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Noether invariants for constant mean curvature surfaces in 3-dimensional homogeneous spaces

Sébastien Cartier

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Abstract

We give explicit formulæ for Noether invariants associated to Killing vector fields for the variational problem of minimal and constant mean curvature surfaces in 3-manifolds. In the case of homogeneous spaces, such invariants are the flux (associated to translations) and the torque (associated to rotations). Then we focus on homogeneous spaces with isometry groups of dimensions 3 or 4 and study the behavior of these invariants under the action of isometries. Finally, we give examples of actual computations and of interpretations of these invariants in different situations.

Mathematics Subject Classification: Primary 53C42; Secondary 53A55.

1 Introduction

The differential Noether theorem [6] describes an isomorphism between the Lie algebra of infinitesimal generators of the *variational symmetries* associated to a variational problem and a space of *conservation laws* for the related Euler-Lagrange equations. In particular, it can be applied to the variational problem of minimal or constant mean curvature (*CMC* for short) surfaces in a homogeneous space using the isometries of the ambient space as symmetries – for the isometries preserve the mean curvature. In the case of minimal surfaces in the euclidean 3-space, Noether theorem leads to the notions of flux and torque, which are geometric invariants of the surfaces. And these geometric constrains can be used to find alignment conditions on the catenoidal ends of a surface [9].

The present paper gives tools to use Noether invariants related to minimal and CMC surfaces in homogeneous manifolds. In Section 2, we give explicit formulæ for Noether forms associated to Killing fields, see Theorem 2.6 and Proposition 2.8. In Sections 3 and 4, we focus on minimal and CMC surfaces in homogeneous spaces $\mathbb{E}^3(\kappa, \tau)$ and Sol_3 respectively, corresponding to isometries of the ambient space. As in the euclidean case, these forms lead to invariants, namely the *flux* and *torque*, related to the geometry of the surface. And in Section 5, we give examples of actual computations of Noether invariants in different situations.

The construction can be written in coordinates using jet bundles [7] or more abstractly using basic tools of contact geometry [2, 8]. We choose the second approach, which is coordinate-free and allows us to provide general formulæ.

2 General results

2.1 Contact structure and lagrangians

The present subsection deals with classical results on contact structures and lagrangians. Details on the notions introduced can be found in [2].

Let $(M, \langle \cdot, \cdot \rangle)$ be a 3-dimensional riemannian manifold and consider the following fibration:

$$FM \xrightarrow{\pi'} \mathcal{C} \xrightarrow{\pi} M,$$

where the contact manifold \mathcal{C} is the unit fiber bundle UM of M – or equivalently the Grassmannian of oriented 2-planes tangent to M – and FM the orthonormal frame bundle. Since the study is local, we consider a local chart on M with generic point x . An element of \mathcal{C} is a couple (x, e_0) with $e_0 \in \mathbb{S}^2$ and a point of FM writes (x, e) where $e = (e_0, e_1, e_2)$ is an orthonormal family with respect to $\langle \cdot, \cdot \rangle_x$. Finally, the projections π' and π are respectively:

$$\pi'(x, e) = (x, e_0) \quad \text{and} \quad \pi(x, e_0) = x.$$

In the sequel, we work in FM to facilitate computations, but actually the quantities we define are *basic*, i.e. they are liftings of quantities defined on \mathcal{C} . To ease the understanding, we use the same notation for a quantity and its liftings. Also, we do not distinguish an infinitesimal generator of an action on M from its *extensions* to \mathcal{C} or FM , i.e. the infinitesimal generators of the natural extension of the action.

If $e = (e_0, e_1, e_2)$ is an orthonormal frame on M , denote $(\theta^0, \theta^1, \theta^2)$ dual basis composed of 1-forms and consider elements $(\omega_j^i)_{0 \leq i, j \leq 2}$ of $\Omega^1(FM)$ such that:

$$\begin{aligned} d\theta^0 &= -\omega_1^0 \wedge \theta^1 + \omega_0^2 \wedge \theta^2, & d\theta^1 &= \omega_1^0 \wedge \theta^0 - \omega_2^1 \wedge \theta^2, \\ d\theta^2 &= -\omega_0^2 \wedge \theta^0 + \omega_2^1 \wedge \theta^1 & \text{and} & \quad \omega_j^i = -\omega_i^j. \end{aligned}$$

The *structure forms* $\theta^0, \theta^1, \theta^2, \omega_1^0, \omega_2^1$ and ω_0^2 are independent and generate $\Omega^1(FM)$.

Proposition 2.1. *Let $\theta^0 \in \Omega^1(\mathcal{C})$ be defined as follows:*

$$\forall (x, e_0) \in \mathcal{C}, \quad \forall (u, \xi) \in T_{(x, e_0)}\mathcal{C}, \quad \theta_{(x, e_0)}^0(u, \xi) = \langle e_0, u \rangle_x.$$

If I is the line subfiber bundle of $T^\mathcal{C}$ generated by θ^0 , then (\mathcal{C}, I) is a contact structure and in the sequel we call θ^0 the contact form.*

The *contact ideal* $\mathcal{I} \subset \Omega^*(\mathcal{C})$ is the ideal – with respect to the exterior product – generated by $\{\theta^0, d\theta^0\}$. Lifting e_0 to an element (e_0, e_1, e_2) of FM with dual basis $(\theta^0, \theta^1, \theta^2)$, the 1-form θ^0 on FM coincides with the lifting of the contact form, which is why they are denoted the same.

If $f : \Sigma \rightarrow M$ is an immersion of a simply connected surface Σ , there exists a *legendrian lift* $N : \Sigma \rightarrow \mathcal{C}$ of f to \mathcal{C} , which means that N verifies $N^*\theta^0 = 0$ and $f = \pi \circ N$. Note that, by construction of θ^0 , the lift N is unique up to sign and it is a normal vector to f . Moreover, the condition $N^*\theta^0 = 0$ implies $N^*d\theta^0 = 0$, and thus $N^*\mathcal{I} = \{0\}$.

The study is local, so we can assume Σ is compact, eventually with boundary. Consider the functional \mathcal{A} such that:

$$\mathcal{A}(\Sigma) = \int_{\Sigma} N^*\Lambda_0 \quad \text{with} \quad \Lambda_0 = e_0 \lrcorner \text{vol}_M,$$

where vol_M is the volume form on M . We call Λ_0 the *lagrangian* of the functional. Actually, \mathcal{A} is the area functional, since an expression of Λ_0 in FM is $\Lambda_0 = \theta^1 \wedge \theta^2$, the volume form being $\text{vol}_M = \theta^0 \wedge \theta^1 \wedge \theta^2$. A classical result on the area functional is the following:

Proposition 2.2. *Let $f : \Sigma \rightarrow M$ be an immersion with legendrian lift $N : \Sigma \rightarrow \mathcal{C}$. Then f is a critical point of the functional \mathcal{A} if and only if the Euler-Lagrange condition $N^*\Psi_0$ is satisfied, where:*

$$\Psi_0 = -\omega_0^2 \wedge \theta^1 - \omega_1^0 \wedge \theta^2$$

is the Euler-Lagrange operator. Moreover, if f is a critical point of \mathcal{A} , then the Euler-Lagrange condition means that it is a minimal immersion.

In the following, fix H a constant, eventually zero. The variational characterization of CMC- H immersions in M can be deduced from the previous result on minimal immersions by adding a Lagrange multiplier to grasp the volume constraint when $H \neq 0$. Remark first that:

Lemma 2.3. *In any riemannian manifold (M, g) of finite dimension, there exists locally a vector field $\Xi \in \mathfrak{X}(M)$, which we call a volume field, such that:*

$$\text{div}_M \Xi = 1.$$

We use such a vector field to write the lagrangian involved in the variational characterization of CMC- H immersion:

Proposition 2.4. *Let Ξ be a volume field on M . The lagrangian Λ defined on \mathcal{C} by:*

$$\Lambda = \Lambda_0 + 2H\Lambda' \quad \text{with} \quad \Lambda' = \Xi \lrcorner \text{vol}_M, \quad (1)$$

is associated to the variational problem of CMC- H immersions in M . In other words, an immersion $f : \Sigma \rightarrow M$ with legendrian lift $N : \Sigma \rightarrow \mathcal{C}$ is a critical point of

the functional associated to Λ if and only if it is a CMC-H immersion. Furthermore, the Euler-Lagrange operator writes $\Psi = \Psi_0 + 2H\Lambda_0$ and the Euler-Lagrange equation is $N^*\Psi = 0$.

We define the *Euler-Lagrange system* as the differential ideal $\mathcal{E} \subset \Omega^*(\mathcal{C})$ generated by $\{\theta^0, d\theta^0, \Psi\}$. Hence, to determine a Noether form related to a minimal or CMC immersion $f : \Sigma \rightarrow M$, we only need to compute a class of forms on \mathcal{C} modulo the ideal \mathcal{E} and pull it back in $\Omega^1(\Sigma)$.

2.2 Symmetries and Noether forms

We call a *divergence symmetry* of the variational problem with lagrangian Λ defined by (1), any element $S \in \mathfrak{X}(\mathcal{C})$ for which there exists a class $\Phi_S \in H^1(\mathcal{C})$ such that $\mathcal{L}_S\Lambda \equiv d\varphi \pmod{\mathcal{E}}$, for an arbitrary $\varphi \in \Phi_S$. Noether theorem states then:

Theorem 2.5 (Noether, 1918 [6]). *Any divergence symmetry S is in one-to-one correspondence with a class of 1-forms $\mu_S \in H^1(\mathcal{C})/\mathcal{E}$ defined by:*

$$\mu_S = S \lrcorner \Lambda - \varphi \quad \text{in } H^1(\mathcal{C})/\mathcal{E} \quad \text{with } \varphi \in \Phi_S.$$

Moreover, if $N : \Sigma \rightarrow \mathcal{C}$ is the legendrian lift of a critical point of Λ – i.e. a CMC-H immersion –, then the pull back $N^*\mu_S$ is a closed form on Σ and the quantity:

$$\sigma_S(c) = \int_c N^*\mu_S$$

is the Noether invariant or conserved quantity associated to S along the cycle $c \in H_1(\Sigma)$.

Note that the closure condition for $N^*\mu_S$ is the *conservation law* mentioned in the introduction. The expression of the Noether form can be made completely explicit in the case of Killing fields:

Theorem 2.6. *Let $S \in \mathfrak{X}(M)$ be a Killing field. Then the extension of S to \mathcal{C} is a divergence symmetry. Furthermore, if $F \in \mathfrak{X}(\mathcal{C})$ is the extension of a potential vector of S – i.e. a field $F \in \mathfrak{X}(M)$ such that $\text{curl}_M F = S$ –, then the corresponding Noether form μ_S writes:*

$$\mu_S = S \lrcorner \Lambda_0 - 2HF^b \quad \text{in } H^1(\mathcal{C})/\mathcal{E},$$

and this expression does not depend on the choice of the potential vector.

Proof. Since S is Killing field on M , we have $\text{div}_M S = 0$. Thus, there exists a field $F \in \mathfrak{X}(M)$ such that $S = \text{curl}_M F$. Moreover, we have $\mathcal{L}_S\Lambda \equiv 2H\mathcal{L}_S\Lambda' \pmod{\mathcal{E}}$, with by definition $\mathcal{L}_S\Lambda' = d(S \lrcorner \Lambda') + S \lrcorner d\Lambda'$. If $*$ denotes the Hodge operator and \cdot^b, \cdot^\sharp the musical isomorphisms, we know that:

$$S = \text{curl}_M F = (*dF^b)^\sharp \quad \text{and} \quad S \lrcorner d\Lambda' = S \lrcorner \text{vol}_M = *S^b = *^2 dF^b = dF^b.$$

Hence:

$$\mathcal{L}_S \Lambda \equiv 2Hd(S \lrcorner \Lambda' + F^b) \pmod{\mathcal{E}},$$

and S is indeed a divergence symmetry. The Noether form associated to S is:

$$\mu_S = S \lrcorner \Lambda - 2H(S \lrcorner \Lambda' + F^b) = S \lrcorner \Lambda_0 - 2HF^b \quad \text{in } H^1(\mathcal{C})/\mathcal{E},$$

and this expression does not depend on the choice of the potential vector F since $d(F^b - \tilde{F}^b) = 0$ for any other choice \tilde{F} of a potential vector of S . \square

Corollary 2.7. *If M is a homogeneous space, the extensions of infinitesimal generators of 1-parameter families of isometries are divergence symmetries.*

Let $f : \Sigma \rightarrow M$ be an (oriented) CMC- H immersion. We choose its legendrian lift $N : \Sigma \rightarrow \mathcal{C}$ so that it coincides with the unit normal to f . Since $df = e_1\theta^1 + e_2\theta^2$ and $*df = -e_2\theta^1 + e_1\theta^2$, we have:

Proposition 2.8. *The pullback $N^*\mu_S$ is well defined in $H^1(\Sigma)$ and writes:*

$$N^*\mu_S = N^*\mu_S^0 - 2HN^*\mu'_S \quad \text{with} \quad \mu_S^0 = \langle S, *df \rangle \quad \text{and} \quad \mu'_S = \langle F, df \rangle.$$

We call μ_S^0 the minimal part of the Noether form and μ'_S its CMC part.

In the cases of homogeneous spaces $\mathbb{E}^3(\kappa, \tau)$ and Sol_3 (Sections 3 and 4 respectively), we denote $\sigma_i(\cdot)$, with $i = 1, 2, 3, R$, the Noether invariants corresponding to isometries (1, 2, 3 for translations and R for the rotation when it exists). The *flux* through a cycle $c \in H_1(\Sigma)$ is the vector $\sigma(c) = (\sigma_1(c), \sigma_2(c), \sigma_3(c))$ and the *torque* is the number $\sigma_R(c)$ when it exists.

If $\mathcal{S}(t)$ is a 1-parameter family of isometries with infinitesimal generator S , we know from Proposition 2.8 that determining the corresponding Noether form μ_S restricted to f means actually computing S and a potential vector F .

It is also interesting to study the relations between Noether invariant of congruent immersions – i.e. immersions deduced from one another by the action of an isometry. Namely, considering $\mathcal{S}'(t)$ a(nother) 1-parameter family of isometries, we want to compare the form μ_S in restriction to an immersion f and the corresponding form denoted $\mu_S(\mathcal{S}'(t))$ in restriction to the immersion $\mathcal{S}'(t) \circ f$. Remark that:

$$\mu_S(\mathcal{S}'(t)) = \langle d\mathcal{S}'(t)^{-1} \cdot S(\mathcal{S}(t)), *df \rangle - 2H \langle d\mathcal{S}'(t)^{-1} \cdot F(\mathcal{S}(t)), df \rangle.$$

3 Noether forms in $\mathbb{E}^3(\kappa, \tau)$

The spaces $\mathbb{E}^3(\kappa, \tau)$ are simply connected 3-dimensional homogeneous spaces. They are characterized by real parameters κ and τ such that $\kappa - 4\tau^2 \neq 0$. The model considered is $\Omega_\kappa \times \mathbb{R} \subset \mathbb{R}^3$, with generic coordinates $(w = x_1 + ix_2, x_3)$ and:

$$\Omega_\kappa = \begin{cases} \mathbb{C} & \text{if } \kappa \geq 0 \\ \mathbb{D}(2|\kappa|^{-1/2}) & \text{if } \kappa < 0 \end{cases},$$

endowed with the metric:

$$ds^2 = \lambda^2 |dw|^2 + (\tau \lambda (x_2 dx_1 - x_1 dx_2) + dx_3)^2 \quad \text{with} \quad \lambda = \frac{1}{1 + \kappa' |w|^2} \quad \text{and} \quad \kappa' = \frac{\kappa}{4}.$$

These spaces are riemannian fibrations of the *base* Ω_κ for the natural projection $\mathbb{E}^3(\kappa, \tau) \rightarrow \Omega_\kappa$ on the first two coordinates. The parameter κ can be interpreted as the curvature of the base and τ as the one of the fibration. Thus, the space $\mathbb{E}^3(\kappa, \tau)$ has the geometry of Berger spheres if $\kappa > 0$, the one of the Heisenberg group if $\kappa = 0$ and the geometry of the universal cover of $\text{PSL}_2(\mathbb{R})$ if $\kappa < 0$ – in the latter however, the model used corresponds to the universal cover of $\text{PSL}_2(\mathbb{R})$ minus a fiber. In Section 5, we focus on the Heisenberg group $\text{Nil}_3 = \mathbb{E}^3(0, 1/2)$ and the product space $\mathbb{H}^2 \times \mathbb{R} = \mathbb{E}^3(-1, 0)$.

We consider the orthonormal frame (E_1, E_2, E_3) defined when $\tau \neq 0$ by:

$$\begin{aligned} E_1 &= \frac{1}{\lambda} \left(\cos(\sigma x_3) \frac{\partial}{\partial x_1} + \sin(\sigma x_3) \frac{\partial}{\partial x_2} \right) + \tau (x_1 \sin(\sigma x_3) - x_2 \cos(\sigma x_3)) \frac{\partial}{\partial x_3}, \\ E_2 &= \frac{1}{\lambda} \left(-\sin(\sigma x_3) \frac{\partial}{\partial x_1} + \cos(\sigma x_3) \frac{\partial}{\partial x_2} \right) + \tau (x_1 \cos(\sigma x_3) + x_2 \sin(\sigma x_3)) \frac{\partial}{\partial x_3} \\ &\quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3} \quad \text{with} \quad \sigma = \frac{\kappa}{2\tau}, \end{aligned}$$

and if $\tau = 0$:

$$E_1 = \frac{1}{\lambda} \frac{\partial}{\partial x_1}, \quad E_2 = \frac{1}{\lambda} \frac{\partial}{\partial x_2} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3}.$$

The space spanned by E_1, E_2 is said to be *horizontal* and the vector field E_3 is a Killing field.

3.1 Isometries and curl operator

A natural volume field is $\Xi = x_3 E_3$. Note that in the case of Berger spheres ($\kappa > 0$ and $\tau \neq 0$), this field is not globally defined.

The isometry group of $\mathbb{E}^3(\kappa, \tau)$ is 4-dimensional, generated by three 1-parameter families of translations et one of rotations:

$$\begin{aligned} \mathcal{S}_1(t)(w, x_3) &= \left(\frac{t+w}{1-\kappa' t w}, x_3 + \frac{4}{\sigma} \arctan \left(\frac{\kappa' t x_2}{1-\kappa' t x_1 + |1-\kappa' t w|} \right) \right), \\ \mathcal{S}_2(t)(w, x_3) &= \left(\frac{it+w}{1+i\kappa' t w}, x_3 - \frac{4}{\sigma} \arctan \left(\frac{\kappa' t x_1}{1-\kappa' t x_2 + |1+i\kappa' t w|} \right) \right), \\ \mathcal{S}_3(t)(w, x_3) &= (w, x_3 + t) \quad \text{and} \quad \mathcal{S}_R(t)(w, x_3) = (w e^{it}, x_3) \quad \text{with} \quad \sigma = \frac{\kappa}{2\tau}. \end{aligned}$$

Infinitesimal generators of these families are respectively:

$$\begin{aligned} S_1 &= \left(1 + \kappa'(x_1^2 - x_2^2)\right) \frac{\partial}{\partial x_1} + 2\kappa'x_1x_2 \frac{\partial}{\partial x_2} + \tau x_2 \frac{\partial}{\partial x_3}, \\ S_2 &= 2\kappa'x_1x_2 \frac{\partial}{\partial x_1} + \left(1 - \kappa'(x_1^2 - x_2^2)\right) \frac{\partial}{\partial x_2} - \tau x_1 \frac{\partial}{\partial x_3}, \\ S_3 &= \frac{\partial}{\partial x_3} \quad \text{and} \quad S_R = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}. \end{aligned}$$

Let $X \in \mathfrak{X}(\mathbb{E}^3(\kappa, \tau))$ be written as $X = X^1E_1 + X^2E_2 + X^3E_3$. The expression of $\text{curl } X$ depends on τ . If $\tau \neq 0$:

$$\begin{aligned} \text{curl } X &= \left(dX^3(E_2) - dX^2(E_3) - \sigma X^1\right) E_1 + \left(dX^1(E_3) - dX^3(E_1) - \sigma X^2\right) E_2 \\ &\quad + \left(dX^2(E_1) - dX^1(E_2) - 2\tau X^3\right) E_3, \end{aligned}$$

and if $\tau = 0$ we have:

$$\begin{aligned} \text{curl } X &= \left(dX^3(E_2) - dX^2(E_3)\right) E_1 + \left(dX^1(E_3) - dX^3(E_1)\right) E_2 \\ &\quad + \left(dX^2(E_1) - dX^1(E_2) + 2\kappa'(x_2X^1 - x_1X^2)\right) E_3. \end{aligned}$$

For the horizontal translations and the rotation, the potential vectors are, if $\kappa \neq 0$:

$$F_1 = \frac{1}{\sigma} S_1^h + \lambda x_2 E_3, \quad F_2 = \frac{1}{\sigma} S_2^h - \lambda x_1 E_3 \quad \text{and} \quad F_R = \frac{1}{\sigma} S_R^h + \frac{\lambda}{2\kappa'} E_3,$$

where \cdot^h denotes the horizontal part, and if $\kappa = 0$:

$$F_1 = (\tau x_1 x_2 - x_3) E_2, \quad F_2 = (\tau x_1 x_2 + x_3) E_1 \quad \text{and} \quad F_R = x_1 x_3 E_1 + x_2 x_3 E_2.$$

The case of the vertical translation is discriminated by τ :

$$F_3 = \begin{cases} -\frac{1}{2\tau} E_3 & \text{if } \tau \neq 0 \\ -\frac{x_2}{2} E_1 + \frac{x_1}{2} E_2 & \text{if } \tau = 0 \end{cases}.$$

3.2 Evolution under the action of isometries

3.2.1 If $\tau \neq 0$

We have the following behavior for the Noether forms:

$$\begin{aligned} \mu_1(\mathcal{S}_1(t)) &= \mu_1(\mathcal{S}_3(t)) = \mu_1, \quad \mu_1(\mathcal{S}_2(t)) = \frac{1 - \kappa' t^2}{1 + \kappa' t^2} \mu_1 + \frac{2t}{1 + \kappa' t^2} (2\kappa' \mu_R + \tau \mu_3) \\ &\quad \text{and} \quad \mu_1(\mathcal{S}_R(t)) = \cos t \mu_1 - \sin t \mu_2, \\ \mu_2(\mathcal{S}_1(t)) &= \frac{1 - \kappa' t^2}{1 + \kappa' t^2} \mu_2 - \frac{2t}{1 + \kappa' t^2} (2\kappa' \mu_R + \tau \mu_3), \quad \mu_2(\mathcal{S}_2(t)) = \mu_2(\mathcal{S}_3(t)) = \mu_2, \\ &\quad \text{and} \quad \mu_2(\mathcal{S}_R(t)) = \cos t \mu_2 + \sin t \mu_1, \end{aligned}$$

$$\begin{aligned}
\mu_3(\mathcal{S}_1(t)) &= \mu_3(\mathcal{S}_2(t)) = \mu_3(\mathcal{S}_3(t)) = \mu_3(\mathcal{S}_R(t)) = \mu_3, \\
\mu_R(\mathcal{S}_1(t)) &= \frac{1 - \kappa' t^2}{1 + \kappa' t^2} \mu_R + \frac{t}{1 + \kappa' t^2} (\mu_2 - \tau t \mu_3), \\
\mu_R(\mathcal{S}_2(t)) &= \frac{1 - \kappa' t^2}{1 + \kappa' t^2} \mu_R - \frac{t}{1 + \kappa' t^2} (\mu_1 + \tau t \mu_3) \\
&\text{and } \mu_R(\mathcal{S}_3(t)) = \mu_R(\mathcal{S}_R(t)) = \mu_R.
\end{aligned}$$

We deduce the values of Noether forms depending on the symmetries of the surface:

Proposition 3.1. *Let $f : \Sigma \rightarrow \mathbb{E}^3(\kappa, \tau)$, $\tau \neq 0$, be a minimal or CMC immersion. We have the following assertions:*

- (i) *Suppose f is invariant under the action of a translation $\mathcal{S}_1(t)$ (resp. $\mathcal{S}_2(t)$). Then $\mu_2 = \mu_3 = 0$ (resp. $\mu_1 = \mu_3 = 0$) if $\kappa = 0$, and $\mu_2 = \kappa \mu_R + 2\tau \mu_3 = 0$ (resp. $\mu_1 = \kappa \mu_R + 2\tau \mu_3 = 0$) if $\kappa \neq 0$.*
- (ii) *Suppose f is invariant for a rotation $\mathcal{S}_R(t)$. If c is a cycle homologous to its image $\mathcal{S}_R(t) \cdot c$, then $\mu_1 = \mu_2 = 0$.*

3.2.2 If $\tau = 0$

Noether forms associated to horizontal translations and the rotation evolve the same:

$$\begin{aligned}
\mu_1(\mathcal{S}_1(t)) &= \mu_1(\mathcal{S}_3(t)) = \mu_1, \quad \mu_1(\mathcal{S}_2(t)) = \frac{1 - \kappa' t^2}{1 + \kappa' t^2} \mu_1 + \frac{4\kappa' t}{1 + \kappa' t^2} \mu_R, \\
&\text{and } \mu_1(\mathcal{S}_R(t)) = \cos t \mu_1 - \sin t \mu_2, \\
\mu_2(\mathcal{S}_1(t)) &= \frac{1 - \kappa' t^2}{1 + \kappa' t^2} \mu_2 - \frac{4\kappa' t}{1 + \kappa' t^2} \mu_R, \quad \mu_2(\mathcal{S}_2(t)) = \mu_2(\mathcal{S}_3(t)) = \mu_2, \\
&\text{and } \mu_2(\mathcal{S}_R(t)) = \cos t \mu_2 + \sin t \mu_1, \\
\mu_R(\mathcal{S}_1(t)) &= \frac{1 - \kappa' t^2}{1 + \kappa' t^2} \mu_R + \frac{t}{1 + \kappa' t^2} \mu_2, \quad \mu_R(\mathcal{S}_2(t)) = \frac{1 - \kappa' t^2}{1 + \kappa' t^2} \mu_R - \frac{t}{1 + \kappa' t^2} \mu_1 \\
&\text{and } \mu_R(\mathcal{S}_3(t)) = \mu_R(\mathcal{S}_R(t)) = \mu_R.
\end{aligned}$$

However, for the form corresponding to \mathcal{S}_3 , the minimal part remains the same and the CMC part verifies:

$$\begin{aligned}
\mu'_3(\mathcal{S}_1(t)) &= \frac{1}{|1 - \kappa' t w|^2} \mu'_3 + \frac{t}{2\lambda |1 - \kappa' t w|^2} \langle \mathcal{S}_2, df \rangle, \\
\mu'_3(\mathcal{S}_2(t)) &= \frac{1}{|1 + i\kappa' t w|^2} \mu'_3 - \frac{t}{2\lambda |1 + i\kappa' t w|^2} \langle \mathcal{S}_1, df \rangle, \\
\mu'_3(\mathcal{S}_3(t)) &= \mu'_3 \quad \text{and} \quad \mu'_3(\mathcal{S}_R(t)) = \mu'_3.
\end{aligned}$$

Proposition 3.2. *Let $f : \Sigma \rightarrow \mathbb{E}^3(\kappa, 0)$ be a minimal or CMC immersion. We have the following assertions:*

(i) Suppose f is invariant under the action of a translation $\mathcal{S}_1(t)$ (resp. $\mathcal{S}_2(t)$). Then $\mu_2 = \mu_R = 0$ (resp. $\mu_1 = \mu_R = 0$).

(ii) Suppose f is invariant for a rotation $\mathcal{S}_R(t)$. If c is a cycle homologous to $\mathcal{S}_R(t) \cdot c$, then $\mu_1 = \mu_2 = 0$.

4 Noether forms in Sol_3

The space Sol_3 is the 3-dimensional Lie group:

$$\text{Sol}_3 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \right\}, \quad ds^2 = e^{2x_3} dx_1^2 + e^{-2x_3} dx_2^2 + dx_3^2.$$

We can also define a canonical frame (E_1, E_2, E_3) on Sol_3 , given by:

$$E_1 = e^{-x_3} \frac{\partial}{\partial x_1}, \quad E_2 = e^{x_3} \frac{\partial}{\partial x_2} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3}.$$

4.1 Isometries and curl operator

As in the $\mathbb{E}^3(\kappa, \tau)$, a natural volume field is $\Xi = x_3 E_3$.

The isometry group of Sol_3 is of dimension 3, generated by the following three 1-parameter families of translations:

$$\begin{aligned} \mathcal{S}_1(t)(x_1, x_2, x_3) &= (x_1 + t, x_2, x_3), & \mathcal{S}_2(t)(x_1, x_2, x_3) &= (x_1, x_2 + t, x_3) \\ \text{and } \mathcal{S}_3(t)(x_1, x_2, x_3) &= (e^{-t}x_1, e^t x_2, x_3 + t). \end{aligned}$$

The infinitesimal generators are respectively:

$$S_1 = \frac{\partial}{\partial x_1}, \quad S_2 = \frac{\partial}{\partial x_2} \quad \text{and} \quad S_3 = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}.$$

Consider $X \in \mathfrak{X}(\text{Sol}_3)$ written $X = X^1 E_1 + X^2 E_2 + X^3 E_3$ in the canonical frame. The curl of X is:

$$\begin{aligned} \text{curl } X &= \left(dX^3(E_2) - dX^2(E_3) + X^2 \right) E_1 + \left(dX^1(E_3) - dX^3(E_1) + X^1 \right) E_2 \\ &\quad + \left(dX^2(E_1) - dX^1(E_2) \right) E_3. \end{aligned}$$

We deduce expressions of potential vectors:

$$F_1 = x_2 E_3, \quad F_2 = -x_1 E_3 \quad \text{and} \quad F_3 = -\frac{x_2 e^{-x_3}}{2} E_1 + \frac{x_1 e^{x_3}}{2} E_2 - x_1 x_2 E_3.$$

4.2 Evolution under the action of isometries

The expressions of the Noether forms are simpler in the case of Sol_3 than in the $\mathbb{E}^3(\kappa, \tau)$. We have directly:

$$\begin{aligned}\mu_1(\mathcal{S}_1(t)) &= \mu_1(\mathcal{S}_2(t)) = \mu_1 & \text{and} & & \mu_1(\mathcal{S}_3(t)) &= e^t \mu_1, \\ \mu_2(\mathcal{S}_1(t)) &= \mu_2(\mathcal{S}_2(t)) = \mu_2 & \text{and} & & \mu_2(\mathcal{S}_3(t)) &= e^{-t} \mu_2, \\ \mu_3(\mathcal{S}_1(t)) &= \mu_3 - t\mu_1, & \mu_3(\mathcal{S}_2(t)) &= \mu_3 + t\mu_2 & \text{and} & \mu_3(\mathcal{S}_3(t)) = \mu_3.\end{aligned}$$

Proposition 4.1. *Let $f : \Sigma \rightarrow Sol_3$ be a minimal or CMC immersion. If f is invariant under the action of a horizontal translation $\mathcal{S}_1(t)$ (resp. $\mathcal{S}_2(t)$), then $\mu_1 = 0$ (resp. $\mu_2 = 0$). And if f is invariant for a vertical translation $\mathcal{S}_3(t)$, then $\mu_1 = \mu_2 = 0$.*

5 Examples

5.1 Vertical catenoids in Nil_3

In the Heisenberg group, using notations of P. Bérard and M. P. Calvacante in [1], the vertical catenoids come as a 1-parameter family (\mathcal{C}_a) of rotationally invariant minimal, where a is a positive parameter. A catenoid \mathcal{C}_a , for some $a > 0$, can be parametrized as:

$$X_a : (t, \theta) \in \mathbb{R} \times [0, 2\pi] \mapsto (f(a, t) \cos \theta, f(a, t) \sin \theta, t) \in Nil_3,$$

where $f(a, \cdot)$ is a positive function which is a global solution of the Cauchy problem:

$$f(f^2 + 4)f_{tt} = 4(1 + f_t^2), \quad f(0) = a \quad \text{and} \quad f_t(0) = 0.$$

The parameter a is indeed the *size of the neck*. Consider for any fixed $t \in \mathbb{R}$, the closed curve $\mathcal{C}_a \cap \{x_3 = t\}$, parametrized by:

$$\theta \in [0, 2\pi] \mapsto (f(a, t) \cos \theta, f(a, t) \sin \theta, t) \in Nil_3$$

An orthonormal basis (e_1, e_2) of the tangent space to \mathcal{C}_a is:

$$\begin{aligned}e_1 &= \frac{2}{\sqrt{4 + f^2}} \left(-\sin \theta E_1 + \cos \theta E_2 - \frac{f}{2} E_3 \right) \\ \text{and } e_2 &= -\frac{\sqrt{4 + f^2}}{\sqrt{4 + f_t^2(4 + f^2)}} \left[\left(f_t \cos \theta - \frac{2f}{4 + f^2} \sin \theta \right) E_1 \right. \\ &\quad \left. + \left(f_t \sin \theta + \frac{2f}{4 + f^2} \cos \theta \right) E_2 + \frac{4}{4 + f^2} E_3 \right],\end{aligned}$$

with e_1 tangent to the curve $\mathcal{C}_a \cap \{x_3 = t\}$.

If S is the infinitesimal generator of a 1-parameter family of isometries, we have:

$$\sigma_S = -\frac{f}{2}\sqrt{4+f^2}\int_0^{2\pi}\langle S, e_2\rangle d\theta,$$

and, along the curve, the infinitesimal generators S_1, S_2, S_3, S_R are:

$$\begin{aligned} S_1 &= E_1 + f \sin \theta E_3, & S_2 &= E_2 - f \cos \theta E_3, & S_3 &= E_3 \\ \text{and } S_R &= f(-\sin \theta E_1 + \cos \theta E_2) - \frac{f^2}{2}E_3. \end{aligned}$$

We obtain $\sigma_1 = \sigma_2 = \sigma_R = 0$ and:

$$\sigma_3 = 2\pi \frac{2f}{\sqrt{4+f_t^2(4+f^2)}}.$$

This expression of σ_3 is given in [1, Proposition 2.2] up to the 2π factor. Moreover, as σ_3 is independent of t , we can make $t = 0$ in its expression to get $\sigma_3 = 2\pi a$. Hence, it appears that the vertical flux controls the size of the neck.

5.2 Horizontal catenoids in Nil_3

We follow the notations of B. Daniel and L. Hauswirth in [4]. Consider the coordinates (y_1, y_2, y_3) on Nil_3 defined by:

$$y_1 = x_1, \quad y_2 = x_2 \quad \text{and} \quad y_3 = x_3 + \frac{x_1 x_2}{2}.$$

The metric is $dy_1^2 + dy_2^2 + (y_1 dy_2 - dy_3)^2$ and the change of basis on the tangent space writes:

$$\frac{\partial}{\partial y_1} = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial y_2} = \frac{\partial}{\partial x_2} - \frac{x_1}{2} \frac{\partial}{\partial x_3} \quad \text{and} \quad \frac{\partial}{\partial y_3} = \frac{\partial}{\partial x_3}.$$

In these coordinates, the immersion $f_\alpha = (F_1, F_2, h) : \mathbb{C} \rightarrow \text{Nil}_3$ describing the catenoid \mathcal{C}_α of parameter $\alpha > 0$ is:

$$\begin{aligned} F_1(u, v) &= \frac{G'(u)}{\alpha} \cos \varphi(u) \sinh A(u, v) - \frac{C}{\alpha} \sin \varphi(u) \cosh A(u, v), \\ F_2(u, v) &= \frac{C}{\alpha} A(u, v) - \frac{C}{\alpha} \beta(u) - G(u) \end{aligned}$$

$$\begin{aligned} \text{and } h(u, v) &= \frac{C}{\alpha} \left(\frac{G'(u)}{\alpha} - 1 \right) \cos \varphi(u) \cosh A(u, v) \\ &\quad - \frac{1}{\alpha} \left(\frac{C^2}{\alpha} + G'(u) \right) \sin \varphi(u) \sinh A(u, v), \end{aligned}$$

with C, φ, β, A, G defined as in [4], i.e. $C = \sin(2\theta)/(2\alpha)$, φ is solution of the ODE:

$$\varphi'^2 = \alpha^2 + \cos(2\theta) \cos^2 \varphi - C^2 \cos^4 \varphi,$$

β and G are respectively defined by:

$$\begin{cases} \beta' = C \cos^2 \varphi \\ \beta(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} G' = \frac{C^2 \cos^2 \varphi - \cos(2\theta)}{\alpha - \varphi'} \\ G(0) = 0 \end{cases},$$

$A = \alpha v + \beta(u)$ and the parameter θ is chosen as solution of the equation:

$$\int_{-1}^1 \frac{2\alpha C^2 t^2 - \alpha \cos(2\theta) + C^2 t^2 \sqrt{P(t)}}{\sqrt{(1-t^2)P(t)} (\alpha + \sqrt{P(t)})} dt = 0 \quad \text{with} \quad P(t) = \alpha^2 + \cos(2\theta)t^2 - C^2 t^4.$$

Consider the closed convex curve $\mathcal{C}_\alpha \cap \{y_2 = t\}$ which is of period $2U$, naturally parametrized by:

$$u \in [0, 2U] \mapsto (F_1(u, v), t, h(u, v)) \in \text{Nil}_3,$$

with the following expression of A on the curve:

$$A(u, v) = \frac{\alpha}{C}t + \beta(u) + \frac{\alpha}{C}G(u).$$

An orthonormal basis (e_1, e_2) of the tangent space to \mathcal{C}_α is:

$$\begin{aligned} e_1 &= \cos \varphi E_1 - \sin \varphi E_3 \\ \text{and } e_2 &= \frac{1}{\cosh A} (-(\sin \varphi \sinh A)E_1 + E_2 - (\cos \varphi \sinh A)E_3), \end{aligned}$$

with e_1 tangent to the curve $\mathcal{C}_\alpha \cap \{y_2 = t\}$.

If S is the infinitesimal generator of a 1-parameter family of isometries, we have:

$$\sigma_S = -\frac{1}{C} \int_0^{2U} (C^2 + G'^2) \langle S, \cosh A e_2 \rangle du,$$

and the infinitesimal generators S_1, S_2, S_3, S_R write as follow along the curve:

$$\begin{aligned} S_1 &= E_1 + tE_3, \quad S_2 = E_2 - F_1 E_3, \quad S_3 = E_3 \\ \text{and } S_R &= -tE_1 + F_1 E_2 - \frac{F_1^2 + t^2}{2} E_3. \end{aligned}$$

We obtain:

$$\begin{aligned} \sigma_1 &= \frac{1}{C} \int_0^{2U} (C^2 + G'^2) (\sin \varphi + t \cos \varphi) \sinh A \, du, \\ \sigma_2 &= -\frac{1}{C} \int_0^{2U} (C^2 + G'^2) (1 + F_1 \cos \varphi \sinh A) \, du, \\ \sigma_3 &= \frac{1}{C} \int_0^{2U} (C^2 + G'^2) \cos \varphi \sinh A \, du \\ \text{and } \sigma_R &= -\frac{1}{C} \int_0^{2U} (C^2 + G'^2) \left[t \sin \varphi \sinh A + F_1 + \frac{F_1^2 + t^2}{2} \cos \varphi \sinh A \right] du. \end{aligned}$$

These quantities are homological invariant and thus independent of the parameter t . It implies:

$$\sigma_1 = \sigma_3 = \sigma_R = 0 \quad \text{and} \quad \sigma_2 = \frac{1}{2\alpha C} \int_0^{2U} (C^2 + G'^2)(G' \cos^2 \varphi - 2\alpha) du.$$

Moreover, using relations between G and φ [4, page 14], we get:

$$\sigma_2 = \frac{\cos(2\tilde{\theta}_\alpha)}{\alpha C} G(U) - 2CU.$$

5.3 CMC-1/2 vertical ends in $\mathbb{H}^2 \times \mathbb{R}$

In a recent paper [3], the author and L. Hauswirth have constructed entire graphs and annuli of constant mean curvature 1/2 in $\mathbb{H}^2 \times \mathbb{R}$ with prescribed asymptotic behavior seen as deformations of rotational examples, and it appears that the existence conditions are flux conditions. The constructed surfaces have vertical ends, which means the ends are properly immersed topological annuli with asymptotically horizontal normal vector.

Following the notations in [3], the surfaces are parametrized in the Poincaré disk model of $\mathbb{H}^2 \times \mathbb{R}$ by:

$$X_\beta^\eta : re^{i\theta} \in \Omega \mapsto \left(\frac{2r}{1+r^2} e^{i\theta}, e^{\eta(r,\theta)} h_\beta(r) \right) \in \mathbb{H}^2 \times \mathbb{R},$$

where $\beta > 0$ is a real parameter, (r, θ) are the polar coordinates on \mathbb{R}^2 , Ω is the subset of the unit circle \mathbb{D} given by:

$$\Omega = \{w \in \mathbb{D} | R < |w| < 1\} \quad \text{with} \quad R > \left| \frac{\sqrt{\beta} - 1}{\sqrt{\beta} + 1} \right|,$$

η is a $\mathcal{C}^{2,\alpha}$ -function on $\Omega \cup \mathbb{S}^1$ for some $\alpha \in (0, 1)$ and h_β is the function:

$$h_\beta(r) = \int_{|\log \beta|}^{2 \log \left(\frac{1+r}{1-r} \right)} \frac{\cosh t - \beta}{\sqrt{2\beta \cosh t - 1 - \beta^2}} dt.$$

Note that when $\eta \equiv 0$, the 1-parameter family (X_β^0) indexed by β is the family of CMC-1/2 rotational examples – see [5] for details – and if $\beta = 1$, X_1^η is the end of an entire graph. We also have a simpler expression of h_1 :

$$h_1(r) = 2 \frac{1+r^2}{1-r^2}.$$

We compute the vertical flux σ_3 on a circle $\{r = t\}$ with $R < t < 1$. The infinitesimal generator of the vertical translations and the associated potential vector are respectively:

$$S_3 = E_3 \quad \text{and} \quad F_3 = \frac{2r}{1+r^2} (-\sin \theta E_1 + \cos \theta E_2),$$

and making $t \rightarrow 1$, we obtain:

$$\sigma_3 = 2\pi \left(1 - \beta |e^{-\gamma}|_{L^2(\mathbb{S}^1)}^2\right) \quad \text{with} \quad \gamma = \eta|_{r=1}.$$

If $\beta = 1$, the ends X_1^η are ends of entire graphs, which implies in particular $\sigma_3 = 0$ i.e. $|e^{-\gamma}|_{L^2(\mathbb{S}^1)}^2 = 1$. It is precisely the necessary and sufficient condition of [3, Theorem 3.8]. And in the case of annuli, $\beta \neq 1$, the condition on the values at infinity in the definition of a β -deformable annulus at the beginning of [3, Subsection 5.2] is also the conservation of the vertical flux.

It is worth mentioning that in general the Noether invariants express only necessary conditions on the existence of a surface and not sufficient conditions as in the present case.

Focus now on rotational annuli, the ends of which are parametrized by the immersions X_β^0 with $\beta \neq 1$. Similarly as for the computation of σ_3 , we have $\sigma_1 = \sigma_2 = \sigma_R = 0$ and we already know $\sigma_3 = 2\pi(1 - \beta)$.

The expressions of the behaviors of Noether invariants under the action of isometries computed in Subsection 3.2 show that the values of the flux and torque do not change when translating a rotational annulus. This point differs from the situation of minimal catenoids in \mathbb{R}^3 , see for instance [9]. Indeed, in the space form \mathbb{R}^3 , the torque has two more components and these components carry the information on the axis of the ends. It is no longer the case in $\mathbb{H}^2 \times \mathbb{R}$, which underlies the existence result [3, Theorem 5.10] of annuli without axis.

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