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Frederic Bernicot, Taoufik Hmidi

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ON THE GLOBAL WELL-POSEDNESS FOR EULER EQUATIONS WITH UNBOUNDED VORTICITY

FRÉDÉRIC BERNICOT AND TAOUFIK HMIDI

ABSTRACT. In this paper, we are interested in the global persistence regularity for the 2D incompressible Euler equations in some function spaces allowing unbounded vorticities. More precisely, we prove the global propagation of the vorticity in some weighted Morrey-Campanato spaces and in this framework the velocity field is not necessarily Lipschitz but belongs to the log-Lipschitz class $L^\alpha L$, for some $\alpha \in (0, 1)$.

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1. INTRODUCTION

The motion of incompressible perfect flows evolving in the whole space is governed by the Euler system described by the equations

$$(1) \quad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = 0, & x \in \mathbb{R}^d, t > 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

Here, the vector field $u : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ denotes the velocity of the fluid particles and the scalar function P stands for the pressure. It is a classical fact that the incompressibility condition leads to a closed system and the pressure can be recovered from the velocity through some singular operator. The literature on the well-posedness theory for Euler system is very abundant and a lot of results were obtained in various function spaces. For instance, it is well-known according to the work of Kato and Ponce [15] that the system (1) admits a maximal unique solution in the framework of Sobolev spaces, namely $u_0 \in W^{s,p}$, with $s > \frac{d}{p} + 1$.

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This result was extended to Hölder spaces C^s , $s > 1$ by Chemin [8] and later by Chae [7] in the critical and sub-critical Besov spaces, see also [20]. We point out that the common technical ingredient of these contributions is the use of the commutator theory but with slightly different difficulties. Even though, the local theory for classical solutions is well-achieved, the global existence of such solutions is still now an outstanding open problem due to the poor knowledge of the conservation laws. However this problem is affirmatively solved for some special cases like the dimension two and the axisymmetric flows without swirl. It is worthy pointing out that for these known cases the geometry of the initial data plays a central role through the special structure of their vorticities. Historically, we can fairly say that Helmholtz was the first to point out in the seminal paper [13] the importance of the vorticity $\omega \triangleq \text{curl } u$ in the study of the incompressible inviscid flows. In that paper he provided the foundations of the vortex motion theory by the establishment of some basic laws governing the vorticity. Some decades later in the thirties of the last century, Wolibner proved in [29] the global existence of sufficiently smooth solutions in space dimension two. Very later in the mid of the eighties, a rigorous connection between the vorticity and the global existence was performed by Beale, Kato and Majda in [3]. They proved the following blow up criterion: let $u_0 \in H^s$, with $s > \frac{d}{2} + 1$ and denote by T^* the lifespan of the solution, then

$$T^* < +\infty \implies \int_0^{T^*} \|\omega(\tau)\|_{L^\infty} d\tau = +\infty.$$

An immediate consequence of this criterion is the global existence of Kato's solutions in space dimension two. This follows from the conservation of the vorticity along the particle trajectories, namely the vorticity satisfies the Helmholtz equation

$$(2) \quad \partial_t \omega + u \cdot \nabla \omega = 0.$$

Recall that in this case the vorticity can be assimilated to the scalar $\omega = \partial_1 u^2 - \partial_2 u^1$ and we derive from the equation (2) an infinite family of conservation laws. For instance, for every $p \in [1, \infty]$

$$\forall t \geq 0, \quad \|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}.$$

It seems that the standard methods used for the local theory cease to work in the limiting space $H^{\frac{d}{2}+1}$ due to the lack of embedding in the Lipschitz class. Nevertheless the well-posedness theory can be successfully implemented in a slight modification of this space in order to guarantee this embedding, take for example Besov spaces of type $B_{p,1}^{\frac{d}{2}+1}$, for more details see for instance [7]. In this critical framework the BKM criterion cited before is not known to work and should be replaced by the following one,

$$T^* < +\infty \implies \int_0^{T^*} \|\omega(\tau)\|_{B_{\infty,1}^0} d\tau = +\infty.$$

In this class of initial data the global well-posedness in dimension two is not a trivial task and was proved by Vishik in [26] through the use in an elegant way of the conservation of the Lebesgue measure by the flow. We mention that a simple proof of Vishik's result, which has the advantage to work in the viscous case, was given in [14]. By using the formal L^p conservation laws it seems that we can go beyond the limitation fixed by the general theory of hyperbolic systems and construct global weak solution for $p > 1$ but for the uniqueness we require in general the vorticity to be bounded. This was carefully done by Yudovich in

his paper [27] following the tricky remark that the gradient of the velocity belongs to all L^p with slow growth with respect to p :

$$\sup_{p \geq 2} \frac{\|\nabla v(t)\|_{L^p}}{p} < \infty.$$

The uniqueness part is obtained by performing energy estimate and choosing suitably the parameter p . In this new pattern the velocity belongs to the class of log-Lipschitz functions and this is sufficient to establish the existence and uniqueness of the flow map, see for instance [8]. The real matter at this level of regularity concerns only the uniqueness part which requires minimal regularity for the velocity and the assumption of bounded vorticity is almost necessary in the scale of Lebesgue spaces. However slight improvements have been carried out during the last decades by allowing the vorticity to be unbounded. For example, in [28] Yudovich proved the uniqueness when the L^p -norms of the initial vorticity do not grow much faster than $\ln p$:

$$\sup_{p \geq 2} \frac{\|\omega_0\|_{L^p}}{\ln p} < \infty.$$

We refer also to [9, 10] for other extensions on the construction of global weak solutions. In [25], Vishik accomplished significant studies for the existence and uniqueness problem with unbounded vorticities. He gave various results when the vorticity lies in the space $B_\Gamma \cap L^{p_0} \cap L^{p_1}$, where $p_0 < 2 < p_1$ and B_Γ is the borderline Besov spaces defined by

$$(3) \quad \sup_{n \geq 1} \frac{1}{\Gamma(n)} \sum_{q=-1}^n \|\Delta_q \omega_0\|_{L^\infty} < \infty.$$

As an example, it was shown that for $\Gamma(n) = O(\ln n)$ there exists a unique local existence but the global existence is only proved when $\Gamma(n) = O(\ln^{\frac{1}{2}} n)$. Nevertheless the propagation of the initial regularity is not well understood and Vishik were only able to prove that for the positive times the vorticity belongs to the big class B_{Γ_1} with $\Gamma_1(n) = n\Gamma(n)$. We point out that the persistence regularity for spaces which are not embedded either in the Lipschitz class or in the spaces of conservation laws is in general a difficult subject. Recently, in [5] the first author and Keraani were able to find a suitable space of initial data called log- BMO space for which there is global existence and uniqueness without any loss of regularity. This space is strictly larger than the L^∞ space and much smaller than the usual BMO space.

The main goal of this paper is to continue this investigation and try to generalize the result of [5] to a new collection of spaces which are not comparable to the bounded class. To state our main result we need to introduce the following spaces.

Definition 1. Let $\alpha \geq 0$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a locally integrable function.

(1) We say that f belongs to the space $L^\alpha mo$ if

$$\|f\|_{L^\alpha mo} \triangleq \sup_{\substack{B \text{ ball} \\ 0 < r \leq \frac{1}{2}}} |\ln r|^\alpha \int_B \left| f - \int_B f \right| < \infty.$$

(2) Let $F : [1, +\infty[\rightarrow [0, +\infty[$. We say that f belongs to $L^\alpha mo_F$ if

$$\|f\|_{L^\alpha mo_F} \triangleq \|f\|_{L^\alpha mo} + \sup_{\substack{B_1, B_2 \\ 2r_2 \leq r_1 \leq \frac{1}{2}}} \frac{|\int_{B_2} f - \int_{B_1} f|}{F\left(\frac{|\ln(r_2)|}{|\ln(r_1)|}\right)} < \infty,$$

where r_i denotes the radius of the ball B_i , $|B|$ denotes the Lebesgue measure of the ball B and the average $\frown_B f$ is defined by

$$\frown_B f \triangleq \frac{1}{|B|} \int_B f(x) dx.$$

For the sake of a clear presentation we will first state a partial result and the general one will be given in Section 3, Theorem 3.

Theorem 1. *Take $F(x) = \ln x$ and assume that $\omega_0 \in L^p \cap L^\alpha mo_F$ with $p \in]1, 2[$ and $\alpha \in]0, 1[$. Then the 2d Euler equations admit a unique global solution*

$$\omega \in L_{loc}^\infty([0, +\infty[, L^p \cap L^\alpha mo_{1+F}).$$

Some remarks are in order.

Remark 1. *The regularity of the initial vorticity measured in the space $L^\alpha mo$ is preserved globally in time. However we bring up a slight loss of regularity in the second part of the $L^\alpha mo_F$ norm. Instead of F we need $1+F$. This appears as a technical artefact and we believe that we can remove it.*

Remark 2. *The case $\alpha = 0$ is not included in our statement since it corresponds to the result of [5]. However for $\alpha > 1$ the vorticity must be bounded and the velocity is Lipschitz and in this case the propagation in the space $L^\alpha mo$ can be done without the use of the second part of the space $L^\alpha mo_F$. The limiting case $\alpha = 1$ is omitted in our main result for the sake of simplicity but our computations can be performed as well with slight modifications especially when we deal with the regularity of the flow in Proposition 6.*

The proof of Theorem 1 will be done in the spirit of the work of [5]. We establish a crucial logarithmic estimate for the composition in the space $L^\alpha mo_F$ with a flow which preserves Lebesgue measure. We prove in particular the key estimate

$$\|\omega(t)\|_{L^\alpha mo} \leq C \|\omega_0\|_{L^\alpha mo_F} (1 + V(t)) \ln(2 + V(t)),$$

where $V(t) = \int_0^t \|u(\tau)\|_{L^{1-\alpha} L} d\tau$ and the space $L^{1-\alpha} L$ is defined in Section 2.4. We observe from the preceding estimate that we can propagate globally in time the regularity in the space $L^\alpha mo$ and the second part of the space $L^\alpha mo_F$ is not involved for the positive times.

The remainder of this paper is organized as follows. In the next section we introduce some functional spaces and prove some of their basic properties. We shall also examine the regularity of the flow map associated to a vector field belonging to the class $L^\alpha L$. In Section 3 we shall establish a logarithmic estimate for a transport model and we will see how to derive some of their consequences in the study of the inviscid flows. The proof of the main results will be given at the end of this section. We close this paper with an appendix covering the proof of some technical lemmata.

2. FUNCTIONAL TOOLS

This section is devoted to some useful tools. We will firstly recall some classical spaces like Besov spaces and BMO spaces and give a short presentation of Littlewood-Paley operators. Secondly, we introduce the spaces $L^\alpha mo$ and $L^\alpha mo_F$ and discuss some of their important

properties. We end this section with the study of log-Lipschitz spaces.

In the sequel we denote by C any positive constant that may change from line to line and C_0 a real positive constant depending on the size of the initial data. We will use the following notations: for any non-negative real numbers A and B , the notation $A \lesssim B$ means that there exists a positive constant C independent of A and B and such that $A \leq CB$.

2.1. Littlewood-Paley operators. To define Besov spaces we first introduce the dyadic partition of the unity, for more details see for instance [8]. There are two non-negative radial functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2 \setminus \{0\})$ such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2,$$

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\},$$

$$|p - q| \geq 2 \Rightarrow \text{supp } \varphi(2^{-p}\cdot) \cap \text{supp } \varphi(2^{-q}\cdot) = \emptyset,$$

$$q \geq 1 \Rightarrow \text{supp } \chi \cap \text{supp } \varphi(2^{-q}\cdot) = \emptyset.$$

Let $u \in \mathcal{S}'(\mathbb{R}^2)$, the Littlewood-Paley operators are defined by

$$\Delta_{-1}u = \chi(D)u, \quad \forall q \geq 0, \quad \Delta_q u = \varphi(2^{-q}D)u \quad \text{and} \quad S_q u = \sum_{-1 \leq p \leq q-1} \Delta_p u.$$

We can easily check that in the distribution sense we have the identity

$$u = \sum_{q \in \mathbb{Z}} \Delta_q u, \quad \forall u \in \mathcal{S}'(\mathbb{R}^2).$$

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality: for any $u, v \in \mathcal{S}'(\mathbb{R}^2)$,

$$\begin{aligned} \Delta_p \Delta_q u &= 0 & \text{if } |p - q| \geq 2 \\ \Delta_p (S_{q-1} u \Delta_q v) &= 0 & \text{if } |p - q| \geq 5. \end{aligned}$$

Let us note that the above operators Δ_q and S_q map continuously L^p into itself uniformly with respect to q and p . We also notice that these operators are of convolution type. For example for $q \in \mathbb{Z}$, we have

$$\Delta_{-1}u = h * u, \quad \Delta_q u = 2^{2q} g(2^q \cdot) * u, \quad \text{with } g, h \in \mathcal{S}, \quad \widehat{h}(\xi) = \chi(\xi), \quad \widehat{g}(\xi) = \varphi(\xi).$$

Now we recall Bernstein inequalities, see for example [8].

Lemma 1. *There exists a constant $C > 0$ such that for all $q \in \mathbb{N}$, $k \in \mathbb{N}$ and for any tempered distribution u we have*

$$\begin{aligned} \sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} &\leq C^k 2^{qk} \left(k + 2 \left(\frac{1}{a} - \frac{1}{b} \right) \right) \|S_q u\|_{L^a} \quad \text{for } b \geq a \geq 1 \\ C^{-k} 2^{qk} \|\Delta_q u\|_{L^a} &\leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \leq C^k 2^{qk} \|\Delta_q u\|_{L^a}. \end{aligned}$$

Using Littlewood-Paley operators, we can define Besov spaces as follows. For $(p, r) \in [1, +\infty]^2$ and $s \in \mathbb{R}$, the Besov space $B_{p,r}^s$ is the set of tempered distributions u such that

$$\|u\|_{B_{p,r}^s} := \left(2^{qs} \|\Delta_q u\|_{L^p} \right)_{\ell^r} < +\infty.$$

We remark that the usual Sobolev space H^s coincides with $B_{2,2}^s$ for $s \in \mathbb{R}$ and the Hölder space C^s coincides with $B_{\infty,\infty}^s$ when s is not an integer.

The following embeddings are an easy consequence of Bernstein inequalities,

$$B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s+2(\frac{1}{p_2}-\frac{1}{p_1})}, \quad p_1 \leq p_2 \quad \text{and} \quad r_1 \leq r_2.$$

Our next task is to introduce some new function spaces and to study some of their useful properties that will be frequently used along this paper.

2.2. The $L^\alpha mo$ space. Here the abbreviation Lmo stands for *logarithmic bounded mean oscillation*.

Definition 2. Let $\alpha \in [0, 1]$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a locally integrable function. We say that f belongs to $L^\alpha mo$ if

$$\|f\|_{L^\alpha mo} := \sup_{0 < r \leq \frac{1}{2}} |\ln r|^\alpha \left| \int_B f - \int_B f \right| + \left(\sup_{|B|=1} \int_B |f(x)| dx \right) < \infty,$$

where the supremum is taken over all the balls B of radius $r \leq \frac{1}{2}$.

We observe that for $\alpha = 0$ the space $L^\alpha mo$ reduces to the usual Bmo space (the local version of BMO). It is also plain that the space $L^\alpha mo$ contains the class of continuous functions f such that

$$\sup_{0 < |x-y| \leq \frac{1}{2}} |\ln |x-y||^\alpha |f(x) - f(y)| < +\infty,$$

that is the functions of modulus of continuity $\mu(r) = |\ln r|^{-\alpha}$. There are two elementary properties that we wish to mention:

- For $\alpha \in]0, 1[$, consider a ball B of radius r and take $k \geq 0$ with $2^k r \leq \frac{1}{2}$, then

$$\begin{aligned} \left| \int_B f - \int_{2^k B} f \right| &\lesssim \|f\|_{L^\alpha mo} \sum_{\ell=0}^k (|\ln r| - \ell)^{-\alpha} \\ (4) \qquad \qquad \qquad &\lesssim \|f\|_{L^\alpha mo} |\ln r|^{1-\alpha}. \end{aligned}$$

- For a ball B of radius 1 and $k \geq 1$, $2^k B$ can be covered by 2^{2k} balls of radius 1, so

$$(5) \qquad \int_{2^k B} |f| \lesssim \|f\|_{L^\alpha mo}.$$

Next, we discuss some relations between the $L^\alpha mo$ spaces and the frequency cut-offs.

Proposition 1. *The following assertions hold true.*

- (1) Let $f \in L^\alpha mo$, $\alpha \in [0, 1]$ and $n \in \mathbb{N}^*$, then

$$\|\Delta_n f\|_{L^\infty} \lesssim n^{-\alpha} \|f\|_{L^\alpha mo},$$

and if $\alpha \in (0, 1)$

$$\|S_n f\|_{L^\infty} \lesssim n^{1-\alpha} \|f\|_{L^\alpha mo}.$$

(2) We denote by $\mathcal{R}_{ij} := \partial_{x_i} \partial_{x_j} \Delta^{-1}$ the “iterated” Riesz transform. Then for every function $f \in L^\alpha mo \cap L^p$, with $p \in (1, \infty)$ and $\alpha \in (0, 1)$,

$$\|S_n \mathcal{R}_{ij} f\|_{L^\infty} \lesssim n^{1-\alpha} \|f\|_{L^\alpha mo \cap L^p}.$$

This proposition yields easily to the following corollary.

Corollary 1. We have the embedding $L^\alpha mo \hookrightarrow B_\Gamma$, see the definition (3), with

- $\Gamma(N) = \ln(N)$ if $\alpha = 1$
- $\Gamma(N) = N^{1-\alpha}$ if $\alpha \in (0, 1)$.

Proof of Proposition 1. (1) The Littlewood-Paley operator Δ_n corresponds to a convolution by $2^{2n}g(2^n \cdot)$ with g a smooth function such that its Fourier transform is compactly supported away from zero. Therefore using the cancellation property of g , namely, $\int_{\mathbb{R}^2} g(x) dx = 0$, we obtain

$$\begin{aligned} \Delta_n f(x) &= \int_{\mathbb{R}^2} 2^{2n} g(2^n(x-y)) f(y) dy \\ &= \int_{\mathbb{R}^2} 2^{2n} g(2^n(x-y)) \left[f(y) - \int_{B(x, 2^{-n})} f \right] dy, \end{aligned}$$

Denote by $B \triangleq B(x, 2^{-n})$ the ball of center x and radius 2^{-n} . Hence, due to the fast decay of g , it comes for every integer M

$$\begin{aligned} |\Delta_n f(x)| &\lesssim |B|^{-1} \sum_{k=0}^{n-1} 2^{-kM} \int_{2^k B} \left| f(y) - \int_B f \right| dy \\ &\quad + |B|^{-1} \sum_{k=-1}^{\infty} 2^{-(n+k)M} \int_{2^{n+k} B} \left| f(y) - \int_B f \right| dy \\ &\triangleq \text{I} + \text{II}. \end{aligned}$$

To estimate the first sum I we use the first inequality of (4),

$$\begin{aligned} \text{I} &\leq \sum_{k=0}^{n-1} 2^{-k(M-2)} \int_{2^k B} \left| f - \int_B f \right| \\ &\lesssim \sum_{k=0}^{n-1} 2^{-k(M-2)} \int_{2^k B} \left| f - \int_{2^k B} f \right| + \sum_{k=0}^{n-1} 2^{-k(M-2)} \left| \int_B f - \int_{2^k B} f \right| \\ &\lesssim \sum_{k=0}^{n-1} 2^{-k(M-2)} (|\ln(2^{k-n})|)^{-\alpha} \|f\|_{L^\alpha mo} + \sum_{k=0}^n 2^{-k(M-2)} \sum_{\ell=0}^k \frac{1}{|n-\ell|^\alpha} \|f\|_{L^\alpha mo} \\ &\lesssim \sum_{k=0}^{n-1} 2^{-k(M-2)} \frac{1}{|k-n|^\alpha} \|f\|_{L^\alpha mo} + \sum_{k=0}^{n-1} 2^{-k(M-2)} \sum_{\ell=0}^k \frac{1}{|n-\ell|^\alpha} \|f\|_{L^\alpha mo} \\ &\lesssim (1+n)^{-\alpha} \|f\|_{L^\alpha mo}. \end{aligned}$$

As to the second sum we combine (4) and (5)

$$\begin{aligned} \int_{2^{n+k}B} \left| f - \int_B f \right| &\leq \int_{2^{n+k}B} \left| f - \int_{2^n B} f \right| + \int_{2^n B} \left| f - \int_B f \right| \\ &\lesssim \|f\|_{L^\alpha m_0} + n^{1-\alpha} \|f\|_{L^\alpha m_0} \\ &\lesssim \|f\|_{L^\alpha m_0} n^{1-\alpha}. \end{aligned}$$

Consequently,

$$\begin{aligned} \text{II} &\leq \|f\|_{L^\alpha m_0} \sum_{k \geq -1} 2^{-(n+k)(M-2)} n^{1-\alpha} \\ &\lesssim n^{-\alpha} \|f\|_{L^\alpha m_0}. \end{aligned}$$

The proof is now achieved by combining these two estimates.

Now let us focus on the estimate of $S_n f$. We write according to the first estimate (1) of the proposition

$$\begin{aligned} \|S_n f\|_{L^\infty} &\leq \|\Delta_{-1} f\|_{L^\infty} + \sum_{q=0}^{n-1} \|\Delta_q f\|_{L^\infty} \\ &\lesssim \|\Delta_{-1} f\|_{L^\infty} + \sum_{q=0}^{n-1} \frac{1}{(1+q)^\alpha} \|f\|_{L^\alpha m_0} \\ &\lesssim \|\Delta_{-1} f\|_{L^\infty} + n^{1-\alpha} \|f\|_{L^\alpha m_0}. \end{aligned}$$

So it remains to estimate the low frequency part. For this purpose we imitate the proof of $\|\Delta_n f\|_{L^\infty}$ with the following slight modification

$$\begin{aligned} \Delta_{-1}(f)(x) &= \int_{\mathbb{R}^2} h(x-y) f(y) dy \\ &= \int_{\mathbb{R}^2} h(x-y) \left(f(y) - \int_{B(x,1)} f \right) dy + \int_{B(x,1)} f. \end{aligned}$$

Therefore we get

$$\begin{aligned} \|\Delta_{-1} f\|_{L^\infty} &\lesssim \|f\|_{L^\alpha m_0} + \sup_{x \in \mathbb{R}^2} \int_{B(x,1)} |f| \\ &\lesssim \|f\|_{L^\alpha m_0}. \end{aligned}$$

(2) This can be easily obtained by combining the first part of Proposition 1 with the continuity on the L^p space of the localized Riesz transforms $\Delta_n \partial_i \partial_j \Delta^{-1}$ together with the help of Bernstein inequality, for $n \geq 1$:

$$\begin{aligned} \|S_n \mathcal{R}_{ij} f\|_{L^\infty} &\lesssim \|\Delta_{-1} \mathcal{R}_{ij} f\|_{L^\infty} + \sum_{q=0}^{n-1} \|\Delta_q f\|_{L^\infty} \\ &\lesssim \|\mathcal{R}_{ij} f\|_{L^p} + \sum_{q=0}^{n-1} \frac{1}{(1+q)^\alpha} \|f\|_{L^\alpha m_0} \\ &\lesssim \|f\|_{L^p} + n^{1-\alpha} \|f\|_{L^\alpha m_0}. \end{aligned}$$

The proof of the desired result is now completed. \square

Now we will introduce closed subspaces of the space $L^\alpha mo$ which play a crucial role in the study of Euler equations as we will see later in the concerned section.

2.3. The $L^\alpha mo_F$ space. It seems that the establishment of the local well-posedness for Euler equations in the framework of $L^\alpha mo$ spaces is quite difficult and cannot be easily reached by the usual methods. What we are able to do here is to construct the solutions in some weighted $L^\alpha mo$ spaces whose study will be the subject of this section.

Before stating the definition of these spaces we need the following concepts.

Definition 3. Let $F : [1, +\infty[\rightarrow [0, +\infty[$ be a non-decreasing continuous function.

• We say that F belongs to the class \mathcal{A} if there exists $C > 0$ such that:

(1) Divergence at infinity: $\lim_{x \rightarrow +\infty} F(x) = +\infty$.

(2) Slow growth: $\forall x, y \geq 1$

$$F(xy) \leq C(1 + F(x))(1 + F(y)).$$

(3) Lipschitz condition: F is differentiable and

$$\sup_{x > 1} |F'(x)| \leq C.$$

(4) Cancellation at 1:

$$\forall x \in [0, 1], \quad F(1+x) \leq Cx.$$

• We say that F belongs to the class \mathcal{A}' if it belongs to \mathcal{A} and satisfies

$$\int_2^{+\infty} \frac{1}{x F(x)} dx = +\infty.$$

Remark 3. (1) From the slow growth assumption we see that necessarily the function F should have at most a polynomial growth.

(2) The assumption (3) is only used through Lemma 4 and could be in fact relaxed for example to $\|F^{(k)}\|_{L^\infty} < \infty$ for some $k \in \mathbb{N}$. But for the sake of simple presentation we limited our discussion to the case $k = 1$.

Example. (1) For any $\beta \in]0, 1]$, the function $x \mapsto x^\beta - 1$ belongs to the class $\mathcal{A} \setminus \mathcal{A}'$.

(2) For any $\beta \geq 1$, the function $x \mapsto \ln^\beta(x)$ belongs to the class \mathcal{A} and this function belongs to the class \mathcal{A}' only for $\beta = 1$.

(3) The function $x \mapsto \ln x \ln \ln(e+x)$ belongs to the class \mathcal{A}' .

We can now introduce the weighted $L^\alpha mo$ spaces.

Definition 4. Let $\alpha \in [0, 1]$ and F be in the class \mathcal{A} . We define the space $L^\alpha mo_F$ as the set of locally integrable functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\|f\|_{L^\alpha mo_F} \triangleq \|f\|_{L^\alpha mo} + \sup_{B_1, B_2} \frac{|\int_{B_2} f - \int_{B_1} f|}{F\left(\frac{|\ln(r_2)|}{|\ln(r_1)|}\right)} < +\infty,$$

where the supremum is taken over all the pairs of balls $B_2(x_2, r_2)$ and $B_1(x_1, r_1)$ in \mathbb{R}^2 with $0 < r_1 \leq \frac{1}{2}$ and $2B_2 \subset B_1$. Here, for a ball B and $\lambda > 0$, λB denotes the ball that is concentric with B and whose radius is λ times the radius of B .

Now we list some useful properties of these spaces that will be used later.

Remark 4. (1) The space LBMO introduced in [5] corresponds to $\alpha = 0$ and $F = \ln$.

(2) Let $F_1, F_2 \in \mathcal{A}$ such that $F_1 \lesssim F_2$. Then we have the embedding

$$L^\alpha mo_{F_1} \hookrightarrow L^\alpha mo_{F_2}.$$

(3) For every $g \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and $f \in L^\alpha mo_F$ one has

$$\|g * f\|_{L^\alpha mo_F} \leq \|g\|_{L^1} \|f\|_{L^\alpha mo_F}.$$

Indeed, this property is just the consequence of Minkowski inequality and that the $L^\alpha mo_F$ -norm is invariant by translation.

The main goal of the following proposition is to discuss the link between the space of bounded functions and the space $L^\alpha mo_F$. We will see in particular that under suitable assumptions on F these spaces are not comparable. More precisely we get the following.

Proposition 2. Let $\alpha \in [0, 1]$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the radial function defined by

$$f(x) = \begin{cases} \ln(1 - \ln|x|) & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

The following properties hold true.

- (1) The function f belongs to $L^\alpha mo$.
- (2) For $F(x) = \ln x, x \geq 1$, then $f \in L^\alpha mo_F$.
- (3) For $\alpha \in]0, 1[$ and $F \in \mathcal{A}$ with $\ln \lesssim F$, the spaces L^∞ and $L^\alpha mo_F$ are not comparable.

Proof. (1) There are at least two ways to get this result. The first one uses Spanne's criterion, see Theorem 2 of [23] and we omit here the details. However the second one is related to Poincaré inequality which states that for any ball for B we have

$$\int_B \left| f - \int_B f \right| \lesssim r \int_B |\nabla f|.$$

For the example, it is obvious that $|\nabla f(x)| \lesssim \frac{1}{|x|(1-\ln|x|)}$. So the quantity of the right-hand side in the Poincaré inequality is maximal for a ball B centered at 0 and consequently it comes

$$\begin{aligned} r \int_B |\nabla f| &\lesssim r^{-1} \int_0^r \frac{1}{1 - \ln \eta} d\eta \\ &\lesssim \frac{1}{1 - \ln r}, \end{aligned}$$

which concludes the proof of $f \in L^\alpha mo$.

(2) We reproduce the arguments developed in [5, Proposition 3], where it is proven that

$$(6) \quad \left| \int_{B_2} f - \int_{B_1} f \right| \lesssim \ln \left(\frac{1 + |\ln(r_2)|}{1 + |\ln(r_1)|} \right) + \mathcal{O}(|\ln(r_1)|^{-1}) + \mathcal{O}(|\ln(r_2)|^{-1}).$$

If $A \triangleq \frac{1 + |\ln(r_2)|}{1 + |\ln(r_1)|} \geq 2$ then $\mathcal{O}(|\ln(r_1)|^{-1}) + \mathcal{O}(|\ln(r_2)|^{-1})$ is bounded by $\ln(A)$ and so

$$\left| \int_{B_2} f - \int_{B_1} f \right| \lesssim \ln(A).$$

If $A \leq 2$, then $\ln(A)$ is equivalent to $A - 1 = \frac{\ln(r_1/r_2)}{1 + |\ln(r_1)|} \gtrsim (1 + |\ln(r_1)|)^{-1}$. The latter inequality follows from the fact $r_2 \leq r_1/2$. Therefore we get

$$\mathcal{O}(|\ln(r_1)|^{-1}) + \mathcal{O}(|\ln(r_2)|^{-1}) \lesssim A - 1 \approx \ln(A),$$

which also gives

$$\left| \int_{B_2} f - \int_{B_1} f \right| \lesssim \ln(A).$$

Finally this ensures that $f \in L^\alpha mo_F$, for every $\alpha \in [0, 1]$.

(3) According to Remark 4 we get the embedding $L^\alpha mo_{\ln} \hookrightarrow L^\alpha mo_F$. Now by virtue of the second claim of Proposition 2 the function f which is clearly not bounded belongs to the space $L^\alpha mo_F$. It remains to construct a function which is bounded but does not belong to the space $L^\alpha mo_F$. Let D_+ be the upper half unit disc defined by

$$D_+ \triangleq \{(x, y); x^2 + y^2 \leq 1, y \geq 0\}.$$

By the same way we define the lower half unit disc D_- . Let $r \leq 1$, denote by B_r the disc of center zero and radius r and let $g = \mathbf{1}_{D_+}$ be the characteristic function of D_+ . Easy computations yield

$$g(x) - \int_{B_r} g = \begin{cases} \frac{1}{2}, & x \in B_r \cap D_+ \\ -\frac{1}{2}, & x \in B_r \cap D_- \end{cases}$$

Thus we find for every $r \in (0, 1)$

$$\int_{B_r} |g - \int_{B_r} g| = \frac{1}{2}.$$

This shows that the function g does not belong to $L^\alpha mo$ for every $\alpha > 0$. \square

Our next aim is to go over some refined properties of the weighted lmo spaces. One result that we will prove and which seems to be surprising says that all the spaces $L^\alpha mo_F$ are contained in the space $L^\alpha mo_{\ln}$. This rigidity follows from the cancellation property of F at the point 1. More precisely, we shall show the following.

Proposition 3. *Let $\alpha \in (0, 1]$ and $F \in \mathcal{A}$. Then $L^\alpha mo_F \hookrightarrow L^\alpha mo_{\ln}$.*

Let $F : [1, +\infty) \rightarrow \mathbb{R}_+$ defined by $F(x) = \ln(1 + \ln x)$ then $F \in \mathcal{A}$ and $L^\alpha mo_F \subsetneq L^\alpha mo_{\ln}$.

Proof. Fix a function $f \in L^\alpha mo_F$ and a point x and set $\phi(r) = \int_{B(x,r)} f$. From the definition of the space $L^\alpha mo_F$ combined with the cancellation property of F and its polynomial growth we get that for every $r \in (0, \frac{1}{2})$ and $k \geq 1$

$$\begin{aligned} |\phi(r) - \phi(2^{-k}r)| &\leq \sum_{\ell=0}^{k-1} |\phi(2^{-\ell}r) - \phi(2^{-\ell-1}r)| \\ &\lesssim \sum_{\ell=0}^{k-1} F\left(\frac{1 + \ell + |\ln r|}{\ell + |\ln r|}\right) \\ &\lesssim \sum_{\ell=0}^{k-1} \frac{1}{\ell + |\ln r|} \\ &\lesssim \ln\left(\frac{k + |\ln r|}{|\ln r|}\right). \end{aligned}$$

Then for $s < \frac{r}{2}$ choose $k \geq 1$ such that $2^{-k-1}r \leq s < 2^{-k}r$ and so

$$|\phi(r) - \phi(s)| \leq |\phi(r) - \phi(2^{-k}r)| + |\phi(s) - \phi(2^{-k}r)|.$$

As we have just seen, the first term is bounded by

$$\ln \left(\frac{k + |\ln r|}{|\ln r|} \right) \approx \ln \left(\frac{1 + |\ln s|}{|\ln r|} \right).$$

The second term is bounded as follows :

$$\begin{aligned} |\phi(s) - \phi(2^{-k}r)| &\leq |\phi(s) - \phi(2^{-k+1}r)| + |\phi(2^{-k-1}r) - \phi(2^{-k}r)| \\ &\lesssim F \left(\frac{\ln s}{\ln(2^{-k+1}r)} \right) + F \left(\frac{1 + k + |\ln(r)|}{k + |\ln(r)|} \right) \\ &\lesssim \frac{1}{|\ln s|} \\ &\lesssim \ln \left(\frac{|\ln s|}{|\ln r|} \right), \end{aligned}$$

where we used the cancellation property of F and the fact that both $B(x, 2s)$ and $B(x, 2 \cdot 2^{-k}r)$ are included into $B(x, 2^{-k+1}r)$. So combining these two previous estimates, it comes for every x , $r < \frac{1}{2}$ and $s \leq \frac{r}{2}$

$$(7) \quad \left| \int_{B(x,r)} f - \int_{B(x,s)} f \right| \lesssim \ln \left(\frac{|\ln s|}{|\ln r|} \right).$$

Now let $B_2 = B(x_2, r_2)$ and $B_1 = B(x_1, r_1)$ two balls with $0 < r_1 \leq \frac{1}{2}$ and $2B_2 \subset B_1$. We wish to estimate $\left| \int_{B_2} f - \int_{B_1} f \right|$. First, it is clear that the interesting case is when at least the radius r_2 is small, otherwise r_1 and r_2 are equivalent to 1 and there is nothing to prove. So assume that $r_2 \leq \frac{1}{100}$, then it is only sufficient to study the case where $r_1 \leq \frac{1}{10}$. So let us only consider this situation : $r_2 \leq \frac{1}{100}$ and $r_1 \leq \frac{1}{10}$. Then we have

$$\left| \int_{B_2} f - \int_{B_1} f \right| \lesssim \left| \int_{B_2} f - \int_{\frac{r_1}{r_2}B_2} f \right| + \left| \int_{\frac{r_1}{r_2}B_2} f - \int_{B_1} f \right|.$$

Applying (7), the first term is bounded by $\ln \left(\frac{\ln r_2}{\ln r_1} \right)$. The second term can be easily bounded by $|\ln(r_1)|^{-1}$. Indeed, the two balls $\frac{r_1}{r_2}B_2$ and B_1 are comparable and of radius $r_1 \leq \frac{1}{10}$. So there exists a ball B of radius $r = 5r_1$, such that $\frac{2r_1}{r_2}B_2 \cup 2B_1 \subset B$. Then we have

$$\begin{aligned} \left| \int_{\frac{r_1}{r_2}B_2} f - \int_{B_1} f \right| &\leq \left| \int_{\frac{r_1}{r_2}B_2} f - \int_B f \right| + \left| \int_B f - \int_{B_1} f \right| \\ &\lesssim F \left(\frac{\ln r_1}{\ln r} \right) \lesssim \frac{1}{|\ln r_1|}. \end{aligned}$$

Now, since $r_2 \leq \frac{1}{2}r_1$ then

$$\begin{aligned} \ln \left(\frac{\ln r_2}{\ln r_1} \right) &= \ln \left(1 + \frac{\ln(r_2/r_1)}{\ln r_1} \right) \\ &\geq \ln \left(1 + \frac{\ln 2}{|\ln r_1|} \right) \\ &\geq C \frac{1}{|\ln r_1|}. \end{aligned}$$

This concludes the proof of

$$\left| \int_{B_2} f - \int_{B_1} f \right| \lesssim \ln \left(\frac{1 - \ln(r_2)}{1 - \ln(r_1)} \right).$$

Hence we get the inclusion $L^\alpha mo_F \subset L^\alpha mo_{\ln}$.

Then consider the specific function $F(\cdot) = \ln(1 + \ln(|\cdot|))$. It is easy to check that $F \in \mathcal{A}$ and the function f defined in Proposition 2 belongs to $L^\alpha mo_{\ln} \setminus L^\alpha mo_F$. Indeed, (6) becomes an equality for this specific function f with balls B_1, B_2 centered at 0. \square

The next proposition shows that for $\alpha = 1$, the cancellation property of F at 1 (in $L^\alpha mo_F$ space) can be “forgotten”, since it is already encoded in the Lmo space:

Proposition 4. *Let $\alpha = 1$ and $F \in \mathcal{A}$. Then $L^\alpha mo_F = Lmo_{1+F}$. Moreover, we have $Lmo = Lmo_{\ln}$.*

Proof. Since $F \leq 1 + F$, it follows that $L^\alpha mo_F \subset Lmo_{1+F}$. Reciprocally, since for $t \geq 1$, $F(t) \simeq 1 + F(t)$ (due to $F \in \mathcal{A}$), following the proof of Proposition 3 to prove that $Lmo_{1+F} \subset L^\alpha mo_F$ it is sufficient to check that for every function $f \in Lmo$, every ball B of radius $r < \frac{1}{4}$ then

$$(8) \quad \left| \int_B f - \int_{2B} f \right| \lesssim \frac{1}{|\ln(r)|}.$$

Indeed, the only difference between Lmo_{1+F} and $L^\alpha mo_F$ (where F is replaced by $1 + F$) is the loss of the cancellation property of F at the point 1 and this property was used in the previous proposition to check (8).

However, here since $\alpha = 1$ (8) automatically holds since the function belongs to Lmo . Then producing the same reasoning as for Proposition 3, we deduce that $Lmo \subset Lmo_{\ln}$, which yields $Lmo = Lmo_{\ln}$, since the other embedding is obvious. \square

2.4. Regularity of the flow map. We shall continue in this section our excursion into function spaces by introducing the log-Lipschitz class with exponent $\beta \in (0, 1]$, denoted by $L^\beta L$ and showing some links with the foregoing $L^\alpha mo$ spaces. We next examine the regularity of the flow map associated to a vector field belonging to this class $L^\beta L$. We start with the following definition. We say that a function f belongs to the class $L^\beta L$ if

$$\|f\|_{L^\beta L} \triangleq \sup_{0 < |x-y| < \frac{1}{2}} \frac{|f(x) - f(y)|}{|x-y| |\ln|x-y||^\beta} < \infty.$$

Take now a smooth divergence-free vector field $u = (u^1, u^2)$ on \mathbb{R}^2 and $\omega = \partial_1 u^2 - \partial_2 u^1$ its vorticity. It is apparent from straightforward computations that

$$(9) \quad \Delta u = \nabla^\perp \omega.$$

This identity leads through the use of the fundamental solution of the Laplacian to the so-called Biot-Savart law. Now we shall solve the equation (9) when the source term belongs to the space $L^\alpha mo \cap L^p$. Without going further into the details we restrict ourselves to the a priori estimates required for the resolution of this equation.

Proposition 5. *Let $\alpha \in (0, 1)$, $p \in (1, \infty)$ and $\omega \in L^\alpha mo \cap L^p$ be the vorticity of the velocity u given by the equation (9). Then $u \in L^{1-\alpha}L$ and there exists an absolute constant $C > 0$ such that*

$$\|u\|_{L^{1-\alpha}L} \leq C \|\omega\|_{L^\alpha mo \cap L^p}.$$

Proof. Let $N \in \mathbb{N}^*$ be a given number that will be fixed later and $0 < |x - y| < \frac{1}{2}$. Using the mean value theorem combined with Bernstein inequality give

$$\begin{aligned} |u(x) - u(y)| &\leq |S_N u(x) - S_N u(y)| + 2 \sum_{q \geq N} \|\Delta_q u\|_{L^\infty} \\ &\lesssim |x - y| \|\nabla S_N u\|_{L^\infty} + \sum_{q \geq N} 2^{-q} \|\Delta_q \omega\|_{L^\infty}. \end{aligned}$$

From Proposition 1, it follows

$$\begin{aligned} |u(x) - u(y)| &\lesssim N^{1-\alpha} \|\omega\|_{L^\alpha mo \cap L^p} |x - y| + \|\omega\|_{L^\alpha mo} \sum_{q \geq N} 2^{-q} q^{-\alpha} \\ &\lesssim \|\omega\|_{L^\alpha mo \cap L^p} N^{1-\alpha} (|x - y| + 2^{-N}). \end{aligned}$$

By choosing $2^{-N} \approx |x - y|$ we find

$$|u(x) - u(y)| \lesssim |x - y| |\ln |x - y||^{1-\alpha} \|\omega\|_{L^\alpha mo \cap L^p}.$$

This completes the proof of the proposition. \square

We recall Osgood Lemma whose proof can be found for instance in [1], page 128.

Lemma 2 (Osgood Lemma). *Let $a, A > 0$, $\Gamma : [a, +\infty[\rightarrow \mathbb{R}_+$ be a non-decreasing function and $\gamma : [t_0, T] \rightarrow \mathbb{R}_+$ be a locally integrable function. Let $\rho : [t_0, T] \rightarrow [a, +\infty[$ be a measurable function such that*

$$\rho(t) \leq A + \int_{t_0}^t \gamma(\tau) \Gamma(\rho(\tau)) \rho(\tau) d\tau.$$

Let $\mathcal{M}(y) = \int_a^y \frac{1}{x\Gamma(x)} dx$ and assume that $\lim_{y \rightarrow +\infty} \mathcal{M}(y) = +\infty$. Then

$$\forall t \in [t_0, T], \quad \rho(t) \leq \mathcal{M}^{-1} \left(\mathcal{M}(A) + \int_0^t \gamma(\tau) d\tau \right).$$

In what follows we discuss the regularity of the flow map associated to a vector field belonging to the log-Lipschitz class. This precise description will be of great interest in the proof of the main result.

Proposition 6. *Let u be a smooth divergence-free vector field belonging to $L^{1-\alpha}L$, with $\alpha \in (0, 1)$ and ψ be its flow, that is the solution of the differential equation,*

$$\partial_t \psi(t, x) = u(t, \psi(t, x)), \quad \psi(0, x) = x.$$

Then, there exists $C \triangleq C(\alpha) > 1$ such that for every $t \geq 0$

$$|x - y| < \ell(t) \implies |\psi^{\pm 1}(t, x) - \psi^{\pm 1}(t, y)| \leq |x - y| e^{CV(t) |\ln |x - y||^{1-\alpha}},$$

where $\ell(t) \in (0, \frac{1}{2})$ is given by

$$\ell(t)e^{CV(t)|\ln(\ell(t))|^{1-\alpha}} = \frac{1}{2} \quad \text{and} \quad V(t) \triangleq \int_0^t \|u(\tau)\|_{L^{1-\alpha}L} d\tau.$$

Here we denote by ψ^1 the flow ψ and ψ^{-1} its inverse.

Proof. It is well-known that for every $t \geq 0$ the mapping $x \mapsto \psi(t, x)$ is a Lebesgue measure preserving homeomorphism (see [8] for instance). We fix $x \neq y$ such that $|x - y| < \frac{1}{2}$ and we define for $t \geq 0$,

$$z(t) \triangleq |\psi(t, x) - \psi(t, y)|.$$

Clearly the function z is strictly positive and satisfies

$$z(t) \leq z(0) + C \int_0^t \|u(\tau)\|_{L^{1-\alpha}L} |\ln z(\tau)|^{1-\alpha} z(\tau) d\tau,$$

as soon as $z(\tau) \leq \frac{1}{2}$, for all $\tau \in [0, t]$. Let $T > 0$ and $I \triangleq \{t \in [0, \ell(T)] \mid \forall \tau \in [0, t], z(\tau) \leq \frac{1}{2}\}$, where the value of $\ell(T)$ has been defined in Proposition 6. We aim to show that the set I is the full interval $[0, \ell(T)]$. First I is a non-empty set since $0 \in I$ and it is an interval according to its definition. The continuity in time of the flow guarantees that I is closed. It remains to show that I is an open set of $[0, \ell(T)]$. From the differential equation,

$$\forall t \in I, \quad z(t) \leq z(0) + C \int_0^t \|u(\tau)\|_{L^{1-\alpha}L} (-\ln z(\tau))^{1-\alpha} z(\tau) d\tau.$$

Accordingly, we infer

$$-|\ln z(t)|^\alpha + |\ln z(0)|^\alpha \leq C\alpha V(t),$$

and this yields

$$|\ln z(t)| \geq (|\ln z(0)|^\alpha - C\alpha V(t))^{\frac{1}{\alpha}}.$$

despite that

$$(10) \quad C\alpha V(t) \leq |\ln z(0)|^\alpha.$$

Consequently

$$z(t) \leq e^{-\left(|\ln z(0)|^\alpha - C\alpha V(t)\right)^{\frac{1}{\alpha}}}.$$

By virtue of Taylor formula and since $\frac{1}{\alpha} - 1 > 0$ we get

$$\begin{aligned} -\left(|\ln z(0)|^\alpha - C\alpha V(t)\right)^{\frac{1}{\alpha}} &= -|\ln z(0)| + \frac{1}{\alpha} \int_0^{C\alpha V(t)} \left(|\ln z(0)|^\alpha - x\right)^{\frac{1}{\alpha}-1} dx \\ &\leq \ln z(0) + CV(t) |\ln z(0)|^{1-\alpha}. \end{aligned}$$

It follows that

$$z(t) \leq z(0)e^{CV(t)|\ln z(0)|^{1-\alpha}}.$$

Therefore to show that I is open it suffices to make the assumption

$$z(0)e^{CV(t)|\ln z(0)|^{1-\alpha}} < \frac{1}{2},$$

which is satisfied when $z(0) < \ell(T)$. This last claim follows from the increasing property of the function $x \mapsto xe^{CV(t)|\ln x|^{1-\alpha}}$ on the interval $[0, x_c]$ where $x_c < 1$ is the unique real number satisfying $|\ln x_c|^\alpha = C(1 - \alpha)V(t)$. From the definition of $\ell(t)$ we can easily check that $\ell(T) < x_c$ and (10) is satisfied.

The proof of the assertion for ψ^{-1} can be derived by performing similar computations for the generalized flow defined by

$$\partial_t \psi(t, s, x) = u(t, \psi(t, s, x)), \quad \psi(s, s, x) = x$$

and the flow ψ^{-1} is nothing but $x \mapsto \psi(0, t, x)$. □

3. REGULARITY PERSISTENCE

The main object of this section is to examine the propagation of the initial regularity measured in the spaces $L^\alpha mo_F$ for the following transport model governed by a divergence-free vector field,

$$(11) \quad \begin{cases} \partial_t w + u \cdot \nabla w = 0, & x \in \mathbb{R}^2, t > 0, \\ \operatorname{div} u = 0, \\ w|_{t=0} = f. \end{cases}$$

Along the first part of this study we shall not prescribe any relationship between the solution w and the vector field u . Once this study is achieved, we will apply this result for the inviscid vorticity where the vector field is induced by the vorticity. This will enable us not only to prove Theorem 1 but also to state more general results on the local and global theory extending the special case of $F(x) = \ln x$.

3.1. Composition in the space $L^\alpha mo_F$. We begin with the following observation concerning the structure of the solutions to (11). Under reasonable assumptions on the regularity of the velocity, the solution can be recovered from its initial data and the flow ψ according to the formula $w(t) = f \circ \psi^{-1}(t)$. Thus the study of the propagation in the space $L^\alpha mo$ reduces to the composition by a measure preserving map in this space. We should note that this latter problem can be easily solved as soon as the map is bi-Lipschitz (see [4] for composition in some BMO-type spaces by a bi-Lipschitz measure preserving map). In our context the flow is not necessarily Lipschitz but in some sense very close to this class. It is apparent according to Proposition 6 that ψ belongs to the class C^s for every $s < 1$. It turns out that working with a flow under the Lipschitz class has a profound effect and makes the composition in the space $L^\alpha mo$ very hard to get. This is the principal reason why we need to use the weighted subspace $L^\alpha mo_F$ in order to compensate this weak regularity and consequently to well-define the composition. Our result reads as follows.

Theorem 2. *Let $\alpha \in (0, 1)$, $F \in \mathcal{A}$ and consider a smooth solution w of the equation (11) defined on $[0, T]$. Then there exists a constant $C \triangleq C(\alpha) > 0$ such that the following holds true:*

(1) For every $t \in [0, T]$

$$\|w(t)\|_{L^\alpha mo} \leq C \|f\|_{L^\alpha mo_F} (1 + V(t)) F(2 + V^{\frac{1}{\alpha}}(t)),$$

$$\text{with } V(t) \triangleq \int_0^t \|u(\tau)\|_{L^{1-\alpha} L} d\tau.$$

(2) For every $t \in [0, T]$

$$\|w(t)\|_{L^\alpha mo_{1+F}} \leq C \|f\|_{L^\alpha mo_F} F(2 + V^{\frac{1}{\alpha}}(t)).$$

Before giving the proof, some remarks are in order.

Remark 5. (1) According to the first result of the foregoing theorem, the estimate of the solution in the space $L^\alpha mo$ does not involve the weighted part of the space $L^\alpha mo_F$, which is only required for the initial data.

(2) The estimate of the second part of Theorem 2 is subjected to a slight loss. Indeed, instead of F we put $1 + F$. This is due to the fact that we need some cancellations for the difference of two averages and to avoid this loss more sophisticated analysis should be carried out and we believe that this loss is a technical artifact.

Proof of Theorem 2. (1) The proof will be done in the spirit of the recent work [5]. First we observe that the solution is given by $w(t) = f \circ \psi^{-1}(t)$, where ψ is the flow associated to the vector field u . Therefore the estimate in the space $L^\alpha mo$ reduces to the stability by the right composition with a homeomorphism preserving Lebesgue measure with the prescribed regularity given in Proposition 6. Since the flow ψ and its inverse share the same properties and the estimates that will be involved along the proof, we prefer for the sake of simple notation to use in the composition with ψ instead of ψ^{-1} . Let $B = B(x_0, r)$ be the ball of center x_0 and radius $r \in (0, \frac{1}{2})$. We intend to give a suitable estimate for the quantity

$$\mathcal{I}_r \triangleq |\ln r|^\alpha \int_B |f \circ \psi - \int_B (f \circ \psi)| dx.$$

To reach this goal we use in a crucial way the local regularity of the flow stated before in Proposition 6. The estimate of \mathcal{I}_r will require some discussions depending on a threshold value for r denoted by r_t . The identification of r_t is related to hidden arguments that will be clarified during the proof. To begin with, fix a sufficiently large constant $\delta > \max(\sqrt{2}, 2C)$ (where C is given by Proposition 6) and define r_t as the unique solution in the interval $(0, \frac{1}{2})$ of the following equation

$$(12) \quad \delta r_t e^{\delta V(t) |\ln(r_t)|^{1-\alpha}} = r_t^{\frac{1}{2}}.$$

The existence and uniqueness can be easily proven by studying the variations of the function

$$(13) \quad h(r) \triangleq \delta r^{\frac{1}{2}} e^{\delta V(t) |\ln(r)|^{1-\alpha}}, \quad r \in (0, 1/2)$$

and using the fact that $h(\frac{1}{2}) > 1$ since $\delta > \sqrt{2}$. We point out that h is non-decreasing in the interval $[0, r_t]$ and $h(r) \leq 1$ in this range. We have also the bound

$$(14) \quad C' + C' V^{\frac{1}{\alpha}}(t) \leq |\ln r_t| \leq C + C V^{\frac{1}{\alpha}}(t), \quad \text{for some } C, C' > 0.$$

Indeed, set $X = -\ln r_t$ then from (12) and Young inequality

$$\begin{aligned} X &= 2 \ln \delta + 2\delta V(t) X^{1-\alpha} \\ &\leq C_1 + C_1 V^{\frac{1}{\alpha}}(t) + \frac{1}{2} X. \end{aligned}$$

This gives the estimate of the right-hand side of (14). For the left one it is apparent from the equation on X and its positivity that

$$X \geq 2 \ln \delta \quad \text{and} \quad X \geq 2\delta V(t) X^{1-\alpha}.$$

Thus

$$X \geq \ln \delta + \frac{1}{2} (2\delta)^{\frac{1}{\alpha}} V^{\frac{1}{\alpha}}(t),$$

which concludes the proof of (14). Before starting the computations for \mathcal{I}_r we need to introduce the radius

$$(15) \quad r_\psi \triangleq \delta r e^{\delta V(t) |\ln(r)|^{1-\alpha}}.$$

Now we will check that for $r \in (0, r_t]$

$$(16) \quad 1 \leq \frac{|\ln r|}{|\ln r_\psi|} \leq 2.$$

The inequality of the left-hand side can be deduced as follows. First it is obvious that $0 < r < r_\psi$ and it remains to show that $r_\psi < \frac{1}{2}$ whenever $r \in]0, r_t]$. For this purpose we show by using simple arguments that the function $k : r \mapsto r_\psi$ is non-decreasing in the interval $(0, r_t]$. From this latter fact and (12) we find $r_\psi \leq k(r_t) = r_t^{\frac{1}{2}} \leq \frac{1}{2}$. Let us now move to the second inequality of (16) and for this aim we start with studying the function

$$g(x) = \frac{-x}{-x + a + bx^{1-\alpha}}, x \geq -\ln r_t; \quad a \triangleq \ln \delta, b \triangleq \delta V(t).$$

We observe that the quotient $\frac{|\ln r|}{|\ln r_\psi|}$ coincides with $g(-\ln r)$. By easy computations we get

$$g'(x) = -\frac{a + b\alpha x^{1-\alpha}}{(-x + a + bx^{1-\alpha})^2} < 0.$$

This yields in view of (12) and (14)

$$g(x) \leq g(-\ln r_t) \leq \frac{\ln r_t}{\frac{\ln r_t}{2}} \leq 2.$$

The estimate of \mathcal{I}_r depends whether the radius r is smaller or larger than the critical value r_t .

Case 1: $0 < r \leq r_t$.

As ψ is a homeomorphism which preserves Lebesgue measure then $\psi(B)$ is an open connected set with $|\psi(B)| = |B|$. Let us consider a Whitney covering of this open set $\psi(B)$, that consists in a collection of balls $(O_j)_j$ such that:

- The collection of double balls is a bounded covering:

$$\psi(B) \subset \bigcup_j 2O_j.$$

- The collection is disjoint and for all j ,

$$O_j \subset \psi(B).$$

- The Whitney property is verified: the radius r_j of O_j satisfies

$$r_j \approx d(O_j, \psi(B)^c).$$

We set $\tilde{B} \triangleq B(\psi(x_0), r_\psi)$ then according to Proposition 6 (and $\delta > 2\rho(\alpha)$) we have $\psi(B) \subset \tilde{B}$. It is easy to see from the invariance of the Lebesgue measure by the flow that

$$\begin{aligned} \int_B |f \circ \psi - \int_B (f \circ \psi)| dx &= \int_{\psi(B)} |f - \int_{\psi(B)} f| dx \\ &\leq 2 \int_{\psi(B)} |f - \int_{\tilde{B}} f| dx. \end{aligned}$$

Using the preceding notations

$$\begin{aligned} |\ln r|^\alpha \int_{\psi(B)} |f - \int_{\tilde{B}} f| &\lesssim \frac{|\ln r|^\alpha}{|B|} \sum_j |O_j| \int_{2O_j} |f - \int_{\tilde{B}} f| \\ &\lesssim I_1 + I_2, \end{aligned}$$

with

$$\begin{aligned} I_1 &\triangleq \frac{|\ln r|^\alpha}{|B|} \sum_j |O_j| \int_{2O_j} |f - \int_{2O_j} f| \\ I_2 &\triangleq \frac{|\ln r|^\alpha}{|B|} \sum_j |O_j| \left| \int_{2O_j} f - \int_{\tilde{B}} f \right|. \end{aligned}$$

On one hand, since $\sum |O_j| \leq |B|$ and $r_j \leq r < \frac{1}{2}$ (due to $|O_j| \leq |\psi(B)| = |B|$) then

$$\begin{aligned} I_1 &\leq \frac{1}{|B|} \sum_j |O_j| \frac{|\ln r|^\alpha}{|\ln(r_j)|^\alpha} \|f\|_{L^\alpha mo} \\ &\leq \|f\|_{L^\alpha mo}. \end{aligned}$$

On the other hand, since $d(O_j, \tilde{B}) \leq r_\psi$ and $r_{\tilde{B}} = r_\psi$, it ensures that

$$O_j \subset \frac{r_\psi}{r_j} O_j$$

and hence the two balls $Q_1 \triangleq \frac{r_\psi}{r_j} O_j$ and \tilde{B} are comparable¹. This entails

$$\left| \int_{Q_1} f - \int_{\tilde{B}} f \right| \lesssim \frac{1}{|\ln(r_\psi)|^\alpha} \|f\|_{L^\alpha mo}.$$

We point out that we have used the fact that for $0 < r < r_t$ the radius r_ψ of \tilde{B} is smaller than $\frac{1}{2}$. Moreover according to the definition of the space $L^\alpha mo_F$, it comes since $r_{Q_1} \leq \frac{1}{2}$

$$\begin{aligned} \left| \int_{2O_j} f - \int_{Q_1} f \right| &\lesssim \|f\|_{L^\alpha mo_F} F\left(\frac{\ln r_j}{\ln r_{Q_1}}\right) \\ &\lesssim \|f\|_{L^\alpha mo_F} F\left(\frac{\ln r_j}{\ln r_\psi}\right). \end{aligned}$$

It follows that

$$\left| \int_{2O_j} f - \int_{\tilde{B}} f \right| \lesssim \|f\|_{L^\alpha mo_F} \left(|\ln r_\psi|^{-\alpha} + F\left(\frac{\ln r_j}{\ln r_\psi}\right) \right).$$

Together with (16) this estimate yields

$$\left| \int_{2O_j} f - \int_{\tilde{B}} f \right| \lesssim \|f\|_{L^\alpha mo_F} \left(|\ln r|^{-\alpha} + F\left(\frac{\ln r_j}{\ln r_\psi}\right) \right).$$

Consequently,

$$I_2 \lesssim \|f\|_{L^\alpha mo_F} + \|f\|_{L^\alpha mo_F} \frac{|\ln r|^\alpha}{|B|} \left(\sum_j |O_j| F\left(\frac{\ln r_j}{\ln r_\psi}\right) \right).$$

¹Here we say that two balls Q_1 and Q_2 are comparable if $Q_1 \subset 4Q_2$ and $Q_2 \subset 4Q_1$.

For every $k \in \mathbb{N}$ we set

$$u_k \triangleq \sum_{e^{-(k+1)r} < r_j \leq e^{-kr}} |O_j|,$$

so that

$$(17) \quad \begin{aligned} \mathbb{I}_2 &\lesssim \|f\|_{L^\alpha m o_F} \left(1 + \frac{|\ln r|^\alpha}{|B|} \sum_{k \geq 0} u_k F \left(\frac{-1 - k + \ln r}{\ln(r) + a + b |\ln r|^{1-\alpha}} \right) \right) \\ &\triangleq \|f\|_{L^\alpha m o_F} + \mathbb{I}_3. \end{aligned}$$

with

$$(18) \quad a \triangleq \ln \delta, \quad b \triangleq \delta V(t).$$

The numbers a and b appeared before in the definition of r_ψ given in (15). Let N be a real number that will be judiciously fixed later. We split the sum in the right-hand side of (17) into two parts

$$\mathbb{I}_3 = \sum_{k \leq N} (\dots) + \sum_{k > N} (\dots) \triangleq \mathbb{II}_1 + \mathbb{II}_2.$$

Since $\sum u_k \leq |B|$ and F is non-decreasing then

$$(19) \quad \mathbb{II}_1 \lesssim |\ln r|^\alpha F \left(\frac{1 + N + |\ln r|}{|\ln(r)| - A - B |\ln r|^{1-\alpha}} \right).$$

To estimate the term \mathbb{II}_2 we need a refined bound for u_k given below and whose proof will be postponed to the end of this paper in Lemma 3 of the Appendix.

$$(20) \quad u_k \lesssim \delta r^2 e^{-k} e^{\delta(V(t)+1)(k-\ln(r))^{1-\alpha}}.$$

By virtue of (20) and Lemma 4 we get

$$(21) \quad \begin{aligned} \mathbb{II}_2 &\lesssim |\ln r|^\alpha e^{-N} e^{b(N-\ln r)^{1-\alpha}} F \left(\frac{1 + N + |\ln r|}{|\ln r| - a - b |\ln r|^{1-\alpha}} \right) \\ &+ e^{-N} e^{b(N-\ln r)^{1-\alpha}} \frac{|\ln r|^\alpha}{|\ln r| - a - b |\ln r|^{1-\alpha}}. \end{aligned}$$

So we choose $N = N(r)$ such that $e^{-N} e^{b(N-\ln r)^{1-\alpha}} = 1$. Under this assumption we get

$$\mathbb{II}_1 + \mathbb{II}_2 \lesssim |\ln r|^\alpha F \left(\frac{1 + N + |\ln r|}{|\ln r| - a - b |\ln r|^{1-\alpha}} \right) + \frac{|\ln r|^\alpha}{|\ln r| - a - b |\ln r|^{1-\alpha}}.$$

The condition on N is also equivalent to

$$N = 1 + b(N + |\ln r|)^{1-\alpha}.$$

Then from Young inequality

$$(22) \quad N \lesssim 1 + b^{\frac{1}{\alpha}} + b |\ln r|^{1-\alpha}$$

and therefore

$$\begin{aligned} \frac{1 + N + |\ln r|}{|\ln r| - a - b |\ln r|^{1-\alpha}} - 1 &= \frac{1 + N + a + b |\ln r|^{1-\alpha}}{|\ln r_\psi|} \\ &\lesssim \frac{1 + b^{\frac{1}{\alpha}} + b |\ln r|^{1-\alpha}}{|\ln r_\psi|}. \end{aligned}$$

Using the *cancellation property* of F at the point 1, that is $\sup_{x \in (0,1)} \frac{F(1+x)}{x} < \infty$, together with (16), it comes

$$\begin{aligned} |\ln r|^\alpha F \left(\frac{1+N+|\ln r|}{|\ln r| - a - b|\ln r|^{1-\alpha}} \right) &\lesssim |\ln r|^\alpha \frac{1+b\frac{1}{\alpha} + b|\ln r|^{1-\alpha}}{|\ln r_\psi|} \\ &\lesssim \frac{|\ln r|^\alpha + |\ln r|^\alpha b\frac{1}{\alpha} + b|\ln r|}{|\ln r|} \\ &\lesssim 1 + b + b\frac{1}{\alpha} |\ln r|^{\alpha-1}. \end{aligned}$$

Since $r \in (0, r_t]$ and according to (14) we find

$$1 + b + b\frac{1}{\alpha} |\ln r|^{\alpha-1} \lesssim 1 + V(t)$$

and so

$$|\ln r|^\alpha F \left(\frac{1+N+|\ln r|}{|\ln r| - a - b|\ln r|^{1-\alpha}} \right) \lesssim (1 + V(t)).$$

It follows that

$$\text{II}_1 + \text{II}_2 \lesssim 1 + V(t) + \frac{|\ln r|^\alpha}{|\ln r| - a - b|\ln r|^{1-\alpha}}.$$

To estimate the last term we use (16)

$$\begin{aligned} \frac{|\ln r|^\alpha}{|\ln r| - a - b|\ln r|^{1-\alpha}} &= \frac{|\ln r|^\alpha}{|\ln r_\psi|} \\ &\leq |\ln r|^{\alpha-1} \\ &\lesssim 1. \end{aligned}$$

Finally, we get

$$(23) \quad \sup_{0 < r \leq r_t} \left(|\ln r|^\alpha \int_{\psi(B)} \left| f - \int_{\tilde{B}} f \right| + \mathcal{I}_r \right) \lesssim \|f\|_{L^{\alpha \text{ mo } F}} (1 + V(t)).$$

Let us now move to the second case.

Case 2: $r_t \leq r \leq \frac{1}{2}$.

According to (14)

$$(24) \quad \begin{aligned} |\ln r| &\leq |\ln r_t| \\ &\lesssim 1 + V_\alpha^{\frac{1}{\alpha}}(t) \end{aligned}$$

which yields in turn

$$(25) \quad |\ln r|^\alpha \int_B \left| f \circ \psi - \int_B f \circ \psi \right| \lesssim (1 + V(t)) \int_{\psi(B)} |f|.$$

Let \tilde{O}_j denote the ball which is concentric to O_j and whose radius is equal to $1/2$. We can write by the definitions,

$$\begin{aligned} \int_{\psi(B)} |f| &\leq \frac{1}{|B|} \sum_j |O_j| \int_{2O_j} |f - \int_{\tilde{O}_j} f| + \frac{1}{|B|} \sum_{j \in \mathbb{N}} |O_j| \int_{\tilde{O}_j} |f| \\ &\leq \frac{1}{|B|} \sum_j |O_j| \int_{2O_j} |f - \int_{\tilde{O}_j} f| + \sup_{|B|=1} \int_B |f| \\ &\leq \|f\|_{L^\alpha mo_F} \frac{1}{|B|} \sum_j |O_j| F(-\ln r_j) + \|f\|_{L^\alpha mo}. \end{aligned}$$

Now reproducing the same computations as for the first case leads to

$$\begin{aligned} \frac{1}{|B|} \sum_j |O_j| F(-\ln r_j) &\leq \frac{1}{|B|} \sum_{k \in \mathbb{N}} u_k F(k - \ln r) \\ &\lesssim F(N - \ln r) \left(1 + e^{-N} e^{b(N - \ln(r))^{1-\alpha}}\right). \end{aligned}$$

This computation still holds as soon as $e^{-N}r \leq \ell(t)$, since we use Proposition 6 for the scales $e^{-k}r$ with $k \geq N$. We choose N such that $e^{-N} e^{b(N - \ln(r))^{1-\alpha}} = 1$ and we check that this choice legitimates the previous estimate since $e^{-N}r \leq \ell(t)$. Consequently, we get from (22) and (24)

$$N + |\ln r| \lesssim 1 + V^{\frac{1}{\alpha}}(t).$$

We then obtain

$$(26) \quad \int_{\psi(B)} |f| \lesssim F(2 + V^{\frac{1}{\alpha}}(t)) \|f\|_{L^\alpha mo_F}$$

and

$$(27) \quad \begin{aligned} \mathcal{I}_r &\lesssim |\ln r|^\alpha \int_{\psi(B)} \left| f \circ \psi - \int_{\tilde{B}} f \right| \\ &\lesssim (1 + V(t)) F(2 + V^{\frac{1}{\alpha}}(t)) \|f\|_{L^\alpha mo_F}. \end{aligned}$$

Finally, we have obtained for $r \in [r_t, 1/2]$

$$\mathcal{I}_r \lesssim (1 + V(t)) F(2 + V^{\frac{1}{\alpha}}(t)) \|f\|_{L^\alpha mo_F}.$$

Putting together the estimates of the case 1 and the case 2 yields

$$(28) \quad \|f \circ \psi\|_{L^\alpha mo} \lesssim (1 + V(t)) F(2 + V^{\frac{1}{\alpha}}(t)) \|f\|_{L^\alpha mo_F}.$$

(2) To deal with the second term in the $L^\alpha mo_F$ -norm we will make use of the arguments developed above for $L^\alpha mo$ part. Take $B_2 = B(x_2, r_2)$ and $B_1 = B(x_1, r_1)$ two balls with $r_1 \leq 1$ and $2B_2 \subset B_1$ and let us see how to estimate the quantity

$$\mathcal{J} \triangleq \frac{|\int_{B_2} f \circ \psi - \int_{B_1} f \circ \psi|}{1 + F\left(\frac{\ln r_2}{\ln r_1}\right)}.$$

There are different cases to consider.

$$\textbf{Case 1: } r_t \leq r_1 \leq \frac{1}{2}.$$

Using (26)

$$\begin{aligned} \frac{|\int_{B_1} f \circ \psi|}{1 + F(\frac{\ln r_2}{\ln r_1})} &\leq \int_{\psi(B_1)} |f| \\ &\lesssim F(2 + V^{\frac{1}{\alpha}}(t)) \|f\|_{L^{\alpha} mo_F}. \end{aligned}$$

If $r_2 > r_t$ then by repeating the same arguments for the quantity involving B_2 , it comes

$$\mathcal{J} \lesssim F(2 + V^{\frac{1}{\alpha}}(t)) \|f\|_{L^{\alpha} mo_F}.$$

If $r_2 \leq r_t$ then we estimate the average on $\psi(B_2)$ by using (23) and (14)

$$\begin{aligned} \int_{\psi(B_2)} |f| &\leq \int_{\psi(B_2)} \left| f - \int_{\tilde{B}_2} f \right| + \left| \int_{\tilde{B}_2} f \right| \\ &\lesssim |\ln r_2|^{-\alpha} (1 + V(t)) \|f\|_{L^{\alpha} mo_F} + \left| \int_{\tilde{B}_2} f \right| \\ &\lesssim \|f\|_{L^{\alpha} mo_F} + \left| \int_{\tilde{B}_2} f \right|, \end{aligned}$$

where $\tilde{B}_i \triangleq B(\psi(x_i), r_{i,\psi})$, $i = 1, 2$ and $r_{i,\psi}$ is the radius associated to r_i , which was introduced in (15). It remains to treat the last term of the above inequality. For this goal we write

$$\begin{aligned} \left| \int_{\tilde{B}_2} f \right| &\lesssim \left| \int_{\tilde{B}_2} f - \int_{B(\psi(x_2), 1/2)} f \right| + \sup_{B, r=\frac{1}{2}} \left| \int_B f \right| \\ &\lesssim F(|\ln r_{2,\psi}|) \|f\|_{L^{\alpha} mo_F} + \|f\|_{L^{\alpha} mo}. \end{aligned}$$

This yields in view of (16) and the Definition 3

$$\begin{aligned} \frac{|\int_{B_2} f \circ \psi|}{1 + F(\frac{\ln r_2}{\ln r_1})} &\lesssim \left(1 + \frac{F(|\ln r_{2,\psi}|)}{1 + F(\frac{\ln r_2}{\ln r_1})} \right) \|f\|_{L^{\alpha} mo_F} \\ &\lesssim \left(1 + \frac{F(|\ln r_2|)}{1 + F(\frac{\ln r_2}{\ln r_1})} \right) \|f\|_{L^{\alpha} mo_F} \\ &\lesssim (1 + F(\ln r_1)) \|f\|_{L^{\alpha} mo_F}. \end{aligned}$$

Since $r_1 \in (r_t, \frac{1}{2})$ then using (14) we find

$$\frac{|\int_{B_2} f \circ \psi|}{1 + F(\frac{\ln r_2}{\ln r_1})} \lesssim F(2 + V^{\frac{1}{\alpha}}(t)) \|f\|_{L^{\alpha} mo_F}.$$

Finally we get for $r_2 \leq r_t$

$$\mathcal{J} \lesssim F(2 + V^{\frac{1}{\alpha}}(t)) \|f\|_{L^{\alpha} mo_F}.$$

To achieve the proof of the second part of Theorem 2, it remains to analyze the last case:

Case 2: $0 < r_1 \leq r_t$.

We decompose \mathcal{J} as follows:

$$\mathcal{J} = \frac{\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3}{1 + F(\frac{\ln r_2}{\ln r_1})},$$

with

$$\begin{aligned}\mathcal{J}_1 &\triangleq \left| \int_{\psi(B_2)} f - \int_{\tilde{B}_2} f \right| + \left| \int_{\psi(B_1)} f - \int_{\tilde{B}_1} f \right| \\ \mathcal{J}_2 &\triangleq \left| \int_{\tilde{B}_2} f - \int_{2\tilde{B}_1} f \right| \\ \mathcal{J}_3 &\triangleq \left| \int_{\tilde{B}_1} f - \int_{2\tilde{B}_1} f \right|.\end{aligned}$$

The first term \mathcal{J}_1 can be handled as for (23) and we get by (14)

$$\begin{aligned}\left| \int_{\psi(B_2)} f - \int_{\tilde{B}_2} f \right| + \left| \int_{\psi(B_1)} f - \int_{\tilde{B}_1} f \right| &\lesssim \|f\|_{L^\alpha mo_F} (1 + V(t)) (|\ln r_2|^{-\alpha} + |\ln r_1|^{-\alpha}) \\ &\lesssim \|f\|_{L^\alpha mo_F} (1 + V(t)) |\ln r_t|^{-\alpha} \\ &\lesssim \|f\|_{L^\alpha mo_F}\end{aligned}$$

which gives in turn

$$\frac{\mathcal{J}_1}{1 + F\left(\frac{\ln r_2}{\ln r_1}\right)} \lesssim \|f\|_{L^\alpha mo_F}.$$

Since $\tilde{B}_2 \subset 2\tilde{B}_1$ and $r_{2\tilde{B}_1} \leq \frac{1}{2}$, then

$$\mathcal{J}_2 \lesssim F\left(\frac{\ln r_{2,\psi}}{\ln r_{1,\psi}}\right) \|f\|_{L^\alpha mo_F}.$$

Hence we get from the property (2) of the Definition 3 combined with (16)

$$\begin{aligned}\frac{\mathcal{J}_2}{1 + F\left(\frac{\ln r_2}{\ln r_1}\right)} &\leq \frac{F\left(\frac{\ln r_{2,\psi}}{\ln r_{1,\psi}}\right)}{1 + F\left(\frac{\ln r_2}{\ln r_1}\right)} \|f\|_{L^\alpha mo_F} \\ &\lesssim \left(1 + F\left(\frac{\ln r_{2,\psi}}{\ln r_2} \frac{\ln r_1}{\ln r_{1,\psi}}\right)\right) \|f\|_{L^\alpha mo_F} \\ &\lesssim \|f\|_{L^\alpha mo_F}.\end{aligned}$$

Since \tilde{B}_1 and $2\tilde{B}_1$ are comparable and $r_{2\tilde{B}_1} \leq \frac{1}{2}$ we easily have

$$\frac{\mathcal{J}_3}{1 + F\left(\frac{\ln r_2}{\ln r_1}\right)} \lesssim \mathcal{J}_3 \lesssim \|f\|_{L^\alpha mo_F}.$$

The proof of Theorem 2 is now achieved. \square

3.2. Application to Euler equations. In this section we shall deal with the local and global well-posedness theory for the two dimensional Euler equations in the space $L^\alpha mo_F$. This project will be performed through the use of the logarithmic estimate developed in Theorem 2. We shall now state a more general result than Theorem 1.

Theorem 3. *Let $\omega_0 \in L^\alpha mo_F \cap L^p$ with $\alpha \in (0, 1)$ and $p \in (1, 2)$. Then,*

- (1) *If F belongs to the class \mathcal{A} , there exists $T > 0$ such that the system (1) admits a unique local solution*

$$\omega \in L^\infty([0, T]; L^\alpha mo_{1+F}).$$

(2) If F belongs to the class \mathcal{A}' , the system (1) admits a unique global solution

$$\omega \in L_{\text{loc}}^\infty(\mathbb{R}_+; L^\alpha m_{0_{1+F}}).$$

Proof. The proof is based on the establishment of the *a priori estimates* which are the cornerstone for the existence and the uniqueness parts. Here we omit the details about the existence and the uniqueness which are classical and some of their elements can be found for example in the paper [5].

(1) Using Theorem 2 one has

$$(29) \quad \|\omega(t)\|_{L^\alpha m_{0_{1+F}} \cap L^p} \lesssim \|\omega_0\|_{L^\alpha m_{0_F} \cap L^p} \left(1 + V(t)F(2 + V^{\frac{1}{\alpha}}(t))\right)$$

with $V(t) = \int_0^t \|u(\tau)\|_{L^{1-\alpha}L} d\tau$. Combining this estimate with Proposition 5 implies after integration in time

$$(30) \quad V(t) \lesssim \|\omega_0\|_{L^\alpha m_{0_F} \cap L^p} \left(t + \int_0^t V(\tau)F(2 + V^{\frac{1}{\alpha}}(\tau))d\tau\right).$$

According to the Remark 3 the function F has at most a polynomial growth: $F(2+x) \lesssim 1+x^\beta$. Therefore

$$V(t) \lesssim \|\omega_0\|_{L^\alpha m_{0_F} \cap L^p} \left(t + tV(t) + tV^{1+\frac{\beta}{\alpha}}(t)\right)$$

and consequently we can find $T \triangleq T(\|\omega_0\|_{L^\alpha m_{0_F} \cap L^p}) > 0$ such that

$$\forall t \in [0, T], \quad V(t) \leq 1.$$

Plugging this estimate into (29) gives

$$\|\omega(t)\|_{L^\alpha m_{0_{1+F}} \cap L^p} \lesssim \|\omega_0\|_{L^\alpha m_{0_F} \cap L^p}.$$

Now from Theorem 2 we get also

$$\|\omega(t)\|_{L^\alpha m_{0_{1+F}} \cap L^p} \lesssim \|\omega_0\|_{L^\alpha m_{0_F} \cap L^p}.$$

(2) Fix $T > 0$ an arbitrary number, then from (30) we deduce

$$\forall t \in [0, T], \quad V(t) \leq C\|\omega_0\|_{L^\alpha m_{0_F} \cap L^p}T + C\|\omega_0\|_{L^\alpha m_{0_F} \cap L^p} \left(\int_0^t V(\tau)F(2 + V^{\frac{1}{\alpha}}(\tau))d\tau\right)$$

and introduce the function $\mathcal{M} : [a, +\infty[\rightarrow [0, +\infty[$ defined by

$$\mathcal{M}(y) = \int_a^y \frac{1}{x F(2 + x^{\frac{1}{\alpha}})} dx, \quad a = \inf(A_T, 1) \quad A_T = C\|\omega_0\|_{L^\alpha m_{0_F} \cap L^p} T.$$

Since F belongs to the class \mathcal{A}' we can easily check that

$$\int_{A_T}^{+\infty} \frac{1}{x F(2 + x^{\frac{1}{\alpha}})} dx = +\infty.$$

Therefore applying Lemma 2

$$\forall t \in [0, T], \quad V(t) \leq \mathcal{M}^{-1}\left(\mathcal{M}(A_T) + C\|f\|_{L^\alpha m_{0_F}} t\right).$$

This gives the global a priori estimates

$$\forall t \geq 0, \quad V(t) \leq \mathcal{M}^{-1}\left(\mathcal{M}(A_t) + C\|f\|_{L^\alpha m_{0_F}} t\right).$$

Inserting this estimate into (29) allows to get a global estimate for vorticity. Hence, there exists a continuous function $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ related to \mathcal{M} such that

$$(31) \quad \|\omega(t)\|_{L^\alpha m_0 \cap L^p} \leq G(t).$$

According to Theorem 2 and the preceding estimate

$$\|\omega(t)\|_{L^\alpha m_0 + F} \leq G(t).$$

This concludes the a priori estimates. \square

APPENDIX A. TECHNICAL LEMMATA

We will prove the following lemma used before in the inequality (20).

Lemma 3. *There exists a universal implicit constant such that for $r \in (0, r_t]$ and for $k \in \mathbb{N}$,*

$$u_k \lesssim \delta r^2 e^{-k} e^{\delta(V(t)+1)(k-\ln(r))^{1-\alpha}}.$$

Proof. If we denote by $c_0 \geq 1$ the implicit constant appearing in Whitney Lemma, then

$$u_k \leq \left| \left\{ y \in \psi(B) \setminus d(y, \psi(B)^c) \leq c_0 e^{-k} r \right\} \right|.$$

The preservation of Lebesgue measure by ψ yields

$$\left| \left\{ y \in \psi(B) \setminus d(y, \psi(B)^c) \leq c_0 e^{-k} r \right\} \right| = \left| \left\{ x \in B \setminus d(\psi(x), \psi(B)^c) \leq c_0 e^{-k} r \right\} \right|.$$

Since $\psi(B)^c = \psi(B^c)$ then

$$u_k \leq \left| \left\{ x \in B \setminus d(\psi(x), \psi(B^c)) \leq c_0 e^{-k} r \right\} \right|.$$

We set

$$D_k = \left\{ x \in B \setminus d(\psi(x), \psi(B^c)) \leq c_0 e^{-k} r \right\}.$$

Since $\psi(\partial B)$ is the frontier of $\psi(B)$ and $d(\psi(x), \psi(B^c)) = d(\psi(x), \partial\psi(B))$ then

$$D_k \subset \left\{ x \in B \setminus \exists y \in \partial B \text{ with } |\psi(x) - \psi(y)| \leq c_0 e^{-k} r \right\}.$$

The regularity of ψ^{-1} (Proposition 6) implies since $c_0 e^{-k} r \leq c_0 r_t \lesssim \ell(t)$

$$D_k \subset \left\{ x \in B \setminus \exists y \in \partial B : |x - y| \leq c_0 e^{-k} r e^{\delta(V(t)+1)(k-\ln(r))^{1-\alpha}} \right\}.$$

Here we choose δ large enough such that $\delta > \ln(c_0)$ and $\delta > c_0$. Thus, D_k is contained in the annulus

$$\mathcal{A} = \left\{ x \in B \setminus d(x, \partial B) \leq \delta e^{-k} r e^{\delta(V(t)+1)(k-\ln(r))^{1-\alpha}} \right\}$$

and so (since we are in dimension 2)

$$u_k \leq |D_k| \lesssim \delta r^2 e^{-k} e^{\delta(V(t)+1)(k-\ln(r))^{1-\alpha}},$$

as claimed. \square

We conclude this paper by the following result which has been used in several places.

Lemma 4. Let $\alpha \in]0, 1[$, $A, B, C, D > 0$ and $F : [1, +\infty[\rightarrow \mathbb{R}_+$ be a differentiable nondecreasing function such that

$$C > D \quad \text{and} \quad \|F'\|_{L^\infty} \triangleq M < +\infty.$$

Consider the sequence

$$w_n \triangleq \sum_{k \geq n} e^{-k+A(k+B)^{1-\alpha}} F\left(\frac{k+C}{D}\right).$$

Assume that

$$\frac{A(1-\alpha)}{(n+B)^\alpha} \leq \frac{1}{4},$$

then

$$w_n \leq 4e^{-n+A(n+B)^{1-\alpha}} F\left(\frac{n+C}{D}\right) + \frac{16M}{D} e^{-n+A(n+B)^{1-\alpha}}.$$

Proof. Let $R_n := \sum_{k \geq n} e^{-k}$ and $v_n := e^{A(n+B)^{1-\alpha}} F\left(\frac{n+C}{D}\right)$. According to Abel's formula

$$w_n = R_n v_n + \sum_{k \geq n+1} R_k (v_k - v_{k-1}).$$

It is clear that $0 < R_n \leq \frac{e^{-n}}{1-e^{-1}}$ and

$$w_n \leq \frac{1}{1-e^{-1}} e^{-n+A(n+B)^{1-\alpha}} F\left(\frac{n+C}{D}\right) + \frac{1}{1-e^{-1}} \sum_{k \geq n+1} e^{-k} |v_k - v_{k-1}|.$$

It remains to estimate the last sum. For this purpose we use the mean value theorem combined with the nondecreasing property of F

$$|v_k - v_{k-1}| \leq \frac{A(1-\alpha)}{(k-1+B)^\alpha} e^{A(k+B)^{1-\alpha}} F\left(\frac{k+C}{D}\right) + \frac{M}{D} e^{A(k+B)^{1-\alpha}}.$$

Consequently

$$\begin{aligned} \sum_{k \geq n+1} R_k |v_k - v_{k-1}| &\leq \frac{A(1-\alpha)}{(1-e^{-1})(n+B)^\alpha} w_{n+1} + \frac{M}{(1-e^{-1})D} \sum_{k \geq n+1} e^{-k+A(k+B)^{1-\alpha}} \\ &\leq \frac{A(1-\alpha)}{(1-e^{-1})(n+B)^\alpha} w_n + \frac{M}{(1-e^{-1})D} \sum_{k \geq n} e^{-k+A(k+B)^{1-\alpha}}. \end{aligned}$$

This leads to

$$w_n \leq \frac{1}{1-e^{-1}} e^{-n+A(n+B)^{1-\alpha}} F\left(\frac{n+C}{D}\right) + \frac{A(1-\alpha)}{(1-e^{-1})(n+B)^\alpha} w_n + \frac{M}{(1-e^{-1})D} \sum_{k \geq n} e^{-k+A(k+B)^{1-\alpha}}.$$

Reproducing the same computations with $F(x) = 1$, $M = 0$, we get

$$\sum_{k \geq n} e^{-k+A(k+B)^{1-\alpha}} \leq \frac{1}{1-e^{-1}} e^{-n+A(n+B)^{1-\alpha}} + \frac{A(1-\alpha)}{(1-e^{-1})(n+B)^\alpha} \sum_{k \geq n} e^{-k+A(k+B)^{1-\alpha}}.$$

Assuming that

$$\frac{A(1-\alpha)}{(1-e^{-1})(n+B)^\alpha} \leq \frac{1}{2},$$

we get

$$\sum_{k \geq n} e^{-k+A(k+B)^{1-\alpha}} \leq \frac{2}{1-e^{-1}} e^{-n+A(n+B)^{1-\alpha}}$$

and

$$w_n \leq \frac{2}{1-e^{-1}} e^{-n+A(n+B)^{1-\alpha}} F\left(\frac{n+C}{D}\right) + \frac{4M}{(1-e^{-1})^2 D} e^{-n+A(n+B)^{1-\alpha}}.$$

Since $\frac{1}{1-e^{-1}} \leq 2$ we deduce

$$w_n \leq 4e^{-n+A(n+B)^{1-\alpha}} F\left(\frac{n+C}{D}\right) + \frac{16M}{D} e^{-n+A(n+B)^{1-\alpha}}.$$

□

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CNRS - UNIVERSITÉ DE NANTES, LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, 2, RUE DE LA HOUSSINIÈRE F-44322 NANTES CEDEX 03, FRANCE

E-mail address: frederic.bernicot@univ-nantes.fr

IRMAR, UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, 35 042 RENNES CEDEX, FRANCE

E-mail address: thmidi@univ-rennes1.fr