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Uniqueness of positive periodic solutions with some peaks.

Geneviève Allain. Anne Beaulieu.
March 25, 2013

Abstract. This work deals with the semilinear equation $-\Delta u + u - u^p = 0$ in $\mathbb{R}^N$, $2 \leq p < \frac{N+2}{N-2}$. We consider the positive solutions which are $\frac{2\pi}{\varepsilon}$-periodic in $x_1$ and decreasing to 0 in the other variables, uniformly in $x_1$. Let a periodic configuration of points be given on the $x_1$-axis, which repel each other as the period tends to infinity. If there exists a solution which has these points as peaks, we prove that the points must be asymptotically uniformly distributed on the $x_1$-axis. Then, for $\varepsilon$ small enough, we prove the uniqueness up to a translation of the positive solution with some peaks on the $x_1$-axis, for a given minimal period in $x_1$.

1 Introduction.

We consider the equation

$$-\Delta u + u - u^p = 0 \text{ in } S^1_\varepsilon \times \mathbb{R}^{N-1}$$

where $u_+ = \max(u, 0)$.

By $S^1_\varepsilon$, we mean that

$$u(x_1 + \frac{2\pi}{\varepsilon}, x') = u(x_1, x')$$

and that

$$\frac{\partial u}{\partial x_1}(x_1 + \frac{2\pi}{\varepsilon}, x') = \frac{\partial u}{\partial x_1}(x_1, x').$$

We suppose that

$$u(x_1, x') \to 0, \text{ as } |x'| \to 0, \text{ uniformly in } x_1.$$ 

If $u > 0$, we know that $u$ is radial and decreasing in $x'$ ([7], [2]). We consider the subcritical case

$$2 \leq p < \frac{N+2}{N-2} \text{ for } N \geq 3, \quad p \geq 2 \text{ for } N = 2.$$ 

We assume that $p \geq 2$ instead of $p > 1$ for some technical reasons.

Let $U$ be the groundstate solution in $\mathbb{R}^N$. It verifies

$$-\Delta U + U - U^p = 0 \text{ in } \mathbb{R}^N.$$ 

It is known that $U$ is positive, radial and tending to 0 at infinity. Moreover the behavior at infinity is

$$|x|^\frac{N-1}{N} e^{x^2} U(x) \to L_0 \text{ as } |x| \to +\infty$$
and 

\[ |x|^{\frac{N-1}{2}} e^{x^1} \frac{\partial U}{\partial x_1}(x) \to L_1 \quad \text{as} \quad |x| \to +\infty, \quad x_1 > 0, \]

for some positive limits \( L_0 \) and \( L_1 \). (see [9].)

Several recent articles deal with the construction of positive solutions for the equation

\[-\Delta u + u - u^p = 0 \quad \text{in} \quad \mathbb{R}^N.\]

Let us refer to [5], [10], [6].

Let us call the Dancer solution the positive solution of (1.1) which is \( 2\pi \epsilon \)-periodic in \( x_1 \), tending to 0 as \( |x'| \to +\infty \), even in \( x_1 \) and decreasing in \( x_1 \) in \([-\pi, \pi] \). This solution, that we call \( u_D \), was constructed in [5] by a bifurcation from the ground-state solution in \( \mathbb{R}^{N-1} \). The Dancer solution exists when \( 0 < \epsilon < \epsilon^* \), where \( \epsilon^* \) is a known threshold. We have

\[ \|u_D - U\|_{L^{\infty}([-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}] \times \mathbb{R}^{N-1})} \to 0 \text{ as } \epsilon \to 0. \] (1.2)

For all \( x' \) the fonction \( x_1 \mapsto u_D(x_1, x') \) reaches its maximum value at the points \( \frac{l\pi}{\epsilon}, l \in \mathbb{Z} \) and reaches its minimum value at the points \( \pm \frac{\pi}{\epsilon} \).

Now, for any \( \epsilon > 0 \), for any \( k \geq 2 \), let \( a_i^\epsilon, i = 1, ..., k \), be \( k \) points of \([-\pi, \pi] \) which are such that

\[ \frac{a_{i+1}^\epsilon - a_i^\epsilon}{\epsilon} \to +\infty \quad \text{as} \quad \epsilon \to 0, \quad i = 0, ..., k + 1. \] (1.3)

where we denote \( a_0^\epsilon = a_k^\epsilon - 2\pi \) and \( a_{k+1}^\epsilon = a_0^\epsilon + 2\pi \).

Let us denote

\[ U_i(x_1, x') = U(x_1 - \frac{a_i^\epsilon}{\epsilon}, x').\]

Let us give the following

**Definition 1.1** The solution \( u \) of (1.1) admits the points \( \frac{a_1^\epsilon}{\epsilon}, ..., \frac{a_k^\epsilon}{\epsilon} \) as peaks if \( \frac{a_1^\epsilon}{\epsilon}, ..., \frac{a_k^\epsilon}{\epsilon} \) are \( k \) points of \([-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}] \) verifying (1.3) and if

\[ \|u - \sum_{i=1}^{k} U_i\|_{L^{\infty}([-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}] \times \mathbb{R}^{N-1})} \to 0 \quad \text{as} \quad \epsilon \to 0. \] (1.4)

Let us remark that by the Maximum Principle, any solution of (1.1) verifying (1.4) needs to be positive.

We can ask whether for any configuration of points in a period which repel each other in the sense of (1.3), there exists a solution having these points as peaks. We give a negative answer. In particular it is not possible to consider peaks which repel each other with an infinitely small speed wrt the period.

Our main result is the following uniqueness result for the small values of \( \epsilon \).

**Theorem 1.1** Let \( u \) be a solution of (1.1) that admits the points \( \frac{a_1^\epsilon}{\epsilon}, ..., \frac{a_k^\epsilon}{\epsilon} \) in \([-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}] \) as peaks in the sense of the definition 1.1. Then, for \( \epsilon \) small enough there exists \( \alpha_\epsilon \to 0 \) such that

\[ u(x_1, x') = u_D(x_1 - \frac{a_1^\epsilon}{\epsilon} - \alpha_\epsilon, x'). \]
where $u_D$ is the Dancer solution of period $\frac{2\pi}{k\varepsilon}$.

We can write $u_D$ as

$$u_D(x) = \sum_{l \in \mathbb{Z}} U(x_1 + \frac{2l\pi}{k\varepsilon}, x') + \psi(x)$$

(1.5)

and if we define

$$d_x = \text{dist}(x, \cup_{l \in \mathbb{Z}} \{(\frac{2l\pi}{k\varepsilon}, 0)\}),$$

then for every $0 < \eta' \leq \min\{2 - \eta, 2(p - 1 - \eta)\}$, there exists $C$ independent of $\varepsilon$ such that

$$(|\psi| + |\nabla \psi|)(x) \leq Ce^{-\eta d_x} e^{-\frac{\eta}{k\varepsilon}(\frac{\pi}{\varepsilon})^{\frac{1-N}{2}}}.$$  (1.6)

The most involved part of the proof of Theorem 1.1 is to prove that the peaks are asymptotically uniformly distributed. More precisely, we will begin by the proof of the following

**Proposition 1.1** Let $u$ be a solution of (1.1) admitting the points $\frac{a_1}{\varepsilon}, \ldots, \frac{a_k}{\varepsilon}$ in $[-\pi\varepsilon, \pi\varepsilon]$ as peaks.

Then we have necessarily

$$\frac{a^{i+1}}{\varepsilon} - \frac{2\pi}{k\varepsilon} \rightarrow 0, \quad i = 0, \ldots, k + 1.$$  (1.7)

In [10], part 3, Malchiodi gives a construction of a periodic solution with one peak, using a Lyapunov-Schmitt method.

Let us quote the following

**Proposition 1.2** (Malchiodi, [10], Corollary 3.2.) For $1 < p < \frac{N+2}{N-2}$, there exists a solution of (1.1), even in $x_1$, of the form

$$v = \sum_{i \in \mathbb{Z}} U(x_1 + i\frac{2\pi}{\varepsilon}, x') + \overline{w}$$

(1.8)

where

$$||\overline{w}||_{H^1([\frac{a_i}{\varepsilon}, \frac{a_{i+1}}{\varepsilon}] \times \mathbb{R}^{N-1})} \rightarrow 0$$

and

$$|\overline{w}(x)| + |\nabla \overline{w}(x)| \leq Ce^{-\frac{\pi}{\varepsilon}(1+\xi_0)} e^{-\eta_0 \text{dist}(x, \cup_{l \in \mathbb{Z}} \{(\frac{2l\pi}{k\varepsilon}, 0)\})}$$

(1.9)

for some $\xi_0 > 0$ and $\eta_0 > 0$.

This solution is the Dancer solution, in consideration of the uniqueness of the even $\frac{2\pi}{k\varepsilon}$-periodic solution which verifies (1.2) (see [4], p. 969). In that previous work, the functions are assumed to be even in $x_1$. In ours, we have to overcome some difficulties arising from the lack of evenness. Finally, we prove that the solution is even.

In the course of the proof of Theorem 1.1, we will consider an approximate solution of (1.1).  

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Let us denote
\[ U_{i,l}(x) = U(x_1 - \frac{a^i_j + 2\pi l}{\varepsilon}, x'), \quad i = 1, \ldots, k, \quad l \in \mathbb{Z}, \]
then
\[ U_{i,0} = U_i. \]

Let us define
\[ v_i = \sum_{l \in \mathbb{Z}} U_{i,l}, \quad \pi_k = \sum_{i=1}^k v_i. \]

We will study the linearized operator about this approximate solution, namely
\[ L = -\Delta + 1 - pU_p^{-1}. \]

We will prove that the linearized operator \( L \) has no zero eigenvalue and we will give an estimate of the eigenvalues which tend to 0.

The operator \((-\Delta + 1)^{-1}L\) is an operator of \( H^1(S_1^1 \times \mathbb{R}^{N-1}) \) into itself of the form \( id - K \), where \( K \) is a compact operator. So \((-\Delta + 1)^{-1}L\) is a Fredholm operator of index 0.

We consider the eigenvalues of the operator \( L \), in the following sense:
there exists \( \xi \in H^1(S_1^1 \times \mathbb{R}^{N-1}), \xi \neq 0, \) verifying
\[ L \xi = \lambda (-\Delta + 1) \xi. \]

The operator \( L \) has a countably infinite discrete set of eigenvalues, \( \lambda_i, \ i = 1, 2, \ldots \). If we designate by \( V_i \) the eigenspace corresponding to \( \lambda_i \), by \( H^1 \) the space \( H^1(S_1^1 \times \mathbb{R}^{N-1}) \) and by \( L^2 \) the space \( L^2(S_1^1 \times \mathbb{R}^{N-1}) \), then
\[ \lambda_i = \inf \left\{ \frac{<L^2 u, u>_{L^2}}{<u, u>_{H^1}}, u \neq 0, <u, v>_{H^1} = 0, \forall v \in V_1 \oplus \ldots \oplus V_{i-1} \right\}, \quad \text{for } i \geq 2, \quad (1.10) \]
and
\[ \lambda_1 = \inf \left\{ \frac{<L^2 u, u>_{L^2}}{<u, u>_{H^1}}, u \neq 0 \right\}, \]
(see [8]). Let us quote the following result concerning the eigenvalues of the operator \(-\Delta + 1 - pU_p^{-1}\) (with the definition above, with \( \mathbb{R}^N \) instead of \( S_1^1 \times \mathbb{R}^{N-1} \) and when \( k = 1 \) and \( a^i_\varepsilon = 0 \))

**Theorem 1.2** The first eigenvalue of \(-\Delta + 1 - pU_p^{-1}\) in \( \mathbb{R}^N \) is \( 1 - p \). The eigenspace associated with the eigenvalue 0 is spanned by the eigenvectors \( \partial U/\partial x_j, \ j = 1, \ldots, N. \)

This theorem follows from [1].

Let us define
\[ \sigma_i = \frac{1}{2} \text{dist} \left( \frac{a^i_j}{\varepsilon}, \cup_{j \neq i} a^j_{\varepsilon} \right) + 1, \quad i = 1, \ldots, k, \quad (1.11) \]
and
\[ \bar{\sigma} = \min_{i=1}^k \sigma_i. \]

Let us summarize the properties of the eigenvalues and of the eigenvectors of \( L \) in the following.
Theorem 1.3

(i) The eigenvalues of $L$ are less than 1. There exists a sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ such that each eigenvalue of $L$ tends either to 1 or to an eigenvalue of $-\Delta + 1 - pU_p - 1$ as $\varepsilon_m \to 0$.

(ii) Let $F$ be the vector space associated with the eigenvalues tending to 0. Then the dimension of $F$ is $k$ and $F$ is spanned by $k$ eigenvectors $\varphi_i$, $i = 1, \ldots, k$ such that there exist $k$ real numbers $\alpha_i \neq 0$, independent of $\varepsilon$, verifying

\[ < \varphi_i, \varphi_j >_{H^1(S^1_\varepsilon \times \mathbb{R}^{N-1})} = 0 \quad i \neq j \quad ; \quad \| \varphi_i \|_\infty = 1 \]  

and

\[ \| \varphi_i - \alpha_i \frac{\partial v_i}{\partial x_1} \|_{L^q(S^1_\varepsilon \times \mathbb{R}^{N-1})} + \| \nabla (\varphi_i - \alpha_i \frac{\partial v_i}{\partial x_1}) \|_{L^q(S^1_\varepsilon \times \mathbb{R}^{N-1})} \to 0 \]  

for all $1 \leq q \leq \infty$.

(iii) If $\lambda_i(\varepsilon_m) \to 0$, then $\lambda_i(\varepsilon_m) \neq 0$ and

\[ \lambda_i(\varepsilon_m) e^{-2\alpha_i \frac{\varepsilon_m}{\varepsilon}} \to H \]  

where $H \neq 0$.

The paper is organized as follows. In section 2, we study the eigenvalues of the operator $L$ which tend to 0 and the associated eigenvectors. We give the proof of Theorem 1.3. In section 3, we use a Lyapunov-Schmitt method to give the proof of Proposition 1.1. In section 4, we conclude the proof of Theorem 1.1.

In sections 2 and 3, we will refer to some technical results, which are reported in the appendix (section 5).

2 An analysis of the eigenvalues.

In this part, we prove the theorem 1.3.

Proof of (i).

Let $\varphi$ be such that

\[ L \varphi = \lambda (-\Delta \varphi + \varphi) \quad \text{in} \quad S^1_\varepsilon \times \mathbb{R}^{N-1}. \]

We suppose that there exists $c$ such that $\varphi(c) = 1$ and that $\| \varphi \|_\infty = 1$. We denote

\[ \phi(x) = \varphi(x + c). \]

By standard elliptic estimates, there exists a subsequence such that $\phi \to \bar{\varphi}$ uniformly on the compact sets of $\mathbb{R}^N$. Let us suppose that $\lambda \neq 1$. First, if $|c - (\frac{\alpha_i}{\varepsilon}, 0)| \to +\infty$ for all $i$, then

\[ -\Delta \bar{\varphi} + \bar{\varphi} = 0 \quad \text{in} \quad \mathbb{R}^N \quad ; \quad \| \bar{\varphi} \|_\infty = \bar{\varphi}(0) = 1. \]
This is in contradiction with the maximum principle, so this case does not occur. So there exists $\bar{c}$ and $i$ such that $(c - (\frac{a_i}{\epsilon}, 0)) \to \bar{c}$. Then,

$$(-\Delta + 1 - pU^{p-1})(x + \bar{c})\bar{\phi} = \bar{\lambda}(-\Delta \bar{\phi} + \bar{\phi}) \quad \text{in } \mathbb{R}^N$$

(2.15)

and $\bar{\phi}$ is non zero and even in $x'$. Thus $\bar{\lambda}$ is an eigenvalue of $-\Delta + 1 - pU^{p-1}$.

By a diagonal process, we can construct a subsequence $(\epsilon_m)$ such that any eigenvalue of $L$ which does not tend to 1 converges to an eigenvalue of $-\Delta + 1 - pU^{p-1}$.

Proof of (ii).

We divide the proof into three parts.

Firstly, let us prove that if \( \varphi \in F \setminus \{0\} \), then there exists $I \subset \{1, \ldots, k\}$ and some real numbers $\alpha_i \neq 0$ and independent of $\epsilon$ such that

$$\|\varphi - \sum_{i \in I} \alpha_i \frac{\partial v_i}{\partial x_1}\|_{\infty} + \|\nabla (\varphi - \sum_{i \in I} \alpha_i \frac{\partial v_i}{\partial x_1})\|_{\infty} \to 0$$

(2.16)

We follow the proof of (i) to get (2.15) with $\lambda = 0$. Then we can denote $c$ instead of \((c, 0) \in \mathbb{R} \times \mathbb{R}^N\) and there exists some real number $\alpha \neq 0$ such that

$$\varphi(x) = \alpha \frac{\partial U}{\partial x_1}(x_1 + \bar{c}, x').$$

We get

$$\varphi(x + c) - \alpha \frac{\partial U}{\partial x_1}(x + \bar{c}) \to 0 \quad \text{uniformly on the compact sets,}$$

that is

$$\varphi(x + c) - \alpha \frac{\partial U}{\partial x_1}(x + c - \frac{a_i}{\epsilon}) \to 0 \quad \text{uniformly on the compact sets,}$$

that leads to

$$\varphi(x) - \alpha \frac{\partial U_i}{\partial x_1}(x) \to 0 \quad \text{uniformly for } x \text{ such that } (x_1 - \frac{a_i}{\epsilon}) \text{ is bounded.}$$

Finally for each $i$, either there exists $\alpha_i \neq 0$ such that

$$\varphi(x) - \alpha_i \frac{\partial v_i}{\partial x_1} \to 0 \quad \text{uniformly for } x \text{ such that } (x_1 - \frac{a_i}{\epsilon}) \text{ is bounded}$$

or

$$\varphi \to 0 \quad \text{uniformly for } x \text{ such that } (x_1 - \frac{a_i}{\epsilon}) \text{ is bounded.}$$

Moreover, the first case occurs for at least one $i$. By the beginning of the present proof,

$$\varphi(x) \to 0 \quad \text{if } |x - (\frac{a_i}{\epsilon}, 0)| \to +\infty \forall i.$$
Thus there exists $J \subset \{1, ..., k\}$ and $\alpha_i \neq 0$ and independent of $\varepsilon$ such that

$$\|\varphi - \sum_{i \in J} \alpha_i \frac{\partial v_i}{\partial x_1}\|_\infty \to 0. \quad (2.17)$$

We deduce that

$$\|\nabla(\varphi - \sum_{i \in J} \alpha_i \frac{\partial v_i}{\partial x_1})\|_\infty \to 0 \quad (2.18)$$

by standard elliptic arguments. Since the functions $\frac{\partial v_i}{\partial x_1}$ are linearly independent, we deduce that $\dim F \leq k$.

Secondly, let us assume that $\varphi_1 \in F$ and $\varphi_2 \in F$ are such that

$$\langle \varphi_1, \varphi_2 \rangle_{H^1(\mathbb{R}^N)} = 0 \quad \text{and} \quad \|\varphi_1\|_\infty = \|\varphi_2\|_\infty = 1.$$  

We write

$$\varphi_1 = \sum_{i \in J_1} \alpha_i \frac{\partial v_i}{\partial x_1} + o(1) \quad \text{and} \quad \varphi_2 = \sum_{i \in J_2} \beta_i \frac{\partial v_i}{\partial x_1} + o(1)$$

in the sense of (2.17) and (2.18). Taking the scalar product in $H^1$ we obtain

$$0 = \sum_{i \in J_1, j \in J_2} \alpha_i \beta_j \langle \frac{\partial v_i}{\partial x_1}, \frac{\partial v_j}{\partial x_1} \rangle_{H^1} + o(1), \quad \sum_{i \in J_1} \alpha_i \frac{\partial v_i}{\partial x_1} > H^1 + o(1), \quad \sum_{i \in J_2} \beta_i \frac{\partial v_i}{\partial x_1} > H^1$$

$$+ < o(1), o(1) >_{H^1}.$$  

In view of Proposition 5.7, the Lebesgue Theorem leads to

$$0 = \sum_{i \in J_1 \cap J_2} \alpha_i \beta_i \frac{\partial U}{\partial x_1} H^1(\mathbb{R}^N). \quad (2.19)$$

We deduce that $J_1 \cap J_2 = \emptyset$.

Thirdly, let us assume that $F \neq \{0\}$. We define a finite set $J$ and eigenvalues $\lambda_j, \ j \in J$ such that $\lambda_j(\varepsilon_m) \to 0$. Let $\varphi_j$ be an eigenvector associated with $\lambda_j$. Let us assume that

$$\langle \varphi_i, \varphi_j \rangle_{H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})} = 0 \quad i \neq j; \quad \|\varphi_i\|_{L^\infty([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})} = 1.$$  

We write

$$\frac{\partial v_i}{\partial x_1} = \sum_{j \in J} c_j \varphi_j + \xi; \quad < \xi, \varphi_j >= 0 \text{ for } j \in J.$$  

We have

$$I \xi = I \frac{\partial v_i}{\partial x_1} - \sum_j c_j \lambda_j (-\Delta \varphi_j + \varphi_j). \quad (2.20)$$
In view of (5.61), we deduce that

$$\|\xi\|_\infty \leq C \|L\xi\|_\infty$$

and consequently that

$$\|\xi\|_\infty \to 0.$$  

We conclude that there exists at least one $j$ such that $c_j \neq 0$ and such that $\varphi_j((\frac{\pi}{\varepsilon}, 0)) \neq 0$. By the second step, this $j$ is unique. Let us call it $i$ and then we have that

$$\frac{\partial v_i}{\partial x_1} - c_i \varphi_i \to 0 \quad \text{if } |x_1 - \frac{a_i}{\varepsilon_m}| \text{ is bounded.}$$

Now, since $\|\frac{\partial v_i}{\partial x_1}\|_\infty \neq 0$, we deduce that $F \neq \{0\}$ and that $\dim F \geq k$. Consequently

$$\dim F = k$$

and we define $(\varphi_1, \ldots, \varphi_k)$ a basis of $F$ verifying (1.12) and such that

$$\|\varphi_i - c_i \frac{\partial v_i}{\partial x_1}\|_\infty + \|\nabla (\varphi_i - c_i \frac{\partial v_i}{\partial x_1})\|_\infty \to 0$$

with $\alpha_i \neq 0$, independent of $\varepsilon$.

Finally we write

$$\frac{\partial v_i}{\partial x_1} = \sum_{j=1}^{k} c_j \varphi_j + \xi : \quad <\xi, \varphi_j>_{H^1} = 0 \text{ for all } j$$  \hspace{1cm} (2.21)$$

and $c_i \neq 0$ and $\|\xi\|_\infty \to 0$. For each $j \neq i$ we have, by our convention, $\varphi_j((\frac{\pi}{\varepsilon}, 0)) \neq 0$. Since $\frac{\partial v_i}{\partial x_1}$ and $\xi$ tend to 0 at $\frac{\pi}{\varepsilon}$, we infer that

$$c_j \to 0 \quad \text{for all } j \neq i.$$  

Moreover, by Proposition 5.7, $\varphi_i$ is bounded in $L^q$ and $\|\xi\|_{L^q} \to 0$ for all $q \geq 1$, that gives (1.13).

Proof of (iii).

Let us adopt the case where $k \geq 2$ and where $|\frac{\pi}{\varepsilon} - \sigma_i| \to +\infty$. Otherwise, Proposition 1.1 is irrelevant, and the estimate of $\lambda_i$ is true, but must be done for the period $\frac{4\pi}{\varepsilon}$ instead of $\frac{2\pi}{\varepsilon}$.

We have by (2.21)

$$\int \mathbb{L} \frac{\partial v_i}{\partial x_1} \varphi_i dx = c_i \lambda_i \|\varphi_i\|_{H^1}^2$$

and $\frac{1}{|c_i\|\varphi_i\|_{H^1}}$ is bounded from below, in view of (5.77). We write

$$\int \mathbb{L} \frac{\partial v_i}{\partial x_1} \varphi_i dx = \frac{1}{c_i} \int \mathbb{L} \frac{\partial v_i}{\partial x_1} \frac{\partial v_i}{\partial x_1} dx - \int \mathbb{L} \frac{\partial v_i}{\partial x_1} (\frac{1}{c_i} \frac{\partial v_i}{\partial x_1} - \varphi_i) dx$$

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Let us define, for $j = 0, ..., k + 1$

$$
\Omega_j = \{ x \in [-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1} : \text{dist}(x, \cup_{l=0}^{k+1} \{ (\frac{a_l^i}{\varepsilon}, 0) \}) = |x - (\frac{a_l^j}{\varepsilon}, 0)| \}. \tag{2.22}
$$

We have

$$
[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1} = \bigcup_{i=0}^{k+1} \Omega_i.
$$

We write

$$
< L \frac{\partial v_i}{\partial x_1}, \frac{\partial v_i}{\partial x_1} >_{L^2} = p \sum_i \int_{\Omega_i} (U^{p-1}_{i,l} - \bar{w}^{p-1}_{i,l}) \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx \tag{2.23}
$$

We are going to prove on one hand that

$$
\int_{\Omega_i} (U^{p-1}_{i,l} - \bar{w}^{p-1}_{i,l}) \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx = (1 - p) \int_{\Omega_i} U^{p-2}_{i,l} \sum_{j \neq i} v_j \left( \frac{\partial U_{i,l}}{\partial x_1} \right)^2 dx + o(e^{-2\sigma_i} \sigma_i^{\frac{1-N}{2}}) \tag{2.24}
$$

and on the other hand that

$$
\sum_{j=0}^{k+1} \sum_{l \neq i} \left| \int_{\Omega_j} (U^{p-1}_{i,l} - \bar{w}^{p-1}_{i,l}) \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx \right| + \sum_{l \neq 0} \int_{\Omega_i} (U^{p-1}_{i,l} - \bar{w}^{p-1}_{i,l}) \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx \tag{2.25}
$$

$$
= o(e^{-2\sigma_i} \sigma_i^{\frac{1-N}{2}}).
$$

Using Lemma 5.2, we write, if $p > 2$,

$$
\int_{\Omega_i} (U^{p-1}_{i,l} - \bar{w}^{p-1}_{i,l}) \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx = (1 - p) \int_{\Omega_i} U^{p-2}_{i,l} \sum_{j \neq i} v_j + \sum_{l \neq 0} U_{i,l} \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx
$$

$$
+ O\int_{\Omega_i} \left( \sum_{j \neq i} v_j + \sum_{l \neq 0} U_{i,l} \right)^{\min\{p-1,2\}} \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx
$$

while, if $p = 2$

$$
\int_{\Omega_i} (U^{p-1}_{i,l} - \bar{w}^{p-1}_{i,l}) \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx = (1 - p) \int_{\Omega_i} U^{p-2}_{i,l} \left( \sum_{j \neq i} v_j + \sum_{l \neq 0} U_{i,l} \right) \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx.
$$

Now, we use Proposition 5.9 to get, when $p - 1 > 1$

$$
\int_{\Omega_i} \left( \sum_{j \neq i} v_j + \sum_{l \neq 0} U_{i,l} \right)^{\min\{p-1,2\}} \frac{\partial v_i}{\partial x_1} \frac{\partial U_{i,l}}{\partial x_1} dx = \mathcal{O}(e^{-2\sigma_i} \min\{p-1,2\} \sigma_i^{\frac{1-N}{2}}).
$$

Since $\frac{\pi}{\varepsilon} - \sigma_i \to +\infty$, we have

$$
e^{-\frac{2\pi}{\varepsilon}} = o(e^{-2\sigma_i}).$$
Using Proposition 5.9 again, we obtain for all $p \geq 2$

$$\int_{\Omega_i} U_i^{p-2} \left( \sum_{l \neq 0} U_{i,l} \right) \frac{\partial v_i}{\partial x_1} \frac{\partial U_i}{\partial x_1} \, dx = o(e^{-2\sigma_i \sigma_i^{1-N}})$$

and

$$\int_{\Omega_i} U_i^{p-2} \left( \sum_{j \neq i} v_j \right) \frac{\partial v_i}{\partial x_1} \frac{\partial U_i}{\partial x_1} \, dx = \int_{\Omega_i} U_i^{p-2} \left( \sum_{j \neq i} v_j \right) \left( \frac{\partial U_i}{\partial x_1} \right)^2 \, dx + o(e^{-2\sigma_i \sigma_i^{1-N}}).$$

We have proved (2.24).

Let us turn now to the proof of (2.25).

We estimate, for $l \neq 0$ and using Proposition 5.9

$$\int_{\Omega_i} \left( U_i^{p-1} - \overline{u}_e^{p-1} \right) \frac{\partial v_i}{\partial x_1} \frac{\partial U_i}{\partial x_1} \, dx = o(e^{-2\sigma_i \sigma_i^{1-N}}).$$

Now, let $0 < \beta < 1$ be given. We have for all $l$ and for $j = 0, \ldots, k+1, j \neq i$

$$\left| \int_{\Omega_j} \left( U_i^{p-1} - \overline{u}_e^{p-1} \right) \frac{\partial v_i}{\partial x_1} \frac{\partial U_i}{\partial x_1} \, dx \right| \leq \int_{\Omega_j} \left( U_i^{p-1} + \overline{u}_e^{p-1} \right) \left| \frac{\partial v_i}{\partial x_1} \right| \left| \frac{\partial U_i}{\partial x_1} \right| \, dx \| \frac{\partial v_i}{\partial x_1} \|_{L^\infty(\Omega_j)}^{1-\beta}.$$

But

$$\left| \frac{a_i}{\varepsilon} + \frac{2\pi l}{\varepsilon} - \frac{a_j}{\varepsilon} \right| \geq 2\sigma_i.$$

Choosing $\beta$ such that $p - 1 + \beta > 1$, we get, using Proposition 5.9

$$\int_{\Omega_j} \left( U_i^{p-1} + \overline{u}_e^{p-1} \right) \left| \frac{\partial v_i}{\partial x_1} \right| \left| \frac{\partial U_i}{\partial x_1} \right| \, dx = O(e^{-2\sigma_i \sigma_i^{1-N}}).$$

Since

$$\| \frac{\partial v_i}{\partial x_1} \|_{L^\infty(|x-(\frac{a_j}{\varepsilon},0)|<\sigma_j)}^{1-\beta} \to 0$$

we deduce (2.25).

Now (2.24) and (2.25) give

$$< L \frac{\partial v_i}{\partial x_1}, \frac{\partial v_i}{\partial x_1} >_{L^2} = \int_{\Omega_i} U_i^{p-2} \left( \sum_{j \neq i} v_j \right) \left( \frac{\partial U_i}{\partial x_1} \right)^2 \, dx + o(e^{-2\sigma_i \sigma_i^{1-N}}). \tag{2.26}$$

Using Corollary 5.1, we get

$$e^{2\sigma_i \sigma_i^{1-N}} \int_{\Omega_i} U_i^{p-2} \left( \sum_{j \neq i} v_j \right) \left( \frac{\partial U_i}{\partial x_1} \right)^2 \, dx \to H_i$$

where $H_i \neq 0$ is a real number.
It remains now to prove that
\[
\int L \frac{\partial v_i}{\partial x_1} \left( \frac{1}{c_i} \partial v_i - \varphi_i \right) dx = o(e^{-2\sigma_i \frac{1-N}{2}}). \tag{2.27}
\]
We write
\[
| \int_{\Omega_i} \frac{\partial v_i}{\partial x_1} (\varphi_i - \frac{1}{c_i} \partial v_i) dx | = (1-p) \int_{\Omega_i} U_i^{p-2} \sum_{j \neq i} U_j \frac{\partial v_i}{\partial x_1} (\varphi_i - \frac{1}{c_i} \partial v_i) dx |
\]
\[
+ o(e^{-2\sigma_i \frac{1-N}{2}})
\]
\[
\leq \int_{\Omega_i} \sum_{j \neq i} e^{-|x-(\frac{a_j}{\varepsilon},0)|} e^{-|x-(\frac{a_j}{\varepsilon},0)|} |\varphi_i - \frac{1}{c_i} \partial v_i| dx + o(e^{-2\sigma_i \frac{1-N}{2}})
\]
\[
= O(e^{-2\sigma_i \frac{1-N}{2}}) \int_{\Omega_i} |\varphi_i - \frac{1}{c_i} \partial v_i| dx + o(e^{-2\sigma_i \frac{1-N}{2}}) = o(e^{-2\sigma_i \frac{1-N}{2}}).
\]

For \( j \neq i \), we write
\[
| \int_{\Omega_j} \frac{\partial v_i}{\partial x_1} (\varphi_i - \frac{1}{c_i} \partial v_i) dx | \leq C \sum_{l} \int_{\Omega_j} |(U_{i,l}^{p-1} + U_{j,l}^{p-1}) \frac{\partial U_{i,l}}{\partial x_1} (\varphi_i - \frac{1}{c_i} \partial v_i)| dx
\]
\[
\leq C \int_{\Omega_j} e^{-2\sigma_i \frac{1-N}{2}} |\varphi_i - \frac{1}{c_i} \partial v_i| dx = o(e^{-2\sigma_i \frac{1-N}{2}})
\]
We obtain (2.27) and consequently, we have proved (1.14).

### 3 The Lyapunov-Schmitt reduction.

In this part, we prove the proposition 1.1.

Let us define
\[
\mathcal{M}(u) = -\Delta u + u - u^p_T.
\]

To begin with, we have

**Lemma 3.1**
\[
||\mathcal{M}(\pi_\varepsilon)||_\infty \leq C e^{-2\sigma_\varepsilon \frac{1-N}{2}}. \tag{3.28}
\]

**Proof.**
\[
\mathcal{M}(\pi_\varepsilon) = \sum_{i,l} U_{i,l}^p \frac{\partial U_{i,l}}{\partial x_1} (\varphi_i - \frac{1}{c_i} \partial v_i)
\]
We have for all \( i \)
\[
 \sum_{j \neq i} U_{j,l} + \sum_{l \in \mathbb{Z}^*} U_{i,l} \leq C e^{-|x-(\frac{a_i}{\varepsilon},0)|} \left( \frac{a_i}{2\varepsilon} \right) e^{-|x-\frac{a_i}{\varepsilon}|} \text{ in } \Omega_i.
\]
In $\Omega_i$, we write by Lemma 5.2
\[
\mathcal{M}(\bar{u}_i) = -pU_i^{p-1}(\sum_{j \neq i, l \in \mathbb{Z}} U_{j,l} + \sum_{l \in \mathbb{Z}} U_{i,l}) + 0(\sum_{j \neq i, l \in \mathbb{Z}} U_{j,l} + \sum_{l \in \mathbb{Z}} U_{i,l})^2 + \sum_{j \neq i, l \in \mathbb{Z}} U_{j,l}^p
\]
while $p - 1 \geq 1$. We easily deduce the proof of the Lemma.

Let $k$ real numbers $\delta_1, \ldots, \delta_k$ and $v \in H^1(\frac{\Omega_i}{\varepsilon} \times \mathbb{R}^{N-1})$ be given. Let us suppose that
\[
u = \bar{u}_i + v + \sum_{i=1}^{k} \delta_i \varphi_i; \quad <v, \varphi_i >_{H^1} = 0, \quad i = 1, \ldots, k
\]
is a solution of (1.1) and that
\[
\|v\|_{\infty} + \sum_{i=1}^{k} |\delta_i| \to 0.
\]

We define
\[
h = -\mathcal{M}(\bar{u}_i + v + \sum_{i=1}^{k} \delta_i \varphi_i) + \mathcal{L}(v + \sum_{i=1}^{k} \delta_i \varphi_i).
\]
Then $v$ and $\delta_1, \ldots, \delta_k$ are such that
\[
\mathcal{L}(v + \sum_{i=1}^{k} \delta_i \varphi_i) = h.
\]

We denote
\[
h = h^\perp + h^\top, \quad h^\top \in \text{Vect}\{\varphi_1, \ldots, \varphi_k\}, \quad h^\perp \in (\text{Vect}\{\varphi_1, \ldots, \varphi_k\})^\perp,
\]
relatively to the Hilbert space $H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$. First, $v$ is a $\frac{2\pi}{\varepsilon}$-periodic solution of the equation
\[
\begin{cases}
\mathcal{L}v = h^\perp \\
<v, \varphi_i >_{H^1} = 0, \quad i = 1, \ldots, k.
\end{cases}
\]
Then $(\delta_1, \ldots, \delta_k)$ verifies
\[
\mathcal{L}(\sum_{i=1}^{k} \delta_i \varphi_i) = h^\top.
\]

We have the following

**Proposition 3.3** Let $v$ be a solution of (3.32). Then there exists $C$ independent of $\varepsilon$ such that
\[
\text{if } p > 2 \quad \|v\|_{H^1} \leq C(e^{-2\sigma \frac{1-N}{p}} + \sum_{i=1}^{k} |\delta_i|^2)
\]
\[
\text{if } p = 2 \forall \eta \in [0,1[, \quad \|v\|_{H^1} \leq C(e^{-2\sigma \frac{1-N}{2}} + \sum_{i=1}^{k} |\delta_i|^2)
\]
and for all $p$
\[
\|v\|_{\infty} + \|\nabla v\|_{\infty} \leq C(e^{-2\sigma \frac{1-N}{p}} + \sum_{i=1}^{k} |\delta_i|^2).
\]
Proof. 
We write 
\[ h = \Delta \pi_e - \pi_e + (\pi_e + v + \sum \delta_i \varphi_i)_+ - p \pi_e^{p-1}(v + \sum \delta_i \varphi_i) \]

and Lemma 5.2 gives 
\[ |(\pi_e + v + \sum \delta_i \varphi_i)_+ - \pi_e - p \pi_e^{p-1}(v + \sum \delta_i \varphi_i)| \leq C|v + \sum \delta_i \varphi_i|^2. \]

We deduce that 
\[ |h + M(\pi_e)| \leq C|v + \sum \delta_i \varphi_i|^2. \tag{3.35} \]

Since 
\[ h^+ = h - \sum_{i=1}^k <h, \varphi_i> (-\Delta \varphi_i + \varphi_i), \]

then 
\[ \|h^+\|_{L^2} \leq C\|h\|_{L^2} \quad \text{and} \quad \|h^+\|_{\infty} \leq C\|h\|_{\infty}. \]

Now, we use (5.61) to obtain 
\[ \|v\|_{\infty} \leq C\|h^+\|_{\infty}. \]

Using (3.28), we deduce the estimate 
\[ \|v\|_{\infty} \leq C\|h^+\|_{L^2} \] 
\[ \text{and the estimate (3.34) follows in the standard way.} \]

We have also by (5.64) 
\[ \|v\|_{H^1} \leq C\|h^+\|_{L^2}. \]

We write 
\[ \|h\|_{L^2}^2 \leq C(\|M(\pi_e)\|_{L^2}^2 + \|v\|_{L^2}^4 + \sum_j |\delta_j|^4). \]

Using (3.36), we deduce, for \( \varepsilon \) small enough 
\[ \|v\|_{H^1} \leq C(\|M(\pi_e)\|_{L^2} + \sum_j |\delta_j|^2). \]

Now we use (3.29) and Lemma 5.2 to obtain 
\[ \|M(\pi_e)\|_{L^2}^2 \leq \sum_{i=0}^{k+1} \int_{\Omega_i} pu_i^{2(p-1)} (\sum_{j \neq i; l \in \mathbb{Z}} U_{j,l} + \sum_{l \in \mathbb{Z}^*} U_{i,l})^2 + C \int_{\Omega_i} (\sum_{j \neq i; l \in \mathbb{Z}} U_{j,l} + \sum_{l \in \mathbb{Z}^*} U_{i,l})^4. \]

Proposition 5.9 gives 
\[ \|M(\pi_e)\|_{L^2} \leq C(e^{-2N\frac{1-N}{2}}) \quad \text{if} \ p > 2 \tag{3.37} \]
and for all $\eta \in [0, 1[$
\[
\|\mathcal{M}(\overline{u}_\varepsilon)\|_{L^2} \leq C(e^{-2\eta \sigma \frac{1-N}{2}}) \quad \text{if } p = 2.
\]

We deduce (3.33).
We have proved the proposition.

Now let $d_i$ be defined by
\[
h^T = \sum_{i=1}^k d_i (-\Delta \varphi_i + \varphi_i).
\]
We have the following

**Proposition 3.4** For $i = 1, ..., k$
\[
d_i = \frac{p}{\|\varphi_i\|_{H^1}^2} \int_{\Omega_i} U_i^{p-1} \sum_{j \neq i} v_j \frac{\partial U_i}{\partial x_1} dx + O(\sum_{j=1}^k \delta_j^2) + o(e^{-2\eta \sigma \frac{1-N}{2}}).
\]

**Proof.** We have
\[
d_i = \frac{1}{\|\varphi_i\|_{H^1}^2} \int_{\frac{\partial S}{\varepsilon} \times \mathbb{R}^{N-1}} (h \varphi_i) dx.
\]

The coefficient $\|\varphi_i\|_{H^1}$ does not matter, thanks to (5.77). We deduce from (3.34) and (3.35) that
\[
\int_{\frac{\partial S}{\varepsilon} \times \mathbb{R}^{N-1}} (h + \mathcal{M}(\overline{u}_\varepsilon)) \varphi_i dx = O(e^{-4\eta \sigma \frac{1-N}{2}} + \sum_{j} \delta_j^2).
\]

Now let us estimate the second integral, for $i = 1, ..., k$. Without loss of generality, we let $i = 1$. We write
\[
\int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} \mathcal{M}(\overline{u}_\varepsilon) \varphi_1 dx = \int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} \mathcal{M}(\overline{u}_\varepsilon)(\varphi_1 - \alpha_1 \frac{\partial v_1}{\partial x_1}) dx + \alpha_1 \int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} \mathcal{M}(\overline{u}_\varepsilon) \frac{\partial v_1}{\partial x_1} dx.
\]

The estimate (3.28) gives directly
\[
\int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} \mathcal{M}(\overline{u}_\varepsilon)(\varphi_1 - \alpha_1 \frac{\partial v_1}{\partial x_1}) dx = o(e^{-2\eta \sigma \frac{1-N}{2}}),
\]
since
\[
\|\varphi_1 - \alpha_1 \frac{\partial v_1}{\partial x_1}\|_{L^1} \to 0.
\]

Now, as in the proof of Theorem 1.3, (iii), we write
\[
\int_{-\frac{\pi}{\varepsilon}}^{\frac{\pi}{\varepsilon}} \mathcal{M}(\overline{u}_\varepsilon) \frac{\partial v_1}{\partial x_1} dx = \int_{\Omega_1} \mathcal{M}(\overline{u}_\varepsilon) \frac{\partial v_1}{\partial x_1} dx + \sum_{j=0}^{k+1} p \int_{\Omega_j} \mathcal{M}(\overline{u}_\varepsilon) \frac{\partial v_1}{\partial x_1} dx.
\]
We have
\[ M(\pi_\varepsilon) = \sum_{i,l} U_{i,l}^p - (\sum_{i,l} U_{i,l})^p. \]

By Lemma 5.2 we have, in \( \Omega_j \),
\[ M(\pi_\varepsilon) = -pU_{j,l}^{p-1} \left( \sum_{i \neq j,l \in \mathbb{Z}} U_{i,l} + \sum_{l \neq 0} U_{j,l} \right) + O(e^{-2\sigma_j \sigma_1^{1-N}}) \]

We get, for \( j \neq 1 \)
\[ \int_{\Omega_j} M(\pi_\varepsilon) \frac{\partial v_1}{\partial x_1} dx = O\left( \left\| \frac{\partial v_1}{\partial x_1} \right\|_{L^1(\Omega_j)} \right) \int_{\Omega_j} \frac{\partial v_1}{\partial x_1} U_j^{p-1} \left( \sum_{i \neq j,l} U_{i,l} + \sum_{l \neq 0} U_{j,l} \right) + O(e^{-2\sigma_j \sigma_1^{1-N}}) dx \]
\[ = o(e^{-2\sigma_j \sigma_1^{1-N}}) \]  
by Proposition 5.9.

Now we write
\[ \int_{\Omega_1} M(\pi_\varepsilon) \frac{\partial v_1}{\partial x_1} dx = o(e^{-2\sigma_1 \sigma_1^{1-N}}) - p \int_{\Omega_1} U_1^{p-1} \left( \sum_{j \neq 1} v_j + \sum_{l \neq 0} U_{1,l} \right) \frac{\partial v_1}{\partial x_1} dx. \]  
(3.44)

Moreover, since \( k \geq 2 \), we have
\[ \frac{\pi}{\varepsilon} - \sigma_1 \rightarrow +\infty \]
and consequently
\[ e^{-2\sigma_1} = o(e^{-2\sigma_1}). \]

So, we deduce from Proposition 5.9 that
\[ \int_{\Omega_1} U_1^{p-1} \sum_{l \neq 0} U_{1,l} \sum_l \frac{\partial U_{1,l}}{\partial x_1} dx = o(e^{-2\sigma_1 \sigma_1^{1-N}}) \]
and
\[ \int_{\Omega_1} U_1^{p-1} \sum_{j \neq 1} v_j \sum_{l \neq 0} \frac{\partial U_{1,l}}{\partial x_1} dx = o(e^{-2\sigma_1 \sigma_1^{1-N}}). \]

Finally
\[ \int_{\Omega_1} U_1^{p-1} \left( \sum_{j \neq 1} v_j + \sum_{l \neq 0} U_{1,l} \right) \frac{\partial v_1}{\partial x_1} dx = \int_{\Omega_1} U_1^{p-1} \sum_{j \neq 1} v_j \frac{\partial U_{1,l}}{\partial x_1} dx + o(e^{-2\sigma_1 \sigma_1^{1-N}}). \]  
(3.45)

Now (3.40), (3.42), (3.43) and (3.44) give the proof of the proposition.

**Proposition 3.5** Let \( u \) be given as in Proposition 1.1 and let \( \delta_1, \ldots, \delta_k \) be defined in (3.30). We can possibly replace the \( k \) given points \( a_1^k / \varepsilon, \ldots, a_k^k / \varepsilon \) by \( k \) points \( b_1^k / \varepsilon, \ldots, b_k^k / \varepsilon \) verifying
\[ \frac{a_i^k}{\varepsilon} - \frac{b_i^k}{\varepsilon} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0 \]
in order to have
\[ \delta_i = 0, \quad i = 1, \ldots, k. \]
Proof. Let
\[ u = \bar{u}_\varepsilon + \sum_{j=1}^{k} \delta_j \varphi_j + v \]
be the given solution of (1.1).
Let us give \((\alpha_1, ..., \alpha_k)\) depending on \(\varepsilon\), such that \((\alpha_1, ..., \alpha_k) \to 0\). We can replace the points \(\frac{a_i}{\varepsilon}\) by the points \(\frac{a_i}{\varepsilon} + \alpha_i\). In other words, we write
\[ u = \tilde{u}_\varepsilon + \sum_{j=1}^{k} \tilde{\delta}_j \tilde{\varphi}_j + \tilde{v} \]
where
\[ \tilde{u}_\varepsilon(x) = \sum_{j=1}^{k} \sum_{l \in \mathbb{Z}} U_{j,l}(x_1 - \alpha_j, x') \]
and \(\tilde{\varphi}_j, \ j = 1, ..., k\)
are the eigenfunctions corresponding to the eigenvalues tending to 0, for the configuration of points \(\frac{a_i}{\varepsilon} + \alpha_i\), and
\[ < \tilde{v}, \tilde{\varphi}_j > = 0, \ j = 1, ..., k. \]
Soustraying the expressions of \(u\) and performing the scalar product in \(H^1\) by \(\varphi_i\), we get
\[ \delta_i \| \varphi_i \|_{H^1}^2 + \sum_{j} \delta_j < \tilde{\varphi}_j - \varphi_j, \varphi_i > = < \mu_\varepsilon - \bar{u}_\varepsilon, \varphi_i > + \delta_i \| \varphi_i \|_{H^1}^2 + < v - \bar{v}, \varphi_i >. \quad (3.46) \]
First we remark that we have
\[ < v - \tilde{v}, \varphi_i > = < \tilde{v}, \tilde{\varphi}_i - \varphi_i > \]
while by (3.33)
\[ \| \tilde{v} \|_{H^1} \leq C(e^{-2\eta}a^{\frac{1-N}{2}} + \sum_j \tilde{\delta}_j^2), \]
with \(\eta = 1\), for \(p > 2\), thus
\[ | < v - \tilde{v}, \varphi_i > | \leq C(e^{-2\eta}a^{\frac{1-N}{2}} + \sum_j \delta_j^2). \quad (3.47) \]
Moreover
\[ | < \mu_\varepsilon - \bar{u}_\varepsilon, \varphi_i > | \leq C \sum_j |\alpha_j| \]
and, as a consequence of (1.13)
\[ \| \tilde{\varphi}_j - \varphi_j \|_{H^1} \to 0. \]
Thanks to (3.46), we deduce that
\[ \sum_i |\delta_i| \leq C(\sum_j |\delta_j| + \sum_j |\alpha_j| + e^{-2\eta}a^{\frac{1-N}{2}}). \quad (3.48) \]
and consequently
\[ \| \tilde{v} \|_{H^1} \leq C(e^{-2 \eta \sigma \frac{1-N}{2}} + \sum_j |\delta_j|^2 + \sum_j |\alpha_j|^2), \]

thus
\[ | < v - \tilde{v}, \varphi_i > | \leq C(e^{-2 \eta \sigma \frac{1-N}{2}} + \sum_j |\delta_j|^2 + \sum_j |\alpha_j|^2) \tag{3.49} \]

for some \( C \) independent of \( \varepsilon \).

Now let us prove that we can choose \((\alpha_1, \ldots, \alpha_k)\) such that
\[ < u_\varepsilon - \tilde{u}_\varepsilon, \varphi_i > + \delta_i \| \varphi_i \|^2 + < v - \tilde{v}, \varphi_i > = 0. \]

We define
\[ \mathcal{F}(\alpha_1, \ldots, \alpha_k) = ( < u_\varepsilon - \tilde{u}_\varepsilon, \varphi_i > )_{i=1, \ldots, k}. \]

This definition gives, for \( i \) and \( j = 1, \ldots, k \)
\[ \frac{\partial \mathcal{F}_i}{\partial \alpha_j} = \sum_{l \in \mathbb{Z}} \int_{[-\pi \varepsilon, \pi \varepsilon] \times \mathbb{R}^{N-1}} \frac{\partial U_{i,l}}{\partial x_1} (x_1 - \alpha_j, x') \varphi_i(x) dx + \sum_{l \in \mathbb{Z}} \int_{[-\pi \varepsilon, \pi \varepsilon] \times \mathbb{R}^{N-1}} \nabla \frac{\partial U_{i,l}}{\partial x_1} (x_1 - \alpha_j, x') \nabla \varphi_i(x) dx. \]

We deduce that, as \( \varepsilon \to 0 \)
\[ \frac{\partial \mathcal{F}_i}{\partial \alpha_i}(0) \to || \frac{\partial U}{\partial x_1} ||_{H^1}^2 \quad \text{and} \quad \frac{\partial \mathcal{F}_j}{\partial \alpha_j}(0) \to 0 \quad \text{for} \ j \neq i. \]

Thus \( d\mathcal{F}(0) \) is an isomorphism, for \( \varepsilon \) small enough.

Let us define \( \alpha = (\alpha_1, \ldots, \alpha_k) \). We have to solve
\[ \mathcal{F}(\alpha) + (\delta_i \| \varphi_i \|^2 + < v - \tilde{v}, \varphi_i >)_{i=1, \ldots, k} = 0. \]

We define
\[ Q(\alpha) = \mathcal{F}(\alpha) - \mathcal{F}(0) - d\mathcal{F}(0)(\alpha) \]
and
\[ \mathcal{G}(\alpha) = (-d\mathcal{F}(0))^{-1}(Q(\alpha) + (\delta_i \| \varphi_i \|^2 + < v - \tilde{v}, \varphi_i >)_{i=1, \ldots, k}). \]
Since we have together
\[ |Q(\alpha)| = O(|\alpha|^2) \]
and (3.49), we can use the Brouwer fixed point Theorem in a standard way. We find a real number \( R \),
\[ R \leq C(e^{-2 \eta \sigma \frac{1-N}{2}} + \sum_j |\delta_j|) \]
such that
\[ (|\alpha| \leq R) \Rightarrow |\mathcal{G}(\alpha)| \leq R. \]
So we find \( \alpha, |\alpha| \leq R \), such that \( \mathcal{G}(\alpha) = \alpha \), that is
\[ < u_\varepsilon - \tilde{u}_\varepsilon, \varphi_i > + \delta_i \| \varphi_i \|^2 + < v - \tilde{v}, \varphi_i > = 0. \]
Returning to (3.46), we deduce that $\tilde{\delta}_i = 0$, $i = 1, \ldots, k$.

**Proof of Proposition 1.1.** The points are asymptotically uniformly distributed.

From now on, we suppose that $\delta_i = 0$, $i = 1, \ldots, k$.

Let $i_0 \in \{1, \ldots, k\}$ be such that $\sigma_{i_0} - \sigma \to 0$ (we know that there exists at least one $i$ such that $\sigma_i = \sigma$). We have

$$L(\delta_1 \varphi_1 + \ldots + \delta_k \varphi_k) = h^T,$$

that is

$$\sum_{i=1}^{k} \delta_i \lambda_i (-\Delta \varphi_i + \varphi_i) = \sum_{i=1}^{k} d_i (-\Delta \varphi_i + \varphi_i)$$

thus

$$\delta_i \lambda_i = d_i \quad \text{for all } i.$$

In particular

$$d_{i_0} = 0.$$

Since, by Theorem 1.3 we have

$$|\lambda_{i_0}| \geq H e^{-2\sigma \frac{1-a}{2}}$$

we deduce from (3.39) that

$$\int_{\Omega_{i_0}} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx = o(\lambda_{i_0}).$$

We deduce that

$$\int_{\Omega_{i_0} : x_1 > a_{i_0}^1} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx$$

$$= - \int_{\Omega_{i_0} : x_1 < a_{i_0}^1} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx + o(\lambda_{i_0}).$$

But let us suppose that

$$2\sigma = \frac{a_{i_0}}{\varepsilon} - \frac{a_{\varepsilon}}{\varepsilon}.$$

We use Corollary 5.1 to get a positive real number $D_0$ such that

$$e^{2\sigma \frac{N-1}{2}} \int_{\Omega_{i_0} : x_1 > a_{i_0}^1} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx \to D_0$$

and

$$-e^{\frac{i_0}{\varepsilon} - \frac{a_{i_0}}{\varepsilon}} \int_{\Omega_{i_0} : x_1 < a_{i_0}^1} U_{i_0}^{p-1} \sum_{j \neq i_0} v_j \frac{\partial U_{i_0}}{\partial x_1} dx \to D_0.$$
and consequently
\[ 2\sigma - \left( \frac{a_{i_0}}{\varepsilon} - \frac{a_{i_0-1}}{\varepsilon} \right) \to 0. \]

This property is valid for all exponent \( i \) such that \( \sigma - \sigma \to 0 \) instead of \( i_0 \). Thus we have (1.7).

4 The proof of Theorem 1.1 completed.

The uniqueness.

Now, we have \( \frac{a_i}{\varepsilon} - \left( \frac{a_1}{\varepsilon} + \frac{2\pi}{k\varepsilon} \right) \to 0 \). Replacing the points \( \frac{a_i}{\varepsilon} \) by \( \frac{a_1}{\varepsilon} + \frac{2\pi}{k\varepsilon} \), \( i = 1, \ldots, k \), we can write \( u \) as

\[ u = \sum_{l \in \mathbb{Z}} U(x_1 - \frac{a_1}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') + \sum_{i=1}^{k} \tilde{\delta}_i \tilde{\varphi}_i + \tilde{v}, \quad <\tilde{v}, \tilde{\varphi}_i> = 0, \quad i = 1, \ldots, k. \]

By the definition of \( \tilde{\varphi}_i \) given in section 2 (analogue to that of \( \varphi_i \)), \( \tilde{\varphi}_i \) is \( \frac{2\pi}{k\varepsilon} \)-periodic in \( x_1 \). But now, the corresponding operator \( L \) is of minimal period \( \frac{2\pi}{k\varepsilon} \), since now \( \pi \) is replaced by \( \sum_{l \in \mathbb{Z}} U(x_1 - \frac{a_1}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') \). So we have

\[ \tilde{\varphi}_i(x) = \tilde{\varphi}_1(x_1 + \frac{2\pi l}{k\varepsilon}, x') \]

and \( \tilde{\varphi}_1 \) is \( \frac{2\pi}{k\varepsilon} \)-periodic. Let us denote \( \varphi_1 = \tilde{\varphi} \).

Now we recall that
\[ \mathbb{L}v = h \perp \]

with \( h = -M(\pi) + O(v^2 + (\sum_i \tilde{\delta}_i \tilde{\varphi}_i)^2) \). We can use the Banach fixed point theorem in \( L^\infty \) to deduce that \( v \) is of minimal period \( \frac{2\pi}{k\varepsilon} \).

Consequently, \( u \) is \( \frac{2\pi}{k\varepsilon} \)-periodic and in the space \( H^1(\mathbb{S}^1_{k\varepsilon} \times \mathbb{R}^{N-1}) \) we write

\[ u = \sum_{l \in \mathbb{Z}} U(x_1 - \frac{a_1}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') + \tilde{\delta} \tilde{\varphi} + \tilde{v}, \quad <\tilde{v}, \tilde{\varphi}> = 0. \]

Thanks to Proposition 3.5, we can perform a translation in \( x_1 \) to get \( \tilde{\delta} = 0 \). We get some \( \frac{a_1}{\varepsilon} \to 0 \) such that

\[ u = \sum_{l \in \mathbb{Z}} U(x_1 - \frac{a_1}{\varepsilon} - \frac{a_\varepsilon}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') + v, \quad <v, \tilde{\varphi}(x_1 - \frac{a_\varepsilon}{\varepsilon})> = 0. \]

Let \( u_D \) be the Dancer solution of period \( \frac{2\pi}{k\varepsilon} \). Then

\[ u_D(x_1 - \frac{a_1}{\varepsilon}, x') = \sum_{l \in \mathbb{Z}} U(x_1 - \frac{a_1}{\varepsilon} + \frac{2\pi l}{k\varepsilon}, x') + \delta \tilde{\varphi} + v \]
for some $v$ such that $\langle v, \tilde{\phi} \rangle = 0$ and some $\delta \to 0$.

Exactly as for $u$, we find some point $\frac{b_x}{\varepsilon} \to 0$ such that

$$ u_D(x_1 - \frac{a_1^1}{\varepsilon}, x') = \sum_{l \in \mathbb{Z}} U(x_1 - \frac{a_2^1}{\varepsilon} - \frac{b_x}{\varepsilon} + \frac{2\pi l}{\varepsilon}k\varepsilon\cdot x') + v_D, \quad \langle v_D, \tilde{\phi}(x_1 - \frac{b_x}{\varepsilon}) \rangle = 0. $$

Now we can prove that for $\varepsilon$ small enough,

$$ u = u_D(x_1 + \frac{b_x - a_x - a_1^1}{\varepsilon}, x'). $$

The proof is the same as for the case of a solution which is even in $x_1$. Let us write it for the sake of completeness.

Without loss of generality, let $a_1^1 = 0$.

We define

$$ \overline{u}_D(x) = u_D(x_1 + \frac{b_x - a_x}{\varepsilon}, x') \quad \text{and} \quad w = u - \overline{u}_D. $$

Let us suppose that $w \neq 0$, at least for a sequence $\varepsilon \to 0$. Then $\|w\|_\infty$ is attained at a point $c = (c_1, c')$, with $c'$ obviously bounded independently of $\varepsilon$ and $c_1 \in ]-\frac{\pi}{k\varepsilon}, \frac{\pi}{k\varepsilon}[$. Now $c_1$ is bounded. To see that, we write

$$ -\Delta w + w(1 - \frac{u^p - v^p}{u - v}) = 0. \quad (4.50) $$

If $w(c) > 0$, then

$$ \frac{u^p - \overline{u}_D^p}{u - \overline{u}_D}(c) \leq pu^{p-1} $$

thus

$$ \Delta w(c) \geq w(c)(1 - pu^{p-1}(c)). $$

But if $|c_1| \to +\infty$, we have $u^{p-1}(c) \to 0$, thus

$$ 1 - pu^{p-1}(c) > 0 $$

for $\varepsilon$ small enough, that is in contradiction with the Maximum Principle. So we may extract a subsequence such that $c \to \overline{c}$ for some $\overline{c}$.

Let us define

$$ z(x) = \frac{w}{\|w\|_\infty}. $$

It verifies (4.50). By standard arguments $z \to \overline{z}$ uniformly on the compact sets. Moreover

$$ \overline{z}(\overline{c}) = 1 $$

and

$$ pu^{p-1} \leq \frac{u^p - \overline{u}_D^p}{u - \overline{u}_D} \leq pu^{p-1} \quad \text{if} \ u > \overline{u}_D $$

and we have the reverse inequality if $u < \overline{u}_D$. More,

$$ \lim u = \lim \overline{u}_D = U(x). $$
So
\[
\lim \frac{u^p - \bar{u}_D^p}{u - \bar{u}_D} = pU.
\]

Thus
\[
-\Delta \bar{z} + \bar{z}(1 - pU^{p-1}) = 0.
\]

We deduce that \( \bar{z} = \alpha \frac{\partial U}{\partial x_1} \), for some \( \alpha \neq 0 \).

We have
\[
<z, \tilde{\varphi}(x_1 - \frac{a_\varepsilon}{\varepsilon}, x')>_H^1 = \frac{1}{\|u - \bar{u}_D\|_\infty} <v - v_D(x_1 + \frac{b_\varepsilon - a_\varepsilon}{\varepsilon}, x'), \tilde{\varphi}(x_1 - \frac{a_\varepsilon}{\varepsilon}, x')>_H^1 = 0.
\]

Moreover
\[
|z| \leq 1, \quad \tilde{\varphi} \to \frac{\partial U}{\partial x_1},
\]
and by Proposition 5.7,
\[
|\nabla \tilde{\varphi}(x)| + |\tilde{\varphi}(x)| \leq Ce^{-\eta|x|} \text{ in } \left[ -\frac{\pi}{k\varepsilon}, \frac{\pi}{k\varepsilon} \right].
\]

We use the Lebesgue Theorem to infer that
\[
<z, \tilde{\varphi}(x_1 - \frac{a_\varepsilon}{\varepsilon}, x')>_H^1 \to \alpha \|\frac{\partial U}{\partial x_1}\|_{H^1(\mathbb{R}^N)}^2.
\]

So we are led to a contradiction. We conclude that \( u = \bar{u}_D \), for \( \varepsilon \) small enough.

**The proof of the estimate 1.6.**

Without loss of generality, we let \( k = 1 \). We write
\[
u_D = \sum U_l + v
\]
where \( v \) is even in \( x_1 \) and verifies
\[
\mathbb{L}v = h
\]
and \( \bar{u}_\varepsilon \) is replaced by \( \sum U_l \) in the definition of \( \mathbb{L} \). The restriction of \( \mathbb{L} \) to the even functions has no eigenvalue tending to 0. The same proof as for (5.61) gives
\[
\|v\|_\infty + \|\nabla v\|_\infty \leq C\|h\|_\infty
\]
and consequently the same proof as for (3.34) gives
\[
\forall \eta \in ]0, 1[, \quad \|v\|_\infty + \|\nabla v\|_\infty \leq Ce^{-\frac{2\eta}{\varepsilon} \pi k^N} \leq C e^{-\frac{2\eta}{\varepsilon} \pi k^N}.
\]

Let \( R_0 > 0 \) be given. It remains to estimate, for all \( \eta \in ]0, 1[ \)
\[
(|v(y)| + |\nabla v(y)|)e^{\eta d_y}
\]
when \( d_y > R_0 \).
Let $\beta$ be a positive real number, independent of $\varepsilon$, which will be chosen later. Let $y \in [-\frac{\pi}{\varepsilon} + \beta, \frac{\pi}{\varepsilon} - \beta] \times \mathbb{R}^{N-1}$. We follow the course of the proof of Proposition 5.6 from (5.65), $\xi$ being replaced by $v$. With the notations of that proof, we perform the truncation around 0, using the truncature function $\theta$. So we drop the index $i$. By (5.67), we have

\[(\theta v)(y) = \int_{\mathbb{R}^N} G(y - x)(p\theta \overline{\nabla})^{-1}v + \theta h - \Delta \theta v - 2\nabla \theta \nabla v)(x)dx\]

Now

\[h = -\mathcal{M}(\pi_e) + O(\varepsilon^2)\]

We have

\[|v(y)| \leq C(\int_{\mathbb{R}^N} vG(y - x)p\theta e^{(-p+1)|x|}dx + \int_{\mathbb{R}^N} G(y - x)\theta|\mathcal{M}(\pi_e)|dx + \int_{\mathbb{R}^N} G(y - x)\theta^2 dx \]

\[+ \int_{\mathbb{R}^N; \frac{\pi}{\varepsilon} - \beta < x_1 < \frac{\pi}{\varepsilon}} G(y - x)(|v| + |\nabla v|)dx)\]

Now In Supp$\theta$,

\[d_x = |x| \quad \text{and} \quad U \leq Ce^{-|x|/\varepsilon^{1-N}}\]

Let us recall that

\[\mathcal{M}(\pi_e) = pU^{p-1}(\sum_{l \neq 0} U_l) + O((\sum_{l \neq 0} U_l)^2)\]

We deduce that for all $\eta \in [0, 1[$

\[\int_{\mathbb{R}^N} G(y - x)\theta|\mathcal{M}(\pi_e)|dx \leq C \sum_{l \in \mathbb{Z}^*} \int_{[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}} e^{-\eta|y|} e^{\eta|x-y|} G(y - x)e^{\eta|x|}(U^{p-1}U_l + U_l^2)\theta dx.\]

Using Proposition 5.9, we obtain for all $\eta'$ such that $0 < \eta' < p - 1 - \eta$

\[\sum_{l \in \mathbb{Z}^*} \int_{[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}} e^{-\eta|y|} e^{\eta|x-y|} G(y - x)e^{\eta|x|}U^{p-1}U_l\theta dx \leq Ce^{-\eta'd_{\theta}} e^{-\eta'\varepsilon^{\frac{1-N}{2}}}\]

Moreover

\[\sum_{l \in \mathbb{Z}^*} \int_{[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}} e^{-\eta|y|} e^{\eta|x-y|} G(y - x)e^{\eta|x|}U_l^2\theta dx \leq Ce^{-\eta'd_{\theta}} e^{-(2+\eta')\frac{\pi}{\varepsilon}(\frac{\pi}{\varepsilon})\frac{1-N}{2}}\]

Now Proposition 5.9 gives also, since $p - 1 + \eta > 1$

\[\int_{\mathbb{R}^N} |v|G(y - x)p\theta e^{(-p+1)|x|}dx \leq \|ve^{\eta'd_{\theta}}\|_{\infty} \int_{\mathbb{R}^N} G(y - x)p\theta e^{(-p+1-\eta)|x|}dx \]

\[\leq \|ve^{\eta'd_{\theta}}\|_{\infty} e^{-|y|/\varepsilon^{1-N}} \leq \|ve^{\eta'd_{\theta}}\|_{\infty} e^{-\eta_1 R_0},\]

where we define $\eta_1 = 1 - \eta$, and

\[\int_{\mathbb{R}^N} G(y - x)v^2\theta dx \leq \|ve^{\eta'd_{\theta}}\|^2_{\infty} \int_{\mathbb{R}^N} G(y - x)\theta e^{-2\eta|x|}dx \]

(4.54)
We terminate by a barrier function argument, as in the proof of (5.62). Let us define $y$.

The same proof gives $x$. Choosing if necessary $C$ we choose $\beta$.

We have

$$\int_{\mathbb{R}^N} G(y-x)(|v|+|\nabla v|)dx \leq (\|ve^{\eta d_y}\|_{\infty} + \|e^{\eta d_y} \nabla v\|_{\infty}) e^{-\eta\|d_y\|} \int_{\mathbb{R}^N} e^{\eta|y-x|} G(y-x)dx.$$ (4.55)

Finally, we obtain, for $y \in [-\frac{n}{\epsilon} + \beta, \frac{n}{\epsilon} - \beta] \times \mathbb{R}^{N-1}$ and for all $0 < \eta' \leq \min\{2-\eta, 2(p-1-\eta)\}$

$$|v(y)e^{\eta|y|}| \leq C_1 e^{-\frac{2\pi}{\epsilon} \frac{1-N}{\pi} y} + \frac{C_2}{\beta} (\|ve^{\eta d_y}\|_{\infty} + \|e^{\eta d_y} \nabla v\|_{\infty}) + C_1 (\|ve^{\eta d_y}\|_{\infty}^2 + \|ve^{\eta d_y}\|_{\infty} e^{-\eta R_0})$$

where the constants are independent of $\beta$ and of $R_0$.

We obtain the same estimate for $\|e^{\eta d_y} \nabla v\|_{\infty}$, using the proof of (5.73).

We terminate by a barrier function argument, as in the proof of (5.62). Let us define

$$\overline{\phi} = C_1 e^{-\frac{2\pi}{\epsilon} \frac{1-N}{\pi} y} + \frac{C_2}{\beta} (\|ve^{\eta d_y}\|_{\infty} + \|e^{\eta d_y} \nabla v\|_{\infty}) + C_1 (\|ve^{\eta d_y}\|_{\infty}^2 + \|ve^{\eta d_y}\|_{\infty} e^{-\eta R_0}).$$

We have

$$-\Delta(v - \overline{\phi}) + (1 - p\eta^{-1})(v - \overline{\phi}) = -(1 - p\eta^{-1})\overline{\phi} + h$$

If $x \in \overline{[\frac{n}{\epsilon} - \beta, \frac{n}{\epsilon} + \beta]} \times \mathbb{R}^{N-1}$

$$|h + \mathcal{M}(\overline{\phi})| \leq C\epsilon^2 \leq C\|ve^{\eta d_y}\|_{\infty}^2 e^{-2\eta(\frac{n}{\epsilon} - \beta)}$$

and

$$\mathcal{M}(\overline{\phi}) \leq Ce^{-(p-1)|x|^2} e^{-|x-(\frac{2\pi}{\epsilon})0|} |x - (\frac{2\pi}{\epsilon})0| \leq C \frac{(\frac{2\pi}{\epsilon})0}{\frac{1-N}{\pi} e^{-\frac{2\pi}{\epsilon} e^{-(p-2)|x|^2}}.$$ Choosing if necessary $C_1$ large enough, we deduce that

$$-(1 - p\eta^{-1})\overline{\phi} + h < 0 \quad \text{for} \quad y \in \overline{[\frac{n}{\epsilon} - \beta, \frac{n}{\epsilon} + \beta]} \times \mathbb{R}^{N-1}.$$ The same proof gives

$$-(1 - p\eta^{-1})\overline{\phi} + h < 0 \quad \text{for} \quad y \in \overline{[0, \frac{n}{\epsilon} + \beta]} \times \mathbb{R}^{N-1}.$$ Finally, the Maximum Principle gives

$$|v(y)| \leq \overline{\phi} \leq 0 \quad \text{for} \quad y \in \overline{[\frac{n}{\epsilon} - \beta, \frac{n}{\epsilon} + \beta]} \times \mathbb{R}^{N-1}.$$ Thus, in $\overline{[-\frac{n}{\epsilon}, \frac{n}{\epsilon}]} \times \mathbb{R}^{N-1}$, we have when $d_y \geq R_0$

$$|\nabla v(y)| + |v(y)|/\eta \leq C e^{-\frac{2\pi}{\epsilon} \frac{1-N}{\pi} y} + \frac{C}{\beta} (\|ve^{\eta d_y}\|_{\infty} + \|e^{\eta d_y} \nabla v\|_{\infty}) + C\|ve^{\eta d_y}\|_{\infty}^2 e^{-\eta R_0}$$

and when $d_y \leq R_0$

$$|\nabla v(y)| + |v(y)|/\eta \leq C e^{-\frac{2\pi}{\epsilon} \frac{1-N}{\pi} y} e^{\eta R_0}.$$ We choose $\beta$ and $R_0$ large enough to obtain (1.6).
5 Appendix.

Let $\eta > 0$ be given. Let $h$ be a function, defined on $[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}$, which has the property

$$\text{he}^{\eta \text{dist}(x, \cup_{j=0}^{k+1} \{ (\frac{y_j}{\varepsilon}, 0) \})} \text{ is bounded in } L^\infty([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$$

(5.56)

independently of $\varepsilon$.

Then $h$ belongs to $H^{-1}([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$ (the dual space of $H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$ in the following sense

$$(h, \psi)_{H^{-1}, H^1} = \int_{[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1}} h \psi \, dx \quad \text{for } \psi \in H^1.$$ We will denote $H^1$ in place of $H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$.

By the Lax-Milgram Theorem, there exists $u \in H^1$ such that

$$-\Delta u + u = h.$$ (5.57)

It is classical that $u \in L^\infty$ (see [8], Theorem 9.13 and use the Sobolev embedding Theorem).

As a consequence of the maximum principle,

$$\|u\|_{L^\infty} \leq \|h\|_{L^\infty}.$$ (5.58)

More we have the following

Proposition 5.6 Let $h \in H^{-1}([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$. There exists a unique $\xi \in H^1([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$ which verifies

$$\mathbb{L} \xi = h.$$ (5.59)

Let us suppose that $h \in L^\infty([-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1})$ and that

$$(h, \varphi_i)_{H^{-1}, H^1} = 0, \quad i = 1, \ldots, k.$$ (5.60)

Then

$$\|\xi\|_\infty + \|\nabla \xi\|_\infty \leq C \|h\|_\infty$$ (5.61)

where $C$ is independent of $\varepsilon$.

Let $\eta \in [0, 1]$ be given. Let us suppose that $h$ verifies (5.60) and has the additional property (5.56) for all $\eta \in [0, 1]$.

Then for all $\eta \in [0, 1]$ there exists $C$ independent of $\varepsilon$ and dependent of $\eta$ such that

$$\|e^{\eta \text{dist}(x, \cup_{j=0}^{k+1} \{ (\frac{y_j}{\varepsilon}, 0) \})} \|_{\infty} + \| \nabla e^{\eta \text{dist}(x, \cup_{j=0}^{k+1} \{ (\frac{y_j}{\varepsilon}, 0) \})} \|_\infty$$

$$\leq C \|h e^{\eta \text{dist}(x, \cup_{j=0}^{k+1} \{ (\frac{y_j}{\varepsilon}, 0) \})} \|_\infty.$$ (5.62)
**Proof** Let \( \xi \in H^1(\mathbb{R}^N) \). First, we deduce from (1.14) that
\[
(L \xi = 0) \Rightarrow (\xi = 0).
\]
Moreover, the operator \( L \) is a Fredholm operator, so we have the existence of a unique solution \( \xi \) of (5.59) when \( h \in H^{-1} \). The property (5.60) for \( h \) implies the existence of \( \xi \), without knowing the property (1.14). In this case, \( \xi \) verifies
\[
\int_{\mathbb{R}^N} \xi(-\Delta + 1)(\varphi_i) dx = 0, \quad i=1,...,k.
\]
(5.63)
Moreover we have in this case
\[
\|\xi\|_{H^1} \leq C \|u\|_{H^1},
\]
(5.64)
where \( u \) is defined in (5.57) and \( C \) is independent of \( \varepsilon \). Indeed, this can be proved using the expansion of \( \xi \) on a basis of eigenvectors of the operator \((-\Delta + 1)^{-1}L\).
Now, let \( h \in L^\infty \) verifying (5.56) and (5.60). Let us prove (5.61).
Let us assume that \( \|h\|_\infty \to 0 \) and that \( \|\xi\|_\infty = 1 \). There exists \( c \) such that \( \xi(c) = 1 \).
Let \( \xi(x) = \xi(x+c) \). By the standard elliptic estimates, \( \xi \) tends to a limit \( \tilde{\xi} \), uniformly on the compact sets of the operator \(-\Delta + 1\) and we have either
\[
(-\Delta + 1)\tilde{\xi} = 0 \quad \text{in } \mathbb{R}^N \text{ if } |c - \frac{a_i}{\varepsilon}| \to +\infty \text{ for all } i
\]
or
\[
(-\Delta + 1 - pU^{p-1}(x + \overline{c}))\tilde{\xi} = 0 \quad \text{in } \mathbb{R}^N \text{ if there exists } i \text{ and } \overline{c} \text{ such that } (c - \frac{a_i}{\varepsilon}) \to \overline{c}.
\]
The first case is in contradiction with the maximum principle, so it does not occur. In the second case, we have that
\[
\xi(x+c) \to \frac{\partial U}{\partial x_1}(x_1 + \overline{c}, x') \quad \text{uniformly on the compact sets.}
\]
We use (5.63). Since \((1 - \lambda_i)(-\Delta + 1)\varphi_i = pU^{p-1}\varphi_i\), we use the Lebesgue Theorem to get a contradiction.

We have proved that
\[
\|\xi\|_\infty \leq C \|h\|_\infty.
\]
The inequality for \( \|\nabla \xi\|_\infty \) follows from standard elliptic estimates ([8],Theorem 9.13). So we have proved (5.61).

In what follows, we denote
\[
d_x = \text{dist}(x, \bigcup_{j=0}^{k+1} \{ (a_j^i \varepsilon, 0) \}).
\]
We define
\[
\tilde{\sigma}_i = \frac{a_i^{j+1} - a_i^j}{2\varepsilon} \quad \text{for } i = 0, ..., k.
\]
Let \( \beta \) be a positive real number, independent of \( \varepsilon \). Let \( i = 1, \ldots, k \) and let us consider the domain \( D \) defined by
\[
D = \{ y; -\tilde{\sigma}_{i-1} + \beta \leq y_1 - \frac{a_i}{\varepsilon} \leq \tilde{\sigma}_i - \beta \}.
\]
The number \( \beta \) will be chosen later.

Let \( R_0 > 0 \) be given. We are going to estimate
\[
|\xi(y)e^{\eta dy}| + |\nabla \xi(y)e^{\eta dy}|
\]
when \( y \in D \) is such that \( d_y > R_0 \).

For \( i = 1, \ldots, k \), let \( \theta_i \) be a function which is \( \frac{2\pi}{\varepsilon} \)-periodic in \( x_1 \) and which verifies in \( [-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}] \times \mathbb{R}^{N-1} \)
\[
\theta_i(x) = 1 \quad \text{for} \quad -\tilde{\sigma}_{i-1} + \beta \leq x_1 - \frac{a_i}{\varepsilon} \leq \tilde{\sigma}_i - \beta \quad (5.65)
\]
\[
\theta_i(x) = 0 \quad \text{for} \quad x_1 - \frac{a_i}{\varepsilon} \geq \tilde{\sigma}_i \quad \text{or} \quad x_1 - \frac{a_i}{\varepsilon} \leq -\tilde{\sigma}_{i-1}.
\]
Moreover, we suppose that \( \theta_i \) is \( C^2 \) in \( x_1 \). More precisely, we build \( \theta_i \) from the function \( \tilde{\theta} \) defined in \([0, \beta] \), \( \tilde{\theta}(x_1) = -\frac{6}{\beta}x_1^5 + \frac{15}{\beta}x_1^4 - \frac{10}{\beta}x_1^3 + 1 \), by \( \theta_i(x) = \tilde{\theta}(x_1 - \frac{a_i}{\varepsilon} - \tilde{\sigma}_i + \beta) \), if \( -\tilde{\sigma}_i - \beta \leq x_1 - \frac{a_i}{\varepsilon} \leq \tilde{\sigma}_i \) and \( \theta_i(x) = \tilde{\theta}(\frac{a_i}{\varepsilon} - \tilde{\sigma}_i + \beta - x_1) \), if \( -\tilde{\sigma}_{i-1} \leq x_1 - \frac{a_i}{\varepsilon} \leq -\tilde{\sigma}_i - \beta \). Thus we have for all \( x \) and for \( M \) independent of \( i \), of \( \beta \) and of \( \varepsilon \)
\[
|\theta_i(x)| \leq M, \quad |\nabla \theta_i(x)| \leq \frac{M}{\beta}, \quad |\Delta \theta_i(x)| \leq \frac{M}{\beta^2}.
\]

Let \( G \) be the Green function of the operator
\[-\Delta + 1 \quad \text{on} \quad \mathbb{R}^N.\]

We have
\[
0 < G(x) \leq C \frac{e^{-|x|}}{|x|^{N-2}}(1 + |x|^\frac{N-4}{2}) \quad \text{if} \quad N \geq 2. \quad (5.66)
\]
We write
\[
(\theta_i \xi)(y) = \int_{\mathbb{R}^N} G(y - x)(p\theta_i \eta e^{-|x|} + \theta_i h - \Delta \theta_i \xi - 2\nabla \theta_i \nabla \xi)(x)dx, \quad (5.67)
\]
Let \( y \in D \), we have \( (\theta_i \xi)(y) = \xi(y) \).

We consider (5.67). Firstly, we have for all \( \eta \in [0, 1] \)
\[
|\int_{\mathbb{R}^N} G(y - x)(p\theta_i \eta e^{-|x|} + \theta_i h)(x)dx| \leq C\|he^{\eta dx}\|_\infty \int_{\mathbb{R}^N} G(y - x)\theta_i(x)e^{-\eta dx}dx \quad (5.68)
\]
\[+C\|\xi e^{\eta dx}\|_\infty \int_{\mathbb{R}^N} G(y - x)\theta_i(x)e^{-\eta dx}\eta e^{-|x|}dx.\]
For \( x \) in \( \text{Supp}\theta_i \) we write
\[
d_x \geq d_y - |x - y|
\]
while
\[
\forall \eta \in ]0,1[ \exists \varepsilon < \varepsilon_0 \quad \varpi_\varepsilon(x) \leq C e^{-\eta d_x}.
\]
So we have
\[
| \int_{\mathbb{R}^N} G(y - x)\theta_i(x)e^{-\eta d_x} dx |
\]
\[
\leq \int_{\mathbb{R}^N} G(y - x)e^{\eta |y - x|} |\theta_i(x)|e^{-\eta d_y} dx \leq C e^{-\eta d_y}
\]
and, for any \( \eta' \) such that \( \eta < \eta' < 1 \)
\[
| \int_{\mathbb{R}^N} G(y - x)\theta_i(x)e^{-\eta d_x} dx |
\]
\[
\leq C e^{-\eta' d_y} \int_{\mathbb{R}^N} G(y - x)e^{\eta'|x - y|} e^{-(p - 1)\eta + n - \eta'} d_x |\theta_i(x)| dx \leq C e^{-\eta' d_y}.
\]
Secondly, we get for all \( \eta \in ]0,1[ \)
\[
| \int_{\mathbb{R}^N} G(y - x)(-\Delta \theta_i \xi - 2\nabla \theta_i \nabla \xi)(x) dx | \leq \frac{C}{\beta}(\|\xi e^{\eta d_x}\|_\infty
\]
\[
+ \|\nabla \xi e^{\eta d_x}\|_\infty) \int_{x \in \text{Supp}\theta_i} G(y - x)e^{-\eta d_x} dx.
\]
But, as above
\[
| \int_{x \in \text{Supp}\theta_i} G(y - x)e^{-\eta d_x} dx | \leq C e^{-\eta d_y}.
\]
Then (5.67)-(5.71) give for all \( \eta \in ]0,1[ \) and for a constant \( C \) independent of \( \beta \) and for some \( \eta_1 > 0 \)
\[
|\xi(y) e^{\eta d_y}| \leq C(\|he^{\eta d_x}\|_\infty + \frac{1}{\beta}(\|\nabla \xi e^{\eta d_x}\|_\infty + \|\xi e^{\eta d_x}\|_\infty) + C e^{-\eta_1 R_0}\|\xi e^{\eta d_x}\|_\infty \quad \text{for all } y \in \mathcal{D}.
\]
We have to prove that
\[
|\nabla \xi(y) e^{\eta d_y}| \leq C(\|he^{\eta d_x}\|_\infty + \frac{1}{\beta}(\|\nabla \xi e^{\eta d_x}\|_\infty + \|\xi e^{\eta d_x}\|_\infty) + C e^{-\eta_1 R_0}\|\xi e^{\eta d_x}\|_\infty \quad \text{for all } y \in \mathcal{D}.
\]
Let \( u = \xi e^{\eta|x - (\frac{a_1}{\varepsilon}, 0)|} \). When \( d_x = |x - (\frac{a_1}{\varepsilon}, 0)| \geq R_0 \), \( u \) satisfies the equation
\[
-\Delta u + u(1 - \eta^2 - m \varpi_\varepsilon^{p-1}) + 2\eta\nabla |x - (\frac{a_1}{\varepsilon}, 0)| \nabla u = e^{\eta d_x} h.
\]
Without loss of generality, we suppose that there exists \( C \) independent of \( \varepsilon \) such that
\[
\text{for all } x \text{ such that } |x - y| \leq 1, \quad |d_x - |x - (\frac{a_1}{\varepsilon}, 0)|| \leq C.
\]
We use Theorem 9.13 of [8], with \( \Omega' = \{ x, |x - y| \leq \frac{1}{2} \} \) and \( \Omega = \{ x, |x - y| < 1 \} \). We get

\[
\| \nabla u \|_{L^\infty(\Omega')} \leq C(\| u \|_{L^\infty(\Omega)} + \| e^{\eta_d} h \|_{L^\infty(\Omega)}
\]

that gives

\[
| \nabla (\xi e^{\eta_d})(y) | \leq C(\| \xi e^{\eta_d} \|_{\infty} + \| he^{\eta_d} \|_{\infty}).
\]

(5.74)

Then (5.72) and (5.74) lead to (5.73). Now (5.73) and (5.72) give

\[
(\| \xi(y) \| + | \nabla \xi(y) |) e^{\eta_d} \leq C_1 \| h e^{\eta_d} \|_{\infty} + \frac{C}{\beta}(\| \xi e^{\eta_d} \|_{\infty} + \| \nabla \xi e^{\eta_d} \|_{\infty}) + C e^{-\eta R_0} \| \xi e^{\eta_d} \|_{\infty} \quad \text{for } y \in D.
\]

(5.75)

It remains to prove (5.75) for \( y \) in \( \Omega \setminus D \). Let us denote

\[
\tilde{D} = \{ y; \frac{a_i}{\epsilon} + \tilde{\sigma}_i - \beta < y < \frac{a_{i+1}}{\epsilon} - \tilde{\sigma}_i + \beta \}
\]

By (5.75), we have on \( \partial \tilde{D} \)

\[
| \xi(y) | \leq C_1 e^{-\eta(\tilde{\sigma}_i - \beta)} \| h e^{\eta_d} \|_{\infty} + \frac{C}{\beta}(\| \xi e^{\eta_d} \|_{\infty} + \| \nabla \xi e^{\eta_d} \|_{\infty}) + C e^{-\eta R_0} \| \xi e^{\eta_d} \|_{\infty}.
\]

Let us denote

\[
\overline{\phi}(y) = C e^{-\eta(\tilde{\sigma}_i - \beta)} \| h e^{\eta_d} \|_{\infty} + \frac{C}{\beta}(\| \xi e^{\eta_d} \|_{\infty} + \| \nabla \xi e^{\eta_d} \|_{\infty}) + C e^{-\eta R_0} \| \xi e^{\eta_d} \|_{\infty}.
\]

We can suppose that \( C_1 > 1 \).

We have

\[
- \Delta (\xi - \overline{\phi}) + (\xi - \overline{\phi})(1 - p m^{-1}_{\epsilon^{-1}}) = -\phi(1 - p m^{-1}_{\epsilon^{-1}}) + h.
\]

In \( \tilde{D} \), we have \( d_x \geq \tilde{\sigma}_i - \beta \), thus

\[
|h| \leq e^{-\eta(\tilde{\sigma}_i - \beta)} \| h e^{\eta_d} \|_{\infty} \quad \text{in } \tilde{D}.
\]

We deduce that for \( \epsilon \) small enough, we have together

\[
-\phi(1 - p m^{-1}_{\epsilon^{-1}}) + h \leq 0 \quad \text{and} \quad 1 - p m^{-1}_{\epsilon^{-1}} > 0 \quad \text{in } \tilde{D}.
\]

Then the Maximum Principle gives

\[
\xi - \overline{\phi} \leq 0 \quad \text{in } \tilde{D}.
\]

and the same proof gives

\[
-\xi - \overline{\phi} \leq 0 \quad \text{in } \tilde{D}.
\]

Moreover, we have

\[
d_x \leq \tilde{\sigma}_i \quad \text{in } \tilde{D}.
\]

Finally

\[
| \xi(x) e^{\eta_d} | \leq C_1 e^{-\eta \beta} \| h e^{\eta_d} \|_{\infty} + \frac{C}{\beta}(\| \xi e^{\eta_d} \|_{\infty} + \| \nabla \xi e^{\eta_d} \|_{\infty}) + C e^{-\eta R_0} \| \xi e^{\eta_d} \|_{\infty} \quad \text{in } \tilde{D}.
\]

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The same proof as for (5.73) gives

\[ |\nabla \xi(x)e^{\eta dx}| \leq C_1 e^{\eta \beta} \|he^{\eta dx}\|_\infty + \frac{C}{\beta} (\|\xi e^{dx}\|_\infty + \|\nabla \xi e^{dx}\|_\infty) + Ce^{-\eta R_0} \|\xi e^{\eta dx}\|_\infty \text{ in } \tilde{D}.\]

We have proved

\[ |\xi(x)e^{\eta dx}| + |\nabla \xi(x)e^{\eta dx}| \leq C_1 e^{\eta \beta} \|he^{\eta dx}\|_\infty \]

when \( d_x \geq R_0 \), and

\[ |\xi(x)e^{\eta dx}| + |\nabla \xi(x)e^{\eta dx}| \leq C_1 e^{\eta \beta} \|he^{\eta dx}\|_\infty e^{-\eta R_0} \]

in \( \Omega_i \), when \( d_x \leq R_0 \), for all \( i \).

Now we choose \( \beta \) and \( R_0 \) large enough to get (5.62). We have proved the proposition.

**Proposition 5.7** Let \( \varphi \) be an eigenfunction of \( L \), associated with an eigenvalue \( \lambda \) which does not tend to 1. Let us suppose that \( \|\varphi\|_{L^\infty([-\frac{\pi}{2}, \frac{\pi}{2}\times \mathbb{R}^{N-1}])} = 1 \). Then for all \( \eta \in ]0, 1[ \) there exists \( C > 0 \), independent of \( \varepsilon \), such that

\[ |\varphi(x)| + |\nabla \varphi(x)| \leq Ce^{-\eta \text{dist}(x, \cup_{i=0}^{k+1}\{(\frac{a_i^k \varepsilon}{2}, 0)\})}, \quad (5.76) \]

where we use the notation : \( a_0^k = a_k^k - 2\pi \) and \( a_{k+1}^k = a_1^k + 2\pi \).

Moreover

\[ C_1 \leq \|\varphi\|_{H^1} \leq C_2 \quad (5.77) \]

where \( C_1 \) and \( C_2 \) are some positive real numbers independent of \( \varepsilon \).

Let \( \xi \) be defined in (2.21). Then

\[ |\xi(x)| + |\nabla \xi(x)| \leq Ce^{-\eta \text{dist}(x, \cup_{i=0}^{k+1}\{(\frac{a_i^k \varepsilon}{2}, 0)\})}e^{-\eta R_0}. \quad (5.78) \]

**Proof** To prove (5.76), we follow the proof of (5.62) in Proposition 5.6, with \( h = 0 \). We find, for \( y \in \Omega_i \) such that \( d_y \geq R_0 \)

\[ |\varphi(y)e^{\eta dy}| + |\nabla \varphi(y)e^{\eta dy}| \leq \frac{C}{\beta} (\|\varphi e^{dy}\|_\infty + \|\nabla \varphi e^{dy}\|_\infty) + Ce^{-\eta R_0} \|\varphi e^{\eta dy}\|_\infty \]

while for \( y \in \Omega_i \) such that \( d_y \leq R_0 \)

\[ |\varphi(y)e^{\eta dy}| + |\nabla \varphi(y)e^{\eta dy}| \leq Ce^{\eta R_0} \]

where the constant \( C \) is independent of \( \beta \) and of \( R_0 \). We choose \( R_0 \) and \( \beta \) large enough to obtain(5.76).

Let us prove (5.77). We have

\[ (1 - \lambda)\|\varphi\|_{H^1}^2 = p \int_{\frac{1}{e}\times \mathbb{R}^{N-1}} \varphi^{p-1} \varphi^2 dx. \]

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In view of (5.76) and of (2.16), we may use the Lebesgue Theorem to obtain that
\[ \|\varphi\|_{H^1} \not\to 0, \quad i = 0, \ldots, k. \]
Now we have
\[ L = \sum_{j=1}^{k} c_j \lambda_j (-\Delta \varphi_j + \varphi_j) = \sum_{l \in \mathbb{Z}} (w_l^0 - 1) \frac{\partial U_{i,l}}{\partial x_1} + p \sum_{j=1}^{k} \frac{\lambda_j}{1 - \lambda_j} w_l^0 \varphi_j. \]
If we write \( L = h \), then, for all \( \eta \in ]0, 1[ \)
\[ |h(x)| \leq C_\eta e^{-\varphi} e^{-\eta x}. \]
We use Proposition 5.6 to obtain (5.78).
We have proved the proposition.

**Proposition 5.8** Let \( C \) and \( A \) be positive real numbers. Let \( f \) and \( g \) be functions which verify the following property, for \( |x| > A \)
\[ |f(x)| \leq Ce^{-|x|}, \quad |g(x)| \leq Ce^{-|x|} |x|^\frac{1-N}{2}. \]
Let \( a > b > 0 \). Let \( y_0 \) be such that \( |y_0| \to +\infty \) and \( \alpha = \frac{|y_0|}{2} \). Then
\[ \left| \int_{\frac{a}{2}}^{\frac{b}{2}} f^a(x)g^b(x - (y_0,0))dx \right| = o(|y_0|^{\frac{b-N}{2}} e^{-b|y_0|}). \quad (5.79) \]

**Proof** We easily see that if \( |x| < \alpha \),
\[ |x - (y_0,0)| = |y_0|\frac{x}{y_0} - (1,0) \geq \frac{1}{2} |y_0|. \]
Thus
\[ \left| \int_{\frac{a}{2}}^{\frac{b}{2}} f^a(x)g^b(x - (y_0,0))dx \right| \leq C|y_0|^{\frac{b-N}{2}} \int_{\frac{a}{2}}^{\frac{b}{2}} e^{-|x|} |x-(y_0,0)|dx \]
\[ \leq C|y_0|^{\frac{b-N}{2}} \int_{\frac{a}{2}}^{\frac{b}{2}} r^{-1-N} \int_{S_{N-1}} e^{-ar} e^{-b\sqrt{r^2 - 2y_0x + y_0^2}} drd\mu(z) \]
\[ \leq C|y_0|^{\frac{b-N}{2}} \int_{\frac{a}{2}}^{\frac{b}{2}} r^{-1-N} \int_{S_{N-1}} e^{-ar} e^{-b\sqrt{r^2 - 2y_0x + y_0^2}} drd\mu(z) \]
\[ \leq C|y_0|^{\frac{b-N}{2}} \int_{\frac{a}{2}}^{\frac{b}{2}} r^{-1-N} \int_{S_{N-1}} e^{-ar} e^{-b|y_0-r|} drd\mu(z) \]
\[ \leq C|y_0|^{\frac{b-N}{2}} e^{-b|y_0|} e^{(b-a)\alpha} \alpha^N \]
and \( e^{(b-a)\alpha} \alpha^N \to 0. \)
Proposition 5.9 Let $f$ and $g$ be smooth functions and $C$, $C_1$, $C_2$ and $A$ be positive real numbers which verify, for $|x| > A$

$$0 \leq f(x) \leq Ce^{-|x|}; \quad C_1 e^{-|x|}\frac{|x|^{1-N}}{2} \leq g(x) \leq C_2 e^{-|x|}\frac{|x|^{1-N}}{2}.$$.

Let us define

$$\Omega_i = \{x \in \left[-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right] \times \mathbb{R}^{N-1}; \ \text{dist}(x, \cup_{l=0}^{k+1}\{(\frac{a_l}{\varepsilon}, 0)\}) = |x - (\frac{a_i}{\varepsilon}, 0)|\}.$$

Let $a > b > 0$. Let $i \neq j$ and let $y_0 = \frac{a_j - a_i}{\varepsilon}$.

If $f(x) \geq 0$ and $f \neq 0$, then there exist positive real numbers $C_1$ and $C_2$ such that, for $i = 0, ..., k + 1$

$$(1+o(1))C_1^b C_0 e^{-b|y_0|}\frac{|x|^{1-N}}{2} \leq \int_{\Omega_i} f^a(x - (\frac{a_j}{\varepsilon}, 0))g^b(x - (\frac{a_i}{\varepsilon}, 0))dx \leq C_2^b C_0 e^{-b|y_0|}\frac{|x|^{1-N}}{2} (1+o(1)),$$

where

$$C_0 = \int_{\mathbb{R}^N} f^a(x)dx.$$

The estimate (5.80) holds true if we replace $\Omega_i$ by the set

$$\Omega_i^+ = \{x \in \Omega_i; \ x_1 > \frac{a_j}{\varepsilon}\}$$

while $C_0$ is replaced by

$$\int_{\mathbb{R}^N, x_1 > 0} f^a(x)dx.$$

Proof. Let

$$\alpha = \frac{|y_0|}{2}.$$

For $x$ such that $|x - (\frac{a_j}{\varepsilon}, 0)| < \alpha^{\frac{1}{2}}$, we have

$$|y_0| - |x - (\frac{a_j}{\varepsilon}, 0)| \leq |x - (\frac{a_j}{\varepsilon}, 0)| \leq |y_0| + |x - (\frac{a_j}{\varepsilon}, 0)|,$$

thus

$$|x - (\frac{a_j}{\varepsilon}, 0)| = (1 + o(1))|y_0|.$$

We write

$$\int_{|x - (\frac{a_j}{\varepsilon}, 0)| < \alpha^{\frac{1}{2}}} f^a(x - (\frac{a_j}{\varepsilon}, 0))g^b(x - (\frac{a_i}{\varepsilon}, 0))dx$$

$$\geq C_1^b (1 + o(1))|y_0|^{b\frac{(1-N)}{2}} e^{-b|y_0|}\int_{|x - (\frac{a_j}{\varepsilon}, 0)| < \alpha^{\frac{1}{2}}} f^a(x - (\frac{a_j}{\varepsilon}, 0))dx$$

$$\geq C_1^b C_0 (1 + o(1))|y_0|^{b\frac{(1-N)}{2}} e^{-b|y_0|}$$

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and

\[
\int_{|x-(\frac{a_j}{\varepsilon},0)|<\alpha \frac{1}{2}} f^a(x-(\frac{a_j}{\varepsilon},0))g^b(x-(\frac{a_j}{\varepsilon},0))dx
\]

\[
\leq C_2^b(1+o(1))|y_0|^{\frac{(1-N)}{2}}e^{-b|y_0|} \int_{|x-(\frac{a_j}{\varepsilon},0)|<\alpha \frac{1}{2}} f^a(x-(\frac{a_j}{\varepsilon},0))dx
\]

\[
\leq C_2^b C_0(1+o(1))|y_0|^{\frac{(1-N)}{2}}e^{-b|y_0|}.
\]

Moreover, in view of Proposition 5.8, we have

\[
\int_{x \in \Omega_i;|x-(\frac{a_j}{\varepsilon},0)|<\alpha} f^a(x-(\frac{a_j}{\varepsilon},0))g^b(x-(\frac{a_j}{\varepsilon},0))dx = o(1)|y_0|^{\frac{(1-N)}{2}}e^{-b|y_0|}.
\]

Last, we have

\[
\int_{x \in \Omega_i;|x-(\frac{a_j}{\varepsilon},0)|>\alpha} f^a(x-(\frac{a_j}{\varepsilon},0))g^b(x-(\frac{a_j}{\varepsilon},0))dx
\]

\[
\leq C_2^b C_0 e^{-b|y_0|} |y_0|^{\frac{(1-N)}{2}} \int_{x \in \Omega_i;|x-(\frac{a_j}{\varepsilon},0)|>\alpha} e^{b|x-(\frac{a_j}{\varepsilon},0)|} f^a(x-(\frac{a_j}{\varepsilon},0))dx
\]

\[
= o(e^{-b|y_0|}|y_0|^{\frac{(1-N)}{2}}),
\]

since \(a > b\).

We have proved the proposition.

**Corollary 5.1** If

\[
\lim_{|x| \to +\infty} g(x)e^{|x|} |x|^{-\frac{|x|}{2}} = L
\]

then

\[
\lim_{|x| \to +\infty} \int_{\Omega_i} f^a(x-(\frac{a_j}{\varepsilon},0))g^b(x-(\frac{a_j}{\varepsilon},0))dx e^{b|y_0|} |y_0|^{\frac{N+1}{2}} = LC_0
\]

where \(C_0\) is defined in Proposition 5.9. Moreover, we can replace \(\Omega_i\) by \(\Omega_i^+\).

**Lemma 5.2** Let \(k\) be a positive integer. Let \(a\) and \(b\) be real numbers, with \(a > 0\). If \(p > 1\) is given and \(k = \lfloor p \rfloor\). There exists \(C\) independent of \(a\) and \(b\) such that

\[
|-(a+b)^p + a^p + \ldots + \frac{p\ldots(p-k+1)}{k!}a^{p-k}b^k| \leq C|b|^p.
\]

**Proof** First, let us suppose that \(b < 0\). If \(a+2b < 0\), then \(a < 2|b|\), and the claim is true. If \(a+2b > 0\), then \(a+b > 0\) and we write

\[
|-(a+b)^p + a^p + \ldots + \frac{p\ldots(p-k+1)}{k!}a^{p-k}b^k| = \frac{p\ldots(p-k)}{(k+1)!} (a+ab)^{p-k-1}b^{k+1} \quad (5.81)
\]

where \(\alpha \in (0,1)\). But \(a+ab > -2b+ab\), thus

\[
|-(a+b)^p + a^p + \ldots + \frac{p\ldots(p-k+1)}{k!}a^{p-k}b^k| \leq \frac{p\ldots(p-k)}{(k+1)!} (2-\alpha)^{p-k-1}|b|^p
\]

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that gives the claim. Secondly, let us suppose that \( b > 0 \). If \( a < b \), the claim is true. If \( a > b \), we use (5.81) again and we obtain the claim.

References


