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REPRESENTATIONS OF WEIL-DELIGNE GROUPS AND FROBENIUS CONJUGACY CLASSES

ABHIJIT LASKAR

ABSTRACT. Let X be a smooth projective algebraic variety over a number field F , with an embedding $\tau : F \hookrightarrow \mathbb{C}$. The action of $\text{Gal}(\bar{F}/F)$ on ℓ -adic cohomology groups $H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell)$, induces Galois representations $\rho_\ell^i : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}(H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell))$. Fix a non-archimedean valuation v on F , of residual characteristic p . Let F_v be the completion of F at v and $'W_v$ be the Weil-Deligne group of F_v . We establish new cases, for which the linear representations ρ_ℓ^i of $'W_v$, associated to ρ_ℓ^i , form a compatible system of representations of $'W_v$ defined over \mathbb{Q} . Under suitable hypotheses, we show that in some cases, these representations actually form a compatible system of representations of $'W_v$, with values in the Mumford-Tate group of $H_B^i(\tau X(\mathbb{C}), \mathbb{Q})$. When X has good reduction at v , we establish a motivic relationship between the compatibility of the system $\{\rho_\ell^i\}_{\ell \neq p}$ and the conjugacy class of the crystalline Frobenius of the reduction of X at v .

1. INTRODUCTION AND NOTATION

Throughout, F is a number field, with an embedding $\tau : F \hookrightarrow \mathbb{C}$, v is a non-archimedean valuation on F and F_v is the completion. By \bar{F} we denote a fixed separable algebraic closure of F , $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ is an extension of τ , \bar{v} is an extension of v to \bar{F} and \bar{F}_v is the localization of \bar{F} at \bar{v} . The residue fields of F_v and \bar{F}_v are denoted by k_v and \bar{k}_v , respectively. Let the characteristic of k_v be $p > 0$ and write $|k_v| := q_v$. We write $\Gamma_v := \text{Gal}(\bar{F}_v/F_v) \subset \Gamma_F := \text{Gal}(\bar{F}/F)$ and $I_v \subset \Gamma_v$ is the inertia group. By an *arithmetic Frobenius* $\Phi_v \in \Gamma_v$, we mean an element which induces the Frobenius automorphism ϕ_v of \bar{k}_v . We denote by W_v the *Weil group* of F_v , i.e., the dense subgroup formed by elements $w \in \Gamma_v$ which induce on \bar{k}_v an integral power $\phi_v^{\alpha(w)}$. The map $\alpha : W_v \rightarrow \mathbb{Z}$ thus defined is a group homomorphism and $\ker(\alpha) = I_v$. The *Weil-Deligne group* $'W_v$ of F_v is the group scheme over \mathbb{Q} defined as the semi-direct product of W_v with the additive group \mathbb{G}_a over \mathbb{Q} , on which W_v acts as $: w \cdot x \cdot w^{-1} = q_v^{\alpha(w)} \cdot x$. For ease of exposition, we shall assume our varieties to be geometrically irreducible.

Consider a smooth projective algebraic variety X over F . The action of Γ_F on the geometric ℓ -adic cohomology groups $V_\ell^i := H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell)$, induces Galois representations $\rho_\ell^i : \Gamma_F \rightarrow \text{GL}(V_\ell^i)$. A fundamental problem in arithmetic geometry, is to determine, how far the properties of ρ_ℓ^i are independent of ℓ . For instance, it has been conjectured [25] that, *if v is any non-archimedean valuation on F , then for every $w \in W_v$, the characteristic polynomial $P_{\ell,v}^i(w, T) := \det(1 - \rho_\ell^i(w)T; V_\ell^i)$, of the \mathbb{Q}_ℓ -linear map $\rho_\ell^i(w)$ has coefficients in \mathbb{Q} and is independent of ℓ ?* By Deligne's result [6] on the Weil conjectures, we know that this conjecture holds true if we assume that variety X has good reduction at v . But the case of bad-reduction is wide open. The starting point of this article is an observation (see Thm. 2.1), which gives a criterion for detecting the rationality and ℓ -independence of $P_{\ell,v}^i(w, T)$, irrespective of the type of reduction at v . This allows us to verify the above conjecture in a large number of new cases; see Cor.2.3. Equipped with these results, we can take a step forward in the analysis of the bad reduction case. In order to do this one attaches to ρ_ℓ^i , a

linear representation $\rho_\ell^i := (V_\ell^i, \rho_\ell^{i'}, N_{i,\ell}')$ of $'W_v$ over \mathbb{Q}_ℓ , where $\rho_\ell^{i'}$ is a continuous representation of W_v and $N_{i,\ell}'$ is the associated monodromy operator (see §3.2). These ρ_ℓ^i are the basic source of linear representations of $'W_v$. The following is a longstanding conjecture of Deligne, Tate et al (cf.[10, 2.4.3])

Conjecture 1.1 ($C_{WD}(X_{F_v}, i)$). *The system $\{\rho_\ell^i\}_{\ell \neq p}$ forms a compatible system (in the sense of [5, 8.8]) of linear representations of $'W_v$ defined over \mathbb{Q} .*

The notion of compatible system considered here is a strong one (see §3.2). The first main result of this article establishes some new cases of the above conjecture.

Theorem 1.2. *Let X a smooth projective variety over F which is a finite product of moduli spaces of stable vector bundles of co-prime rank and degree over smooth projective curves, unirational varieties of dimension ≤ 3 , uniruled surfaces, hyperkähler varieties of $K3^{[n]}$ type, abelian varieties, curves and Fermat hypersurfaces. Then, for every $i \in \mathbb{N}$, the $C_{WD}(X_{F_v}, i)$ Conjecture (1.1) holds true.*

For surfaces and complete intersections, we show a slightly weaker result. Let $\rho_\ell^{i,ss}$ denote the (Frobenius) semi-simplification (see §.3.14) of ρ_ℓ^i .

Theorem 1.3. (1) *Let S be a smooth projective surface over F . Then $C_{WD}(S_{F_v}, i)$ holds true $i \neq 2$. When $i = 2$, the system $\{\rho_\ell^{2,ss}\}_{\ell \neq p}$ forms a compatible system of linear representations of $'W_v$ defined over \mathbb{Q} .*

(2) *Let X be a smooth complete intersection of dimension n in a projective space \mathbb{P}^r , defined over F . Then $C_{WD}(X_{F_v}, i)$ holds true for $i \neq n$. When $i = n$, the system $\{\rho_\ell^{n,ss}\}_{\ell \neq p}$ forms a compatible system of linear representations of $'W_v$ defined over \mathbb{Q} .*

Corollary 3.16 shows that when X is a smooth complete intersection of Hodge level 1, $C_{WD}(X_{F_v}, i)$ holds true for all i . Previously, Conjecture 1.1, was only known to hold for $i = 1$ and X an abelian variety (see [5]). The relevance of Thm. 1.2 and Thm 1.3, in the Langlands program, is described by Cor. 3.13 and Cor. 3.17, respectively.

Now write $\tau X := X \times_{F,\tau} \mathbb{C}$ and let $V^i := H_B^i(\tau X(\mathbb{C}), \mathbb{Q})$ denote the (degree i) Betti cohomology group of the complex algebraic variety τX . Let G_∞^i Mumford-Tate group of the Hodge structure on V^i . Conjecture 1.1 has a sharper reformulation, if one assumes the Hodge conjecture. In that case, ρ_ℓ^i factors through $G_\infty^i(\mathbb{Q}_\ell)$ and it is expected that *the $\{\rho_\ell^i\}_{\ell \neq p}$ forms a compatible system (in the sense of [5, 8.11]) of representations of $'W_v$ with values in the algebraic group G_∞^i .* In its full generality, this is unknown even in the case of $i = 1$ and X an abelian variety of $\dim \geq 2$. Theorem 4.7 shows that, under some mild assumptions (without assuming the Hodge conjecture), this conjecture holds for the varieties in Thm. 1.2. There is another closely related conjecture (in the good reduction case) formulated by Serre [24, 12.6], in terms of *motivic Galois groups*. Recall that Grothendieck's standard conjectures on algebraic cycles predict that the category of motives for homological equivalence of algebraic cycles coincides with the category of numerical(Grothendieck) motives and is Tannakian. This would imply in particular that the action of the Galois group Γ_F on the ℓ -adic realization $H_\ell(M)$ of any motive M over F , factors as $\rho_{M,\ell} : \Gamma_F \rightarrow G_M(\mathbb{Q}_\ell)$, where G_M is the *motivic Galois group* of M . This is the group associated, via the Betti realization functor H_τ , to the Tannakian subcategory generated by M and the Tate motive. Now, let $(\text{Conj}(G_M), \text{Cl})$ be the universal categorical quotient of G_M for its action on itself by conjugation. For every $\ell \neq p$, $\rho_{M,\ell}(\Phi_v)$ defines an element $\text{Cl}(\rho_{M,\ell}(\Phi_v)) \in \text{Conj}(G_M)(\mathbb{Q}_\ell)$.

Conjecture 1.4 (Serre, [24, 12.6]). *If M is a motive with good reduction at v , then $\text{Cl}(\rho_{M,\ell}(\Phi_v)) \in \text{Conj}(G_M)(\mathbb{Q})$, $\forall \ell \neq p$ and is independent of ℓ .*

In [14], we showed this conjecture (under additional assumptions) for motives of the form $M := h^i(X)$, where X is any smooth proper algebraic variety with good reduction at v and $M \in \text{Ob}(\mathcal{M}_{AH}^{av}(F))$ i.e. the full Tannakian subcategory, generated by abelian varieties and Artin motives, inside the category of motives $\mathcal{M}_{AH}(F)$ for absolute Hodge cycles. Now, it is natural to wonder, what happens in the case $\ell = p$. The final section of the paper investigates this issue. To state the principal result in this direction, we need more notation. First note that the advantage of working over $\mathcal{M}_{AH}(F)$, is that it has many desired properties of homological motives, unconditionally. Moreover, it is well known that if the Hodge conjecture is true, then the category of homological motives coincides with the category of motives for absolute Hodge cycles.

Let X/F be a smooth proper algebraic variety with good reduction at v and such that the motive $M := h^i(X) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F))$. Denote the ring of Witt vectors of k_v by $W(k_v)$ and by F_v^0 the fraction field of $W(k_v)$. Let X_v denote the special fiber of a smooth proper model of X over the ring of integers of F_v . Fix a $i \in \mathbb{N}$ and let $\Phi_{\text{Cris}} : H_{\text{Cris}}^i(X_v/W(k_v)) \otimes F_v^0 \rightarrow H_{\text{Cris}}^i(X_v) \otimes F_v^0$, be the degree i crystalline Frobenius of X at v .

The C_{Cris} -conjecture (now a theorem, cf. [11, Th. 3.2.3]) allows us to define a crystalline realization (cf. [27, §4.1]), i.e., a fiber functor $H_{M, \text{Cris}} : \langle M, \mathbb{Q}(1) \rangle^{\otimes} \rightarrow \text{Vect}_{F_v^0}$. We write $G_{M, \text{Cris}}$ for the automorphism group of $H_{M, \text{Cris}}$. The action of $G_{M, \text{Cris}}$ on itself by conjugation, provides a universal categorical quotient $(\text{Conj}(G_{M, \text{Cris}}), \text{Cl}_{\text{Cris}})$. Now, Φ_{Cris} defines an element $\Phi_{M, \text{Cris}} \in G_{M, \text{Cris}}(F_v^0)$ which in turn defines an element $\text{Cl}_{\text{Cris}}(\Phi_{M, \text{Cris}}) \in \text{Conj}(G_{M, \text{Cris}})(F_v^0)$.

Theorem 1.5. *Let M be as above. Assume that G_M is connected and there is prime number ℓ_0 such that $\rho_{M, \ell_0}(\Phi_v)$ is weakly neat. Then, there exists a unique conjugacy class*

$$\text{Frob}_M \in \text{Conj}'(G_M)(F_v^0),$$

such that $\text{Cl}_{\text{Cris}}(\Phi_{M, \text{Cris}}) = \text{Frob}_M$ and $\text{Cl}(\rho_{M, \ell}(\Phi_v)) = \text{Frob}_M, \forall \ell \neq p$.

Roughly speaking, $(\text{Conj}'(G_M), \text{Cl})$ refers to a modification of $(\text{Conj}(G_M), \text{Cl})$, which we need to make when the derived group $G_{M/\mathbb{Q}}^{\text{der}}$ of $G_{M/\mathbb{Q}}$ has certain factors of type D (see §4). Now denote by $\mathcal{M}_{\text{num}}^{av}(F)$ the Tannakian category of motives generated by abelian varieties and zero dimensional varieties, inside the category of numerical Grothendieck motives over F . Since homological equivalence coincides with numerical equivalence for zero dimensional varieties and abelian varieties [16], we can identify $\mathcal{M}_{\text{num}}^{av}(F)$ to a subcategory (a priori not full) of $\mathcal{M}_{AH}^{av}(F)$. Thus, our result also holds for $M \in \text{Ob}(\mathcal{M}_{\text{num}}^{av}(F))$. The special case, where M is the motive $h^1(A)$ for an abelian variety A , Theorem 1.5 was proved by Noot in [17, Th. 4.2]. We also note that as the Hodge conjecture remains unproven, a priori $\mathcal{M}_{AH}^{av}(F)$ has more objects than in $\mathcal{M}_{\text{num}}^{av}(F)$. An example of this phenomenon is the motive of a $K3$ surface.

2. ACTION OF WEIL GROUP ON ℓ -ADIC COHOMOLOGY

We follow the notation of §1. All algebraic cycles and Chow groups are with rational coefficients. For any smooth projective algebraic variety X over the number field F , we write $\tau X := X \times_{F, \tau} \mathbb{C}$. For any complex algebraic variety Y of dimension d , we denote by

$$\gamma_B^d : \text{CH}^d(Y \times Y) \rightarrow \text{H}_B^{2d}(Y(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q})(d),$$

the cycle class map from codimension d algebraic cycles on $Y \times Y$ to the Tate twisted degree $2d$ Betti cohomology of $Y \times Y$. The Künneth isomorphism

$$\text{H}_B^{2d}(Y(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q})(d) \cong \bigoplus_{i=0}^{2d} \text{H}_B^{2d-i}(Y(\mathbb{C}), \mathbb{Q}) \otimes \text{H}_B^i(Y(\mathbb{C}), \mathbb{Q})(d),$$

gives us a decomposition $\gamma_B^d([\Delta]) = \sum_{i=0}^{2d} \pi^i$, of the class of the diagonal sub-variety Δ of $Y \times Y$. To say that the *degree i Künneth standard conjecture* w.r.t. Betti Cohomology is true for Y , means that there is a correspondence $z^i \in \text{CH}^d(Y \times Y)$ such that $\gamma_B^d(z^i) = \pi^i$.

Theorem 2.1. *Let X be any smooth projective variety over F . Fix a $i \in \mathbb{N}$ and assume that the degree i Künneth standard conjecture w.r.t Betti Cohomology holds for τX . Then, for every $\ell \neq p$ and $w \in W_v$ with $\alpha(w) \leq 0$, the polynomial $P_{\ell,v}^i(w, T) \in \mathbb{Z}[T]$ and is independent of ℓ .*

Proof. Let us denote by $\tau\Delta \subset \tau X \times \tau X$ the diagonal subvariety and by d the dimension of X . Let $\gamma_B^d : \text{CH}^d(\tau X \times \tau X) \rightarrow \text{H}_B^{2d}(\tau X(\mathbb{C}) \times \tau X(\mathbb{C}), \mathbb{Q})(d)$. By hypothesis there is a correspondence $\tau z^i \in \text{CH}^d(\tau X \times \tau X)$ such that $\gamma_B^d(\tau z^i) = \tau \pi^i$. Let

$$p^* : \text{CH}^d(X_{\bar{F}} \times X_{\bar{F}}) \rightarrow \text{CH}^d(\tau X \times \tau X)$$

denote the base change map. It is well known (cf.[8, 2.9, a]) that base change induces an isomorphism between algebraic cycle groups over \bar{F} and \mathbb{C} modulo homological equivalence. This implies that there is a correspondence $z_{/\bar{F}}^i \in \text{CH}^d(X_{\bar{F}} \times X_{\bar{F}})$ such that $\gamma_B^d(\tau z^i) = \gamma_B^d(p^*(z_{/\bar{F}}^i))$. Now there is also a diagram

$$(1) \quad \begin{array}{ccc} \text{CH}^d(\tau X \times \tau X) & \xrightarrow{\gamma_B^d \otimes 1} & \text{H}_B^{2d}(\tau X(\mathbb{C}) \times \tau X(\mathbb{C}), \mathbb{Q})(d) \otimes \mathbb{Q}_\ell \\ p^* \uparrow & & \uparrow I_{\ell, \tau} \\ \text{CH}^d(X_{\bar{F}} \times X_{\bar{F}}) & \xrightarrow{\gamma_\ell^d} & \text{H}_{et}^i(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d) \end{array}$$

where γ_ℓ^d is the cycle class map of ℓ -adic cohomology and $I_{\ell, \tau}$ is the comparison isomorphism between Betti and ℓ -adic cohomology groups. It follows from the arguments given in [8, I, page 21], that (1) is commutative.

Let $\Delta_{/\bar{F}} \subset X_{\bar{F}} \times X_{\bar{F}}$ denote the diagonal subvariety. We have the Künneth decomposition of the ℓ -adic cohomology class $\gamma_\ell^d([\Delta_{/\bar{F}}]) = \sum_{i=0}^{2d} \pi_{/\bar{F}}^i$. It follows from the commutativity of (1), that

$$\gamma_\ell^d(z_{/\bar{F}}^i) = I_{\ell, \tau}^{-1}((\gamma_B^d \otimes 1)(\tau z^i)) = I_{\ell, \tau}^{-1}(\tau \pi^i \otimes 1) = \pi_{/\bar{F}}^i.$$

Now, clearly we can suppose that $z_{/\bar{F}}^i$ is defined over some finite extension (say) F_1 of F . By using the natural action of $G := \text{Gal}(F_1/F)$ on $\text{CH}^d(X_{F_1} \times X_{F_1})$, we set

$$z^i := \frac{1}{|G|} \sum_{\sigma \in G} \sigma^* z_{/\bar{F}}^i.$$

As $z^i \in \text{CH}^d(X_{F_1} \times X_{F_1})^G$, we conclude that $z^i \in \text{CH}^d(X \times X)$. Now we have

$$(2) \quad \gamma_\ell^d(z^i) = \frac{1}{|G|} \sum_{\sigma \in G} \gamma_\ell^d(\sigma^* z_{/\bar{F}}^i) = \frac{1}{|G|} \sum_{\sigma \in G} \tilde{\sigma}^* \gamma_\ell^d(z_{/\bar{F}}^i),$$

where $\tilde{\sigma}$ is a lift of σ to $\text{Gal}(\bar{F}/F)$.

Now, by Künneth formula and Poincaré duality for ℓ -adic cohomology, we have the following canonical isomorphisms

$$(3) \quad \begin{aligned} \text{H}_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d) &\cong \bigoplus_{r \geq 0} \text{H}_{et}^{2d-r}(X_{\bar{F}}, \mathbb{Q}_\ell)(d) \otimes \text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell) \\ &\cong \bigoplus_{r \geq 0} \text{Hom}_{\mathbb{Q}_\ell}(\text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \otimes \text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell) \cong \bigoplus_{r \geq 0} \text{End}_{\mathbb{Q}_\ell}(\text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell)) \end{aligned}$$

More precisely, under the above isomorphism, an element $u \in \text{H}_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d)$ is mapped to the element $\underline{u} := (z \mapsto pr_{2*}(pr_{1*}^*(z) \cdot u)) \in \bigoplus_{r \geq 0} \text{End}_{\mathbb{Q}_\ell}(\text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell))$, where pr_1

and pr_2 are projections. As one easily verifies, $\pi_{/\bar{F}}^i$ is the idempotent projection

$$p_i : H_{et}^*(X_{\bar{F}}, \mathbb{Q}_\ell) \rightarrow H_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell) \hookrightarrow H_{et}^*(X_{\bar{F}}, \mathbb{Q}_\ell).$$

Clearly p_i is invariant under the action of $\text{Gal}(\bar{F}/F)$ on $\bigoplus_{i \geq 0} \text{End}_{\mathbb{Q}_\ell}(H_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell))$. Thus, it follows from (3) that $\pi_{/\bar{F}}^i$ is invariant under the action of $\text{Gal}(\bar{F}/F)$ on $H_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d)$. Under the above observations, (2) now reads as

$$(4) \quad \gamma_\ell^d(z^i) = \frac{1}{|G|} \sum_{\sigma \in G} \tilde{\sigma}^* \pi_{/\bar{F}}^i = \frac{1}{|G|} (|G| \cdot \pi_{/\bar{F}}^i) = \pi_{/\bar{F}}^i.$$

Finally, let α^i denote the image of z^i under the canonical injective morphism $\text{CH}^d(X \times X) \rightarrow \text{CH}^d(X_{F_v} \times X_{F_v})$. Let $\gamma_{\ell,v}^d : \text{CH}^d(X_{F_v} \times X_{F_v}) \rightarrow H_{et}^{2d}(X_{\bar{F}_v} \times X_{\bar{F}_v}, \mathbb{Q}_\ell)(d)$ denote the cycle class map. By invariance of ℓ -adic cohomology groups under extension from \bar{F} to algebraically closed over-field, we have $H_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell) \cong H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ and $H_{et}^{2d}(X_{\bar{F}_v} \times X_{\bar{F}_v}, \mathbb{Q}_\ell) \cong H_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)$. We also know that $\gamma_{\ell,v}^d|_{\text{CH}^d(X \times X)} = \gamma_\ell^d$; in particular $\gamma_{\ell,v}^d(\alpha^i) = \gamma_\ell^d(z^i)$. Thus using (4) and (3), it follows that $\gamma_{\ell,v}^d(\alpha^i)$ induces an endomorphism $H_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell) \rightarrow H_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell)$, which is identity on $H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ and 0 otherwise. In particular we see that

$$(5) \quad \text{Tr}((\gamma_{\ell,v}^d(\alpha^i) \circ w_*^m) : H_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell)) = \text{Tr}(\rho_\ell^i(w^m)), \text{ for every } m \geq 0,$$

where $w_*^m := \bigoplus_{i=0}^{2d} \rho_\ell^i(w^m)$ is the \mathbb{Q}_ℓ -linear map induced by w^m on $H_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell)$. Now, pick a $N \geq 1$, such that $N\gamma_{\ell,v}^d(\alpha^i)$ belongs to the image of the Chow group of codimension d algebraic cycles on $X_{F_v} \times X_{F_v}$, with \mathbb{Z} -coefficients. It follows from (5) and [20, Thm 0.1], that

$$(6) \quad \text{Tr}(\rho_\ell^i(w^m)) \in (1/N)\mathbb{Z}, \text{ and is independent of } \ell.$$

Using (6) and applying the next lemma to the eigenvalues of $\rho_\ell^i(w)$, we conclude that $\text{Tr}(\rho_\ell^i(w)) \in \mathbb{Z}$. This combined with the Newton identities relating power sums and symmetric polynomials, we conclude that the characteristic polynomial $P_{\ell,v}^i(w, T)$ of $\rho_\ell^i(w)$ belongs to $\mathbb{Z}[T]$ and is independent of ℓ . \square

Lemma 2.2 (cf. [13, 2.8]). *Let a_1, \dots, a_r and b_1, \dots, b_s be elements of a field of characteristic 0. We put $s_m = \sum_{i=1}^r a_i^m - \sum_{j=1}^s b_j^m$ for an $m \in \mathbb{N}$. Assume there exists an integer $N \geq 1$ such that $Ns_m \in \mathbb{Z}$ for all $m \geq 0$. Then $s_m \in \mathbb{Z}$.*

Corollary 2.3. *Let X be a smooth projective variety over F which is a finite product of hyperkähler varieties of $K3^{[n]}$ type; moduli spaces of stable vector bundles of co-prime rank and degree over smooth projective curves; unirational varieties of dimension ≤ 4 ; uniruled varieties of dimension ≤ 3 ; curves, surfaces, abelian varieties and smooth complete intersections in projective spaces. Then, for every $i \in \mathbb{N}$, $\ell \neq p$ and $w \in W_v$, the polynomial $P_{\ell,v}^i(w, T) \in \mathbb{Q}[T]$ and is independent of ℓ .*

Proof. We begin with an observation. Let Y_1 and Y_2 be varieties over \mathbb{C} . Let $\gamma_B^{\dim Y_*}([\Delta_*]) = \sum_{r_*=0}^{2 \dim Y_*} \pi^{r_*}$, be the Künneth decomposition of the diagonal sub-variety Δ_* of $Y_* \times Y_*$, for $*$ = 1, 2. Let $\gamma_B^{\dim Y_1 \times Y_2}([\Delta_{Y_1 \times Y_2}]) = \sum_{i=0}^{2 \dim Y_1 \times Y_2} \pi^i$, be the Künneth decomposition of the diagonal sub-variety $\Delta_{Y_1 \times Y_2}$ of $Y_1 \times Y_2$. Then we check that the identity $\pi_{Y_1 \times Y_2}^i = \sum_{r_1+r_2=i} \pi^{r_1} \otimes \pi^{r_2}$, holds true for every $0 \leq i \leq 2 \dim Y_1 \times Y_2$. Now, assume that for every r_1 and r_2 , satisfying $r_1 + r_2 = i$, there exists algebraic cycles $z^{r_*} \in \text{CH}^{\dim Y_*}(Y_* \times Y_*)$ such that $\gamma_B^{\dim Y_*}(z^{r_*}) = \pi^{r_*}$, where $*$ = 1, 2. Then, by the above identity, $\pi_{Y_1 \times Y_2}^i =$

$\sum_{r_1+r_2=i} \gamma_B^{\dim Y_1}(z^{r_1}) \otimes \gamma_B^{\dim Y_2}(z^{r_2})$. It now follows from the multiplicativity of the cycle class maps that $\pi_{Y_1 \times Y_2}^i = \sum_{r_1+r_2=i} \gamma_B^{\dim Y_1 \times Y_2}(z^{r_1} \times z^{r_2})$. As $\gamma_B^{\dim Y_1 \times Y_2}$ is a group homomorphism it follows that $\pi_{Y_1 \times Y_2}^i = \gamma_B^{\dim Y_1 \times Y_2}(\sum_{r_1+r_2=i} (z^{r_1} \times z^{r_2}))$. This observation tells us that, in order to apply Thm.2.1 to the variety τX , it suffices to check that the individual varieties in the statement of the corollary satisfies the Künneth Standard conjecture (w.r.t Betti cohomology) in all degrees $\leq i$. Now, for curves, surfaces, abelian varieties and smooth complete intersection in projective spaces, this is well-known (cf.[13]). For hyperkähler varieties of $K3^{[n]}$ type this follows from the results of [2]. For unirational varieties of dimension ≤ 4 and uniruled varieties of dimension ≤ 3 this is shown in [1]. For the moduli spaces $N_C(q, e)$, this follows from [3].

Now, applying Thm. 2.1 to τX , we see that for every $i \in \mathbb{N}$, the characteristic polynomial

$$(7) \quad P_{\ell, v}^i(w, T) \in \mathbb{Z}[T] \text{ and is independent of } \ell, \text{ for every } w \in W_v \text{ such that } \alpha(w) \geq 0.$$

Now, let $w \in W_v$ such that $\alpha(w) > 0$, then $\alpha(w^{-1}) < 0$. As $\rho_\ell^i(w) = (\rho_\ell^i(w^{-1}))^{-1}$, so if $P_{\ell, v}^i(w^{-1}, T) = T^m + \sum_{r=1}^m a_{m-r} T^{m-r}$, then $P_{\ell, v}^i(w, T) = T^m + a_0^{-1}(\sum_{r=1}^{m-1} a_r T^{m-r} + 1)$. By (7), $a_r \in \mathbb{Z}$, hence $P_{\ell, v}^i(w, T) \in \mathbb{Q}[T]$ and is independent of ℓ . □

- Remark 2.4.** (i) Special cases of Cor.2.3, such as abelian varieties (cf.[25]) and curves, had been previously proved using methods quite different from ours. Thanks to the motivic nature of Thm. 2.1, we deduce these known cases and much more, at once.
- (ii) For any smooth projective curve C of genus > 1 over F , the moduli space $N_C(q, e)$ of stable vector bundles of co-prime rank q and degree e over C is known to be a smooth projective fine moduli space.
- (iii) Recall that an algebraic variety Y is said to be a *hyperkähler* variety Y of $K3^{[n]}$ -type if (a) $Y = S^{[n]}$ is the punctual Hilbert scheme which parametrizes closed subschemes of of length n of a $K3$ surface S , or (b) Y is any projective deformation of a hyperkähler variety of type $S^{[n]}$. Any general projective deformation of $S^{[n]}$ is not of the form $S'^{[n]}$ for any other $K3$ surface S' . In dimension 2, hyperkähler varieties are $K3$ surfaces.

3. AROUND THE C_{WD} CONJECTURE

3.1. Monodromy. Let K be a complete discretely valued field with a finite residue field k_v , where v denotes the valuation on K . Let $\text{char}(k_v) = p > 0$. Fix an algebraic closure \bar{K} of K and write $\Gamma_K := \text{Gal}(\bar{K}/K)$. Let \bar{v} be the extension of v to \bar{K} . The residue field of \bar{K} at \bar{v} is denoted by \bar{k}_v (which is also an algebraic closure of k_v).

For any prime number ℓ , we denote by μ_{ℓ^n} the group of ℓ^n -th roots of unity in \bar{k}_v and $\mathbb{Z}_\ell(1) := \varprojlim_n \mu_{\ell^n}$. The inertia group $I_K \subset \Gamma_K$ fits into the following exact sequence

$$1 \rightarrow P \rightarrow I_K \xrightarrow{t} \mathbb{Z}_{(\ell')}(1) \rightarrow 1,$$

where P is a pro- p -group and $\mathbb{Z}_{(\ell')}(1) = \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$. Let $\ell \neq p$ be a prime number. We

denote by $t_\ell : I_K \rightarrow \mathbb{Z}_\ell(1)$, the ℓ -component of t . Explicitly, the surjective map t_ℓ is given as $x \mapsto \left(x(\pi^{\frac{1}{\ell^n}}) / \pi^{\frac{1}{\ell^n}} \right)_n$, where $\pi \in \mathcal{O}_K$ is an uniformizer. The map t_ℓ is unique upto multiplication by an element of \mathbb{Z}_ℓ^\times .

For any \mathbb{Q}_ℓ -vector space U , we write $U(1) := U \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)$, where $\mathbb{Q}_\ell(1) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)$. Grothendieck's ℓ -adic monodromy theorem [25, Appendix], says that any ℓ -adic representation $\xi_\ell : \Gamma_K \rightarrow \text{GL}(U)$ is *quasi-unipotent*, i.e., there exists an open subgroup $J \subseteq I_K$ such

that $\xi_\ell|_J$ is unipotent (i.e. $\xi_\ell(\sigma)$ is a unipotent linear map for every $\sigma \in J$). Moreover, there exists a unique nilpotent morphism $N_\ell : U(1) \rightarrow U$ such that

$$(8) \quad \xi_\ell(x) = \exp(t_\ell(x)N_\ell), \quad \text{for all } x \in J.$$

The morphism N_ℓ is called the *monodromy operator* associated to ξ_ℓ .

3.2. Linear representations of Weil-Deligne groups. The notation is as in §3.1, we assume further that K is a finite extension of \mathbb{Q}_p and $|k_v| = q_v$. Denote by $\phi_v : x \mapsto x^{q_v}$ the arithmetic Frobenius automorphism of \bar{k}_v over k_v .

Definition 3.3. Let E be any field of characteristic 0. A linear representation $'W_K$ over E is a triple $\underline{\xi} = (\Delta, \xi', N')$ consisting of

- (a) A finite dimensional linear representation $\xi' : W_K \rightarrow \text{GL}(\Delta)$ of W_K over E .
- (b) A nilpotent endomorphism N' of Δ , such that $\xi'(w)N'\xi'(w)^{-1} = q_v^{\alpha(w)} \cdot N$ for every $w \in W_K$.

Definition 3.4. A *morphism* $\underline{\xi}_1 \rightarrow \underline{\xi}_2$ between two linear representations of $'W_K$ over E , is a E -linear map $f : \Delta_1 \rightarrow \Delta_2$ such that $f \circ \xi'_1(w) = \xi'_2(w) \circ f$, for all $w \in W_K$, and $f \circ N'_1 = N'_2 \circ f$.

The collection of all linear representations of $'W_K$ over E , forms a neutral Tannakian category $\mathbf{Rep}_E('W_K)$ over E . Any field embedding $\tau : E \rightarrow L$ gives rise to a functor

$$\mathbf{Rep}_E('W_K) \rightarrow \mathbf{Rep}_L('W_K), \quad (\Delta, \xi', N') \mapsto (L \otimes_{\tau, E} \Delta, {}^\tau \xi', 1 \otimes_{\tau, E} N').$$

Definition 3.5 (Deligne [5]). Let $\underline{\xi}$ be as above and E_0 a sub-field of E . We say that $\underline{\xi}$ is *defined over* E_0 , if given any algebraically closed field $\Omega \supset E$ and any $\sigma \in \text{Aut}(\Omega/E_0)$, the representations ${}^\sigma \underline{\xi}_{/\Omega} = (\Omega \otimes_{\sigma, E} \Delta, {}^\sigma \xi'_{/\Omega}, 1 \otimes_{\sigma, E} N')$ of $'W_K$ over Ω , obtained by extension of scalars via σ , are all isomorphic. This condition is independent of the choice of Ω .

Definition 3.6 (Deligne [5]). Let $(E_i)_{i \in I}$ be a family of extensions of E_0 , and for every i , let $\underline{\xi}_i$ be a linear representation of $'W_K$ over E_i . We say that $(\underline{\xi}_i)_{i \in I}$ forms a *compatible family of representations* of $'W_K$ if every $\underline{\xi}_i$ is defined over E_0 , and for any $i, j \in I$, if Ω is an algebraically closed field containing E_i and E_j , then the representations $\underline{\xi}_{i/\Omega}$ and $\underline{\xi}_{j/\Omega}$ obtained by extension of scalars are isomorphic.

Now fix an arithmetic Frobenius $\Phi \in \Gamma_K$ and an isomorphism $\iota : \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell(1)$. To any arbitrary ℓ -adic representation $\xi_\ell : \Gamma_K \rightarrow \text{GL}(U)$ (as in §3.1), we associate a representation $\underline{\xi}_\ell = (U, \xi'_\ell, N'_\ell)$ of $'W_K$ over \mathbb{Q}_ℓ , by setting

- (a) $\xi'_\ell(w) = \xi_\ell(w) \exp(-N_\ell t_\ell(\Phi^{-\alpha(w)} w))$, where N_ℓ is the monodromy operator as in (8).
- (b) $N'_\ell \in \text{End}(U)$ corresponds to N_ℓ , via ι .

The following lemma is well-known.

Lemma 3.7. *The isomorphism class of $\underline{\xi}_\ell$ depends only on ξ_ℓ , and it doesn't depend on the choice of Φ and ι .*

3.8. Proof of Theorem 1.2. The proof will follow from a series of subsidiary results which are of independent interest. We denote by $V_\ell^i := H_{\text{et}}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ and by $\underline{\rho}_\ell^i = (V_\ell^i, \rho_{i,\ell}', N'_{i,\ell})$ the representation of $'W_v$ over \mathbb{Q}_ℓ , associated (see §3.2) to the canonical ℓ -adic representations $\rho_\ell^i : \Gamma_F \rightarrow \text{GL}(V_\ell^i)$.

For simplicity of notation, let us suppress the natural number i for a moment. As $N'_\ell : V_\ell \rightarrow V_\ell$ is a nilpotent endomorphism, it induces a unique filtration \mathfrak{M}_\bullet on V_ℓ called the *local monodromy filtration* [7], characterized by the following properties:

- (1) \mathfrak{M}_\bullet is an increasing filtration $\cdots \mathfrak{M}_{j-1}V_\ell \subset \mathfrak{M}_jV_\ell \subset \mathfrak{M}_{j+1}V_\ell \cdots$ of Γ_v representations, such that $\mathfrak{M}_jV_\ell = 0$ for sufficiently small j and $\mathfrak{M}_jV_\ell = V_\ell$ for sufficiently large j .
- (2) $N_\ell(\mathfrak{M}_jV_\ell(1)) \subseteq \mathfrak{M}_{j-2}V_\ell$ for all j .
- (3) Using the second condition we can define an induced map $N : \mathrm{Gr}_j^{\mathfrak{M}}V_\ell(1) \rightarrow \mathrm{Gr}_{j-2}^{\mathfrak{M}}V_\ell$, where $\mathrm{Gr}_j^{\mathfrak{M}}V_\ell = \mathfrak{M}_jV_\ell/\mathfrak{M}_{j-1}V_\ell$. Then $N^r : \mathrm{Gr}_r^{\mathfrak{M}}V_\ell(r) \rightarrow \mathrm{Gr}_{-r}^{\mathfrak{M}}V_\ell$ is an isomorphism for each $r \geq 0$.

Explicitly, the filtration \mathfrak{M}_\bullet is defined as the convolution $\mathfrak{F} * \mathfrak{G}$ of the Kernel filtration \mathfrak{F}_\bullet and image filtration \mathfrak{G}^\bullet on V_ℓ , induced by N'_ℓ , i.e.,

$$\mathfrak{M}_rV_\ell := \sum_{p-q=r} \mathfrak{F}_pV_\ell \cap \mathfrak{G}^qV_\ell \text{ for } r \in \mathbb{Z},$$

Let $\bar{\rho}_{\ell,j} : \Gamma_v \rightarrow \mathrm{GL}(\mathrm{Gr}_j^{\mathfrak{M}}V_\ell)$ denote the representation induced by ρ_ℓ on the graded parts of \mathfrak{M}_\bullet . For every $w \in W_v$, we write $P_\ell(w, T) := \det(1 - \rho_\ell(w)T; V_\ell)$ the characteristic polynomial of $\rho_\ell(w)$ and $\bar{P}_{\ell,j}(w, T) := \det(1 - \bar{\rho}_{\ell,j}(w)T; \mathrm{Gr}_j^{\mathfrak{M}}V_\ell)$ the characteristic polynomial of $\bar{\rho}_{\ell,j}(w)$.

Main Lemma 3.9. *Fix a $w \in W_v$, then for every j , $\bar{P}_{\ell,j}(w, T) \in \mathbb{Q}[T]$ and is independent of ℓ .*

Proof. Let L be the sub-field of an algebraic closure $\bar{\mathbb{Q}}_\ell$ of \mathbb{Q}_ℓ , generated by the roots of $P_\ell(w, T)$. By Cor.2.3, $P_\ell(w, T) \in \mathbb{Q}[T]$ and has coefficients independent of ℓ . This implies that if $P_\ell(w, \beta) = 0$, then $P_\ell(w, \sigma(\beta)) = 0$, for every $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$. Now, from linear algebra, we know that $P_\ell(w, T) = \prod_j \bar{P}_{\ell,j}(w, T)$. Let η be an eigenvalue $\bar{\rho}_{\ell,j}(w)$ and hence of $\rho_\ell(w)$. By [15], the weight monodromy conjecture (WMC) holds for V_ℓ , hence the complex absolute value

$$(9) \quad |\eta|_{\mathbb{C}} = q_v^{(i+j)\alpha(w)/2} \text{ for every } j$$

(recall that i is the degree of the cohomology group V_ℓ). As $|\beta|_{\mathbb{C}} = |\sigma(\beta)|_{\mathbb{C}}$, it follows from (9), that β and $\sigma(\beta)$ occurs as the roots of a same factor (say $\bar{P}_{\ell,j_0}(w, T)$ of $P_\ell(w, T)$). Now suppose $\bar{P}_{\ell,j_0}(w, T) = T^m + \sum_{r=1}^m a_{m-r}T^{m-r}$. As the coefficients a_r 's are symmetric polynomials in the roots of $\bar{P}_{\ell,j_0}(w, T)$, it follows from the previous observation that $\sigma(a_r) = a_r$, for every $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$. In other words, $a_r \in \mathbb{Q}$, i.e., $\bar{P}_{\ell,j_0}(w, T) \in \mathbb{Q}[T]$ and is independent of ℓ . By varying β over all roots of $P_\ell(w, T)$, we conclude the $\bar{P}_{\ell,j}(w, T) \in \mathbb{Q}[T]$ and is independent of ℓ , for every j . \square

The proof of the next two results would employ some motivic ideas, we review some of the necessary facts very briefly, for details see [8, II].

We denote by $\mathcal{V}(F)$ the category of smooth projective algebraic varieties over F . Recall that the category of (pure) motives $\mathcal{M}_{AH}(F)$, defined by absolute Hodge cycles is a \mathbb{Q} -linear semisimple neutral Tannakian category. There exists a contravariant functor for $\mathcal{M}_{AH}(F)$) $h : \mathcal{V}(F) \rightarrow \mathcal{M}_{AH}(F)$ such that Betti, ℓ -adic or deRham cohomology on $\mathcal{V}(F)$ factorizes through h . This provides fiber functors H_τ , H_ℓ and H_{dR} on $\mathcal{M}_{AH}(F)$. We call these functors as *realizations*. There also exists a natural grading $h(Z) = \bigoplus h^i(Z)$, $i \in \mathbb{N}$ for every $Z \in \mathrm{Ob}(\mathcal{V}(F))$, which extends to all of $\mathcal{M}_{AH}(F)$. We denote by $\mathcal{M}_{AH}^{av}(F)$ the Tannakian subcategory of $\mathcal{M}_{AH}(F)$ generated by abelian varieties and zero dimensional varieties. Everything stated above holds more generally for any arbitrary field of characteristic 0, with an embedding in \mathbb{C} .

Lemma 3.10. *Let X/F be as in Thm. 1.2. Then there exists a finite extension F' of F such that $h(X_{F'}) \in \mathrm{Ob}(\mathcal{M}_{AH}^{av}(F'))$.*

Proof. First, we claim that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$. Let $X = \prod_r X_r$, where each X_r is one of the varieties in the statement of the theorem. As $h(X_{\bar{F}}) = \otimes_r h(X_{r,\bar{F}})$ and $\mathcal{M}_{AH}^{av}(\bar{F})$ is closed \otimes , in order to verify the claim it suffices to do so for the individual X_r 's (which by abuse of notation we denote by X).

First suppose that X is the moduli space $N_C(q, e)$. By [3] we have a decomposition $\oplus_b h(C)^{\otimes a}(b) = h(X) \oplus N$, for some $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ and some motive N . Now, let $J(C)$ denote the Jacobian of C . We know that $h(C) = \mathbb{1} \oplus h^1(J(C)) \oplus \mathbb{L}$, where $\mathbb{1}$ denotes the unit object of $\mathcal{M}_{AH}(\bar{F})$ and \mathbb{L} is the Lefschetz motive. As $\mathcal{M}_{AH}^{av}(F)$ is closed under direct summands, $h(X) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F))$.

Now consider the case where X is a hyperkähler variety of $K3^{[n]}$ -type. The cohomology group $H_B^2(\tau X, \mathbb{Q})$ carries a weight 2 Hodge structure. It is well-known that in this case, the Kuga-Satake morphism $H_B^2(\tau X, \mathbb{Q})(1) \hookrightarrow H^1(KX, \mathbb{Q}) \otimes H^1(KX, \mathbb{Q})$, is an absolute Hodge correspondence, where KX is the Kuga-Satake abelian variety associated to $H_B^2(\tau X, \mathbb{Q})$. This implies that there is a monomorphism of motives

$$h^2(\tau X)(1) \hookrightarrow h^1(KX) \otimes h^1(KX).$$

Moreover, since X is defined over F , one can show (as in the case of $K3$ surfaces) that KX is defined over a finite extension F' of F . The main result of [21] shows that $h(X_{\bar{F}}) \in \text{Ob}(\langle h^2(X_{\bar{F}}) \rangle^{\otimes})$ i.e. the smallest Tannakian category generated by $h^2(X_{\bar{F}})$. It follows that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$.

Finally, when X is a uniruled surface, we know (cf.[1]) that there exists a curve C' over \bar{F} and a decomposition $\oplus_{b'} h(C')^{\otimes a'}(b') = h(X_{\bar{F}}) \oplus N'$, for some $a' \in \mathbb{Z}$, $b' \in \mathbb{N}$ and some motive N' . Thus, as before, it follows that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$.

Next, when X is a Fermat hypersurface or a unirational variety of dimension ≤ 3 , it follows from [8, II, 6.26] that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$.

Thus in each case we see that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$. By using [8, I, 2.9] we conclude there exists a finite extension F' of F such that $h(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$. □

Now, for X as in Thm. 1.2, we denote by M the motive $h^i(X)$. As M is a direct summand of $h(X)$, by Lemma 3.10 there is a finite extension F' of F such that $M_{F'} := h^i(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$. Let us denote by v' an extension of the valuation v to F' ; n the residual degree; $\Phi_{v'} := \Phi_v^n$ an arithmetic Frobenius corresponding to this extension. We denote by $\langle M_{F'}, \mathbb{Q}(1) \rangle^{\otimes}$ the Tannakian subcategory of $\mathcal{M}_{AH}^{av}(F')$ which is tensor generated by $M_{F'}$ and the Tate motive $\mathbb{Q}(1)$ and we write $G_{M_{F'}} := \underline{\text{Aut}}^{\otimes}(H_{\tau} |_{\langle M_{F'}, \mathbb{Q}(1) \rangle^{\otimes}})$. We recall, that the ℓ -adic representations $\rho_{M_{F'}, \ell} : \Gamma_{F'} \rightarrow \text{GL}(H_{\ell}(M_{F'}))$, arising from the action of $\Gamma_{F'}$ on the ℓ -adic realization $H_{\ell}(M_{F'})$ of $M_{F'}$, factorizes through $G_{M_{F'}}(\mathbb{Q}_{\ell})$.

Proposition 3.11. *The continuous representation $\rho_{\ell}^{i'} : W_v \rightarrow \text{GL}(V_{\ell}^i)$ is a semisimple representation of W_v*

Proof. We know that there exists (possibly after passing to another finite extension, which we again denote by F') an abelian variety A over F' and a unique homomorphism of algebraic groups $\theta : G_A \rightarrow G_{M_{F'}}$ where $G_A := \underline{\text{Aut}}^{\otimes}(H_{\tau} |_{\langle h^1(A), \mathbb{Q}(1) \rangle^{\otimes}})$. Moreover, $\theta(\mathbb{Q}_{\ell}) \circ \rho_{A, \ell} = \rho_{M_{F'}, \ell}$, where $\theta(\mathbb{Q}_{\ell})$ denotes the induced map on \mathbb{Q}_{ℓ} -valued points and $\rho_{A, \ell}$ is the Galois representation associated to the motive $h^1(A)$. For details on the above facts see [14, 3.10 & 3.11]. In particular, we have $\theta(\mathbb{Q}_{\ell})(\rho_{A, \ell}(\Phi_{v'})) = \rho_{M_{F'}, \ell}(\Phi_{v'})$.

Now, it is well known that $\rho_{A, \ell}(\Phi_{v'})$ is a semisimple automorphism. As θ is a homomorphism of algebraic groups, the image of $\rho_{A, \ell}(\Phi_{v'})$, under $\theta(\mathbb{Q}_{\ell})$ must be semisimple. In other words, $\rho_{M_{F'}, \ell}(\Phi_{v'})$ is semisimple. Finally, let $\rho_M : \Gamma_F \rightarrow \text{GL}(H_{\ell}(M))$ be the ℓ -adic representation associated M . As $\rho_{M_{F'}, \ell}(\Phi_{v'}) = (\rho_{M, \ell}(\Phi_v))^n$, so $\rho_{M, \ell}(\Phi_v)$ is semisimple.

Now, by definition

$$H_\ell(M) = \pi^{i*} \oplus_r H_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell),$$

where π^i is the (i -th) Künneth component of the diagonal subvariety $\Delta \subset X_{\bar{F}} \times X_{\bar{F}}$ and π^{i*} is the image of π^i under the isomorphism (3). As we've seen before, π^{i*} is the idempotent projection p_i . We conclude that $H_\ell(M) = H_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell)$ and $\rho_{M,\ell} = \rho_\ell^i$. Thus, it follows from the above discussion that $\rho_\ell^i(\Phi_v)$ is semisimple.

Now, we note that $\rho_\ell^{i'}(\Phi_v)$ ($= \rho_\ell^i(\Phi_v)$) generates a subgroup of finite index in $\rho_\ell^{i'}(W_v)$. We know that in characteristic 0 a representation (in the ordinary sense) of a group is semisimple if and only if its restriction to a subgroup of finite index is semisimple. As $\rho_\ell^i(\Phi_v)$ is a semisimple automorphism, it follows that $\rho_\ell^{i'}$ is a semisimple representation of W_v . \square

Now, by the main lemma 3.9, the character of the representations $\overline{\rho}_{\ell,j}^i$ (on the graded parts of the local monodromy filtration on V_ℓ^i) has values in \mathbb{Q} and is independent of ℓ . It now follows from Prop. 3.11 and [5, Prop. 8.9], that $\{(V_\ell^i, \rho_\ell^{i'}, N_{i,\ell}^i)\}_{\ell \neq p}$ forms a compatible system of linear representations of $'W_v$ defined over \mathbb{Q} . This completes the proof of Theorem 1.2. \square

Remark 3.12. Let Y be an algebraic variety over F and v a non-archimedean valuation of F . Let Φ_v be any arithmetic Frobenius element of Γ_v . A conjecture of Serre (cf.[24, 12.4]) predicts that *the operators $\rho_\ell^i(\Phi_v)$ are semisimple*, where $\rho_\ell^i : \Gamma_F \rightarrow \text{GL}(H_{et}^i(Y_{\bar{F}}, \mathbb{Q}_\ell))$ are the canonical ℓ -adic representations associated to Y . Prop. 3.11, says that this is true for the algebraic varieties in the statement of Theorem 1.2.

We now discuss how our result fits in the context of Langlands program. Let us review some notation. For any $m \in \mathbb{N}$, we denote by $\mathcal{G}_m(F_v)$ the set of equivalence classes of m -dimensional complex semi-simple representations of $'W_v$. We denote by $\mathcal{A}_m(F_v)$ the set of equivalence class of irreducible admissible representations of $\text{GL}_m(F_v)$. The local Langlands correspondence for F_v gives a bijection

$$\text{rec}_{F_v,m} : \mathcal{A}_m(F_v) \rightarrow \mathcal{G}_m(F_v).$$

Corollary 3.13. *Let X be as in the theorem, $i \in \mathbb{N}$ and let $b_i := \dim V_\ell^i$ denote the i -th Betti number of X . Then, there exists a representation $\underline{\rho}^i$ of $'W_v$ defined over \mathbb{Q} and a unique class $[\pi_i] \in \mathcal{A}_{b_i}(F_v)$ such that*

$$\text{rec}_{F_v,b_i}([\pi_i]) = [\underline{\rho}^i_{/\mathbb{C}}] = [\sigma \underline{\rho}^i_{/\mathbb{C}}] \text{ for every } \ell \neq p \text{ and embeddings } \sigma : \mathbb{Q}_\ell \hookrightarrow \mathbb{C},$$

where $\sigma \underline{\rho}^i_{/\mathbb{C}} := (\mathbb{C} \otimes_{\sigma, \mathbb{Q}_\ell} V_\ell^i, \sigma \rho_\ell^{i'}_{/\mathbb{C}}, 1 \otimes_{\sigma, \mathbb{Q}_\ell} N_{i,\ell}^i)$.

Proof. First note that the Betti number b_i is independent of ℓ . It follows from the theorem and Prop.3.11 that there exists a representation $\underline{\rho}^i$ of $'W_v$ defined over \mathbb{Q} such that class $[\underline{\rho}^i_{/\mathbb{C}}] \in \mathcal{G}_{b_i}(F_v)$ and

$$\underline{\rho}^i_{/\mathbb{C}} \cong \sigma \underline{\rho}^i_{/\mathbb{C}} \text{ for every } \ell \neq p \text{ and embeddings } \sigma : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}.$$

Now we apply the local Langlands correspondence to get a unique class $[\pi_i] \in \mathcal{A}_{b_i}(F_v)$ verifying the required relation. \square

3.14. The case of surfaces and complete intersections. Let the notation be as in §3.2. Let $\underline{\xi} = (\Delta, \xi', N')$ be any linear representation of $'W_K$ over E . Pick an arithmetic Frobenius element $\Phi \in W_K$. Consider the multiplicative Jordan-Chevalley decomposition of the endomorphism $\xi'(\Phi) = \xi'(\Phi)_{ss} \cdot \xi'(\Phi)_u$, where $\xi'(\Phi)_{ss}$ is semisimple and $\xi'(\Phi)_u$ is unipotent.

Definition 3.15. The *semi-simplification* $\underline{\xi}^{ss}$ of $\underline{\xi}$ is the linear representation of $'W_K$ over E , obtained from $\underline{\xi}$ by keeping Δ , $\xi'|_{I_K}$ and N' unchanged, and replacing $\xi'(\Phi)$ by $\xi'(\Phi)_{ss}$.

It can be shown that $\underline{\xi}^{ss}$ is independent of the initial choice of the arithmetic Frobenius Φ .

The semi-simplification of the representations $\underline{\rho}_\ell^i = (V_\ell^i, \rho_\ell^i, N'_{i,\ell})$ of the Weil-Deligne group $'W_v$ of F_v , over \mathbb{Q}_ℓ , are denoted by $\underline{\rho}_\ell^{i,ss}$.

Proof of Theorem 1.3. (1) The motive of S has a decomposition $h(S) = \bigoplus_{i=0}^4 h^i(S)$, such that $h^0(S) \cong h^4(S)(2) \cong \mathbb{1}$ and

$$h^1(S) \cong h^3(S)(1) \cong h^1(\text{Pic}_S^0) \cong h^1(\text{Alb}_S) \cong h^1(S)^\vee(-1),$$

where Pic_S^0 and Alb_S are the Picard and Albanese variety of S , respectively. This observation allows us to deduce $C_{WD}(S_{F_v}, i)$, for $i \neq 2$, directly from $h^1(\text{Alb}_S)$ [5].

Now let $i = 2$. Let $\overline{\rho}_{\ell,j}^2 : \Gamma_v \rightarrow \text{GL}(\text{Gr}_j^{\mathfrak{M}} V_\ell^2)$ denote the representation induced by ρ_ℓ^2 on the graded parts $\text{Gr}_j^{\mathfrak{M}} V_\ell^2$ of the local monodromy filtration \mathfrak{M}_\bullet on V_ℓ^2 . By Cor.2.3, we know that the characteristic polynomial of $\rho_\ell^2(w) \in \mathbb{Q}[T]$ and is independent of ℓ . We also know from [19], that WMC holds for V_ℓ^2 , i.e., for every $w \in W_v$, the eigenvalues of $\overline{\rho}_{\ell,j}^2(w)$ are all algebraic integers of complex absolute value $q_v^{(2+j)\alpha(w)/2}$. It follows, as in the proof of Lemma 3.9, that the character of the representations $\overline{\rho}_{\ell,j}^2$ has values in \mathbb{Q} and is independent of ℓ . We now apply [5, Prop.8.9] to conclude the proof in this case.

- (2) First consider the case $i \neq n$. If $i < n$, then by a repeated application of the weak Lefschetz theorem [7], for ℓ -adic cohomology, we get Γ_F -equivariant isomorphisms $V_\ell^i \cong H_{\text{et}}^i(\mathbb{P}^r, \mathbb{Q}_\ell)$. It follows that $V_\ell^i = \mathbb{Q}_\ell(-i/2)$ for even $i < n$ and $V_\ell^i = 0$ for odd $i < n$. Now, by Poincaré duality, for every i we have Γ_F -equivariant isomorphisms $V_\ell^i = (V_\ell^{2n-i}(n))^\vee$. Therefore by the previous observation, we deduce that for i even and $n < i \leq 2n$, we have $V_\ell^i = (\mathbb{Q}_\ell(-n + i/2) \otimes \mathbb{Q}_\ell(n))^\vee = \mathbb{Q}_\ell(-i/2)$ and $V_\ell^i = 0$ for i odd and $n < i \leq 2n$. Thus, in order to prove our claim, we only need to consider the cases of even positive integers $i \neq n$. Now, fix such a i and let $x \in \mathbb{Z}_\ell(1)$ be a generator, then $\mathbb{Q}_\ell(-i/2) = \mathbb{Q}_\ell \cdot x^{-i/2}$. Let $\chi_\ell : \Gamma_F \rightarrow \mathbb{Q}_\ell^\times$ be the ℓ -adic cyclotomic character of Γ_F , then the ℓ -adic representations $\rho_\ell^i : \Gamma_F \rightarrow \text{GL}(V_\ell^i) = \text{GL}(\mathbb{Q}_\ell(-i/2))$ are given by

$$\lambda \mapsto (x^{-i/2} \mapsto \chi_\ell^{-i/2}(\lambda) \cdot x^{-i/2}).$$

Now note that for any non-archimedean valuation v (of residual characteristic p) of F , $\chi_\ell(\lambda) = 1$ for every $\ell \neq p$ and $\lambda \in I_v$ (inertia subgroup of Γ_v). This implies that $\rho_\ell^i|_{I_v} = 1$ and the ℓ -adic monodromy operators $N'_{i,\ell}$'s associated to ρ_ℓ^i 's are all trivial. In other words, the representation of $'W_v$ associated to ρ_ℓ^i is given as

$$\underline{\rho}_\ell^i = (\mathbb{Q}_\ell(-i/2), \rho_\ell^i|_{W_v}, 0).$$

This combined with the fact that $\mathbb{Q}_\ell(-i/2)$ is a 1-dimensional representation, implies that in order to show that $\{\underline{\rho}_\ell^i\}_{\ell \neq p}$ forms a compatible system of representations of

¹This decomposition also holds in the category of Chow motives.

' W_v defined over \mathbb{Q} , we need to show that for every $w \in W_v$, the traces $\text{Tr}(\rho_\ell^i(w)) \in \mathbb{Q}$ and is independent of ℓ . As $\rho_\ell^i|_{I_v} = \{1\}$, it suffices to show that for any arithmetic Frobenius element $\Phi_v \in \Gamma_v$, the trace $\text{Tr}(\rho_\ell^i(\Phi_v)) \in \mathbb{Q}$ and is independent of ℓ . Now, $\text{Tr}(\rho_\ell^i(\Phi_v)) = \chi_\ell^{-i/2}(\Phi_v) = q_v^{-i/2} \in \mathbb{Q}$ and is independent of ℓ . Hence, $\{\underline{\rho}_\ell^i\}_{\ell \neq p}$ forms a compatible system of representations of ' W_v defined over \mathbb{Q} .

Now consider the case $i = n$. Let $\overline{\rho}_{\ell,j}^n : \Gamma_v \rightarrow \text{GL}(\text{Gr}_j^{\mathfrak{M}} V_\ell^n)$ denote the representation induced by ρ_ℓ^n on the graded parts $\text{Gr}_j^{\mathfrak{M}} V_\ell^n$ of the local monodromy filtration \mathfrak{M}_\bullet on V_ℓ^n . By Cor.2.3, we know that the characteristic polynomial of $\rho_\ell^n(w) \in \mathbb{Q}[T]$ and is independent of ℓ . We also know from [23], that WMC holds for V_ℓ^n , i.e., for every $w \in W_v$, the eigenvalues of $\overline{\rho}_{\ell,j}^n(w)$ are all algebraic integers of complex absolute value $q_v^{(n+j)\alpha(w)/2}$. It follows, as before, that the character of the representations $\overline{\rho}_{\ell,j}^n$ has values in \mathbb{Q} and is independent of ℓ . We now apply [5, Prop.8.9] to conclude the proof in this case. \square

Corollary 3.16. *Let X be a smooth complete intersection of odd dimension $n = 2m + 1$ and of Hodge level 1, defined over F . Then $C_{WD}(X_{F_v}, i)$ holds true for all i .*

Proof. In view of Thm. 1.3, we only need to establish that for $i = n$, Φ_v acts semi-simply on V_ℓ^n . By a result of Deligne [4], we know that there is an abelian variety $J(X)$ (intermediate Jacobian of X) defined over F and an isomorphism of Γ -modules, $H_{\text{et}}^n(X_{\overline{F}}, \mathbb{Q}_\ell)(m) \cong H_{\text{et}}^1(J(X)_{\overline{F}}, \mathbb{Q}_\ell)$. Thus our claim follows from the corresponding fact for $H_{\text{et}}^1(J(X)_{\overline{F}}, \mathbb{Q}_\ell)$. \square

The next result is the analog of Cor. 3.13, in case of surfaces and smooth complete intersections.

Corollary 3.17. *Let the notation be as in Thm. 1.3*

- (1) *If S is a smooth projective surface over F , then for every $i \neq 2$, there exists a representation $\underline{\rho}^i$ of ' W_v defined over \mathbb{Q} and a unique class $[\pi_i] \in \mathcal{A}_{b_i}(F_v)$ such that*

$$\text{rec}_{F_v, b_i}([\pi_i]) = [\underline{\rho}^i_{/\mathbb{C}}] = [{}^\sigma \underline{\rho}_{\ell/\mathbb{C}}^i] \text{ for every } \ell \neq p \text{ and embeddings } \sigma : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}.$$

When $i = 2$, there exists a representation $\underline{\rho}^2$ of ' W_v defined over \mathbb{Q} and a unique class $[\pi_2] \in \mathcal{A}_{b_2}(F_v)$ such that

$$\text{rec}_{F_v, b_2}([\pi_2]) = [\underline{\rho}^2_{/\mathbb{C}}] = [{}^\sigma \underline{\rho}_{\ell/\mathbb{C}}^{2ss}] \text{ for every } \ell \neq p \text{ and embeddings } \sigma : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}.$$

- (2) *If X is a smooth complete intersection of dimension n in a projective space \mathbb{P}^r , defined over F , then for every $i \neq n$, there exists a representation $\underline{\rho}^i$ of ' W_v defined over \mathbb{Q} and a unique class $[\pi_i] \in \mathcal{A}_{b_i}(F_v)$ such that*

$$\text{rec}_{F_v, b_i}([\pi_i]) = [\underline{\rho}^i_{/\mathbb{C}}] = [{}^\sigma \underline{\rho}_{\ell/\mathbb{C}}^i] \text{ for every } \ell \neq p \text{ and embeddings } \sigma : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}.$$

When $i = n$, there exists a representation $\underline{\rho}^n$ of ' W_v defined over \mathbb{Q} and a unique class $[\pi_n] \in \mathcal{A}_{b_n}(F_v)$ such that

$$\text{rec}_{F_v, b_n}([\pi_n]) = [\underline{\rho}^n_{/\mathbb{C}}] = [{}^\sigma \underline{\rho}_{\ell/\mathbb{C}}^{nss}] \text{ for every } \ell \neq p \text{ and embeddings } \sigma : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}.$$

Proof. This follows from Thm. 1.3 and the local Langlands correspondence for F_v . \square

4. CONJUGACY CLASS OF FROBENIUS

The principal result of this section is Theorem 4.7, first we need some preliminaries.

4.1. The Algebraic Group $H^{\natural ad}$ and weakly neat elements. Let H be a reductive algebraic group over a field K of characteristic 0 and \bar{K} an algebraic closure of K . The adjoint action of H^{ad} on H gives a universal categorical quotient $(\text{Conj}(H), \text{Cl})$, where $\text{Conj}(H)$ is an algebraic variety over K and $\text{Cl} : H \rightarrow \text{Conj}(H)$ is the quotient map. The pair $(\text{Conj}(H), \text{Cl})$ is also the universal categorical quotient for the action of H on itself by conjugation.

The derived group $H_{\bar{K}}^{der}$ is the almost direct product of almost simple subgroups H_i , for $i \in I$ a finite indexing set. If $H_{\bar{K}}^{der}$ has any almost direct factor of type $\text{SO}(2k_i)_{\bar{K}}$, then there is an action which extends the above action. We briefly recall this. For details see [14] and [17]. Let $J \subset I$ such that for $i \in J$, $H_i \cong \text{SO}(2k_i)_{\bar{K}}$ with $k_i \geq 4$ and set $H_i^{\natural} := \text{O}(2k_i)_{\bar{K}}$. Define an algebraic group

$$H^{\natural ad} = \prod_{i \in J} H_i^{\natural ad} \times \prod_{i \notin J} H_i^{ad} \supset H_{\bar{K}}^{ad}.$$

There is a natural action of $H^{\natural ad}$ on $H_{\bar{K}}$, which extends the adjoint action on $H_{\bar{K}}^{der}$ and is trivial on the center of $H_{\bar{K}}$. Denote by $\text{Conj}'(H)_{\bar{K}}$ the categorical quotient of $H_{\bar{K}}$ under this action of $H^{\natural ad}$. Now, using the properties of categorical quotients, we can descend from $\text{Conj}'(H)_{\bar{K}}$ to an algebraic variety $\text{Conj}'(H)$ over K . We also get a quotient map (as above), which we again denote by $\text{Cl} : H \rightarrow \text{Conj}'(H)$. Now, let $\Omega \supseteq K$ be any algebraically closed field. From the functorial properties of the above action we deduce an *adjoint map*

$$\text{Ad} : H^{\natural ad}(\Omega) \rightarrow \text{Aut}(\mathfrak{h} \otimes \Omega),$$

where \mathfrak{h} denotes the Lie algebra of H .

Now we discuss the notion of *weakly neat* elements. Fix a faithful K -linear representation V of H . A semisimple element $g \in H(\Omega)$ is said to be *weakly neat* if 1 is the only root of unity among the quotients $\lambda\mu^{-1}$, with λ, μ being the eigenvalues of g .

For any positive integer n we denote by $\varphi_n : H \rightarrow H$ the n -th power map. Let Y/K denote any one of the algebraic varieties $\text{Conj}(H)$ or $\text{Conj}'(H)$. The map φ_n is equivariant for the action of conjugation. Since Y is a universal categorical quotient, this implies that φ_n induces a map $\bar{\varphi}_n : Y \rightarrow Y$ such that $\text{Cl} \circ \varphi_n = \bar{\varphi}_n \circ \text{Cl}$. The proof of the following lemma (see for example [14, 2.7]) is left to the reader.

Lemma 4.2. *Let $\Omega \supseteq K$ be an algebraically closed field. Let V be a finite dimensional faithful K -linear representation of H and $\alpha, \beta \in H(\Omega)$ be two weakly neat elements having same characteristic polynomial in the representation V . If $\bar{\varphi}_n(\text{Cl}(\alpha)) = \bar{\varphi}_n(\text{Cl}(\beta))$ for some positive integer n , then $\text{Cl}(\alpha) = \text{Cl}(\beta)$.*

4.3. Representations of Weil-Deligne groups with values in algebraic groups. We generalize some of the notions about linear representations of Weil-Deligne groups, described in §3.2.

Let H be an algebraic group over \mathbb{Q} and $\xi_\ell : \Gamma_v \rightarrow H(\mathbb{Q}_\ell)$ be an ℓ -adic representation. By Grothendieck's ℓ -adic monodromy theorem, there exists a nilpotent element $N'_\ell \in \text{Lie}(H/\mathbb{Q}_\ell)(-1)$ and an open subgroup $J \subseteq I_v$ such that for $\lambda \in J$, we have $\xi_\ell(\lambda) = \exp(N'_\ell t_\ell(\lambda))$. Now, by using the identification $\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(1)$, we interpret N'_ℓ as an element of $\text{Lie}(H/\mathbb{Q}_\ell)$ and we define as in §3.2,

$$(10) \quad \xi'_\ell : W_v \rightarrow H(\mathbb{Q}_\ell), \quad w \mapsto \xi_\ell(w) \exp(-N'_\ell t_\ell(\Phi_v^{-\alpha(w)} w)).$$

A triple $(H/\mathbb{Q}_\ell, \xi'_\ell, N'_\ell)$ as above is called a *representation of the Weil-Deligne group $'W_v$ of F_v with values in H/\mathbb{Q}_ℓ* . For a fixed ℓ , we say that $(H/\mathbb{Q}_\ell, \xi'_\ell, N'_\ell)$ is *rational over \mathbb{Q}* (as a representation with values in H), if for every algebraically closed field $\Omega \supset \mathbb{Q}_\ell$ and for every

$\sigma \in \text{Aut}_{\mathbb{Q}}(\Omega)$, there exists a $g \in H(\Omega)$ such that

$$(11) \quad \sigma \xi'_{\ell/\Omega} = g \cdot \xi'_{\ell/\Omega} \cdot g^{-1} \quad \text{and} \quad N'_{\ell} \otimes_{\sigma, \mathbb{Q}_{\ell}} 1 = \text{Ad}(g)(N'_{\ell} \otimes_{\mathbb{Q}_{\ell}} 1),$$

where $\xi'_{\ell/\Omega}$ denote the extension of scalars $\xi'_{\ell} \otimes \Omega : W_v \rightarrow H_{/\Omega}(\Omega)$ and $N'_{\ell} \otimes_{\mathbb{Q}_{\ell}} 1$ is the image of N'_{ℓ} in $(\text{Lie}(H) \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}_{\ell}} \Omega$.

If in addition to (11), we have for every pair (ℓ, ℓ') of prime numbers (different from p) and every algebraically closed field $\Omega \supset \mathbb{Q}_{\ell}, \mathbb{Q}_{\ell'}$, an element $g \in H(\Omega)$ such that

$$(12) \quad \xi'_{\ell/\Omega} = g \cdot \xi'_{\ell'/\Omega} \cdot g^{-1} \quad \text{and} \quad N'_{\ell} \otimes_{\mathbb{Q}_{\ell}} 1 = \text{Ad}(g)(N'_{\ell'} \otimes_{\mathbb{Q}_{\ell'}} 1),$$

then we say that the representations $(H_{/\mathbb{Q}_{\ell}}, \rho^i_{\ell}, N'_{\ell})$, form a *compatible system of representations of $'W_v$ (with values in H)*.

Note that, as the action of $H(\Omega)$ factors through $H(\Omega) \rightarrow H^{ad}(\Omega)$, $H(\Omega)$ -conjugacy may be replaced by $H^{ad}(\Omega)$ -conjugacy everywhere. Now, suppose that H is a reductive algebraic group, then as above we can define the algebraic group $H^{\natural ad}$. To say that the representation $(H_{/\mathbb{Q}_{\ell}}, \xi'_{\ell}, N'_{\ell})$ is *rational over \mathbb{Q} modulo the action of $H^{\natural ad}$* , means that in (11) we require that $g \in H^{\natural ad}(\Omega)$. In a similar manner we make sense of the phrase *compatible system of representations of $'W_v$ modulo the action of $H^{\natural ad}$*

4.4. Conjugacy class in Mumford-Tate group. Let G_{∞}^i denotes the Mumford-Tate group of the Hodge structure on the Betti cohomology group $V^i := H_B^i(\tau X(\mathbb{C}), \mathbb{Q})$. The comparison isomorphism $V^i \otimes \mathbb{Q}_{\ell} \cong V_{\ell}^i$ between Betti and ℓ -adic cohomology, allows us to identify $G_{\infty/\mathbb{Q}_{\ell}}^i$ to an algebraic subgroup of $\text{GL}(V_{\ell}^i)$. We denote by G_{ℓ}^i the global algebraic monodromy group, defined as the Zariski closure of $\rho_{\ell}^i(\Gamma_F)$ in $\text{GL}(V_{\ell}^i)$.

Proposition 4.5. *Let X be as in Thm. 1.2. Fix $i \in \mathbb{N}$ and assume that all absolute Hodge cycles in all tensor spaces formed by $V^i, V^{i \vee}, \mathbb{Q}(1)$ and $\mathbb{Q}(1)^{\vee}$, are defined over F . Then, G_{ℓ}^i is subgroup of $G_{\infty/\mathbb{Q}_{\ell}}^i$ for every ℓ .*

Proof. Let $X = \prod_r X_r$, where each X_r is one of the varieties of the the type mentioned in the statement of the proposition. As seen in the proof of Lemma 3.10, $h(X_{r, \bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$, for each r . As $h(X_{\bar{F}}) = \otimes_r h(X_{r, \bar{F}})$ and $\mathcal{M}_{AH}^{av}(\bar{F})$ is closed under tensor products, $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$.

Write $M := h^i(X)$; the algebraic groups $G_{M_{\bar{F}}}$ and G_M are stabilizers of all absolute Hodge classes (AHC) and AHC that are defined over F , respectively, in all tensor spaces formed by $V^i, V^{i \vee}, \mathbb{Q}(1)$ and $\mathbb{Q}(1)^{\vee}$ (see [8, I, Principle A]). So, it follows from hypothesis that $G_{M_{\bar{F}}} = G_M$.

Now the Mumford-Tate group G_{∞}^i of the Hodge structure V^i is the stabilizer of Hodge classes relative to the embedding τ , in all tensor spaces formed by $V^i, V^{i \vee}, \mathbb{Q}(1)$ and $\mathbb{Q}(1)^{\vee}$. As $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$, all Hodge classes relative to τ are AHC (see [8, II, 6.27]). This implies that $G_{\infty}^i = G_{M_{\bar{F}}} = G_M$.

Finally, as $\rho_{M, \ell} : \Gamma_F \rightarrow \text{GL}(H_{\ell}(M))$ factorizes through $G_M(\mathbb{Q}_{\ell})$ and $\rho_{M, \ell} = \rho_{\ell}^i$, we conclude that G_{ℓ}^i is a subgroup of $G_{\infty/\mathbb{Q}_{\ell}}^i$. □

Remark 4.6. (1) By [8, I, 2.9] there exists number fields verifying the hypothesis of Proposition 4.5.

(2) The Proposition above pertains to the *Mumford-Tate conjecture*, which claims that $G_{\infty/\mathbb{Q}_{\ell}}^i = G_{\ell}^{i \circ}$, for any smooth projective variety over F .

Now, let X be as in Prop. 4.5. Then, the triple $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{\ell}^{i'}, N'_{i,\ell})$ is a representation of the Weil-Deligne group $'W_v$ of F_v , with values in $G_{\infty/\mathbb{Q}_\ell}^i$.

For any finite extension F' of F we denote by $\Gamma_{F'}$ the absolute Galois group and by v' an extension of v to F' . Let $I_{v'}$, $W_{v'}$ and $'W_{v'}$ denote the inertia, Weil and Weil-Deligne group of $F'_{v'}$, respectively. We also have a group homomorphism $\alpha' : W_{v'} \rightarrow \mathbb{Z}$ as in §1. By $\Phi'_{v'}$ we mean an arithmetic Frobenius for this extension.

If $X_{F'}$ denotes base change of X to F' , then the ℓ -adic representation $\rho_{\ell}^i|_{\Gamma_{F'}}$ associated to $X_{F'}$ induces a representation $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{\ell}^{i'}|_{\Gamma_{F'}}, N'_{i,\ell})$ of the Weil-Deligne group $'W_{v'}$ of $F'_{v'}$, with values in $G_{\infty/\mathbb{Q}_\ell}^i$.

Theorem 4.7. *Let the notation and hypotheses be as in Proposition 4.5.*

- (i) *If $\rho_{\ell}^i|_{\Gamma_v}$ is potentially unramified representation for every $\ell \neq p$ and $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat for some prime number $\ell_0 \neq p$, then there exists a unique conjugacy class*

$$\text{Frob}_{v,i} \in \text{Conj}'(G_{\infty}^i)(\mathbb{Q})$$

such that $\text{Cl}(\rho_{\ell}^i(\Phi_v)) = \text{Frob}_{v,i}, \forall \ell \neq p$.

- (ii) *If $\rho_{\ell}^i|_{\Gamma_v}$ is a semi-stable representation for every $\ell \neq p$ and $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat for some prime number $\ell_0 \neq p$, then the representations $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{\ell}^{i'}, N'_{i,\ell})$ are rational over \mathbb{Q} and form a compatible system of representations of $'W_v$ modulo the action of $G_{\infty}^{i, \text{had}}$.*

Proof. (i) As before let us denote by M the motive $h^i(X)$. Let F' be a finite extension of F such that $M_{F'} := h^i(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{qv}(F'))$, as in Lemma 3.10. Let v' be an extension of the valuation v to F' and $\Phi_{v'}$ an arithmetic Frobenius for this extension. Now, we have seen in the proof of Prop. 4.5 that $G_{\infty}^i = G_{M_{\bar{F}}} = G_M$, in particular $G_{M_{\bar{F}}} = G_{M_{F'}}$. We also have $\text{Conj}'(G_{\infty}^i) = \text{Conj}'(G_{M_{F'}}) = \text{Conj}'(G_M)$ and $\rho_{M_{F'}/\ell} = \rho_{\ell}^i|_{\Gamma_{v'}}$. Now, using the fact that $\rho_{\ell}^i|_{\Gamma_v}$ is potentially unramified for all $\ell \neq p$, it can be checked (cf. the proof of [14, Thm. 1.1]) that there exists an element² $\text{Frob}_{v',i} \in \text{Conj}'(G_{\infty}^i)(\mathbb{Q})$ such that,

$$(13) \quad \text{Cl}(\rho_{\ell}^i(\Phi_{v'})) = \text{Frob}_{v',i}, \forall \ell \neq p.$$

Now, let the residual degree of the extension $F'_{v'}/F_v$ be n . We denote by $\varphi_n : G_{\infty}^i \rightarrow G_{\infty}^i$ the n -th power map and by $\bar{\varphi}_n : \text{Conj}'(G_{\infty}^i) \rightarrow \text{Conj}'(G_{\infty}^i)$ the induced map on $\text{Conj}'(G_{\infty}^i)$. Then it follows easily from (13) and the fact that $\bar{\varphi}_n$ is a finite morphism, that

$$\text{Cl}(\rho_{\ell}^i(\Phi_v)) \in \text{Conj}'(G_{\infty}^i)(\bar{\mathbb{Q}}), \forall \ell \neq p.$$

Now, by hypothesis there exists a ℓ_0 such that $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat. So, by Corollary 2.3, $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat for every $\ell \neq p$. Again, Corollary 2.3 and a repeated use of Lemma 4.2, shows that $\text{Cl}(\rho_{\ell}^i(\Phi_v))$ (as an element of $\text{Conj}'(G_{\infty}^i)(\bar{\mathbb{Q}})$) is independent of ℓ . We denote this element by $\text{Frob}_{v,i}$. As $\text{Frob}_{v,i} \in \text{Conj}'(G_{\infty}^i)(\mathbb{Q}_\ell)$ for every $\ell \neq p$, $\text{Frob}_{v,i}$ is a \mathbb{Q} -rational point of $\text{Conj}'(G_{\infty}^i)$.

- (ii) Now let the notation be as in part (i). By, applying [14, 3.9] to the motive $M_{F'}$, we see that for $\ell \neq p$ the representations $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{\ell}^{i'}|_{\Gamma_{F'}}, N'_{i,\ell})$ are rational over \mathbb{Q} and form a compatible system of representations of $'W_v$, modulo the action of $G_{\infty}^{i, \text{had}}$. In particular by setting $\Phi_{v'} := \Phi_v^n$, we get that for any fixed $\ell \neq p$, algebraically closed field $\Omega \supset \mathbb{Q}_\ell$ and $\sigma \in \text{Aut}_{\mathbb{Q}}(\Omega)$, there exists a $g_{\sigma} \in G_{\infty}^{i, \text{had}}(\Omega)$ such that

²In [14, eqn. 14] this was denoted by $\text{Cl}_M \text{Fr}_{v'}$

$$(14) \quad {}^\sigma \rho_{\ell/\Omega}^{i'}(\Phi_v^n) = g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_v^n)) \cdot g_\sigma^{-1} \text{ and } N'_{i,\ell} \otimes_{\sigma, \mathbb{Q}_\ell} 1 = \text{Ad}(g_\sigma)(N'_{i,\ell} \otimes_{\mathbb{Q}_\ell} 1).$$

Thus we have $({}^\sigma \rho_{\ell/\Omega}^{i'}(\Phi_v^n))^n = (g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_v^n)) \cdot g_\sigma^{-1})^n$. Now it follows from Corollary 2.3 that ${}^\sigma \rho_{\ell/\Omega}^{i'}(\Phi_v)$ and $g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_v)) \cdot g_\sigma^{-1}$ have the same characteristic polynomial. It now follows from a property of weakly neat elements (cf. [18, 7.4]), that ${}^\sigma \rho_{\ell/\Omega}^{i'}(\Phi_v) = g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_v)) \cdot g_\sigma^{-1}$.

As the monodromy operator $N'_{i,\ell}$ is unchanged by finite base extension, it now follows from (14) that $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{\ell/\Omega}^{i'}, N'_{i,\ell})$ is rational over \mathbb{Q} modulo the action of $G_\infty^{i, \text{bad}}$.

The compatibility condition is checked in a similar manner. \square

Remark 4.8. The main differences between Theorem 4.7 and [14, Theorem 1.1 & corollary 1.3], are the following :

- (a) in part (i) (resp. (ii)) of Theorem 4.7 we just need that $\rho_\ell^i|_{\Gamma_v}$ is potentially unramified (resp. semi-stable) representation for every $\ell \neq p$, while in [loc.cit.] a stronger condition was used namely X has good (resp. semi-stable) reduction at v ;
- (b) here we only know that there is a finite extension F' of F such that the motive $h^i(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$ (by Lemma 3.10), while in [loc.cit.] we assumed this property over the base field F ;

5. CONJUGACY CLASS OF CRYSTALLINE FROBENIUS

We follow the notation of §1 and we introduce some more notation. The ring of Witt vectors of k_v is denoted by $W(k_v)$ and $\sigma : W(k_v) \rightarrow W(k_v)$ is the morphism induced by the Frobenius automorphism of k_v . For any finite extension F' of F , we denote by v' an extension of v to F' ; $F'_{v'}$ is the completion of F' at v' ; $k_{v'}$ the residue field of $F'_{v'}$. The ring of Witt vectors of $k_{v'}$ is denoted by $W(k_{v'})$ and $\sigma' : W(k_{v'}) \rightarrow W(k_{v'})$ is the morphism induced by the Frobenius automorphism of $k_{v'}$. The fraction field of $W(k_{v'})$ is denoted by $F_{v'}^0$. Note that F_v^0 and $F_{v'}^0$ naturally identifies to the maximal unramified extension of \mathbb{Q}_p inside F_v and $F_{v'}$, respectively.

Consider a smooth projective algebraic variety Y over F with potential good reduction at v . Fix a $i \in \mathbb{N}$ and denote by N the motive $h^i(Y) \in \text{Ob}(\mathcal{M}_{AH}(F))$. Assume that the extension of scalars $N_{\bar{F}} \in \text{Ob}(\mathcal{M}_{AH}^{av}(F))$. Thus, by taking a large enough finite extension F' of F , we may assume that Y has good reduction over F' and $N_{F'} \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$. Let $Y_{v'}$ denote the reduction of $Y_{F'}$ at v' and $\langle N_{F'}, \mathbb{Q}(1) \rangle^\otimes$ denote the Tannakian subcategory of $\mathcal{M}_{AH}(F')$, \otimes -generated by $N_{F'}$ and the Tate motive. It is well known from the C_{Cris} -conjecture (cf.[11, Th.3.2.3]) that $H_p(N_{F'}) = H_{\text{et}}^i(Y_{\bar{F}}, \mathbb{Q}_p)$ is a crystalline representation of $\Gamma_{v'}$ and its associated filtered Dieudonné module is isomorphic to $H_{\text{Cris}}^i(Y_{v'}) \otimes F_{v'}^0$. This allows us to define the crystalline realization (cf.[27, §4.1]), i.e., a fiber functor

$$H_{N_{F'}, \text{Cris}} : \langle N_{F'}, \mathbb{Q}(1) \rangle^\otimes \rightarrow \text{Vect}_{F_{v'}^0},$$

which associates to any motive $L \in \langle N_{F'}, \mathbb{Q}(1) \rangle^\otimes$ its associated filtered Dieudonné $F_{v'}$ -module $(H_p(L) \otimes_{\mathbb{Q}_p} B_{\text{Cris}})^{\Gamma_{v'}}$, where B_{Cris} is the $F_{v'}^0[\Gamma_{v'}]$ -algebra introduced by Fontaine (cf. [9]). We call this functor the *crystalline realization* for the subcategory of motives $\langle N_{F'}, \mathbb{Q}(1) \rangle^\otimes$. We write $G_{N_{F'}, \text{Cris}} := \underline{\text{Aut}}^\otimes(H_{N_{F'}, \text{Cris}})$ for the automorphism group of the fiber functor $H_{N_{F'}, \text{Cris}}$.

The absolute Frobenius endomorphism Fr_{abs} of $Y_{v'}$ induces a σ' -linear endomorphism $Fr_{\text{Cris}} : H_{\text{Cris}}^i(Y_{v'}/W(k_{v'})) \rightarrow H_{\text{Cris}}^i(Y_{v'}/W(k_{v'}))$, on crystalline cohomology groups of $Y_{v'}$.

Now, let $|k_{v'}| = p^{s'}$, then $Fr_{\text{Cris}}^{s'}$ is a $W(k_{v'})$ -linear endomorphism of $H_{\text{Cris}}^i(Y_{v'}/W(k_{v'}))$. Let us denote by

$$\Phi_{\text{Cris}} : H_{\text{Cris}}^i(Y_{v'}/W(k_{v'})) \otimes F_{v'}^0 \rightarrow H_{\text{Cris}}^i(Y_{v'}) \otimes F_{v'}^0,$$

the *degree i crystalline Frobenius* of $Y_{F'}$ at v' , i.e., the $F_{v'}^0$ -linear endomorphism $Fr_{\text{Cris}}^{s'} \otimes 1$. Now, Φ_{Cris} defines an element $\Phi'_{N, \text{Cris}} \in G_{N_{F'}, \text{Cris}}(F_{v'}^0)$.

As before the conjugation action of $G_{N_{F'}, \text{Cris}}$ gives us a universal categorical quotient $(\text{Conj}'(G_{N_{F'}, \text{Cris}}), \text{Cl}_{\text{Cris}})$. We also have the universal categorical quotient $(\text{Conj}'(G_{N_{F'}}), \text{Cl})$ for the conjugation action of $G_{N_{F'}} := \underline{\text{Aut}}^{\otimes}(H_{\tau}|_{\langle N_{F'}, \mathbb{Q}(1) \rangle^{\otimes}})$. For every $\ell \neq p$ the ℓ -adic representation $\rho_{N_{F'}, \ell} : \Gamma_{F'} \rightarrow G_{N_{F'}}(\mathbb{Q}_{\ell})$ defines an element $\text{Cl}(\rho_{N_{F'}, \ell}(\Phi_{v'})) \in \text{Conj}'(G_{N_{F'}})(\mathbb{Q}_{\ell})$.

Now, it is well known (cf. [8, II, Th.3.2]) from the general theory of Tannakian categories that $G_{N_{F'}, \text{Cris}}$ is an inner form of $G_{N_{F'}} \otimes_{\mathbb{Q}} F_{v'}^0$. It follows (see for example [26]) that there is an isomorphism $\text{Conj}'(G_{N_{F'}, \text{Cris}}) \cong (\text{Conj}'(G_{N_{F'}})) \otimes_{\mathbb{Q}} F_{v'}^0$, which allows us to make sense of the following

Theorem 5.1. *Let Y be any smooth projective algebraic variety over F with potential good reduction at a non-archimedean valuation v of F . For any fixed $i \in \mathbb{N}$ write $N := h^i(Y)$ and assume that $N_{\bar{F}} \in \text{Ob}(\mathcal{M}_{\text{AH}}^{av}(\bar{F}))$. Then, there exists a finite extension F' of F and a unique conjugacy class*

$$\text{Frob}'_N \in \text{Conj}'(G_{N_{F'}})(F_{v'}^0),$$

such that $\text{Cl}_{\text{Cris}}(\Phi'_{N, \text{Cris}}) = \text{Frob}'_N$ and $\text{Cl}(\rho_{N_{F'}, \ell}(\Phi_{v'})) = \text{Frob}'_N, \forall \ell \neq p$.

Proof of Theorem 5.1. We shall first find a finite extension of F and then show that it is the right one. To avoid confusion, we shall denote all successive extensions of F by F' .

Recall, as in the proof of Propn. 3.11, there is a finite extension F' of F , an abelian variety A over F' such that $N_{F'} \in \langle h^1(A), \mathbb{Q}(1) \rangle^{\otimes}$. Further, this induces a unique homomorphism of algebraic groups $\theta : G_A \rightarrow G_{N_{F'}}$ satisfying $\theta(\mathbb{Q}_{\ell}) \circ \rho_{A, \ell} = \rho_{N_{F'}, \ell}$ for every prime number ℓ (where $\theta(\mathbb{Q}_{\ell})$ is the induced map on \mathbb{Q}_{ℓ} -valued points).

Lemma 5.2. *There exists an abelian variety A over F' as above with potential good reduction at v' .*

Proof. First we note that for any abelian variety A , by Grothendieck's ℓ -adic monodromy theorem, $I_{v'}$ (possibly after passing over to another finite extension of F) acts unipotently on $H_{\text{et}}^1(A/\bar{F}, \mathbb{Q}_{\ell}) \cong W \otimes \mathbb{Q}_{\ell}$, where $W := H_B^1(\tau A, \mathbb{Q})$. As Y has potential good reduction at v , we may assume that $I_{v'}$ acts trivially on $H_{\ell}(N)$. Now, by [14, Lemma 3.10] we may choose A such that $G_A^{\text{ad}} \cong G_{N_{F'}}^{\text{ad}}$. This combined with the fact that $\theta(\mathbb{Q}_{\ell}) \circ \rho_{A, \ell} = \rho_{N_{F'}, \ell}$, implies that $I := \rho_{A, \ell}(I_{v'}) \subseteq \mathcal{Z}_A(\mathbb{Q}_{\ell})$, where \mathcal{Z}_A denotes the center of G_A . Fix an algebraic closure Ω of \mathbb{Q}_{ℓ} . For any $\sigma \in I \subseteq \mathcal{Z}_A(\mathbb{Q}_{\ell})$, denote by $\bar{\sigma}$ the image of σ in $\mathcal{Z}_{A/\Omega}(\Omega)$. As G_A is reductive, $\mathcal{Z}_{A/\Omega}$ is a diagonalizable algebraic group. Since G_A is a subgroup of $\text{GL}(W)$, there exists a matrix $P \in \text{GL}_m(\Omega)$ such that $P \cdot \bar{\sigma} \cdot P^{-1}$ is of the form $\text{diag}(\lambda_1, \dots, \lambda_m)$, where $m := \dim W$. As $\bar{\sigma}$ is unipotent, $\lambda_1 = \dots = \lambda_m = 1$, i.e. $\bar{\sigma}$ is the identity matrix. Since $\sigma \in I$ was an arbitrary element, I is trivial. Now, using the Néron-Ogg-Shafarevich condition [25, Thm. 1], we conclude that A has potential good reduction at v' . \square

End of the proof of Theorem 5.1: By passing over to another finite extension we may suppose that A has good reduction at v' . Now, we have the crystalline realization

$$H_{A, \text{Cris}} : \langle h^1(A), \mathbb{Q}(1) \rangle^{\otimes} \rightarrow \text{Vect}_{F_{v'}^0}$$

and we write $G_{A, \text{Cris}} := \underline{\text{Aut}}^{\otimes}(H_{A, \text{Cris}})$. As before, the crystalline Frobenius element for A at v' defines an element $\Phi_{A, \text{Cris}} \in G_{A, \text{Cris}}(F_{v'}^0)$. By passing over to another finite extension we may suppose that $\Phi_{A, \text{Cris}}$ is weakly neat. Again, as $G_{A, \text{Cris}}$ is an inner form of $G_A \otimes_{\mathbb{Q}} F_{v'}^0$, there

is an isomorphism $\text{Conj}'(G_{A,\text{Cris}}) \cong (\text{Conj}'(G_A)) \otimes_{\mathbb{Q}} F_{v'}^0$. For our purpose we identify these two varieties. Now, by [17, 4.2] we know that there is an element $\text{Cl}_A \text{Fr}_v \in \text{Conj}'(G_A)(\mathbb{Q})$ such that $\text{Cl}(\rho_{A,\ell}(\Phi_{v'})) = \text{Cl}_A \text{Fr}_v \forall \ell \neq p$ and its image in $\text{Conj}'(G_{A,\text{Cris}})(F_v^0)$ coincides with $\text{Cl}_{\text{Cris}}(\Phi_{A,\text{Cris}})$. Let us write $\text{Frob}_A \in \text{Conj}'(G_A)(F_{v'}^0) = \text{Conj}'(G_{A,\text{Cris}})(F_{v'}^0)$, for this image.

By universal property of (universal) categorical quotients we have the following commutative diagram:

$$(15) \quad \begin{array}{ccc} G_A & \xrightarrow{\text{Cl}} & \text{Conj}' G_A \\ \theta \downarrow & & \downarrow \bar{\theta} \\ G_{N_{F'}} & \xrightarrow{\text{Cl}} & \text{Conj}' G_{N_{F'}} \end{array}$$

Again, the inclusion functor $\langle N_{F'}, \mathbb{Q}(1) \rangle^{\otimes} \rightarrow \langle h^1(A), \mathbb{Q}(1) \rangle^{\otimes}$, induces a unique homomorphism

$$\theta_{\text{cris}} : G_{A,\text{Cris}} \rightarrow G_{N_{F'},\text{Cris}}.$$

As before, this gives us the following commutative diagram :

$$(16) \quad \begin{array}{ccc} G_{A,\text{Cris}} & \xrightarrow{\text{Cl}_{\text{Cris}}} & \text{Conj}' G_{A,\text{Cris}} \\ \theta_{\text{cris}} \downarrow & & \downarrow \bar{\theta}_{\text{Cris}} \\ G_{N_{F'},\text{Cris}} & \xrightarrow{\text{Cl}_{\text{Cris}}} & \text{Conj}' G_{N_{F'},\text{Cris}} \end{array}$$

We set $\text{Frob}'_N := \text{Cl}_{\text{Cris}}(\Phi'_{N,\text{Cris}}) \in \text{Conj}'(G_{N_{F'},\text{Cris}})(F_{v'}^0)$. We shall show that

$$\text{Cl}(\rho_{N_{F'},\ell}(\Phi_{v'})) = \text{Frob}'_N \text{ for every } \ell \neq p.$$

By functoriality, $\theta_{\text{cris}}(F_{v'}^0)(\Phi_{A,\text{Cris}}) = \Phi'_{N,\text{Cris}}$, where $\theta_{\text{cris}}(F_{v'}^0)$ is the induced map on $F_{v'}^0$ -valued points. Thus, by commutativity of (16) we obtain $\text{Frob}'_N = \bar{\theta}_{\text{Cris}}(F_{v'}^0)(\text{Frob}_A)$. Now, as $\theta(\mathbb{Q}_{\ell}) \circ \rho_{A,\ell} = \rho_{N_{F'},\ell}$ and $\text{Cl}(\rho_{A,\ell}(\Phi_{v'})) = \text{Frob}_A$ for all $\ell \neq p$, it follows that $\text{Cl}(\rho_{N_{F'},\ell}(\Phi_{v'})) = \bar{\theta}(F_{v'}^0)(\text{Frob}_A)$ as elements of $\text{Conj}'(G_{N_{F'}})(F_{v'}^0) = \text{Conj}'(G_{N_{F'},\text{Cris}})(F_{v'}^0)$. Thus the proposition will be proved if we can show that $\text{Frob}'_N = \bar{\theta}(F_{v'}^0)(\text{Frob}_A)$. In other words we need to show that $\bar{\theta}_{\text{Cris}}(F_{v'}^0)(\text{Frob}_A) = \bar{\theta}(F_{v'}^0)(\text{Frob}_A)$. This follows from the next lemma. □

Lemma 5.3. *The morphism $\bar{\theta}_{/F_{v'}^0} := \bar{\theta} \otimes_{\mathbb{Q}} \text{id} : \text{Conj}'(G_A) \otimes_{\mathbb{Q}} F_{v'}^0 \rightarrow \text{Conj}'(G_{N_{F'}}) \otimes_{\mathbb{Q}} F_{v'}^0$ coincides with $\bar{\theta}_{\text{Cris}}$.*

Proof. As $G_{*,\text{Cris}}$ is an inner form of $G_* \otimes_{\mathbb{Q}} F_{v'}^0$, where $*$ = $N_{F'}$ or A we can identify $G_{*,\text{Cris}} \otimes_{F_{v'}^0} \bar{\mathbb{Q}}_p$ with $G_* \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p$. Then, under this identification and by definition of θ and θ_{Cris} , it follows that the homomorphisms of algebraic groups

$$(17) \quad \begin{aligned} \theta_{/\bar{\mathbb{Q}}_p} &:= \theta \otimes_{\mathbb{Q}} \text{id} : G_A \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \rightarrow G_{N_{F'}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \text{ and} \\ \theta_{\text{Cris}/\bar{\mathbb{Q}}_p} &:= \theta_{\text{Cris}} \otimes_{F_{v'}^0} \text{id} : G_{A,\text{Cris}} \otimes_{F_{v'}^0} \bar{\mathbb{Q}}_p \rightarrow G_{N_{F'},\text{Cris}} \otimes_{F_{v'}^0} \bar{\mathbb{Q}}_p, \end{aligned} \text{ are identical.}$$

It follows from (17), that $\bar{\theta}_{/\bar{\mathbb{Q}}_p} = \bar{\theta}_{\text{Cris}/\bar{\mathbb{Q}}_p}$. As $G_{A,\text{Cris}}$ is geometrically reducible (as algebraic variety over $F_{v'}^0$), so is $\text{Conj}'(G_{A,\text{Cris}})$. Now, it is a well known fact that if two morphisms $f, g : Y \rightarrow Z$ of algebraic varieties Y and Z are identical over the algebraic closure of the base field and Y is geometrically reducible, then f and g are identical. By using this fact we conclude that $\bar{\theta}_{/F_{v'}^0} = \bar{\theta}_{\text{Cris}}$. □

Proof of Theorem 1.5. By applying Theorem 5.1 to X we obtain an extension F' of F and an element $\text{Frob}'_M \in \text{Conj}'(G_{M_{F'}})(F_v^0)$. Let n be the residual degree of the $F'_{v'}/F_v$. By hypothesis G_M is connected, which implies that it is geometrically connected because G_M is an algebraic group. In particular $G_M \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \cong G_{M, \text{Cris}} \otimes_{F_v^0} \bar{\mathbb{Q}}_p$ is connected. This implies that $G_{M, \text{Cris}}$ is connected. Now, all Hodge cycles relative to the embedding $\tau : F \hookrightarrow \mathbb{C}$ on M are absolute Hodge cycles, because $M \in \text{Ob}(\mathcal{M}_{AH}^{av}(F))$. This implies that $G_M^0 = G_{M_{\bar{F}}}$ (see [8, II, 6.23]). Thus, in our situation $G_M = G_{M_{\bar{F}}} = G_{M_{F'}}$ and similarly $G_{M_{F'}, \text{Cris}} = G_{M, \text{Cris}}$. We also get $\text{Conj}'(G_{M_{F'}}) = \text{Conj}'(G_M)$ and $\text{Conj}'(G_{M_{F'}, \text{Cris}}) = \text{Conj}'(G_{M, \text{Cris}})$.

As the crystalline Frobenius element for $X_{F'}$ at v' is the n -th power of the crystalline Frobenius element for X at v , so $(\Phi_{M, \text{Cris}})^n = \Phi'_{M, \text{Cris}}$ (as elements of $G_{M, \text{Cris}}(F_v^0)$).

Now, let $\varphi_n : G_{M, \text{Cris}} \rightarrow G_{M, \text{Cris}}$ be the n -th power map. The map φ_n is equivariant for the action of conjugation. By the universal property of universal categorical quotient, φ_n induces a map $\bar{\varphi}_n : \text{Conj}'(G_{M, \text{Cris}}) \rightarrow \text{Conj}'(G_{M, \text{Cris}})$ such that $\text{Cl}_{\text{Cris}} \circ \varphi_n = \bar{\varphi}_n \circ \text{Cl}_{\text{Cris}}$. Write $\text{Frob}_M := \text{Cl}_{\text{Cris}}(\Phi_{M, \text{Cris}}) \in \text{Conj}'(G_{M, \text{Cris}})(F_v^0) \subset \text{Conj}'(G_{M, \text{Cris}})(F_{v'}^0)$, then it follows from the above discussion that

$$(18) \quad \bar{\varphi}_n(F_{v'}^0)(\text{Frob}_M) = \text{Frob}'_M.$$

By [14, 1.1] there is an element

$$\text{Cl}_M \text{Fr}_v \in \text{Conj}'(G_M)(\mathbb{Q}) \text{ such that } \text{Cl}(\rho_{M, \ell}(\Phi_v)) = \text{Cl}_M \text{Fr}_v \ \forall \ell \neq p.$$

Using the inclusions

$$\text{Conj}'(G_M)(\mathbb{Q}) \subset \text{Conj}'(G_M)(F_v^0) \cong \text{Conj}'(G_{M, \text{Cris}})(F_v^0) \subset \text{Conj}'(G_{M, \text{Cris}})(F_{v'}^0),$$

we regard $\text{Cl}_M \text{Fr}_v$ as an element of $\text{Conj}'(G_{M, \text{Cris}})(F_{v'}^0)$. By taking $\Phi_{v'} := \Phi_v^n$ for the arithmetic Frobenius for the extension of F' , we have $\text{Cl}((\rho_{M, \ell}(\Phi_v))^n) = \text{Cl}(\rho_{M_{F'}, \ell}(\Phi_{v'}))$. It follows from this and (18), that $\bar{\varphi}_n(F_{v'}^0)(\text{Cl}_M \text{Fr}_v) = \bar{\varphi}_n(F_{v'}^0)(\text{Frob}_M)$ as elements of $\text{Conj}'(G_{M, \text{Cris}})(F_{v'}^0)$. Since $\text{Cl}_M \text{Fr}_v, \text{Frob}_M \in \text{Conj}'(G_{M, \text{Cris}})(F_v^0)$, we get

$$(19) \quad \bar{\varphi}_n(F_v^0)(\text{Cl}_M \text{Fr}_v) = \bar{\varphi}_n(F_v^0)(\text{Frob}_M) \in \text{Conj}'(G_{M, \text{Cris}})(F_v^0).$$

Now, fix a prime number ℓ and let $\Omega \supset \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell$ be an algebraically closed field. As $M = h^i(X)$, we have $H_\ell(M) = H_{\text{et}}^i(X_{\bar{F}}, \mathbb{Q}_\ell)$. Again by the C_{Cris} conjecture (cf. [11, Th.3.2.3]), we have $H_{M/\text{Cris}}(M) = H_{\text{Cris}}^i(X_v/W(k_v)) \otimes F_v^0$. As X has good reduction at v , we can use Deligne's result on Weil conjectures [6, I.6] and smooth specialization (proper-smooth base change) to conclude that for every $\ell \neq p$, the characteristic polynomial of $\rho_{M, \ell}(\Phi_v)$ (seen as an element of $\text{GL}(H_\ell(M))$) has coefficients in \mathbb{Q} and is independent of ℓ . Next by a result of Katz and Messing [12, Theorem 1], we conclude that $\Phi_{M, \text{Cris}}$ (seen as an element of $\text{GL}(H_{\text{Cris}}(M))$) has the same characteristic polynomial as $\rho_{M, \ell}(\Phi_v)$. This implies in particular that if $\rho_{M, \ell}(\Phi_v)$ is weakly neat then $\Phi_{M, \text{Cris}}$ is also weakly neat. Now we have two weakly neat elements $\rho_{M, \ell}(\Phi_v)$ and $\Phi_{M, \text{Cris}}$ in $G_M(\Omega) \cong G_{M, \text{Cris}}(\Omega)$ which have same characteristic polynomial. Therefore, by Lemma 4.2 and (19) we conclude that $\text{Frob}_M = \text{Cl}_M \text{Fr}_v$. In other words, $\text{Cl}(\rho_{M, \ell}(\Phi_v)) = \text{Frob}_M$. □

By using the previous theorem (and its proof) combined with Lemma 3.10, we easily relate the element $\text{Frob}_{v, i} \in \text{Conj}'(G_\infty^i)(\mathbb{Q})$ of Theorem 4.7 to the conjugacy class of the crystalline Frobenius $\Phi_{\text{Cris}, i}$ associated to the motive $h^i(X)$.

Corollary 5.4. *Let the notation and hypothesis be as in Proposition 4.5. Let v be a non-archimedean valuation and $i \in \mathbb{N}$. Assume that X has good reduction at v and there is a prime number ℓ_0 such that $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat. Then, the image of the element $\text{Frob}_{v, i}$ in $\text{Conj}'(G_\infty^i)(F_v^0)$ coincides with $\text{Cl}_{\text{Cris}}(\Phi_{\text{Cris}, i})$.*

Remark 5.5. It is natural to wonder about what one can say concerning Serre’s conjecture 1.4, for motives which do not belong to $\mathcal{M}_{AH}^{qv}(F)$ and whether there also exists an analogue of Thm. 1.5 for them. We discuss a potential candidate. Let f be a normalized newform of weight $w > 2$, level N and character χ (for ease of exposition, we shall assume that the field generated by the Fourier coefficients of f is \mathbb{Q}). Let $M(f)$ be the Grothendieck motive associated to f , as constructed by Scholl [22]. It is well-known that if f is not of CM type, then $M(f)$ is not an object of $\mathcal{M}_{AH}^{qv}(\mathbb{Q})$. Moreover, by considering $M(f)$ as an element of $\mathcal{M}_{AH}(\mathbb{Q})$, we can construct as before, the motivic Galois group, corresponding to the Betti-realization of $M(f)$. Now let $p \nmid N$ be a prime number, then by [22], we know that $M(f)$ has good reduction at p . This allows us to frame Conjecture 1.4 for $M(f)$ at such a prime p . Finally, by [ibid.], one can also define the crystalline realization of $M(f)$ at p and hence the motivic (Tannakian) group associated to this realization. Then we may ask whether there exists an analogue of Thm. 1.5 for $M(f)$.

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