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On Frobenius conjugacy class of some algebraic varieties over number fields

Abhijit Laskar ¹

Abstract

Let X be smooth projective algebraic variety over a number field F and \bar{F} an algebraic closure of F . The action of $\text{Gal}(\bar{F}/F)$ on the ℓ -adic etale cohomology groups $H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell)$, induces Galois representations $\rho_\ell^i : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}(H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell))$. Fix a non-archimedean valuation v on F and let Φ_v be an arithmetic Frobenius element at v . In this article we answer, for many algebraic varieties, classical semisimplicity and ℓ -independence questions related to $\rho_\ell^i(\Phi_v)$ and its conjugacy class in certain natural subgroups of $\text{GL}(H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell))$. We deal with both the good reduction and bad reduction case. We also treat the case where ℓ is equal to the residual characteristic at v , by studying crystalline realization of motives.

1 Introduction and Main results

Consider a smooth projective algebraic variety X over a number field F , with an embedding $\tau : F \hookrightarrow \mathbb{C}$. Fix an algebraic closure \bar{F} of F . The action of $\Gamma_F := \text{Gal}(\bar{F}/F)$ on the ℓ -adic etale cohomology groups $V_\ell^i := H_{et}^i(X/\bar{F}, \mathbb{Q}_\ell)$, induces Galois representations $\rho_\ell^i : \Gamma_F \rightarrow \text{GL}(V_\ell^i)$. Now, fix a non-archimedean valuation v on F and let Φ_v be an arithmetic Frobenius element at v . The aim of this article is to study $\rho_\ell^i(\Phi_v)$ and its conjugacy class in certain natural subgroups of $\text{GL}(V_\ell^i)$, for many algebraic varieties. We shall study both the good reduction and bad reduction case.

We begin with some results of geometric nature. For any complex algebraic variety Y , let $H_B^i(Y(\mathbb{C}), \mathbb{Q})$ denote the Betti cohomology groups of the complex analytic variety $Y(\mathbb{C})$. We shall say that the Lefschetz standard conjecture holds for Y if there exists an algebraic self-correspondence on Y that give an inverse to the operations

$$H_B^i(Y(\mathbb{C}), \mathbb{Q}) \rightarrow H_B^{2 \dim Y - i}(Y(\mathbb{C}), \mathbb{Q})$$

given by the cup product $2 \dim Y - i$ times with a hyperplane section for all $i \leq \dim Y$.

We put $\tau X := X \times_{F, \tau} \mathbb{C}$ and $V^i := H_B^i(\tau X(\mathbb{C}), \mathbb{Q})$. For any element w of the Weil group W_v of the completion F_v , let $\alpha(w)$ denote the corresponding exponent of the Frobenius automorphism of the absolute Galois group of the residue field of F_v . We denote by

$$P_{\ell, v}^i(w, T) := \det(1 - w^* T; V_\ell^i)$$

the characteristic polynomial of the \mathbb{Q}_ℓ -linear map $\rho_\ell^i(w)$.

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Theorem 1.1. *Let X be a smooth projective algebraic variety defined over F , v a non-archimedean valuation on F and $i \in \mathbb{N}$. Assume that the Lefschetz standard conjecture holds for τX . Then for every $\ell \neq p$ and $w \in W_v$ with $\alpha(w) \leq 0$, the polynomial $P_{\ell,v}^i(w, T) \in \mathbb{Z}[T]$ and is independent of ℓ .*

Corollary 1.2. *Let X be a smooth projective variety over F which is a finite product of hyperkähler varieties of $K3^{[n]}$ type; unirational varieties of dimension ≤ 4 ; uniruled varieties of dimension ≤ 3 ; moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves. Then for every $\ell \neq p$ and $w \in W_v$ with $\alpha(w) \leq 0$, the polynomial $P_{\ell,v}^i(w, T) \in \mathbb{Z}[T]$ and is independent of ℓ .*

Corollary 1.2 gives a positive response to a classical conjecture of Serre-Tate [23, Problem 2]. Note that the case where X has good reduction at v , follows from the results of Deligne [8] on Weil Conjectures. But, here we do not make any assumption on the reduction of Y at v . Other (non-trivial) known cases of 1.2 are abelian varieties, curves and surfaces (see [19]).

Theorem 1.3. *Let X be a smooth projective variety over F which is a finite product of hyperkähler varieties of $K3^{[n]}$ type, unirational varieties of dimension ≤ 3 , uniruled surfaces, Fermat hypersurfaces, moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves. For every non-archimedean valuation v and prime number $\ell \neq p$, $\rho_\ell^i(\Phi_v) : V_\ell^i \rightarrow V_\ell^i$ is a semisimple automorphism.*

This gives a positive response to another question of Serre [22, 12.4]. Again the only known case of 1.3 is where $i = 1$ and X is an abelian variety or a curve (cf. [22]).

Now, let $G_\infty^i \subseteq \mathrm{GL}(V^i)$ denote the Mumford-Tate group of the natural Hodge structure on V^i . By using the comparison isomorphism $V^i \otimes \mathbb{Q}_\ell \cong V_\ell^i$ between Betti and ℓ -adic cohomology, we identify $G_{\infty/\mathbb{Q}_\ell}^i$ to an algebraic subgroup of $\mathrm{GL}(V_\ell^i)$. Let G_ℓ^i be the global algebraic monodromy group defined as the Zariski closure of $\rho_\ell^i(\Gamma_F)$ in $\mathrm{GL}(V_\ell^i)$.

Proposition 1.4. *Let X be as in Theorem 1.3. Fix $i \in \mathbb{N}$ and assume that all absolute Hodge cycles in all tensor spaces formed by $V^i, V^{i^\vee}, \mathbb{Q}(1)$ and $\mathbb{Q}(1)^\vee$, are defined over F . Then, G_ℓ^i is subgroup of $G_{\infty/\mathbb{Q}_\ell}^i$ for every ℓ .*

The Proposition above pertains to the *Mumford-Tate conjecture*, which claims that $G_{\infty/\mathbb{Q}_\ell}^i = G_\ell^{i^\circ}$, for any smooth projective variety over F .

Now, if X is as above, then it follows from Theorem 1.2 that the conjugacy class of $\rho_\ell^i(\Phi_v)$ in $\mathrm{GL}(V^i)$ is rational over \mathbb{Q} and independent of ℓ , in the sense of [21, I, §2.4]. By, using Proposition 1.4, we can get deeper results concerning the conjugacy class of $\rho_\ell^i(\Phi_v)$, which we now describe.

As G_∞^i is a reductive algebraic group, we can construct the universal categorical quotient $(\mathrm{Conj}(G_\infty^i), \mathrm{Cl})$ for the action of G_∞^i on itself by conjugation. The variety $\mathrm{Conj}(G_\infty^i)$ is defined over \mathbb{Q} and $\mathrm{Cl} : G_\infty^i \rightarrow \mathrm{Conj}(G_\infty^i)$ denotes the canonical quotient map. For any algebraically closed field $\Omega \supset \mathbb{Q}$, the map Cl induces a bijection between $\mathrm{Conj}(G_\infty^i)(\Omega)$ and the semi-simple conjugacy classes of $G_\infty^i(\Omega)$. If we are in the situation of Proposition 1.4, then $\rho_\ell^i(\Phi_v)$ defines an element $\mathrm{Cl}(\rho_\ell^i(\Phi_v)) \in \mathrm{Conj}(G_\infty^i)(\mathbb{Q}_\ell)$ for every $\ell \neq p$.

We also note that by using Grothendieck's ℓ -adic monodromy theorem we can associate to ρ_ℓ^i , a representation of $'W_v$ with values in $G_{\infty/\mathbb{Q}_\ell}^i$ (in the sense of [7, 8.11]). We denote this representation by $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_\ell^{i'}, N_{i,\ell}')$ (see §3 for details), where $\rho_\ell^{i'} : W_v \rightarrow G_\infty^i(\mathbb{Q}_\ell) \subset \mathrm{GL}(V_\ell^i)$ is a representation of the Weil-group W_v of F_v trivial on an open subgroup of the inertia group and $N_{i,\ell}'$ is the corresponding *monodromy operator*. The notion of rationality and compatibility for representations of $'W_v$ with values in an algebraic group has been defined in [7, 8.11].

Theorem 1.5. *Let the notations and hypotheses be as in Proposition 1.4 and v be a non-archimedean valuation on F .*

- (i) *If $\rho_\ell^i|_{\Gamma_v}$ is potentially unramified representation for every $\ell \neq p$ and $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat for some prime number $\ell_0 \neq p$, then there exists a unique conjugacy class*

$$\mathrm{Frob}_{v,i} \in \mathrm{Conj}'(G_\infty^i)(\mathbb{Q})$$

such that $\mathrm{Cl}(\rho_\ell^i(\Phi_v)) = \mathrm{Frob}_{v,i}, \forall \ell \neq p$.

- (ii) *The representation $\rho_\ell^{i'}$ is a semisimple representation of W_v . If $\rho_\ell^i|_{\Gamma_v}$ is a semi-stable representation for every $\ell \neq p$ and $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat for some prime number $\ell_0 \neq p$, then the representations $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_\ell^{i'}, N_{i,\ell}')$ are rational over \mathbb{Q} and form a compatible system of representations of $'W_v$ modulo the action of $G_\infty^{i \text{ had}}$.*

Roughly speaking, $(\mathrm{Conj}'(G_\infty^i), \mathrm{Cl})$ refers to a modification of $(\mathrm{Conj}(G_\infty^i), \mathrm{Cl})$, which we need to make when the derived group $G_{\infty/\mathbb{Q}}^{i \text{ der}}$ of $G_{\infty/\mathbb{Q}}^i$ has certain factors of type D . It is the quotient for the action of a slightly larger group $G_\infty^{i \text{ h}}$ on G_∞^i that extends the conjugation action. If $G_{\infty/\mathbb{Q}}^{i \text{ der}}$ has no factors of the above type, then $(\mathrm{Conj}'(G_\infty^i), \mathrm{Cl})$ coincides with $(\mathrm{Conj}(G_\infty^i), \mathrm{Cl})$. A similar modification is made in part (ii) of the theorem (see §3 for more on this and for the technical condition of weak neatness)

By assuming a stronger hypothesis that the variety itself has good (resp. semi-stable) reduction at v , results similar to Theorem 1.5 was obtained for any i and X any finite product of $K3$ surfaces, abelian varieties, Fermat hypersurfaces and curves, in [15]. Under those hypothesis the case of $i = 1$ and X an abelian variety was shown by Noot in [17] and [18].

We now move to a topic of motivic nature and discuss an issue closely related to the above topics. Recall that Grothendieck's standard conjectures on algebraic cycles predict that the category of motives for homological equivalence of algebraic cycles coincides with the category of numerical(Grothendieck) motives and is Tannakian (cf.[12, Cor.2]). This would imply in particular that the action of the Galois group Γ_F on the ℓ -adic realization $H_\ell(M)$ of any motive M over F , factors as

$$\rho_{M,\ell} : \Gamma_F \rightarrow G_M(\mathbb{Q}_\ell),$$

where G_M is the *motivic Galois group* of M . This is the group associated, via the Betti realization functor H_τ , to the Tannakian subcategory generated by M and the Tate

motive. Let $(\text{Conj}(G_M), \text{Cl})$ be the universal categorical quotient of G_M for its action on itself by conjugation. For every $\ell \neq p$, $\rho_{M,\ell}(\Phi_v)$ defines an element $\text{Cl}(\rho_{M,\ell}(\Phi_v)) \in \text{Conj}(G_M)(\mathbb{Q}_\ell)$. In [22, 12.6], Serre conjectures that: *If M is a motive with good reduction at v , then $\text{Cl}(\rho_{M,\ell}(\Phi_v)) \in \text{Conj}(G_M)(\mathbb{Q})$, $\forall \ell \neq p$ and is independent of ℓ .* Special cases of this conjecture were studied in [15] and [17]. It is natural to ask that what one may expect from this conjecture of Serre in the case $\ell = p$. The main result of the last section of this article gives an answer to this question. As the standard conjectures remain unsolved, we work in the framework of Tannakian category of motives defined by absolute Hodge cycles, as in Deligne [9]. This category has many desired properties of Grothendieck motives, unconditionally. Of course, it is well known that if the Hodge conjecture is true, then these categories coincide.

We introduce some more notation, denote the ring of Witt vectors of k_v by $W(k_v)$ and by $\sigma : W(k_v) \rightarrow W(k_v)$ the morphism induced by the Frobenius automorphism of k_v . Let us denote by F_v^0 the fraction field of $W(k_v)$. Note that F_v^0 naturally identifies to the maximal unramified extension of \mathbb{Q}_p inside F_v .

Let X/F be a smooth proper algebraic variety with good reduction at v . Let X_v denote the special fiber of a smooth proper model of X over the ring of integers of F_v . Fix $i \in \mathbb{N}$. The absolute Frobenius endomorphism Fr_{abs} of X_v induces a σ -linear map $Fr_{\text{Cris}} : H_{\text{Cris}}^i(X_v/W(k_v)) \rightarrow H_{\text{Cris}}^i(X_v/W(k_v))$, where $H_{\text{Cris}}^i(X_v/W(k_v))$ refers to crystalline cohomology of X_v . Now, let $|k_v| = p^s$, then Fr_{Cris}^s is a $W(k_v)$ -linear endomorphism of $H_{\text{Cris}}^i(X_v/W(k_v))$. Let us denote by

$$\Phi_{\text{Cris}} : H_{\text{Cris}}^i(X_v/W(k_v)) \otimes F_v^0 \rightarrow H_{\text{Cris}}^i(X_v) \otimes F_v^0,$$

the *crystalline Frobenius* element for X at v i.e. the F_v^0 -linear endomorphism $Fr_{\text{Cris}}^s \otimes 1$.

We denote by $\mathcal{M}_{AH}(F)$ the category of motives (over F) for absolute Hodge cycles and by $\mathcal{M}_{AH}^{ab}(F)$ the full Tannakian subcategory generated by abelian varieties and Artin motives. Henceforth, we assume that the motive $M := h^i(X) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(F))$. Let $\langle M, \mathbb{Q}(1) \rangle^\otimes$ denote the full Tannakian subcategory of $\mathcal{M}_{AH}^{ab}(F)$ tensor generated by M and the Tate motive $\mathbb{Q}(1)$. By the C_{Cris} -conjecture (cf. [11, Th. 3.2.3]) proved by Fontaine-Messing (partially) and Faltings (in full generality) $H_p(M) = H_{\text{et}}^i(X_{/\bar{F}}, \mathbb{Q}_p)$ is a crystalline representation of Γ_v and its associated filtered Dieudonné module is isomorphic to $H_{\text{Cris}}^i(X_v) \otimes F_v^0$. This allows us to define the crystalline realization (cf. [24, §4.1]), i.e., a fiber functor

$$H_{M,\text{Cris}} : \langle M, \mathbb{Q}(1) \rangle^\otimes \rightarrow \text{Vect}_{F_v^0},$$

which associates to any object in $\langle M, \mathbb{Q}(1) \rangle^\otimes$ its associated filtered Dieudonné module (see §4 for details).

We put $G_{M,\text{Cris}} := \underline{\text{Aut}}^\otimes(H_{M,\text{Cris}})$ for the automorphism group of $H_{M,\text{Cris}}$. The action of $G_{M,\text{Cris}}$ on itself by conjugation, provides a universal categorical quotient $(\text{Conj}(G_{M,\text{Cris}}), \text{Cl}_{\text{Cris}})$. The algebraic variety $\text{Conj}(G_{M,\text{Cris}})$ is defined over F_v^0 and $\text{Cl}_{\text{Cris}} : G_{M,\text{Cris}} \rightarrow \text{Conj}(G_{M,\text{Cris}})$ denotes the quotient map. By a result of Blasius [4, Th. 5.3] Φ_{Cris} defines an element $\Phi_{M,\text{Cris}} \in G_{M,\text{Cris}}(F_v^0)$ which in turn defines an element $\text{Cl}_{\text{Cris}}(\Phi_{\text{Cris}}) \in \text{Conj}(G_{M,\text{Cris}})(F_v^0)$. As $G_{M,\text{Cris}}$ and G_M are inner forms of each other, hence there is an isomorphism $\text{Conj}(G_{M,\text{Cris}}) \cong \text{Conj}(G_M)_{/F_v^0}$, which allows to make sense of the following result.

Theorem 1.6. *Let M be as above. Assume that G_M is connected and there is prime number ℓ_0 such that $\rho_{M,\ell_0}(\Phi_v)$ is weakly neat. Then, there exists a unique conjugacy class*

$$\text{Frob}_M \in \text{Conj}'(G_M)(F_v^0),$$

such that $\text{Cl}_{\text{Cris}}(\Phi_{\text{Cris}}) = \text{Frob}_M$ and $\text{Cl}(\rho_{M,\ell}(\Phi_v)) = \text{Frob}_M, \forall \ell \neq p$.

Finally, let us denote by $\mathcal{M}_{\text{num}}^{\text{ab}}(F)$ (cf.[12]) the Tannakian category of motives generated by abelian varieties and zero dimensional varieties, inside the category of numerical motives over F . Since homological equivalence coincides with numerical equivalence for abelian varieties [16], we can identify $\mathcal{M}_{\text{num}}^{\text{ab}}(F)$ to a subcategory (a priori not full) of $\mathcal{M}_{\text{AH}}^{\text{ab}}(F)$. Thus, our result also holds for $M \in \text{Ob}(\mathcal{M}_{\text{num}}^{\text{ab}}(F))$. Here we should remark that in the special case, where M is the motive $h^1(A) \in \text{Ob}(\mathcal{M}_{\text{num}}^{\text{ab}}(F))$ for an abelian variety A , Theorem 1.6 was proved by Noot in [17, Th. 4.2]. We also note that as the Hodge conjecture remains unproven, a priori $\mathcal{M}_{\text{AH}}^{\text{ab}}(F)$ contains more objects than $\mathcal{M}_{\text{num}}^{\text{ab}}(F)$. An example of this phenomenon is the motive of a $K3$ surface.

1.7 Notation

In the introduction and in the main text of the article the following notation is used, unless otherwise specified : \bar{F} is a fixed algebraic closure of a number field F ; $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ is an extension of the embedding $\tau : F \hookrightarrow \mathbb{C}$; \bar{v} is an extension of v to \bar{F} ; \bar{F}_v is the localization of \bar{F} at \bar{v} . The residue fields of F_v and \bar{F}_v are denoted by k_v and \bar{k}_v , respectively. The characteristic of k_v is $p > 0$. We write $\Gamma_v := \text{Gal}(\bar{F}_v/F_v) \subset \Gamma_F := \text{Gal}(\bar{F}/F)$ and $I_v \subset \Gamma_v$ is the inertia group. An arithmetic Frobenius $\Phi_v \in \Gamma_v$ is an element which induces the Frobenius automorphism ϕ_v of \bar{k}_v .

We denote by W_v the *Weil group* of F_v i.e. the dense subgroup formed by elements $w \in \Gamma_v$ which induce on \bar{k}_v an integral power $\phi_v^{\alpha(w)}$. The map $\alpha : W_v \rightarrow \mathbb{Z}$ thus defined is a group homomorphism and $\ker(\alpha) = I_v$. The *Weil-Deligne group* $'W_v$ of F_v is the group scheme over \mathbb{Q} defined as the semi-direct product of W_v with the additive group \mathbb{G}_a over \mathbb{Q} , on which W_v acts as : $w \cdot x \cdot w^{-1} = |k_v|^{\alpha(w)} \cdot x$.

The ring of Witt vectors of k_v is denoted by $W(k_v)$; $\sigma : W(k_v) \rightarrow W(k_v)$ the morphism induced by the Frobenius automorphism of k_v ; F_v^0 the fraction field of $W(k_v)$.

2 Action of Frobenius on ℓ -adic cohomology groups

In the following all the algebraic cycles and Chow groups are with rational coefficients.

Proof of Theorem 1.1. Let us denote by $\tau\Delta \subset \tau X \times \tau X$ the diagonal subvariety and by d the dimension of X . Let $\gamma_B^d : \text{CH}^d(\tau X \times \tau X) \rightarrow \text{H}_B^{2d}(\tau X \times \tau X)(d)$, be the cycle class map from codimension d algebraic cycles on $\tau X \times \tau X$ to the Tate twisted degree $2d$ Betti cohomology of $\tau X \times \tau X$.

The Künneth isomorphism $\text{H}_B^{2d}(\tau X \times \tau X)(d) \cong \bigoplus_{i=0}^{2d} \text{H}_B^{2d-i}(\tau X) \otimes \text{H}_B^i(\tau X)(d)$, gives us a decomposition $\gamma_B^d([\tau\Delta]) = \sum_{i=0}^{2d} \tau\pi^i$. Now, by hypothesis the Lefschetz standard

conjecture is true for τX , which by [14, Thm 4-1] implies that for each $i \in [0, 2d]$ there is a correspondence $\tau z^i \in \text{CH}^d(\tau X \times \tau X)$ such that $\gamma_B^d(\tau z^i) = \tau \pi^i$.

Let $p^* : \text{CH}^d(X_{\bar{F}} \times X_{\bar{F}}) \rightarrow \text{CH}^d(\tau X \times \tau X)$ denote the base change map. It is well known (cf. [9, 2.9, a]) that base change induces an isomorphism between algebraic cycle groups over \bar{F} and \mathbb{C} modulo homological equivalence. This implies that there is a correspondence $z^i_{/\bar{F}} \in \text{CH}^d(X_{\bar{F}} \times X_{\bar{F}})$ such that $\gamma_B^d(\tau z^i) = \gamma_B^d(p^*(z^i_{/\bar{F}}))$. Now there is also a diagram

$$\begin{array}{ccc} \text{CH}^d(\tau X \times \tau X) & \xrightarrow{\gamma_B^d \otimes 1} & \text{H}_B^{2d}(\tau X \times \tau X)(d) \otimes \mathbb{Q}_\ell & (1) \\ p^* \uparrow & & I_{\ell, \tau} \uparrow & \\ \text{CH}^d(X_{\bar{F}} \times X_{\bar{F}}) & \xrightarrow{\gamma_\ell^d} & \text{H}_{et}^i(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d) & \end{array}$$

where γ_ℓ^d is the cycle class map of ℓ -adic cohomology and $I_{\ell, \tau}$ is the comparison isomorphism between Betti and ℓ -adic cohomology groups. It follows from the arguments given in [9, I, page 21], that (1) is commutative.

Let $\Delta_{/\bar{F}} \subset X_{\bar{F}} \times X_{\bar{F}}$ denote the diagonal subvariety. We have the Künneth decomposition of the ℓ -adic cohomology class $\gamma_\ell^d([\Delta_{/\bar{F}}]) = \sum_{i=0}^{2d} \pi^i_{/\bar{F}}$. It follows from the commutativity of (1), that

$$\gamma_\ell^d(z^i_{/\bar{F}}) = I_{\ell, \tau}^{-1}((\gamma_B^d \otimes 1)(\tau z^i)) = I_{\ell, \tau}^{-1}(\tau \pi^i \otimes 1) = \pi^i_{/\bar{F}}.$$

Now, clearly we can suppose that $z^i_{/\bar{F}}$ is defined over some finite extension (say) F_1 of F . By using the natural action of $G := \text{Gal}(F_1/F)$ on $\text{CH}^d(X_{F_1} \times X_{F_1})$, we set

$$z^i := \frac{1}{|G|} \sum_{\sigma \in G} \sigma^* z^i_{/\bar{F}}.$$

As $z^i \in \text{CH}^d(X_{F_1} \times X_{F_1})^G$, we conclude that $z^i \in \text{CH}^d(X \times X)$. Now we have

$$\gamma_\ell^d(z^i) = \frac{1}{|G|} \sum_{\sigma \in G} \gamma_\ell^d(\sigma^* z^i_{/\bar{F}}) = \frac{1}{|G|} \sum_{\sigma \in G} \tilde{\sigma}^* \gamma_\ell^d(z^i_{/\bar{F}}), \quad (2)$$

where $\tilde{\sigma}$ is a lift of σ to $\text{Gal}(\bar{F}/F)$.

Now, by Künneth formula and Poincaré duality for ℓ -adic cohomology, we have the following canonical isomorphisms

$$\begin{aligned} \text{H}_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d) &\cong \bigoplus_{r \geq 0} \text{H}_{et}^{2d-r}(X_{\bar{F}}, \mathbb{Q}_\ell)(d) \otimes \text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell) \\ &\cong \bigoplus_{r \geq 0} \text{Hom}_{\mathbb{Q}_\ell}(\text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \otimes \text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell) \cong \bigoplus_{r \geq 0} \text{End}_{\mathbb{Q}_\ell}(\text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell)) \end{aligned} \quad (3)$$

More precisely, under the above isomorphism, an element $u \in \text{H}_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d)$ is mapped to the element $\underline{u} := (z \mapsto pr_{2*}(pr_1^*(z) \cdot u)) \in \bigoplus_{r \geq 0} \text{End}_{\mathbb{Q}_\ell}(\text{H}_{et}^r(X_{\bar{F}}, \mathbb{Q}_\ell))$, where pr_1 and pr_2 are projections. As one easily verifies, $\pi^i_{/\bar{F}}$ is the idempotent projection

$$p_i : \text{H}_{et}^*(X_{\bar{F}}, \mathbb{Q}_\ell) \rightarrow \text{H}_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell) \hookrightarrow \text{H}_{et}^*(X_{\bar{F}}, \mathbb{Q}_\ell).$$

Clearly p_i is invariant under the action of $\text{Gal}(\bar{F}/F)$ on $\bigoplus_{i \geq 0} \text{End}_{\mathbb{Q}_\ell}(\text{H}_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell))$. Thus, it follows from (3) that $\pi_{/\bar{F}}^i$ is invariant under the action of $\text{Gal}(\bar{F}/F)$ on $\text{H}_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d)$. Under the above observations, (2) now reads as

$$\gamma_\ell^d(z^i) = \frac{1}{|G|} \sum_{\sigma \in G} \tilde{\sigma}^* \pi_{/\bar{F}}^i = \frac{1}{|G|} (|G| \cdot \pi_{/\bar{F}}^i) = \pi_{/\bar{F}}^i. \quad (4)$$

Finally, let α^i denote the image of z^i under the canonical injective morphism $\text{CH}^d(X \times X) \rightarrow \text{CH}^d(X_{F_v} \times X_{F_v})$. Let $\gamma_{\ell,v}^d : \text{CH}^d(X_{F_v} \times X_{F_v}) \rightarrow \text{H}_{et}^{2d}(X_{\bar{F}_v} \times X_{\bar{F}_v}, \mathbb{Q}_\ell)(d)$ denote the cycle class map. By invariance of ℓ -adic cohomology groups under extension from \bar{F} to algebraically closed overfield, we have $\text{H}_{et}^i(X_{\bar{F}}, \mathbb{Q}_\ell) \cong \text{H}_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ and $\text{H}_{et}^{2d}(X_{\bar{F}_v} \times X_{\bar{F}_v}, \mathbb{Q}_\ell) \cong \text{H}_{et}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)$. We also know that $\gamma_{\ell,v}^d|_{\text{CH}^d(X \times X)} = \gamma_\ell^d$; in particular $\gamma_{\ell,v}^d(\alpha^i) = \gamma_\ell^d(z^i)$. Thus using (4) and (3), it follows that $\gamma_{\ell,v}^d(\alpha^i)$ induces an endomorphism $\text{H}_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell) \rightarrow \text{H}_{et}^*(X_{\bar{F}_v}, \mathbb{Q}_\ell)$, which is identity on $\text{H}_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ and 0 otherwise. By [19, cor.0.6, 1] we conclude the proof the theorem. \square

Before proceeding further, let us recall few definitions. A *hyperkähler* variety over any field K of characteristic 0 (see for example [1]) is a simply connected smooth projective K -variety Y of even dimension $2n$, with the property that there exists a section ω of Ω_Y^2 , unique up to multiplication by a constant, such that ω^n vanishes nowhere. The variety Y is said to be of $K3^{[n]}$ -type if it is one of the following:

1. For any $K3$ surface S , take $Y = S^{[n]}$ the punctual Hilbert scheme which parametrizes closed subschemes of S of length n .
2. Any projective deformation of a hyperkähler variety of type $S^{[n]}$.

Let us remark that any general projective deformation of $S^{[n]}$ is not of the form $S'^{[n]}$ for any other $K3$ surface S' . Hyperkähler varieties of $K3^{[n]}$ -type, form one of the two series of examples of hyperkähler varieties constructed by Beauville [3]. Apart from these only two exceptional examples has been discovered by O'Grady.

Now, let C be a smooth projective curve of genus > 1 over F . It is well known that the moduli space $N_C(q, e)$ of stable vector bundles of coprime rank q and degree e over C is a smooth projective fine moduli space.

Proof of Corollary 1.2. By [14, Prop. 4-3] the Lefschetz standard conjecture is stable under products. This implies that in order to establish our result, it suffices to show that the Lefschetz standard conjecture is true for τX , where X is now one of the varieties in the statement of the corollary. But, this follows from the results of [5] and [2] for hyperkähler varieties of $K3^{[n]}$ type and unirational varieties of dimension ≤ 4 and uniruled varieties of dimension ≤ 3 , respectively. For the moduli spaces $N_C(q, e)$ as above this follows from [6]. The corollary now follows from Theorem 1.1. \square

Corollary 2.1. *Let X be as in corollary 1.2. Then for every $\ell \neq p$ and $w \in W_v$, the polynomial $P_{\ell,v}^i(w, T) \in \mathbb{Q}[T]$ and is independent of ℓ .*

Proof. This an immediate consequence of Cor. 1.2 \square

The proofs of our next results employs some motivic ideas. We review these very briefly. Denote by $\mathcal{V}(F)$ the category of smooth projective algebraic varieties over F . Recall that the category of (pure) motives $\mathcal{M}_{AH}(F)$, defined by absolute Hodge cycles is a \mathbb{Q} -linear semisimple neutral Tannakian category. There exists a contravariant functor for $\mathcal{M}_{AH}(F)$) $h : \mathcal{V}(F) \rightarrow \mathcal{M}_{AH}(F)$ such that Betti, ℓ -adic or deRham cohomology on $\mathcal{V}(F)$ factorizes through h . This provides fibre functors H_τ, H_ℓ and H_{dR} on $\mathcal{M}_{AH}(F)$. We call these functors as *realizations*. There also exists a natural grading $h(Z) = \bigoplus h^i(Z)$, $i \in \mathbb{N}$ for every $Z \in \text{Ob}(\mathcal{V}(F))$, which extends to all of $\mathcal{M}_{AH}(F)$. We denote by $\mathcal{M}_{AH}^{ab}(F)$ the Tannakian subcategory of $\mathcal{M}_{AH}(F)$ generated by abelian varieties and zero dimensional varieties. Everything stated above holds more generally for any arbitrary field of characteristic 0, with an embedding in \mathbb{C} . For details on the above facts see [9, II].

Lemma 2.2. *Let X/F be as in Theorem 1.3. Then there exists a finite extension F' of F such that $h(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(F'))$.*

Proof. First, we claim that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$. We observe that as $\mathcal{M}_{AH}^{ab}(\bar{F})$ is a Tannakian category, it is closed under tensor products. Thus, in order to verify the claim it suffices to do so for each of the individual varieties in the statement of Theorem 1.3, since X is a finite products of these varieties.

So, first assume that X is a hyperkähler variety of $K3^{[n]}$ -type. As the Kuga-Satake Abelian variety associated to X is defined over a finite extension of F (see [1]) and $h(X_{\bar{F}})$ belongs to the smallest Tannakian subcategory generated by $h^2(X_{\bar{F}})$ (cf. [20]), it follows that in this case $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$.

Next, when X is a Fermat hypersurfaces or a unirational variety of dimension ≤ 3 , it follows from [9, II, 6.26] that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$.

When X is the moduli space $N_C(q, e)$ as above, then it follows from [6], that $h(X)$ is a direct summand of a motive of the form $\bigoplus_b h(C)^{\otimes a}(b)$. Now, let $J(C)$ denote the Jacobian of C . We know that $h(C) = \mathbb{1} \oplus h^1(J(C)) \oplus \mathbb{L}$, where $\mathbb{1}$ denotes the unit object of $\mathcal{M}_{AH}(\bar{F})$ and \mathbb{L} is the Lefschetz motive. It follows that $h(X) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(F))$.

Finally, when X is a uniruled surface, we know (see [2]) that there exists a curve C' over \bar{F} such that $h(X_{\bar{F}})$ is a direct summand of a motive of the form $\bigoplus_{b'} h(C')^{\otimes a'}(b')$. Thus, as above it follows that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$.

Now, for each individual variety as in the statement of the corollary, we see that $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$. Hence, the same holds for any product of these varieties. By using [9, I, 2.9] we conclude there exists a finite extension F' of F such that $h(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(F'))$. □

Proof of Theorem 1.3. Let us denote by M the motive $h^i(X)$. As M is a direct summand of $h(X)$, by Lemma 2.2 there is a finite extension F' of F such that $M_{F'} := h^i(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(F'))$. Let us denote by v' an extension of the valuation v to F' ; n the residual degree; $\Phi_{v'} := \Phi_v^n$ an arithmetic Frobenius corresponding to this extension. We denote by $\langle M_{F'}, \mathbb{Q}(1) \rangle^\otimes$ the Tannakian subcategory of $\mathcal{M}_{AH}^{ab}(F')$ which is tensor generated by $M_{F'}$ and the Tate motive $\mathbb{Q}(1)$ and we write $G_{M_{F'}} := \underline{\text{Aut}}^\otimes(H_\tau|_{\langle M_{F'}, \mathbb{Q}(1) \rangle^\otimes})$. We recall, that the ℓ -adic representations $\rho_{M_{F'}, \ell} : \Gamma_{F'} \rightarrow$

$\mathrm{GL}(H_\ell(M_{F'}))$, arising from the action of $\Gamma_{F'}$ on the ℓ -adic realization $H_\ell(M_{F'})$ of $M_{F'}$, factorizes through $G_{M_{F'}}(\mathbb{Q}_\ell)$.

We know that there exists (possibly after passing to another finite extension, which we again denote by F') an abelian variety A over F' and a unique homomorphism of algebraic groups $\theta : G_A \rightarrow G_{M_{F'}}$ where $G_A := \underline{\mathrm{Aut}}^\otimes(H_\tau|_{\langle h^1(A), \mathbb{Q}(1) \rangle^\otimes})$. Moreover, $\theta(\mathbb{Q}_\ell) \circ \rho_{A,\ell} = \rho_{M_{F'},\ell}$, where $\theta(\mathbb{Q}_\ell)$ denotes the induced map on \mathbb{Q}_ℓ -valued points and $\rho_{A,\ell}$ is the Galois representation associated to the motive $h^1(A)$. For details on the above facts see [15, 3.10 & 3.11]. In particular, we have $\theta(\mathbb{Q}_\ell)(\rho_{A,\ell}(\Phi_{v'})) = \rho_{M_{F'},\ell}(\Phi_{v'})$.

Now, it is well known that $\rho_{A,\ell}(\Phi_{v'})$ is semisimple i.e. becomes diagonalizable after a base change. As θ is a homomorphism of algebraic groups, the image of $\rho_{A,\ell}(\Phi_{v'})$, under $\theta(\mathbb{Q}_\ell)$ must be semisimple. In other words, $\rho_{M_{F'},\ell}(\Phi_{v'})$ is semisimple.

Finally, let $\rho_{M,\ell} : \Gamma_F \rightarrow \mathrm{GL}(H_\ell(M))$ be the ℓ -adic representation associated M . As $\rho_{M_{F'},\ell}(\Phi_{v'}) = (\rho_{M,\ell}(\Phi_v))^n$, so $\rho_{M,\ell}(\Phi_v)$ is semisimple. Now, by definition $H_\ell(M) = \pi^{i*} \mathrm{H}_{\mathrm{et}}^*(X_{\bar{F}}, \mathbb{Q}_\ell)$, where π^i is the Künneth projector (as before) and π^{i*} is the image of π^i under the isomorphism of (3). As we have already seen, π^{i*} is the idempotent projection p_i . We conclude that $H_\ell(M) = \mathrm{H}_{\mathrm{et}}^i(X_{\bar{F}}, \mathbb{Q}_\ell)$ and $\rho_{M,\ell} = \rho_\ell^i$. Thus, it follows from the above discussion that $\rho_\ell^i(\Phi_v)$ is semisimple. \square

3 Conjugacy class of Frobenius

The principal aim of this section is to show Theorem 1.5, but for this we shall need some preliminaries.

The Algebraic Group $H^{\natural ad}$. Let H be a reductive algebraic group over a field K of characteristic 0 and \bar{K} an algebraic closure of K . The adjoint action of H^{ad} on H gives a universal categorical quotient $(\mathrm{Conj}(H), \mathrm{Cl})$, where $\mathrm{Conj}(H)$ is an algebraic variety over K and $\mathrm{Cl} : H \rightarrow \mathrm{Conj}(H)$ is the quotient map. The pair $(\mathrm{Conj}(H), \mathrm{Cl})$ is also the universal categorical quotient for the action of H on itself by conjugation.

The derived group $H_{\bar{K}}^{\mathrm{der}}$ is the almost direct product of almost simple subgroups H_i , for $i \in I$ a finite indexing set. If $H_{\bar{K}}^{\mathrm{der}}$ has any almost direct factor of type $\mathrm{SO}(2k_i)_{\bar{K}}$, then there is an action which extends the above action. We briefly recall this. For details see [15] and [17].

Let $J \subset I$ such that for $i \in J$, $H_i \cong \mathrm{SO}(2k_i)_{\bar{K}}$ with $k_i \geq 4$ and set $H_i^{\natural} := \mathrm{O}(2k_i)_{\bar{K}}$. Define an algebraic group

$$H^{\natural ad} = \prod_{i \in J} H_i^{\natural ad} \times \prod_{i \notin J} H_i^{ad} \supset H_{\bar{K}}^{ad}.$$

There is a natural action of $H^{\natural ad}$ on $H_{\bar{K}}$, which extends the adjoint action on $H_{\bar{K}}^{\mathrm{der}}$ and is trivial on the center of $H_{\bar{K}}$. Denote by $\mathrm{Conj}'(H)_{\bar{K}}$ the categorical quotient of $H_{\bar{K}}$ under this action of $H^{\natural ad}$. Now, using the properties of categorical quotients, we can descend from $\mathrm{Conj}'(H)_{\bar{K}}$ to an algebraic variety $\mathrm{Conj}'(H)$ over K . We also get a quotient map (as above), which we again denote by

$$\mathrm{Cl} : H \rightarrow \mathrm{Conj}'(H).$$

Now, let $\Omega \supseteq K$ be any algebraically closed field. From the functorial properties of the above action we deduce an *adjoint map*

$$\text{Ad} : H^{\text{bad}}(\Omega) \rightarrow \text{Aut}(\mathfrak{h} \otimes \Omega),$$

where \mathfrak{h} denotes the Lie algebra of H .

Weakly neat elements. Fix a faithful K -linear representation V of H . A semisimple element $g \in H(\Omega)$ is said to be *weakly neat* if 1 is the only root of unity among the quotients $\lambda\mu^{-1}$, with λ, μ being the eigenvalues of g .

For any positive integer n we denote by $\varphi_n : H \rightarrow H$ the n -th power map. Let Y/K denote any one of the algebraic varieties $\text{Conj}(H)$ or $\text{Conj}'(H)$. The map φ_n is equivariant for the action of conjugation. Since Y is a universal categorical quotient, this implies that φ_n induces a map $\bar{\varphi}_n : Y \rightarrow Y$ such that $\text{Cl} \circ \varphi_n = \bar{\varphi}_n \circ \text{Cl}$. The proof of the following lemma (see for example [15, 2.7]) is left to the reader .

Lemma 3.1. *Let $\Omega \supseteq K$ be an algebraically closed field. Let V be a finite dimensional faithful K -linear representation of H and $\alpha, \beta \in H(\Omega)$ be two weakly neat elements having same characteristic polynomial in the representation V . If $\bar{\varphi}_n(\text{Cl}(\alpha)) = \bar{\varphi}_n(\text{Cl}(\beta))$ for some positive integer n , then $\text{Cl}(\alpha) = \text{Cl}(\beta)$.*

Compatible systems. We briefly recall the notion of compatible system of representations of Weil-Deligne groups with values in an algebraic group as sketched by Deligne [7, 8.11], see also [18].

Let H be an algebraic group over \mathbb{Q} and $\xi_\ell : \Gamma_v \rightarrow H(\mathbb{Q}_\ell)$ be an ℓ -adic representation. By Grothendieck's monodromy theorem [23, Appendix], there exists a nilpotent element $N'_\ell \in \text{Lie}(H/\mathbb{Q}_\ell)(-1)$ and an open subgroup $J \subseteq I_v$ such that for $\lambda \in J$, we have $\xi_\ell(\lambda) = \exp(N'_\ell t_\ell(\lambda))$. Now, by using the identification $\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(1)$, we interpret N'_ℓ as an element of $\text{Lie}(H/\mathbb{Q}_\ell)$ and we define

$$\xi'_\ell : W_v \rightarrow H(\mathbb{Q}_\ell), \quad w \mapsto \xi_\ell(w) \exp(-N'_\ell t_\ell(\Phi_v^{-\alpha(w)} w)). \quad (5)$$

A triple $(H/\mathbb{Q}_\ell, N'_\ell, \xi'_\ell)$ as above is called a *representation of the Weil-Deligne group* $'W_v$ of F_v with values in H/\mathbb{Q}_ℓ .

For a fixed ℓ , we say that $(H/\mathbb{Q}_\ell, N'_\ell, \xi'_\ell)$ is *rational over \mathbb{Q}* (as a representation with values in H), if for every algebraically closed field $\Omega \supset \mathbb{Q}_\ell$ and for every $\sigma \in \text{Aut}_{\mathbb{Q}}(\Omega)$, there exists a $g \in H(\Omega)$ such that

$$\sigma \xi'_{\ell/\Omega} = g \cdot \xi'_{\ell/\Omega} \cdot g^{-1} \quad \text{and} \quad \sigma(N'_\ell \otimes_{\mathbb{Q}_\ell} 1) = \text{Ad}(g)(N'_\ell \otimes_{\mathbb{Q}_\ell} 1), \quad (6)$$

where $\xi'_{\ell/\Omega}$ denote the extension of scalars $\xi'_\ell \otimes_{\mathbb{Q}_\ell} \Omega : W_v \rightarrow H/\Omega(\Omega)$ and $N'_\ell \otimes_{\mathbb{Q}_\ell} 1$ is the image of N'_ℓ in $(\text{Lie}(H) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \Omega$.

If in addition to (6), we have for every pair (ℓ, ℓ') of prime numbers (different from p) and every algebraically closed field $\Omega \supset \mathbb{Q}_\ell, \mathbb{Q}_{\ell'}$, an element $g \in H(\Omega)$ such that

$$\xi'_{\ell/\Omega} = g \cdot \xi'_{\ell'/\Omega} \cdot g^{-1} \quad \text{and} \quad N'_\ell \otimes_{\mathbb{Q}_\ell} 1 = \text{Ad}(g)(N'_{\ell'} \otimes_{\mathbb{Q}_{\ell'}} 1), \quad (7)$$

then we say that the representations $(H/\mathbb{Q}_\ell, N'_\ell, \rho^{i'}_\ell)$, form a *compatible system of representations of $'W_v$* (with values in H).

Note that, as the action of $H(\Omega)$ factors through $H(\Omega) \rightarrow H^{ad}(\Omega)$, $H(\Omega)$ -conjugacy may be replaced by $H^{ad}(\Omega)$ -conjugacy everywhere. Now, suppose that H is a reductive algebraic group, then as above we define the algebraic group H^{bad} . To say that the representation $(H/\mathbb{Q}_\ell, N'_\ell, \xi'_\ell)$ is *rational over \mathbb{Q} modulo the action of H^{bad}* , means that in (6) we require that $g \in H^{bad}(\Omega)$. In a similar manner we make sense of the phrase *compatible system of representations of $'W_v$ modulo the action of H^{bad}*

3.2 Proof of Proposition 1.4 & Theorem 1.5

Proof of 1.4. Let $X = \prod_r X_r$, where each X_r is one of the varieties of the type mentioned in the statement of the proposition. As seen in the proof of Lemma 2.2, $h(X_{r,\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$, for each r . As $h(X_{\bar{F}}) = \otimes_r h(X_{r,\bar{F}})$ and $\mathcal{M}_{AH}^{ab}(\bar{F})$ is closed under tensor products, $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$.

Write $M := h^i(X)$; the algebraic groups $G_{M_{\bar{F}}}$ and G_M are stabilizers of all absolute Hodge classes (AHC) and AHC that are defined over F , respectively, in all tensor spaces formed by $V^i, V^{i^\vee}, \mathbb{Q}(1)$ and $\mathbb{Q}(1)^\vee$ (see [9, I, Principle A]). So, it follows from hypothesis that $G_{M_{\bar{F}}} = G_M$.

Now the Mumford-Tate group G_∞^i of the Hodge structure V^i is the stabilizer of Hodge classes relative to the embedding τ , in all tensor spaces formed by $V^i, V^{i^\vee}, \mathbb{Q}(1)$ and $\mathbb{Q}(1)^\vee$. As $h(X_{\bar{F}}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$, all Hodge classes relative to τ are AHC (see [9, II, 6.27]). This implies that $G_\infty^i = G_{M_{\bar{F}}} = G_M$.

Finally, as $\rho_{M,\ell} : \Gamma_F \rightarrow \text{GL}(H_\ell(M))$ factorizes through $G_M(\mathbb{Q}_\ell)$ and $\rho_{M,\ell} = \rho_\ell^i$, we conclude that G_ℓ^i is a subgroup of $G_{\infty/\mathbb{Q}_\ell}^i$. □

Remark 3.3. By [9, I, 2.9] there exists number fields verifying the hypothesis of Proposition 1.4.

Now, let X be as in Theorem 1.4. For every non-archimedean valuation v on F and prime number $\ell \neq p$ (the residual characteristic at v), the ℓ -adic representation ρ_ℓ^i gives rise to a representation

$$\rho_\ell^{i'}(w) = \rho_\ell^i(w) \exp(-N'_{i,\ell} t_\ell(\Phi_v^{-\alpha(w)} w)) \quad (8)$$

of W_v , as in (5). The triple $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_\ell^{i'}, N'_{i,\ell})$ is a representation of the Weil-Deligne group $'W_v$ of F_v , with values in $G_{\infty/\mathbb{Q}_\ell}^i$.

For any finite extension F' of F we denote by $\Gamma_{F'}$ the absolute Galois group and by v' an extension of v to F' . Let $I_{v'}$, $W_{v'}$ and $'W_{v'}$ denote the inertia, Weil and Weil-Deligne group of $F'_{v'}$, respectively. We also have a group homomorphism $\alpha' : W_{v'} \rightarrow \mathbb{Z}$ as in §1.7. By $\Phi'_{v'}$ we mean an arithmetic Frobenius for this extension.

If $X_{F'}$ denotes base change of X to F' , then the ℓ -adic representation $\rho_\ell^i|_{\Gamma_{F'}}$, associated to $X_{F'}$ induces a representation $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_\ell^{i'}|_{\Gamma_{F'}}, N'_{i,\ell})$ of the Weil-Deligne group $'W_{v'}$ of $F'_{v'}$, with values in $G_{\infty/\mathbb{Q}_\ell}^i$

Remark 3.4. Although Theorem 1.5 is similar in spirit to [15, Theorem 1.1 & corollary 1.3], there are two important differences between [loc.cit.] and our case, which are as following :

- (a) here we only know that there is a finite extension F' of F such that the motive $h^i(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(F'))$ (by Lemma 2.2), while in [loc.cit.] we assumed this property over the base field F ;
- (b) in part (i) (resp. (ii)) of Theorem 1.5 we just need that $\rho_\ell^i|_{\Gamma_v}$ is potentially unramified (resp. semi-stable) representation for every $\ell \neq p$, while in [loc.cit.] a stronger condition was used namely X has good (resp. semi-stable) reduction at v .

Proof of Theorem 1.5. (i) The idea of the proof is similar to that of [15, Thm. 1.1], so we only present the main arguments and sketch the rest. As before let us denote by M the motive $h^i(X)$. Let F' be a finite extension of F such that $M_{F'} := h^i(X_{F'}) \in \text{Ob}(\mathcal{M}_{AH}^{ab}(F'))$, as in Lemma 2.2. Let v' be an extension of the valuation v to F' and $\Phi_{v'}$ an arithmetic Frobenius for this extension. Now, we have seen in the proof of Prop. 1.4 that $G_\infty^i = G_{M_{\bar{F}}} = G_M$, in particular $G_{M_{\bar{F}}} = G_{M_{F'}}$. We also have $\text{Conj}'(G_\infty^i) = \text{Conj}'(G_{M_{F'}}) = \text{Conj}'(G_M)$ and $\rho_{M_{F'}/\ell} = \rho_\ell^i|_{\Gamma_{v'}}$. Now, using the fact that $\rho_\ell^i|_{\Gamma_v}$ is potentially unramified for all $\ell \neq p$, it can be checked (as at the beginning of the proof of [ibid.]) that $M_{F'}$ verifies [ibid. eqn. 14], in particular there exists a

$${}^2\text{Frob}_{v',i} \in \text{Conj}'(G_\infty^i)(\mathbb{Q}), \text{ such that } \text{Cl}(\rho_\ell^i(\Phi_{v'})) = \text{Frob}_{v',i}, \forall \ell \neq p. \quad (9)$$

Now, let the residual degree of the extension $F'_{v'}/F_v$ be n . We denote by $\varphi_n : G_\infty^i \rightarrow G_\infty^i$ the n -th power map and by $\bar{\varphi}_n : \text{Conj}'(G_\infty^i) \rightarrow \text{Conj}'(G_\infty^i)$ the induced map on $\text{Conj}'(G_\infty^i)$. Then it follows easily from (9) and the fact that $\bar{\varphi}_n$ is a finite morphism, that

$$\text{Cl}(\rho_\ell^i(\Phi_v)) \in \text{Conj}'(G_\infty^i)(\bar{\mathbb{Q}}), \forall \ell \neq p.$$

Now, by hypothesis there exists a ℓ_0 such that $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat. So, by Corollary 2.1, $\rho_\ell^i(\Phi_v)$ is weakly neat for every $\ell \neq p$. Again, Corollary 2.1 and a repeated use of Lemma 3.1, shows that $\text{Cl}(\rho_\ell^i(\Phi_v))$ (as an element of $\text{Conj}'(G_\infty^i)(\bar{\mathbb{Q}})$) is independent of ℓ . We denote this element by $\text{Frob}_{v,i}$. As $\text{Frob}_{v,i} \in \text{Conj}'(G_\infty^i)(\mathbb{Q}_\ell)$ for every $\ell \neq p$, $\text{Frob}_{v,i}$ is a \mathbb{Q} -rational point of $\text{Conj}'(G_\infty^i)$.

- (ii) First note that $\rho_\ell^{i'}(\Phi_v) (= \rho_\ell^i(\Phi_v))$ generates a subgroup of finite index in $\rho_\ell^{i'}(W_v)$. Now, we know that in characteristic 0 a representation (in the ordinary sense) of a group is semisimple if and only if its restriction to a subgroup of finite index is semisimple. Thus, it follows from Theorem 1.3 that $\rho_\ell^{i'}$ is a semisimple representation of W_v .

²In [15, eqn. 14] this element is denoted by $\text{Cl}_M \text{Fr}_{v'}$

Now let the notation be as in part (i). By, applying [15, 3.9] to the motive $M_{F'}$, we see that for $\ell \neq p$ the representations $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{\ell/\Omega}^{i'}|_{\Gamma_{F'}}, N'_{i,\ell})$ are rational over \mathbb{Q} and form a compatible system of representations of $'W_v$, modulo the action of $G_{\infty}^{i \text{ } \text{had}}$. In particular by setting $\Phi_{v'} := \Phi_v^n$, we get that for any fixed $\ell \neq p$, algebraically closed field $\Omega \supset \mathbb{Q}_\ell$ and $\sigma \in \text{Aut}_{\mathbb{Q}}(\Omega)$, there exists a $g_\sigma \in G_{\infty}^{i \text{ } \text{had}}(\Omega)$ such that

$$\sigma(\rho_{\ell/\Omega}^{i'}(\Phi_v^n)) = g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_v^n)) \cdot g_\sigma^{-1} \text{ and } \sigma(N'_{i,\ell} \otimes_{\mathbb{Q}_\ell} 1) = \text{Ad}(g_\sigma)(N'_{i,\ell} \otimes_{\mathbb{Q}_\ell} 1). \quad (10)$$

Thus we have $(\sigma(\rho_{\ell/\Omega}^{i'}(\Phi_v^n)))^n = (g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_v^n)) \cdot g_\sigma^{-1})^n$. Now it follows from Corollary 2.1 that $\sigma(\rho_{\ell/\Omega}^{i'}(\Phi_v^n))$ and $g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_v^n)) \cdot g_\sigma^{-1}$ have the same characteristic polynomial. This now allows us to use a property of weakly neat elements (cf. [18, 7.4]) to conclude that $\sigma(\rho_{\ell/\Omega}^{i'}(\Phi_v^n)) = g_\sigma \cdot (\rho_{\ell/\Omega}^{i'}(\Phi_v^n)) \cdot g_\sigma^{-1}$.

As the monodromy operator $N'_{i,\ell}$ is unchanged by finite base extension, it now follows from (10) that $(G_{\infty/\mathbb{Q}_\ell}^i, \rho_{\ell/\Omega}^{i'}, N'_{i,\ell})$ is rational over \mathbb{Q} modulo the action of $G_{\infty}^{i \text{ } \text{had}}$.

The compatibility condition is checked in a similar manner (cf. proof of [15, Thm. 1.2]). □

4 Conjugacy class of Crystalline Frobenius

We follow the notation of §1.7 and we introduce some more notation. For any finite extension F' of F , we denote by v' an extension of v to F' ; $F'_{v'}$ is the completion of F' at v' ; $k_{v'}$ the residue field of $F'_{v'}$; $\Gamma_{v'}$ is the absolute Galois group of $\text{Gal}(\bar{F}_{v'}/F'_{v'})$; $\Phi_{v'}$ an arithmetic Frobenius at v' . The fraction field of the ring of Witt vectors $W(k_{v'})$ is denoted by $F_{v'}^0$. Note that $F_{v'}^0$ naturally identifies to the maximal unramified extension of \mathbb{Q}_p inside $F_{v'}$.

Now, let Y be any smooth projective algebraic variety over F with potential good reduction at v . For any $i \in \mathbb{N}$, we denote by N the motive $h^i(Y) \in \text{Ob}(\mathcal{M}_{AH}(F))$. Assume that the extension of scalars $N_{\bar{F}} \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$. Thus, by taking a large enough finite extension F' of F , we may assume that Y has good reduction over F' and $N_{F'} \in \text{Ob}(\mathcal{M}_{AH}^{ab}(F'))$. Let $Y_{v'}$ denote the reduction of $Y_{F'}$ at v' and $\langle N_{F'}, \mathbb{Q}(1) \rangle^{\otimes}$ denote the Tannakian subcategory of $\mathcal{M}_{AH}(F')$, \otimes -generated by $N_{F'}$ and the Tate motive. It is well known (cf.[11, Th.3.2.3]) from the C_{Cris} -conjecture proved by Fontaine-Messing (partially) and Faltings (in full generality) that $H_p(N_{F'}) = H_{\text{et}}^i(Y_{\bar{F}}, \mathbb{Q}_p)$ is a crystalline representation of $\Gamma_{v'}$ and its associated filtered Dieudonné module is isomorphic to $H_{\text{Cris}}^i(Y_{v'}) \otimes F_{v'}^0$. This allows us to define the crystalline realization (cf.[24, §4.1]), i.e., a fiber functor

$$H_{N_{F'}, \text{Cris}} : \langle N_{F'}, \mathbb{Q}(1) \rangle^{\otimes} \rightarrow \text{Vect}_{F_{v'}^0},$$

which associates to any motive $L \in \langle N_{F'}, \mathbb{Q}(1) \rangle^\otimes$ its associated filtered Dieudonné $F_{v'}$ -module $(H_p(L) \otimes_{\mathbb{Q}_p} B_{\text{Cris}})^{\Gamma_{v'}}$, where B_{Cris} is the $F_{v'}^0[\Gamma_{v'}]$ -algebra introduced by Fontaine (cf. [10]). We call this functor the *crystalline realization* for the subcategory of motives $\langle N_{F'}, \mathbb{Q}(1) \rangle^\otimes$.

We write $G_{N_{F'}, \text{Cris}} := \underline{\text{Aut}}^\otimes(H_{N_{F'}, \text{Cris}})$ for the automorphism group of the fibre functor $H_{N_{F'}, \text{Cris}}$. Let

$$\Phi'_{\text{Cris}} : H_{\text{Cris}}^i(Y_{v'}/W(k_{v'})) \otimes F_{v'}^0 \rightarrow H_{\text{Cris}}^i(Y_{v'}) \otimes F_{v'}^0,$$

be the (i -th) crystalline Frobenius element for $Y_{F'}$ at v' . Then, it follows from a result of Blasius [4, Th.5.3] that Φ'_{Cris} defines an element $\Phi'_{N, \text{Cris}} \in G_{N_{F'}, \text{Cris}}(F_{v'}^0)$.

Now, as before the conjugation action of $G_{N_{F'}, \text{Cris}}$ gives us a universal categorical quotient $(\text{Conj}'(G_{N_{F'}, \text{Cris}}), \text{Cl}_{\text{Cris}})$. We also have the universal categorical quotient $(\text{Conj}'(G_{N_{F'}}), \text{Cl})$ for the conjugation action of $G_{N_{F'}} := \underline{\text{Aut}}^\otimes(H_\tau|_{\langle N_{F'}, \mathbb{Q}(1) \rangle^\otimes})$. For every $\ell \neq p$ the ℓ -adic representation $\rho_{N_{F'}, \ell} : \Gamma_{F'} \rightarrow G_{N_{F'}}(\mathbb{Q}_\ell)$ defines an element $\text{Cl}(\rho_{N_{F'}, \ell}(\Phi_{v'})) \in \text{Conj}'(G_{N_{F'}})(\mathbb{Q}_\ell)$.

It is well known (see [9, II, Th.3.2]) from the general theory of Tannakian categories that $G_{N_{F'}, \text{Cris}}$ is an inner form of $G_{N_{F'}} \otimes_{\mathbb{Q}} F_{v'}^0$. It follows that there is an isomorphism $\text{Conj}'(G_{N_{F'}, \text{Cris}}) \cong (\text{Conj}'(G_{N_{F'}})) \otimes_{\mathbb{Q}} F_{v'}^0$, which allows us to make sense of the following

Theorem 4.1. *let Y be any smooth projective algebraic variety over F with potential good reduction at a non-archimidean valuation v of F . For any fixed $i \in \mathbb{N}$ write $N := h^i(Y)$ and assume that $N_{\bar{F}} \in \text{Ob}(\mathcal{M}_{AH}^{ab}(\bar{F}))$. Then, there exists a finite extension F' of F and a unique conjugacy class*

$$\text{Frob}'_N \in \text{Conj}'(G_{N_{F'}})(F_{v'}^0),$$

such that $\text{Cl}_{\text{Cris}}(\Phi'_{N, \text{Cris}}) = \text{Frob}'_N$ and $\text{Cl}(\rho_{N_{F'}, \ell}(\Phi_{v'})) = \text{Frob}'_N, \forall \ell \neq p$.

Proof of Theorem 4.1. We shall first find a finite extension of F and then show that it is the right one. To avoid confusion, we shall denote all successive extensions of F by F' .

Recall, as in the proof of Theorem 1.3, there is a finite extension F' of F , an abelian variety A over F' such that $N_{F'} \in \langle h^1(A), \mathbb{Q}(1) \rangle^\otimes$. Further, this induces a unique homomorphism of algebraic groups $\theta : G_A \rightarrow G_{N_{F'}}$ satisfying $\theta(\mathbb{Q}_\ell) \circ \rho_{A, \ell} = \rho_{N_{F'}, \ell}$ for every prime number ℓ (where $\theta(\mathbb{Q}_\ell)$ is the induced map on \mathbb{Q}_ℓ -valued points).

Lemma 4.2. *There exists an abelian variety A over F' as above with potential good reduction at v' .*

Proof. First we note that for any abelian variety A , by Grothendieck's ℓ -adic monodromy theorem, $I_{v'}$ (possibly after passing over to another finite extension of F) acts unipotently on $H_{\text{et}}^1(A/\bar{F}, \mathbb{Q}_\ell) \cong W \otimes \mathbb{Q}_\ell$, where $W := H_B^1(\tau A, \mathbb{Q})$. As Y has potential good reduction at v , we may assume that $I_{v'}$ acts trivially on $H_\ell(N)$. Now, by [15, Lemma 3.10] we may choose A such that $G_A^{\text{ad}} \cong G_{N_{F'}}^{\text{ad}}$. This combined with the fact that $\theta(\mathbb{Q}_\ell) \circ \rho_{A, \ell} = \rho_{N_{F'}, \ell}$, implies that $I := \rho_{A, \ell}(I_{v'}) \subseteq \mathcal{Z}_A(\mathbb{Q}_\ell)$, where \mathcal{Z}_A denotes the centre of G_A . Fix an algebraic closure Ω of \mathbb{Q}_ℓ . For any $\sigma \in I \subseteq \mathcal{Z}_A(\mathbb{Q}_\ell)$, denote by

$\bar{\sigma}$ the image of σ in $\mathcal{Z}_{A/\Omega}(\Omega)$. As G_A is reductive, $\mathcal{Z}_{A/\Omega}$ is a diagonalizable algebraic group. Since G_A is a subgroup of $\mathrm{GL}(W)$, there exists a matrix $P \in \mathrm{GL}_m(\Omega)$ such that $P \cdot \bar{\sigma} \cdot P^{-1}$ is of the form $\mathrm{diag}(\lambda_1, \dots, \lambda_m)$, where $m := \dim W$. As $\bar{\sigma}$ is unipotent, $\lambda_1 = \dots = \lambda_m = 1$, i.e. $\bar{\sigma}$ is the identity matrix. Since $\sigma \in I$ was an arbitrary element, I is trivial. Now, using the Néron-Ogg-Shafarevich condition [23, Thm. 1], we conclude that A has potential good reduction at v' . \square

End of the proof of Theorem 4.1: By passing over to another finite extension we may suppose that A has good reduction at v' . Now, we have the crystalline realization $H_{A,\mathrm{Cris}} : \langle h^1(A), \mathbb{Q}(1) \rangle^\otimes \rightarrow \mathrm{Vect}_{F_{v'}^0}$, and we write $G_{A,\mathrm{Cris}} := \underline{\mathrm{Aut}}^\otimes(H_{A,\mathrm{Cris}})$. As before, the crystalline Frobenius element for A at v' defines an element $\Phi_{A,\mathrm{Cris}} \in G_{A,\mathrm{Cris}}(F_{v'}^0)$. By passing over to another finite extension we may suppose that $\Phi_{A,\mathrm{Cris}}$ is weakly neat. Again, as $G_{A,\mathrm{Cris}}$ is an inner form of $G_A \otimes_{\mathbb{Q}} F_{v'}^0$, there is an isomorphism $\mathrm{Conj}'(G_{A,\mathrm{Cris}}) \cong (\mathrm{Conj}'(G_A)) \otimes_{\mathbb{Q}} F_{v'}^0$. For our purpose we identify these two varieties. Now, by [17, 4.2] we know that there is an element $\mathrm{Cl}_A \mathrm{Fr}_v \in \mathrm{Conj}'(G_A)(\mathbb{Q})$ such that $\mathrm{Cl}(\rho_{A,\ell}(\Phi_{v'})) = \mathrm{Cl}_A \mathrm{Fr}_v \forall \ell \neq p$ and its image in $\mathrm{Conj}'(G_{A,\mathrm{Cris}})(F_{v'}^0)$ coincides with $\mathrm{Cl}_{\mathrm{Cris}}(\Phi_{A,\mathrm{Cris}})$. Let us write $\mathrm{Frob}_A \in \mathrm{Conj}'(G_A)(F_{v'}^0) = \mathrm{Conj}'(G_{A,\mathrm{Cris}})(F_{v'}^0)$, for this image.

By universal property of (universal) categorical quotients we have the following commutative diagram:

$$\begin{array}{ccc} G_A & \xrightarrow{\mathrm{Cl}} & \mathrm{Conj}' G_A \\ \theta \downarrow & & \downarrow \bar{\theta} \\ G_{N_{F'}} & \xrightarrow{\mathrm{Cl}} & \mathrm{Conj}' G_{N_{F'}} \end{array} \quad (11)$$

Again, the inclusion functor $\langle N_{F'}, \mathbb{Q}(1) \rangle^\otimes \rightarrow \langle h^1(A), \mathbb{Q}(1) \rangle^\otimes$, induces a unique homomorphism

$$\theta_{\mathrm{cris}} : G_{A,\mathrm{Cris}} \rightarrow G_{N_{F'},\mathrm{Cris}}.$$

As before, this gives us the following commutative diagram :

$$\begin{array}{ccc} G_{A,\mathrm{Cris}} & \xrightarrow{\mathrm{Cl}_{\mathrm{Cris}}} & \mathrm{Conj}' G_{A,\mathrm{Cris}} \\ \theta_{\mathrm{cris}} \downarrow & & \downarrow \bar{\theta}_{\mathrm{Cris}} \\ G_{N_{F'},\mathrm{Cris}} & \xrightarrow{\mathrm{Cl}_{\mathrm{Cris}}} & \mathrm{Conj}' G_{N_{F'},\mathrm{Cris}} \end{array} \quad (12)$$

We set $\mathrm{Frob}'_N := \mathrm{Cl}_{\mathrm{Cris}}(\Phi'_{N,\mathrm{Cris}}) \in \mathrm{Conj}'(G_{N_{F'},\mathrm{Cris}})(F_{v'}^0)$. We shall show that $\mathrm{Cl}(\rho_{N_{F'},\ell}(\Phi_{v'})) = \mathrm{Frob}'_N$ for every $\ell \neq p$.

By functoriality, $\theta_{\mathrm{cris}}(F_{v'}^0)(\Phi_{A,\mathrm{Cris}}) = \Phi'_{N,\mathrm{Cris}}$, where $\theta_{\mathrm{cris}}(F_{v'}^0)$ is the induced map on $F_{v'}^0$ -valued points. Thus, by commutativity of (12) we obtain $\mathrm{Frob}'_N = \bar{\theta}_{\mathrm{Cris}}(F_{v'}^0)(\mathrm{Frob}_A)$. Now, as $\theta(\mathbb{Q}_\ell) \circ \rho_{A,\ell} = \rho_{N_{F'},\ell}$ and $\mathrm{Cl}(\rho_{A,\ell}(\Phi_{v'})) = \mathrm{Frob}_A$ for all $\ell \neq p$, it follows that $\mathrm{Cl}(\rho_{N_{F'},\ell}(\Phi_{v'})) = \bar{\theta}(F_{v'}^0)(\mathrm{Frob}_A)$ as elements of $\mathrm{Conj}'(G_{N_{F'}})(F_{v'}^0) = \mathrm{Conj}'(G_{N_{F'},\mathrm{Cris}})(F_{v'}^0)$. Thus the proposition will be proved if we can show that $\mathrm{Frob}'_N = \bar{\theta}(F_{v'}^0)(\mathrm{Frob}_A)$. In other words we need to show that $\bar{\theta}_{\mathrm{Cris}}(F_{v'}^0)(\mathrm{Frob}_A) = \bar{\theta}(F_{v'}^0)(\mathrm{Frob}_A)$. This follows from the next lemma. \square

Lemma 4.3. *The morphism $\bar{\theta}_{/F_v^0} := \bar{\theta} \otimes_{\mathbb{Q}} \text{id} : \text{Conj}'(G_A) \otimes_{\mathbb{Q}} F_v^0 \rightarrow \text{Conj}'(G_{N_{F'}}) \otimes_{\mathbb{Q}} F_v^0$ coincides with $\bar{\theta}_{\text{Cris}}$.*

Proof. As $G_{*,\text{Cris}}$ is an inner form of $G_* \otimes_{\mathbb{Q}} F_v^0$, where $*$ = $N_{F'}$ or A we can identify $G_{*,\text{Cris}} \otimes_{F_v^0} \bar{\mathbb{Q}}_p$ with $G_* \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p$. Then, under this identification and by definition of θ and θ_{Cris} , it follows that the homomorphisms of algebraic groups

$$\begin{aligned} \theta_{/\bar{\mathbb{Q}}_p} &:= \theta \otimes_{\mathbb{Q}} \text{id} : G_A \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \rightarrow G_{N_{F'}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \text{ and} \\ \theta_{\text{Cris}/\bar{\mathbb{Q}}_p} &:= \theta_{\text{Cris}} \otimes_{F_v^0} \text{id} : G_{A,\text{Cris}} \otimes_{F_v^0} \bar{\mathbb{Q}}_p \rightarrow G_{N_{F'},\text{Cris}} \otimes_{F_v^0} \bar{\mathbb{Q}}_p \text{ are identical.} \end{aligned} \quad (13)$$

It follows from (13), that $\bar{\theta}_{/\bar{\mathbb{Q}}_p} = \bar{\theta}_{\text{Cris}/\bar{\mathbb{Q}}_p}$. As $G_{A,\text{Cris}}$ is geometrically reducible (as algebraic variety over F_v^0), so is $\text{Conj}'(G_{A,\text{Cris}})$. Now, it is a well known fact that if two morphisms $f, g : Y \rightarrow Z$ of algebraic varieties Y and Z are identical over the algebraic closure of the base field and Y is geometrically reducible, then f and g are identical. By using this fact we conclude that $\bar{\theta}_{/F_v^0} = \bar{\theta}_{\text{Cris}}$. \square

Proof of Theorem 1.6. By applying Theorem 4.1 to X we obtain an extension F' of F and an element $\text{Frob}'_M \in \text{Conj}'(G_{M_{F'}})(F_v^0)$. Let n be the residual degree of the F'_v/F_v . By hypothesis G_M is connected, which implies that it is geometrically connected because G_M is an algebraic group. In particular $G_M \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \cong G_{M,\text{Cris}} \otimes_{F_v^0} \bar{\mathbb{Q}}_p$ is connected. This implies that $G_{M,\text{Cris}}$ is connected. Now, all Hodge cycles relative to the embedding $\tau : F \hookrightarrow \mathbb{C}$ on M are absolute Hodge cycles, because $M \in \text{Ob}(\mathcal{M}_{AH}^{ab}(F))$. This implies that $G_M^0 = G_{M_{\bar{F}}}$ (see [9, II, 6.23]). Thus, in our situation $G_M = G_{M_{\bar{F}}} = G_{M_{F'}}$ and similarly $G_{M_{F'},\text{Cris}} = G_{M,\text{Cris}}$. We also get $\text{Conj}'(G_{M_{F'}}) = \text{Conj}'(G_M)$ and $\text{Conj}'(G_{M_{F'},\text{Cris}}) = \text{Conj}'(G_{M,\text{Cris}})$.

As the crystalline Frobenius element for $X_{F'}$ at v' is the n -th power of the crystalline Frobenius element for X at v , so $(\Phi_{M,\text{Cris}})^n = \Phi'_{M,\text{Cris}}$ (as elements of $G_{M,\text{Cris}}(F_v^0)$).

Now, let $\varphi_n : G_{M,\text{Cris}} \rightarrow G_{M,\text{Cris}}$ be the n -th power map. The map φ_n is equivariant for the action of conjugation. By the universal property of universal categorical quotient, φ_n induces a map $\bar{\varphi}_n : \text{Conj}'(G_{M,\text{Cris}}) \rightarrow \text{Conj}'(G_{M,\text{Cris}})$ such that $\text{Cl}_{\text{Cris}} \circ \varphi_n = \bar{\varphi}_n \circ \text{Cl}_{\text{Cris}}$. Write $\text{Frob}_M := \text{Cl}_{\text{Cris}}(\Phi_{M,\text{Cris}}) \in \text{Conj}'(G_{M,\text{Cris}})(F_v^0) \subset \text{Conj}'(G_{M,\text{Cris}})(F_v^0)$, then it follows from the above discussion that

$$\bar{\varphi}_n(F_v^0)(\text{Frob}_M) = \text{Frob}'_M. \quad (14)$$

By [15, 1.1] there is an element

$$\text{Cl}_M \text{Fr}_v \in \text{Conj}'(G_M)(\mathbb{Q}) \text{ such that } \text{Cl}(\rho_{M,\ell}(\Phi_v)) = \text{Cl}_M \text{Fr}_v \forall \ell \neq p.$$

Using the inclusions

$$\text{Conj}'(G_M)(\mathbb{Q}) \subset \text{Conj}'(G_M)(F_v^0) \cong \text{Conj}'(G_{M,\text{Cris}})(F_v^0) \subset \text{Conj}'(G_{M,\text{Cris}})(F_v^0),$$

we regard $\text{Cl}_M \text{Fr}_v$ as an element of $\text{Conj}'(G_{M,\text{Cris}})(F_v^0)$. By taking $\Phi_{v'} := \Phi_v^n$ for the arithmetic Frobenius for the extension of F' , we have $\text{Cl}((\rho_{M,\ell}(\Phi_v))^n) = \text{Cl}(\rho_{M_{F'},\ell}(\Phi_{v'}))$.

It follows from this and (14), that $\bar{\varphi}_n(F_{v'}^0)(\text{Cl}_M \text{Fr}_v) = \bar{\varphi}_n(F_{v'}^0)(\text{Frob}_M)$ as elements of $\text{Conj}'(G_{M, \text{Cris}})(F_v^0)$. Since $\text{Cl}_M \text{Fr}_v, \text{Frob}_M \in \text{Conj}'(G_{M, \text{Cris}})(F_v^0)$, we get

$$\bar{\varphi}_n(F_v^0)(\text{Cl}_M \text{Fr}_v) = \bar{\varphi}_n(F_v^0)(\text{Frob}_M) \in \text{Conj}'(G_{M, \text{Cris}})(F_v^0). \quad (15)$$

Now, fix a prime number ℓ and let $\Omega \supset \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell$ be an algebraically closed field. As $M = h^i(X)$, we have $H_\ell(M) = H_{\text{et}}^i(X_{\bar{F}}, \mathbb{Q}_\ell)$. Again by Falting's result (cf. [11, Th.3.2.3]) on the C_{Cris} conjecture $H_{M/\text{Cris}}(M) = H_{\text{Cris}}^i(X_v/W(k_v)) \otimes F_v^0$. As X has good reduction at v , we can use Deligne's result on Weil conjectures [8, I.6] and smooth specialization (proper-smooth base change) to conclude that for every $\ell \neq p$, the characteristic polynomial of $\rho_{M, \ell}(\Phi_v)$ (seen as an element of $\text{GL}(H_\ell(M))$) has coefficients in \mathbb{Q} and is independent of ℓ . Next by a result of Katz and Messing [13, Theorem 1], we conclude that $\Phi_{M, \text{Cris}}$ (seen as an element of $\text{GL}(H_{\text{Cris}}(M))$) has the same characteristic polynomial as $\rho_{M, \ell}(\Phi_v)$. This implies in particular that if $\rho_{M, \ell}(\Phi_v)$ is weakly neat then $\Phi_{M, \text{Cris}}$ is also weakly neat. Now we have two weakly neat elements $\rho_{M, \ell}(\Phi_v)$ and $\Phi_{M, \text{Cris}}$ in $G_M(\Omega) \cong G_{M, \text{Cris}}(\Omega)$ which have same characteristic polynomial. Therefore, by Lemma 3.1 and (15) we conclude that $\text{Frob}_M = \text{Cl}_M \text{Fr}_v$. In other words, $\text{Cl}(\rho_{M, \ell}(\Phi_v)) = \text{Frob}_M$. □

By using the previous theorem (and its proof) combined with Lemma 2.2, we easily relate the element $\text{Frob}_{v, i} \in \text{Conj}'(G_\infty^i)(\mathbb{Q})$ of Theorem 1.5 to the conjugacy class of the crystalline Frobenius $\Phi_{\text{Cris}, i}$ associated to the motive $h^i(X)$.

Corollary 4.4. *Let the notation and hypothesis be as in Theorem 1.4. Let v be a non-archimedean valuation and $i \in \mathbb{N}$. Assume that X has good reduction at v and there is a prime number ℓ_0 such that $\rho_{\ell_0}^i(\Phi_v)$ is weakly neat. Then, the image of the element $\text{Frob}_{v, i}$ in $\text{Conj}'(G_\infty^i)(F_v^0)$ coincides with $\text{Cl}_{\text{Cris}}(\Phi_{\text{Cris}, i})$.*

Remark 4.5. We may also include abelian varieties and curves in the previous corollary.

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