Stability of isotropic steady states for the relativistic Vlasov-Poisson system

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STABILITY OF ISOTROPIC STEADY STATES FOR THE RELATIVISTIC VLASOV-POISSON SYSTEM

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Abstract. In this work, we study the orbital stability of stationary solutions to the relativistic Vlasov-Manev system. This system is a kinetic model describing the evolution of a stellar system subject to its own gravity with some relativistic corrections. For this system, the orbital stability was proved for isotropic models constructed as minimizers of the Hamiltonian under a subcritical condition. We obtain here this stability for all isotropic models by a non-variationnal approach. We use here a new method developed in [23] for the classical Vlasov-Poisson system. We derive the stability from the monotonicity of the Hamiltonian under suitable generalized symmetric rearrangements and from a Antonov type coercivity property. We overcome here two new difficulties : the first one is the *a priori* non-continuity of the potentials, from which a greater control of the rearrangements is necessary. The second difficulty is related to the homogeneity breaking which does not give the boundedness of the kinetic energy. Indeed, in this paper, we does not suppose any subcritical condition satisfied by the steady states.

1. Introduction and main results

1.1. **Introduction to the relativistic Vlasov-Poisson system.** The relativistic Vlasov-Poisson system in dimension three reads:

$$\begin{cases} \partial_t f + \dfrac{v}{\sqrt{1+|v|^2}} \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, \quad \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ \\ f(t=0,x,v) = f_0(x,v) \geq 0, \end{cases} \tag{1.1}$$

where the gravitational Poisson field ϕ_f is defined for all $t \in \mathbb{R}^+$ by

$$\begin{cases} \Delta \phi_f(t,x) = \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v)dv, \\ \phi_f(t,x) \to 0 \text{ as } |x| \to +\infty. \end{cases} \tag{1.2}$$

This expression is equivalent to:

$$\phi_f(x) = -\frac{1}{4\pi|x|} * \rho. \tag{1.3}$$

This nonlinear transport system describes the evolution of a stellar system subject to its own gravity with some relativistic corrections.

The Cauchy problem of equations of Vlasov-Poisson type is not yet well understood for weak solutions: although the existence of local weak solutions has been proved , the question of its uniqueness remains unknown (see [7]). Moreover, for the relativistic case (1.1), in the most recent works about the Cauchy problem of smooth solutions, only radial smooth initial data have given results (see [12] and [26]). In

particular, Glassey and Schaeffer [12] have proved for radial smooth solutions that a blow-up in finite time T is characterized by the blow-up of the kinetic energy

$$\int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1}-1\right) f(t,x,v)dxdv \to +\infty \text{ as } t \to T.$$

There holds for smooth enough solutions of system (1.1) some important conservation properties: first, the total Hamiltonian

$$\mathcal{H}(f(t)) = \int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1}-1\right) f(t,x,v)dxdv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi_f(t,x)|^2\,dx, \tag{1.4}$$

is preserved in time and second we have the conservation of all Casimir functions: for all $\beta \in \mathcal{C}^1(\mathbb{R}_+,\mathbb{R}_+)$ such that $\beta(0)=0$,

$$\int_{\mathbb{R}^6} \beta(f(t,x,v))dxdv = \int_{\mathbb{R}^6} \beta(f_0(x,v))dxdv. \tag{1.5}$$

The property (1.5) is equivalent to the equimeasurability:

$$\forall t \geq 0, \quad \mu_{f(t)} = \mu_{f_0}, \tag{1.6}$$

where the distribution function μ_f is defined by

$$\forall s \geq 0, \quad \mu_f(s) = \text{meas}\{(x,v) \in \mathbb{R}^6,\ f(x,v) > s\}. \tag{1.7}$$

Remark that in particular, the L^p norms of f are conserved.

In this paper, we will consider weak solutions to (1.1) in the natural energy space

$$\mathcal{E}_p = \left\{ f \geq 0 \text{ with } f \in L^1(\mathbb{R}^6) \cap L^p(\mathbb{R}^6) \text{ and } \sqrt{1+|v|^2} f \in L^1(\mathbb{R}^6) \right\}. \tag{1.8}$$

For all $f_0 \in \mathcal{E}_p$, from classical kinetic Cauchy theory, the system (1.1) admits a local renormalized solution $f(t)$ in the sens of Diperna-Lions [9, 10]. This solution satisfies (1.5) and (1.6) but the conservation of Hamiltonian does not occur in general: it only remains

$$\forall t \geq 0, \quad \mathcal{H}(f(t)) \leq \mathcal{H}(f_0). \tag{1.9}$$

Note that, from this nonincreasing property and from (1.5), the kinetic energy will be bounded if f_0 satisfies a certain subcritical condition

$$C_p \left\|f_0\right\|_{L^1}^{\frac{2p-3}{3(p-1)}} \left\|f_0\right\|_{L^p}^{\frac{p}{3(p-1)}} < 1. \tag{1.10}$$

Indeed, from Hardy-Littlewood-Sobolev inequality, the potential energy satisfies

$$\text{for all } f \in \mathcal{E}_p, \quad \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi_f|^2\,dx \leq C_p \left\|f\right\|_{L^1}^{\frac{2p-3}{3(p-1)}} \left\|f\right\|_{L^p}^{\frac{p}{3(p-1)}} \left\|\sqrt{1+|v|^2} f\right\|_{L^1} \tag{1.11}$$

where C_p is defined as the best corresponding constant and thus the kinetic energy can be controled thanks to

$$\mathcal{H}(f) \geq \left(1 - C_p \left\|f\right\|_{L^1}^{\frac{2p-3}{3(p-1)}} \left\|f\right\|_{L^p}^{\frac{p}{3(p-1)}}\right) \left\|\sqrt{1+|v|^2} f\right\|_{L^1} - \left\|f\right\|_{L^1}. \tag{1.12}$$

Hence we have global existence for the system (1.1) as soon as f_0 satisfies the subcritical condition (1.10).

1.2. **Main results and strategy of the proof.** Our main result gives the nonlinear stability of a large class of stationary solutions for the relativistic Vlasov-Poisson equation. We adapt here some techniques introduced for the classical Vlasov-Poisson equation [23].

Note (see [4] for the Vlasov-Poisson system) that the isotropic steady states of the system (1.1) are the functions of the form

$$Q(x,v) = F(e),$$

where e is the miscroscopic energy given by

$$e(x,v) = \sqrt{|v|^2+1} - 1 + \phi_Q(x).$$

An important question which has been the subject of several works is the question of nonlinear stability of these steady states when F satisfies

$$\frac{\partial F}{\partial e} < 0.$$

For some of these stationary solutions, built as minimizers of suitable functional, the concentration-compactness Lemma ([27]- [28]) allows to prove their stability under a subcritical condition of the type (1.10), see [21]. Here we get the nonlinear stability of all these stationary solutions in the energy space $\mathcal{E}_p$, defined by (1.8), with respect to its natural norm:

$$\|f\|_{\mathcal{E}_p} = \|f\|_{L^1} + \|f\|_{L^p} + \|\sqrt{1+|v|^2} f\|_{L^1}.$$

Theorem 1.1 (Stability of spherical models)**.** *Let Q be a spherical, continuous, nonnegative, non zero, compactly supported steady solution to* (1.1)*. Assume that Q is a nonincreasing function of its microscopic energy, i.e. there exists a continuous function $F : \mathbb{R} \to \mathbb{R}_+$ such that for all $(x,v) \in \mathbb{R}^6$*

$$Q(x,v) = F\left(\sqrt{|v|^2+1} - 1 + \phi_Q(x)\right), \tag{1.13}$$

and there exists $e_0 < 0$ such that $F(e) = 0$ for $e \geq e_Q$, F is decreasing and $\mathcal{C}^1$ on $(-\infty, e_Q)$. Let $p > \frac{3}{2}$. Then Q is orbitally stable in the $\mathcal{E}_p$-norm by the flow (1.1)*: for all $\varepsilon > 0$ there exists $\eta > 0$ such that the following holds true. Let $f_0 \in \mathcal{E}_p$ be such that*

$$\|f_0 - Q\|_{\mathcal{E}_p} \leq \eta \tag{1.14}$$

and let $f(t)$ be a corresponding renormalized solution to (1.1) *on $[0,T)$. Then for all $t \in [0,T)$ there exists a continuous translation shift $z(t)$ such that*

$$\|f(t,x,v) - Q(x - z(t), v)\|_{\mathcal{E}_p} \leq \varepsilon. \tag{1.15}$$

In the radial case, since a blow-up in finite time is equivalent to a blow-up of the kinetic energy, Theorem 1.1 provides global existence when the initial data is near the (non necessary subcritical) function Q. The existence of such Q away from all subcritical class of functions is an open problem.

Strategy of the proof: This proof uses arguments developed in [23] for the classical gravitational Vlasov-Poisson system. But here two new main difficulties appear: the first one is that the studied class of functions in the energy space $\mathcal{E}_p$ does not imply the boundedness of the potentials and the second one is related to the breaking homogeneity character of the relativistic problem. Let us give the global strategy of the proof.

For $f \in \mathcal{E}_p$, we know that there exists a nonnegative function on $\mathbb{R}_+$, f^*, which is the Schwarz symmetrization of f, such that

$$\forall s \geq 0, \quad \mu_f(s) = \mu_{f^*}(s).$$

We can similarly build a rearrangement with respect to the microscopic energy

$$e(x,v) = \sqrt{|v|^2+1} - 1 + \phi(x),$$

for a non zero potential ϕ. This rearrangement $f^{*\phi}$ is defined by

$$f^{*\phi} = f^* \circ a_\phi \text{ with } a_\phi(\lambda) = \text{meas}\{(x,v),\ e(x,v) < \lambda\}$$

and satisfies

$$\mu_f = \mu_{f^{*\phi}}, \quad \mathcal{H}(f^{*\phi_f}) \leq \mathcal{H}(f) \text{ and } Q = Q^{*\phi_Q}.$$

Note that, in the classical Vlasov-Poisson case, this monotonicity of the Hamiltonian was observed in the physics literature [25, 11, 31, 1]. Now by defining the functionnal J by

$$J(\phi) = \mathcal{H}(Q^{*\phi}) + \frac{1}{2}\left\|\nabla\phi_{Q^{*\phi}} - \nabla\phi\right\|_{L^2}^2,$$

we shall prove that

$$\mathcal{H}(f) - \mathcal{H}(Q) \geq J(\phi_f) - J(\phi_Q) + \int_0^{+\infty} a_{\phi_f}^{-1}(s)(f^*(s) - Q^*(s))ds. \tag{1.16}$$

Hence we can reduce our problem to the study of the functionnal J which only depends on the potential ϕ. Moreover, from the Pólya-Szegö inequality, we get

$$J(\phi^*) \leq J(\phi)$$

where ϕ^* is the Schwarz symmetrization of ϕ, and therefore the study of J can be restricted to radial modes. We then use the Burchard-Guo compactness result [8] to get the compactness of ϕ from that of ϕ^*.

Our proof is performed in two steps:
(i) Proof of the local coercivity of J on radial potentials ϕ near ϕ_Q.
(ii) Proof of Theorem 1.1 via a local compactness proposition thanks to (1.16).
This second step will be deduced from (1.16) and equimeasurability arguments derived from the rigidity of the flow. The local coercivity stated in the first step will follow from the Taylor expansion near ϕ_Q for radial potentials

$$J(\phi) - J(\phi_Q) = \frac{1}{2}D^2J(\phi_Q)(\phi-\phi_Q, \phi-\phi_Q) + o\left(\|\nabla\phi - \nabla\phi_Q\|_{L^2}^2\right)$$

and from the strict coercivity of the quadratic form $D^2J(\phi_Q)$. To prove this coercivity, we follow the same lines as [23] where a Poincaré-like inequality was proved for the classical Vlasov-Poisson system. This inequality is a generalisation of Antonov type coercivity estimate and is based on a Hörmander approach [18, 19]. The main new difficulties here, lie in the control of the jacobian a_ϕ which can't be bounded in general because of the non boundedness of the potential and the homogeneity breaking.

This papers is organized as follows. Section 2 deals with the proof of the local coercivity of the functional J near ϕ_Q stated in Proposition 2.2. By differentiating the functional J at ϕ_Q in Subsection 2.1, we obtain the second order Taylor expansion stated in Lemma 2.3. Then, in Subsection 2.2, from Hardy type control argument,

we deduce the Proposition 2.2 for radially symmetric potential and we finally generalize this proposition for all potential by using a compactness argument [8].Section 3 is devoted to the proof of Theorem 1.1 from the coercivity result stated in Section 2. First, in Subsection 3.1, we prove the local compactness of local minimizing sequences, Proposition 3.1. Then, in Subsection 3.2, by using this compactness result combined with a contradiction argument, we finally deduce the orbital stability, Theorem 1.1. The several property about the generalized rearrangement used in these both sections can be found in the completed study of the rearrangement with respect of the microscopic energy stated in Appendix A and B.

2. Coercivity of the functional J

The aim of this section is to prove the following proposition 2.2 below. Our study is based on rearrangements with respect to the microscopic energy. To define these rearrangements, we introduce, for $q > 3$,

$$\Phi_q = \left\{ \phi \in L^q(\mathbb{R}^3) \text{ s.t. } \phi \leq 0,\ m(\phi) > 0,\ \nabla\phi \in L^2(\mathbb{R}^3),\ \lim_{|x|\to+\infty} \phi(x) = 0 \right\}, \quad (2.1)$$

with

$$m(\phi) = \inf_{x\in\mathbb{R}^3} (1 + |x|)\,|\phi(x)|, \quad (2.2)$$

and the norm on Φ_q

$$\|\phi\|_{\Phi_q} = \|\nabla\phi\|_{L^2(\mathbb{R}^3)} + \|\phi\|_{L^q(\mathbb{R}^3)}\,. \quad (2.3)$$

The space Φ_q is a natural space for the potential of distribution functions in $\mathcal{E}_p$, as given by the following lemma.

Lemma 2.1. *Let $f \in \mathcal{E}_p$ non zero with $p > \frac{3}{2}$. Then the potential ϕ_f belongs to Φ_q for all $3 < q \leq \frac{3(4p-3)}{p} (< 12)$.*

Proof. From classical interpolation methods and from the Hardy-Littlewood-Sobolev inequality, we have for $f \in \mathcal{E}_p$ with $p > 1$

$$\rho_f \in L^r(\mathbb{R}^3),\ 1 \leq r \leq \frac{4p-3}{3p-2} \text{ and } \phi_f \in L^q(\mathbb{R}^3),\ \frac{3}{2} < q \leq \frac{3(4p-3)}{p}.$$

Moreover, for $p > \frac{3}{2}$ the Hardy-Littlewood-Sobolev inequality implies that $\nabla\phi_f$ belongs to $L^2(\mathbb{R}^3)$. We prove now that $m(\phi_f)$ defined by (2.2) is positive. Since f is non zero, the density ρ_f is non zero, too. Hence there exists $R > 0$ such that

$$M := \int_{|x|<R} \rho_f(x)dx > 0.$$

We have then

$$|\phi_f(x)| = \int_{\mathbb{R}^3} \frac{\rho_f(y)}{4\pi|x-y|}dy \geq \int_{|y|<R} \frac{\rho_f(y)}{4\pi|x-y|}dy \geq \int_{|y|<R} \frac{\rho_f(y)}{4\pi(|x|+R)}dy.$$

Finally,

$$(|x| + R)\,|\phi_f(x)| \geq M,$$

which concludes the proof of lemma 2.1. □

Now for $\phi \in \Phi_q$ and $f \in \mathcal{E}_p$ with $p > \frac{3}{2}$, we define the jacobian a_ϕ by

$$\forall e < 0, \quad a_\phi(e) = \mathrm{meas}\left\{(x,v) \in \mathbb{R}^6 \ : \ \sqrt{|v|^2+1} - 1 + \phi(x) < e\right\}, \tag{2.4}$$

and the rearrangement with respect to the microscopic energy by

$$f^{*\phi} = \begin{cases} f^*\left(a_\phi\left(\sqrt{|v|^2+1} - 1 + \phi(x)\right)\right) & \text{if} \quad \sqrt{|v|^2+1} - 1 + \phi(x) < 0 \\ \\ 0 & \text{if} \quad \sqrt{|v|^2+1} - 1 + \phi(x) \geq 0, \end{cases} \tag{2.5}$$

where f^* is the Schwarz-symmetrization of f in $\mathbb{R}^6$. In Appendix A we recall some properties about the Schwarz symmetrization and we give all the properties we need about the Jacobian and the rearrangement with respect to the microscopic energy. In particular, $f^{*\phi}$ is well defined in $\mathcal{E}_p$ and the function Q defined in Theorem 1.1 satisfies $Q = Q^{*\phi_Q}$.

Using these definitions, we introduce the functional J on Φ_q defined by

$$J(\phi) = \int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1} - 1 + \phi(x)\right) Q^{*\phi} dxdv + \frac{1}{2}\left\|\nabla\phi\right\|_{L^2}^2, \tag{2.6}$$

which is equivalent to

$$J(\phi) = \mathcal{H}(Q^{*\phi}) + \frac{1}{2}\left\|\nabla\phi_{Q^{*\phi}} - \nabla\phi\right\|_{L^2}^2. \tag{2.7}$$

We claim now the following Proposition which we prove in the next subsections.

Proposition 2.2 (Local coercivity of the functional J)**.** *There exists a constant $\delta_0 > 0$ such that, for all $q > 3$, the following holds true. Let a sequence ϕ_n of Φ_q such that*

$$\forall n \in \mathbb{N}, \ \left\|\nabla\phi_n - \nabla\phi_Q\right\|_{L^2} \leq \delta_0 \quad \textit{and} \quad \lim_{n\to+\infty} J(\phi_n) \leq J(\phi_Q). \tag{2.8}$$

Then there exists a sequence of translation shifts in space x_n such that

$$\left\|\nabla\phi_n - \nabla\phi_Q(\cdot - x_n)\right\|_{L^2} \to 0 \ \textit{as} \ n \to +\infty. \tag{2.9}$$

This coercivity of the functional J near ϕ_Q is the first step to prove the stability of Q stated in Theorem 1.1. To obtain it, on the one hand, we will look for a second order Taylor expansion of J around ϕ_Q and, on the second hand, we will control the second derivative of J at ϕ_Q thanks to a Poincaré-type inequality.

2.1. **Second order Taylor expansion of J at ϕ_Q.** In order to prove Proposition 2.2 we give first a Taylor expansion of the functional J near the potential ϕ_Q.

Lemma 2.3 (Taylor expansion of J)**.** *Let $6 < q < 12$ and $\phi, \tilde{\phi} \in \Phi_q$. Then the function*

$$\lambda \mapsto J(\phi + \lambda(\tilde{\phi} - \phi))$$

is twice differentiable on $[0,1]$.
Moreover, for ϕ in Φ_q radially symmetric, there holds the Taylor expansion near ϕ_Q:

$$J(\phi) - J(\phi_Q) = \frac{1}{2}D^2J(\phi_Q)(\phi - \phi_Q, \phi - \phi_Q) + \varepsilon(\left\|\nabla\phi - \nabla\phi_Q\right\|_{L^2}), \tag{2.10}$$

where $\varepsilon(\delta) = \circ(\delta^2)$ *as* $\delta \to 0$.
Finally the second derivative of J *at* ϕ_Q *in the direction* h *is given by:*

$$D^2J(\phi_Q)(h,h) = \int_{\mathbb{R}^3} |\nabla h|^2 \, dx - \int_{\mathbb{R}^6} \left|F'(e_{\phi_Q}(x,v))\right| (h(x) - \Pi h(x))^2 \, dxdv, \quad (2.11)$$

where $e_{\phi_Q}(x,v) = \sqrt{1+|v|^2} - 1 - \phi_Q(x)$ *and* Πh *is the projector defined by*

$$\Pi h(x) = \frac{\displaystyle\int_{\mathbb{R}^3} \left(\left(1 + e_{\phi_Q}(x,v) - \phi_Q(y)\right)_+^2 - 1\right)_+^{\frac{1}{2}} \left(1 + e_{\phi_Q}(x,v) - \phi_Q(y)\right) h(y) dy}{\displaystyle\int_{\mathbb{R}^3} \left(\left(1 + e_{\phi_Q}(x,v) - \phi_Q(y)\right)_+^2 - 1\right)_+^{\frac{1}{2}} \left(1 + e_{\phi_Q}(x,v) - \phi_Q(y)\right) dy}. \quad (2.12)$$

Remark that the function Πh can be seen as the projection of h on the functions of the microscopic energy $e_{\phi_Q}(x,v)$. To prove the lemma 2.3 we will first prove that J is two times differentiable on Φ_q, then we will evaluate its derivatives on ϕ_Q and finally we will control the rest of the expansion for radially symmetric potentials.

Proof. Let J_0 be a functionnal on Φ_q, defined by

$$J_0(\phi) = \int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1} - 1 + \phi(x)\right) Q^{*\phi} dxdv = \int_{\mathbb{R}^6} e_\phi(x,v) Q^{*\phi} dxdv. \quad (2.13)$$

Then J is given by

$$J(\phi) = J_0(\phi) + \frac{1}{2} \|\nabla \phi\|_{L^2}^2 . \quad (2.14)$$

To differentiate J, we just have to differentiate J_0. Let $\phi, \tilde{\phi} \in \Phi_q$ and $h = \tilde{\phi} - \phi$. We study then the function $\lambda \mapsto J_0(\phi + \lambda h)$ on $[0,1]$.
First derivative of J_0. From the change of variable (A.11), we get

$$J_0(\phi) = \int_{\inf \phi}^0 eQ^*(a_\phi(e)) a'_\phi(e) de = \int_{\inf \phi}^0 e \left(G \circ a_\phi\right)'(e) de,$$

where G is the $\mathcal{C}^1$ bounded function, with bounded derivative, defined by

$$G(s) = \int_0^s Q^*(\sigma) d\sigma. \quad (2.15)$$

Moreover, from the property (A.8) in lemma A.2, the jacobian $a_\phi(e)$ converges to 0 as $e \to \inf \phi$. Thus we have $[eG \circ a_\phi(e)]_{\inf \phi}^0 = 0$ and an integration by parts gives

$$J_0(\phi) = -\int_{-\infty}^0 G \circ a_\phi(e) de. \quad (2.16)$$

Step 1 : first derivative of J_0.

To differentiate J_0 given by (2.16), we shall use the Lebesgue's derivation theorem. From formula (A.9) in the Appendix, we have for all $e < 0$

$$\frac{\partial}{\partial \lambda} G(a_{\phi+\lambda h}(e)) = Q^*(a_{\phi+\lambda h}(e)) \frac{\partial}{\partial \lambda} a_{\phi+\lambda h}(e) \quad (2.17)$$

with

$$\frac{\partial}{\partial \lambda} a_{\phi+\lambda h}(e) = -4\pi \int_{\mathbb{R}^3} K\left(e - \phi(x) - \lambda h(x)\right) h(x) dx,$$

and

$$K(\eta) = \left((1+\eta)_+^2 - 1\right)_+^{\frac{1}{2}} (1+\eta). \tag{2.18}$$

Note that the uniform inequality

$$a_{\phi+\lambda h}(e) \geq \frac{4\pi}{3} \int_{\mathbb{R}^3} \left(\left(1 + e + \frac{\min\{m(\phi), m(\tilde{\phi})\}}{1+|x|} \right)_+^2 - 1 \right)_+^{\frac{3}{2}} dx$$

and the compact support of Q^* imply that there exists $e_0 < 0$ such that

$$\forall e \geq e_0, \ \forall \lambda \in [0,1], \quad Q^*(a_{\phi+\lambda h}(e)) = 0. \tag{2.19}$$

Moreover, we have for all $e < e_0$, for all $\lambda \in [0,1]$,

$$\left| \frac{\partial}{\partial \lambda} a_{\phi+\lambda h}(e) \right| \leq 4\pi \int_{\mathbb{R}^3} K\left(e_0 - \phi(x) - \tilde{\phi}(x) \right) |h(x)| \, dx,$$

and $Q^*(a_{\phi+\lambda h}(e)) \leq \|Q\|_\infty$. Finally we obtain, by noting $\phi_\lambda = \phi + \lambda h$

$$\frac{\partial}{\partial \lambda} J_0(\phi_\lambda) = 4\pi \int_{-\infty}^0 \int_{\mathbb{R}^3} Q^*(a_{\phi_\lambda}(e)) K\left(e - \phi(x) - \lambda h(x)\right) h(x) dx de. \tag{2.20}$$

Step 2 : second derivative of J_0.

We keep the previous notations. An integration by parts with respect to e yields

$$\frac{\partial}{\partial \lambda} J_0(\phi_\lambda) = -\frac{4\pi}{3} \int_{-\infty}^0 \int_{\mathbb{R}^3} Q^{*\prime}(a_{\phi_\lambda}(e)) a'_{\phi_\lambda}(e) \left((1 + e - \phi_\lambda(x))_+^2 - 1 \right)_+^{\frac{3}{2}} h(x) dx de.$$

We perform now the change of variable $s = a_{\phi_\lambda}(e)$, which gives

$$\frac{\partial}{\partial \lambda} J_0(\phi_\lambda) = -\frac{4\pi}{3} \int_0^{L_0} \int_{\mathbb{R}^3} Q^{*\prime}(s) \left(\left(1 + a_{\phi_\lambda}^{-1}(s) - \phi_\lambda(x) \right)_+^2 - 1 \right)_+^{\frac{3}{2}} h(x) dx ds, \tag{2.21}$$

where L_0 is the measure of the support of Q and thus satisfies $Supp(Q^*) = [0, L_0)$. Define

$$g(\lambda, x, s) := \left(\left(1 + a_{\phi_\lambda}^{-1}(s) - \phi_\lambda(x) \right)_+^2 - 1 \right)_+^{\frac{3}{2}}.$$

From the first step, for all $\lambda \in [0,1]$ and for all $s \in [0, L_0]$, we have $a_{\phi_\lambda}^{-1}(s) \leq e_0$ and thus the set

$$\{x \in \mathbb{R}^3, \, g(\lambda, x, s) \neq 0\} \subset \{x \in \mathbb{R}^3, \, \phi_\lambda(x) \leq e_0\} \subset \{x \in \mathbb{R}^3, \, \phi(x) + \tilde{\phi}(x) \leq e_0\}$$

is uniformaly contained in the bounded set $\Omega = \{x \in \mathbb{R}^3, \, \phi(x) + \tilde{\phi}(x) \leq e_0\}$. Moreover, by recalling the notations $h = \phi - \tilde{\phi}$ and

$$K(\eta) = \left((1+\eta)_+^2 - 1\right)_+^{\frac{1}{2}} (1+\eta), \tag{2.22}$$

we have for $(\lambda, x, s) \in [0,1] \times \Omega \times [0, L_0]$

$$\frac{\partial g}{\partial \lambda}(\lambda, x, s) = 3K\left(a_{\phi_\lambda}^{-1}(s) - \phi(x) - \lambda h(x) \right) \left(-h(x) + \frac{\partial}{\partial \lambda} a_{\phi_\lambda}^{-1}(s) \right), \tag{2.23}$$

where, from lemma (A.2) in Appendix,

$$\frac{\partial}{\partial\lambda}a_{\phi_\lambda}^{-1}(s)=\frac{\displaystyle\int_\Omega K\left(a_{\phi_\lambda}^{-1}(s)-\phi(x)-\lambda h(x)\right)h(x)dx}{\displaystyle\int_\Omega K\left(a_{\phi_\lambda}^{-1}(s)-\phi(x)-\lambda h(x)\right)dx}.$$

To differentiate (2.21) with respect to λ, we first prove at fixed $s\in(0,L_0)$

$$\frac{\partial}{\partial\lambda}\int_\Omega g(\lambda,x,s)h(x)dx=\int_\Omega\frac{\partial g}{\partial\lambda}(\lambda,x,s)h(x)dx. \tag{2.24}$$

Let fixed $s\in(0,L_0)$. We begin by bounding $\frac{\partial}{\partial\lambda}a_{\phi_\lambda}^{-1}(s)$. For all $x\in\mathbb{R}^3$

$$\phi(x)+\lambda h(x)\geq\phi(x)+\tilde{\phi}(x)\ \text{ and }\ a_{\phi_\lambda}^{-1}(s)\leq e_0, \tag{2.25}$$

which provides

$$0\leq\int_\Omega K\left(a_{\phi_\lambda}^{-1}(s)-\phi(x)-\lambda h(x)\right)h(x)dx\leq\int_\Omega K\left(e_0-\phi(x)-\tilde{\phi}(x)\right)h(x)dx. \tag{2.26}$$

Now denote

$$\Omega_\lambda=\left\{x\in\mathbb{R}^3,\ \phi_\lambda(x)<a_{\phi_\lambda}^{-1}\left(\frac{s}{2}\right)\right\}.$$

Then the set Ω_λ is included in Ω and

$$\text{meas }(\Omega_\lambda)=a_{\phi_\lambda}\circ a_{\phi_\lambda}^{-1}\left(\frac{s}{2}\right)=\frac{s}{2}.$$

Moreover, for all $x\in\Omega_\lambda$, we have

$$a_{\phi_\lambda}^{-1}(s)-\phi_\lambda(x)\geq a_{\phi_\lambda}^{-1}(s)-a_{\phi_\lambda}^{-1}\left(\frac{s}{2}\right),$$

which, combined with

$$\forall\eta>0,\ K(\eta)=\left(\eta\left(2+\eta\right)\right)_+^{\frac{1}{2}}\left(1+\eta\right)\geq\sqrt{2\eta},$$

implies

$$\int_\Omega K\left(a_{\phi_\lambda}^{-1}(s)-\phi_\lambda(x)\right)dx\geq\frac{s}{\sqrt{2}}\left(a_{\phi_\lambda}^{-1}(s)-a_{\phi_\lambda}^{-1}\left(\frac{s}{2}\right)\right)^{\frac{1}{2}}. \tag{2.27}$$

We claim that, at fixed $s>0$, there exists a constant $C>0$ such that for all $\lambda\in[0,1]$

$$a_{\phi_\lambda}^{-1}(s)-a_{\phi_\lambda}^{-1}\left(\frac{s}{2}\right)\geq C. \tag{2.28}$$

Indeed, assume that the property (2.28) does not hold, then there exist $\lambda\in[0,1]$ and a sequence (λ_n) such that, as $n\to+\infty$,

$$\lambda_n\to\lambda\ \text{ and }\ a_{\phi_{\lambda_n}}^{-1}(s)-a_{\phi_{\lambda_n}}^{-1}\left(\frac{s}{2}\right)\to 0.$$

From lemma A.2 in Appendix, we conclude that

$$a_{\phi_\lambda}^{-1}(s)-a_{\phi_\lambda}^{-1}\left(\frac{s}{2}\right)=0,$$

which is not possible since $s>0$ and $a_{\phi_\lambda}^{-1}$ is strictly increasing on $\mathbb{R}_+^*$. Finally from the inequalities (2.26) and (2.27), for $s\in(0,L_0)$ there exists a constant $C_s>0$ such that for all $\lambda\in[0,1]$,

$$\frac{\partial}{\partial\lambda}a_{\phi_\lambda}^{-1}(s)\leq C_s.$$

We can thus uniformly bound

$$\left|\frac{\partial}{\partial\lambda}g\right| \leq 3\left(1+e_0-\phi(x)-\tilde{\phi}(x)\right)_+^2\left(-\phi(x)-\tilde{\phi}(x)+C_s\right),$$

where we used (2.25) and the fact that $K(\eta)\leq(1+\eta)_+^2$. By noticing that the function $\phi+\tilde{\phi}$ belongs to $L^3_{loc}(\mathbb{R}^3)$ (since q>3), Lebesgue derivation theorem provides (2.24).

Now aim to integrate with respect to $s\in(0,L_0)$. We define for all $s\in(0,L_0)$ and for all $\lambda\in[0,1]$

$$I(\lambda,s):=\frac{\partial}{\partial\lambda}\int_\Omega g(\lambda,x,s)h(x)dx=\int_\Omega\frac{\partial}{\partial\lambda}g(\lambda,x,s)h(x)dx,$$

which is continuous function of the variable λ. Remark that

$$I(\lambda,s)=3\int_\Omega K\left(a^{-1}_{\phi_\lambda}(s)-\phi_\lambda(x)\right)\left(-h(x)+\frac{\partial}{\partial\lambda}a^{-1}_{\phi_\lambda}(s)\right)h(x)dx,$$

where, from Cauchy-Schwarz inequality,

$$\begin{aligned}\int_\Omega K\left(a^{-1}_{\phi_\lambda}(s)-\phi_\lambda(x)\right)\frac{\partial a^{-1}_{\phi_\lambda}}{\partial\lambda}h(x)dx &= \frac{\left(\int_{\mathbb{R}^3}K\left(a^{-1}_{\phi_\lambda}(s)-\phi_\lambda(x)\right)h(x)dx\right)^2}{\int_{\mathbb{R}^3}K\left(a^{-1}_{\phi_\lambda}(s)-\phi_\lambda(x)\right)dx}\\ &\leq \int_{\mathbb{R}^3}K\left(a^{-1}_{\phi_\lambda}(s)-\phi_\lambda(x)\right)(h(x))^2\,dx.\end{aligned} \tag{2.29}$$

Thus, we have

$$|I(\lambda,s)|\leq 3\int_{\mathbb{R}^3}K\left(a^{-1}_{\phi_\lambda}(s)-\phi_\lambda(x)\right)(h(x))^2\,dx.$$

Moreover, since Q^* is decreasing from $\|Q\|_{L^\infty}$ to 0, the function $Q^{*\prime}$ belongs to $L^1(0,L_0)$ and finally, from Lebesgue's derivation theorem, we get

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}J_0(\phi_\lambda) = \;& 4\pi\int_0^{L_0}\int_{\mathbb{R}^3}Q^{*\prime}(s)K\left(a^{-1}_{\phi_\lambda}(s)-\phi_\lambda(x)\right)(h(x))^2dxds\\ & -4\pi\int_0^{L_0}Q^{*\prime}(s)\frac{\left(\int_{\mathbb{R}^3}K\left(a^{-1}_{\phi_\lambda}(s)-\phi_\lambda(x)\right)h(x)dx\right)^2}{\int_{\mathbb{R}^3}K\left(a^{-1}_{\phi_\lambda}(s)-\phi_\lambda(x)\right)dx}ds.\end{aligned} \tag{2.30}$$

Using the change of variable $e=a^{-1}_{\phi_\lambda}(s)$, we get

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}J_0(\phi_\lambda) = \;& 4\pi\int_{-\infty}^0\int_{\mathbb{R}^3}Q^{*\prime}(a_{\phi_\lambda}(e))a'_{\phi_\lambda}(e)K\left(e-\phi_\lambda(x)\right)(h(x))^2dxde\\ & -4\pi\int_{-\infty}^0Q^{*\prime}(a_{\phi_\lambda}(e))a'_{\phi_\lambda}(e)\frac{\left(\int_{\mathbb{R}^3}K\left(e-\phi_\lambda(x)\right)h(x)dx\right)^2}{\int_{\mathbb{R}^3}K\left(e-\phi_\lambda(x)\right)dx}de.\end{aligned} \tag{2.31}$$

Step 3: derivatives of J at ϕ_Q:

Let $\phi \in \Phi_q$ and $h = \phi - \phi_Q$. Then, by (2.20),

$$DJ_0(\phi_Q)(h) = 4\pi \int_{-\infty}^{0} \int_{\mathbb{R}^3} Q^*(a_{\phi_Q}(e)) K\left(e - \phi_\lambda(x)\right) h(x) dx de$$

Note that, from lemma B.2, the function F defined by theorem 1.1 satisfies

$$Q^*(a_{\phi_Q}(e)) = F(e).$$

Now perform the change of variable $u = \left((1 + e - \phi_Q(x))_+^2 - 1\right)_+^{\frac{1}{2}}$ with respect to e. Then

$$\begin{aligned} DJ_0(\phi_Q)(h) &= 4\pi \int_0^{+\infty} \int_{\mathbb{R}^3} F\left(\sqrt{1+|v|^2} - 1 + \phi_Q(x)\right) u^2 h(x) dx du \\ &= \int_{\mathbb{R}^6} Q(x,v) h(x) dx dv. \end{aligned}$$

Hence, from (2.14),

$$DJ(\phi_Q)(h) = DJ_0(\phi_Q)(h) + \int_{\mathbb{R}^3} \nabla \phi_Q \cdot \nabla h dx = 0,$$

where we used the Poisson equation satisfied by ϕ_Q.

We now give the explicit expression of the second derivative of J_0 at ϕ_Q. Remark first that $F'(e) = Q^{*\prime}(a_{\phi_Q}(e)) a'_{\phi_Q}(e)$ and thus from (2.31),

$$\begin{aligned} D^2 J_0(\phi_Q)(h,h) = \; & 4\pi \int_{-\infty}^{0} \int_{\mathbb{R}^3} F'(e) K\left(e - \phi_\lambda(x)\right) \left(h(x)\right)^2 dx de \\ & -4\pi \int_{-\infty}^{0} F'(e) \frac{\left(\int_{\mathbb{R}^3} K\left(e - \phi_\lambda(x)\right) h(x) dx\right)^2}{\int_{\mathbb{R}^3} K\left(e - \phi_\lambda(x)\right) dx} de. \end{aligned}$$

We apply the change of variable $u = \left((1 + e - \phi_Q(x))_+^2 - 1\right)_+^{\frac{1}{2}}$ with respect to e to get

$$D^2 J_0(\phi_Q)(h,h) = \int_{\mathbb{R}^6} F'\left(e\right) \left(h(x)\right)^2 dx dv - \int_{\mathbb{R}^6} F'\left(e\right) \Pi h\left(e\right) h(x) dx dv,$$

where $e = e(x,v) = \sqrt{1+|v|^2} - 1 + \phi_Q(x)$ and Πh is the projector on the space of functions depending only on $e(x,v)$, defined by (2.12). Hence we have

$$D^2 J_0(\phi_Q)(h,h) = \int_{\mathbb{R}^6} F'\left(e(x,v)\right) \left(h(x) - \Pi h\left(e(x,v)\right)\right)^2 dx dv,$$

and the decomposition (2.14) provides

$$D^2 J(\phi_Q)(h,h) = D^2 J_0(\phi_Q)(h,h) + \int_{\mathbb{R}^3} |\nabla h|^2 dx, \tag{2.32}$$

which concludes the proof of (2.11).

Step 4: proof of the Taylor expansion (2.10)*:*

Let $\phi \in \Phi_q$ radially symmetric and $h = \phi - \phi_Q$. We note for $\lambda \in [0,1]$, $\phi_\lambda := \phi_Q + \lambda h$. Then, using $DJ(\phi_Q)(h) = 0$, we have

$$J(\phi_Q+h)-J(\phi_Q)=\frac{1}{2}D^2J(\phi_Q)(h,h)$$
$$+\|\nabla h\|_{L^2}^2\int_0^1(1-\lambda)\left(D^2J_0(\phi_\lambda)-D^2J_0(\phi_Q)\right)\left(\frac{h}{\|\nabla h\|_{L^2}},\frac{h}{\|\nabla h\|_{L^2}}\right)d\lambda. \quad (2.33)$$

It is sufficient to prove that

$$\sup_{\lambda\in[0,1]}\sup_{\|\nabla\hat{h}\|_{L^2}=1}\left|\left(D^2J_0(\phi_\lambda)-D^2J_0(\phi_Q)\right)\left(\hat{h},\hat{h}\right)\right|\to 0 \quad (2.34)$$

as $\|\nabla\phi-\nabla\phi_Q\|_{L^2}\to 0$ to obtain the Taylor expansion (2.10). Note that the functions $\hat{h}$ in (2.34) are taken to be radially symmetric. In order to prove (2.34) we argue by contradiction. Let $\varepsilon>0$, $\psi_n\in\Phi_q$, $\hat{h}_n\in\Phi_q$ and $\lambda_n\in[0,1]$ such that

$$\|\nabla\psi_n-\nabla\phi_Q\|_{L^2(\mathbb{R}^3)}\leq\frac{1}{n},\qquad \|\nabla\hat{h}_n\|_{L^2(\mathbb{R}^3)}=1, \quad (2.35)$$

and

$$\left|\left(D^2J_0(\phi_n)-D^2J_0(\phi_Q)\right)\left(\hat{h}_n,\hat{h}_n\right)\right|>\varepsilon, \quad (2.36)$$

where $\phi_n=(1-\lambda_n)\phi_Q+\lambda_n\psi_n$. The sequence ϕ_n satisfies

$$\|\nabla\phi_n-\nabla\phi_Q\|_{L^2(\mathbb{R}^3)}\leq\frac{1}{n}. \quad (2.37)$$

We recall from (2.30) that

$$D^2J_0(\phi_n)\left(\hat{h}_n,\hat{h}_n\right)=4\pi\int_0^{L_0}Q^{*\prime}(s)\gamma_n(s)ds, \quad (2.38)$$

where

$$\gamma_n(s)=\int_{\mathbb{R}^3}g_n(x,s)(\hat{h}_n(x))^2dx-\frac{\left(\int_{\mathbb{R}^3}g_n(x,s)\hat{h}_n(x)dx\right)^2}{\int_{\mathbb{R}^3}g_n(x,s)dx}, \quad (2.39)$$

and

$$g_n(x,s)=\left(\left(1+a_{\phi_n}^{-1}(s)-\phi_n(x)\right)_+^2-1\right)_+^{\frac{1}{2}}(1+a_{\phi_n}^{-1}(s)-\phi_n(x)).$$

Notice first that the convergence (2.37) implies the convergence of ϕ_n to ϕ_Q in $L^6(\mathbb{R}^3)$. Thus, from (A.8), we have

$$a_{\phi_n}^{-1}(s)\to a_{\phi_Q}^{-1}(s). \quad (2.40)$$

Moreover, we have

$$|\phi_n(r)|\leq\int_r^{+\infty}\left|\phi_n'(r)\right|dr\leq\|r\phi_n'(r)\|_{L^2(\mathbb{R}_+)}\left(\int_r^{+\infty}\frac{1}{r^2}dr\right)^{\frac{1}{2}},$$

where we used the convergence $\phi_n(r)$ to 0 as $r\to+\infty$ from the definition of Φ_q. It gives

$$\forall r\in\mathbb{R}_+^*,\quad |\phi_n(r)|\leq\frac{\|\nabla\phi_n\|_{L^2(\mathbb{R}^3)}}{\sqrt{4\pi r}}\leq\frac{C}{r^{\frac{1}{2}}}. \quad (2.41)$$

Thus the set of integration in x in the integral (2.39) can be restricted to a bounded domain Ω uniformly with respect to $s \in [0, L_0]$. Indeed, from the increase of $a_{\phi_n}^{-1}$, for all $s \in [0, L_0]$,

$$D_n(s) := \{x \in \mathbb{R}^3 \ : \ \phi_n(x) < a_{\phi_n}^{-1}(s)\} \subset \{x \in \mathbb{R}^3 \ : \ \phi_n(x) < a_{\phi_n}^{-1}(L_0)\}, \quad (2.42)$$

and, since $\phi_n \in \Phi_q$ is nonpositive,

$$D_n(s) \subset \left\{x \in \mathbb{R}^3 \ : \ |x| \leq \frac{C^2}{e_0^2}\right\} =: \Omega \quad (2.43)$$

where $e_0 = \sup_{n\in\mathbb{N}} a_{\phi_n}^{-1}(L_0) < 0$.

Now, from the local compactness of the Sobolev embedding $\dot{H}^1 \hookrightarrow L^p_{loc}$ for $1 \leq p < 6$, there exists $\hat{h} \in \dot{H}^1_{rad}$ such that, up to a subsequence,

$$\phi_n \to \phi_Q \text{ and } \hat{h}_n \to \hat{h} \text{ in } L^p(\Omega) \text{ as } n \to +\infty.$$

At fixed $s \in [0, L_0]$, these convergences combined with the convergence (2.40) provide, on the one hand, the convergence for all $i \in \{0, 1, 2\}$,

$$\hat{h}_n^i \to \hat{h}^i \text{ in } L^2(\Omega_s),$$

and, on the other hand, the convergence of

$$x \mapsto g_n(x, s)^2 = 2\eta + 5\eta^2 + 4\eta^3 + \eta^4 \ \text{ with } \eta = \left(a_{\phi_n}^{-1}(s) - \phi_n(x)\right)_+$$

in $L^1_x(\Omega)$ to $g(\cdot, s)^2$ where

$$g(x, s) = \left(\left(1 + a_{\phi_Q}^{-1}(s) - \phi_Q(x)\right)_+^2 - 1\right)_+^{\frac{1}{2}} (1 + a_{\phi_Q}^{-1}(s) - \phi_Q(x)).$$

Thus, for all $i \in \{0, 1, 2\}$, as $n \to +\infty$

$$g_n(x, s)\hat{h}_n^i \to g(x, s)\hat{h}^i \text{ in } L^1_x(\mathbb{R}^3).$$

The convergence of $\gamma_n(s)$, at fixed s, follows. Now, by Cauchy-Schwarz, we have

$$\left|\frac{\left(\int_{\mathbb{R}^3} g_n(x, s)\hat{h}_n(x)dx\right)^2}{\int_{\mathbb{R}^3} g_n(x, s)dx}\right| \leq \int_{\mathbb{R}^3} g_n(x, s)(\hat{h}_n(x))^2 dx,$$

which provides

$$\begin{aligned} 0 \ \leq \ \gamma_n(s) \ &\leq \ \int_{\mathbb{R}^3} g_n(x, s)(\hat{h}_n(x))^2 dx \\ &\leq \ \int_{\mathbb{R}^3} g_n(x, L_0)(\hat{h}_n(x))^2 dx. \end{aligned}$$

Thus γ_n is uniformly bounded on $[0, L_0]$ and from standard dominated convergence theorem,

$$D^2 J_0(\phi_n)\left(\hat{h}_n, \hat{h}_n\right) \to D^2 J_0(\phi_Q)\left(\hat{h}, \hat{h}\right) \text{ as } n \to +\infty, \quad (2.44)$$

and similarly

$$D^2 J_0(\phi_Q)\left(\hat{h}_n, \hat{h}_n\right) \to D^2 J_0(\phi_Q)\left(\hat{h}, \hat{h}\right) \text{ as } n \to +\infty. \quad (2.45)$$

These convergences contradict (2.36), which proves the Taylor expansion (2.10) and concludes the proof of Lemma 2.3. □

2.2. **Proof of Proposition 2.2.** We use now the Taylor expansion stated in lemma 2.3 to obtain the proposition 2.2. In a first step we prove the local coercivity of the functionnal J_0 near ϕ_Q for radially symmetric potentials by using a Hardy type control, obtained in a second step. In a third step, we finally pass from radially symmetric modes to general modes by using a compactness argument [8] which concludes the proof of the proposition 2.2.

Step 1: coercivity of the quadratic form $D^2J(\phi_Q)$

In this step, our aim is to prove that there exists an universal constant $C_0 > 0$ such that

$$\forall h \in \dot{H}^1_{rad} \quad D^2J(\phi_Q)(h,h) \geq C_0 \|\nabla h\|^2_{L^2}. \tag{2.46}$$

where the space

$$\dot{H}^1_{rad} = \left\{ h \in L^2_{loc}(\mathbb{R}^3),\ \text{radially symmetric, s.t. } \nabla\phi \in L^2(\mathbb{R}^3) \text{ and } \lim_{|x|\to+\infty} \phi(x) = 0 \right\}.$$

is a Banch space. We consider the linear operator generated by the Hessian $D^2(\phi_Q)$:

$$\mathcal{L}h = -\Delta h - \int_{\mathbb{R}^3} |F'(e)|(h - \Pi h)dv.$$

Remark that the compactness of the quadratic form $D^2J(\phi_Q)$ on $\dot{H}^1_{rad}$ is given by the previous proof of the Taylor expansion. From the Fredholm alternative, we have only to prove the strict posivity

$$\forall h \in \dot{H}^1_{rad},\ h \neq 0,\ (\mathcal{L}h, h) > 0. \tag{2.47}$$

to obtain the coercivity (2.46). From the Taylor expansion (2.11), this inequality can be seen as a Poincaré inequality with an explicit constant, and we shall adapt the Hörmander's proof [18, 19] to obtain it.

Let us introduce the following operator T defined by

$$Tf(e,r) = \frac{\partial_r f}{r^2 \left((1+e-\phi_Q(r))^2 - 1\right)^{\frac{1}{2}} (1+e-\phi_Q(r))} = \frac{\partial_r f}{r^2 u\sqrt{1+u^2}},$$

where

$$u(r,e) = \left((1+e-\phi_Q(r))^2_+ - 1\right)^{\frac{1}{2}}_+. \tag{2.48}$$

Recalling that $\phi_Q(r)$ is strictly increasing and that $Supp(F) = [0, e_Q)$, we shall denote the space

$$\mathfrak{U} = \{(r,e),\ u > 0\} = \{(r,e),\ e \in (\phi_Q(0), 0),\ r \in (0, r(e))\} \ \text{ with } r(e) = \phi_Q^{-1}(e),$$

Then we define on $\tilde{\mathfrak{U}} = \mathfrak{U} \cap (0, r(e_Q)) \times (\phi_Q(0), e_Q)$, for a given $h \in \dot{H}^1_{rad}$, the function

$$f(r,e) = \int_0^r (h(\tau) - \Pi h(e)) \left((1+e-\phi_Q(\tau))^2 - 1\right)^{\frac{1}{2}}_+ (1+e-\phi_Q(\tau))\tau^2 d\tau. \tag{2.49}$$

We can differentiate f and get, in particular,

$$Tf = h - \Pi h. \tag{2.50}$$

Now let $\varepsilon > 0$ and study the behavior of $f(e,r)$ for $r \to 0$ and $r \to r(e)$ when e belongs to $(\phi_Q(0) + \varepsilon, -\varepsilon)$. Notice first that, from (2.41), for all $\tau > 0$

$$\tau^{\frac{1}{2}} |h(\tau)| \leq \|\nabla h\|_{L^2}, \tag{2.51}$$

and

$$\forall e \in (\phi_Q(0)+\varepsilon, -\varepsilon), \quad |\Pi h(e)| \leq C_\varepsilon. \tag{2.52}$$

These inequalities combined with the continuity of ϕ_Q imply the existence of $C_\varepsilon > 0$ such that, for all $e \in (\phi_Q(0)+\varepsilon, -\varepsilon)$ and for all $r \in (0, r(e))$,

$$|f(e,r)| \leq C_e r^{\frac{5}{2}}. \tag{2.53}$$

Moreover the function f satisfies

$$f(e, r(e)) = \int_0^{+\infty} (h(\tau) - \Pi h(e)) u(r,e)(1+e-\phi_Q(\tau))\tau^2 d\tau = 0, \tag{2.54}$$

where we used

$$\Pi h(e) = \frac{\displaystyle\int_0^{+\infty} u(\tau,e)(1+e-\phi_Q(\tau))h(\tau)\tau^2 d\tau}{\displaystyle\int_0^{+\infty} u(\tau,e)(1+e-\phi_Q(\tau))\tau^2 d\tau}.$$

Hence, using (2.51),(2.52) and the increase of ϕ_Q, we get for $e \in (\phi_Q(0)+\varepsilon, -\varepsilon)$ and $r \in (0, r(e))$

$$\begin{aligned} |f(r,e)| &= \left| \int_r^{r(e)} (h(\tau) - \Pi h(e)) \left((1+e-\phi_Q(\tau))^2 - 1 \right)_+^{\frac{1}{2}} (1+e-\phi_Q(\tau))\tau^2 d\tau \right| \\ &\leq C_\varepsilon \left((1+e-\phi_Q(r))^2 - 1 \right)_+^{\frac{1}{2}} (1+e-\phi_Q(r)) \int_r^{r(e)} \tau^{\frac{3}{2}} d\tau, \end{aligned}$$

in which

$$\left((1+e-\phi_Q(r))^2 - 1 \right)_+^{\frac{1}{2}} (1+e-\phi_Q(r)) \leq (e-\phi_Q(r))_+^{\frac{1}{2}} (2-\phi_Q(0))^{\frac{1}{2}} (1-\phi_Q(0)),$$

and

$$\int_r^{r(e)} \tau^{\frac{3}{2}} d\tau \leq C(r(e)-r) \lesssim (e-\phi_Q(r)).$$

since $\phi_Q'(r) \geq \inf\{\phi_Q'(\tau), \tau \in [r(-\varepsilon), r(\phi_Q(0)+\varepsilon)]\} > 0$. Finally there exists $C_\varepsilon > 0$ such that for all $e \in (\phi_Q(0)+\varepsilon, -\varepsilon)$ and for all $r \in (0, r(e))$

$$|f(e,r)| \leq C_\varepsilon (e-\phi_Q(r))^{\frac{3}{2}}. \tag{2.55}$$

Now denote

$$I(h) = \int_{\mathbb{R}^6} |F'(e)| (h - \Pi h)^2 dx dv.$$

First, passing to the spherical coordinates and performing the change of variable $e = \sqrt{1+|v|^2} - 1 + \phi_Q(r)$, we get from (2.50)

$$\begin{aligned} I(h) &= 16\pi^2 \int |F'(e)| (h-\Pi h)^2 r^2 u(e,r) \sqrt{1+u(e,r)^2} dr de \\ &= 16\pi^2 \int_{\phi_Q(0)}^0 |F'(e)| de \int_0^{r(e)} (h(r) - \Pi h(e)) \partial_r f dr, \end{aligned}$$

where $u(r,e)$ is defined by (2.48). Now, from (2.55) and (2.53), we have

$$(h(r(e)) - \Pi h(e)) f(r(e), e) = 0 \ \text{ and } \ \lim_{r\to 0} (h(r) - \Pi h(e)) f(r,e) = 0,$$

from which an integration by parts gives

$$I(h) \leq -16\pi^2 \int_{\tilde{\mathcal{U}}} |F'(e)| f \partial_r h de dr.$$

This inequality, combined with the identity

$$\rho_Q(r) = \frac{4\pi}{3}\int |F'(e)|u(e,r)^3 de,$$

leads to:

$$\begin{aligned} I(h) &\leq (4\pi)^{\frac{3}{2}}\|\nabla h\|_{L^2(\mathbb{R}^3)}\left(\int_0^{r(e_Q)}\frac{dr}{r^2}\left(\int_{\phi_Q(r)}^{e_Q}|F'(e)|f de\right)^2\right)^{\frac{1}{2}} \\ &\leq (4\pi)^{\frac{3}{2}}\|\nabla h\|_{L^2(\mathbb{R}^3)}\left(\frac{3}{4\pi}\int_0^{r(e_Q)}\frac{\rho_Q(r)}{r^2}dr\int_{\phi_Q(r)}^{e_Q}|F'(e)|\frac{f^2}{u(e,r)^3}de\right)^{\frac{1}{2}}, \end{aligned}$$

where we used Cauchy-Schwarz inequalities. Performing the change of variable $u = u(e,r)$ with respect to e, we finally obtain

$$I(h) \leq \|\nabla h\|_{L^2(\mathbb{R}^3)}\left(3\int \rho_Q(r)|F'(e)|\frac{f^2}{r^4u^4\sqrt{1+u^2}}dxdv\right)^{\frac{1}{2}}. \tag{2.56}$$

Now we claim the following Hardy type control:

$$I(h) \geq 3\int\left(\rho_Q(r) + \frac{\phi_Q'(r)}{r(1+u^2)}\right)|F'(e)|\frac{f^2}{r^4u^4\sqrt{1+u^2}}dxdv. \tag{2.57}$$

Assume (2.57), then (2.56) yields

$$I(h) + 3\int\frac{\phi_Q'(r)}{r(1+u^2)}|F'(e)|\frac{f^2}{r^4u^4\sqrt{1+u^2}}dxdv \leq \|\nabla h\|^2_{L^2(\mathbb{R}^3)}.$$

Thus, letting $\varepsilon \to 0$ yields $(\mathcal{L}h, h) \geq 0$. Moreover, if $(\mathcal{L}h,h) = 0$, then $f = 0$ on $\tilde{\mathcal{U}}$, which implies $h(r) = \Pi h(e)$ on $\tilde{\mathcal{U}}$, also $0 = (\mathcal{L}h,h) = \|\nabla h\|^2_{L^2(\mathbb{R}^3)}$ and finally $h = 0$. This concludes the proof of (2.47).

Step 2: Hardy type control.

Let us prove now the Hardy type control (2.57). Let g be a given smooth function in $\tilde{\mathcal{U}}$ and q such that $f = qg$. After easy computations, we get

$$(Tf)^2 = g^2(Tq)^2 + T(q^2gTg) - \frac{T^2g}{g}f^2 \geq T(q^2gTg) - \frac{T^2g}{g}f^2. \tag{2.58}$$

We take $g(e,r) = r^3u(e,r)^3$. Then, remarking that

$$\frac{\partial u}{\partial r} = -\phi_Q'(r)\frac{\sqrt{1+u^2}}{u},$$

we have

$$Tg = \frac{\partial_r g}{r^2u\sqrt{1+u^2}} = 3\frac{u^2}{\sqrt{1+u^2}} - 3r\phi_Q'(r),$$

and therefore

$$T^2g = \frac{-3}{ru\sqrt{1+u^2}}\left(\phi_Q''(r) + \frac{\phi_Q'(r)}{r}\left(2 + \frac{1}{1+u^2}\right)\right).$$

Since $\Delta\phi_Q = \rho_Q$, it implies

$$\frac{T^2g}{g} = -\frac{3}{r^4u^4\sqrt{1+u^2}}\left(\rho_Q(r) + \phi_Q'(r)\frac{1}{r(1+u^2)}\right). \tag{2.59}$$

Injecting this into (2.58) and integrating on $\tilde{\mathfrak{U}}$ yield:

$$\begin{aligned} I(h) \quad \geq \quad & 3\int \left(\rho_Q(r) + \frac{\phi_Q'(r)}{r(1+u^2)}\right) |F'(e)| \frac{f^2}{r^4u^4\sqrt{1+u^2}} dxdv \\ & + \int |F'(e)| T\left(f^2 \frac{Tg}{g}\right) dxdv. \end{aligned}, \tag{2.60}$$

where we used $q^2 gTg = f^2 \frac{Tg}{g}$. Now

$$\int |F'(e)| T\left(f^2 \frac{Tg}{g}\right) dxdv = 16\pi^2 \int_{\phi_Q(0)}^{e_Q} |F'(e)| de \int_0^{r(e)} \partial_r \left(f^2 \frac{Tg}{g}\right) dr.$$

From (2.55) and (2.53), we can perform an integration by parts and we get

$$\int |F'(e)| T\left(f^2 \frac{Tg}{g}\right) dxdv = 0.$$

We obtain thus the Hardy type control (2.57) which concludes the proof of the strict positivity of the Hessian $D^2(\phi_Q)$ on radial modes (2.47) and therefore the coercivity (2.46).

Step 3: end of the proof: Let us first prove the proposition 2.2 for radially symmetric potential. Let ϕ_n a sequence of Φ_q, radially symmetric, such that

$$\forall n \in \mathbb{N},\ \|\nabla\phi_n - \nabla\phi_Q\|_{L^2} \leq \delta_1,\ \ \limsup_{n\to+\infty} J(\phi_n) \leq J(\phi_Q), \tag{2.61}$$

where δ_1 will be defined later. Then we use the Taylor expansion (2.10) and the coercivity (2.46), which holds only for radially symmetric potential:

$$J(\phi_n) - J(\phi_Q) \geq \frac{C_0}{2} \|\nabla\phi_n - \nabla\phi_Q\|_{L^2}^2 + \varepsilon(\|\nabla\phi_n - \nabla\phi_Q\|_{L^2}).$$

Now, since $\varepsilon(\delta) = \circ(\delta^2)$ as $\delta \to 0$, we can choose δ_1 such that

$$\forall \delta \in [0, \delta_1],\ \ \varepsilon(\delta) \leq \frac{C_0}{4}\delta^2.$$

Thus

$$J(\phi_n) - J(\phi_Q) \geq \frac{C_0}{4} \|\nabla\phi_n - \nabla\phi_Q\|_{L^2}^2,$$

which finally provides, from (2.61),

$$\lim J(\phi_n) = J(\phi_Q) \text{ and } \|\nabla\phi_n - \nabla\phi_Q\|_{L^2} \to 0 \text{ as } n \to +\infty. \tag{2.62}$$

Let pass now at the general case. We consider a sequence ϕ_n of Φ_q. We define then $\phi_n^* = -(-\phi_n)^*$, the opposite of the Schwarz rearrangement of $(-\phi_n)$. We introduce, for all $n \in \mathbb{N}$, the potential $\phi_n^\#$ given by

$$\forall x \in \mathbb{R}^3,\ \phi_n^\#(x) = \phi_n^*\left(\frac{4\pi}{3}|x|^3\right).$$

It is the Schwarz rearrangement of ϕ_n, defined as a function of $\mathbb{R}^3$. Then we claim that we have

$$J(\phi_n^\#) - J(\phi_n) = \frac{1}{2}\|\nabla\phi_n^\#\|_{L^2}^2 - \frac{1}{2}\|\nabla\phi_n\|_{L^2}^2 \leq 0, \tag{2.63}$$

and that there exists a constant $\delta_0 > 0$ such that

$$\|\nabla\phi_n - \nabla\phi_Q\|_{L^2} \leq \delta_0 \text{ implies } \left\|\nabla\phi_n^\# - \nabla\phi_Q\right\|_{L^2} \leq \delta_1. \tag{2.64}$$

Assume first that these claims are true and take the sequence ϕ_n such that

$$\forall n \in \mathbb{N},\ \|\nabla\phi_n - \nabla\phi_Q\|_{L^2} \leq \delta_0,\quad \lim_{n\to+\infty} J(\phi_n) \leq J(\phi_Q). \tag{2.65}$$

Then, from (2.63) and (2.64), the sequence $\phi_n^\#$ satisfies (2.61) and therefore satisfies (2.62). Moreover we obtain the equality

$$\lim_{n\to+\infty} J(\phi_n^\#) = \lim_{n\to+\infty} J(\phi_n) = J(\phi_Q),$$

which implies from (2.63)

$$\lim_{n\to+\infty} \|\nabla\phi_n^\#\|_{L^2} = \lim_{n\to+\infty} \|\nabla\phi_n\|_{L^2}. \tag{2.66}$$

From (2.62) and (2.66), we are in the equality case of the Polya-Szego inequality. Thanks to Theorem 2 in [8], we obtain the following compactness result : there exists a sequence of translation shifts in space, $x_n \in \mathbb{R}$, such that

$$\|\nabla\phi_n - \nabla\phi_Q(\cdot - x_n)\|_{L^2} \to 0 \text{ as } n \to +\infty.$$

Let us now prove the claims (2.63) and (2.64). Remark that, from Schwarz rearrangement classical properties, for all $n \in \mathbb{N}$, $\phi_n^\#$ belongs to Φ_q and satisfies the Polya-Szego inequality

$$\|\nabla\phi_n^\#\|_{L^2} \leq \|\nabla\phi_n\|_{L^2}. \tag{2.67}$$

Moreover, for all C^1 function β such that $\beta(0) = 0$, one has

$$\int_{\mathbb{R}^3} \beta(\phi_n(x))dx = \int_{\mathbb{R}^3} \beta(\phi_n^\#(x))dx,$$

which for $e < 0$ and for $\beta(t) = \big((1+e-t)_+^2 - 1\big)_+^{\frac{3}{2}}$ implies $a_{\phi_n} = a_{\phi_n^*}$ given by the expression (A.2). Thus, from (2.16),

$$\begin{aligned} J(\phi_n) &= \frac{1}{2}\|\nabla\phi_n\|_{L^2}^2 - \int_{-\infty}^0 G(a_{\phi_n}(e))de \\ &= \frac{1}{2}\|\nabla\phi_n\|_{L^2}^2 - \int_{-\infty}^0 G(a_{\phi_n^\#}(e))de, \end{aligned}$$

which provides directly the claim (2.63). Let us prove the claim (2.64) by a contradiction argument. We assume that there exists a sequence ψ_n in $\dot{H}^1$ such that

$$\|\nabla\psi_n - \nabla\phi_Q\|_{L^2} \leq \frac{1}{n} \text{ and } \left\|\nabla\psi_n^\# - \nabla\phi_Q\right\|_{L^2} > \delta_1. \tag{2.68}$$

The contractivity property of the rearrangement in L^p-norms and a Sobolev embedding give

$$\|\psi_n^\# - \phi_Q\|_{L^6} \leq \|\psi_n - \phi_Q\|_{L^6} \leq C\|\nabla\psi_n - \nabla\phi_Q\|_{L^2} \leq \frac{C}{n}.$$

Moreover by (2.67) and (2.68), we have

$$\|\nabla\psi_n^\#\|_{L^2} \leq \|\nabla\psi_n\|_{L^2} \leq \|\nabla\phi_Q\|_{L^2} + \frac{1}{n}. \tag{2.69}$$

Hence $\psi_n^\#$ is bounded in $\dot{H}_r^1$ and the sequence $\nabla\psi_n^\#$ converges to $\nabla\phi_Q$ in the L^2 weak topology. In fact, from Fatou lemma, the inequality (2.69) implies

$$\lim_{n\to+\infty} \|\nabla\psi_n^\#\|_{L^2} = \|\nabla\phi_Q\|_{L^2}.$$

Thus the sequence $\nabla\psi_n^{\#}$ converges to $\nabla\phi_Q$ in $L^2(\mathbb{R}^3)$ which contradicts (2.68). Notice that, for all $q \in (6, 12)$, the space Φ_q is included in $\dot{H}^1$ and thus δ_0 does not depend on q. The proof of Proposition 2.2 is now complete.

3. Nonlinear stability of Q

We are now ready to prove Theorem 1.1 in this section. The proof is based on the two following arguments : i) The local coercivity property of J stated in Proposition 2.2 which ensures the compactness of the potential field, and ii) The compactness of the whole distribution function in the energy space.

3.1. Local compactness of the distribution function.

We will prove in this subsection that the Proposition 2.2 implies the following compactness result:

Proposition 3.1 (Local compactness of local minimizing sequences)**.** *Let $p > \frac{3}{2}$. Let $\delta_0 > 0$ the constant defined in Proposition 2.2 and let f_n be a sequence of $\mathcal{E}_p$ such that*

$$\|f_n^* - Q^*\|_{L^1(\mathbb{R})} \to 0,\ \|f_n^* - Q^*\|_{L^p(\mathbb{R})} \to 0,\ \limsup_{n\to+\infty} \mathcal{H}(f_n) \leq \mathcal{H}(Q), \tag{3.1}$$

and

$$\|\nabla\phi_{f_n} - \nabla\phi_Q\|_{L^2} < \delta_0. \tag{3.2}$$

Then there exists a translation shift x_n such that

$$\|f_n - Q(\cdot - x_n)\|_{\mathcal{E}_p} \to 0 \text{ as } n \to +\infty. \tag{3.3}$$

Proof of Proposition 3.1. Let (f_n) a sequence of $\mathcal{E}_p$ satisfying (3.1) and (3.2).

Step 1: Compactness of the potential. We first remark that, from inequality (B.4), we have

$$\mathcal{H}(f_n) \geq \int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1} - 1 + \phi_{f_n}(x)\right) f_n^{*\phi_{f_n}} dxdv + \frac{1}{2}\|\nabla\phi_{f_n}\|_{L^2}^2,$$

which implies, from the change of variable (A.11),

$$\begin{aligned}\mathcal{H}(f_n) - J(\phi_{f_n}) &\geq \int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1} - 1 + \phi_{f_n}(x)\right)\left(f_n^{*\phi_{f_n}} - Q^{*\phi_{f_n}}\right) dxdv \\ &\geq \int_0^{+\infty} a_{\phi_{f_n}}^{-1}(s)(f_n^*(s) - Q^*(s))ds.\end{aligned}$$

Finally

$$\mathcal{H}(f_n) - H(Q) \geq J(\phi_{f_n}) - J(\phi_Q) + \int_0^{+\infty} a_{\phi_{f_n}}^{-1}(s)(f_n^*(s) - Q^*(s))ds. \tag{3.4}$$

Now, since (f_n) satisfies (3.1) and (3.2), the sequence (f_n) is bounded in the energy space $\mathcal{E}_p$. From classical interpolation inequalities, for

$$q = \frac{3(4p-3)}{p} \in (6, 12),$$

the sequence ϕ_{f_n} belongs to Φ_q and is bounded in $L^q(\mathbb{R}^3)$. Thus, from (A.3) we have

$$\left|a_{\phi_{f_n}}^{-1}(s)\right| \leq C\left(\frac{1}{s^{\frac{1}{q-3}}} + \frac{1}{s^{\frac{1}{q}}}\right). \tag{3.5}$$

Hence we obtain from Hölder inequalities

$$\left|\int_1^{+\infty} a^{-1}_{\phi_{f_n}}(s)(f_n^*(s)-Q^*(s))ds\right| \leq 2C\|f_n^*-Q^*\|_{L^1(\mathbb{R})},$$

and

$$\left|\int_0^{1} a^{-1}_{\phi_{f_n}}(s)(f_n^*(s)-Q^*(s))ds\right| \leq \|a^{-1}_{\phi_{f_n}}\|_{L^{p'}(0,1)}\|f_n^*-Q^*\|_{L^p(\mathbb{R})},$$

where $p'=\frac{p}{p-1}<3$. Notice that $\frac{p'}{q-3}<1$ which gives, from (3.5), the boundedness :

$$\|a^{-1}_{\phi_{f_n}}\|^{p'}_{L^{p'}(0,1)} \leq 2C\int_0^1 \frac{1}{s^{\frac{p'}{q-1}}}ds = 2C\left(1-\frac{p'}{q-1}\right).$$

We have then the convergence

$$\int_0^{+\infty} a^{-1}_{\phi_{f_n}}(s)(f_n^*(s)-Q^*(s))ds \to 0$$

as $n\to+\infty$ and, injecting this in (3.4), we conclude from (3.1) that

$$\lim_{n\to+\infty} J(\phi_{f_n}) \leq J(\phi_Q). \tag{3.6}$$

Together with the condition (3.2), this allows us to apply Proposition (2.2) and we conclude that there exists a sequence of translation shifts in space x_n such that

$$\|\nabla\phi_{f_n}-\nabla\phi_Q(\cdot-x_n)\|_{L^2}\to 0 \ \text{ as } n\to+\infty. \tag{3.7}$$

Step 2: Convergence of $f_n(\cdot+x_n,\cdot)$ in $\mathcal{E}_p$. To obtain the convergence in the energy space $\mathcal{E}_p$, the method that we chose is very similar with the method developed in [23]. We renote $f_n := f_n(\cdot+x_n,\cdot)$. We remark first that since $Q=Q^{*\phi_Q}$

$$\begin{aligned}\left|\int_{\mathbb{R}^6}\left(\sqrt{|v|^2+1}-1+\phi_Q\right)\left(f_n^{*\phi_Q}-Q\right)dxdv\right| &= \left|\int_0^\infty a^{-1}_{\phi_Q}(s)\left(f_n^*(s)-Q^*(s)\right)ds\right| \\ &\leq \|\phi_Q\|_{L^\infty}\|f_n^*-Q^*\|_{L^1},\end{aligned}$$

and thus

$$\int_{\mathbb{R}^6}\left(\sqrt{|v|^2+1}-1+\phi_Q(x)\right)\left(f_n^{*\phi_Q}-Q\right)dxdv\to 0 \tag{3.8}$$

Now, from the inequality (B.4), it implies that

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^6}\left(\sqrt{|v|^2+1}-1+\phi_Q(x)\right)(f_n-Q)\,dxdv\geq 0.$$

Hence, since

$$\mathcal{H}(f_n)=\mathcal{H}(Q)+\frac{1}{2}\|\nabla\phi_{f_n}-\nabla\phi_Q\|^2_{L^2}+\int_{\mathbb{R}^6}\left(\sqrt{|v|^2+1}-1+\phi_Q(x)\right)(f_n-Q)\,dxdv,$$

in which

$$\limsup_{n\to+\infty}\mathcal{H}(f_n)\leq\mathcal{H}(Q) \text{ and } \lim_{n\to+\infty}\|\nabla\phi_{f_n}-\nabla\phi_Q\|_{L^2}=0,$$

we obtain, as $n\to+\infty$,

$$\int_{\mathbb{R}^6}\left(\sqrt{|v|^2+1}-1+\phi_Q(x)\right)(f_n-Q)\,dxdv\to 0. \tag{3.9}$$

The two convergences (3.8) and (3.9) yield

$$T_n := \int_{\mathbb{R}^6} e_Q(x,v)\left(f_n - f_n^{*\phi_Q}\right)dxdv \to 0 \text{ as } n \to +\infty, \tag{3.10}$$

where $e_Q(x,v) := \sqrt{|v|^2+1} - 1 + \phi_Q(x)$. As in the proof of (B.4), we write T_n in the following equivalent form

$$T_n = \int_{t=0}^{+\infty} dt \left(\int_{S_1^n(t)} e_Q(x,v)dxdv - \int_{S_2^n(t)} e_Q(x,v)dxdv \right),$$

where

$$S_1^n(t) = \{(x,v) \in \mathbb{R}^6, \quad f_n^{*\phi_Q}(x,v) \le t < f_n(x,v)\},$$
$$S_2^n(t) = \{(x,v) \in \mathbb{R}^6, \quad f_n(x,v) \le t < f_n^{*\phi_Q}(x,v)\}.$$

From (B.14), we have

$$\forall (x,v) \in S_1^n(t), \quad e_Q(x,v) \ge (f_n^* \circ a_{\phi_Q})^{-1}(t).$$

Thus

$$T_n \ge \int_{t=0}^{+\infty} dt \left(\int_{S_1^n(t)} (f_n^* \circ a_{\phi_Q})^{-1}(t)dxdv - \int_{S_2^n(t)} e_Q(x,v)dxdv \right),$$

and since $\text{meas}(S_1^n(t)) = \text{meas}(S_2^n(t))$ for all $t \in \mathbb{R}_+$,

$$T_n \ge \int_{t=0}^{+\infty} dt \int_{S_2^n(t)} \left[(f_n^* \circ a_{\phi_Q})^{-1}(t) - e_Q(x,v) \right] dxdv.$$

Remark from (B.13), that the right term is nonnegative and thus, from (3.10), we get as $n \to +\infty$

$$A_n := \int_{t=0}^{+\infty} dt \int_{S_2^n(t)} \left[(f_n^* \circ a_{\phi_Q})^{-1}(t) - e_Q(x,v) \right] dxdv \to 0 \tag{3.11}$$

We now claim that this implies

$$B_n := \int_{t=0}^{+\infty} dt \int_{\Omega_2^n(t)} \left[(Q^* \circ a_{\phi_Q})^{-1}(t) - e_Q(x,v) \right] dxdv \to 0 \tag{3.12}$$

as $n \to +\infty$ where

$$\Omega_2^n(t) = \{(x,v) \in \mathbb{R}^6, \quad f_n(x,v) \le t < Q(x,v)\}.$$

To prove it, we decompose

$$S_2^n = (S_2^n \backslash \Omega_2^n) \cup (S_2^n \cap \Omega_2^n) \text{ and } \Omega_2^n = (\Omega_2^n \backslash S_2^n) \cup (S_2^n \cap \Omega_2^n).$$

Thus

$$\begin{aligned} A_n - B_n = & \int_{t=0}^{+\infty} dt \int_{\Omega_2^n(t)\backslash S_2^n(t)} \left[e_Q(x,v) - (Q^* \circ a_{\phi_Q})^{-1}(t) \right] dxdv \\ & + \int_{t=0}^{+\infty} dt \int_{S_2^n(t)\backslash \Omega_2^n(t)} \left[(f_n^* \circ a_{\phi_Q})^{-1}(t) - e_Q(x,v) \right] dxdv \\ & + \int_{t=0}^{+\infty} dt \int_{S_2^n(t)\cap \Omega_2^n(t)} \left[(f_n^* \circ a_{\phi_Q})^{-1}(t) - (Q^* \circ a_{\phi_Q})^{-1}(t) \right] dxdv \end{aligned} \tag{3.13}$$

Now let us examine each term as $n \to +\infty$. We first observe that for $g, h \in L^6(\mathbb{R}^6)$ we have

$$\int_0^{+\infty} \mathrm{meas}(\{g < t \leq h\})dt = \int_{\mathbb{R}^6} (h-g)_+ dxdv, . \tag{3.14}$$

Thus we obtain

$$\int_0^{+\infty} \mathrm{meas}(S_2^n(t) \backslash \Omega_2^n(t))dt \leq \int_0^{+\infty} \mathrm{meas}(\{Q < t \leq f_n^{*\phi_Q}\})dt = \int_{\mathbb{R}^6} (f_n^{*\phi_Q} - Q)_+ dxdv,$$

which gives, from (3.1),

$$\int_0^{+\infty} \mathrm{meas}(S_2^n(t) \backslash \Omega_2^n(t))dt \leq \|f_n^{*\phi_Q} - Q\|_{L^1} = |f_n^* - Q^*|_{L^1} \to 0,$$

and similarly for $\mathrm{meas}(\Omega_2^n(t) \backslash S_2^n(t))$. Using in addition the estimate

$$|e_Q(x,v)| \leq |\phi_Q(0)|, \quad \left|(f_n^* \circ a_{\phi_Q})^{-1}(t)\right| \leq |\phi_Q(0)|, \quad \left|(Q^* \circ a_{\phi_Q})^{-1}(t)\right| \leq |\phi_Q(0)|,$$

we deduce that the first two terms of (3.13) converge to 0 as $n \to +\infty$. We now deal with the third term. Combining the strong L^1 convergence in (3.1), the monotonicity of f_n^* and the continuity of Q^*, we get

$$\forall e \in (\phi_Q(0), 0), \quad f_n^* \circ a_{\phi_Q}(e) \to Q^* \circ a_{\phi_Q}(e) \text{ as } n \to +\infty.$$

Thus for $e \in (\phi_Q(0), 0)$ such that $Q^* \circ a_{\phi_Q}(e) > t$ we have for n large enough

$$f_n^* \circ a_{\phi_Q}(e) > t,$$

which from the definition of the pseudoinverse $(f_n^* \circ a_{\phi_Q})^{-1}$ provides

$$e \leq \liminf_{n \to +\infty} (f_n^* \circ a_{\phi_Q})^{-1}(t).$$

From the definition of $(Q^* \circ a_{\phi_Q})^{-1}$, we conlude that

$$\liminf_{n \to +\infty} (f_n^* \circ a_{\phi_Q})^{-1}(t) \geq (Q^* \circ a_{\phi_Q})^{-1}(t).$$

We just inject it into the third term of (3.13) to obtain

$$\liminf_{n \to +\infty} (A_n - B_n) \geq 0.$$

Moreover, from (3.11), from the definition of Ω_2^n and from (B.13), we have

$$A_n \to 0 \text{ and } B_n \geq 0.$$

We conclude that the convergence (3.12) holds true:

$$\int_{t=0}^{+\infty} dt \int_{\{f_n \leq t < Q\}} \left[(Q^* \circ a_{\phi_Q})^{-1}(t) - e_Q(x,v)\right] dxdv \to 0. \tag{3.15}$$

Since $e \mapsto F(e)$ is continuous and strictly decreasing with respect to $e = \sqrt{|v|^2+1} - 1 + \phi_Q(x)$ for $(x,v) \in \{Q(x,v) > 0\}$, we have

$$t < Q(x,v) \text{ implies } Q^* \circ a_{\phi_Q})^{-1}(t) - e_Q(x,v) > 0.$$

Thus, up to a subsequence,

$$\text{for a.e. } (t,x,v) \in \mathbb{R}_+^* \times \mathbb{R}^6, \quad \mathbf{1}_{\{f_n \leq t < Q\}} \to 0, \text{ as } n \to +\infty.$$

Since $\mathbf{1}_{\{f_n \leq t < Q\}} \leq \mathbf{1}_{\{t < Q\}}$ and

$$\int_{t=0}^{+\infty} dt \int_{\mathbb{R}^6} \mathbf{1}_{\{t<Q\}} dxdvdt = \|Q\|_{L^1} < +\infty,$$

we may apply the dominated convergence theorem to get:

$$\int_{t=0}^{+\infty} dt \int_{\mathbb{R}^6} \mathbf{1}_{\{f_n \le t < Q\}} dxdvdt \to 0 \text{ as } n \to +\infty,$$

which, from (3.14) is equivalent to

$$\int_{\mathbb{R}^6} (Q - f_n)_+ dxdv \to 0 \text{ as } n \to +\infty. \tag{3.16}$$

Now we write

$$\begin{aligned} \int_{\mathbb{R}^6} (f_n - Q)_+ dxdv &\le \int_{\mathbb{R}^6} (f_n - f_n^{*\phi_Q})_+ dxdv + \int_{\mathbb{R}^6} (f_n^{*\phi_Q} - Q)_+ dxdv \\ &\le \int_0^{+\infty} \text{meas}(\{f_n^{*\phi_Q} \le t < f_n\}) dt + \|f_n^{*\phi_Q} - Q\|_{L^1}, \end{aligned}$$

where, from the equimeasurability of f_n and $f_n^{*\phi_Q}$, we have

$$\begin{aligned} \int_0^{+\infty} \text{meas}(\{f_n^{*\phi_Q} \le t < f_n\}) dt &= \int_0^{+\infty} \text{meas}(\{f_n \le t < f_n^{*\phi_Q}\}) dt \\ &= \int_{\mathbb{R}^6} (f_n^{*\phi_Q} - f_n)_+ dxdv \\ &\le \int_{\mathbb{R}^6} (Q - f_n)_+ dxdv + \int_{\mathbb{R}^6} (f_n^{*\phi_Q} - Q)_+ dxdv. \end{aligned}$$

We finally get

$$\int_{\mathbb{R}^6} (f_n - Q)_+ dxdv \le \int_{\mathbb{R}^6} (Q - f_n)_+ dxdv + 2\|f_n^{*\phi_Q} - Q\|_{L^1}.$$

Now, since $\|f_n^{*\phi_Q} - Q\|_{L^1} = \|f_n^* - Q^*\|_{L^1} \to 0$, we obtain from (3.16) the L^1 convergence

$$\|f_n - Q\|_{L_1(\mathbb{R}^6)} \to 0 \text{ as } n \to +\infty. \tag{3.17}$$

Now, from the convergence of $\nabla\phi_{f_n}$ to $\nabla\phi_Q$ in $L^2(\mathbb{R}^3)$ and from (3.1) we have

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1} - 1\right)(f_n - Q)\, dxdv \le 0. \tag{3.18}$$

But f_n converges almost everywhere in $\mathbb{R}^6$ to Q, thus we have an equality in (3.18) and thus the convergence is strong:

$$\int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1}\right) |f_n - Q|\, dxdv \to 0 \text{ as } n \to +\infty. \tag{3.19}$$

Similarly, we remark that

$$\|f_n\|_{L^p(\mathbb{R}^6)} = \|f_n^*\|_{L^p(\mathbb{R}_+)} \to \|Q^*\|_{L^p(\mathbb{R}_+)} = \|Q\|_{L^p(\mathbb{R}^6)},$$

when $n \to +\infty$ and we obtain the strong L^p convergence and the proof of Proposition 3.1 is completed.

□

3.2. **Stability from the local compactness.** Let $p > \frac{3}{2}$. From the Hardy-Littlewood-Sobolev inequality and Hölder inequalities, we have from the classical interpolation's inequality:

$$\|\nabla\phi_f - \nabla\phi_g\|_{L^2} \leq K \|\rho_f - \rho_g\|_{L^{\frac{6}{5}}} \leq K \|f-g\|_{L^1}^{\frac{2p-3}{6(p-1)}} \|f-g\|_{L^p}^{\frac{p}{6(p-1)}} \||v|(f-g)\|_{L^1}^{\frac{1}{2}}. \tag{3.20}$$

Thus for all $f \in \mathcal{E}_p$

$$\|f - Q\|_{\mathcal{E}_p} < \varepsilon \text{ implies } \|\nabla\phi_f - \nabla\phi_Q\|_{L^2} < K\varepsilon. \tag{3.21}$$

Let fixed $\varepsilon_0 > 0$ such that

$$K\varepsilon_0 < \frac{\delta_0}{2},$$

where K is the constant in (3.21) and δ_0 is defined by Proposition 3.1. Let $\varepsilon \in (0, \varepsilon_0)$. Then an equivalent reformulation of Proposition 3.1 is the following: there exists

$$0 < \eta < \varepsilon_0$$

such that the following sentence holds true: if $f \in \mathcal{E}_p$ is such that

$$\|f^* - Q^*\|_{L^1(\mathbb{R})} \leq \eta,\ \|f^* - Q^*\|_{L^p(\mathbb{R})} \leq \eta,\ \mathcal{H}(f) \leq \mathcal{H}(Q) + \eta, \tag{3.22}$$

and

$$\|\nabla\phi_f - \nabla\phi_Q\|_{L^2} < \delta_0, \tag{3.23}$$

then there exists a translation shift $y \in \mathbb{R}^3$ such that

$$\|f - Q(\cdot - y)\|_{\mathcal{E}_p} < \varepsilon. \tag{3.24}$$

Remark that the assumption (3.23) can be replaced by

$$\inf_{z\in\mathbb{R}^3} \|\nabla\phi_f(\cdot + z) - \nabla\phi_Q\|_{L^2} < \delta_0. \tag{3.25}$$

Indeed, on the first hand, if the condition (3.23) is satisfied, the condition (3.25) is satisfied too. In the other hand, if a function f satisfies (3.22) and (3.25), then there exists $z \in \mathbb{R}^3$ such that $\tilde{f} = f(\cdot + z, \cdot)$ satisfies (3.23). But $\tilde{f}$ satisfies (3.22) too. Thus we have (3.24) for $\tilde{f}$ and also for f.

Now we prove theorem 1.1. Let $f_0 \in \mathcal{E}_p$ such that

$$\|f_0 - Q\|_{\mathcal{E}_p} < \eta. \tag{3.26}$$

Let $f(t) \in \mathcal{F}([0,T), \mathcal{E}_p)$ a corresponding renormalized solution to (1.1) as stated in [9, 10]. We want to show that $f(t)$ satisfies (3.24) for all $t \in [0,T)$.

Let us first prove that $f(t)$ satisfies (3.22) for all t. From (3.26), we have

$$\|f_0 - Q\|_{L^1(\mathbb{R}^6)} \leq \eta,\ \|f_0 - Q\|_{L^p(\mathbb{R}^6)} \leq \eta,\ \mathcal{H}(f_0) \leq \mathcal{H}(Q) + \eta,$$

and from the property of contraction of symmetric rearrangement:

$$\|f^* - Q^*\|_{L^p(\mathbb{R}_+)} \leq \|f - Q\|_{L^p(\mathbb{R}^6)},$$

we deduce that f_0 satisfies (3.22). By conservation of the flow, we have

$$\forall t \in (0,T),\quad f(t)^* = f_0^* \text{ and } \mathcal{H}(f(t)) \leq \mathcal{H}(f_0).$$

We conclude that (3.22) is satisfied for all $t \in (0,T)$.

Let us now prove that (3.25) is satisfied for all $t \in (0,T)$. At $t = 0$, since $\eta < \varepsilon_0$, from (3.21), we have

$$\|\nabla\phi_{f_0} - \nabla\phi_Q\|_{L^2} < \frac{\delta_0}{2}.$$

Moreover, from the regularity of the flow, the potential satisfies

$$t \mapsto \nabla\phi_{f(t)} \in \mathcal{C}([0,T), L^2(\mathbb{R}^3)),$$

which provides the continuity on $[0,T)$ of the function

$$\beta : t \mapsto \inf_{z\in\mathbb{R}^3} \left\|\nabla\phi_{f(t)}(\cdot+z) - \nabla\phi_Q\right\|_{L^2},$$

Indeed for all $z \in \mathbb{R}^3$, we have the uniform boundedness

$$\begin{aligned}\Big|\left\|\nabla\phi_{f(t)}(\cdot+z) - \nabla\phi_Q\right\|_{L^2} - \left\|\nabla\phi_{f(t_0)}(\cdot+z) - \nabla\phi_Q\right\|_{L^2}\Big| \\ \leq \left\|\nabla\phi_{f(t)}(\cdot+z) - \nabla\phi_{f(t_0)}(\cdot+z)\right\|_{L^2} = \left\|\nabla\phi_{f(t)} - \nabla\phi_{f(t_0)}\right\|_{L^2},\end{aligned}$$

which provides the continuity of β. Now, by a contradiction argument, assume that there exists $t_1 > 0$ such that $\beta(t_1) > \delta_0$. Since $\beta(0) < \delta_0/2$ there exists $t_2 > 0$ such that $\beta(t_2) = 3\delta_q/4$. In particular, $f(t_2)$ satisfies (3.22) and (3.25) and thus there exists $z(t_2) \in \mathbb{R}^3$ such that

$$\|f(t_2) - Q(\cdot - x(t_2))\|_{\mathcal{E}_p} < \varepsilon < \varepsilon_0.$$

Injecting it into (3.21), we conclude that $\beta(t_2) < \delta_0/2$, which contradicts our assumption. The proof of Theorem (1.1) is complete.

Appendix A. Jacobian of the microscopic energy

We give some useful properties of the jacobian a_ϕ given by (2.4) and the rearrangement with respect to the microscopic energy defined in (2.5). We recall that the space Φ_q is defined for $q > 3$ by (2.1).

We first gather in the following two lemmas some important properties of the jacobian a_ϕ.

Lemma A.1 (Properties of the Jacobian a_ϕ)**.** *Let $\phi \in \Phi_q$ with $q > 3$. We recall that the Jacobian a_ϕ is defined as*

$$\forall e < 0, \quad a_\phi(e) = meas\left\{(x,v) \in \mathbb{R}^6 \ : \ \sqrt{|v|^2+1} - 1 + \phi(x) < e\right\}. \tag{A.1}$$

Then:

(i) We have the explicit formula:

$$\forall e < 0, \quad a_\phi(e) = \frac{4\pi}{3}\int_{\mathbb{R}^3} \left((1+e-\phi(x))_+^2 - 1\right)_+^{\frac{3}{2}} dx. \tag{A.2}$$

Notice that $\left((1+e-\phi(x))_+^2 - 1\right)_+ = \left((1+e-\phi(x))^2 - 1\right)\mathbf{1}_{e-\phi(x)>0}(x)$.

(ii) a_ϕ is $\mathcal{C}^1$ on $(-\infty, 0)$ and is a strictly increasing $\mathcal{C}^1$ diffeomorphism from $(\inf\phi, 0)$ onto $\mathbb{R}_+^*$, which defines a_ϕ^{-1}. Moreover there exist two positive constants C and $\tilde{C}$ such that for all $e < 0$ and for all $s > 0$

$$a_\phi(e) \leq \frac{C}{|e|^{q-3}}\left(1 + \frac{1}{|e|^3}\right)\|\phi\|_{L^q}^q \quad \text{and} \quad a_\phi^{-1}(s) \geq -\tilde{C}\left(\frac{\|\phi\|_{L^q}^{\frac{q}{q-3}}}{s^{\frac{1}{q-3}}} + \frac{\|\phi\|_{L^q}}{s^{\frac{1}{q}}}\right). \tag{A.3}$$

The quantity $\inf\phi$ is the essential infimum of the measurable function ϕ.

Let prove these properties.

Proof of Lemma A.1. To prove *(i)*, remark that for $e<0$

$$\sqrt{|v|^2+1}-1+\phi(x)<e \Leftrightarrow \left\{\phi(x)<e \text{ and } |v|<\left((1+e-\phi(x))_+^2-1\right)_+^{\frac12}\right\}.$$

Since $\lim_{|x|\to+\infty}\phi=0$, the set $\left\{x\in\mathbb{R}^3\ :\ \phi(x)<e\right\}$ is bounded and since ϕ belongs to $L^q(\mathbb{R}^3)$ with $q>3$, we have $\phi\in L^3_{loc}(\mathbb{R}^3)$ which implies

$$x\mapsto \left((1+e-\phi(x))_+^2-1\right)_+^{\frac12} \text{ belongs to } L^3(\mathbb{R}^3).$$

Thus for all $e<0$, $a_\phi(e)$ is finite and after passing to the spherical coordinates in velocity we obtain the formula (A.2).

Proof of (ii). We define $g(e,x):=\left((1+e-\phi(x))_+^2-1\right)_+^{\frac32}$. Then, for $e<e_0<0$,

$$0\le\frac{\partial g}{\partial e}(e,x)\le 3\left((1+e_0-\phi(x))_+^2-1\right)_+^{\frac12}(1+e_0-\phi(x)),$$

which, as function of the variable x, belongs to $L^1(\mathbb{R}^3)$. Indeed its support is included in the bounded set $\left\{x\in\mathbb{R}^3\ :\ \phi(x)<e_0\right\}$ and $\phi\in L^2_{loc}(\mathbb{R}^3)$. Hence we may apply the dominated convergence theorem and get that a_ϕ is a $\mathcal{C}^1$ function on $\mathbb{R}^*_-$, nul on $(-\infty,\inf\phi]$ (if $\inf\phi$ is finite) and strictly increasing on $(\inf\phi,0)$.
We now look for the limit of $a_\phi(e)$ when $e\to0$. Since $\phi\in\Phi_q$,

$$a_\phi(e)\ge\frac{4\pi}{3}\int_{\mathbb{R}^3}\left(\left(1+e+\frac{m(\phi)}{1+|x|}\right)_+^2-1\right)_+^{\frac32}dx\to+\infty \text{ as } e\to0.$$

To conclude the proof of *(ii)*, let us study the behavior of $a_\phi(e)$ as $e\to-\infty$ in the case $\inf\phi=-\infty$. We observe, from Hölder inequality, that

$$a_\phi(e)\le\left(\text{meas}\{x\in\mathbb{R}^3\ :\ \phi(x)<e\}\right)^{\frac{q-3}{q}}\left(\int_{\phi(x)<e}\left((1+e-\phi(x))^2-1\right)^{\frac q2}dx\right)^{\frac3q}, \tag{A.4}$$

where both terms can be controled. The first term satisfies

$$\text{meas}\{x\in\mathbb{R}^3\ :\ \phi(x)<e\}\le\int_{\phi(x)<e}\left(\frac{\phi(x)}{e}\right)^q dx\le\frac{\|\phi\|_{L^q}^q}{|e|^q}, \tag{A.5}$$

and the second term

$$\int_{\phi(x)<e}\left((1+e-\phi(x))^2-1\right)^{\frac q2}dx\le\int_{\phi(x)<e}C\left(1+|\phi(x)|^q\right)dx\le C\left(\frac{\|\phi\|_{L^q}^q}{|e|^q}+\|\phi\|_{L^q}^q\right),$$

where we use the inequality (A.5). Now, injecting these both inequalities in (A.4), we conclude that there exists a constant $C>0$ such that for all $e<0$

$$a_\phi(e)\le\frac{C}{|e|^{q-3}}\left(1+\frac{1}{|e|^3}\right)\|\phi\|_{L^q}^q. \tag{A.6}$$

Finally the inverse a_ϕ^{-1} satisfies for all $s>0$

$$a_\phi^{-1}(s)\ge-\tilde C\left(\frac{\|\phi\|_{L^q}^{\frac{q}{q-3}}}{s^{\frac{1}{q-3}}}+\frac{\|\phi\|_{L^q}}{s^{\frac1q}}\right), \tag{A.7}$$

and the properties *(ii)* are proved. □

The first lemma A.1 gives first properties about the jacobian a_ϕ at fixed $\phi \in \Phi_q$. The regularity of the jacobian with respect to the potential ϕ is studied in the following lemma.

Lemma A.2 (Regularity of the Jacobian a_ϕ with respect to ϕ)**.** *Let $q > 3$. Then*

(iii) Let (ϕ_n), (e_n) and (s_n) sequences of respectively Φ_q, $\mathbb{R}_-^*$ and R_+^*. Assume that there exist $\phi \in \Phi_q$, $e \in \mathbb{R}_- \cup \{-\infty\}$ and $s \in \mathbb{R}_+^* \cup \{+\infty\}$ such that

$$\phi_n \to \phi \text{ in } L^q(\mathbb{R}^3),\ e_n \to e \text{ and } s_n \to s.$$

Then by denoting $a_\phi(-\infty) = 0$, $a_\phi(0) = +\infty$ and $a_\phi^{-1}(+\infty) = 0$, we have

$$a_{\phi_n}(e_n) \to a_\phi(e) \text{ and } a_{\phi_n}^{-1}(s_n) \to a_\phi^{-1}(s). \tag{A.8}$$

(iv) Let $\phi, \tilde{\phi} \in \Phi_q$ and let $h = \phi - \tilde{\phi}$. Then the function $(\lambda, e) \mapsto a_{\phi+\lambda h}(e)$ is a $\mathcal{C}^1$ function on $[0,1] \times \mathbb{R}_-^*$ and we have

$$\frac{\partial}{\partial \lambda} a_{\phi+\lambda h}(e) = -4\pi \int_{\mathbb{R}^3} K\left(e - \phi(x) - \lambda h(x)\right) h(x) dx. \tag{A.9}$$

where the function K, defined by

$$K(\eta) = \left((1+\eta)_+^2 - 1\right)_+^{\frac{1}{2}} (1+\eta),$$

is non decreasing and has its support in $\mathbb{R}_+^*$.

(v) With the same notation as (iv). Let $s \in \mathbb{R}_+^*$. Then the function $\lambda \mapsto a_{\phi+\lambda h}^{-1}(s)$ is a $\mathcal{C}^1$ function on $[0,1]$ and we have

$$\frac{\partial}{\partial \lambda} a_{\phi+\lambda h}^{-1}(s) = \frac{\displaystyle\int_{\mathbb{R}^3} K\left(a_{\phi+\lambda h}^{-1}(s) - \phi(x) - \lambda h(x)\right) h(x) dx}{\displaystyle\int_{\mathbb{R}^3} K\left(a_{\phi+\lambda h}^{-1}(s) - \phi(x) - \lambda h(x)\right) dx}. \tag{A.10}$$

Proof. Proof of (iii): From the control of the jacobian (A.3), we have directly for $e = -\infty$,

$$a_{\phi_n}(e_n) \to 0 = a_\phi(-\infty) \text{ as } n \to +\infty.$$

Let us now treat the case $e < 0$ such that $e \neq -\infty$. Up to a subsequence, we have the convergence

$$\left((1 + e_n + \phi_n(x))_+^2 - 1\right)_+^{\frac{3}{2}} \to \left((1 + e + \phi(x))_+^2 - 1\right)_+^{\frac{3}{2}}$$

almost everywhere in $\mathbb{R}^3$. To obtain the convergence in $L^1(\mathbb{R}^3)$, from a generalized dominated convergence theorem, we have just to prove the following L^1-convergence,

$$g_n := \left((1 + e_0 + \phi_n)_+^2 - 1\right)_+^{\frac{3}{2}} \to \left((1 + e_0 + \phi)_+^2 - 1\right)_+^{\frac{3}{2}} =: g$$

where $e_0 = \inf_{n \in \mathbb{N}} e_n < 0$. In order to do it, we define the set

$$\Omega = \left\{x \in \mathbb{R}^3,\ \phi(x) < \frac{e_0}{2}\right\}.$$

Since Ω is included in a compact set, the convergence of g_n to g in $L^1(\Omega)$ comes from the convergence of ϕ_n to ϕ in $L^3_{loc}(\mathbb{R}^3)$. Out of Ω we have $g(x) = 0$ and, from

a similar Hölder inequality as (A.4), we get

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\Omega} g_n(x)dx &\leq C\left(\frac{\|\phi_n\|_{L^q}^q}{|e|^q}+\|\phi_n\|_{L^q}^q\right)^{\frac{3}{q}}\left(\mathrm{meas}\{x\in\mathbb{R}^3:\phi_n(x)<e_0,\phi(x)\geq\frac{e_0}{2}\}\right)^{\frac{q-3}{q}}\\ &\leq \overline{C}\left(\mathrm{meas}\{x\in\mathbb{R}^3:\ |\phi_n(x)-\phi(x)|\geq\frac{e_0}{2}\}\right)^{\frac{q-3}{q}}\\ &\leq \overline{C}\left(\frac{2\|\phi_n-\phi\|_{L^q}}{e_0}\right)^{q-3}.\end{aligned}$$

Thus the sequence g_n converges to g in $L^1(\mathbb{R}^3\setminus\Omega)$ and finally in $L^1(\mathbb{R}^3)$. By consequence, we have the convergence $a_{\phi_n}(e_n)\to a_\phi(e)$ for $e\in]-\infty,0[$.

Let us now treat the case $e=0$. Let $M>0$ be an arbitrary constant. Denote for all $n\in\mathbb{N}$ the space $\Omega_n=\left\{x\in\mathbb{R}^3,\ |\phi_n(x)|<\frac{m(\phi)}{2(1+|x|)}\right\}$ and let $e_0<0$ such that

$$\frac{4\pi}{3}\int_{\mathbb{R}^3}\left(\left(1+e_0+\frac{m(\phi)}{2(1+|x|)}\right)_+^2-1\right)_+^{\frac{3}{2}}dx>2M.$$

Remark that this integral is well defined since the inner function is zero out of $B(0,R)$ for

$$R=\max\left\{0,\frac{m(\phi)}{2|e_0|}-1\right\}.$$

For n large enough, we have $e_n>e_0$ and thus

$$\begin{aligned}a_{\phi_n}(e_n) &\geq \frac{4\pi}{3}\int_{\mathbb{R}^3\setminus\Omega_n}\left(\left(1+e_0+\frac{m(\phi)}{2(1+|x|)}\right)_+^2-1\right)_+^{\frac{3}{2}}dx\\ &\geq 2M-\frac{4\pi}{3}\int_{\Omega_n}\left(\left(1+e_0+\frac{m(\phi)}{2(1+|x|)}\right)_+^2-1\right)_+^{\frac{3}{2}}dx\end{aligned}$$

To prove that the second term converges to 0 as $n\to+\infty$, we remark that the set of integration of this term is $\Omega_n\cap B(0,R)$. Now, from the definitions of $m(\phi)$ and Ω_n, we have for all $x\in\Omega_n$

$$\phi_n(x)-\phi(x)\geq-\frac{m(\phi)}{2(1+|x|)}+\frac{m(\phi)}{1+|x|}\geq\frac{m(\phi)}{2(1+|x|)},$$

and thus

$$\|\phi_n-\phi\|_{L^q(\mathbb{R}^3)}^q\geq\int_{\Omega_n}\left(\frac{m(\phi)}{2(1+|x|)}\right)^q dx\geq\int_{\Omega_n\cap B(0,R)}\left(\frac{m(\phi)}{2(1+|x|)}\right)^q dx.$$

Since ϕ_n converges to ϕ in $L^q(\mathbb{R}^3)$, we deduce that the measure of the set $\Omega_n\cap B(0,R)$ converges to 0, which implies that the integral

$$\int_{\Omega_n}\left(\left(1+e_0+\frac{m(\phi)}{2(1+|x|)}\right)_+^2-1\right)_+^{\frac{3}{2}}dx$$

converges to 0 as $n\to+\infty$. Hence for n large enough $a_{\phi_n}(e_n)\geq M$, which concludes the proof of the convergence of $a_{\phi_n}(e_n)$.

To prove that $a_{\phi_n}^{-1}(s_n) \to a_\phi^{-1}(s)$, we denote $e_n := a_{\phi_n}^{-1}(s_n)$. We know from the above result that, if e_n converges to $e \in [0, +\infty]$, then

$$s_n = a_{\phi_n}(e_n) \to a_\phi(e).$$

Hence, any subsequence of $(e_n)_{n\in\mathbb{N}}$ converges to $e = a_\phi^{-1}(s)$, which gives the convergence of the whole sequence $(e_n)_{n\in\mathbb{N}}$ to $e = a_\phi^{-1}(s)$. The proof of *(iii)* is complete.

Proof of (iv): We recall that $h = \tilde{\phi} - \phi$ with $\phi, \tilde{\phi} \in \Phi_q$ and that $\lambda \in [0,1]$. From the convexity of Φ_q, the function $\phi + \lambda h$ belongs to Φ_q and $a_{\phi+\lambda h}$ is well-defined. For $e < 0$ fixed, we aim to differentiate on $(0,1)$ the function

$$\lambda \mapsto a_{\phi+\lambda h}(e) = \frac{4\pi}{3} \int_{\mathbb{R}^3} \left((1 + e - \phi(x) - \lambda h(x))_+^2 - 1\right)_+^{\frac{3}{2}} dx.$$

First, the set of integration satisfies

$$\left\{x \in \mathbb{R}^3 \ : \ (\phi + \lambda h)(x) < e\right\} \subset \left\{x \in \mathbb{R}^3 \ : \ \phi(x) < e\right\} \cup \left\{x \in \mathbb{R}^3 \ : \ \tilde{\phi}(x) < e\right\},$$

which is included in a compact set of $\mathbb{R}^3$. Now, we have for all x in this set

$$\frac{\partial}{\partial\lambda}\left[\left((1 + e - \phi(x) - \lambda h(x))_+^2 - 1\right)_+^{\frac{3}{2}}\right] = 3K\left(e - \phi(x) - \lambda h(x)\right) h(x),$$

where $K(\eta) = \left((1+\eta)_+^2 - 1\right)_+^{\frac{1}{2}} (1+\eta)$. This derivative can be bound uniformly with respect to λ by

$$\begin{aligned} K\left(e - \phi(x) - \lambda h(x)\right) |h(x)| &\leq K\left(e - \phi(x) - \tilde{\phi}(x)\right) |\phi(x) + \tilde{\phi}(x)| \\ &\leq C\left(1 + \left|\phi(x) - \tilde{\phi}(x)\right|^3\right), \end{aligned}$$

which, combined with the fact that $\phi + \tilde{\phi} \in L^3_{loc}(\mathbb{R}^3)$ allows the Lebesgue dominated convergence theorem and thus implies *(iv)*.

Proof of (v): Let $s \in \mathbb{R}_+^*$. The continuity of $\lambda \mapsto a_{\phi+\lambda h}^{-1}(s)$ comes directly from *(iii)*. Let us differentiate this function on $\lambda_0 \in (0,1)$. Denote $\phi_0 = \phi + \lambda_0 h$ and $\phi_\lambda = \phi + \lambda h$ with $\lambda \neq \lambda_0$. Then we rewrite

$$\frac{a_{\phi_\lambda}^{-1}(s) - a_{\phi_0}^{-1}(s)}{\lambda - \lambda_0} = \frac{a_{\phi_\lambda}^{-1}(s) - a_{\phi_0}^{-1}(s)}{a_{\phi_0}(a_{\phi_\lambda}^{-1}(s)) - a_{\phi_0}(a_{\phi_0}^{-1}(s))} \frac{a_{\phi_0}(a_{\phi_\lambda}^{-1}(s)) - a_{\phi_\lambda}(a_{\phi_\lambda}^{-1}(s))}{\lambda - \lambda_0}.$$

Since $a_{\phi_\lambda}^{-1}(s)$ converges to $a_{\phi_0}^{-1}(s)$ as $\lambda \to \lambda_0$, the first term satisfies

$$\frac{a_{\phi_\lambda}^{-1}(s) - a_{\phi_0}^{-1}(s)}{a_{\phi_0}(a_{\phi_\lambda}^{-1}(s)) - a_{\phi_0}(a_{\phi_0}^{-1}(s))} \to \frac{1}{a'_{\phi_0}(a_{\phi_0}^{-1}(s))} \quad \text{as } \lambda \to \lambda_0,$$

and the second term satisfies

$$\frac{a_{\phi_0}(a_{\phi_\lambda}^{-1}(s)) - a_{\phi_\lambda}(a_{\phi_\lambda}^{-1}(s))}{\lambda - \lambda_0} \to \frac{\partial}{\partial\lambda} a_{\phi+\lambda h}\left(a_{\phi_0}^{-1}(s)\right)\Big|_{\lambda=\lambda_0}.$$

Using (A.9), we finally get the expression (A.10).

□

Lemma A.3 (Changes of variables)**.** *Let $\alpha \in \mathcal{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $G \in L^1(\mathbb{R}_+)$ and for all $(x,v) \in \mathbb{R}^6$*

$$e(x,v) = \sqrt{|v|^2+1} - 1 + \phi(x).$$

Then

$$\begin{aligned} \int_{e(x,v)<0} \alpha\left(e(x,v)\right) G\left(a_\phi\left(e(x,v)\right)\right) dxdv &= \int_{\inf \phi}^0 \alpha(e) G(a_\phi(e)) a'_\phi(e) de \\ &= \int_0^{+\infty} \alpha(a_\phi^{-1}(s)) G(s) ds, \end{aligned} \tag{A.11}$$

where $\inf \phi$ *is the essential infimum of* ϕ*.*

Proof. We perform the change of variable $e = \sqrt{|v|^2+1} - 1 + \phi(x)$, with respect to the velocity variable v to get

$$\begin{aligned} &\int_{e(x,v)<0} \alpha\left(e(x,v)\right) G\left(a_\phi\left(e(x,v)\right)\right) dxdv \\ &= 4\pi \int_{\mathbb{R}^3} dx \int_{\phi(x)}^0 \alpha(e) G(a_\phi(e)) \left((1+e-\phi(x))_+^2 - 1\right)_+^{\frac{1}{2}} (1+e-\phi(x)) de \\ &= \int_{\inf \phi}^0 \alpha(e) G(a_\phi(e)) a'_\phi(e) de. \end{aligned}$$

We have then directly (A.11) since a_ϕ is a $\mathcal{C}^1$-diffeomorphism from $(\inf \phi, 0)$ onto $\mathbb{R}_+^*$. □

Appendix B. Rearrangement with respect to the microscopic energy

We use now this jacobian to define a new rearrangement of any $f \in \mathcal{E}_p$ with respect to the microsopic energy $\sqrt{|v|^2+1}-1+\phi(x)$, where ϕ belongs to Φ_q given by (2.1). We first recall some basic properties of the classical Schwarz symmetrization.

Lemma B.1 (Schwarz symmetrization)**.** *Let $f \in \mathcal{E}_p$, nonzero, with $p > \frac{3}{2}$. We define the Schwarz symmetrization f^* of f on $\mathbb{R}_+^*$ by*

$$\forall t > 0, \quad f^*(t) = \inf\{s \geq 0, \quad \mu_f(s) \geq t\},$$

where μ_f is the distribution function of f defined by (1.7)*. Then f^* is the unique nonincreasing function on $\mathbb{R}_+^*$ such that f and f^* have the same distribution function*

$$\forall s \geq 0, \mu_f(s) = \mu_{f^*}(s).$$

Moreover, if f is continuous then f^* is continuous. In particular, Q^* is continuous.

Now, from this Schwarz symmetrisation, we define a new rearrangement with respect to the microscopic energy.

Lemma B.2 (Symmetric rearrangement of f with respect to the microscopic energy)**.** *Let $f \in \mathcal{E}_p$, nonzero, with $p > \frac{3}{2}$ and $\phi \in \Phi_q$. Let f^* be the Schwarz*

rearrangement of f in $\mathbb{R}^6$. We recall that the function $f^{*\phi}$ is defined by

$$f^{*\phi}(x,v) = \begin{cases} f^*\left(a_\phi\left(\sqrt{|v|^2+1}-1+\phi(x)\right)\right) & \text{if } \sqrt{|v|^2+1}-1+\phi(x)<0, \\ \\ 0 & \text{if } \sqrt{|v|^2+1}-1+\phi(x)\geq 0. \end{cases} \tag{B.1}$$

Then,

(i) $f^{*\phi}$ is equimeasurable with f, which means

$$f^{*\phi} \in Eq(f) = \{g \in L^1_+ \cap L^p \text{ with } \mu_g = \mu_f\}. \tag{B.2}$$

(ii) $f^{*\phi}$ belongs to $\mathcal{E}_p$ with

$$\left\| |v| f^{*\phi} \right\|_{L^1} \leq C\left(\|\nabla\phi\|_{L^2}^2 \, \|f\|_{L^1}^{\frac{2(2p-3)}{6(p-1)}} \, \|f\|_{L^p}^{\frac{2p}{6(p-1)}} + \|f\|_{L^1} \right). \tag{B.3}$$

(iii) A function Q as defined in Theorem 1.1 satisfies

$$F = Q^* \circ a_{\phi_Q} \text{ on } \mathbb{R}^*_- \text{ and } Q = Q^{*\phi_Q} \text{ on } \mathbb{R}^6.$$

(iv) Let $f, g \in \mathcal{E}_p$ satisfying $\mu_g \leq \mu_f$ and $\phi \in \Phi$, then

$$\int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1}-1+\phi(x)\right)\left(g - f^{*\phi}\right) dxdv \geq 0. \tag{B.4}$$

with equality if and only if $g = f^{*\phi}$. In particular

$$\mathcal{H}(g) \geq \mathcal{H}(f^{*\phi_g}) + \frac{1}{2}\left\| \nabla\phi_g - \nabla\phi_{f^{*\phi_g}} \right\|_{L^2}^2 \geq \mathcal{H}(f^{*\phi_g}), \tag{B.5}$$

with equality if and only if $g = f^{*\phi_g}$.

Proof of Lemma B.2. (i) Equimeasurability: Let $\beta \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\beta(0) = 0$. Using the change of variable given by (A.11) we have

$$\int_{\mathbb{R}^6} \beta\left(f^{*\phi}(x,v)\right) dxdv = \int_0^{+\infty} \beta\left(f^*(s)\right) ds = \int_{\mathbb{R}^6} \beta\left(f(x,v)\right) dxdv,$$

which gives the equimeasurability of f and $f^{*\phi}$.

(ii) Control of the kinetic energy: From the definition of $f^{*\phi}$, see (2.5), we have

$$\int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1}-1+\phi(x)\right) f^{*\phi} \leq 0,$$

and

$$-\int_{\mathbb{R}^6} \phi(x) f^{*\phi}(x,v) dxdv = \int_{\mathbb{R}^3} \nabla\phi(x) \cdot \nabla\phi_{f^{*\phi}}(x) dx.$$

Thus, from the Cauchy-Schwartz inequality

$$\begin{aligned} \int_{\mathbb{R}^6} |v| f^{*\phi} &\leq \int_{\mathbb{R}^6} \left(\sqrt{|v|^2+1}-1\right) f^{*\phi} + \left\| f^{*\phi} \right\|_{L^1} \\ &\leq \|\nabla\phi\|_{L^2} \left\| \nabla\phi_{f^{*\phi}} \right\|_{L^2} + \|f\|_{L^1}. \end{aligned} \tag{B.6}$$

Moreover, the interpolation inequality (1.11) and the equimesurability of f and $f^{*\phi}$ yield

$$\left\| \nabla\phi_{f^{*\phi}} \right\|_{L^2} \leq C \, \|f\|_{L^1}^{\frac{2p-3}{6(p-1)}} \, \|f\|_{L^p}^{\frac{p}{6(p-1)}} \left\| |v| f^{*\phi} \right\|_{L^1}^{\frac{1}{2}}. \tag{B.7}$$

Hence the inequalities (B.6) and (B.7) imply

$$\int_{\mathbb{R}^6} |v| f^{*\phi} - K(\phi, f) \left(\int_{\mathbb{R}^6} |v| f^{*\phi} \right)^{\frac{1}{2}} - \|f\|_{L^1} \leq 0,$$

with

$$K(\phi, f) = C \, \|\nabla\phi\|_{L^2} \, \|f\|_{L^1}^{\frac{2p-3}{6(p-1)}} \, \|f\|_{L^p}^{\frac{p}{6(p-1)}} .$$

The control of the kinetic energy (B.3) follows.

(iii) The profil Q, fixed point for the rearrangement $f^{*\phi_f}$: From the assumptions on Q in theorem 1.1, the measure of the support of Q is $a_{\phi_Q}(e_Q)$ and the function F is a strictly decreasing $\mathcal{C}^1$-diffeomorphism from $[\min \phi_Q, e_Q]$ onto $[0, \|Q\|_\infty]$. Thus we have for all $e \in [\min \phi_Q, e_0]$,

$$\begin{aligned} a_{\phi_Q}(e) &= \text{meas}\left\{(x,v) \in \mathbb{R}^6 \ : \ F\left(\sqrt{|v|^2+1} - 1 + \phi_Q(x)\right) > F(e)\right\} \\ &= \text{meas}\left\{(x,v) \in \mathbb{R}^6 \ : \ Q(x,v) > F(e)\right\} \\ &= \text{meas}\left\{s \in \mathbb{R}_+^* \ : \ Q^*(s) > F(e)\right\}, \end{aligned}$$

from the definition of the Schwarz symmetrization Q^*. We rewrite this equality: for all $a \in [0, \|Q\|_\infty]$

$$a_{\phi_Q} \circ F^{-1}(a) = \text{meas}\left\{s \in \mathbb{R}_+^* \ : \ Q^*(s) > a\right\}. \tag{B.8}$$

We know that Q^* is a decreasing function on $[0, meas(Supp(Q))] = [0, a_{\phi_Q}(e_Q)]$. The equality (B.8) and the continuity of $a_{\phi_Q} \circ F^{-1}$ imply the strict decrease of Q^*. Moreover, since the Schwarz symmetrisation conserves the continuity, Q^* is continuous on $[0, a_{\phi_Q}(e_Q)]$. We conclude that, for all $a \in [0, \|Q\|_\infty]$,

$$Q^*\left(\text{meas}\left\{s \in \mathbb{R}_+^* \ : \ Q^*(s) > a\right\}\right) = a.$$

We deduce that for all $e \in [\min \phi_Q, e_Q]$, $F(e) = Q^* \circ a_{\phi_Q}(e)$, which implies in particular $0 = F(e_Q) = Q^* \circ a_{\phi_Q}(e_Q)$. But $Q^* \circ a_{\phi_Q}$ is discreasing on R_-^* and we also have

$$F = Q^* \circ a_{\phi_Q} \text{ on } [\min \phi_Q, 0). \tag{B.9}$$

Let now for all $(x,v) \in \mathbb{R}^6$

$$e(x,v) := \sqrt{|v|^2+1} - 1 + \phi_Q(x) \in [\min \phi_Q, +\infty).$$

If $e(x,v) \leq 0$, from (B.9), we have $Q(x,v) = F(e(x,v)) = Q^{*\phi_Q}(x,v)$. If $e(x,v) > 0$, from the definition of $Q^{*\phi_Q}$, we have $Q^{*\phi_Q}(x,v) = 0 = F(e(x,v)) = Q(x,v)$.

(iv) The Hamiltonian of the rearrangement: Let $f, g \in \mathcal{E}_p$. Then

$$\begin{aligned} \|\nabla\phi_g - \nabla\phi_f\|_{L^2}^2 &= -\int_{\mathbb{R}^6} (\phi_g - \phi_f)(g-f) dx dv \\ &= -\int_{\mathbb{R}^6} \phi_g g dx dv - \int_{\mathbb{R}^6} \phi_f f dx dv + 2\int_{\mathbb{R}^6} \phi_g f dx dv, \end{aligned}$$

where we use $\int \phi_f g = -\int \nabla\phi_f \nabla\phi_g = \int \phi_g f$. Finally we have

$$\|\nabla\phi_g - \nabla\phi_f\|_{L^2}^2 = -2\int_{\mathbb{R}^6} \phi_g (g-f) dx dv + \|\nabla\phi_f\|_{L^2}^2 - \|\nabla\phi_g\|_{L^2}^2 .$$

Thus we obtain

$$\begin{aligned}\mathcal{H}(g)-\mathcal{H}(f) &= \int_{\mathbb{R}^6}\left(\sqrt{|v|^2+1}-1\right)(g-f)\,dxdv+\frac{1}{2}\left(\|\nabla\phi_f\|_{L^2}^2-\|\nabla\phi_g\|_{L^2}^2\right)\\ &= \int_{\mathbb{R}^6}\left(\sqrt{|v|^2+1}-1+\phi_g\right)(g-f)\,dxdv+\frac{1}{2}\|\nabla\phi_g-\nabla\phi_f\|_{L^2}^2,\end{aligned}$$

where the Hamiltonian is defined by (1.4). We apply this equality to $g, f^{*\phi_g} \in \mathcal{E}_p$ with g nonzero and $f \in \mathcal{E}_p$ to get

$$\mathcal{H}(g)=\mathcal{H}(f^{*\phi_g})+\frac{1}{2}\left\|\nabla\phi_g-\nabla\phi_{f^{*\phi_g}}\right\|_{L^2}^2+\int_{\mathbb{R}^6}\left(\sqrt{|v|^2+1}-1+\phi_g\right)\left(g-f^{*\phi_g}\right)dxdv.$$

Hence, to prove the inequality (B.5), it is sufficient to prove (B.4).

Let $f \in \mathcal{E}_p$, $g \in \mathcal{E}_p$, $\phi \in \Phi_q$ and let T defined by

$$T=\int_{\mathbb{R}^6}\left(\sqrt{|v|^2+1}-1+\phi(x)\right)\left(g-f^{*\phi}\right)dxdv. \tag{B.10}$$

The claim $T \geq 0$ is a classical inequality for the rearrangements (see [24] for the Schwarz rearrangement for example and [23] for the new rearrangement). To simplify the notation we define

$$e(x,v)=\sqrt{|v|^2+1}-1+\phi(x),$$

and we use the layer cake representation

$$f(x,v)=\int_{\mathbb{R}_+}\mathbf{1}_{t<f(x,y)}dt$$

to find

$$T=\int_0^{+\infty}dt\left(\int_{S_1(t)}e(x,v)dxdv-\int_{S_2(t)}e(x,v)dxdv\right), \tag{B.11}$$

where

$$\begin{aligned}S_1(t)&=\left\{(x,v)\in\mathbb{R}^6,\ f^{*\phi}(x,v)\leq t<g(x,v)\right\},\\ S_2(t)&=\left\{(x,v)\in\mathbb{R}^6,\ g(x,v)\leq t<f^{*\phi}(x,v)\right\}.\end{aligned}$$

Now, from the properties about the pseudo-inverse of $f^*\circ a_\phi$, we deduce that

$$e(x,v)\geq\left(f^*\circ a_\phi\right)^{-1}(t),\quad \text{for all } (x,v)\in S_1(t),$$

and, from $\mu_g \leq \mu_f$, we have for all $t \in \mathbb{R}^+$, $meas(S_1(t)) \leq meas(S_2(t))$. Thus, since $\left(f^*\circ a_\phi\right)^{-1} \leq 0$,

$$T\geq\int_0^{+\infty}dt\int_{S_2(t)}\left(\left(f^*\circ a_\phi\right)^{-1}(t)-e(x,v)\right)dxdv.$$

But,

$$e(x,v)<\left(f^*\circ a_\phi\right)^{-1}(t),\quad \text{for all } (x,v)\in S_2(t),$$

which implies on the one hand that $T \geq 0$ and in the other hand that, if $T = 0$, we have, for almost all $t \in \mathbb{R}_+$, $meas(S_2(t)) = 0$. Thus we have, almost everywhere $meas(S_1(t)) = meas(S_2(t)) = 0$ which gives $g = f^{*\phi}$.

□

Now we give a lemma in which states a method to inverse clearly the function $f^* \circ a_\phi$.

Lemma B.3 (Pseudo inverse of $f^* \circ a_\phi$)**.** *Let $f \in \mathcal{E}_p$, nonzero, with $p > \frac{3}{2}$ and $\phi \in \Phi_q$. Let f^* the Schwartz rearrangement of f in $\mathbb{R}^6$. We define the pseudo inverse of $f^* \circ a_\phi$ for $s \in (0, \|f\|_{L^\infty})$ as*

$$(f^* \circ a_\phi)^{-1}(s) = \sup\{e \in (\inf \phi, 0) \ : \ f^* \circ a_\phi(e) > s\}. \tag{B.12}$$

Then $(f^* \circ a_\phi)^{-1}$ is a nonincreasing function from $(0, \|f\|_{L^\infty})$ to $(\inf \phi, 0)$ and for all $(x, v) \in \mathbb{R}^6$ and $s \in (0, \|f\|_{L^\infty})$,

$$f^{*\phi}(x, v) > s \Rightarrow \sqrt{|v|^2 + 1} - 1 + \phi(x) \leq (f^* \circ a_\phi)^{-1}(s), \tag{B.13}$$

and

$$f^{*\phi}(x, v) \leq s \Rightarrow \sqrt{|v|^2 + 1} - 1 + \phi(x) \geq (f^* \circ a_\phi)^{-1}(s). \tag{B.14}$$

For the proof of this lemma, we refer to [23].

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