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ON FROBENIUS CONJUGACY CLASS OF SOME ALGEBRAIC VARIETIES OVER NUMBER FIELDS

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ABSTRACT. Let X be a smooth projective algebraic variety over a number field F . The action of $\text{Gal}(\bar{F}/F)$ on ℓ -adic cohomology groups $H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell)$, induces Galois representations $\rho_\ell^i : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}(H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell))$. Fix a non-archimedean valuation v on F . Let F_v be the completion of F at v , Φ_v be any arithmetic Frobenius element at v and W_v be the Weil group of F_v . First, we exhibit new cases of X for which the characteristic polynomial of $\rho_\ell^i(w)$ has coefficients in \mathbb{Q} and independent of ℓ , for any $w \in W_v$. Then we establish new cases, for which the representations $(\rho_\ell^{i'}, N_{i,\ell}')$ of the Weil-Deligne group $'W_v$ of F_v , are Frobenius semisimple and form a compatible system of representations of $'W_v$, thus giving partial answer to a conjecture of Deligne, Tate, et al. Finally, we treat the question of ℓ -independence of the conjugacy classes of $\rho_\ell^i(\Phi_v)$ in certain natural subgroups of $\text{GL}(H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell))$, both when ℓ is different or equal, to the residual characteristic at v .

1. INTRODUCTION

This paper is concerned with certain (related) questions, about the action of Frobenius elements on the etale cohomology groups of algebraic varieties, defined over number fields. We've tried to answer these questions in a significant number of new cases and in the process we gained new insight into these questions.

1.1. Background and notation. Throughout, F is a number field, with an embedding $\tau : F \hookrightarrow \mathbb{C}$, v is a non-archimedean valuation on F and F_v is the completion. By \bar{F} we denote a fixed separable algebraic closure of F , $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ is an extension of τ , \bar{v} is an extension of v to \bar{F} and \bar{F}_v is the localization of \bar{F} at \bar{v} . The residue fields of F_v and \bar{F}_v are denoted by k_v and \bar{k}_v , respectively, and the characteristic of k_v is $p > 0$. We write $\Gamma_v := \text{Gal}(\bar{F}_v/F_v) \subset \Gamma_F := \text{Gal}(\bar{F}/F)$ and $I_v \subset \Gamma_v$ is the inertia group. An arithmetic Frobenius $\Phi_v \in \Gamma_v$ is an element which induces the Frobenius automorphism ϕ_v of \bar{k}_v . We denote by W_v the *Weil group* of F_v i.e. the dense subgroup formed by elements $w \in \Gamma_v$ which induce on \bar{k}_v an integral power $\phi_v^{\alpha(w)}$. The map $\alpha : W_v \rightarrow \mathbb{Z}$ thus defined is a group homomorphism and $\ker(\alpha) = I_v$. The *Weil-Deligne group* $'W_v$ of F_v is the group scheme over \mathbb{Q} defined as the semi-direct product of W_v with the additive group \mathbb{G}_a over \mathbb{Q} , on which W_v acts as $w \cdot x \cdot w^{-1} = |k_v|^{\alpha(w)} \cdot x$

For simplicity of the proofs, we'll assume all our varieties to be geometrically irreducible; this condition can be loosened. Consider a smooth projective algebraic variety X over F . The action of Γ_F on the geometric ℓ -adic cohomology groups $V_\ell^i := H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell)$, induces Galois representations $\rho_\ell^i : \Gamma_F \rightarrow \text{GL}(V_\ell^i)$. A fundamental question in arithmetic geometry is the following,

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Problem 1.2 (Serre-Tate, [30]). *Is it true that for every $w \in W_v$, the characteristic polynomial $P_{\ell,v}^i(w, T) := \det(1 - \rho_\ell^i(w)T; V_\ell^i)$, of the \mathbb{Q}_ℓ -linear map $\rho_\ell^i(w)$ has coefficients in \mathbb{Q} and is independent of ℓ ?*

By Deligne’s result on the Weil conjectures [9], we know that the answer to Prob.1.2 is affirmative, if we assume that variety X has good reduction at v . But the case of bad-reduction is wide open. We show in Thm. 1.6, that the answer to 1.2 is always true, if we assume that the complex algebraic variety $\tau X := X \times_{F,\tau} \mathbb{C}$ verifies the degree i Künneth standard conjecture for Betti cohomology theory. This criterion is then used to deduce many new cases, where 1.2 holds true, such as hyperkähler varieties of $K3^{[n]}$ type, moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves and unirational varieties of dimension ≤ 4 ; see Cor.1.7 for the full list of new cases.

Another fundamental question, asked by Serre [29, 12.4], is whether the operators $\rho_\ell^i(\Phi_v)$ are semisimple. This is relevant, for instance, to the local Langlands correspondence for general linear groups, which state that *there are unique bijections between the equivalence classes of Frobenius semisimple n -dimensional linear representations of $'W_v$ over \mathbb{Q}_ℓ to the equivalence classes of irreducible smooth representations of $\mathrm{GL}_n(F_v)$ over \mathbb{Q}_ℓ* . Deligne has shown how to attach to ρ_ℓ^i , a linear representation $\underline{\rho}_\ell^i := (V_\ell^i, \rho_{i,\ell}^i, N_{i,\ell}^i)$ of $'W_v$ over \mathbb{Q}_ℓ , where $\rho_{i,\ell}^i$ is a continuous representation of W_v and $N_{i,\ell}^i$ is the associated monodromy operator. These $\underline{\rho}_\ell^i$ are the basic source of linear representations of $'W_v$. But, whether they are Frobenius semisimple is unknown in general. Thm. 1.8 establishes the semi-simplicity question for a class of algebraic varieties, which includes for example, hyperkähler varieties of $K3^{[n]}$ type, moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves and unirational varieties of dimension ≤ 3 .

The representations $\underline{\rho}_\ell^i$ are also subject to the following conjecture of Deligne, Tate, et. al , cf.[13, 2.4.3]

Conjecture 1.3 ($C_{WD}(X_{F_v}, i)$). *The system $\{\underline{\rho}_\ell^i\}_{\ell \neq p}$ forms a compatible system (in the sense of [8, 8.8]) of linear representations of $'W_v$.*

The notion of compatible system considered here is a strong one (see §3.2). Theorem 1.8 shows that Conjecture 1.3 holds for the above mentioned class of algebraic varieties which satisfies Frobenius semi-simplicity. Theorem 1.9 establishes a weaker version of Conj. 1.3 for surfaces and complete intersections.

Now, let G_∞^i denotes the Mumford-Tate group of the Hodge structure on the Betti cohomology group $V^i := H_B^i(\tau X(\mathbb{C}), \mathbb{Q})$. Conjecture 1.3 has a sharper reformulation, if one assumes the Hodge conjecture. In that case, $\underline{\rho}_\ell^i$ factors through $G_\infty^i(\mathbb{Q}_\ell)$ and it is expected that *the $\{\underline{\rho}_\ell^i\}_{\ell \neq p}$ forms a compatible system (in the sense of [8, 8.11]) of representations of $'W_v$ with values in the algebraic group G_∞^i* . In its full generality, this is unknown even for abelian varieties. Theorem 1.11 shows that, after a finite base extension, this conjecture holds in certain special cases, without assuming the Hodge conjecture. A conjecture, similar to the one above, can be also be formulated in terms of *motivic Galois groups*. First, recall that Grothendieck’s standard conjectures on algebraic cycles predict that the category of motives for homological equivalence of algebraic cycles coincides with the category of numerical(Grothendieck) motives and is Tannakian [15, Cor.2]. This would imply in particular that the action of the Galois group Γ_F on the ℓ -adic realization $H_\ell(M)$ of any motive M over F , factors as

$$\rho_{M,\ell} : \Gamma_F \rightarrow G_M(\mathbb{Q}_\ell),$$

where G_M is the *motivic Galois group* of M . This is the group associated, via the Betti realization functor H_τ , to the Tannakian subcategory generated by M and the Tate motive.

Now, let $(\text{Conj}(G_M), \text{Cl})$ be the universal categorical quotient of G_M for its action on itself by conjugation. For every $\ell \neq p$, $\rho_{M,\ell}(\Phi_v)$ defines an element $\text{Cl}(\rho_{M,\ell}(\Phi_v)) \in \text{Conj}(G_M)(\mathbb{Q}_\ell)$.

Conjecture 1.4 (Serre[29, 12.6]). *If M is a motive with good reduction at v , then $\text{Cl}(\rho_{M,\ell}(\Phi_v)) \in \text{Conj}(G_M)(\mathbb{Q})$, $\forall \ell \neq p$ and is independent of ℓ .*

Special cases of this conjecture were studied in [18] and [21]. The case when $\ell = p$, is more mysterious. One of our main result, Thm.1.12, sheds some light on this mystery, in certain special cases which includes motives of hyperkähler variety of $K3^{[n]}$ type, moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves and unirational varieties of dimension ≤ 3 .

1.5. Main results. In what follows, all algebraic cycles and Chow groups are with rational coefficients. Let Y be any complex algebraic variety Y of dimension d .

Let $\gamma_B^d : \text{CH}^d(Y \times Y) \rightarrow \text{H}_B^{2d}(Y(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q})(d)$, be the cycle class map from codimension d algebraic cycles on $Y \times Y$ to the Tate twisted degree $2d$ Betti cohomology of $Y \times Y$. The Künneth isomorphism $\text{H}_B^{2d}(Y(\mathbb{C}) \times Y(\mathbb{C}), \mathbb{Q})(d) \cong \bigoplus_{i=0}^{2d} \text{H}_B^{2d-i}(Y(\mathbb{C}), \mathbb{Q}) \otimes \text{H}_B^i(Y(\mathbb{C}), \mathbb{Q})(d)$, gives us a decomposition $\gamma_B^d([\Delta]) = \sum_{i=0}^{2d} \pi^i$, of the class of the diagonal sub-variety Δ of $Y \times Y$. To say that the *degree i Künneth standard conjecture* (w.r.t Betti Cohomology) is true for Y , means that there is a correspondence $z^i \in \text{CH}^d(Y \times Y)$ such that $\gamma_B^d(z^i) = \pi^i$.

Theorem 1.6. *Let X be any smooth projective variety over F . Fix a $i \in \mathbb{N}$ and assume that the the degree i Künneth standard conjecture (w.r.t Betti Cohomology) holds for τX . Then, for every $\ell \neq p$ and $w \in W_v$ with $\alpha(w) \leq 0$, the polynomial $P_{\ell,v}^i(w, T) \in \mathbb{Z}[T]$ and is independent of ℓ .*

The above result is surprising, since a priori, the automorphisms $\rho_\ell^i(w)$ are not induced by algebraic cycles on $X \times X$. As a consequence of the theorem, we find positive response to Prob.1.2 in the following cases :

Corollary 1.7. *Let X be a smooth projective variety over F which is a finite product of hyperkähler varieties of $K3^{[n]}$ type; moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves; unirational varieties of dimension ≤ 4 ; uniruled varieties of dimension ≤ 3 ; curves, surfaces, abelian varieties and smooth complete intersections in projective spaces. Then, for every $i \in \mathbb{N}$, $\ell \neq p$ and $w \in W_v$ with $\alpha(w) \leq 0$, the polynomial $P_{\ell,v}^i(w, T) \in \mathbb{Z}[T]$ and is independent of ℓ .*

Special cases of Cor.1.7, such as abelian varieties (cf.[30]), surfaces (cf.[23]) and smooth complete intersection in projective spaces (cf.[23]), had been previously proved using methods quite different from ours. Thanks to the motivic nature of Thm. 1.6, we deduce these known cases and much more, at once.

Equipped with Cor.1.7, we can now handle the questions about representations of the Weil-Deligne groups.

Theorem 1.8. *Let X a smooth projective variety over F which is a finite product of hyperkähler varieties of $K3^{[n]}$ type, unirational varieties of dimension ≤ 3 , uniruled surfaces, moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves, Fermat hypersurfaces, curves and abelian varieties. Then, for every $i \in \mathbb{N}$ and prime number $\ell \neq p$*

- (i) *the continuous representation $\rho_\ell^{i'} : W_v \rightarrow \text{GL}(V_\ell^i)$ is a semisimple representation of W_v and ;*
- (ii) *the $C_{WD}(X_{F_v}, i)$ conjecture 1.3, holds true.*

Previously, Thm.1.8, was only known to hold for $i = 1$ and X a curve or an abelian variety, cf. [8]. For surfaces and complete intersections, we show the following result.

- Theorem 1.9.** (1) Let S be a smooth projective surface over F . Then $C_{WD}(S_{F_v}, i)$ holds true $i \neq 2$. For $i = 2$, the system $\{\underline{\rho}_\ell^{2,ss}\}_{\ell \neq p}$ forms a compatible system of linear representations of $'W_v$ defined over \mathbb{Q} .
- (2) Let X be a smooth complete intersection of dimension n in a projective space \mathbb{P}^r , defined over F . Then $C_{WD}(X_{F_v}, i)$ holds true for $i \neq n$. For $i = n$, the system $\{\underline{\rho}_\ell^{n,ss}\}_{\ell \neq p}$ forms a compatible system of linear representations of $'W_v$ defined over \mathbb{Q} .

In the previous theorem, $\underline{\rho}_\ell^{*ss}$ denotes the semi-simplification of $\underline{\rho}_\ell^*$ (see §3.11). Cor. 3.13, shows that for complete intersection of Hodge level 1, $C_{WD}(X_{F_v}, i)$ holds true for all i .

Now, when X is as in Cor.1.7, then we see that the conjugacy class of $\rho_\ell^i(\Phi_v)$ in $\mathrm{GL}(V^i)$ is rational over \mathbb{Q} and independent of ℓ , in the sense of [28, I, §2.4]. We can sharpen this last conclusion, as well as Theorem 1.8(ii), by studying the properties of the Mumford-Tate group G_∞^i of the Hodge structure V^i . First note that the comparison isomorphism $V^i \otimes \mathbb{Q}_\ell \cong V_\ell^i$ between Betti and ℓ -adic cohomology, allows us to identify $G_{\infty/\mathbb{Q}_\ell}^i$ to an algebraic subgroup of $\mathrm{GL}(V_\ell^i)$. We denote by G_ℓ^i the global algebraic monodromy group, defined as the Zariski closure of $\rho_\ell^i(\Gamma_F)$ in $\mathrm{GL}(V_\ell^i)$.

Proposition 1.10. Let X be as in Thm. 1.8. Fix $i \in \mathbb{N}$ and assume that all absolute Hodge cycles in all tensor spaces formed by $V^i, V^{i \vee}, \mathbb{Q}(1)$ and $\mathbb{Q}(1)^\vee$, are defined over F . Then, G_ℓ^i is subgroup of $G_{\infty/\mathbb{Q}_\ell}^i$ for every ℓ .

The Proposition above pertains to the *Mumford-Tate conjecture*, which claims that $G_{\infty/\mathbb{Q}_\ell}^i = G_\ell^{i \circ}$, for any smooth projective variety over F .

As G_∞^i is a reductive algebraic group, we can construct the universal categorical quotient $(\mathrm{Conj}(G_\infty^i), \mathrm{Cl})$ for the action of G_∞^i on itself by conjugation. If we are in the situation of Proposition 1.10, then $\rho_\ell^i(\Phi_v)$ defines an element

$$\mathrm{Cl}(\rho_\ell^i(\Phi_v)) \in \mathrm{Conj}(G_\infty^i)(\mathbb{Q}_\ell) \text{ for every } \ell \neq p.$$

In this situation, we can also associate to ρ_ℓ^i , a representation $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{i,\ell}^i, N'_{i,\ell})$ of $'W_v$ with values in $G_{\infty/\mathbb{Q}_\ell}^i$ (see §4 for details). The notion of rationality and compatibility for representations of $'W_v$ with values in an algebraic group can be defined (see [8, 8.11]), similar to the case of linear representations; see §4.

Theorem 1.11. Let the notation and hypotheses be as in Proposition 1.10.

- (i) If $\rho_\ell^i|_{\Gamma_v}$ is potentially unramified representation for every $\ell \neq p$ and $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat for some prime number $\ell_0 \neq p$, then there exists a unique conjugacy class

$$\mathrm{Frob}_{v,i} \in \mathrm{Conj}'(G_\infty^i)(\mathbb{Q})$$

such that $\mathrm{Cl}(\rho_\ell^i(\Phi_v)) = \mathrm{Frob}_{v,i}, \forall \ell \neq p$.

- (ii) If $\rho_\ell^i|_{\Gamma_v}$ is a semi-stable representation for every $\ell \neq p$ and $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat for some prime number $\ell_0 \neq p$, then the representations $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{i,\ell}^i, N'_{i,\ell})$ are rational over \mathbb{Q} and form a compatible system of representations of $'W_v$ modulo the action of $G_\infty^{i \text{ ad}}$.

Roughly speaking, $(\mathrm{Conj}'(G_\infty^i), \mathrm{Cl})$ refers to a modification of $(\mathrm{Conj}(G_\infty^i), \mathrm{Cl})$, which we need to make when the derived group $G_{\infty/\mathbb{Q}}^{i \text{ der}}$ of $G_{\infty/\mathbb{Q}}^i$ has certain factors of type D . It is the quotient for the action of a slightly larger group $G_\infty^{i \text{ h}}$ on G_∞^i that extends the conjugation action. If $G_{\infty/\mathbb{Q}}^{i \text{ der}}$ has no factors of the above type, then $(\mathrm{Conj}'(G_\infty^i), \mathrm{Cl})$ coincides with $(\mathrm{Conj}(G_\infty^i), \mathrm{Cl})$. A similar modification is made in part (ii) of the theorem (see §4 for more on this and for the technical condition of weak neatness)

By assuming a stronger hypothesis that the variety itself has good (resp. semi-stable) reduction at v , results similar to Thm.1.11, were obtained for any $i \in \mathbb{N}$ and X any finite product of $K3$ surfaces, abelian varieties, Fermat hypersurfaces and curves, in [18]. Under those hypothesis the case of $i = 1$ and X an abelian variety was shown by Noot in [21] and [22]. For other improvements over these results, see Remark 4.6.

In the final section of the paper we investigate conjecture 1.4, for the case $\ell = p$. As the standard conjectures remain unsolved, we work in the framework of Tannakian category of motives $\mathcal{M}_{AH}(F)$ for absolute Hodge cycles [11]. This category has many desired properties of Grothendieck motives, unconditionally. Of course, it is well known that if the Hodge conjecture is true, then these categories coincide.

Let X/F be a smooth proper algebraic variety with good reduction at v . Suppose that the motive $M := h^i(X) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F))$ i.e. the full Tannakian subcategory of $\mathcal{M}_{AH}(F)$ generated by abelian varieties and Artin motives. The C_{Cris} -conjecture (cf. [14, Th. 3.2.3]) proved by Fontaine-Messing, Faltings and Tsuji, allows us to define the crystalline realization (cf.[31, §4.1]), i.e., a fibre functor $H_{M,\text{Cris}} : \langle M, \mathbb{Q}(1) \rangle^{\otimes} \rightarrow \text{Vect}_{F_v^0}$. We put $G_{M,\text{Cris}}$ for the automorphism group of $H_{M,\text{Cris}}$. The action of $G_{M,\text{Cris}}$ on itself by conjugation, provides a universal categorical quotient $(\text{Conj}(G_{M,\text{Cris}}), \text{Cl}_{\text{Cris}})$.

Let X_v denote the special fibre of a smooth proper model of X over the ring of integers of F_v . Denote the ring of Witt vectors of k_v by $W(k_v)$ and by F_v^0 the fraction field of $W(k_v)$. Fix a $i \in \mathbb{N}$ and let $\Phi_{\text{Cris}} : H_{\text{Cris}}^i(X_v/W(k_v)) \otimes F_v^0 \rightarrow H_{\text{Cris}}^i(X_v) \otimes F_v^0$, be the degree i crystalline Frobenius of X at v . Now, Φ_{Cris} defines an element $\Phi_{M,\text{Cris}} \in G_{M,\text{Cris}}(F_v^0)$ which in turn defines an element $\text{Cl}_{\text{Cris}}(\Phi_{M,\text{Cris}}) \in \text{Conj}(G_{M,\text{Cris}})(F_v^0)$. As $\text{Conj}(G_{M,\text{Cris}}) \cong \text{Conj}(G_M)_{/F_v^0}$, we make sense of the following theorem

Theorem 1.12. *Let M be as above. Assume that G_M is connected and there is prime number ℓ_0 such that $\rho_{M,\ell_0}(\Phi_v)$ is weakly neat. Then, there exists a unique conjugacy class*

$$\text{Frob}_M \in \text{Conj}'(G_M)(F_v^0),$$

such that $\text{Cl}_{\text{Cris}}(\Phi_{M,\text{Cris}}) = \text{Frob}_M$ and $\text{Cl}(\rho_{M,\ell}(\Phi_v)) = \text{Frob}_M, \forall \ell \neq p$.

Corollary 5.4 relates the above result to part (i) of Thm. 1.11. Denote by $\mathcal{M}_{num}^{av}(F)$ (cf.[15]) the Tannakian category of motives generated by abelian varieties and zero dimensional varieties, inside the category of numerical Grothendieck motives over F . Since homological equivalence coincides with numerical equivalence for zero dimensional varieties and abelian varieties [20], we can identify $\mathcal{M}_{num}^{av}(F)$ to a subcategory (a priori not full) of $\mathcal{M}_{AH}^{av}(F)$. Thus, our result also holds for $M \in \text{Ob}(\mathcal{M}_{num}^{av}(F))$. The special case, where M is the motive $h^1(A)$ for an abelian variety A , Theorem 1.12 was proved by Noot in [21, Th. 4.2]. We also note that as the Hodge conjecture remains unproven, a priori $\mathcal{M}_{AH}^{av}(F)$ has more objects than in $\mathcal{M}_{num}^{av}(F)$. An example of this phenomenon is the motive of a $K3$ surface.

Finally, a word about the proofs of our results. We've freely used motivic arguments in many of our proofs, not only because this makes the proofs more elegant, but also due to the fact that they represent the most general context in which our questions could be pursued.

2. ACTION OF FROBENIUS ON ℓ -ADIC COHOMOLOGY GROUPS

Proof of Theorem 1.6. Let us denote by $\tau\Delta \subset \tau X \times \tau X$ the diagonal subvariety and by d the dimension of X . Let $\gamma_B^d : \text{CH}^d(\tau X \times \tau X) \rightarrow \text{H}_B^{2d}(\tau X(\mathbb{C}) \times \tau X(\mathbb{C}), \mathbb{Q})(d)$, be the cycle class map from codimension d algebraic cycles on $\tau X \times \tau X$ to the Tate twisted degree $2d$ Betti cohomology of $\tau X \times \tau X$. By hypothesis the degree i Künneth standard conjecture is true for τX , which means that there is a correspondence $\tau z^i \in \text{CH}^d(\tau X \times \tau X)$ such that $\gamma_B^d(\tau z^i) = \tau\pi^i$.

Let $p^* : \mathrm{CH}^d(X_{\bar{F}} \times X_{\bar{F}}) \rightarrow \mathrm{CH}^d(\tau X \times \tau X)$ denote the base change map. It is well known (cf.[11, 2.9, a]) that base change induces an isomorphism between algebraic cycle groups over \bar{F} and \mathbb{C} modulo homological equivalence. This implies that there is a correspondence $z_{/\bar{F}}^i \in \mathrm{CH}^d(X_{\bar{F}} \times X_{\bar{F}})$ such that $\gamma_B^d(\tau z^i) = \gamma_B^d(p^*(z_{/\bar{F}}^i))$. Now there is also a diagram

$$(1) \quad \begin{array}{ccc} \mathrm{CH}^d(\tau X \times \tau X) & \xrightarrow{\gamma_B^d \otimes 1} & \mathrm{H}_B^{2d}(\tau X(\mathbb{C}) \times \tau X(\mathbb{C}), \mathbb{Q})(d) \otimes \mathbb{Q}_\ell \\ p^* \uparrow & & \uparrow I_{\ell, \tau} \\ \mathrm{CH}^d(X_{\bar{F}} \times X_{\bar{F}}) & \xrightarrow{\gamma_\ell^d} & \mathrm{H}_{et}^i(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d) \end{array}$$

where γ_ℓ^d is the cycle class map of ℓ -adic cohomology and $I_{\ell, \tau}$ is the comparison isomorphism between Betti and ℓ -adic cohomology groups. It follows from the arguments given in [11, I, page 21], that (1) is commutative.

Let $\Delta_{/\bar{F}} \subset X_{\bar{F}} \times X_{\bar{F}}$ denote the diagonal subvariety. We have the Künneth decomposition of the ℓ -adic cohomology class $\gamma_\ell^d([\Delta_{/\bar{F}}]) = \sum_{i=0}^{2d} \pi_{/\bar{F}}^i$. It follows from the commutativity of (1), that

$$\gamma_\ell^d(z_{/\bar{F}}^i) = I_{\ell, \tau}^{-1}((\gamma_B^d \otimes 1)(\tau z^i)) = I_{\ell, \tau}^{-1}(\tau \pi^i \otimes 1) = \pi_{/\bar{F}}^i.$$

Now, clearly we can suppose that $z_{/\bar{F}}^i$ is defined over some finite extension (say) F_1 of F . By using the natural action of $G := \mathrm{Gal}(F_1/F)$ on $\mathrm{CH}^d(X_{F_1} \times X_{F_1})$, we set

$$z^i := \frac{1}{|G|} \sum_{\sigma \in G} \sigma^* z_{/\bar{F}}^i.$$

As $z^i \in \mathrm{CH}^d(X_{F_1} \times X_{F_1})^G$, we conclude that $z^i \in \mathrm{CH}^d(X \times X)$. Now we have

$$(2) \quad \gamma_\ell^d(z^i) = \frac{1}{|G|} \sum_{\sigma \in G} \gamma_\ell^d(\sigma^* z_{/\bar{F}}^i) = \frac{1}{|G|} \sum_{\sigma \in G} \tilde{\sigma}^* \gamma_\ell^d(z_{/\bar{F}}^i),$$

where $\tilde{\sigma}$ is a lift of σ to $\mathrm{Gal}(\bar{F}/F)$.

Now, by Künneth formula and Poincaré duality for ℓ -adic cohomology, we have the following canonical isomorphisms

$$(3) \quad \begin{aligned} \mathrm{H}_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d) &\cong \bigoplus_{r \geq 0} \mathrm{H}_{et}^{2d-r}(X_{\bar{F}}, \mathbb{Q}_\ell)(d) \otimes \mathrm{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell) \\ &\cong \bigoplus_{r \geq 0} \mathrm{Hom}_{\mathbb{Q}_\ell}(\mathrm{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \otimes \mathrm{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell) \cong \bigoplus_{r \geq 0} \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell)) \end{aligned}$$

More precisely, under the above isomorphism, an element $u \in \mathrm{H}_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d)$ is mapped to the element $\underline{u} := (z \mapsto pr_{2*}(pr_{1*}^{-1}(z) \cdot u)) \in \bigoplus_{r \geq 0} \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell))$, where pr_1 and pr_2 are projections. As one easily verifies, $\pi_{/\bar{F}}^i$ is the idempotent projection

$$p_i : \mathrm{H}_{et}^*(X_{\bar{F}}, \mathbb{Q}_\ell) \rightarrow \mathrm{H}_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell) \hookrightarrow \mathrm{H}_{et}^*(X_{\bar{F}}, \mathbb{Q}_\ell).$$

Clearly p_i is invariant under the action of $\mathrm{Gal}(\bar{F}/F)$ on $\bigoplus_{i \geq 0} \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell))$. Thus, it follows from (3) that $\pi_{/\bar{F}}^i$ is invariant under the action of $\mathrm{Gal}(\bar{F}/F)$ on $\mathrm{H}_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d)$.

Under the above observations, (2) now reads as

$$(4) \quad \gamma_\ell^d(z^i) = \frac{1}{|G|} \sum_{\sigma \in G} \tilde{\sigma}^* \pi_{/\bar{F}}^i = \frac{1}{|G|} (|G| \cdot \pi_{/\bar{F}}^i) = \pi_{/\bar{F}}^i.$$

Finally, let α^i denote the image of z^i under the canonical injective morphism $\mathrm{CH}^d(X \times X) \rightarrow \mathrm{CH}^d(X_{F_v} \times X_{F_v})$. Let $\gamma_{\ell, v}^d : \mathrm{CH}^d(X_{F_v} \times X_{F_v}) \rightarrow \mathrm{H}_{et}^{2d}(X_{\bar{F}_v} \times X_{\bar{F}_v}, \mathbb{Q}_\ell)(d)$ denote

the cycle class map. By invariance of ℓ -adic cohomology groups under extension from \bar{F} to algebraically closed over-field, we have $H_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell) \cong H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ and $H_{et}^{2d}(X_{\bar{F}_v} \times X_{\bar{F}_v}, \mathbb{Q}_\ell) \cong H_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)$. We also know that $\gamma_{\ell,v}^d|_{\text{CH}^d(X \times X)} = \gamma_\ell^d$; in particular $\gamma_{\ell,v}^d(\alpha^i) = \gamma_\ell^d(z^i)$. Thus using (4) and (3), it follows that $\gamma_{\ell,v}^d(\alpha^i)$ induces an endomorphism $H_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell) \rightarrow H_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell)$, which is identity on $H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ and 0 otherwise. In particular we see that

$$(5) \quad \text{Tr}((\gamma_{\ell,v}^d(\alpha^i) \circ w_*^m) : H_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell)) = \text{Tr}(\rho_\ell^i(w^m)), \text{ for every } m \geq 0,$$

where $w_*^m := \oplus_{i=0}^{2d} \rho_\ell^i(w^m)$ is the \mathbb{Q}_ℓ -linear map induced by w^m on $H_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell)$. Now, pick a $N \geq 1$, such that $N\gamma_{\ell,v}^d(\alpha^i)$ belongs to the image of the Chow group of codimension d algebraic cycles on $X_{\bar{F}_v} \times X_{\bar{F}_v}$, with \mathbb{Z} -coefficients. It follows from (5) and [25, Th. 0.1] that

$$(6) \quad \text{Tr}(\rho_\ell^i(w^m)) \in (1/N)\mathbb{Z}, \text{ and is independent of } \ell.$$

Using (6) and applying the next lemma to the eigenvalues of $\rho_\ell^i(w)$, we conclude that $\text{Tr}(\rho_\ell^i(w)) \in \mathbb{Z}$. This combined with the Newton identities relating power sums and symmetric polynomials, we conclude that the characteristic polynomial $P_{\ell,v}^i(w, T)$ of $\rho_\ell^i(w)$ belongs to $\mathbb{Z}[T]$ and is independent of ℓ . □

Lemma 2.1 (cf. [17, 2.8]). *Let a_1, \dots, a_r and b_1, \dots, b_s be elements of a field of characteristic 0. We put $s_m = \sum_{i=1}^r a_i^m - \sum_{j=1}^s b_j^m$ for an $m \in \mathbb{N}$. Assume there exists an integer $N \geq 1$ such that $Ns_m \in \mathbb{Z}$ for all $m \geq 0$. Then $s_m \in \mathbb{Z}$.*

Before proceeding further, we need to recall few definitions.

Definition 2.2. A *hyperkähler* variety over any field K of characteristic 0 (cf.[1]) is a simply connected smooth projective K -variety Y of even dimension $2n$, with the property that there exists a section ω of Ω_Y^2 , unique up to multiplication by a constant, such that ω^n vanishes nowhere. The variety Y is said to be of $K3^{[n]}$ -type if it is one of the following:

- (1) For any $K3$ surface S , take $Y = S^{[n]}$ the punctual Hilbert scheme which parametrizes closed subschemes of S of length n .
- (2) Any projective deformation of a hyperkähler variety of type $S^{[n]}$.

Any general projective deformation of $S^{[n]}$ is not of the form $S'^{[n]}$ for any other $K3$ surface S' . Hyperkähler varieties of $K3^{[n]}$ -type, form one of the two series of examples of hyperkähler varieties constructed by Beauville [3]. Apart from these only two exceptional examples has been discovered by O'Grady.

Definition 2.3. Let C be a smooth projective curve of genus > 1 over F . The moduli space $N_C(q, e)$ of stable vector bundles of coprime rank q and degree e over C is a smooth projective fine moduli space.

Proof of Corollary 1.7. It follows from Thm. 1.6, that we need to show that τX satisfies the degree i Künneth standard conjecture for each $i \in \mathbb{N}$. It suffices to show this for each one of the varieties in the statement. Indeed, if X_1 and X_2 are any two algebraic varieties over \mathbb{C} , then the Künneth components of diagonal sub-varieties, satisfies the identity $\pi_{X_1 \times X_2}^i = \sum_{r+s=i} \pi_{X_1}^r \otimes \pi_{X_2}^s$. Therefore, if X_1 and X_2 verifies degree i Künneth standard conjecture for each $i \in \mathbb{N}$, then so does their product $X_1 \times X_2$.

Now, for curves, surfaces, abelian varieties and smooth complete intersection in projective spaces, it is a classical fact, cf.[17]. For hyperkähler varieties of $K3^{[n]}$ type this follows from the results of [5]. For unirational varieties of dimension ≤ 4 and uniruled varieties of dimension ≤ 3 this is shown in [2]. For the moduli spaces $N_C(q, e)$ as above this follows from [6]. □

Corollary 2.4. *Let X be as in corollary 1.7. Then for every $\ell \neq p$ and $w \in W_v$, the polynomial $P_{\ell,v}^i(w, T) \in \mathbb{Q}[T]$ and is independent of ℓ .*

Proof. Let $w \in W_v$ such that $\alpha(w) > 0$, then $\alpha(w^{-1}) < 0$. As $\rho_\ell^i(w) = (\rho_\ell^i(w^{-1}))^{-1}$, so if $P_{\ell,v}^i(w^{-1}, T) = T^m + \sum_{r=1}^m a_{m-r} T^{m-r}$, then $P_{\ell,v}^i(w, T) = T^m + a_0^{-1} (\sum_{r=1}^{m-1} a_r T^{m-r} + 1)$. By, Cor. 1.7 $a_r \in \mathbb{Z}$, hence $P_{\ell,v}^i(w, T) \in \mathbb{Q}[T]$. \square

3. AROUND THE C_{WD} CONJECTURE

3.1. Monodromy. Let K be a complete discretely valued field with a finite residue field k_v , where v denotes the valuation on K . Let $\text{char}(k_v) = p > 0$. Fix an algebraic closure \bar{K} of K and write $\Gamma_K := \text{Gal}(\bar{K}/K)$. Let \bar{v} be the extension of v to \bar{K} . The residue field of \bar{K} at \bar{v} is denoted by \bar{k}_v (which is also an algebraic closure of k_v).

For any prime number ℓ , we denote by μ_{ℓ^n} the group of ℓ^n -th roots of unity in \bar{k}_v and $\mathbb{Z}_\ell(1) := \varprojlim_n \mu_{\ell^n}$. The inertia group $I_K \subset \Gamma_K$ fits into the following exact sequence

$$1 \rightarrow P \rightarrow I_K \xrightarrow{t} \mathbb{Z}_{(p')}(1) \rightarrow 1,$$

where P is a pro- p -group and $\mathbb{Z}_{(p')}(1) = \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$. Let $\ell \neq p$ be a prime number. We

denote by $t_\ell : I_K \rightarrow \mathbb{Z}_\ell(1)$, the ℓ -component of t . Explicitly, the surjective map t_ℓ is given as $x \mapsto \left(x(\pi^{\frac{1}{\ell^n}})/\pi^{\frac{1}{\ell^n}} \right)_n$, where $\pi \in \mathcal{O}_K$ is a uniformizer. The map t_ℓ is unique upto multiplication by an element of \mathbb{Z}_ℓ^\times .

For any \mathbb{Q}_ℓ -vector space U , we write $U(1) := U \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)$, where $\mathbb{Q}_\ell(1) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)$. Grothendieck's ℓ -adic monodromy theorem [30, Appendix], says that any ℓ -adic representation $\xi_\ell : \Gamma_K \rightarrow \text{GL}(U)$ is *quasi-unipotent*, i.e., there exists an open subgroup $J \subseteq I_K$ such that $\xi_\ell|_J$ is unipotent (i.e. $\xi_\ell(\sigma)$ is a unipotent linear map for every $\sigma \in J$). Moreover, there exists a unique nilpotent morphism $N_\ell : U(1) \rightarrow U$ such that

$$(7) \quad \xi_\ell(x) = \exp(t_\ell(x)N_\ell), \quad \text{for all } x \in J.$$

The morphism N_ℓ is called the *monodromy operator* associated to ξ_ℓ .

3.2. Linear representations of Weil-Deligne groups. The notation is as in §3.1, we assume further that K is a finite extension of \mathbb{Q}_p and $|k_v| = q_v$. Denote by $\phi_v : x \mapsto x^{q_v}$ the arithmetic Frobenius automorphism of \bar{k}_v over k_v .

Definition 3.3. Let E be any field of characteristic 0. A linear representation $'W_K$ over E is a triple $\underline{\xi} = (\Delta, \xi', N')$ consisting of

- (a) A finite dimensional linear representation $\xi' : W_K \rightarrow \text{GL}(\Delta)$ of W_K over E .
- (b) A nilpotent endomorphism N' of Δ , such that $\xi'(w)N'\xi'(w)^{-1} = q_v^{\alpha(w)} \cdot N$ for every $w \in W_K$.

Definition 3.4. A *morphism* $\underline{\xi}_1 \rightarrow \underline{\xi}_2$ between two linear representations of $'W_K$ over E , is a E -linear map $f : \Delta_1 \rightarrow \Delta_2$ such that $f \circ \xi'_1(w) = \xi'_2(w) \circ f$, for all $w \in W_K$, and $f \circ N'_1 = N'_2 \circ f$.

The collection of all linear representations of $'W_K$ over E , forms a neutral Tannakian category $\mathbf{Rep}_E('W_K)$ over E . Any field embedding $\tau : E \rightarrow L$ gives rise to a functor

$$\mathbf{Rep}_E('W_K) \rightarrow \mathbf{Rep}_L('W_K), \quad (\Delta, \xi', N') \mapsto (L \otimes_{\tau, E} \Delta, \tau \xi', 1 \otimes_{\tau, E} N').$$

Definition 3.5. Let $\underline{\xi}$ be as above and E_0 a sub-field of E . We say that $\underline{\xi}$ is rational over E_0 , if given any algebraically closed field $\Omega \supset E$ and any $\sigma \in \text{Aut}(\Omega/E_0)$, the representations ${}^\sigma \underline{\xi}_{/\Omega} = (\Omega \otimes_{\sigma, E} \Delta, {}^\sigma \xi'_{/\Omega}, 1 \otimes_{\sigma, E} N')$ of $'W_K$ over Ω , obtained by extension of scalars via σ , are all isomorphic. This condition is independent of the choice of Ω .

Definition 3.6. Let $(E_i)_{i \in I}$ be a family of extensions of E_0 , and for every i , let $\underline{\xi}_i$ be a linear representation of $'W_K$ over E_i . We say that $(\underline{\xi}_i)_{i \in I}$ forms a *compatible family of representations* of $'W_K$ if every $\underline{\xi}_i$ is rational over E_0 , and for any $i, j \in I$, if Ω is an algebraically closed field containing E_i and E_j , then the representations $\underline{\xi}_i_{/\Omega}$ and $\underline{\xi}_j_{/\Omega}$ obtained by extension of scalars are isomorphic.

Now fix an arithmetic Frobenius element $\Phi \in \Gamma_K$ and an isomorphism $\iota : \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell(1)$. To any arbitrary ℓ -adic representation $\xi_\ell : \Gamma_K \rightarrow \text{GL}(U)$ (as in §3.1), we associate a representation $\underline{\xi}_\ell = (U, \xi'_\ell, N'_\ell)$ of $'W_K$ over \mathbb{Q}_ℓ , by setting

- (a) $\xi'_\ell(w) = \xi_\ell(w) \exp(-N_\ell t_\ell(\Phi^{-\alpha(w)} w))$, where N_ℓ is the monodromy operator as in (7).
- (b) $N'_\ell \in \text{End}(U)$ corresponds to N_ℓ , via ι .

The following lemma is well-known.

Lemma 3.7. *The isomorphism class of $\underline{\xi}_\ell$ depends only on ξ_ℓ , and it doesn't depend on the choice of Φ and ι .*

3.8. Proof of Theorem 1.8. The proof will occupy the rest of this subsection. Let Y be any smooth and proper algebraic variety over F_v . As before, to the canonical ℓ -adic representations $\rho'_\ell : \Gamma_v \rightarrow \text{GL}(H_{\text{et}}^i(\bar{Y}, \mathbb{Q}_\ell))$, we associate the representation $\underline{\rho}'_\ell = (V_\ell^i, \rho_{i,\ell}^i, N'_{i,\ell})$ of $'W_v$ over \mathbb{Q}_ℓ , where $V_\ell^i := H_{\text{et}}^i(\bar{Y}, \mathbb{Q}_\ell)$.

We first show the semisimplicity of the representation $\rho_{i,\ell}^i$, i.e. part (i) of the theorem. This proof would employ some motivic ideas. We review some of the necessary facts very briefly, for details see [11, II]. Denote by $\mathcal{V}(F)$ the category of smooth projective algebraic varieties over F . Recall that the category of (pure) motives $\mathcal{M}_{AH}(F)$, defined by absolute Hodge cycles is a \mathbb{Q} -linear semisimple neutral Tannakian category. There exists a contravariant functor for $\mathcal{M}_{AH}(F)$ $h : \mathcal{V}(F) \rightarrow \mathcal{M}_{AH}(F)$ such that Betti, ℓ -adic or deRham cohomology on $\mathcal{V}(F)$ factorizes through h . This provides fibre functors H_τ, H_ℓ and H_{dR} on $\mathcal{M}_{AH}(F)$. We call these functors as *realizations*. There also exists a natural grading $h(Z) = \bigoplus h^i(Z)$, $i \in \mathbb{N}$ for every $Z \in \text{Ob}(\mathcal{V}(F))$, which extends to all of $\mathcal{M}_{AH}(F)$. We denote by $\mathcal{M}_{AH}^{av}(F)$ the Tannakian subcategory of $\mathcal{M}_{AH}(F)$ generated by abelian varieties and zero dimensional varieties. Everything stated above holds more generally for any arbitrary field of characteristic 0, with an embedding in \mathbb{C} .

Lemma 3.9. *Let X/F be as in Thm. 1.8. Then there exists a finite extension F' of F such that $h(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$.*

Proof. First, we claim that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$. We observe that as $\mathcal{M}_{AH}^{av}(\bar{F})$ is a Tannakian category, it is closed under tensor products. Thus, in order to verify the claim it suffices to do so for each of the individual varieties in the statement of Thm. 1.8, since X is a finite products of these varieties.

So, first assume that X is a hyperkähler variety of $K3^{[n]}$ -type. As the Kuga-Satake abelian variety associated to X is defined over a finite extension of F (see [1]) and $h(X_{\bar{F}})$ belongs to the smallest Tannakian subcategory generated by $h^2(X_{\bar{F}})$ (cf. [26]), it follows that in this case $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$.

Next, when X is a Fermat hypersurfaces or a unirational variety of dimension ≤ 3 , it follows from [11, II, 6.26] that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$.

When X is the moduli space $N_C(q, e)$ as above, then it follows from [6], that $h(X)$ is a direct summand of a motive of the form $\oplus_b h(C)^{\otimes a}(b)$. Now, let $J(C)$ denote the Jacobian of C . We know that $h(C) = \mathbf{1} \oplus h^1(J(C)) \oplus \mathbb{L}$, where $\mathbf{1}$ denotes the unit object of $\mathcal{M}_{AH}(\bar{F})$ and \mathbb{L} is the Lefschetz motive. It follows that $h(X) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F))$.

Finally, when X is a uniruled surface, we know (see [2]) that there exists a curve C' over \bar{F} such that $h(X_{\bar{F}})$ is a direct summand of a motive of the form $\oplus_{b'} h(C')^{\otimes a'}(b')$. Thus, as above it follows that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$.

Now, for each individual variety as in the statement of the corollary, we see that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$. Hence, the same holds for any product of these varieties. By using [11, I, 2.9] we conclude there exists a finite extension F' of F such that $h(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$. \square

Now, let us denote by M the motive $h^i(X)$. As M is a direct summand of $h(X)$, by Lemma 3.9 there is a finite extension F' of F such that $M_{F'} := h^i(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$. Let us denote by v' an extension of the valuation v to F' ; n the residual degree; $\Phi_{v'} := \Phi_{\bar{v}}^n$ an arithmetic Frobenius corresponding to this extension. We denote by $\langle M_{F'}, \mathbb{Q}(1) \rangle^{\otimes}$ the Tannakian subcategory of $\mathcal{M}_{AH}^{av}(F')$ which is tensor generated by $M_{F'}$ and the Tate motive $\mathbb{Q}(1)$ and we write $G_{M_{F'}} := \underline{\text{Aut}}^{\otimes}(H_{\tau}|_{\langle M_{F'}, \mathbb{Q}(1) \rangle^{\otimes}})$. We recall, that the ℓ -adic representations $\rho_{M_{F'}, \ell} : \Gamma_{F'} \rightarrow \text{GL}(H_{\ell}(M_{F'}))$, arising from the action of $\Gamma_{F'}$ on the ℓ -adic realization $H_{\ell}(M_{F'})$ of $M_{F'}$, factorizes through $G_{M_{F'}}(\mathbb{Q}_{\ell})$.

We know that there exists (possibly after passing to another finite extension, which we again denote by F') an abelian variety A over F' and a unique homomorphism of algebraic groups $\theta : G_A \rightarrow G_{M_{F'}}$ where $G_A := \underline{\text{Aut}}^{\otimes}(H_{\tau}|_{\langle h^1(A), \mathbb{Q}(1) \rangle^{\otimes}})$. Moreover, $\theta(\mathbb{Q}_{\ell}) \circ \rho_{A, \ell} = \rho_{M_{F'}, \ell}$, where $\theta(\mathbb{Q}_{\ell})$ denotes the induced map on \mathbb{Q}_{ℓ} -valued points and $\rho_{A, \ell}$ is the Galois representation associated to the motive $h^1(A)$. For details on the above facts see [18, 3.10 & 3.11]. In particular, we have $\theta(\mathbb{Q}_{\ell})(\rho_{A, \ell}(\Phi_{v'})) = \rho_{M_{F'}, \ell}(\Phi_{v'})$.

Now, it is well known that $\rho_{A, \ell}(\Phi_{v'})$ is semisimple i.e. becomes diagonalizable after a base change. As θ is a homomorphism of algebraic groups, the image of $\rho_{A, \ell}(\Phi_{v'})$, under $\theta(\mathbb{Q}_{\ell})$ must be semisimple. In other words, $\rho_{M_{F'}, \ell}(\Phi_{v'})$ is semisimple.

Finally, let $\rho_{M, \ell} : \Gamma_F \rightarrow \text{GL}(H_{\ell}(M))$ be the ℓ -adic representation associated to M . As $\rho_{M_{F'}, \ell}(\Phi_{v'}) = (\rho_{M, \ell}(\Phi_v))^n$, so $\rho_{M, \ell}(\Phi_v)$ is semisimple. Now, by definition $H_{\ell}(M) = \pi^{i*} \oplus_r H_{et}^r(X_{\bar{F}}, \mathbb{Q}_{\ell})$, where π^i is the (i -th) Künneth component of the diagonal subvariety of $X_{\bar{F}} \times X_{\bar{F}}$ and π^{i*} is the image of π^i under the isomorphism (3). As we've seen before, π^{i*} is the idempotent projection p_i . We conclude that $H_{\ell}(M) = H_{et}^i(X_{\bar{F}}, \mathbb{Q}_{\ell})$ and $\rho_{M, \ell} = \rho_{\ell}^i$. Thus, it follows from the above discussion that $\rho_{\ell}^i(\Phi_v)$ is semisimple.

Now, we note that $\rho_{\ell}^{i'}(\Phi_v)$ ($= \rho_{\ell}^i(\Phi_v)$) generates a subgroup of finite index in $\rho_{\ell}^{i'}(W_v)$. We know that in characteristic 0 a representation (in the ordinary sense) of a group is semisimple if and only if its restriction to a subgroup of finite index is semisimple. As $\rho_{\ell}^i(\Phi_v)$ is a semisimple automorphism, it follows that $\rho_{\ell}^{i'}$ is a semisimple representation of W_v . This completes the proof of part (i) of the theorem.

We now move to the part (ii) of the theorem. For simplicity of notation, henceforth we'll suppress the natural number i , which will be fixed in the rest of the proof.

As, $N_{\ell}' : V_{\ell} \rightarrow V_{\ell}$ is a nilpotent endomorphism, it induces a unique filtration \mathfrak{M}_{\bullet} on V_{ℓ} called the *local monodromy filtration* [10], characterized by the following properties:

- (1) \mathfrak{M}_{\bullet} is an increasing filtration $\cdots \mathfrak{M}_{j-1} V_{\ell} \subset \mathfrak{M}_j V_{\ell} \subset \mathfrak{M}_{j+1} V_{\ell} \cdots$ of Γ_v representations, such that $\mathfrak{M}_j V_{\ell} = 0$ for sufficiently small j and $\mathfrak{M}_j V_{\ell} = V_{\ell}$ for sufficiently large j .
- (2) $N_{\ell}(\mathfrak{M}_j V_{\ell}(1)) \subseteq \mathfrak{M}_{j-2} V_{\ell}$ for all j .

- (3) Using the second condition we can define an induced map $N : \mathrm{Gr}_j^{\mathfrak{M}} V_\ell(1) \rightarrow \mathrm{Gr}_{j-2}^{\mathfrak{M}} V_\ell$, where $\mathrm{Gr}_j^{\mathfrak{M}} V_\ell = \mathfrak{M}_j V_\ell / \mathfrak{M}_{j-1} V_\ell$. Then $N^r : \mathrm{Gr}_r^{\mathfrak{M}} V_\ell(r) \rightarrow \mathrm{Gr}_{-r}^{\mathfrak{M}} V_\ell$ is an isomorphism for each $r \geq 0$.

Explicitly, the filtration \mathfrak{M}_\bullet is defined as the convolution of the Kernel and image filtration on V_ℓ induced by N'_ℓ . Let $\bar{\rho}_{\ell,j} : \Gamma_v \rightarrow \mathrm{GL}(\mathrm{Gr}_j^{\mathfrak{M}} V_\ell)$ denote the representation induced by ρ_ℓ on the graded parts of \mathfrak{M}_\bullet . For every $w \in W_v$, let us denote by $P_\ell(w, T)$ the characteristic polynomial of $\rho_\ell(w)$ and by $\bar{P}_{\ell,j}(w, T)$ the characteristic polynomial of $\bar{\rho}_{\ell,j}(w)$.

Lemma 3.10. *Fix a $w \in W_v$, then for every j , $\bar{P}_{\ell,j}(w, T) \in \mathbb{Q}[T]$ and is independent of ℓ .*

Proof. Let L be the sub-field of an algebraic closure $\bar{\mathbb{Q}}_\ell$ of \mathbb{Q}_ℓ , generated by the roots of $P_\ell(w, T)$. By Cor.2.4, $P_\ell(w, T) \in \mathbb{Q}[T]$ and has coefficients independent of ℓ . This implies that if β is a root of $P_\ell(w, T)$, then so is $\sigma(\beta)$, for every $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$. Now, from linear algebra, we know that $P_\ell(w, T) = \prod_j \bar{P}_{\ell,j}(w, T)$. By [19], we know that the Weight Monodromy conjecture (WMC) holds for V_ℓ , i.e., for every $w \in W_v$, the eigenvalues of $\bar{\rho}_{\ell,j}(w)$ are all algebraic integers of complex absolute value $q_v^{(i+j)\alpha(w)/2}$ (recall that i is the degree of the cohomology group V_ℓ). This, combined with the fact that the complex absolute value of β and $\sigma(\beta)$ are same, implies that β and $\sigma(\beta)$ occurs as the roots of a same factor (say) $\bar{P}_{\ell,j}(w, T)$ of $P_\ell(w, T)$. This implies that the coefficients of each $\bar{P}_{\ell,j}(w, T)$ are stable under $\mathrm{Gal}(L/\mathbb{Q})$, i.e., $\bar{P}_{\ell,j}(w, T) \in \mathbb{Q}[T]$ and is independent of ℓ . \square

The previous lemma says that for every j , the character of the representations $\bar{\rho}_{\ell,j}$ has values in \mathbb{Q} and is independent of ℓ . It now follows from [8, Prop. 8.9] and part (i) of the theorem, that $\{(V_\ell, \rho'_\ell, N'_\ell)\}_{\ell \neq p}$ forms a compatible system of linear representations of $'W_v$ (defined over \mathbb{Q}).

3.11. The case of surfaces and complete intersections. Let the notation be as in §3.2. Let $\underline{\xi} = (\Delta, \xi', N')$ be any linear representation of $'W_v$ over E . Pick an arithmetic Frobenius element $\Phi_v \in W_v$. Consider the multiplicative Jordan-Chevalley decomposition of the endomorphism $\xi'(\Phi_v) = \xi'(\Phi_v)_{ss} \cdot \xi'(\Phi_v)_u$, where $\xi'(\Phi_v)_{ss}$ is semisimple and $\xi'(\Phi_v)_u$ is unipotent.

Definition 3.12. The *semi-simplification* $\underline{\xi}^{ss}$ of $\underline{\xi}$ is the linear representation of $'W_K$ over E , obtained from $\underline{\xi}$, by keeping Δ , $\xi'|_{I_v}$ and N' unchanged and replacing $\xi'(\Phi_v)$ by $\xi'(\Phi_v)_{ss}$.

It can be shown that $\underline{\xi}^{ss}$ is independent of the initial choice of Φ_v . The semi-simplification of the representations $\underline{\rho}_\ell^i = (V_\ell^i, \rho_\ell^i, N'_{i,\ell})$ is denoted by $\underline{\rho}_\ell^{i,ss}$

Proof of Theorem 1.9. (1) The motive of S has a decomposition $^1 h(S) = \bigoplus_{i=0}^4 h^i(S)$, such that $h^0(S) \cong h^4(S)(2) \cong \mathbb{1}$ and

$$h^1(S) \cong h^3(S)(1) \cong h^1(\mathrm{Pic}_S^0) \cong h^1(\mathrm{Alb}_S) \cong h^1(S)^\vee(-1),$$

where Pic_S^0 and Alb_S are the Picard and Albanese variety of S , respectively. This observation allows us to deduce $C_{WD}(S_{F_v}, i)$, for $i \neq 2$, directly from $h^1(\mathrm{Alb}_S)$.

Now let $i = 2$. Let $\bar{\rho}_{\ell,j}^2 : \Gamma_v \rightarrow \mathrm{GL}(\mathrm{Gr}_j^{\mathfrak{M}} V_\ell^2)$ denote the representation induced by ρ_ℓ^2 on the graded parts $\mathrm{Gr}_j^{\mathfrak{M}} V_\ell^2$ of the local monodromy filtration \mathfrak{M}_\bullet on V_ℓ^2 . By Cor.2.4, we know that the characteristic polynomial of $\rho_\ell^2(w) \in \mathbb{Q}[T]$ and is independent of ℓ . We also know from [24] (see also [25]), that WMC holds for V_ℓ^2 , i.e., for every $w \in W_v$, the eigenvalues of $\bar{\rho}_{\ell,j}^2(w)$ are all algebraic integers of complex absolute value $q_v^{(2+j)\alpha(w)/2}$. It follows, as in the proof of Lemma 3.10, that

¹This decomposition also holds in the category of Chow motives.

the character of the representations $\overline{\rho}_{\ell,j}^2$ has values in \mathbb{Q} and is independent of ℓ . We now apply [8, Prop.8.9] to conclude the proof in this case.

- (2) Let us first consider the case $i \neq n$. By using the fact that the Lefschetz standard conjecture is true for \mathbb{P}^r and a repeated application of the weak Lefschetz theorem (for classical cohomology theories), we see that the motive

$$h^i(X) \cong h^i(\mathbb{P}^r).$$

Thus in order to see the validity of our claim, we only need to note that the ℓ -adic realization $H_\ell(\mathbb{T}) = \mathbb{Q}_\ell(1)$ of the Tate motive \mathbb{T} (which is the dual of Lefschetz motive \mathbb{L}), gives rise to a compatible system of linear representations of W_v defined over \mathbb{Q} .

Now consider the case $i = n$. Let $\overline{\rho}_{\ell,j}^n : \Gamma_v \rightarrow \mathrm{GL}(\mathrm{Gr}_j^{\mathfrak{M}} V_\ell^n)$ denote the representation induced by ρ_ℓ^n on the graded parts $\mathrm{Gr}_j^{\mathfrak{M}} V_\ell^n$ of the local monodromy filtration \mathfrak{M}_\bullet on V_ℓ^n . By Cor.2.4, we know that the characteristic polynomial of $\rho_\ell^n(w) \in \mathbb{Q}[T]$ and is independent of ℓ . We also know from [27], that WMC holds for V_ℓ^n , i.e., for every $w \in W_v$, the eigenvalues of $\overline{\rho}_{\ell,j}^n(w)$ are all algebraic integers of complex absolute value $q_v^{(n+j)\alpha(w)/2}$. It follows, as before, that the character of the representations $\overline{\rho}_{\ell,j}^n$ has values in \mathbb{Q} and is independent of ℓ . We now apply [8, Prop.8.9] to conclude the proof in this case. \square

Corollary 3.13. *Let X be a smooth complete intersection of odd dimension $n = 2m + 1$ and of Hodge level 1, defined over F . Then $C_{WD}(X_{F_v}, i)$ holds true for all i .*

Proof. In view of Thm. 1.9, we only need to establish that for $i = n$, Φ_v acts semi-simply on V_ℓ^n . By a result of Deligne [7], we know that there is an abelian variety $J(X)$ (intermediate Jacobian of X) defined over F and an isomorphism of Γ -modules, $H_{\mathrm{et}}^n(X_{\bar{F}}, \mathbb{Q}_\ell)(m) \cong H_{\mathrm{et}}^1(J(X)_{\bar{F}}, \mathbb{Q}_\ell)$. Thus our claim follows from the corresponding fact for $H_{\mathrm{et}}^1(J(X)_{\bar{F}}, \mathbb{Q}_\ell)$. \square

4. CONJUGACY CLASS OF FROBENIUS

The principal aim of this section is to show Theorem 1.11, but for this we shall need some preliminaries.

4.1. The Algebraic Group $H^{\natural ad}$. Let H be a reductive algebraic group over a field K of characteristic 0 and \bar{K} an algebraic closure of K . The adjoint action of H^{ad} on H gives a universal categorical quotient $(\mathrm{Conj}(H), \mathrm{Cl})$, where $\mathrm{Conj}(H)$ is an algebraic variety over K and $\mathrm{Cl} : H \rightarrow \mathrm{Conj}(H)$ is the quotient map. The pair $(\mathrm{Conj}(H), \mathrm{Cl})$ is also the universal categorical quotient for the action of H on itself by conjugation.

The derived group $H_{\bar{K}}^{\mathrm{der}}$ is the almost direct product of almost simple subgroups H_i , for $i \in I$ a finite indexing set. If $H_{\bar{K}}^{\mathrm{der}}$ has any almost direct factor of type $\mathrm{SO}(2k_i)_{\bar{K}}$, then there is an action which extends the above action. We briefly recall this. For details see [18] and [21].

Let $J \subset I$ such that for $i \in J$, $H_i \cong \mathrm{SO}(2k_i)_{\bar{K}}$ with $k_i \geq 4$ and set $H_i^\natural := \mathrm{O}(2k_i)_{\bar{K}}$. Define an algebraic group

$$H^{\natural ad} = \prod_{i \in J} H_i^{\natural ad} \times \prod_{i \notin J} H_i^{ad} \supset H_{\bar{K}}^{ad}.$$

There is a natural action of $H^{\natural ad}$ on $H_{\bar{K}}$, which extends the adjoint action on $H_{\bar{K}}^{\mathrm{der}}$ and is trivial on the center of $H_{\bar{K}}$. Denote by $\mathrm{Conj}'(H)_{\bar{K}}$ the categorical quotient of $H_{\bar{K}}$ under this action of $H^{\natural ad}$. Now, using the properties of categorical quotients, we can descend from

$\text{Conj}'(H)_{\bar{K}}$ to an algebraic variety $\text{Conj}'(H)$ over K . We also get a quotient map (as above), which we again denote by

$$\text{Cl} : H \rightarrow \text{Conj}'(H).$$

Now, let $\Omega \supseteq K$ be any algebraically closed field. From the functorial properties of the above action we deduce an *adjoint map*

$$\text{Ad} : H^{\text{ad}}(\Omega) \rightarrow \text{Aut}(\mathfrak{h} \otimes \Omega),$$

where \mathfrak{h} denotes the Lie algebra of H .

Weakly neat elements. Fix a faithful K -linear representation V of H . A semisimple element $g \in H(\Omega)$ is said to be *weakly neat* if 1 is the only root of unity among the quotients $\lambda\mu^{-1}$, with λ, μ being the eigenvalues of g .

For any positive integer n we denote by $\varphi_n : H \rightarrow H$ the n -th power map. Let Y/K denote any one of the algebraic varieties $\text{Conj}(H)$ or $\text{Conj}'(H)$. The map φ_n is equivariant for the action of conjugation. Since Y is a universal categorical quotient, this implies that φ_n induces a map $\bar{\varphi}_n : Y \rightarrow Y$ such that $\text{Cl} \circ \varphi_n = \bar{\varphi}_n \circ \text{Cl}$. The proof of the following lemma (see for example [18, 2.7]) is left to the reader.

Lemma 4.2. *Let $\Omega \supseteq K$ be an algebraically closed field. Let V be a finite dimensional faithful K -linear representation of H and $\alpha, \beta \in H(\Omega)$ be two weakly neat elements having same characteristic polynomial in the representation V . If $\bar{\varphi}_n(\text{Cl}(\alpha)) = \bar{\varphi}_n(\text{Cl}(\beta))$ for some positive integer n , then $\text{Cl}(\alpha) = \text{Cl}(\beta)$.*

4.3. Representations of Weil-Deligne groups with values in algebraic groups. We generalize some of the notions about linear representations of Weil-Deligne groups, described in §3.2.

Let H be an algebraic group over \mathbb{Q} and $\xi_\ell : \Gamma_v \rightarrow H(\mathbb{Q}_\ell)$ be an ℓ -adic representation. By Grothendieck's ℓ -adic monodromy theorem, there exists a nilpotent element $N'_\ell \in \text{Lie}(H/\mathbb{Q}_\ell)(-1)$ and an open subgroup $J \subseteq I_v$ such that for $\lambda \in J$, we have $\xi_\ell(\lambda) = \exp(N'_\ell t_\ell(\lambda))$. Now, by using the identification $\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(1)$, we interpret N'_ℓ as an element of $\text{Lie}(H/\mathbb{Q}_\ell)$ and we define as in §3.2,

$$(8) \quad \xi'_\ell : W_v \rightarrow H(\mathbb{Q}_\ell), \quad w \mapsto \xi_\ell(w) \exp(-N'_\ell t_\ell(\Phi_v^{-\alpha(w)} w)).$$

A triple $(H/\mathbb{Q}_\ell, \xi'_\ell, N'_\ell)$ as above is called a *representation of the Weil-Deligne group* $'W_v$ of F_v with values in H/\mathbb{Q}_ℓ . For a fixed ℓ , we say that $(H/\mathbb{Q}_\ell, \xi'_\ell, N'_\ell)$ is *rational over \mathbb{Q}* (as a representation with values in H), if for every algebraically closed field $\Omega \supset \mathbb{Q}_\ell$ and for every $\sigma \in \text{Aut}_{\mathbb{Q}}(\Omega)$, there exists a $g \in H(\Omega)$ such that

$$(9) \quad \sigma \xi'_{\ell/\Omega} = g \cdot \xi'_{\ell/\Omega} \cdot g^{-1} \quad \text{and} \quad N'_{\ell} \otimes_{\sigma, \mathbb{Q}_\ell} 1 = \text{Ad}(g)(N'_{\ell} \otimes_{\mathbb{Q}_\ell} 1),$$

where $\xi'_{\ell/\Omega}$ denote the extension of scalars $\xi'_\ell \otimes_{\mathbb{Q}_\ell} \Omega : W_v \rightarrow H/\Omega(\Omega)$ and $N'_{\ell} \otimes_{\mathbb{Q}_\ell} 1$ is the image of N'_ℓ in $(\text{Lie}(H) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \Omega$.

If in addition to (9), we have for every pair (ℓ, ℓ') of prime numbers (different from p) and every algebraically closed field $\Omega \supset \mathbb{Q}_\ell, \mathbb{Q}_{\ell'}$, an element $g \in H(\Omega)$ such that

$$(10) \quad \xi'_{\ell/\Omega} = g \cdot \xi'_{\ell'/\Omega} \cdot g^{-1} \quad \text{and} \quad N'_{\ell} \otimes_{\mathbb{Q}_\ell} 1 = \text{Ad}(g)(N'_{\ell'} \otimes_{\mathbb{Q}_{\ell'}} 1),$$

then we say that the representations $(H/\mathbb{Q}_\ell, \rho^i_{\ell}, N'_\ell)$, form a *compatible system of representations of* $'W_v$ (with values in H).

Note that, as the action of $H(\Omega)$ factors through $H(\Omega) \rightarrow H^{\text{ad}}(\Omega)$, $H(\Omega)$ -conjugacy may be replaced by $H^{\text{ad}}(\Omega)$ -conjugacy everywhere. Now, suppose that H is a reductive algebraic

group, then as above we can define the algebraic group H^{bad} . To say that the representation $(H/\mathbb{Q}_\ell, \xi'_\ell, N'_\ell)$ is *rational over \mathbb{Q} modulo the action of H^{bad}* , means that in (9) we require that $g \in H^{\text{bad}}(\Omega)$. In a similar manner we make sense of the phrase *compatible system of representations of $'W_v$ modulo the action of H^{bad}*

4.4. Proof of Proposition 1.10 & Theorem 1.11.

Proof of 1.10. Let $X = \prod_r X_r$, where each X_r is one of the varieties of the the type mentioned in the statement of the proposition. As seen in the proof of Lemma 3.9, $h(X_{r,\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$, for each r . As $h(X_{\bar{F}}) = \otimes_r h(X_{r,\bar{F}})$ and $\mathcal{M}_{AH}^{av}(\bar{F})$ is closed under tensor products, $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$.

Write $M := h^i(X)$; the algebraic groups $G_{M_{\bar{F}}}$ and G_M are stabilizers of all absolute Hodge classes (AHC) and AHC that are defined over F , respectively, in all tensor spaces formed by $V^i, V^{i^\vee}, \mathbb{Q}(1)$ and $\mathbb{Q}(1)^\vee$ (see [11, I, Principle A]). So, it follows from hypothesis that $G_{M_{\bar{F}}} = G_M$.

Now the Mumford-Tate group G_∞^i of the Hodge structure V^i is the stabilizer of Hodge classes relative to the embedding τ , in all tensor spaces formed by $V^i, V^{i^\vee}, \mathbb{Q}(1)$ and $\mathbb{Q}(1)^\vee$. As $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$, all Hodge classes relative to τ are AHC (see [11, II, 6.27]). This implies that $G_\infty^i = G_{M_{\bar{F}}} = G_M$.

Finally, as $\rho_{M,\ell} : \Gamma_F \rightarrow \text{GL}(H_\ell(M))$ factorizes through $G_M(\mathbb{Q}_\ell)$ and $\rho_{M,\ell} = \rho_\ell^i$, we conclude that G_ℓ^i is a subgroup of $G_{\infty/\mathbb{Q}_\ell}^i$. □

Remark 4.5. By [11, I, 2.9] there exists number fields verifying the hypothesis of Proposition 1.10.

Now, let X be as in Theorem 1.10. Then, the triple $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{\ell}^{i'}, N'_{i,\ell})$ is a representation of the Weil-Deligne group $'W_v$ of F_v , with values in $G_{\infty/\mathbb{Q}_\ell}^i$.

For any finite extension F' of F we denote by $\Gamma_{F'}$ the absolute Galois group and by v' an extension of v to F' . Let $I_{v'}, W_{v'}$ and $'W_{v'}$ denote the inertia, Weil and Weil-Deligne group of $F'_{v'}$, respectively. We also have a group homomorphism $\alpha' : W_{v'} \rightarrow \mathbb{Z}$ as in §1. By $\Phi_{v'}$ we mean an arithmetic Frobenius for this extension.

If $X_{F'}$ denotes base change of X to F' , then the ℓ -adic representation $\rho_\ell^i|_{\Gamma_{F'}}$ associated to $X_{F'}$ induces a representation $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{\ell}^{i'}|_{\Gamma_{F'}}, N'_{i,\ell})$ of the Weil-Deligne group $'W_{v'}$ of $F'_{v'}$, with values in $G_{\infty/\mathbb{Q}_\ell}^i$.

Remark 4.6. Although Theorem 1.11 is similar in spirit to [18, Theorem 1.1 & corollary 1.3], there are two important differences between [loc.cit.] and our case, which are as following :

- (a) here we only know that there is a finite extension F' of F such that the motive $h^i(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$ (by Lemma 3.9), while in [loc.cit.] we assumed this property over the base field F ;
- (b) in part (i) (resp. (ii)) of Theorem 1.11 we just need that $\rho_\ell^i|_{\Gamma_v}$ is potentially unramified (resp. semi-stable) representation for every $\ell \neq p$, while in [loc.cit.] a stronger condition was used namely X has good (resp. semi-stable) reduction at v .

Proof of Theorem 1.11. (i) The principal idea of the proof is similar to that of [18, Thm. 1.1], so we only present the main arguments and sketch the rest. As before let us denote by M the motive $h^i(X)$. Let F' be a finite extension of F such that $M_{F'} := h^i(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$, as in Lemma 3.9. Let v' be an an extension of the valuation v to F' and $\Phi_{v'}$ an arithmetic Frobenius for this extension. Now, we have seen in the proof of Prop. 1.10 that $G_\infty^i = G_{M_{\bar{F}}} = G_M$, in particular $G_{M_{\bar{F}}} = G_{M_{F'}}$. We also have

$\text{Conj}'(G_\infty^i) = \text{Conj}'(G_{M_{F'}}) = \text{Conj}'(G_M)$ and $\rho_{M_{F'}/\ell} = \rho_\ell^i|_{\Gamma_{v'}}$. Now, using the fact that $\rho_\ell^i|_{\Gamma_v}$ is potentially unramified for all $\ell \neq p$, it can be checked (as at the beginning of the proof of [ibid.]) that there exists an element² $\text{Frob}_{v',i} \in \text{Conj}'(G_\infty^i)(\mathbb{Q})$ such that,

$$(11) \quad \text{Cl}(\rho_\ell^i(\Phi_{v'})) = \text{Frob}_{v',i}, \forall \ell \neq p.$$

Now, let the residual degree of the extension $F_{v'}/F_v$ be n . We denote by $\varphi_n : G_\infty^i \rightarrow G_\infty^i$ the n -th power map and by $\bar{\varphi}_n : \text{Conj}'(G_\infty^i) \rightarrow \text{Conj}'(G_\infty^i)$ the induced map on $\text{Conj}'(G_\infty^i)$. Then it follows easily from (11) and the fact that $\bar{\varphi}_n$ is a finite morphism, that

$$\text{Cl}(\rho_\ell^i(\Phi_v)) \in \text{Conj}'(G_\infty^i)(\bar{\mathbb{Q}}), \forall \ell \neq p.$$

Now, by hypothesis there exists a ℓ_0 such that $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat. So, by Corollary 2.4, $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat for every $\ell \neq p$. Again, Corollary 2.4 and a repeated use of Lemma 4.2, shows that $\text{Cl}(\rho_\ell^i(\Phi_v))$ (as an element of $\text{Conj}'(G_\infty^i)(\bar{\mathbb{Q}})$) is independent of ℓ . We denote this element by $\text{Frob}_{v,i}$. As $\text{Frob}_{v,i} \in \text{Conj}'(G_\infty^i)(\mathbb{Q}_\ell)$ for every $\ell \neq p$, $\text{Frob}_{v,i}$ is a \mathbb{Q} -rational point of $\text{Conj}'(G_\infty^i)$.

- (ii) Now let the notation be as in part (i). By, applying [18, 3.9] to the motive $M_{F'}$, we see that for $\ell \neq p$ the representations $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_\ell^{i'}|_{\Gamma_{F'}}, N_{i,\ell}')$ are rational over \mathbb{Q} and form a compatible system of representations of $'W_v$, modulo the action of $G_\infty^{i \text{ bad}}$. In particular by setting $\Phi_{v'} := \Phi_v^n$, we get that for any fixed $\ell \neq p$, algebraically closed field $\Omega \supset \mathbb{Q}_\ell$ and $\sigma \in \text{Aut}_{\mathbb{Q}}(\Omega)$, there exists a $g_\sigma \in G_\infty^{i \text{ bad}}(\Omega)$ such that

$$(12) \quad \sigma \rho_{\ell/\Omega}^{i'}(\Phi_{v'}^n) = g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_{v'}^n)) \cdot g_\sigma^{-1} \text{ and } N_{i,\ell}' \otimes_{\sigma, \mathbb{Q}_\ell} 1 = \text{Ad}(g_\sigma)(N_{i,\ell}' \otimes_{\mathbb{Q}_\ell} 1).$$

Thus we have $(\sigma \rho_{\ell/\Omega}^{i'}(\Phi_{v'}^n))^n = (g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_{v'}^n)) \cdot g_\sigma^{-1})^n$. Now it follows from Corollary 2.4 that $\sigma \rho_{\ell/\Omega}^{i'}(\Phi_{v'}^n)$ and $g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_{v'}^n)) \cdot g_\sigma^{-1}$ have the same characteristic polynomial. It now follows from a property of weakly neat elements (cf. [22, 7.4]), that $\sigma \rho_{\ell/\Omega}^{i'}(\Phi_{v'}^n) = g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_{v'}^n)) \cdot g_\sigma^{-1}$.

As the monodromy operator $N_{i,\ell}'$ is unchanged by finite base extension, it now follows from (12) that $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_\ell^{i'}, N_{i,\ell}')$ is rational over \mathbb{Q} modulo the action of $G_\infty^{i \text{ bad}}$.

The compatibility condition is checked in a similar manner. □

5. CONJUGACY CLASS OF CRYSTALLINE FROBENIUS

We follow the notation of §1 and we introduce some more notation. The ring of Witt vectors of k_v is denoted by $W(k_v)$ and $\sigma : W(k_v) \rightarrow W(k_v)$ is the morphism induced by the Frobenius automorphism of k_v . For any finite extension F' of F , we denote by v' an extension of v to F' ; $F_{v'}$ is the completion of F' at v' ; $k_{v'}$ the residue field of $F_{v'}$. The ring of Witt vectors of $k_{v'}$ is denoted by $W(k_{v'})$ and $\sigma' : W(k_{v'}) \rightarrow W(k_{v'})$ is the morphism induced by the Frobenius automorphism of $k_{v'}$. The fraction field of $W(k_{v'})$ is denoted by $F_{v'}^0$. Note that F_v^0 and $F_{v'}^0$ naturally identifies to the maximal unramified extension of \mathbb{Q}_p inside F_v and $F_{v'}$, respectively.

Consider a smooth projective algebraic variety Y over F with potential good reduction at v . Fix a $i \in \mathbb{N}$ and denote by N the motive $h^i(Y) \in \text{Ob}(\mathcal{M}_{AH}(F))$. Assume that the extension of scalars $N_{\bar{F}} \in \text{Ob}(\mathcal{M}_{AH}^{av}(F))$. Thus, by taking a large enough finite extension F' of F , we may assume that Y has good reduction over F' and $N_{F'} \in \text{Ob}(\mathcal{M}_{AH}^{av}(F'))$. Let

²In [18, eqn. 14] this was denoted by $\text{Cl}_M \text{Fr}_{v'}$

$Y_{v'}$ denote the reduction of $Y_{F'}$ at v' and $\langle N_{F'}, \mathbb{Q}(1) \rangle^{\otimes}$ denote the Tannakian subcategory of $\mathcal{M}_{AH}(F')$, \otimes -generated by $N_{F'}$ and the Tate motive. It is well known from the C_{Cris} -conjecture (cf. [14, Th.3.2.3]) that $H_p(N_{F'}) = H_{\text{et}}^i(Y_{\bar{F}}, \mathbb{Q}_p)$ is a crystalline representation of $\Gamma_{v'}$ and its associated filtered Dieudonné module is isomorphic to $H_{\text{Cris}}^i(Y_{v'}) \otimes F_{v'}^0$. This allows us to define the crystalline realization (cf. [31, §4.1]), i.e., a fibre functor

$$H_{N_{F'}, \text{Cris}} : \langle N_{F'}, \mathbb{Q}(1) \rangle^{\otimes} \rightarrow \text{Vect}_{F_{v'}^0},$$

which associates to any motive $L \in \langle N_{F'}, \mathbb{Q}(1) \rangle^{\otimes}$ its associated filtered Dieudonné $F_{v'}$ -module $(H_p(L) \otimes_{\mathbb{Q}_p} B_{\text{Cris}})^{\Gamma_{v'}}$, where B_{Cris} is the $F_{v'}^0[\Gamma_{v'}]$ -algebra introduced by Fontaine (cf. [12]). We call this functor the *crystalline realization* for the subcategory of motives $\langle N_{F'}, \mathbb{Q}(1) \rangle^{\otimes}$. We write $G_{N_{F'}, \text{Cris}} := \underline{\text{Aut}}^{\otimes}(H_{N_{F'}, \text{Cris}})$ for the automorphism group of the fibre functor $H_{N_{F'}, \text{Cris}}$.

The absolute Frobenius endomorphism Fr_{abs} of $Y_{v'}$ induces a σ' -linear endomorphism $Fr_{\text{Cris}} : H_{\text{Cris}}^i(Y_{v'}/W(k_{v'})) \rightarrow H_{\text{Cris}}^i(Y_{v'}/W(k_{v'}))$, on crystalline cohomology groups of $Y_{v'}$. Now, let $|k_{v'}| = p^{s'}$, then $Fr_{\text{Cris}}^{s'}$ is a $W(k_{v'})$ -linear endomorphism of $H_{\text{Cris}}^i(Y_{v'}/W(k_{v'}))$. Let us denote by

$$\Phi_{\text{Cris}} : H_{\text{Cris}}^i(Y_{v'}/W(k_{v'})) \otimes F_{v'}^0 \rightarrow H_{\text{Cris}}^i(Y_{v'}) \otimes F_{v'}^0,$$

the *degree i crystalline Frobenius* of $Y_{F'}$ at v' , i.e., the $F_{v'}^0$ -linear endomorphism $Fr_{\text{Cris}}^{s'} \otimes 1$. It follows from a result of Blasius [4, Th.5.3] that Φ'_{Cris} defines an element $\Phi'_{N, \text{Cris}} \in G_{N_{F'}, \text{Cris}}(F_{v'}^0)$.

Now, as before the conjugation action of $G_{N_{F'}, \text{Cris}}$ gives us a universal categorical quotient $(\text{Conj}'(G_{N_{F'}, \text{Cris}}), \text{Cl}_{\text{Cris}})$. We also have the universal categorical quotient $(\text{Conj}'(G_{N_{F'}}), \text{Cl})$ for the conjugation action of $G_{N_{F'}} := \underline{\text{Aut}}^{\otimes}(H_{\tau}|_{\langle N_{F'}, \mathbb{Q}(1) \rangle^{\otimes}})$. For every $\ell \neq p$ the ℓ -adic representation $\rho_{N_{F'}, \ell} : \Gamma_{F'} \rightarrow G_{N_{F'}, \ell}(\mathbb{Q}_{\ell})$ defines an element $\text{Cl}(\rho_{N_{F'}, \ell}(\Phi_{v'})) \in \text{Conj}'(G_{N_{F'}})(\mathbb{Q}_{\ell})$.

Now, it is well known [11, II, Th.3.2] from the general theory of Tannakian categories that $G_{N_{F'}, \text{Cris}}$ is an inner form of $G_{N_{F'}} \otimes_{\mathbb{Q}} F_{v'}^0$. It follows that there is an isomorphism $\text{Conj}'(G_{N_{F'}, \text{Cris}}) \cong (\text{Conj}'(G_{N_{F'}})) \otimes_{\mathbb{Q}} F_{v'}^0$, which allows us to make sense of the following

Theorem 5.1. *Let Y be any smooth projective algebraic variety over F with potential good reduction at a non-archimedean valuation v of F . For any fixed $i \in \mathbb{N}$ write $N := h^i(Y)$ and assume that $N_{\bar{F}} \in \text{Ob}(\mathcal{M}_{AH}^{av}(\bar{F}))$. Then, there exists a finite extension F' of F and a unique conjugacy class*

$$\text{Frob}'_N \in \text{Conj}'(G_{N_{F'}})(F_{v'}^0),$$

such that $\text{Cl}_{\text{Cris}}(\Phi'_{N, \text{Cris}}) = \text{Frob}'_N$ and $\text{Cl}(\rho_{N_{F'}, \ell}(\Phi_{v'})) = \text{Frob}'_N, \forall \ell \neq p$.

Proof of Theorem 5.1. We shall first find a finite extension of F and then show that it is the right one. To avoid confusion, we shall denote all successive extensions of F by F' .

Recall, as in the proof of part (i) of Thm. 1.8, there is a finite extension F' of F , an abelian variety A over F' such that $N_{F'} \in \langle h^1(A), \mathbb{Q}(1) \rangle^{\otimes}$. Further, this induces a unique homomorphism of algebraic groups $\theta : G_A \rightarrow G_{N_{F'}}$ satisfying $\theta(\mathbb{Q}_{\ell}) \circ \rho_{A, \ell} = \rho_{N_{F'}, \ell}$ for every prime number ℓ (where $\theta(\mathbb{Q}_{\ell})$ is the induced map on \mathbb{Q}_{ℓ} -valued points).

Lemma 5.2. *There exists an abelian variety A over F' as above with potential good reduction at v' .*

Proof. First we note that for any abelian variety A , by Grothendieck's ℓ -adic monodromy theorem, $I_{v'}$ (possibly after passing over to another finite extension of F) acts unipotently on $H_{\text{et}}^1(A_{\bar{F}}, \mathbb{Q}_{\ell}) \cong W \otimes \mathbb{Q}_{\ell}$, where $W := H_B^1(\tau A, \mathbb{Q})$. As Y has potential good reduction at v , we may assume that $I_{v'}$ acts trivially on $H_{\ell}(N)$. Now, by [18, Lemma 3.10] we may choose A such that $G_A^{\text{ad}} \cong G_{N_{F'}}^{\text{ad}}$. This combined with the fact that $\theta(\mathbb{Q}_{\ell}) \circ \rho_{A, \ell} = \rho_{N_{F'}, \ell}$,

implies that $I := \rho_{A,\ell}(I_{v'}) \subseteq \mathcal{Z}_A(\mathbb{Q}_\ell)$, where \mathcal{Z}_A denotes the centre of G_A . Fix an algebraic closure Ω of \mathbb{Q}_ℓ . For any $\sigma \in I \subseteq \mathcal{Z}_A(\mathbb{Q}_\ell)$, denote by $\bar{\sigma}$ the image of σ in $\mathcal{Z}_{A/\Omega}(\Omega)$. As G_A is reductive, $\mathcal{Z}_{A/\Omega}$ is a diagonalizable algebraic group. Since G_A is a subgroup of $\mathrm{GL}(W)$, there exists a matrix $P \in \mathrm{GL}_m(\Omega)$ such that $P \cdot \bar{\sigma} \cdot P^{-1}$ is of the form $\mathrm{diag}(\lambda_1, \dots, \lambda_m)$, where $m := \dim W$. As $\bar{\sigma}$ is unipotent, $\lambda_1 = \dots = \lambda_m = 1$, i.e. $\bar{\sigma}$ is the identity matrix. Since $\sigma \in I$ was an arbitrary element, I is trivial. Now, using the Néron-Ogg-Shafarevich condition [30, Thm. 1], we conclude that A has potential good reduction at v' . \square

End of the proof of Theorem 5.1: By passing over to another finite extension we may suppose that A has good reduction at v' . Now, we have the crystalline realization

$$H_{A,\mathrm{Cris}} : \langle h^1(A), \mathbb{Q}(1) \rangle^\otimes \rightarrow \mathrm{Vect}_{F_{v'}^0}$$

and we write $G_{A,\mathrm{Cris}} := \underline{\mathrm{Aut}}^\otimes(H_{A,\mathrm{Cris}})$. As before, the crystalline Frobenius element for A at v' defines an element $\Phi_{A,\mathrm{Cris}} \in G_{A,\mathrm{Cris}}(F_{v'}^0)$. By passing over to another finite extension we may suppose that $\Phi_{A,\mathrm{Cris}}$ is weakly neat. Again, as $G_{A,\mathrm{Cris}}$ is an inner form of $G_A \otimes_{\mathbb{Q}} F_{v'}^0$, there is an isomorphism $\mathrm{Conj}'(G_{A,\mathrm{Cris}}) \cong (\mathrm{Conj}'(G_A)) \otimes_{\mathbb{Q}} F_{v'}^0$. For our purpose we identify these two varieties. Now, by [21, 4.2] we know that there is an element $\mathrm{Cl}_A \mathrm{Fr}_v \in \mathrm{Conj}'(G_A)(\mathbb{Q})$ such that $\mathrm{Cl}(\rho_{A,\ell}(\Phi_{v'})) = \mathrm{Cl}_A \mathrm{Fr}_v \forall \ell \neq p$ and its image in $\mathrm{Conj}'(G_{A,\mathrm{Cris}})(F_v^0)$ coincides with $\mathrm{Cl}_{\mathrm{Cris}}(\Phi_{A,\mathrm{Cris}})$. Let us write $\mathrm{Frob}_A \in \mathrm{Conj}'(G_A)(F_{v'}^0) = \mathrm{Conj}'(G_{A,\mathrm{Cris}})(F_{v'}^0)$, for this image.

By universal property of (universal) categorical quotients we have the following commutative diagram:

$$(13) \quad \begin{array}{ccc} G_A & \xrightarrow{\mathrm{Cl}} & \mathrm{Conj}' G_A \\ \theta \downarrow & & \downarrow \bar{\theta} \\ G_{N_{F'}} & \xrightarrow{\mathrm{Cl}} & \mathrm{Conj}' G_{N_{F'}} \end{array}$$

Again, the inclusion functor $\langle N_{F'}, \mathbb{Q}(1) \rangle^\otimes \rightarrow \langle h^1(A), \mathbb{Q}(1) \rangle^\otimes$, induces a unique homomorphism

$$\theta_{\mathrm{cris}} : G_{A,\mathrm{Cris}} \rightarrow G_{N_{F'},\mathrm{Cris}}.$$

As before, this gives us the following commutative diagram :

$$(14) \quad \begin{array}{ccc} G_{A,\mathrm{Cris}} & \xrightarrow{\mathrm{Cl}_{\mathrm{Cris}}} & \mathrm{Conj}' G_{A,\mathrm{Cris}} \\ \theta_{\mathrm{cris}} \downarrow & & \downarrow \bar{\theta}_{\mathrm{Cris}} \\ G_{N_{F'},\mathrm{Cris}} & \xrightarrow{\mathrm{Cl}_{\mathrm{Cris}}} & \mathrm{Conj}' G_{N_{F'},\mathrm{Cris}} \end{array}$$

We set $\mathrm{Frob}'_N := \mathrm{Cl}_{\mathrm{Cris}}(\Phi'_{N,\mathrm{Cris}}) \in \mathrm{Conj}'(G_{N_{F'},\mathrm{Cris}})(F_{v'}^0)$. We shall show that

$$\mathrm{Cl}(\rho_{N_{F'},\ell}(\Phi_{v'})) = \mathrm{Frob}'_N \text{ for every } \ell \neq p.$$

By functoriality, $\theta_{\mathrm{cris}}(F_{v'}^0)(\Phi_{A,\mathrm{Cris}}) = \Phi'_{N,\mathrm{Cris}}$, where $\theta_{\mathrm{cris}}(F_{v'}^0)$ is the induced map on $F_{v'}^0$ -valued points. Thus, by commutativity of (14) we obtain $\mathrm{Frob}'_N = \bar{\theta}_{\mathrm{Cris}}(F_{v'}^0)(\mathrm{Frob}_A)$. Now, as $\theta(\mathbb{Q}_\ell) \circ \rho_{A,\ell} = \rho_{N_{F'},\ell}$ and $\mathrm{Cl}(\rho_{A,\ell}(\Phi_{v'})) = \mathrm{Frob}_A$ for all $\ell \neq p$, it follows that $\mathrm{Cl}(\rho_{N_{F'},\ell}(\Phi_{v'})) = \bar{\theta}(F_{v'}^0)(\mathrm{Frob}_A)$ as elements of $\mathrm{Conj}'(G_{N_{F'}})(F_{v'}^0) = \mathrm{Conj}'(G_{N_{F'},\mathrm{Cris}})(F_{v'}^0)$. Thus the proposition will be proved if we can show that $\mathrm{Frob}'_N = \bar{\theta}(F_{v'}^0)(\mathrm{Frob}_A)$. In other words we need to show that $\bar{\theta}_{\mathrm{Cris}}(F_{v'}^0)(\mathrm{Frob}_A) = \bar{\theta}(F_{v'}^0)(\mathrm{Frob}_A)$. This follows from the next lemma. \square

Lemma 5.3. *The morphism $\bar{\theta}_{/F_{v'}^0} := \bar{\theta} \otimes_{\mathbb{Q}} \text{id} : \text{Conj}'(G_A) \otimes_{\mathbb{Q}} F_{v'}^0 \rightarrow \text{Conj}'(G_{N_{F'}}) \otimes_{\mathbb{Q}} F_{v'}^0$ coincides with $\bar{\theta}_{\text{Cris}}$.*

Proof. As $G_{*,\text{Cris}}$ is an inner form of $G_* \otimes_{\mathbb{Q}} F_{v'}^0$, where $*$ = $N_{F'}$ or A we can identify $G_{*,\text{Cris}} \otimes_{F_{v'}^0} \bar{\mathbb{Q}}_p$ with $G_* \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p$. Then, under this identification and by definition of θ and θ_{Cris} , it follows that the homomorphisms of algebraic groups

$$(15) \quad \theta_{/\bar{\mathbb{Q}}_p} := \theta \otimes_{\mathbb{Q}} \text{id} : G_A \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \rightarrow G_{N_{F'}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \text{ and} \\ \theta_{\text{Cris}/\bar{\mathbb{Q}}_p} := \theta_{\text{Cris}} \otimes_{F_{v'}^0} \text{id} : G_{A,\text{Cris}} \otimes_{F_{v'}^0} \bar{\mathbb{Q}}_p \rightarrow G_{N_{F'},\text{Cris}} \otimes_{F_{v'}^0} \bar{\mathbb{Q}}_p, \text{ are identical.}$$

It follows from (15), that $\bar{\theta}_{/\bar{\mathbb{Q}}_p} = \bar{\theta}_{\text{Cris}/\bar{\mathbb{Q}}_p}$. As $G_{A,\text{Cris}}$ is geometrically reducible (as algebraic variety over $F_{v'}^0$), so is $\text{Conj}'(G_{A,\text{Cris}})$. Now, it is a well known fact that if two morphisms $f, g : Y \rightarrow Z$ of algebraic varieties Y and Z are identical over the algebraic closure of the base field and Y is geometrically reducible, then f and g are identical. By using this fact we conclude that $\bar{\theta}_{/F_{v'}^0} = \bar{\theta}_{\text{Cris}}$. \square

Proof of Theorem 1.12. By applying Theorem 5.1 to X we obtain an extension F' of F and an element $\text{Frob}'_M \in \text{Conj}'(G_{M_{F'}})(F_{v'}^0)$. Let n be the residual degree of the $F'_{v'}/F_v$. By hypothesis G_M is connected, which implies that it is geometrically connected because G_M is an algebraic group. In particular $G_M \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \cong G_{M,\text{Cris}} \otimes_{F_{v'}^0} \bar{\mathbb{Q}}_p$ is connected. This implies that $G_{M,\text{Cris}}$ is connected. Now, all Hodge cycles relative to the embedding $\tau : F \hookrightarrow \mathbb{C}$ on M are absolute Hodge cycles, because $M \in \text{Ob}(\mathcal{M}_{AH}^{av}(F))$. This implies that $G_M^0 = G_{M_{\bar{F}}}$ (see [11, II, 6.23]). Thus, in our situation $G_M = G_{M_{\bar{F}}} = G_{M_{F'}}$ and similarly $G_{M_{F'},\text{Cris}} = G_{M,\text{Cris}}$. We also get $\text{Conj}'(G_{M_{F'}}) = \text{Conj}'(G_M)$ and $\text{Conj}'(G_{M_{F'},\text{Cris}}) = \text{Conj}'(G_{M,\text{Cris}})$.

As the crystalline Frobenius element for $X_{F'}$ at v' is the n -th power of the crystalline Frobenius element for X at v , so $(\Phi_{M,\text{Cris}})^n = \Phi'_{M,\text{Cris}}$ (as elements of $G_{M,\text{Cris}}(F_{v'}^0)$).

Now, let $\varphi_n : G_{M,\text{Cris}} \rightarrow G_{M,\text{Cris}}$ be the n -th power map. The map φ_n is equivariant for the action of conjugation. By the universal property of universal categorical quotient, φ_n induces a map $\bar{\varphi}_n : \text{Conj}'(G_{M,\text{Cris}}) \rightarrow \text{Conj}'(G_{M,\text{Cris}})$ such that $\text{Cl}_{\text{Cris}} \circ \varphi_n = \bar{\varphi}_n \circ \text{Cl}_{\text{Cris}}$. Write $\text{Frob}_M := \text{Cl}_{\text{Cris}}(\Phi_{M,\text{Cris}}) \in \text{Conj}'(G_{M,\text{Cris}})(F_v^0) \subset \text{Conj}'(G_{M,\text{Cris}})(F_{v'}^0)$, then it follows from the above discussion that

$$(16) \quad \bar{\varphi}_n(F_{v'}^0)(\text{Frob}_M) = \text{Frob}'_M.$$

By [18, 1.1] there is an element

$$\text{Cl}_M \text{Fr}_v \in \text{Conj}'(G_M)(\mathbb{Q}) \text{ such that } \text{Cl}(\rho_{M,\ell}(\Phi_v)) = \text{Cl}_M \text{Fr}_v \forall \ell \neq p.$$

Using the inclusions

$$\text{Conj}'(G_M)(\mathbb{Q}) \subset \text{Conj}'(G_M)(F_v^0) \cong \text{Conj}'(G_{M,\text{Cris}})(F_v^0) \subset \text{Conj}'(G_{M,\text{Cris}})(F_{v'}^0),$$

we regard $\text{Cl}_M \text{Fr}_v$ as an element of $\text{Conj}'(G_{M,\text{Cris}})(F_{v'}^0)$. By taking $\Phi_{v'} := \Phi_v^n$ for the arithmetic Frobenius for the extension of F' , we have $\text{Cl}(\rho_{M,\ell}(\Phi_v))^n = \text{Cl}(\rho_{M_{F'},\ell}(\Phi_{v'}))$. It follows from this and (16), that $\bar{\varphi}_n(F_{v'}^0)(\text{Cl}_M \text{Fr}_v) = \bar{\varphi}_n(F_{v'}^0)(\text{Frob}_M)$ as elements of $\text{Conj}'(G_{M,\text{Cris}})(F_{v'}^0)$. Since $\text{Cl}_M \text{Fr}_v, \text{Frob}_M \in \text{Conj}'(G_{M,\text{Cris}})(F_v^0)$, we get

$$(17) \quad \bar{\varphi}_n(F_{v'}^0)(\text{Cl}_M \text{Fr}_v) = \bar{\varphi}_n(F_{v'}^0)(\text{Frob}_M) \in \text{Conj}'(G_{M,\text{Cris}})(F_{v'}^0).$$

Now, fix a prime number ℓ and let $\Omega \supset \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell$ be an algebraically closed field. As $M = h^i(X)$, we have $H_\ell(M) = H_{\text{et}}^i(X_{\bar{F}}, \mathbb{Q}_\ell)$. Again by the C_{Cris} conjecture (cf. [14, Th.3.2.3]), we have $H_{M/\text{Cris}}(M) = H_{\text{Cris}}^i(X_v/W(k_v)) \otimes F_v^0$. As X has good reduction at v , we can use Deligne's result on Weil conjectures [9, I.6] and smooth specialization (proper-smooth base change) to conclude that for every $\ell \neq p$, the characteristic polynomial of $\rho_{M,\ell}(\Phi_v)$

(seen as an element of $\mathrm{GL}(H_\ell(M))$) has coefficients in \mathbb{Q} and is independent of ℓ . Next by a result of Katz and Messing [16, Theorem 1], we conclude that $\Phi_{M,\mathrm{Cris}}$ (seen as an element of $\mathrm{GL}(H_{\mathrm{Cris}}(M))$) has the same characteristic polynomial as $\rho_{M,\ell}(\Phi_v)$. This implies in particular that if $\rho_{M,\ell}(\Phi_v)$ is weakly neat then $\Phi_{M,\mathrm{Cris}}$ is also weakly neat. Now we have two weakly neat elements $\rho_{M,\ell}(\Phi_v)$ and $\Phi_{M,\mathrm{Cris}}$ in $G_M(\Omega) \cong G_{M,\mathrm{Cris}}(\Omega)$ which have same characteristic polynomial. Therefore, by Lemma 4.2 and (17) we conclude that $\mathrm{Frob}_M = \mathrm{Cl}_M \mathrm{Frob}_v$. In other words, $\mathrm{Cl}(\rho_{M,\ell}(\Phi_v)) = \mathrm{Frob}_M$. □

By using the previous theorem (and its proof) combined with Lemma 3.9, we easily relate the element $\mathrm{Frob}_{v,i} \in \mathrm{Conj}'(G_\infty^i)(\mathbb{Q})$ of Theorem 1.11 to the conjugacy class of the crystalline Frobenius $\Phi_{\mathrm{Cris},i}$ associated to the motive $h^i(X)$.

Corollary 5.4. *Let the notation and hypothesis be as in Proposition 1.10. Let v be a non-archimedean valuation and $i \in \mathbb{N}$. Assume that X has good reduction at v and there is a prime number ℓ_0 such that $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat. Then, the image of the element $\mathrm{Frob}_{v,i}$ in $\mathrm{Conj}'(G_\infty^i)(F_v^0)$ coincides with $\mathrm{Cl}_{\mathrm{Cris}}(\Phi_{\mathrm{Cris},i})$.*

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