an invertible function $f(x) \in \mathcal{R}^\times$ for which the following equality holds

$$\frac{f(x)'}{f(x)} = \left(\sum_{-n \leq i \leq -1} a_i x^i \right) - (p^h x^{p^h-1} \sum_{-n \leq i \leq -1} a_i^{\sigma^h} x^{ip^h})$$

2. Robba's exponentials

In this section $z = x^{-1}$. Let $\{\xi_m\}_{m \geq 0}$ be a sequence of (primitive) p^{m+1} -roots of 1 such that $\xi_m^{p^j} = \xi_{m-j}$, $j \geq 0$ and such that ξ_0 is a non trivial p -th root of 1. Let $\pi_m := \xi_m - 1$.

THEOREM 2.1. *For all $m \geq 0$ the function*

$$(2.0.1) \quad E_m(z) := \exp(\pi_m z + \pi_{m-1} \frac{z^p}{p} + \cdots + \pi_0 \frac{z^{p^m}}{p^m}) \in \overline{K}[[z]]$$

has radius of convergence equal to 1.

Proof: Let $E(z) = \exp(z + \frac{z^p}{p} + \frac{z^{p^2}}{p^2} + \cdots)$ be the Artin-Hasse exponential. If $(\lambda_0, \lambda_1, \dots) \in W(\mathcal{O}_K)$, then by a straightforward computation ([**Bou**] exercice 58-b) one shows that

$$\prod_{i \geq 0} E(\lambda_i z^{p^i}) = \exp(\phi_0 z + \phi_1 \frac{z^p}{p} + \phi_2 \frac{z^{p^2}}{p^2} \cdots)$$

where $\phi_k = \lambda_0^{p^k} + p\lambda_1^{p^{k-1}} + \cdots + p^k \lambda_k$ are the phantom components of $(\lambda_0, \lambda_1, \dots)$. Since $|\lambda_i| \leq 1$, hence this infinite product defines a *bounded* analytic function on $D(0, 1^-)$. If $(\lambda_0, \lambda_1, \dots) = (\xi_m, 0, \dots) - (1, 0, \dots)$, then $\phi_i = \pi_{m-i}$. The fact that the radius of convergence of $E_m(z)$ is exactly 1 will be a consequence of the fact that the operator

$$\frac{d}{dx} + E_m(z^{-1})'/E_m(z^{-1}) = \frac{d}{dx} - (\pi_m z^{-2} + \pi_{m-1} z^{-p-1} + \cdots + \pi_0 z^{-p^m-1})$$

is soluble³ and its radius of convergence, for ρ close to 0, is ρ^{p^m+1} (cf. Corollary 3.1). Then, by the log-concavity property of the radius of convergence, we have that this operator has radius of convergence equal to ρ^{p^m+1} , for all $\rho < 1$. \square

REMARK 2.2. Observe that for $|z| < 1$ close to 1, we have $|\pi_m z + \cdots + \pi_0 \frac{z^{p^m}}{p^m}| > |\pi_0|$. On the other hand, the analytic function $\exp(y)$ converges for $|y| < |\pi_0|$, so the convergent composition of $\pi_m z + \cdots + \pi_0 \frac{z^{p^m}}{p^m}$ and $\exp(y)$ does not exist. The precedent theorem asserts that the *formal composition*, after resummation, has radius of convergence equal to 1.

REMARK 2.3. Observe that in the formal case (cf. [**Man**]) a logarithmic derivative of a formal Laurent series has always an x -adic valuation ≥ -1 . But in the p -adic case the Robba-Matsuda's exponentials give an example of logarithmic derivatives of negative x -adic valuation. Then the definitions of p -adic irregularity and formal (x -adic) irregularity must be different (cf. [**Ro**]).

THEOREM 2.4 ([**Ma**]). *Let $p \neq 2$. Then the exponential $E_m^\sigma(z^p)/E_m(z)$ is overconvergent. In other words, if p is different from 2, the differential operator*

$$\frac{d}{dx} + E_m^\sigma(x^{-1})'/E_m(x^{-1}) = \frac{d}{dx} - (\pi_m x^{-2} + \pi_{m-1} x^{-p-1} + \cdots + \pi_0 x^{-p^m-1})$$

has a Frobenius structure of order 1.

³The solubility of this operator is due to the fact that the convergent function $E_m(z^{-1})$ is a solution of this operator at infinity and $E_m(z^{-1})$ converges in the set $\{x \in K \mid |x| > 1\}$.

3. Formal slopes and p -adic slopes

LEMMA 3.1 (Young, cf. [Astx]). *Let $L := \sum_{s=0}^r g_s(x) (\frac{d}{dx})^s$ be a differential operator such that $g_r(x) = 1$, $g_s \in E_\rho$, $s = 0, \dots, r-1$. Let $\rho \in I$, then $R(M, \rho) < |\pi_0| \rho$ if and only if $|g_s|_\rho > \rho^{s-r}$ for some $s < r$, and in this case we have:*

$$R(M, |\cdot|_\rho) = |\pi_0| \min_{0 \leq s < r} (|g_s|_\rho^{-1/r-s})$$

Let M be a soluble p -adic differential module over \mathcal{R} . Then there exist $0 < \varepsilon < 1$ and a rational number $\beta \geq 0$ such that $\text{Ray}(M, \rho) = \rho^{\beta+1}$ for all $\rho \in]1-\varepsilon, 1[$ (cf. [Astx]). If M is defined in some basis by the operator $\frac{d}{dx} + G(x)$, with $G(x) \in M_n(\mathcal{A}([0, 1][1/x]))$, then it is easy to show that there exist $0 < \delta < 1$ and a rational number $\alpha \geq 0$ such that $\text{Ray}(M, \rho) = \rho^{\alpha+1}$ for $\rho \in]0, \delta[$. By log-concavity we have $\alpha \geq \beta$.

DEFINITION 3.2. The number β is called the p -adic slope of M . We set $pt(M) := \beta$. If M is defined by the operator $\frac{d}{dx} + G(x)$, $G(x) \in \mathcal{A}([0, 1][1/x])$ we set $pt_F(M) := \alpha$ and we will call $pt_F(M)$ the formal slope. We have $pt(M) \leq pt_F(M)$.

REMARK 3.3. The precedent definition is justified by the fact that if M is defined by a linear differential operator $L := \sum_{s=0}^r g_s(x) (\frac{d}{dx})^s$, $g_r(x) = 1$, such that $g_s(x) \in \mathcal{A}([0, 1][1/x]) \subset \mathcal{R}$, then $pt_F(M)$ is the usual formal slope defined by

$$(3.0.2) \quad pt_F(L) = \max \left(0, \max_s \left(\frac{s-r-v(g_s)}{r-s} \right) \right)$$

where $v(g_s)$ is the x -adic valuation of $g_s(x)$. The formal slope is the largest slope of the Formal Newton polygon of L .⁴ This follows from lemma 3.1 and some continuity and convexity arguments. Indeed, as M is soluble, observe that, for ρ close to 0, by continuity and log-concavity we have only two cases: $\text{Ray}(M, \rho) = \rho$ or $\text{Ray}(M, \rho) < |\pi_0| \rho$.

LEMMA 3.4. *Let $L = \frac{d}{dx} + \sum_{i \geq -d} a_i x^i$, $a_{-d} \neq 0$, $d \geq 1$ be a soluble rank one differential operator with $\sum_{i \geq -d} a_i x^i \in \mathcal{R}$. Then we have that $\text{Ray}(L, \rho) = |\pi_0| |a_{-d}|^{-1} \rho^d$.*

4. Frobenius structure over \mathcal{R}

4.1. **Reduction to a $K[x, x^{-1}]$ -lattice.** Let $L := \frac{d}{dx} + g(x)$, $g(x) = \sum_i a_i x^i \in \mathcal{R}$, be a rank one soluble differential operator. In this section we show that L is isomorphic over \mathcal{R} to the operator $\tilde{L} = \frac{d}{dx} + \sum_{-d \leq i \leq -1} a_i x^i$ for a suitable $d \geq 1$ (cf. Theorem 4.7). Moreover, if $K = \overline{K}$, then d can be choosed equal to $pt(L) + 1$.

LEMMA 4.1. *The operator $L := \frac{d}{dx} + g(x)$, $g(x) = \sum_i a_i x^i \in \mathcal{R}$ is isomorphic over \mathcal{R} to the operator $\tilde{L} := \frac{d}{dx} + \sum_{-d \leq i \leq \infty} a_i x^i$ for a suitable $d \geq 1$.*

Proof: Let $d \geq 1$ be an integer such that $\sup(|\frac{a_i}{i+1}| \rho^{i+1}) < |\pi_0|$, for all $i < -d$. Such an integer exists because $g(x) \in \mathcal{R}$. Then the series $f(x) := \exp(-\sum_{i < -d} \frac{a_i}{i+1} x^{i+1})$ lies in \mathcal{R} and then L is isomorphic over \mathcal{R} to the operator $\tilde{L} = \frac{d}{dx} + g(x) + f(x)' / f(x)$. \square

⁴We recall that the formal Newton polygon is the convex hull of the set formed by the points of the form $(s, v(a_s) - s)$ and the two additional points $(-\infty, 0)$, $(0, +\infty)$. In particular we observe that in our case the last point is $(r, -r)$.

LEMMA 4.2. Let $L := \frac{d}{dx} + g(x)$, $g(x) = \sum_{-d \leq i \leq \infty} a_i x^i$, $d \geq 2$ be a soluble differential operator over \mathcal{R} . Then $|a_{-d}| \leq |\pi_0|$.

Proof: Let $g_s(x) = g_{s-1}(x)' + g_s(x)g(x)$ be as in the equation 1.1.2. An explicit computation shows that $g_s(x) = a_{-d}^s x^{-sd} + \{\text{terms of degree } \geq -sd + 1\}$. So the equation 1.1.1 shows that $\text{Ray}(L, \rho) \leq \frac{|\pi_0| \rho^d}{|a_{-d}|}$. The solubility implies $1 \leq |\pi_0|/|a_{-d}| \cdot \rho^d$. \square

LEMMA 4.3 ([Ch-Ro] 11.2.4). Let $L = \frac{d}{dx} + a_{-1}x^{-1}$ be a differential equation. Then L is soluble if and only if $a_{-1} \in \mathbb{Z}_p$.

LEMMA 4.4 ([Ch-Ro] 18.4.4). Let $L = \frac{d}{dx} + a_{-1}x^{-1}$ be a differential equation. Then L has a Frobenius structure if and only if $a_{-1} \in \mathbb{Z}_{(p)}$.

LEMMA 4.5. Let $L = \frac{d}{dx} + a_{-1}x^{-1}$. Let

$$\alpha(a_{-1}) := \limsup_n (|a_{-1}(a_{-1}-1)(a_{-1}-2) \cdots (a_{-1}-n+1)|^{\frac{1}{n}}).$$

Then for all $\rho > 0$ we have $\text{Ray}(L, \rho) = \frac{|\pi_0|}{\alpha(a_{-1})} \cdot \rho$.

Proof: A computation shows that $g_n(x) = \alpha_n(a_{-1})x^{-n}$, where $\alpha_n(a_{-1}) := a_{-1}(a_{-1}-1) \cdots (a_{-1}-n+1)$. So $\text{Ray}(L, \rho) = \liminf_n (|\alpha_n(a_{-1})|^{-1/n}) |\pi_0| \rho$. \square

LEMMA 4.6. Let $L = \frac{d}{dx} + g(x)$, $g(x) = \sum_{i \geq -1} a_i x^i \in \mathcal{R}$ be a soluble differential equation. Then $a_{-1} \in \mathbb{Z}_p$. Moreover there exists an analytic function $f(x) \in \mathcal{A}([0, 1[)$ such that $f'(x)/f(x) = \sum_{i \geq 0} a_i x^i$. In other words L is isomorphic to $\frac{d}{dx} + a_{-1}x^{-1}$.

Proof: We proceed by absurd. L is the tensor product of $L_1 := \frac{d}{dx} + a_{-1}x^{-1}$ and $L_2 := \frac{d}{dx} + \sum_{i \geq 0} a_i x^i$. As L_2 has a convergent solution in 0, we have $\text{Ray}(L_2, \rho) = \rho$, for ρ sufficiently close to 0. On the other hand, by the lemma 4.5 we have $\text{Ray}(L_1, \rho) = \frac{|\pi_0|}{\alpha(a_{-1})} \cdot \rho$ for all $\rho > 0$. The radius of the tensor product of two operators with different radius is the minimum of the radius.⁵ So, if L_1 is not soluble, then $\text{Ray}(L_1, 1) = \text{Ray}(L_2, 1) = R < 1$. Then $\text{Ray}(L_1, \rho) = R$ for all $\rho > 0$. But in this case, by convexity, $\text{Ray}(L_2, \rho) > R$ for $\rho < 1$. Then if $\rho < 1$ we would get $\text{Ray}(L, \rho) = \min(\text{Ray}(L_1, \rho), \text{Ray}(L_2, \rho)) = R$. Since L is soluble, hence by continuity of $\text{Ray}(L, \rho)$ we get a contraddiction.

We have shown that $\text{Ray}(L_1, \rho) = \text{Ray}(L_2, \rho) = \text{Ray}(L, \rho) = 1$, for all $0 < \rho \leq 1$. Now by transfert theorem ([Ch-Ro] 9.3.2) there exists a convergent solution $f(x)$ of L_2 in the disk $D(0, 1^-)$. In particular $f \in \mathcal{R}$ and $f'(x)/f(x) = \sum_{i \geq 0} a_i x^i$. \square

THEOREM 4.7. Let $L := \frac{d}{dx} + g(x)$, $g(x) = \sum_i a_i x^i \in \mathcal{R}$ be a rank one soluble differential operator. Then $a_{-1} \in \mathbb{Z}_p$ and there exists a $d \geq \max(1, \text{pt}(M) + 1)$ such that L is isomorphic, over \mathcal{R} , to $\frac{d}{dx} + \sum_{-d_1 \leq i \leq -1} a_i x^i$, for all $d_1 \geq d$. Moreover

⁵Observe that this fact depends on the definition given in the equation 1.1.1. For example $(x-1)(x-1)^{-1} = 1$, so in this case the radius of 1 is not equal to the minimum of the other two radius. This depends on the definition of "radius". In general let $L_1 = \frac{d}{dx} + g(x)$, $L_2 = \frac{d}{dx} + f(x)$ be two differential operators. Suppose that $g, f \in \mathcal{A}(I)$. Let $s_i(x)$ be a power series solution of L_i at $a \in \mathcal{C}(I)$. Let $r = \inf(|a-b|, b \in \overline{K} - C_{\overline{K}}(I))$. Then for all $\rho \leq r$ we have the equality

$$\min(\text{Ray}(s_1 \cdot s_2), \rho) = \min(\min(\text{Ray}(s_1), \rho), \min(\text{Ray}(s_2), \rho)).$$

for all $-pt(M) - 1 \leq i \leq -2$, there exist $b_i \in \overline{K}$ such that L is isomorphic, over $\mathcal{R}_{\overline{K}}$, to $\frac{d}{dx} + \sum_{-pt(M)-1 \leq i \leq -2} b_i x^i + a_{-1} x^{-1}$, and $|b_{-pt(M)-1}| = |\pi_0|$.

Proof: By the lemma 4.1 we can suppose that $L = \frac{d}{dx} + \sum_{-d \leq i \leq \infty} a_i x^i$, with $d \geq 1$. By the lemma 4.6 we can suppose that $d \geq 2$. The theorem will be proved applying the lemma 4.6. To apply this lemma we need that the operator $\frac{d}{dx} + \sum_{i \geq -1} a_i x^i$ is soluble. The solubility is invariant by extension of the field of constants and then we can actually suppose that $K = \overline{K}$.

Let $d - 1 = n \cdot p^m$, $(n, p) = 1$ and let $(\frac{a-d}{n \cdot \pi_0})^{1/p^m}$ be a p^m -th root of $\frac{a-d}{n \cdot \pi_0}$. Let us introduce the following analytic function:

$$f_d(x) := E_m \left(\left(\frac{a-d}{n \cdot \pi_0} \right)^{1/p^m} x^{-n} \right) = \exp \left(\pi_m \left(\frac{a-d}{n \cdot \pi_0} \right)^{1/p^m} x^{-n} + \dots + a_{-d} \frac{x^{-(d-1)}}{d-1} \right).$$

By the lemma 4.2 we have $|a_{-d}| \leq |\pi_0|$. Then, as $|n| = 1$, f_d is an analytic function in $\mathcal{C}([1, \infty[)$. The logarithmic derivative $f'_d/f_d = -a_{-d} x^{-d} + \dots + (-n) \cdot \pi_m \left(\frac{a-d}{n \cdot \pi_0} \right)^{1/p^m} x^{-n-1}$ defines a differential operator

$$L_d := \frac{d}{dx} + f'_d/f_d$$

which is soluble because its solution at infinity is $f_d(x)$.

We proceed now by induction on $d \geq 2$. If $|a_{-d}| < |\pi_0|$, then f_d is an element of \mathcal{R} and then the operator $\frac{d}{dx} + g(x)$ is isomorphic to $\frac{d}{dx} + g(x) + f'_d/f_d$. Now the x -adic valuation of the function $g(x) + f'_d/f_d$ is strictly larger than $-d$.

Otherwise, if $|a_{-d}| = |\pi_0|$, then the operator L_d is soluble and not trivial by the transfert theorem at infinity ([Ch-Ro] 9.3.2).⁶ So the tensor product operator $L \otimes L_d$, defined by $\frac{d}{dx} + g(x) + f'_d/f_d$, is still soluble, and $g(x) + f'_d/f_d$ is of degree $\geq -d+1$ and we can proceed by induction on d . Observe that f'_d/f_d is a polynomial in $x^{-2}\overline{K}[x^{-1}]$. Iterating, we get that there exist functions $f_d, \dots, f_2 \in \mathcal{A}_{\overline{K}}([1, \infty[)$ which are analytic in $\mathcal{C}([1, \infty[)$ and are such that

$$g(x) = f'_d/f_d + \dots + f'_2/f_2 + \left(\sum_{i \geq -1} a_i x^i \right).$$

Then $\frac{d}{dx} + \sum_{i \geq -1} a_i x^i$ is soluble. Observe that, as in the proof of lemma 4.6, if $|a_{-d}| = |\pi_0|$ we have $\text{Ray}(L, \rho) = \min_i \text{Ray}(L_i, \rho) = \text{Ray}(L_d, \rho)$, then

$$\text{Ray}(L, \rho) = \text{Ray} \left(\frac{d}{dx} + f'_d/f_d, \rho \right) = |\pi_0| |a_{-d}|^{-1} \rho^d = \rho^d, \quad \forall 0 < \rho < 1.$$

And in this case $d = pt(M) + 1$. Moreover if $d > pt(M) + 1$, then it must be $|a_{-d}| < |\pi_0|$ and then $f_d \in \mathcal{R}_{\overline{K}}$. Iterating this process we can show that, over \overline{K} , we can obtain $d = \max(1, pt(M) + 1)$. \square

REMARK 4.8. Observe that in the proof of the precedent lemma we show that, over \overline{K} , L is isomorphic to the operator

$$(4.1.1) \quad \frac{d}{dx} + f'_d/f_d + \dots + f'_2/f_2 + a_{-1} x^{-1}.$$

In other words L is the tensor product of the operators $L_d := \frac{d}{dx} + f'_d/f_d, \dots, L_2 := \frac{d}{dx} + f'_2/f_2$, and $L_{-1} := \frac{d}{dx} + a_{-1} x^{-1}$, where $f_i(x)$ are functions obtained from a

⁶Indeed if the operator L_d is trivial over \mathcal{R} then $\text{Ray}(L_d, \rho) = \rho$ for all $\rho > 1 - \varepsilon$, $\exists \varepsilon$ and by the transfert theorem the solution at infinity converges in the disk $\{|x| > 1 - \varepsilon\}$.

Robba's exponential by a substitution of the variable. Moreover L_i is trivial or $\text{Ray}(L_i, \rho) = \rho^i$ for all $\rho < 1$. Over \overline{K} we can choose $d = pt(M) + 1$. This is a kind of canonical form for rank one soluble differential equation over $\mathcal{R}_{\overline{K}}$.

4.2. Frobenius structure over \mathcal{R} .

THEOREM 4.9. *Let k be a perfect field of characteristic $p > 0$. Let $L = \frac{d}{dx} + g(x)$, $g(x) = \sum_i a_i x^i \in \mathcal{R}$ be a soluble differential equation. Then L has a Frobenius structure if and only if $a_{-1} \in \mathbb{Z}_{(p)}$.*

Proof: The operator L is isomorphic to the operator $\frac{d}{dx} + \sum_{-d \leq i \leq -1} a_i x^i$. We can now apply the theorem of Christol-Chiarellotto 1.5. \square

COROLLARY 4.10. *The Robba-Matsuda operator $\frac{d}{dx} + \frac{E'_m(x^{-1})}{E_m(x^{-1})}$ (cf. equation 2.0.1) has a Frobenius structure for all $m \geq 0$.*

Proof: We must show that the formal series $E_m^{\sigma^h}(x^{-p^h})/E_m(x^{-1})$ defines a function of \mathcal{R} for some $h \geq 1$. Since the convergence does not change by extension of the field of constants, hence we can suppose $K = \overline{K}$. We can actually apply the theorem 4.9. \square

REMARK 4.11. The corollary 4.10 works for all $p > 0$. On the other hand, the proof of Matsuda (cf. Theorem 2.4) is a very strong computation and is stronger than our result because it shows that, if $p \neq 2$, the operator $\frac{d}{dx} + E_m(x^{-1})'/E_m(x^{-1})$ has a Frobenius structure of order 1.

5. Appendix: Variation of Radius of convergence by ramifications.

In this section we precise some known (but not published) facts about the variation of the radius of convergence of the pull-back of a module by a covering of the form $x \mapsto x^n$. We study the ramification $\phi_n^* : f(x) \mapsto f(x^n)$ instead of φ , because the application $f(x) \mapsto f^\sigma(x)$ (cf. 1.2) defines an auto-equivalence of the category of differential modules which preserves the radius of convergence.

5.1. Scalar extension. Let I be a (non empty) interval. Let L/K be an extension of valued fields. Let $\mathcal{A}_L(I)$ denote the ring of analytic functions over I , with coefficients in L . Then we have the following diagram

$$\begin{array}{ccc} \mathcal{M}(\mathcal{A}_K(I)) & \xleftarrow{\psi_L} & \mathcal{M}(\mathcal{A}_L(I)) \\ \cup & & \cup \\ \mathcal{C}_K(I) & \subseteq & \mathcal{C}_L(I) \end{array}$$

where the vertical inclusions are the canonical inclusions $a \mapsto | \cdot |_a$ (cf. remark 1.1) and the map ψ_L is the functorial morphism of analytic spaces corresponding to the inclusion $\mathcal{A}_K(I) \subseteq \mathcal{A}_L(I)$. This diagram is commutative in the sense that the inclusion $\mathcal{C}_K(I) \subseteq \mathcal{C}_L(I)$ is a section of the map ψ_L .

5.1.1. By [Ch-Ro] 9.1 there exists a field Ω such that $\psi_\Omega(\mathcal{C}_\Omega(I)) = \mathcal{M}(\mathcal{A}_K(I))$. In this sense all points $| \cdot |_t$ of $\mathcal{M}(\mathcal{A}_K(I))$ have an "incarnation" in a true point of $\mathcal{C}_\Omega(I)$. If $t \in \mathcal{C}_\Omega(I)$ is an incarnation of the seminorm $| \cdot |_t \in \mathcal{M}(\mathcal{A}_K(I))$ (i.e. $\psi_\Omega(t) = | \cdot |_t$) we will say that $D_\Omega(t, \rho(| \cdot |_t)^-)$ is a generic disk for $| \cdot |_t$ (cf. 1.1.1).

5.2. Ramifications and image of a point of Berkovich. Let $\phi_n^* : \mathcal{A}(I^n) \rightarrow \mathcal{A}(I)$ be the morphism $\sum_i a_i x^i \mapsto \sum_i a_i x^{ni}$. Let $\phi_n : \mathcal{M}(\mathcal{A}(I)) \rightarrow \mathcal{M}(\mathcal{A}(I^n))$ be the corresponding morphism of analytic spaces. In this section we compute the image of a point of $\mathcal{M}(\mathcal{A}(I))$ by the ramification ϕ_n .

5.2.1. By a result of Berkovich ([Ber] 1.4.4) we know that every point of $\mathcal{M}(\mathcal{A}(I))$ is a limit of points of the form $|\cdot|_{c,r}$, $c \in K$

$$|f(x)|_{c,r} := \sup_{x \in D(c,r^-)} |f(x)|$$

In other words if $|\cdot|_t$ is a point of $\mathcal{M}(\mathcal{A}(I))$, then $|\cdot|_t$ is the seminorm attached to some disk, or $|\cdot|_t$ is the seminorm attached to a totally ordered⁷ sequence of disks. If K is spherically complete then all points are of the form $|\cdot|_{c,r}$ and, since K is contained in some spherically complete and algebraically closed field K' , hence we can suppose that all points of $\mathcal{M}(\mathcal{A}(I))$ and $\mathcal{M}(\mathcal{A}(I^n))$ are of the form $|\cdot|_{c,r}$ for a suitable $c \in K'$. For simplicity we will suppose that $K = K'$.

REMARK 5.1. Let $t_{c,r}$ be an incarnation of $|\cdot|_{c,r}$. It is clear that $\rho(|\cdot|_{c,r}) = r$ (cf. 1.1), and that $|t_{c,r}|_\Omega = \max(|c|, r)$.

5.2.2. Rational fractions are dense in $\mathcal{A}(I)$, hence to compute the seminorm $|\cdot|_{c',r'} := \phi_n(|\cdot|_{c,r})$ it is enough to know the value $|x - a|_{c',r'}$ for all $a \in K$. On the other hand $\phi_n(|\cdot|_{c,r}) = |\cdot|_{c,r} \circ \phi_n^*$, then $|x - a|_{c',r'} = |\phi_n^*(x - a)|_{c,r} = |x^n - a|_{c,r}$. We write $x^n - a = [(x - c) + c]^n - a = [\sum_{i=1}^n \binom{n}{i} c^{n-i} (x - c)^i] + (c^n - a)$. Then we have that

$$(5.2.1) \quad |x^n - a|_{c,r} = \sup_{1 \leq i \leq n} (|\binom{n}{i}| |c|^{n-i} r^i, |c^n - a|)$$

On the other hand $|x - a|_{c',r'} = \sup(r', |c' - a|)$. Therefore $c' = c^n$ and

$$(5.2.2) \quad r' = \sup_{1 \leq i \leq n} (|\binom{n}{i}| |c|^{n-i} r^i) = |c|^n \sup_{1 \leq i \leq n} (|\binom{n}{i}| (r/|c|)^i).$$

We can compute r' in some particular case:

$$(5.2.3) \quad r' = \begin{cases} \min(r^p, |p| |c|^{p-1} r) & \text{if } n = p \\ \min(r^n, |c|^{n-1} r) & \text{if } (n, p) = 1. \end{cases}$$

This process can be applied to compute the image of $|\cdot|_{c,r}$ under the action of an arbitrary polynomial map instead of ϕ_n . To recover the value of r' it is sufficient to look at the Taylor's development of this polinomial at c .

COROLLARY 5.2 (Deformation of the Generic Disk). *Let $D_\Omega(t_{c,r}, r^-)$ be a generic disk for $|\cdot|_{c,r}$ (cf. 5.1.1). Then $t_{c,r}^n$ is an incarnation of $|\cdot|_{c^n, r'} := \phi_n(|\cdot|_{c,r})$ and $\phi_n(D_\Omega(t_{c,r}, r^-)) \subseteq D_\Omega(t_{c,r}^n, r'^-)$. More precisely let $y \in D_\Omega(t_{c,r}, r^-)$. If $(n, p) = 1$, then*

$$(5.2.4) \quad |y^n - t_{c,r}^n| = |t_{c,r}|^{n-1} \cdot |y - t_{c,r}|.$$

If $n = p$, then

$$(5.2.5) \quad |y^p - t_{c,r}^p| \leq \max(|t_{c,r}|^{p-1} |p| |y - t_{c,r}|, |y - t_{c,r}|^p)$$

and the equality holds if $|t_{c,r}|^{p-1} |p| |y - t_{c,r}| \neq |y - t_{c,r}|^p$.

⁷Ordered by inclusion.

Proof: In all cases we have that $|y - t_{c,r}| < |t_{c,r}|$ (cf. 5.1). If $(n, p) = 1$ we have $|y^n - t_{c,r}^n| = |(y - t_{c,r} + t_{c,r})^n - t_{c,r}^n| = |t_{c,r}|^n \left| \sum_{i=1}^n \binom{n}{i} ((y - t_{c,r})/t_{c,r})^i \right|$. Observe that $|\binom{n}{1}| = 1$. If $n = p$, observe that $|\binom{p}{i}| = |p|$ for all $i = 1, \dots, p-1$, then the same computation gives that $|y^n - t_{c,r}^n| \leq |t_{c,r}|^{p-1} |y - t_{c,r}| \cdot \sup(|p|, |(y - t_{c,r})/t_{c,r}|^{p-1})$. \square

5.3. Variation of the radius of convergence by ramification. Let M be an $\mathcal{A}(I^p)$ -differential module. In this section we compute the radius of convergence at $|\cdot|_{c,r}$ of the pull-back $\mathcal{A}(I^p)$ -differential module $\phi_n^*(M)$. Observe that the radius of $\varphi^*(M)$ and $\phi_p^*(M)$ are equal (cf. 5).

5.3.1. *Frobenius:* Let $s(x) = \sum_{i=0}^{\infty} a_i(x - t_{c,r}^p)^i$ be a convergent analytic function at $t_{c,r}^p$. Let $R = \liminf_i |a_i|^{-1/i}$ be the radius of convergence of $s(x)$ at $t_{c,r}^p$. Let $\phi_p^*(s)(y) := s(y^p) = \sum_i a_i(y^p - t_{c,r}^p)^i$ be its pull-back and R' the radius of $\phi_p^*(s)(y)$ at $t_{c,r}$. By composition we have (cf. corollary 5.2)

$$(5.3.1) \quad R' \geq \min(|p|^{-1} |t_{c,r}|^{1-p} R, R^{1/p})$$

LEMMA 5.3. *Let $x_l \in \mathbb{R}$ be a sequence. Then for all $m \in \mathbb{Z}$ we have:*

$$\liminf_l (x_l) = \min_{0 \leq \varepsilon \leq m-1} (\liminf_{l \in \varepsilon + m\mathbb{Z}} (x_l)) \leq \liminf_{l \in m\mathbb{Z}} (x_l).$$

LEMMA 5.4. *Let $i, n, l \in \mathbb{N}$, $i, n \geq 1$, $i \leq l \leq n \cdot i$. Let*

$$B(i, l, n) := \sum_{\substack{j_1 + \dots + j_i = l \\ 1 \leq j_k \leq n}} \binom{n}{j_1} \cdots \binom{n}{j_i}.$$

Then $B(i, i, n) = n^i$ and $B(i, in, n) = 1$.

LEMMA 5.5. *Let $m \in \mathbb{N}$, $m \geq 1$. Let $\{c_l\}_l$ be a sequence in some ultrametric ring. Let $R := \liminf_l |c_l|^{-1/l}$. If $R > 0$, then we have*

$$(5.3.2) \quad \liminf_{l \in m\mathbb{Z}} |c_{\frac{l}{m}}|^{-\frac{1}{l}} = R^{\frac{1}{m}}$$

Proof: This equation is equivalent to the equation $\liminf_l |c_l|^{-\frac{1}{ml}} = R^{\frac{1}{m}}$. \square

THEOREM 5.6. *We have $R' = \min(R|p|^{-1} |t_{c,r}|^{1-p}, R^{1/p})$.*

Proof: We write $y^p - t_{c,r}^p = (y - t_{c,r} + t_{c,r})^p - t_{c,r}^p$. We have $s(y^p) = \sum_i a_i \cdot (\sum_{j=1}^p \binom{p}{j} t_{c,r}^{p-j} y^j)^i$. After a resommation we get

$$s(y^p) = \sum_{l=0}^{\infty} \left[\sum_{i=\lceil \frac{l}{p} \rceil}^l a_i \cdot B(i, l, p) \cdot t_{c,r}^{ip-l} \right] (y - t_{c,r})^l$$

where $\lceil l/p \rceil := \min(i \in \mathbb{N} \mid i \geq l/p)$. Since the term $\sum_{i=\lceil \frac{l}{p} \rceil}^l a_i \cdot B(i, l, p) \cdot t_{c,r}^{ip-l}$ is a polynomial in $t_{c,r}$ with coefficients in K , hence its valuation is given by $\sup_{i=\lceil \frac{l}{p} \rceil, \dots, l} (|a_i| \cdot |B(i, l, p)| \cdot |t_{c,r}|^{ip-l})$. We have then

$$R' = \liminf_l \left(\sup_{i=\lceil \frac{l}{p} \rceil, \dots, l} (|a_i| \cdot |B(i, l, p)| \cdot |t_{c,r}|^{ip-l}) \right)^{-1/l}.$$

Applying the lemma 5.3 we have

$$(5.3.3) \quad R' \leq \liminf_{l \in p\mathbb{Z}} \left(\sup_{l/p \leq i \leq l} (|a_i| \cdot |B(i, l, p)| \cdot |t_{c,r}|^{ip-l}) \right)^{-1/l}.$$

By the lemma 5.4 we have $B(l/p, l, p) = 1$ and $B(l, l, p) = p^l$ and then clearly $\sup_{l/p \leq i \leq l} (|a_i| \cdot |B(i, l, p)| \cdot |t_{c,r}|^{ip-l}) \geq \sup(|a_{l/p}|, |a_l| \cdot |p|^l \cdot |t_{c,r}|^{l(p-1)})$. This fact and the lemma 5.5 show that

$$R' \leq \liminf_{l \in p\mathbb{Z}} \left(\sup(|a_{l/p}|, |a_l| \cdot |p|^l \cdot |t_{c,r}|^{l(p-1)}) \right)^{-\frac{1}{l}} = \inf(R^{1/p}, R|p|^{-1}|t_{c,r}|^{1-p}). \quad \square$$

Recalling that $|t_{c,r}| = \sup(|c|, r)$ (cf. remark 5.1), we can state the following

COROLLARY 5.7. *Let A be one of the rings E_ρ , $\mathcal{A}(I)$, \mathcal{E} or \mathcal{R} . Let A^p be the ring E_ρ^p , $\mathcal{A}(I^p)$, \mathcal{E} or \mathcal{R} . respectively. Let M be a differential module over A^p , let $\phi_p^*(M)$ be its pull-back over A by the morphism $f(x) \rightarrow f(x^p) : A^p \rightarrow A$. Then we have*

$$\text{Ray}(\phi_p^*(M), |\cdot|_{c,r}) = \min \left(\text{Ray}(M, |\cdot|_{c^p, r'})^{1/p}, |p|^{-1} \sup(|c|, r)^{1-p} \text{Ray}(M, |\cdot|_{c^p, r'}) \right)$$

where $r' = \max(r^p, |p||c|^{p-1}r)$ (cf. equation 5.2.3).

5.3.2. Ramification prime to p . Let $(n, p) = 1$. Following the same method of the precedent discussion we get that $R' = |t_{c,r}|^{1-n}R$, and the following

THEOREM 5.8. *Let A be one of the rings E_ρ , $\mathcal{A}(I)$, \mathcal{E} or \mathcal{R} . Let A^n be one of the rings E_ρ^n , $\mathcal{A}(I^n)$, \mathcal{E} or \mathcal{R} . Let M be a differential module over A^n , let $\phi_n^*(M)$ be its pull-back over A . Then we have*

$$\text{Ray}(\phi_n^*(M), |\cdot|_{c,r}) = \sup(|c|, r)^{1-n} \text{Ray}(M, |\cdot|_{c^n, r'})$$

where $r' = \max(r^n, |c|^{n-1}r)$ (cf. equation 5.2.3).

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