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ABSTRACT. The equivalence between the inequalities of Babuška–Aziz and Friedrichs for sufficiently smooth bounded domains in the plane has been shown by Horgan and Payne 30 years ago. We prove that this equivalence, and the equality between the associated constants, is true without any regularity condition on the domain. For the Horgan–Payne inequality, which is an upper bound of the Friedrichs constant for plane star-shaped domains in terms of a geometric quantity known as the Horgan–Payne angle, we show that it is true for some classes of domains, but not for all bounded star-shaped domains. We prove a weaker inequality that is true in all cases.

1. INTRODUCTION

In 1983, Horgan and Payne published a paper [12] that has since become a classical reference, in which they proved equivalence of three inequalities pertaining to plane domains: the Korn inequality from linear elasticity, the Friedrichs inequality for conjugate harmonic functions, and the Babuška–Aziz inequality that quantifies the inf-sup condition for the divergence. After finding equations between the constants in these inequalities, they estimate the constant in the Friedrichs inequality for star-shaped domains. The estimate involves the minimal angle between the radius vector and the tangent on the boundary, later sometimes called "Horgan–Payne angle" [21].

The present paper evolved from trying to understand the precise hypotheses on the domain that are needed for the proofs in the paper [12]. On one hand, in [12] it is said that "we assume that the domain is simply-connected, with C^1 boundary. It will be clear from our arguments that the results hold for simply-connected Lipschitz domains." Some of the proofs use even higher regularity, however. On the other hand, recently the Babuška–Aziz inequality has been proved [1] for the class of John domains, which is a larger class than Lipschitz domains, including unions of Lipschitz domains, weakly Lipschitz domains, and even some domains with a fractal boundary. It is therefore desirable to know whether the equivalence between the inequalities of Friedrichs and of Babuška–Aziz persists for this larger class of domains. We show that, indeed, this equivalence holds without any regularity assumption on the domain.

For star-shaped domains, we prove that the Horgan–Payne estimate of the Friedrichs constant holds for some domains, including all triangles, rectangles and regular polygons, but that to be true in general, it has to be replaced by a more complicated estimate. We give a counterexample of a domain for which the Horgan–Payne estimate is not true. Finally,

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using the approach of Horgan and Payne, we obtain, for the case of plane star-shaped domains, an improvement of the Babuška–Aziz inequality shown by Durán [10] for bounded domains in any dimension.

2. The inequalities

2.1. **Notation.** Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$. Thus we will assume throughout that Ω is bounded and connected, but we will not impose any a-priori regularity hypothesis. We use the standard definitions of the space of square integrable functions $L^2(\Omega)$ and of the Sobolev space $H^1(\Omega)$. The norm and scalar product in $L^2(\Omega)$ will be denoted by $\|\cdot\|_{0,\Omega}$ and $\langle\cdot,\cdot\rangle_\Omega$. We will need the subspace of functions of mean value zero

$$L^2_\circ(\Omega) = \Big\{ q \in L^2(\Omega) \mid \int_\Omega q(\boldsymbol{x})\, d\boldsymbol{x} = 0 \Big\}.$$

The space $H^1_0(\Omega)$ is the closure of $\mathscr{C}^\infty_0(\Omega)$ in the norm of $H^1(\Omega)$. On account of the Poincaré inequality, the H^1 seminorm is a norm on $H^1_0(\Omega)$, which we will denote by $|\cdot|_{1,\Omega}$. The dual space of $H^1_0(\Omega)$ with $L^2(\Omega)$ as pivot space is $H^{-1}(\Omega)$. The dual norm to $|\cdot|_{1,\Omega}$ is $\|\cdot\|_{-1,\Omega}$, and the duality is again denoted by $\langle\cdot,\cdot\rangle_\Omega$. We will also use the natural extension of these notations to vector functions, so that for instance for $\boldsymbol{v} = (v_1, \ldots, v_d) \in H^1_0(\Omega)^d$

$$|\boldsymbol{v}|_{1,\Omega} = \|\,\mathbf{grad}\,\boldsymbol{v}\,\|_{0,\Omega} = \Big(\sum_{k=1}^{d} \sum_{j=1}^{d} \|\partial_{x_j} v_k\|^2_{0,\Omega} \Big)^{1/2}.$$

If no misunderstanding is possible, we will simply write $\|\cdot\|_0$ and $|\cdot|_1$ for $\|\cdot\|_{0,\Omega}$ and $|\cdot|_{1,\Omega}$. Most of the discussion of this paper will concern plane domains, but one of the new technical tools proved later on (see Lemma 5.4) will be valid for any dimension $d \geq 2$.

2.2. **The Babuška–Aziz inequality.** In [2, Lemma 5.4.3, p. 172], Babuška and Aziz prove the existence of a finite constant C such that for any $q \in L^2_\circ(\Omega)$ there exists a solution $\boldsymbol{u} \in H^1_0(\Omega)^2$ of the equation

$$\operatorname{div} \boldsymbol{u} = q$$

satisfying the estimate

$$|\boldsymbol{u}|^2_{1,\Omega} \leq C\, \|q\|^2_{0,\Omega}. \tag{2.1}$$

Following [12], we call (2.1) the *Babuška–Aziz inequality* and the smallest possible constant C in (2.1), which we will denote by $C(\Omega)$, the *Babuška–Aziz constant* of the domain Ω. Babuška and Aziz give a proof for bounded smooth domains in dimension $d = 2$, and Horgan and Payne formulate the result for bounded Lipschitz domains [12, Theorem 1].

Estimates such as (2.1) for smooth domains have been shown as early as 1961 by Cattabriga [4] in the the context of boundary value problems for the Stokes system, using even L^p norms with $p \neq 2$.

Applying duality and basic Hilbert space theory, one finds the well known [3] equivalence between the Babuška–Aziz inequality and the a-priori estimate for the gradient with a constant $\beta > 0$

$$\forall q \in L^2_\circ(\Omega) \ : \quad \|\,\mathbf{grad}\, q\|_{-1,\Omega} \geq \beta \|q\|_{0,\Omega}, \tag{2.2}$$

as well as the *inf-sup condition* $\beta(\Omega) > 0$, where

$$\beta(\Omega) = \inf_{q \in L^2_\diamond(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^2} \frac{\big\langle \operatorname{div} \boldsymbol{v}, q \big\rangle_\Omega}{|\boldsymbol{v}|_{1,\Omega} \, \|q\|_{0,\Omega}} \,. \tag{2.3}$$

The relation between the *inf-sup constant* $\beta(\Omega)$, which is also the best possible constant in (2.2), and the Babuška–Aziz constant is

$$C(\Omega) = \frac{1}{\beta(\Omega)^2} \,. \tag{2.4}$$

The gradient estimate (2.2) is one of the standard tools in the proof of the Korn inequality and has been proved in this context for bounded Lipschitz domains in any dimension by Nečas [18, Chap. 3, Lemme 3.7.1]. It is sometimes associated with the name of Lions, see [5] and [14, Note (27) p. 320].

The inf-sup condition plays an important role for the pressure stability in hydrodynamics [8], for the rate of convergence of iterative methods such as the Uzawa algorithm [6, 21] and, in a discrete version, for the finite element approximation of the Stokes equation. In the context of mixed variational formulations and their approximation this has been explored since Brezzi's fundamental paper [3]. In this context, it is often referred to as Babuška–Brezzi or *Ladyzhenskaya-Babuška-Brezzi* condition and the inf-sup constant $\beta(\Omega)$ as *LBB constant*, see [15, 19] and many later references. In the paper [13], Ladyzhenskaya and Solonnikov discuss the validity of this estimate — but in the form of the Babuška–Aziz estimate (2.1) — for a class of domains larger than the class of Lipschitz domains.

In a series of recent papers, Durán, Muschietti and coauthors extended the validity of the inf-sup condition to the class of John domains, which contains among others finite unions of bounded Lipschitz domains, weakly Lipschitz domains, and even some domains with fractal boundary, see [1, 9, 10].

2.3. **The Friedrichs inequality.** The Friedrichs inequality is an L^2 estimate between conjugate harmonic functions in dimension $d = 2$. Friedrichs proved it in [11] for a class of piecewise smooth domains and discussed its relation with the Cosserat eigenvalue problem of plane elasticity theory and the Korn inequality. It can be formulated using holomorphic functions in Ω, where $\mathbb{R}^2$ is identified with the complex plane. If w is holomorphic in Ω, $w = h + ig$ with real-valued h and g, then h and g are conjugate harmonic functions and satisfy $\mathbf{grad}\, h = \mathbf{curl}\, g$. One considers the space $\mathfrak{F}_\circ(\Omega)$ of complex valued holomorphic functions that are square integrable on Ω and of mean value zero.

The *Friedrichs inequality* is satisfied for Ω if there is a finite constant Γ such that for all $h + ig \in \mathfrak{F}_\circ(\Omega)$

$$\|h\|^2_{0,\Omega} \le \Gamma \, \|g\|^2_{0,\Omega} \,. \tag{2.5}$$

The smallest possible constant is the *Friedrichs constant* of the domain and will be denoted by $\Gamma(\Omega)$. Friedrichs also gave a counter-example of a domain with an exterior cusp for which $\Gamma(\Omega)$ is infinite.

2.4. **The Horgan–Payne inequality.** Whereas we followed Horgan–Payne [12] for the naming of the inequalities of Babuška–Aziz and Friedrichs, we will now introduce an inequality that appears in [12], but has not so far been named, as far as we know. It involves a geometric quantity $\omega(\Omega)$ that has been called *Horgan–Payne angle* [21]. This angle is defined for a domain $\Omega \subset \mathbb{R}^2$ that is star-shaped with respect to a ball with center $\boldsymbol{x}_0$. In this case, the boundary is Lipschitz continuous, has a tangent almost everywhere, and the ray from $\boldsymbol{x}_0$ passing through $\boldsymbol{x} \in \partial\Omega$ has a positive angle $\omega(\boldsymbol{x}) \leq \frac{\pi}{2}$ with the tangent. The quantity

$$\omega(\Omega) = \inf_{\boldsymbol{x}\in\partial\Omega} \omega(\boldsymbol{x}) \tag{2.6}$$

is also strictly positive. Note that $\omega(\Omega)$ depends not only on the domain Ω, but also on the center $\boldsymbol{x}_0$.

The *Horgan–Payne inequality* is the estimate for the inf-sup constant

$$\beta(\Omega) \geq \sin\frac{\omega(\Omega)}{2}\,. \tag{2.7}$$

In [12, Eq. (6.29)], this inequality is formulated as an estimate for the Friedrichs constant

$$\Gamma(\Omega) \leq \sup_{\boldsymbol{x}\in\partial\Omega} \left(\frac{1}{\cos\gamma(\boldsymbol{x})} + \sqrt{\frac{1}{\cos^2\gamma(\boldsymbol{x})} - 1} \right)^2 \tag{2.8}$$

where $\gamma(\boldsymbol{x}) = \frac{\pi}{2} - \omega(\boldsymbol{x})$ is the positive angle between the ray from $\boldsymbol{x}_0$ passing through $\boldsymbol{x} \in \partial\Omega$ and the normal in $\boldsymbol{x}$. In view of the relation $C(\Omega) = \Gamma(\Omega) + 1$, see Theorem 2.1 below, and (2.4), the estimates (2.8) and (2.7) are equivalent, see also [21, Lemma 1].

2.5. **The main results.** We can now formulate the main results of this paper. The remaining sections will be devoted to the proofs, including some technical lemmas that may be of independent interest, and some examples and counter-examples.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then the Babuška–Aziz constant $C(\Omega)$ is finite if and only if the Friedrichs constant $\Gamma(\Omega)$ is finite, and*

$$C(\Omega) = \Gamma(\Omega) + 1\,. \tag{2.9}$$

This identity was proved in [12, Sec. 5] under additional hypotheses. The proof there requires that Ω is simply connected and satisfies some implicit regularity assumption that amounts basically to C^2 regularity. In Section 3, we give a different proof that does not need any assumptions on Ω. As a corollary we obtain that the Friedrichs inequality is true for the same class of domains as the inf-sup condition for the divergence, in particular for John domains.

In Section 4 we revisit the proof of Horgan–Payne [12, Section 6] and prove that it gives a weaker, more complicated estimate than (2.8), less amenable to a simple geometric interpretation. Nevertheless we show in Section 5 that for a collection of simple plane domains the two estimates coincide. In the opposite direction, we prove an upper bound for the inf-sup constant for domains allowing a "small cut", Lemma 5.4. This can be used to disprove the Horgan–Payne inequality for some domains. Such domains can be constructed from logarithmic spirals, or from segments and circular arcs, or even as polygons.

Theorem 2.2. *(i) Let $\Omega \subset \mathbb{R}^2$ be any triangle, rectangle, rhombus or regular polygon. Then with respect to its barycenter, the Horgan–Payne inequality holds. (ii) There exist domains $\Omega \subset \mathbb{R}^2$ star-shaped with respect to a ball such that the Horgan–Payne inequality (2.7) is not satisfied.*

In Section 6, we use the idea of Horgan–Payne's proof of their inequality to obtain an explicit lower bound of the inf-sup constant for star-shaped domains.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a domain contained in a ball of radius R, star-shaped with respect to a concentric ball of radius ρ. Then*

$$\beta(\Omega) \geq \frac{\rho}{\sqrt{2}R} \left(1 + \sqrt{1 - \frac{\rho^2}{R^2}}\right)^{-\frac{1}{2}} \geq \frac{\rho}{2R}\,. \tag{2.10}$$

This estimate improves a recent result of Durán [10] for the case of dimension 2, where the bound from below has the form (see [10, Remark 3.1])

$$\beta(\Omega) \geq c\,\frac{\rho}{R}\left|\log\frac{\rho}{R}\right|^{-1}.$$

Inequality (2.10) takes a form like (2.7) if we introduce the angles $\tau(\Omega) = \arccos\frac{\rho}{R}$ with best possible (ρ, R), and $\psi(\Omega) = \frac{\pi}{2} - \tau(\Omega)$, as replacement of $\gamma(\Omega)$ and $\omega(\Omega)$: There holds $\beta(\Omega) \geq \sin\frac{\psi(\Omega)}{2}$. In contrast with the Horgan–Payne angle, the angle $\psi(\Omega)$ has a global nature.

3. Equivalence between Babuška–Aziz and Friedrichs

In this section, we prove Theorem 2.1. The proof is divided into two parts.

(i) In a first step, we assume that Ω is a domain in $\mathbb{R}^2$ such that $C(\Omega)$ is finite. We will show that then $\Gamma(\Omega)$ is finite and

$$\Gamma(\Omega) \leq C(\Omega) - 1. \tag{3.1}$$

This part of the proof is basically the same as in [12].

Let $h + ig \in \mathfrak{F}_\circ(\Omega)$. Thus h and g are conjugate harmonic functions in $L^2_\circ(\Omega)$, satisfying

$$\Delta h = 0, \quad \Delta g = 0, \quad \text{and} \quad \mathbf{grad}\, h = \mathbf{curl}\, g \quad \text{in} \quad \Omega.$$

Here $\mathbf{curl}\, g = (\partial_{x_2} g, -\partial_{x_1} g)$. The adjoint of the **curl** operator is the scalar curl: $\operatorname{curl} \boldsymbol{u} = \partial_{x_1} u_2 - \partial_{x_2} u_1$. It follows from integration by parts for all $\boldsymbol{u} \in H^1_0(\Omega)^2$

$$|\boldsymbol{u}|_1^2 = \|\operatorname{div}\boldsymbol{u}\|_0^2 + \|\operatorname{curl}\boldsymbol{u}\|_0^2\,. \tag{3.2}$$

Note that no regularity for Ω is needed here: One has (3.2) first on $\mathscr{C}_0^\infty(\Omega)^2$ and then by continuity on $H^1_0(\Omega)^2$.

From the Babuška–Aziz inequality we get the existence of $\boldsymbol{u} \in H^1_0(\Omega)^2$ such that

$$\operatorname{div}\boldsymbol{u} = h \quad \text{and} \quad \|\operatorname{curl}\boldsymbol{u}\|_0^2 = |\boldsymbol{u}|_1^2 - \|\operatorname{div}\boldsymbol{u}\|_0^2 \leq (C(\Omega) - 1)\,\|h\|_0^2\,.$$

We find

$$\|h\|_0^2 = \langle h, \operatorname{div}\boldsymbol{u}\rangle_\Omega = -\langle \mathbf{grad}\, h, \boldsymbol{u}\rangle_\Omega = -\langle \mathbf{curl}\, g, \boldsymbol{u}\rangle_\Omega = -\langle g, \operatorname{curl}\boldsymbol{u}\rangle_\Omega\,.$$

With the Cauchy-Schwarz inequality and the estimate of $\operatorname{curl} \boldsymbol{u}$, we deduce

$$\|h\|_0^2 \le \sqrt{C(\Omega)-1}\,\|h\|_0\,\|g\|_0\,,$$

hence the estimate

$$\|h\|_0^2 \le (C(\Omega)-1)\,\|g\|_0^2\,,$$

which proves (3.1).

(ii) In a second step, we assume that Ω is such that $\Gamma(\Omega)$ is finite. We will show that $C(\Omega)$ is finite and

$$C(\Omega) \le \Gamma(\Omega) + 1. \tag{3.3}$$

This part of our proof is different from the one given in [12].

Let $p \in L^2_\circ(\Omega)$ be given and define $\boldsymbol{u} \in H^1_0(\Omega)^2$ as the solution of $\Delta \boldsymbol{u} = \mathbf{grad}\, p$, that is $\boldsymbol{u}$ satisfies

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^2 : \ \big\langle \mathbf{grad}\, \boldsymbol{u}, \mathbf{grad}\, \boldsymbol{v} \big\rangle_\Omega = \big\langle p, \operatorname{div} \boldsymbol{v} \big\rangle_\Omega\,. \tag{3.4}$$

We set $q = \operatorname{div} \boldsymbol{u}$ and $g = \operatorname{curl} \boldsymbol{u}$ and observe the following relations as consequences of (3.4):

$$\langle p, q\rangle_\Omega = |\boldsymbol{u}|_1^2 \qquad = \|q\|_0^2 + \|g\|_0^2 \tag{3.5}$$

$$\Delta q = \operatorname{div} \Delta \boldsymbol{u} \qquad = \Delta p \tag{3.6}$$

$$\Delta g = \operatorname{curl} \Delta \boldsymbol{u} \qquad = 0 \tag{3.7}$$

$$\mathbf{curl}\, g - \mathbf{grad}\, q = -\Delta \boldsymbol{u} \qquad = -\,\mathbf{grad}\, p \tag{3.8}$$

$$\|g\|_0^2 = \langle p, q\rangle_\Omega - \|q\|_0^2 = \langle q, p-q\rangle_\Omega\,. \tag{3.9}$$

It follows that g and $q-p$ are conjugate harmonic functions. Note that both belong to $L^2_\circ(\Omega)$, so that we can use the Friedrichs inequality:

$$\|p-q\|_0^2 \le \Gamma(\Omega)\,\|g\|_0^2\,. \tag{3.10}$$

Then we have with (3.9)

$$\|g\|_0^2 \le \|q\|_0 \|p-q\|_0 \le \|q\|_0 \sqrt{\Gamma(\Omega)}\|g\|_0\,,$$

hence

$$\|g\|_0^2 \le \Gamma(\Omega)\|q\|_0^2\,. \tag{3.11}$$

Now we estimate, using (3.5) and both (3.10) and (3.11):

$$\begin{aligned}
\|p\|_0^2 &= \|p-q\|_0^2 - \|q\|_0^2 + 2\langle p, q\rangle_\Omega \\
&= \|p-q\|_0^2 + \|g\|_0^2 + \|q\|_0^2 + \|g\|_0^2 \\
&\le \Gamma(\Omega)\|g\|_0^2 + \Gamma(\Omega)\|q\|_0^2 + \|q\|_0^2 + \|g\|_0^2 \\
&= \big(\Gamma(\Omega)+1\big)\,|\boldsymbol{u}|_1^2\,.
\end{aligned}$$

Now (3.4) shows that the Laplacian is an isometry from $H^1_0(\Omega)^2$ to $H^{-1}(\Omega)^2$, and $|\boldsymbol{u}|_1 = \|\,\mathbf{grad}\, p\|_{-1}$. This gives the estimate

$$\|p\|_0^2 \le \big(\Gamma(\Omega)+1\big)\,\|\,\mathbf{grad}\, p\|_{-1}^2\,,$$

which is the dual or "Lions" version (2.2) of the Babuška–Aziz inequality. Together with (2.4) this gives the desired inequality (3.3).

Theorem 2.1 is proved.

4. Strictly star-shaped domains and the Horgan–Payne inequality

We say that Ω is *strictly star-shaped* if there is an open ball $B \subset \Omega$ such that any segment with one end in B and the other in Ω, is contained in Ω. Let $\boldsymbol{x}_0$ be the center of B and (r, θ) be polar coordinates centered at $\boldsymbol{x}_0$. Let $\theta \mapsto r = f(\theta)$ be the polar parametrization of the boundary $\partial\Omega$, defined on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Then f is Lipschitz continuous in virtue of a result by Maz'ya [16, Lemma 1.1.8].

In this section we follow [12, §6] to construct an upper bound for $\Gamma(\Omega)$ depending on the values of f and its first derivative f' only. Since $\Gamma(\Omega)$ is invariant by dilation, we may normalize f by the condition

$$\boxed{\max_{\theta\in\mathbb{T}} f(\theta) = 1} \tag{4.1}$$

We introduce $P = P(\alpha, \theta)$ as the function defined on $\mathbb{R}_+ \times \mathbb{T}$ by

$$P(\alpha,\theta) = \frac{1}{\alpha f(\theta)^2}\left(1 + \frac{f'(\theta)^2}{f(\theta)^2 - \alpha f(\theta)^4}\right). \tag{4.2}$$

We denote by $M(\Omega)$ and $m(\Omega)$ the following candidates for an upper bound:

Notation 4.1. Under condition (4.1), let $M(\Omega)$ and $m(\Omega)$ be the following two positive numbers

$$M(\Omega) = \inf_{\alpha\in(0,1)} \left\{ \sup_{\theta\in\mathbb{T}} P(\alpha,\theta) \right\} \quad \text{and} \quad m(\Omega) = \sup_{\theta\in\mathbb{T}} \left\{ \inf_{\alpha\in\left(0,\frac{1}{f(\theta)^2}\right)} P(\alpha,\theta) \right\}. \tag{4.3}$$

Note that, unlike $\Gamma(\Omega)$, the quantities $M(\Omega)$ and $m(\Omega)$ depend on the choice of the origin $\boldsymbol{x}_0$ of polar coordinates chosen to parametrize the boundary.

Lemma 4.2. *For any strictly star-shaped domain Ω with center $\boldsymbol{x}_0$, there holds*

$$M(\Omega) \geq m(\Omega). \tag{4.4}$$

Proof. Let us choose $\theta \in \mathbb{T}$ and define P_θ as the function $\alpha \mapsto P(\alpha, \theta)$ for $\alpha \in \left(0, \frac{1}{f(\theta)^2}\right)$. Calculating the second derivative of P_θ, we find that P_θ is strictly convex. The function P_θ tends to $+\infty$ as $\alpha \to 0$, and if $f'(\theta) \neq 0$, as $\alpha \to \frac{1}{f(\theta)^2}$.
In any case, there exists a unique $\alpha(\theta)$ in $\left(0, \frac{1}{f(\theta)^2}\right]$ such that $P(\alpha(\theta), \theta)$ coincides with $\inf_{\alpha\in(0,\frac{1}{f(\theta)^2})} P(\alpha, \theta)$. So,

$$m(\Omega) = \sup_{\theta\in\mathbb{T}} P(\alpha(\theta),\theta)\,. \tag{4.5}$$

Since, in particular, for all $\alpha \in (0, 1)$ and $\theta \in \mathbb{T}$, $P(\alpha(\theta), \theta) \leq P(\alpha, \theta)$, we find (4.4). □

The quantity $m(\Omega)$ is the original bound introduced by Horgan–Payne in [12], cf. (2.8):

Lemma 4.3. *For any strictly star-shaped domain Ω with center $\boldsymbol{x}_0$, there holds*

$$m(\Omega) = \sup_{\boldsymbol{x}\in\partial\Omega} \left(\frac{1}{\cos\gamma(\boldsymbol{x})} + \sqrt{\frac{1}{\cos^2\gamma(\boldsymbol{x})} - 1} \right)^2 \tag{4.6}$$

where we recall that $\gamma(\boldsymbol{x})$ is the angle between the ray $[\boldsymbol{x}_0, \boldsymbol{x}]$ and the normal at $\partial\Omega$ in $\boldsymbol{x}$.

Proof. To prove the lemma, relying on (4.5), it suffices to establish that for any $\theta \in \mathbb{T}$

$$P(\alpha(\theta), \theta) = \left(\frac{1}{\cos\gamma(\boldsymbol{x})} + \sqrt{\frac{1}{\cos^2\gamma(\boldsymbol{x})} - 1} \right)^2, \tag{4.7}$$

where $\boldsymbol{x} = \boldsymbol{x}_0 + (f(\theta)\cos\theta, f(\theta)\sin\theta)$. For this we calculate the value $\alpha(\theta)$ which realizes the minimum of $P(\alpha, \theta)$ for $\alpha \in (0, 1/f(\theta)^2]$: Setting

$$t(\theta) = \frac{f'(\theta)}{f(\theta)}$$

we find

$$P(\alpha, \theta) = \frac{1}{\alpha f(\theta)^2} \left(1 + \frac{t(\theta)^2}{1 - \alpha f(\theta)^2} \right)$$

and

$$\partial_\alpha P(\alpha, \theta) = -\frac{1}{\alpha^2 f(\theta)^2} \left(1 + \frac{t(\theta)^2}{1 - \alpha f(\theta)^2} \right) + \frac{1}{\alpha f(\theta)^2} \frac{t(\theta)^2 f(\theta)^2}{(1 - \alpha f(\theta)^2)^2}\,.$$

Setting $\zeta = \alpha f(\theta)^2$, we see that $\partial_\alpha P(\alpha, \theta) = 0$ if and only if

$$\zeta^2 - 2(1 + t(\theta)^2)\zeta + 1 + t(\theta)^2 = 0. \tag{4.8}$$

Since we look for $\zeta \in (0, 1]$, the convenient root of equation (4.8) is

$$\alpha(\theta) f(\theta)^2 = \zeta = 1 + t(\theta)^2 - t(\theta)\sqrt{1 + t(\theta)^2}\,.$$

Hence we find

$$P(\alpha(\theta), \theta) = \frac{1}{\left(\sqrt{1 + t(\theta)^2} - t(\theta)\right)^2} = \left(\sqrt{1 + t(\theta)^2} + t(\theta)\right)^2.$$

Now (4.7) is a consequence of the latter identity and of the classical formula

$$t(\theta) = \frac{f'(\theta)}{f(\theta)} = \tan\gamma(\boldsymbol{x})$$

valid for the polar parametrization. □

The quantity $M(\Omega)$ is our modified Horgan–Payne like bound.

Theorem 4.4 (Estimate (6.24) in [12])**.** *Let Ω be a bounded strictly star-shaped domain. Its Friedrichs constant satisfies the bound*

$$\Gamma(\Omega) \le M(\Omega)\,. \tag{4.9}$$

The proof of this theorem is due to Horgan and Payne. Unfortunately, instead of simply concluding that $M(\Omega)$ is an upper bound for $\Gamma(\Omega)$, they try to show that $M(\Omega)$ coincides with $m(\Omega)$ and this part of their argument is flawed and invalid, in general. For the convenience of the reader we reproduce here the correct part of [12, §6] leading to the proof of the bound (4.9).

Proof. We assume for simplicity that the origin $\boldsymbol{x}_0$ of polar coordinates coincides with the origin $\mathbf{0}$ of Cartesian coordinates. Let $h \in L^2(\Omega)$ and $g \in L^2(\Omega)$ be a conjugate harmonic functions such that $\mathbf{grad}\, h = \mathbf{curl}\, g$. We normalize h such that $h(\mathbf{0}) = 0$. If we bound the $L^2(\Omega)$ norm of h, we bound a fortiori the $L^2(\Omega)$ norm of $h - \frac{1}{|\Omega|}\int_\Omega h$ which is the harmonic conjugate of g in $L^2_\circ(\Omega)$, hence with minimal $L^2(\Omega)$ norm.

Since $h + ig$ is holomorphic, its square is holomorphic, too, and therefore the functions $H := h^2 - g^2$ and $G := 2gh$ are harmonic conjugate. The equation $\mathbf{grad}\, H = \mathbf{curl}\, G$ leads to the relation in polar coordinates

$$\partial_\rho \widetilde{H} = \frac{1}{\rho}\partial_\theta \widetilde{G}$$

where $\widetilde{H}(r,\theta) = H(\boldsymbol{x})$ and $\widetilde{G}(r,\theta) = G(\boldsymbol{x})$ for $\boldsymbol{x} = (r\cos\theta, r\sin\theta)$. Thus for any $\theta \in \mathbb{T}$ and $r \in (0, f(\theta))$ we have

$$\widetilde{H}(r,\theta) - H(\mathbf{0}) = \int_0^r \partial_\rho \widetilde{H}(\rho,\theta)\,\mathrm{d}\rho = \int_0^r \frac{1}{\rho}\partial_\theta \widetilde{G}(\rho,\theta)\,\mathrm{d}\rho\,.$$

We divide by $f(\theta)^2$ and integrate for $\theta \in \mathbb{T}$ and $r \in (0, f(\theta))$:

$$\begin{aligned}\int_{\mathbb{T}}\int_0^{f(\theta)} \frac{\widetilde{H}(r,\theta) - H(\mathbf{0})}{f(\theta)^2}\, r\mathrm{d}r\mathrm{d}\theta &= \int_{\mathbb{T}}\int_0^{f(\theta)} \frac{1}{f(\theta)^2}\Big\{\int_0^r \frac{1}{\rho}\partial_\theta \widetilde{G}(\rho,\theta)\,\mathrm{d}\rho\Big\}\, r\mathrm{d}r\mathrm{d}\theta \\ &= \int_{\mathbb{T}}\int_0^{f(\theta)} \frac{1}{f(\theta)^2}\frac{1}{\rho}\partial_\theta \widetilde{G}(\rho,\theta)\Big\{\int_\rho^{f(\theta)} r\mathrm{d}r\Big\}\,\mathrm{d}\rho\mathrm{d}\theta \\ &= \frac{1}{2}\int_{\mathbb{T}}\int_0^{f(\theta)} \frac{f(\theta)^2 - \rho^2}{\rho^2 f(\theta)^2}\partial_\theta \widetilde{G}(\rho,\theta)\,\rho\mathrm{d}\rho\mathrm{d}\theta\,.\end{aligned}$$

Since the function $f(\theta)^2 - \rho^2$ is 0 on the boundary, integration by parts yields

$$\int_{\mathbb{T}}\int_0^{f(\theta)} \frac{\widetilde{H}(r,\theta) - H(\mathbf{0})}{f(\theta)^2}\, r\mathrm{d}r\mathrm{d}\theta = -\int_{\mathbb{T}}\int_0^{f(\theta)} \frac{f'(\theta)}{f(\theta)^3}\,\widetilde{G}(\rho,\theta)\,\rho\mathrm{d}\rho\mathrm{d}\theta\,.$$

We recall the notation $t(\theta) = \frac{f'(\theta)}{f(\theta)}$. Coming back to h and g and Cartesian variables $\boldsymbol{x} \in \Omega$ we find:

$$\int_\Omega \frac{h(\boldsymbol{x})^2}{f(\theta)^2}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \Big\{\frac{g(\boldsymbol{x})^2 - g(\mathbf{0})^2}{f(\theta)^2}\,\mathrm{d}\boldsymbol{x} - 2\int_\Omega \frac{t(\theta)h(\boldsymbol{x})g(\boldsymbol{x})}{f(\theta)^2}\Big\}\,\mathrm{d}\boldsymbol{x}. \tag{4.10}$$

In order to take the best advantage of the previous identity we introduce a parameter

$$\alpha \in (0,1)$$

and write for any $\theta \in \mathbb{T}$ (here we use condition (4.1) which ensures that $1 - \alpha f(\theta)^2 > 0$)

$$2\big|t(\theta)h(\boldsymbol{x})g(\boldsymbol{x})\big| \le \big\{1 - \alpha f(\theta)^2\big\}h(\boldsymbol{x})^2 + \frac{t(\theta)^2}{1-\alpha f(\theta)^2}\, g(\boldsymbol{x})^2$$

and deduce from (4.10) that (note that the same α has to be used for all θ)

$$\alpha \int_\Omega h(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \le \int_\Omega \frac{g(\boldsymbol{x})^2}{f(\theta)^2} + \frac{t(\theta)^2}{1-\alpha f(\theta)^2}\,\frac{g(\boldsymbol{x})^2}{f(\theta)^2}\,\mathrm{d}\boldsymbol{x}\,.$$

Thus, for any $\alpha \in (0,1)$

$$\int_\Omega h(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \le \sup_{\theta\in\mathbb{T}} \Big\{\frac{1}{\alpha f(\theta)^2}\Big(1 + \frac{t(\theta)^2}{1-\alpha f(\theta)^2}\Big)\Big\} \int_\Omega g(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x}\,.$$

Optimizing on $\alpha \in (0,1)$ and coming back to the definition of t and P, we find

$$\int_\Omega h(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \le \inf_{\alpha\in(0,1)} \Big\{\sup_{\theta\in\mathbb{T}} P(\alpha,\theta)\Big\} \int_\Omega g(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x},$$

which is nothing else than $\|h\|^2_{0,\Omega} \le M(\Omega)\|g\|^2_{0,\Omega}$, whence the theorem. □

5. Examples and counterexamples

5.1. **Examples.** In this section we exhibit classes of domains Ω for which the Horgan–Payne inequality (2.7) is valid, because the equality $m(\Omega) = M(\Omega)$ holds.

Theorem 5.1. *The equality $m(\Omega) = M(\Omega)$ holds for the following classes of domains Ω*

1. *Ellipses, with $\boldsymbol{x}_0$ at the center of the domain,*
2. *Cyclic polygons containing the center $\mathbf{c}$ of their circumscribed circle, with $\boldsymbol{x}_0 = \mathbf{c}$,*
3. *Circumscribed polygons, with $\boldsymbol{x}_0$ at the center of the inscribed circle.*

Example 5.2. Here are examples corresponding to the three classes above.

1. Discs realize the minimum value 1 of $\Gamma(\Omega)$ out of all plane domains.
2. Cyclic polygons: Rectangles, and all *regular* (convex) polygons.
3. Circumscribed polygons: Triangles, rhombi, (and again, regular polygons).

We are going to prove Theorem 5.1 for each class of domain, successively. We will give formulas for $m(\Omega) = M(\Omega)$ and the corresponding bound of $\beta(\Omega)$.

5.1.1. *Ellipses.* The canonical form of the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with positive coefficients $a \le b$. We take the center $\boldsymbol{x}_0$ of polar coordinates at the origin (center of the ellipse). One can prove the identities, see details in [7, §5.1]

$$m(\Omega) = \frac{b^2}{a^2} \quad \text{and} \quad M(\Omega) = \frac{b^2}{a^2}\,.$$

Moreover the constant $\Gamma(\Omega)$ is analytically known, cf. [11], and finally

$$m(\Omega) = M(\Omega) = \frac{b^2}{a^2} = \Gamma(\Omega)\,. \tag{5.1}$$

In particular, if Ω is a disk $m(\Omega) = M(\Omega) = \Gamma(\Omega) = 1$. Note the corresponding values for the inf-sup constant deduced from the relation $\beta(\Omega) = (1 + \Gamma(\Omega))^{-1/2}$, cf (2.4) and (2.9),

$$\beta(\Omega) = \frac{1}{\sqrt{1 + \frac{b^2}{a^2}}} \text{ in general, and } \beta(\Omega) = \frac{1}{\sqrt{2}} \text{ if } \Omega \text{ is a disk.}$$

5.1.2. *Polygons.* Let Ω be a strictly star-shaped polygon associated with the origin $\boldsymbol{x}_0$. Let $\boldsymbol{c}_j$, $j = 1, \dots, J$ be its vertices. The sides of Ω are the segments $[\boldsymbol{c}_j, \boldsymbol{c}_{j+1}]$ (with the convention $\boldsymbol{c}_{J+1} = \boldsymbol{c}_1$). We denote by

- $r_j = \max\{|\boldsymbol{c}_j - \boldsymbol{x}_0|, |\boldsymbol{c}_{j+1} - \boldsymbol{x}_0|\}$
- $d_j = \operatorname{dist}(\boldsymbol{x}_0, L_j)$ with L_j the line containing the side $[\boldsymbol{c}_j, \boldsymbol{c}_{j+1}]$.

The normalization $\max_{\theta\in\mathbb{T}} f(\theta) = 1$ becomes

$$\max_{j=1}^{J} r_j = 1\,.$$

Let $\theta_j \in \mathbb{T}$ the angle corresponding to the vertex $\boldsymbol{c}_j$ and $\tilde{\theta}_j$ the angle corresponding to the point $\tilde{\boldsymbol{c}}_j \in L_j$ such that $d_j = |\tilde{\boldsymbol{x}}_j - \boldsymbol{x}_0|$. For $\theta \in (\theta_j, \theta_{j+1})$, we find

$$f(\theta) = \frac{d_j}{\cos(\theta - \tilde{\theta}_j)} \quad \text{and} \quad \gamma(\boldsymbol{x}) = \theta - \tilde{\theta}_j.$$

We deduce the formula for P (see also [7, §5.2])

$$P(\alpha, \theta) = \frac{1}{\alpha d_j^2} \frac{1 - \alpha d_j^2}{1 - \alpha f(\theta)^2} \quad \text{for} \quad \theta \in (\theta_j, \theta_{j+1}).$$

For any $\alpha \in (0, 1)$ and $\theta \in (\theta_j, \theta_{j+1})$, the maximal value of P is attained for $f(\theta)$ maximal, i.e., at the most distant end of the segment $[\boldsymbol{c}_j, \boldsymbol{c}_{j+1}]$. Hence

$$M(\Omega) = \inf_{\alpha\in(0,1)} \max_{j=1}^{J} \frac{1}{\alpha d_j^2} \frac{1 - \alpha d_j^2}{1 - \alpha r_j^2} \tag{5.2}$$

$$= \inf_{\alpha\in(0,1)} \max_{j=1}^{J} \frac{1}{\alpha r_j^2} \frac{r_j^2 d_j^{-2} - \alpha r_j^2}{1 - \alpha r_j^2}\,. \tag{5.3}$$

To calculate $m(\Omega)$, we use (4.6) and find

$$m(\Omega) = \max_{j=1}^{J} \left(\frac{r_j}{d_j} + \sqrt{\frac{r_j^2}{d_j^2} - 1} \right)^2. \tag{5.4}$$

The maximum is attained when r_j/d_j is maximal.

Lemma 5.3. *Let Ω be a strictly star-shaped polygon associated with the center $\boldsymbol{x}_0$ and the normalization $\max_j r_j = 1$. Let $d = \min_j d_j$. If Ω is cyclic or circumscribed (with respect to the center $\boldsymbol{x}_0$), then*

$$m(\Omega) = M(\Omega) = \left(\frac{1}{d} + \sqrt{\frac{1}{d^2} - 1} \right)^2.$$

Proof. If Ω is cyclic, all r_j coincide, so are equal to 1. If Ω is circumscribed, all d_j coincide, so are equal to d. In both situations we deduce from (5.3) and (5.2) respectively, that

$$M(\Omega) = \inf_{\alpha\in(0,1)} \frac{1}{\alpha}\,\frac{d^{-2}-\alpha}{1-\alpha}\,. \tag{5.5}$$

Since $d<1$, there is one value α_0 of α realizing the minimum

$$\alpha_0 = \frac{1}{d^2} - \sqrt{\frac{1}{d^4}-\frac{1}{d^2}} \in (0,1). \tag{5.6}$$

This leads to the formula of the lemma for $M(\Omega)$. The formula for $m(\Omega)$ is a consequence of (5.4). □

This concludes the proof of Theorem 5.1 and provides for cyclic or circumscribed polygons the associate lower bound on $\beta(\Omega) = (1+\Gamma(\Omega))^{-1/2}$ in the form

$$\beta(\Omega) \geq \frac{d}{\sqrt{2}}\left(1+\sqrt{1-d^2}\right)^{-\frac{1}{2}}. \tag{5.7}$$

Whereas we have described several classes of polygons for which the two upper bounds $m(\Omega)$ and $M(\Omega)$ coincide, in general they are different. It is indeed not difficult to find domains, even polygons, for which $m(\Omega) \neq M(\Omega)$. Among other examples, a simple convex hexagon that has this property is analyzed in [7].

Now if $m(\Omega) \neq M(\Omega)$, then the proven inequality $\Gamma(\Omega) \leq M(\Omega)$ (4.9) is weaker than the Horgan–Payne inequality $\Gamma(\Omega) \leq m(\Omega)$ (2.8), but this does not yet imply that the latter is not true. In the following section we analyze examples of domains for which the inequality of Horgan–Payne does indeed not hold.

5.2. **Counterexamples.** We will now give examples of strictly star-shaped domains in $\mathbb{R}^2$ that do not satisfy the Horgan–Payne inequality (2.7). We present three examples, a "Cupid's bow" where the boundary is composed of logarithmic spirals, a "double stadium" where the boundary is composed of straight segments and circular arcs, and a polygonal (octagonal) version of the "Cupid's bow". The examples have a common feature, a small passage between two halves of the domain. This means that the domain can be separated into two equal-sized parts by a very short straight cut. Or again, there are points on the boundary where the distance to the origin is much smaller than the Horgan–Payne angle.

The proof that the Horgan–Payne inequality is not satisfied uses a new upper bound for the inf-sup constant proved in Lemma 5.4 below. In the three examples, the domains depend on a small parameter, and we will show that as the parameter tends to zero, the upper bound tends to zero much faster than the lower bound of the Horgan–Payne inequality. This shows that for sufficiently small values of the parameter, the inequality cannot be true. Since our upper bound features an explicit constant, we can provide explicit values of the parameter for which the Horgan–Payne inequality is disproved.

We begin by proving an upper bound for the inf-sup constant $\beta(\Omega)$ in the situation where the bounded domain Ω is separated into two subdomains Ω_+ and Ω_- by a plane cut Σ. Since this estimate may be of independent interest (it can be used to show that $\beta(\Omega) = 0$ for a

large class of domains with outward cusps, for example), we prove it in any dimension $d \geq 2$.

Without loss of generality, we can assume that Σ lies in the plane $\{x_d = 0\}$. Thus we assume with $\mathbb{R}^d_\pm = \{\boldsymbol{x} \in \mathbb{R}^d \mid x_d \gtrless 0\}$

$$\Omega \cap (\mathbb{R}^{d-1} \times \{0\}) = \Sigma \times \{0\} \neq \emptyset \quad \text{and} \quad \Omega_\pm = \Omega \cap \mathbb{R}^d_\pm \,.$$

For simplicity, we assume that Σ is connected. We denote by $|\Omega|$ the d-dimensional measure of Ω and by $|\Sigma|$ the $d-1$-dimensional measure of Σ. By L we denote the width of Σ, that is the minimal distance of two parallel $d-2$-dimensional hypersurfaces in $\mathbb{R}^{d-1}$ that contain Σ between them. If $d = 2$ and Σ is an interval, then $|\Sigma| = L$, the length of the interval.

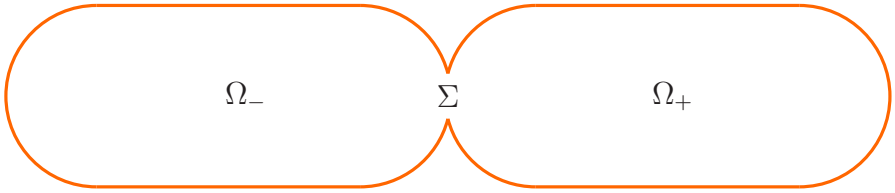

FIGURE 1. Example of configuration for Lemma 5.4: the double stadium

Lemma 5.4. *There exists a constant c_d depending only on the dimension d such that*

$$\beta(\Omega) \leq c_d \left(\frac{|\Omega|}{|\Omega_+||\Omega_-|} L|\Sigma| \right)^{\frac{1}{2}} . \tag{5.8}$$

For $d = 2$, we can take $c_2 = \frac{\sqrt{8}}{\sqrt{3}}$, so that

$$\beta(\Omega) \leq \left(\frac{8}{3} \frac{|\Omega|}{|\Omega_+||\Omega_-|} \right)^{\frac{1}{2}} L \,. \tag{5.9}$$

Remark: This value of c_2 is certainly not optimal; more elaborate methods of proof may give smaller values.

Proof. We choose a piecewise constant function $q \in L^2_\circ(\Omega)$ as follows:

$$q = \frac{1}{|\Omega_+|} \text{ in } \Omega_+ \,, \qquad q = -\frac{1}{|\Omega_-|} \text{ in } \Omega_- \,,$$

and we will obtain an upper bound for $\beta(\Omega)$ from

$$\beta(\Omega) \leq \sup_{\boldsymbol{v} \in \mathscr{C}^\infty_0(\Omega)^d} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x})\, q(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}}{|\boldsymbol{v}|_{1,\Omega}\, \|q\|_{0,\Omega}} \,.$$

We compute explicitly

$$\|q\|^2_{0,\Omega} = \frac{1}{|\Omega_+|} + \frac{1}{|\Omega_-|} = \frac{|\Omega|}{|\Omega_+||\Omega_-|}$$

and for $\boldsymbol{v} \in \mathscr{C}_0^\infty(\Omega)^d$

$$\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x})\, q(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = -\int_\Sigma v_d(x,0)\, \mathrm{d}x \left(\frac{1}{|\Omega_+|} + \frac{1}{|\Omega_-|}\right).$$

This implies

$$\beta(\Omega) \le \left(\frac{|\Omega|}{|\Omega_+||\Omega_-|}\right)^{\frac12} \sup_{\boldsymbol{v}\in\mathscr{C}_0^\infty(\Omega)^d} \frac{\left|\int_\Sigma v_d(x,0)\,\mathrm{d}x\right|}{|\boldsymbol{v}|_{1,\Omega}}.$$

Thus, as soon as we can get an estimate of the mean value for the trace on Σ for any $u \in \mathscr{C}_0^\infty(\Omega)$

$$\left|\int_\Sigma u(x,0)\,\mathrm{d}x\right| \le \tilde{c}(\Sigma)\, |u|_{1,\Omega}, \tag{5.10}$$

we will have an upper bound for $\beta(\Omega)$

$$\beta(\Omega) \le \tilde{c}(\Sigma) \left(\frac{|\Omega|}{|\Omega_+||\Omega_-|}\right)^{\frac12}. \tag{5.11}$$

The rest of the proof is dedicated to the L^1 estimate (5.10). This estimate will be obtained in three steps : First we show a precise version (5.12) of the $H^{1/2}$ estimate of the standard trace lemma. Then, given that $u(\cdot,0)$ vanishes outside of Σ, we deduce from (5.12) a weighted L^2 estimate (5.14). Finally, the Cauchy-Schwarz inequality gives the L^1 estimate (5.10).

The first step is a version with explicit (though not optimal) constant of the standard $H^{1/2}$ estimate of the trace lemma. Namely, we will show: There holds for all $u \in \mathscr{C}_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{|u(x,0)-u(y,0)|^2}{|x-y|^d}\, \mathrm{d}x\, \mathrm{d}y \le 16\, \omega_{d-1} |u|^2_{1,\mathbb{R}^d_+} \tag{5.12}$$

where ω_{d-1} is the surface of the unit sphere in $\mathbb{R}^{d-1}$; $\omega_1 = 2$ for $d = 2$. In order to keep control of the constants, we present a short proof of this classical result, see [17] for this and other variants of the proof.

One writes $h = (y-x)/2$ and

$$u(x,0) - u(y,0) = \big(u(x,0) - u(x+h,|h|)\big) \;-\; \big(u(y,0) - u(y-h,|h|)\big).$$

We only need to estimate the first term on the right hand side, the second term being of the same form. For $x, h \in \mathbb{R}^{d-1}$ we have

$$\begin{aligned} |u(x,0) - u(x+h,|h|)| &= \Big|\int_0^1 \mathbf{grad}\, u(x+sh, s|h|) \cdot \begin{pmatrix} h \\ |h| \end{pmatrix} \mathrm{d}s\Big| \\ &\le \sqrt{2}|h| \int_0^1 |\,\mathbf{grad}\, u(x+sh, s|h|)|\, \mathrm{d}s\,. \end{aligned}$$

Integrating in x on $\mathbb{R}^{d-1}$, this implies

$$\begin{aligned} \|u(\cdot,0) - u(\cdot+h,|h|)\|_{L^2(\mathbb{R}^{d-1})} &\le \sqrt{2}|h| \int_0^1 \|\,\mathbf{grad}\, u(\cdot+sh, s|h|)\|_{L^2(\mathbb{R}^{d-1})}\, \mathrm{d}s \\ &= \sqrt{2}|h| \int_0^1 \|\,\mathbf{grad}\, u(\cdot, s|h|)\|_{L^2(\mathbb{R}^{d-1})}\, \mathrm{d}s\,. \end{aligned}$$

Integrating now in h on $\mathbb{R}^{d-1}$, we obtain

$$\Big\| |h|^{-\frac{d}{2}} \|u(\cdot,0) - u(\cdot+h,|h|)\|_{L^2(\mathbb{R}^{d-1})} \Big\|_{L^2(\mathbb{R}^{d-1},\mathrm{d}h)} \leq \sqrt{2} \int_0^1 \Big\| |h|^{1-\frac{d}{2}} \| \mathbf{grad}\, u(\cdot, s|h|))\|_{L^2(\mathbb{R}^{d-1})} \Big\|_{L^2(\mathbb{R}^{d-1},\mathrm{d}h)} \,\mathrm{d}s\,. \tag{5.13}$$

Using polar coordinates $(|h|, \frac{h}{|h|})$ for the integral in h:

$$\begin{aligned} \Big\| |h|^{1-\frac{d}{2}} \| \mathbf{grad}\, u(\cdot, s|h|))\|_{L^2(\mathbb{R}^{d-1})} \Big\|^2_{L^2(\mathbb{R}^{d-1},\mathrm{d}h)} &= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} |h|^{2-d} |\mathbf{grad}\, u(x, s|h|))|^2 \,\mathrm{d}x\,\mathrm{d}h \\ &= \omega_{d-1} \int_{\mathbb{R}^{d-1}} \int_0^\infty |\mathbf{grad}\, u(x, s|h|))|^2 \,\mathrm{d}|h|\,\mathrm{d}x \\ &= \omega_{d-1}\, s^{-1} \|\mathbf{grad}\, u\|^2_{L^2(\mathbb{R}^d_+)}\,. \end{aligned}$$

Inserting this into (5.13), we obtain with $\int_0^1 s^{-1/2}\mathrm{d}s = 2$

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} |h|^{-d} |u(x,0) - u(x+h,|h|)|^2 \,\mathrm{d}x\,\mathrm{d}h \leq 8\,\omega_{d-1}\, \|\mathbf{grad}\, u\|^2_{L^2(\mathbb{R}^d_+)}$$

and, using $2|x-y|^{-d}\mathrm{d}y = |h|^{-d}\mathrm{d}h$, finally (5.12).

Next we consider $u \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ such that $u(y,0) = 0$ whenever $y \notin \Sigma$ (which is the case for $u \in \mathscr{C}_0^\infty(\Omega)$). Then we find from (5.12) the weighted L^2 estimate

$$\int_{\mathbb{R}^{d-1}} w(x) |u(x,0)|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{|u(x,0) - u(y,0)|^2}{|x-y|^d} \,\mathrm{d}x\,\mathrm{d}y \leq 16\,\omega_{d-1} |u|^2_{1,\mathbb{R}^d_+} \tag{5.14}$$

with the weight function

$$w(x) = \int_{\mathbb{R}^{d-1}\setminus\Sigma} \frac{\mathrm{d}y}{|x-y|^d}.$$

Finally, from the Cauchy-Schwarz inequality,

$$\Big(\int_\Sigma |u(x,0)| \,\mathrm{d}x \Big)^2 \leq \Big(\int_\Sigma \frac{\mathrm{d}x}{w(x)} \Big) \int_{\mathbb{R}^{d-1}} w(x) |u(x,0)|^2 \,\mathrm{d}x\,,$$

we obtain the L^1 estimate

$$\Big(\int_\Sigma |u(x,0)| \,\mathrm{d}x \Big)^2 \leq \Big(\int_\Sigma \frac{\mathrm{d}x}{w(x)} \Big)\, 16\,\omega_{d-1} |u|^2_{1,\mathbb{R}^d_+}\,. \tag{5.15}$$

Noting that the same estimate holds with $|u|^2_{1,\mathbb{R}^d_+}$ replaced by $|u|^2_{1,\mathbb{R}^d_-}$ and using

$$|u|^2_{1,\mathbb{R}^d_+} + |u|^2_{1,\mathbb{R}^d_-} = |u|^2_{1,\mathbb{R}^d}\,,$$

we have proved (5.10) with $\tilde{c}(\Sigma) = \big(8\,\omega_{d-1} \int_\Sigma \frac{\mathrm{d}x}{w(x)}\big)^{1/2}$. It remains to estimate $\int_\Sigma \frac{\mathrm{d}x}{w(x)}$.

In dimension $d = 2$, when Σ is the interval $(0, L)$, we can compute $w(x)$ explicitly:

$$w(x) = \int_{-\infty}^0 \frac{\mathrm{d}y}{(y-x)^2} + \int_L^\infty \frac{\mathrm{d}y}{(y-x)^2} = \frac{1}{x} + \frac{1}{L-x},$$

hence

$$\int_\Sigma \frac{\mathrm{d}x}{w(x)} = \frac{L^2}{6},$$

and we find (5.8) with $c_2 = \sqrt{8\,\omega_1/6} = \sqrt{8}/\sqrt{3}$.

For general $d \geq 2$, if Σ lies between two hyperplanes of distance L, then it is not hard to see that for all $x \in \Sigma$, $w(x) \geq c'_d/L$ with some constant c'_d independent of x, Σ and L. This gives $\int_\Sigma \frac{\mathrm{d}x}{w(x)} \leq L\,|\Sigma|/c'_d$, whence (5.8) with $c_d = \sqrt{8\omega_{d-1}/c'_d}$. □

5.2.1. *First counterexample: Cupid's bow.* Choose a constant $c > 0$ and define the logarithmic spiral by the polar parametrization

$$r = f(\theta) = \mathrm{e}^{-c\theta}\,.$$

To define the domain Ω, we use the logarithmic spiral in the first quadrant and complete the boundary curve by reflections about the x and y axes. Thus the polar parametrization of the boundary curve can be written as

$$f(\theta) = \mathrm{e}^{-c(\frac{\pi}{2} - |\frac{\pi}{2} - |\theta||)}, \quad -\pi \leq \theta \leq \pi\,.$$

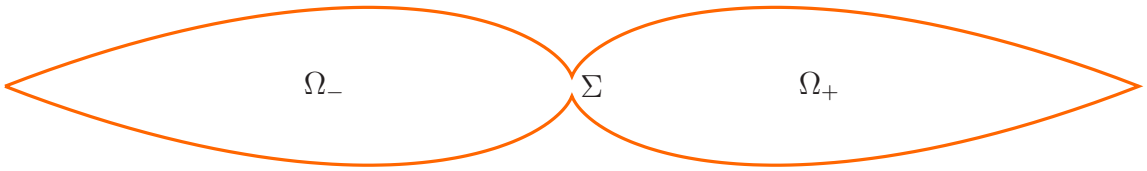

FIGURE 2. Cupid's bow with $c = 2.58$

The important observation is that the angle $\gamma(\theta)$ is constant along the boundary curve, satisfying

$$\tan\gamma(\theta) = \frac{|f'(\theta)|}{f(\theta)} = c\,.$$

Therefore the Horgan–Payne angle is $\omega(\Omega) = \frac{\pi}{2} - \gamma(\theta) = \arctan\frac{1}{c}$. The Horgan–Payne inequality in this case amounts to

$$\beta(\Omega)^2 \geq \sin^2\frac{\omega(\Omega)}{2} = \frac{\sqrt{c^2+1}-c}{2\sqrt{c^2+1}} = \frac{1}{4c^2} + O(c^{-4}) \quad \text{as } c \to \infty\,. \tag{5.16}$$

Now we look at our upper bound from Lemma 5.4. The main observation here is that Ω is separated into equal left and right halves by a vertical cut $\{0\} \times \Sigma$ with

$$\Sigma = (-\mathrm{e}^{-c\frac{\pi}{2}}, \mathrm{e}^{-c\frac{\pi}{2}})$$

which is exponentially small. The quantities appearing in the estimate (5.8) are as follows:

$$|\Omega_+| = |\Omega_-| = 2\int_0^{\frac{\pi}{2}}\int_0^{f(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1-\mathrm{e}^{-c\pi}}{2c}\,, \quad |\Omega| = 2|\Omega_+|\,, \quad L = 2\mathrm{e}^{-c\frac{\pi}{2}}\,.$$

Therefore the estimate (5.9) implies

$$\beta(\Omega)^2 \leq \frac{8}{3}\frac{4c}{1-\mathrm{e}^{-c\pi}}(2\mathrm{e}^{-c\frac{\pi}{2}})^2 = \frac{128}{3}\frac{c\,\mathrm{e}^{-c\pi}}{1-\mathrm{e}^{-c\pi}}\,. \tag{5.17}$$

Clearly, for c large enough, the proven upper bound (5.17) contradicts the Horgan–Payne inequality (5.16). Concretely, for $c = 2.58$ we find numerically for the upper bound

$$\frac{128}{3}\frac{c\,\mathrm{e}^{-c\pi}}{1-\mathrm{e}^{-c\pi}} < 0.0333$$

which is smaller than the lower bound in (5.16)

$$\frac{\sqrt{c^2+1}-c}{2\sqrt{c^2+1}} > 0.0337\,.$$

In this example, without using Lemma 5.4, one can also see that the Friedrichs constant $\Gamma(\Omega)$ must be exponentially large, thus contradicting the version (2.8) of the Horgan–Payne inequality which has a right hand side growing only quadratically in c. Indeed, let $\varepsilon = \mathrm{e}^{-c\frac{\pi}{2}}$ and define the holomorphic function

$$w(z) = \log\frac{i\varepsilon - z}{i\varepsilon + z},$$

which is holomorphic in $\mathbb{C}$ minus two vertical branch cuts $[-i\infty, -i\varepsilon]$ and $[i\varepsilon, i\infty]$. We choose the branch that satisfies $w(0) = 0$. For symmetry reasons, both real and imaginary parts of w belong to $L^2_\circ(\Omega)$, but otherwise these conjugate harmonic functions behave very differently. $\operatorname{Im} w = \arg\frac{i\varepsilon - z}{i\varepsilon + z}$ tends to π in the right half-plane and to $-\pi$ in the left half-plane, on a length scale of the size of ε. Therefore

$$\|\operatorname{Im} w\|^2_{0,\Omega} \sim \pi^2|\Omega| \sim \frac{\pi^2}{c} \quad \text{as } c\to\infty\,.$$

On the other hand, $\operatorname{Re} w = \log|1 + \frac{2i\varepsilon}{z+i\varepsilon}|$ is of the order of ε outside of any disk with a fixed radius $> 2\varepsilon$. It is not hard to see that $\|\operatorname{Re} w\|^2_{0,\Omega}$ tends to zero exponentially fast as $c\to\infty$ and therefore $\Gamma(\Omega) \geq \|\operatorname{Im} w\|^2_{0,\Omega}/\|\operatorname{Re} w\|^2_{0,\Omega}$ grows exponentially, too.

5.2.2. *Second counterexample: Double stadium.* We choose a positive number ε and construct our domain Ω as follows: Take the union of the rectangle $(\sqrt{1-\varepsilon^2}, \frac{\sqrt{1-\varepsilon^2}}{\varepsilon})\times(-1,1)$ and the two circles of radius 1 with centers $(\sqrt{1-\varepsilon^2}, 0)$ and $(\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}, 0)$. This is the "stadium". The domain Ω is the union of the stadium and its reflection with respect to the vertical axis, see Figure 1 in which we have set $\varepsilon = 0.25$. This produces a small passage between the left and right half, and Ω is cut into two by a vertical cut $\{0\}\times\Sigma$ with

$$\Sigma = (-\varepsilon, \varepsilon)\,.$$

To determine the Horgan–Payne angle, we notice that the minimal value of $\omega(\boldsymbol{x})$ is attained at the points $(0,\varepsilon)$ and $(\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}, 1)$. In both cases it satisfies

$$\sin\omega(\boldsymbol{x}) = \varepsilon.$$

The Horgan–Payne inequality amounts to

$$\beta(\Omega)^2 \geq \sin^2\frac{\omega(\Omega)}{2} \sim \frac{\varepsilon^2}{4} \quad \text{as } \varepsilon\to 0\,. \tag{5.18}$$

To determine the upper bound resulting from (5.9), we compute

$$|\Omega_+| \sim 2\sqrt{1-\varepsilon^2}(\frac{1}{\varepsilon} - 1) + \pi \sim \frac{2}{\varepsilon} \quad \text{as } \varepsilon\to 0\,, \qquad L = 2\varepsilon.$$

This leads to an upper bound

$$\beta(\Omega)^2 \le \frac{8}{3}\,4\varepsilon^2\frac{2}{|\Omega_+|} \sim \frac{32}{3}\varepsilon^3 \quad \text{as } \varepsilon \to 0\,. \tag{5.19}$$

It is clear that for sufficiently small ε, the upper bound (5.19) is incompatible with the Horgan–Payne inequality (5.18).

5.2.3. *Third counterexample: Octagon.* We choose a positive number q and define Ω as an octagon with the corners at distance 1 for $\theta \in \{0, \pi\}$, at distance q for $\theta \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$, and at distance q^2 for $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, see Figure 3. Thus the boundary curve is a polygonal interpolation of the Cupid's bow example if we set $q = \mathrm{e}^{-c\frac{\pi}{4}}$. We see that Ω is composed

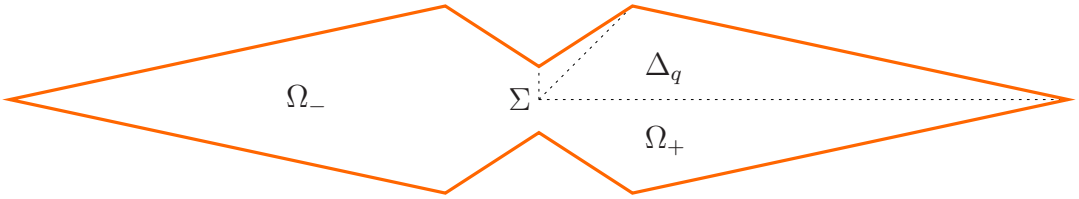

FIGURE 3. Octagon with $q = 0.25$

of 8 triangles that are similar to the triangle Δ_q with corners $(0,0)$, $(1,0)$, $(\frac{q}{\sqrt{2}}, \frac{q}{\sqrt{2}})$. The Horgan–Payne angle is easy to find: It satisfies

$$\tan\omega(\Omega) = \frac{q/\sqrt{2}}{1 - q/\sqrt{2}}\,.$$

Hence the Horgan–Payne inequality amounts to

$$\beta(\Omega)^2 \ge \sin^2\frac{\omega(\Omega)}{2} \sim \frac{q^2}{8} \quad \text{as } q \to 0\,. \tag{5.20}$$

For the quantities in the upper bound (5.9) we obtain with the area $|\Delta_q| = \frac{q}{2\sqrt{2}}$

$$|\Omega_+| = 2(1+q^2)|\Delta_q| = \frac{q(1+q^2)}{\sqrt{2}}, \qquad L = 2q^2\,.$$

Hence the upper bound is

$$\beta(\Omega)^2 \le \frac{8}{3}\frac{2\sqrt{2}}{q(1+q^2)}4q^4 \sim \frac{64\sqrt{2}}{3}q^3 \quad \text{as } q \to 0\,. \tag{5.21}$$

Clearly (5.21) contradicts (5.20) if q is small enough.

6. ESTIMATE INVOLVING THE RATIO OF RADII

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain star-shaped with respect to a ball centered at the origin. Let $r = f(\theta)$, $\theta \in \mathbb{T}$, be the Lipschitz parametrization of the boundary in polar coordinates that exists according to Maz'ya's Lemma quoted at the beginning of Section 4. We need a kind of quantitative version of that Lemma, namely a characterization in terms of this parametrization of the largest ball with respect to which Ω is star-shaped.

Lemma 6.1. *Let $\rho_{\max}$ be the radius of the largest open disk centered at the origin with respect to which Ω is star-shaped. Then*

$$\rho_{\max} = \inf_{\theta\in\mathbb{T}} \frac{f(\theta)^2}{\sqrt{f(\theta)^2 + f'(\theta)^2}}\,. \tag{6.1}$$

Proof. If we introduce the angle $\gamma(\theta)$ between the radius vector and the normal as in Section 4, so that $\tan\gamma(\theta) = \frac{f'(\theta)}{f(\theta)}$, then (6.1) can be written as

$$\rho_{\max} = \inf_{\theta\in\mathbb{T}} f(\theta)\cos\gamma(\theta)\,. \tag{6.2}$$

Considering that $f(\theta)\cos\gamma(\theta)$ is the distance to the origin of the tangent at the boundary point $(r, f(\theta))$, the equality appears rather plausible. We think a detailed proof is still needed, however.

Assume then that Ω is star-shaped with respect to an open ball B_ρ of radius ρ centered at the origin. Fix a point $\boldsymbol{x}$ on $\partial\Omega$. Without loss of generality, we can assume that $\boldsymbol{x}$ corresponds to $\theta = 0$, that is in Cartesian coordinates

$$\boldsymbol{x} = (f(0), 0)\,.$$

Then the open triangle Δ_ρ with corners $\mathbf{0} = (0,0)$, $\boldsymbol{a} = (\rho\cos\tau, \rho\sin\tau)$, and $\boldsymbol{x}$ is contained in Ω and thus does not contain any point on $\partial\Omega$. Here the angle τ is such that $0 < \tau < \pi/2$ and

$$\rho = f(0)\cos\tau,$$

see Figure 4. The side of Δ_ρ from $\boldsymbol{a} = (\rho\cos\tau, \rho\sin\tau)$ to $\boldsymbol{x}$ satisfies the equation in polar coordinates

$$r\cos(\tau - \theta) = \rho, \quad 0 < \theta < \tau\,. \tag{6.3}$$

For any $\theta \in (0, \tau)$, from the fact that the boundary point $r = f(\theta)$ lies outside of Δ_ρ, we therefore get the inequality

$$f(\theta) \geq \frac{\rho}{\cos(\tau - \theta)}, \quad 0 < \theta < \tau\,, \tag{6.4}$$

hence

$$\frac{f(0) - f(\theta)}{\theta} \leq \frac{\rho}{\cos\tau\cos(\tau-\theta)}\frac{\cos(\tau-\theta) - \cos\tau}{\theta}, \quad 0 < \theta < \tau\,. \tag{6.5}$$

If f is differentiable in $\theta = 0$, it follows

$$-f'(0) \leq \frac{\rho\sin\tau}{\cos^2\tau}$$

and from symmetrizing we get

$$|f'(0)| \leq \frac{\rho\sin\tau}{\cos^2\tau} = f(0)\sqrt{\frac{f(0)^2}{\rho^2} - 1}\,. \tag{6.6}$$

Since this is true for any boundary point $\boldsymbol{x}$ where f is differentiable, we get our final estimate, valid for almost every $\theta \in \mathbb{T}$

$$|f'(\theta)| \leq f(\theta)\sqrt{\frac{f(\theta)^2}{\rho^2} - 1}\,. \tag{6.7}$$

This inequality (6.7) is equivalent to

$$\rho \leq \frac{f(\theta)^2}{\sqrt{f(\theta)^2 + f'(\theta)^2}}\,, \tag{6.8}$$

and we have thus shown one half of the relation (6.1). (Note that the Lipschitz continuity of f is also a consequence of (6.5).)

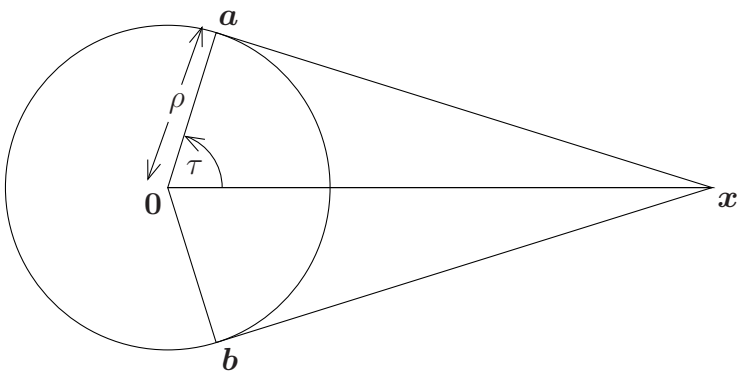

FIGURE 4. Triangles appearing in the proof of Lemma 6.1

It remains to show that if ρ satisfies (6.8) for almost all θ, then Ω is indeed star-shaped with respect to B_ρ. For this, it is sufficient to show that for all $\boldsymbol{x} \in \partial\Omega$ and $\boldsymbol{y} \in B_\rho$ the open segment between $\boldsymbol{y}$ and $\boldsymbol{x}$ is contained in Ω. We can again assume that $\boldsymbol{x}$ corresponds to $\theta = 0$, so that we are in the same configuration as in the first part of the proof. More precisely, we assume that the inequality (6.7) is satisfied almost everywhere on $\mathbb{T}$, and we have to show that the domain Ω_0 which is the interior of the convex hull of $B_\rho \cup \{\boldsymbol{x}\}$ is contained in Ω. This domain Ω_0 is the union of B_ρ and the interior of the triangle $\tilde{\Delta}_\rho$ with corners $\boldsymbol{a} = (\rho\cos\tau, \rho\sin\tau)$, $\boldsymbol{b} = (\rho\cos\tau, -\rho\sin\tau)$, and $\boldsymbol{x}$. Note that the upper half of $\tilde{\Delta}_\rho$ is the triangle Δ_ρ considered in the first part of the proof, and the line joining $\boldsymbol{a} = (\rho\cos\tau, \rho\sin\tau)$ to $\boldsymbol{x}$ satisfies the equation (6.3).

We will show that for $0 < \theta < \tau$, the boundary curve $r = f(\theta)$ does not cross the line (6.3). By symmetry for $0 > \theta > -\tau$ and using the fact that $f(\theta) \geq \rho$ for all θ, this will imply that the boundary curve does not enter Ω_0, which gives the desired result $\Omega_0 \subset \Omega$.

Let $r = g(\theta)$ describe the line (6.3), i.e.

$$g(\theta) = \frac{\rho}{\cos(\tau - \theta)}\,.$$

We want to show that $f(\theta) \geq g(\theta)$ for $\theta \in (0, \tau)$. For this purpose, define the function

$$G : [\rho, \infty) \to [0, \frac{\pi}{2})\,; \quad G(r) = \arccos\frac{\rho}{r}\,.$$

Then G is increasing, satisfies $G \circ g(\theta) = \tau - \theta$ for $\theta \in (0, \tau)$ and has the derivative

$$G'(r) = \frac{1}{r\sqrt{\frac{r^2}{\rho^2} - 1}}\,.$$

The inequality (6.7) is equivalent to $|(G \circ f)'(\theta)| \leq 1$. From

$$(G \circ f)'(\theta) \geq -1 \quad \text{ and } (G \circ f)(0) = \tau$$

we deduce for $\theta \in (0, \tau)$

$$G \circ f(\theta) \geq \tau - \theta = G \circ g(\theta)\,.$$

Due to the monotonicity of G, this implies $f(\theta) \geq g(\theta)$, and the proof is complete. □

Theorem 6.2. *Let $\rho_{\max}$ be the radius of the largest open disk centered at the origin with respect to which Ω is star-shaped. Let $R_{\min}$ the radius of the smallest disk centered at the origin containing Ω. Let*

$$\tau(\Omega) = \arccos \frac{\rho_{\max}}{R_{\min}}\,. \tag{6.9}$$

Then

$$M(\Omega) \leq \left(\frac{R_{\min}}{\rho_{\max}} + \sqrt{\frac{R_{\min}^2}{\rho_{\max}^2} - 1}\right)^2 = \left(\frac{1}{\cos\tau(\Omega)} + \sqrt{\frac{1}{\cos\tau(\Omega)^2} - 1}\right)^2. \tag{6.10}$$

Proof. Without restriction we assume that $R_{\min} = 1$ and we consider the polar coordinates parametrization of $\partial\Omega$ by f. The function P leading to $M(\Omega)$ defined in (4.2) can be written as

$$P(\alpha, \theta) = \frac{1}{\alpha f(\theta)^2}\Big(1 + \frac{\tan^2\gamma(\theta)}{1 - \alpha f(\theta)^2}\Big).$$

There holds

$$P(\alpha, \theta) = \frac{1}{\alpha f(\theta)^2}\Big(\frac{1 + \tan^2\gamma(\theta) - \alpha f(\theta)^2}{1 - \alpha f(\theta)^2}\Big).$$

Defining $d(\theta)$ as the distance to the origin of the line tangent to $\partial\Omega$ at the point of polar coordinates $(f(\theta), \theta)$, we find the relation

$$d(\theta) = f(\theta)\cos\gamma(\theta).$$

Thus

$$P(\alpha, \theta) = \frac{d(\theta)^{-2} - \alpha}{\alpha(1 - \alpha f(\theta)^2)}.$$

Let d be the infimum on $\theta \in \mathbb{T}$ of $d(\theta)$. We deduce that for all $\theta \in \mathbb{T}$ and $\alpha \in (0, 1)$

$$P(\alpha, \theta) \leq \frac{d^{-2} - \alpha}{\alpha(1 - \alpha f(\theta)^2)} \leq \frac{d^{-2} - \alpha}{\alpha(1 - \alpha)}.$$

Therefore $M(\Omega)$ satisfies

$$M(\Omega) \leq \inf_{\alpha \in (0,1)} \frac{d^{-2} - \alpha}{\alpha(1 - \alpha)}.$$

This expression has already been found before, see (5.5), and the optimal value for α is given by (5.6), leading to

$$M(\Omega) \le \left(\frac{1}{d} + \sqrt{\frac{1}{d^2} - 1} \right)^2 .$$

As the identity (6.2) yields

$$d = \rho_{\max} ,$$

the theorem is proved. □

As a consequence of the relation $\beta(\Omega) = (1 + \Gamma(\Omega))^{-1/2}$, we deduce Theorem 2.3 from Theorem 6.2, compare with formula (5.7). Now if we define the new angle

$$\psi(\Omega) = \frac{\pi}{2} - \tau(\Omega) \tag{6.11}$$

we obtain a bound from below for $\beta(\Omega)$ which has the same form as the Horgan–Payne inequality:

$$\beta(\Omega) \ge \sin \frac{\psi(\Omega)}{2} . \tag{6.12}$$

7. Final remarks

The article by Horgan and Payne [12] has the title "On Inequalities of Korn, Friedrichs and Babuška–Aziz". We discussed in Sections 2 and 3 the equivalence between the inequalities of Friedrichs and Babuška–Aziz and the equation $C(\Omega) = \Gamma(\Omega) + 1$ between the associated constants that were shown by Horgan–Payne.

In this paper, we have not mentioned *Korn's inequality*, although Horgan–Payne showed a corresponding equivalence between the inequalities of Korn and of Babuška–Aziz and an equality

$$K(\Omega) = 2C(\Omega) \tag{7.1}$$

between the associated constants. The reason is that we do not know whether this equivalence holds in general. In [12], the proof of this equivalence was reduced to the equivalence between two elliptic eigenvalue problems, an argument that is only valid for smooth domains (at least C^2). For more general domains, it is known that the Babuška–Aziz inequality implies Korn's inequality; this proof of Korn's inequality from the inf-sup condition of the divergence is quite standard. It is, however, an open problem if the converse implication holds, in general, too. It is also an open problem if the equality (7.1) is true for non-smooth domains, even for Lipschitz domains where both inequalities are known to be satisfied.

Let us finally mention another famous problem, which, to our knowledge, is still open: The exact value of the Babuška–Aziz constant $C(\Omega)$ — or, equivalently, the LBB constant $\beta(\Omega)$ or the Friedrichs constant $\Gamma(\Omega)$ — if $\Omega \subset \mathbb{R}^2$ is a *square*. In [12], Horgan–Payne pronounced the conjecture that

$$C(\Omega) = 7/2 \qquad \text{for a square.} \tag{7.2}$$

That this is overly optimistic has been known for quite some time, due to the presence of a continuous spectrum in a related spectral problem already mentioned by Friedrichs [11], see [6, 20]. The explicit knowledge of this continuous spectrum gives a lower bound

$$C(\Omega) \geq \left(\frac{1}{2} - \frac{1}{\pi}\right)^{-1} = 5.5.. \qquad \text{for a square.} \tag{7.3}$$

It is not known, however, whether the inequality (7.3) is strict. The current conjecture is rather that (7.3) is an equality.

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