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Abstract. We consider the inverse problem of determining the potential in the dynamical Schrödinger equation on the interval by the measurement on the boundary. We use the Boundary Control method to recover the spectrum of the problem from the observation at either left or right end points. Using the specificity of the one-dimensional situation we recover the spectral function, reducing the problem to the classical one which could be treated by known methods. We apply the algorithm to the situation when only the finite number of eigenvalues are known and prove the convergence of the method.

1 Introduction

We consider the problem of determining the potential in a one dimensional Schrödinger equation from two boundary measurements. More precisely, given a real potential $q \in L^1(0,1)$ and $a \in H_0^1(0,1)$, we consider the following initial boundary value problem:

$$\begin{cases} iu_t(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0 & t > 0, \quad 0 < x < 1 \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = a(x) & 0 < x < 1. \end{cases} \tag{1.1}$$

Assuming that the initial datum a is unknown, the inverse problem we are interested in is to determine the potential q from the trace of the derivative

of the solution u to (1.1) on the boundary:

$$\{r_0(t), r_1(t)\} := \{u_x(0,t), u_x(1,t)\}, \quad t \in (0, 2T),$$

where $T > 0$ is fixed (it may be arbitrary small). Once the potential has been determined, one can use e.g. the method of iterative observers recently proposed in [17] to recover the initial data.

The multidimensional inverse problems of determining the potential by one measurement were considered in [5, 6, 9, 15]. Using techniques based on Carleman estimates, the authors established global uniqueness and stability results for different geometrical conditions on the domain in arbitrary small time under certain restrictions on the initial source. No reconstruction procedure (even in 1-d case) has been provided. We also mention the approach proposed by Boumenir and Tuan for the heat equation in [10, 11, 12]. Using the boundary observation for $t \in (0, \infty)$, the authors were able to recover the spectrum of the string provided the source is generic (see the definition in the beginning of Section 2). Then choosing another boundary condition, they solved the inverse problem from the two recovered spectra by the classical Levitan-Gasymov method [13, 14].

In this paper we propose a different approach. We introduce the unbounded operator $\mathcal{A}$ on $L^2(0,1)$ defined by

$$\mathcal{A}\phi = -\phi'' + q\phi, \quad \mathcal{D}(\mathcal{A}) := H^2(0,1) \cap H_0^1(0,1). \tag{1.2}$$

Using the Boundary Control (BC) method, see [2, 7], we first show that the eigenvalues of $\mathcal{A}$ can be recovered from the data r_0 (or r_1) in arbitrary small interval provided the source is generic. To achieve this, we derive a generalized eigenvalue problem involving an integral operator (see (2.5)), whose solution leads to the recovery of the eigenvalues of $\mathcal{A}$. Then, using the peculiarity of the one dimensional case, we recover the spectral function associated with $\mathcal{A}$, reducing the original inverse problem to the more "classical" one of recovering an unknown potential from spectral data. At this step we use the observations at both boundary points. Thus we establish an algorithm for recovering both potential and the initial data in a very natural setting: we use the observation on the whole boundary in the arbitrary small time, which corresponds to the uniqueness results from [5, 6, 9]. We do not use the observation in infinite time and do not change the boundary conditions (cf. [10, 11, 12].)

In the last part of the paper we show how to adapt our algorithm to the more realistic situation where not all but only a finite number of eigenvalues of $\mathcal{A}$ are known. We answer a question how many eigenvalues one need to recover in order to achieve prescribed accuracy.

Remark 1.1. *The method of solving the inverse problem presented in this paper could be applied to the case of wave and parabolic equations on the interval.*

The paper is organized as follows. In Section 2, we detail the different steps of our algorithm. In particular, we describe how to recover the spectral data of $\mathcal{A}$ from the boundary data and point out two methods that can be used to recover the potential. In Section 3 we provide the result on the convergence of the method of recovering the potential by a finite number of spectral data.

2 Inverse problem, application of the BC method

2.1 From boundary data to spectral data

It is well known that the selfadjoint operator $\mathcal{A}$ defined by (1.2) admits a family of eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ forming a orthonormal basis in $L^2(0,1)$, and associated sequence of eigenvalues $\lambda_k \to +\infty$. Using the Fourier method, we can represent the solution of (1.1) in the form

$$u(x,t) = \sum_{1}^{\infty} a_k e^{i\lambda_k t}\phi_k(x), \tag{2.1}$$

where a_k are Fourier coefficients:

$$a_k = (a, \phi_k)_{L^2(0,1;\,dx)}\,.$$

Definition 2.1. *We call the initial data $a \in H_0^1(0,1)$ generic if $a_k \neq 0$ for all $k \geq 1$.*

For the boundary data r_0, r_1, we have the representation

$$\{r_0(t), r_1(t)\} = \left\{ \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}\phi_k'(0), \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}\phi_k'(1) \right\}. \tag{2.2}$$

One can prove that $r_0, r_1 \in L^2(0,T)$. This follows from:

(i) the estimates

$$0 < \inf_k \left| \frac{\phi_k'(0)}{k} \right| \leqslant \sup_k \left| \frac{\phi_k'(0)}{k} \right| < \infty, \tag{2.3}$$

(see, e.g. [18]);

(ii) the equivalence of the inclusions $\{ka_k\}_{k=1}^{\infty} \in l^2$ and $a \in H_0^1(0,1)$;

(iii) the fact that the family $\left\{e^{i\lambda_k t}\right\}_{k=1}^{\infty}$ forms a Riesz basis in the closure of its linear span in $L^2(0,T)$ (see, e.g. [3]).

Throughout the paper, we always assume that the source a is generic. The reason for this requirement can be seen from (2.2): if we assume that $a_n = 0$ for some n, then from the representation (2.2) it is easy to see that the pair $\{r_0(t), r_1(t)\}$ does not contain information about the triplet $\{\lambda_n, \phi_n'(0), \phi_n'(1)\}$, and, consequently the potential which corresponds to these data is not unique.

Using the method described in [2] we can recover the spectrum λ_k and products $a_k\phi_k'(0)$ and $a_k\phi_k'(1)$ by the following procedure. We construct the operator $C_0^T : L^2(0,T) \mapsto L^2(0,T)$ by the rule:

$$(C_0^T)f(t) = \int_0^T r_0(2T-t-\tau)f(\tau)\,d\tau, \qquad 0 \leqslant t \leqslant T. \tag{2.4}$$

and consider the following generalized eigenvalue problem : Find $(\mu, f) \in \mathbb{C} \times H_0^1(0,T)$, $C_0^T f \neq 0$, such that

$$\int_0^T \dot{r}_0(2T-t-\tau)f(\tau) = \mu \int_0^T r_0(2T-t-\tau)f(\tau)\,d\tau, \qquad 0 \leqslant t \leqslant T. \tag{2.5}$$

Then, one can prove [2] that the problem above admits a countable set of solutions (μ_n, f_n), $n \geqslant 1$. Moreover, for the eigenvalues we have $\mu_n = i\lambda_n$, where λ_n are the eigenvalues of $\mathcal{A}$; the family $\{f_n(t)\}_{n=1}^{\infty}$ is biorthogonal to $\{e^{i\lambda_k(T-t)}\}_{n=1}^{\infty}$.

A priori we do not suppose that $\dot{r}_0 \in L^2(0,2T)$ and therefore, in general, the integral $\int_0^T \dot{r}_0(2T-t-\tau)f(\tau)\,d\tau$ should be understood as the action of the functional $\dot{r}_0 \in H^{-1}(0,2T)$ on $f \in H_0^1(0,T)$. For algorithmic purposes it is convenient to rewrite the equation (2.5) in the form

$$\int_0^T \left[r_0(2T-t-\tau) - \mu R(2T-t-\tau)\right] h(\tau)\,d\tau = 0, \quad 0 \leqslant t \leqslant T. \tag{2.6}$$

where $R(t) = \int_0^t r(\tau)\,d\tau$, and $f(t) = \int_0^t h(\tau)\,d\tau$.

Next, we consider the equation of the form (2.5) with $r_0(t)$ replaced by its complex conjugate $\overline{r_0(t)}$. This equation yields the sequence $\{-i\lambda_k, g_k(t)\}_{k=1}^{\infty}$, $(-i\lambda_n = \overline{\mu}_n)$. Let us normalize functions f_k, g_k by the rule:

$$\delta_{nk} = \left(C_0^T f_n, g_k\right) = \int_0^T \int_0^T r_0(2T-t-\tau)f_n(\tau)\overline{g_k(t)}\,d\tau\,dt \tag{2.7}$$

and introduce constants γ_k, β_k defined by:

$$\gamma_k = \int_0^T r_0(T-\tau) f_k(\tau)\, d\tau, \tag{2.8}$$

$$\beta_k = \int_0^T r_0(T-\tau) \overline{g_k(\tau)}\, d\tau. \tag{2.9}$$

It was proved in [2] (see formula (2.28)) that the product $a_k \phi_x'(0)$ was given by

$$a_k \phi_k'(0) = \gamma_k \beta_k. \tag{2.10}$$

Similarly we can introduce the integral operator C_1^T associated with the response $r_1(t)$ at the right endpoint, and repeat the procedure described above to find quantities $a_k \phi_k'(1)$.

Summing up, using the method from [2] we are able to recover the eigenvalues λ_k of $\mathcal{A}$ and the products $\phi_k'(0)a_k$ and $\phi_k'(1)a_k$. As a result, we can say that we recovered the spectral data consisting of

$$\mathcal{D} := \left\{ \lambda_k, \frac{\phi_k'(1)}{\phi_k'(0)} \right\}_{k=1}^{\infty}. \tag{2.11}$$

Precisely this data was used in [16], where the authors have shown the uniqueness of the inverse spectral problem and provided the method of recovering the potential. Instead of doing this, we recover the spectral function associated to $\mathcal{A}$ and thus reduce the inverse source problem to the classical one of determining an unknown potential from the spectral data.

Given $\lambda \in \mathbb{C}$, we introduce the solution $y(\cdot, \lambda)$ of the following Cauchy problem on $(0, 1)$:

$$-y''(x,\lambda) + q(x) y(x,\lambda) = \lambda y(x,\lambda), \qquad 0 < x < 1, \tag{2.12}$$

$$y(0,\lambda) = 0, \quad y'(0,\lambda) = 1. \tag{2.13}$$

Then the eigenvalues of the Dirichlet problem of $\mathcal{A}$ are exactly the zeroes of the function $y(1,\lambda)$, while a family of normalized corresponding eigenfunctions is given by $\phi_k(x) = \dfrac{y(x,\lambda_k)}{\|y(\cdot,\lambda_k)\|}$. Thus we can rewrite the second components in $\mathcal{D}$ in the following way:

$$\frac{\phi_k'(1)}{\phi_k'(0)} = \frac{y'(1,\lambda_k)}{y'(0,\lambda_k)} = y'(1,\lambda_k) =: A_k. \tag{2.14}$$

Let us denote by dot the derivative with respect to λ. We use the following fact (see [16, p. 30]):

Lemma 2.2. *If λ_n is an eigenvalue of $\mathcal{A}$, then*

$$\|y(\cdot,\lambda_n)\|^2_{L^2} = y'(1,\lambda_n)\dot{y}(1,\lambda_n).$$

The lemma below can be found in [16] for the case $q \in L^2(0,1)$ holds true for $q \in L^1(0,1)$ as well. The meaning of it is that the function $y(1,\lambda)$ is completely determined by its zeroes, which are precisely the eigenvalues of $\mathcal{A}$.

Lemma 2.3. *For $q \in L^1(0,1)$ the following representations hold*

$$y(1,\lambda) = \prod_{k\geqslant 1} \frac{\lambda_k - \lambda}{k^2\pi^2}$$

$$\dot{y}(1,\lambda_n) = -\frac{1}{n^2\pi^2} \prod_{k\geqslant 1, k\neq n} \frac{\lambda_k - \lambda_n}{k^2\pi^2}.$$

Lemma 2.2 and 2.3 imply that the data $\mathcal{D}$ we have recovered (see (2.11)) allow us to evaluate the norm

$$\|y(\cdot,\lambda_n)\|^2_{L^2} = A_n B_n =: \alpha_n^2, \tag{2.15}$$

where we introduced the notation

$$B_n := -\frac{1}{n^2\pi^2} \prod_{k\geqslant 1, k\neq n} \frac{\lambda_k - \lambda_n}{k^2\pi^2}, \tag{2.16}$$

2.2 Reconstructing the potential from the spectral data

The set of pairs
$\{\lambda_k, \|y(\cdot,\lambda_k)\|^2_{L^2}\}_{k=1}^{\infty}$ is a "classical" spectral data. The potential can thus be recovered by Gelfand-Levitan, Krein method or the BCM (see [4]). Below we outline two possible methods of recovering the potential.

We introduce the spectral function associated with $\mathcal{A}$:

$$\rho(\lambda) = \begin{cases} -\sum\limits_{\lambda\leqslant\lambda_k\leqslant 0} \frac{1}{\alpha_k^2} & \lambda \leqslant 0, \\ \sum\limits_{0<\lambda_k\leqslant\lambda} \frac{1}{\alpha_k^2} & \lambda > 0, \end{cases} \tag{2.17}$$

which is a monotone increasing function having jumps at the points of the Dirichlet spectra. The regularized spectral function is introduced by

$$\sigma(\lambda) = \begin{cases} \rho(\lambda) - \rho_0(\lambda) & \lambda \geqslant 0, \\ \rho(\lambda) & \lambda < 0, \end{cases} \qquad \rho_0(\lambda) = \sum_{0<\lambda_k^0\leqslant\lambda} \frac{1}{(\alpha_k^0)^2} \quad \lambda > 0, \tag{2.18}$$

where ρ_0 is the spectral function associated with the operator $\mathcal{A}$ with $q \equiv 0$. In the definition above eigenvalues and norming coefficients are $\lambda_k^0 = \pi^2 k^2$, $(\alpha_k^0)^2 = \frac{1}{2\pi^2 k^2}$ and the solution to (2.12)–(2.13) for $q \equiv 0$ is $y_0(x,\lambda) = \dfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$.

Let us fix $\tau \in (0,1]$ and introduce the kernel $c^\tau(t,s)$ by the rule (see also [4]):

$$c^\tau(t,s) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}(\tau-t)\sin\sqrt{\lambda}(\tau-s)}{\lambda}\, d\sigma(\lambda), \quad s,t \in [0,\tau], \tag{2.19}$$

Then so-called connecting operator (see [7, 4]) $C^\tau : L^2(0,\tau) \mapsto L^2(0,\tau)$ is defined by the formula

$$(C^\tau f)(t) = (I + K_\tau)f(t) = f(t) + \int_0^\tau c^\tau(t,s)f(s)\,ds\,,\ 0 < t < \tau\,, \tag{2.20}$$

Using the BCM leads to the following: for the fixed $\tau \in (0,1)$ one solves the equation

$$(C^\tau f^\tau)(t) = \tau - t, \quad 0 < t < \tau, \tag{2.21}$$

Setting

$$\mu(\tau) := f^\tau(+0), \tag{2.22}$$

and then varying $\tau \in (0,1)$, the potential at the point τ is recovered by

$$q(\tau) = \frac{\mu''(\tau)}{\mu(\tau)}\,. \tag{2.23}$$

We can also make use of the Gelfand-Levitan theory. According to this approach, for $\tau \equiv 1$, the kernel c^τ satisfies the following integral equation with unknown V:

$$V(y,t) + c^\tau(y,t) + \int_y^\tau c^\tau(t,s)V(y,s)\,ds = 0, \quad 0 < y < t < 1. \tag{2.24}$$

Solving the equation (2.24) for all $y \in (0,1)$ we can recover the potential using

$$q(y) = 2\frac{d}{dy}V(\tau - y, \tau - y). \tag{2.25}$$

Once the potential has been found, we can recover the eigenfunctions ϕ_k, the traces $\phi_k'(0)$ and using (2.10), the Fourier coefficients a_k, $k = 1, \dots, \infty$. Thus, the initial state can be recovered via its Fourier series. We can also use the method of observers (see [17]).

2.3 The algorithm

1) Take $r_0(t) := u_x(0,t)$ and solve the generalized spectral problem (2.5). Denote the solution by $\{\mu_n, f_n(t)\}_{n=1}^{\infty}$ and note the connection with the spectra of $\mathcal{A}$: $\lambda_n = -i\mu_n$.

2) Take the function $\overline{r_0(t)}$ and repeat step one. This yields the sequence $\{\overline{\mu_n}, g_n(t)\}_{n=1}^{\infty}$.

3) Define the operator C_0^T by (2.4) and normalize f_n, g_n according to equation (2.7): $(C_0^T f_n, g_n) = 1$.

4) Find the quantities $a_n\phi_n'(0)$ by (2.8), (2.9) and (2.10).

5) Repeat steps 1)–4) for the function $r_1(t) := u_x(1,t)$ to find $a_k\phi_k'(1)$.

6) Define the spectral data $\mathcal{D}$ by (2.11) and find the norming coefficients α_k by using (2.15) and (2.14), (2.16).

7) Introduce the spectral function $\rho(\lambda)$, the regularized spectral function $\sigma(\lambda)$ and the kernel c^τ respectively defined by (2.17), (2.18) and (2.19).

8) Solve the inverse problem by either BCM using (2.21), (2.22), (2.23) or Gelfand-Levitan method using equations (2.24), (2.25).

9) Use the method of iterative observers described in [17] or Fourier series to recover the initial data.

Our approach yields the following uniqueness result for the inverse problem for 1-d Schrödinger equation:

Theorem 2.4. *Let the source $a \in H_0^1(0,1)$ be generic and T be an arbitrary positive number. Then the potential $q \in L^1(0,1)$ and the initial data are uniquely determined by the observation $\{u_x(0,t), u_x(1,t)\}$ for $t \in (0,T)$.*

The method could be applied to the inverse problem for the wave and parabolic equations with the potential on the interval as well. The details of the recovering the spectrum λ_k and the quantities $a_k\phi_k'(0)$, $a_k\phi_k'(1)$ could be found in [2]. The following important remark, connected with the types of controllability of the corresponding systems, should be taken into the consideration:

Remark 2.5. *The time T of the observation can be arbitrary small for the case of Schrödinger and parabolic equations and it is equal to the double length of the space interval (i.e., $T = 2$ in our case) for the wave equation with the potential.*

For the details see [2].

3 Stability of the scheme : the case of truncated spectral data

In view of applications, in this section we consider the case where only a finite number of eigenvalues of $\mathcal{A}$ are available. More precisely, let us assume that we recovered the exact values of the first N eigenvalues λ_n and traces A_n (see (2.14)), for $n = 1, \dots, N$. Then we can introduce the approximate normalizing coefficients $\widetilde{\alpha}_{n,N}$ by the rule

$$\widetilde{\alpha}_{n,\,N} = A_n \widetilde{B}_{n,\,N}, \quad \text{where} \quad \widetilde{B}_{n,\,N} := -\frac{1}{n^2\pi^2} \prod_{k \geqslant 1, k \neq n}^{N} \frac{\lambda_k - \lambda_n}{k^2\pi^2}. \tag{3.1}$$

We can estimate

$$|\alpha_n - \widetilde{\alpha}_{n,\,N}| \leqslant |A_n||\widetilde{B}_{n,\,N}| \left| 1 - \prod_{k \geqslant N+1}^{\infty} \frac{\lambda_k - \lambda_n}{k^2\pi^2} \right|. \tag{3.2}$$

Since the infinite product (2.16) converges, the right hand side of the above inequality is small as soon as N is big enough, provided n is fixed. But the following remark should be taken into the account. Let us remind the following asymptotic formulas for the eigenvalues and norming coefficients:

$$\lambda_k = \pi^2 k^2 + \int_0^1 q(s)\, ds + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \tag{3.3}$$

$$\alpha_k^2 = \frac{1}{2\pi^2 k^2} + O\left(\frac{1}{k^4}\right), \quad k \to \infty, \tag{3.4}$$

Then the infinite product in the right hand side of (3.2) can be rewritten as

$$\prod_{k \geqslant N+1}^{\infty} \left(1 - \frac{n^2 + O\left(\frac{1}{n^2}\right)}{k^2} + O\left(\frac{1}{k^4}\right) \right) \tag{3.5}$$

We can see that if n is close to N, then the terms $\frac{n^2 + O\,\frac{1}{n^2}}{k^2}$ are close to one, and consequently the first factors in (3.5) and the whole product are small. This implies the infinite product in the right hand side in (3.2) is not close to one. This simple observation yields that we can not guarantee the good estimate in (3.2) when n is close to N.

We set up the following question: how many eigenvalues (we call their number by N) we need to know in order to recover first n approximate normalizing coefficients with a good accuracy, using formula (3.1). In other words, assuming the infinite product in (3.2) or in (3.5) to be close to one, we need to find out the admissible relationship between N and n. Let the parameter γ be such that $|\gamma| < x$, we notice that if $0 < (x+\gamma) < \theta < 1$ then $|\ln(1-x+\gamma)| < 2\frac{|\ln(1-\theta)|}{\theta}x$. Using this observation, we can estimate for $\frac{n^2}{(N+1)^2} \leqslant \theta$:

$$\begin{aligned}
&\left|\ln \prod_{k\geqslant N+1}^{\infty}\left(1-\frac{n^2}{k^2}+\frac{O\left(\frac{1}{n^2}\right)}{k^2}+O\left(\frac{1}{k^4}\right)\right)\right| \\
&\leqslant \sum_{k\geqslant N+1}^{\infty}\left|\ln\left(1-\frac{n^2}{k^2}+\frac{O\left(\frac{1}{n^2}\right)}{k^2}+O\left(\frac{1}{k^4}\right)\right)\right| \\
&\leqslant 2\sum_{k\geqslant N+1}^{\infty}\frac{|\ln(1-\theta)|}{\theta}\frac{n^2}{k^2} = 2n^2\frac{|\ln(1-\theta)|}{\theta}\left(\frac{\pi^2}{6}-\sum_{k=1}^{N}\frac{1}{k^2}\right) \qquad (3.6)
\end{aligned}$$

We fix some $\varepsilon > 0$ and choose N and n such that the right hand side of (3.6) is less than ε, then

$$e^{-\varepsilon} < \prod_{k\geqslant N+1}^{\infty}\left(1-\frac{n^2}{k^2}\right) < 1.$$

Consequently, for such N and n we have (see (3.2)):

$$|\alpha_n - \widetilde{\alpha}_{n,N}| \leqslant |A_n||\widetilde{B}_{n,N}|\left|1-e^{-\varepsilon}\right|. \qquad (3.7)$$

Notice that since $A_n = 1+o(1)$ and $\alpha_n = \frac{1}{\sqrt{2\pi n}}+o(1)$ as $n\to\infty$, the product $|A_n||\widetilde{B}_{n,N}|$ is bounded by some positive $C < 4$. Using formula

$$\frac{\pi^2}{6} = \sum_{k=1}^{N}\frac{1}{k} - \frac{1}{2k^2} + O\left(\frac{1}{N^3}\right)$$

from [19], we summarize all our observations in the lemma:

Lemma 3.1. *Let $0 < \varepsilon < 1$. If n and N satisfy*

$$\frac{n^2}{N} \leqslant \frac{\varepsilon}{2\ln 2}, \qquad (3.8)$$

then there exists an absolute constant $C > 0$ such that

$$|\alpha_k - \widetilde{\alpha}_{k,N}| \leqslant C\left|1-e^{-\varepsilon}\right|, \forall k = 1,\ldots,n. \qquad (3.9)$$

Let $\sigma(\lambda)$ be the regularized spectral function [4]. We recall the representations (see [4]) for the response function and the kernel c^τ (in our case $\tau = 1$):

Lemma 3.2. *Assume that $q \in L^1(0,1)$. Then the following representation for the response function r,*

$$r(t) = \int_{-\infty}^{\infty} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \, d\sigma(\lambda), \tag{3.10}$$

holds for almost all $t \in (0, 2\tau)$. The kernel $c^\tau(t,s)$ admits the representation (2.19) with the integral in the right-hand side converging uniformly on $[0,\tau] \times [0,\tau]$. The function $c^\tau(t,s)$ can also be represented as:

$$c^\tau(t,s) = p(2\tau - t - s) - p(t-s) \tag{3.11}$$

where $p(t)$ is defined by

$$p(t) := \frac{1}{2} \int_0^{|t|} r\,(s)\; ds.$$

The next useful formula follows directly from (3.11):

$$c^\tau(t,t) = \frac{1}{2} \int_0^{2(\tau - t)} r(\tau)\, d\tau. \tag{3.12}$$

If the exact values of the first n eigenvalues λ_k and normalizing factors α_k^2, $k = 1, \ldots, n$, were known, then one could construct the "restricted" response functions and kernels defined by

$$r_n(t) = \sum_{k=1}^{n} \left[\frac{\sin \sqrt{\lambda_k} t}{\sqrt{\lambda_k}} \frac{\operatorname{sign} \lambda_k}{\alpha_k^2} - \frac{\sin \sqrt{\lambda_k^0} t}{\sqrt{\lambda_k^0}} \frac{1}{(\alpha_k^0)^2} \right],$$

$$c_n^\tau(t,s) = \sum_{k=1}^{n} \frac{\sin \sqrt{\lambda_k}(\tau - t) \sin \sqrt{\lambda_k}(\tau - s)}{\lambda_k} \frac{\operatorname{sign} \lambda_k}{\alpha_k^2} - \frac{\sin \sqrt{\lambda_k^0}(\tau - t) \sin \sqrt{\lambda_k^0}(\tau - s)}{\lambda_k^0} \frac{1}{(\alpha_k^0)^2}.$$

from the representation it follows that every $r_n \in C^\infty(0, 2\tau)$ and Lemma 3.2 yields

$$r_n(t) \to r(t), \quad \text{for almost all } t \in (0, 2\tau), \tag{3.13}$$

$$c_n^\tau(t, s) \to c^\tau(t, s), \quad \text{uniformly on } (0, \tau)^2, \tag{3.14}$$

$$c_n^\tau(t, t) \to c^\tau(t, t), \quad \text{uniformly.} \tag{3.15}$$

As we only have access to approximate values of the normalizing factors in practice, we can only compute the approximate restricted kernel defined by

$$\widetilde{c}_{n,\,N}^{\,\tau}(t, s) = \sum_{k=1}^{n} \frac{\sin\sqrt{\lambda_k}(\tau - t)\sin\sqrt{\lambda_k}(\tau - s)}{\lambda_k} \frac{\operatorname{sign}\lambda_k}{\widetilde{\alpha}_{k,\,N}^2} - \frac{\sin\sqrt{\lambda_k^0}(\tau - t)\sin\sqrt{\lambda_k^0}(\tau - s)}{\lambda_k^0} \frac{1}{(\alpha_k^0)^2}. \tag{3.16}$$

We can estimate the difference

$$\|c_n^\tau - \widetilde{c}_{n,\,N}^{\,\tau}\|_\infty \leqslant \sum_{k=1}^{n} \frac{|\widetilde{\alpha}_{k,\,N}^2 - \alpha_k^2|}{\lambda_k \widetilde{\alpha}_{k,\,N}^2 \alpha_k^2} = \sum_{k=1}^{n} \frac{|\widetilde{\alpha}_k + \alpha_k||\widetilde{\alpha}_k - \alpha_k|}{\lambda_k \widetilde{\alpha}_{k,\,N}^2 \alpha_k^2}. \tag{3.17}$$

Using (3.9) and the asymptotic expansions for the eigenvalues and norming coefficients (3.3) (3.4), we deduce from (3.17) that (below, C denotes an absolute constant that might change from line to line):

$$\|c_n^\tau - \widetilde{c}_{n,\,N}^{\,\tau}\|_\infty \leqslant C \sum_{k=1}^{n} \frac{|\widetilde{\alpha}_{k,\,N}^2 - \alpha_k^2|}{\lambda_k \widetilde{\alpha}_{k,\,N}^2 \alpha_k^2} \leqslant C \sum_{k=1}^{n} k|\widetilde{\alpha}_{k,\,N} - \alpha_k|.$$

Let us fix some $\varepsilon > 0$ and $n \in \mathbb{N}$. Applying Lemma 3.2 and choosing N such that estimate (3.9) holds, we finally get

$$\|c_n^\tau - \widetilde{c}_{n,\,N}^{\,\tau}\|_\infty \leqslant C \frac{n(n+1)}{2} \left|1 - e^{-\varepsilon}\right|. \tag{3.18}$$

We fix now some $\delta > 0$. According to (3.14) we can take $n \in \mathbb{N}$ such that

$$\|c_n^\tau - c^\tau\|_\infty \leqslant \frac{\delta}{2}. \tag{3.19}$$

We estimate the difference

$$\|\widetilde{c}_{n,\,N}^{\,\tau} - c^\tau\|_\infty \leqslant \|\widetilde{c}_{n,\,N}^{\,\tau} - c_n^\tau\|_\infty + \|c_n^\tau - c^\tau\|_\infty.$$

Assuming that n is chosen such that (3.18) and (3.19) holds simultaneously, we obtain the existence of a constant $C^* > 0$ such that

$$\|\widetilde{c}^{\tau}_{n,N} - c^{\tau}\|_{\infty} \leqslant C^* n^2 \left|1 - e^{-\varepsilon}\right| + \frac{\delta}{2}. \tag{3.20}$$

Then by choosing an appropriate ε in (3.20) (which results in possible increasing of N, see (3.8)), we achieve

$$\|\widetilde{c}^{\tau}_{n,N} - c^{\tau}\|_{\infty} \leqslant \delta.$$

We summarize the above observations in the following statement.

Proposition 3.3. *Let $\delta > 0$ be fixed. Let n be chosen such that*

$$\|c^{\tau}_{n} - c^{\tau}\|_{\infty} \leqslant \frac{\delta}{2},$$

(this is possible thanks to (3.14)*). Next, take $\varepsilon > 0$ such that*

$$n^2 \left|1 - e^{-\varepsilon}\right| \leqslant \frac{\delta}{2},$$

Finally, choose N such that

$$\frac{n^2}{N} \leqslant \frac{\varepsilon}{2 \ln 2}.$$

Then, there exists an absolute constant $C > 0$ such that the following estimate holds true:

$$\|\widetilde{c}^{\tau}_{n,N} - c^{\tau}\|_{\infty} \leqslant C\delta,$$

where the approximate restricted kernel $\widetilde{c}^{\tau}_{n,N}$ and the approximate normalizing coefficients $\widetilde{\alpha}_{k,N}$, $k = 1, \ldots, n$ are defined respectively by (3.16) and (3.1).

In particular, $\widetilde{c}^{\tau}_{n,N}$ converges uniformly to c^{τ} when n tends to infinity and N is chosen as above.

Along with the equation (2.24) we consider the equation for the approximate restricted kernel $\widetilde{c}^{\tau}_{n,N}$:

$$\widetilde{V}_{n,N}(y,t) + \widetilde{c}^{\tau}_{n,N}(y,t) + \int_{y}^{\tau} \widetilde{c}^{\tau}_{n,N}(t,s)\widetilde{V}_{n,N}(y,s)\,ds = 0, \quad 0 < y < t < \tau. \tag{3.21}$$

Picking $\delta > 0$ we can use Proposition 3.3 to find n and N such that $\|\widetilde{c}^{\tau}_{n,N} - c^{\tau}\|_{\infty} \leqslant \delta$. From now on, we always assume that n and N are chosen

according to Proposition 3.3. Note that in particular we have $N \to +\infty$ as $n \to +\infty$.

We introduce the operator $\widetilde{K}_\tau^{n,\,N}$ given by the integral part of (2.20) with the kernel c^τ substituted by $\widetilde{c}^\tau_{n,\,N}$: $\left(\widetilde{K}_\tau^{n,\,N} f\right)(t) = \int_0^\tau \widetilde{c}^\tau_{n,\,N}(t,s) f(s)\,ds$. The closeness of c^τ and $\widetilde{c}^\tau_{n,\,N}$ implies the operator $I + \widetilde{K}_\tau^{n,\,N}$ to be invertible along with $I + K_\tau$. The latter in turn implies the existence of the potential $\widetilde{q}_{n,\,N}$ that produces the response function

$$\widetilde{r}_{n,\,N}(t) = \sum_{k=1}^{n} \left[\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}} \frac{\operatorname{sign}\lambda_k}{\widetilde{\alpha}^2_{k,\,N}} - \frac{\sin\sqrt{\lambda_k^0}t}{\sqrt{\lambda_k^0}} \frac{1}{(\alpha_k^0)^2} \right],$$

and the unique solvability of (3.21) (see [4], [8], [1]).

We define $M := \|(I + K_\tau)^{-1}\|$. The invertibility of $I + K_\tau$ and $I + \widetilde{K}_\tau^{n,\,N}$ implies the norms of the solutions $\|V(y,\cdot)\|_{L^2}$ and $\|\widetilde{V}_{n,\,N}(y,\cdot)\|_{L^2}$ to be bounded.

Let us write down the difference (2.24) and (3.21) in the form

$$\begin{aligned} &\left[V(y,t) - \widetilde{V}_{n,\,N}(y,t)\right] \qquad (3.22)\\ &+ \int_y^\tau c^\tau(t,s) \left[V(y,s) - \widetilde{V}_{n,\,N}(y,s)\right] ds = \widetilde{c}^\tau_{n,\,N}(y,t) - c^\tau(y,t)\\ &+ \int_y^\tau \left[c^\tau(t,s) - \widetilde{c}^\tau_{n,\,N}(t,s)\right] \widetilde{V}_{n,\,N}(y,s)\,ds, \quad 0 < y < t < \tau. \end{aligned}$$

The equality above and the invertibility of $I + K_\tau$ implies the estimate:

$$\begin{aligned} &\|V(y,\cdot) - \widetilde{V}_{n,\,N}(y,\cdot)\|_{L^2(y,\tau)} \qquad (3.23)\\ &\leqslant M \left(1 + \max_{0\leqslant y\leqslant\tau} \|\widetilde{V}_{n,\,N}(y,\cdot)\|_{L^2}\right) \|c^\tau - \widetilde{c}^\tau_{n,\,N}\|_\infty. \end{aligned}$$

To estimate the L^∞ norm, we write down (3.22) in the form

$$\begin{aligned} &\left[V(y,t) - \widetilde{V}_{n,\,N}(y,t)\right] \qquad (3.24)\\ = \;& \int_y^\tau c^\tau(t,s) \left[\widetilde{V}_{n,\,N}(y,s) - V(y,s)\right] ds + \widetilde{c}^\tau_{n,\,N}(y,t) - c^\tau(y,t)\\ &+ \int_y^\tau \left[c^\tau(t,s) - \widetilde{c}^\tau_{n,\,N}(t,s)\right] V_{n,\,N}(y,s)\,ds, \quad 0 < y < t < \tau. \end{aligned}$$

This leads to the following inequality:

$$\begin{aligned} &\|V(y,t) - \widetilde{V}_{n,\,N}(y,t)\|_\infty \qquad (3.25)\\ &\leqslant \|c^\tau\|_\infty \max\nolimits_{0\leqslant y\leqslant\tau} \int_y^\tau \left|\widetilde{V}_{n,\,N}(y,s) - V(y,s)\right| ds + \|\widetilde{c}^\tau_{n,\,N} - c^\tau\|_\infty\\ &\qquad + \|c^\tau - \widetilde{c}^\tau_{n,\,N}\|_\infty \max\nolimits_{0\leqslant y\leqslant\tau} \int_y^\tau |\widetilde{V}_{n,\,N}(y,s)|\,ds. \end{aligned}$$

Using (3.23) and (3.25) we finally get

$$\|V - \widetilde{V}_{n,N}\|_\infty \leqslant \|c^\tau - \widetilde{c}^\tau_{n,N}\|_\infty \, (M\|c^\tau\|_\infty + 1) \left(1 + \max_{0\leqslant y\leqslant \tau} \|\widetilde{V}_{n,N}(y,\cdot)\|_{L^2}\right).$$

The last equality in particular implies the uniform convergence of $\widetilde{V}_{n,N}(y,y)$ to $V(y,y)$. Using (2.25), we conclude that the potentials $q_{n,N}$ and q corresponding respectively to $\widetilde{V}_{n,N}$ and V satisfy

$$\int_0^t \widetilde{q}_{n,N}(s)\,ds \rightrightarrows_{n\to\infty} \int_0^t q(s)\,ds, \quad \text{uniformly in } t. \tag{3.26}$$

The latter in turn, implies that

$$\widetilde{q}_{n,N} \longrightarrow q, \quad \text{in } H^{-1}(0,1).$$

In fact, (3.26) yields more than that: we have the following

Theorem 3.4. *If n, N satisfy conditions from Proposition 3.3, then*

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \widetilde{q}_{n,N}(s)\,ds \longrightarrow_{n\to\infty} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} q(s)\,ds, \quad \textit{uniformly in } t,\ \varepsilon. \tag{3.27}$$

We remark that (3.27) is still not enough to guarantee the convergence $\widetilde{q}_{n,N}$ to q almost everywhere on $(0,1)$.

On the other hand let us restrict (2.24) to the diagonal $y = t$:

$$V(y,y) + c^T(y,y) + \int_y^T c^T(y,s)V(y,s)\,ds = 0, \quad 0 < y < 1, \tag{3.28}$$

and recalling (2.25), (3.12), we see that the best possible result one can expect is a convergence $\widetilde{q}_{n,N}$ to q almost everywhere on $(0,1)$.

Remark 3.5. *The stability of the scheme crucially depends on the type of the convergence $r_n \to r$. For now we know (see [4]) that the convergence is pointwise almost everywhere on the interval. The significant progress in the proving of the stability could be achieved by the improvement of this result.*

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References

[1] Avdonin, S. A., Belishev, M. I. & Ivanov, S. A. (1992) Matrix inverse problem for the equation $u_{tt} - u_{xx} + Q(x)u = 0$. *Math. USSR Sborn,* **7**, 287–310.

[2] Avdonin, S., Gesztesy, F. & Makarov, K. (2010) Spectral estimation and inverse initial boundary value problems. *Inverse Probl. Imaging,* **4**, 1–9.

[3] Avdonin, S., Lenhart, S. & Protopopescu, V. (2005) Determining the potential in the Schrödinger equation from the Dirichlet to Neumann map by the boundary control method *J. Inverse Ill-Posed Probl,* **13**, 317–330

[4] Avdonin, S. & Mikhaylov, V. (2010) The boundary control approach to inverse spectral theory. *Inverse Problems,* **26**, 045009 19pp.

[5] Baudouin, L. & and Puel, J.-P. (2002) Uniqueness and stability in an inverse problem for the Schrödinger equation. *Inverse Problems,* **18**, 1537–1554.

[6] Baudouin, L. & and Puel, J.-P. (2007) Corrigendum. Uniqueness and stability in an inverse problem for the Schrödinger equation. *Inverse Problems,* **23**, 1327–1328

[7] Belishev M. I. (2007) Recent progress in the boundary control method.*Inverse Problems,* **23**, R1–R67.

[8] Belishev M. I. & Ivanov S. A. (2002) Characterization of data in the dynamic inverse problem for a two-velocity system. *J. Math. Sci,* **109**, 1814–183.

[9] Bellassoued, M. & Choulli, M. (2009) Logarithmic stability in the dynamical inverse problem for the Schrödinger equation by arbitrary boundary observation. *J. Math. Pures Appl.,* **91**, 233–255.

[10] Boumenir, A. & Tuan V. K. (2009) Inverse problems for multidimensional heat equations by measurements at a single point on the boundary. *Numer. Funct. Anal. Optim.,* **30**, 1215–1230.

[11] Boumenir, A. & Tuan V. K. (2010) An inverse problem for the heat equation. *Proc. Amer. Math. Soc.,* **138**, 3911–3921.

[12] Boumenir, A. & Tuan V. K. (2010) Recovery of a heat equation by four measurements at one end. *Numer. Funct. Anal. Optim.,* **31**, 155–163.

[13] Levitan, B. M. (1987) *Inverse Sturm-Liouville Problems.* VNU Science Press, Utrecht.

[14] Levitan, B. M. & Gasymov, M. G. (1964) Determination of a differential equation by two of its spectra. *Russ. Math. Surveys,* **19**, 1–62.

[15] Mercado, A., Axel Osses, A. & Lionel Rosier (2008) Carleman inequalities and inverse problems for the Schrödinger equation. *C. R. Acad. Sci. Paris, Ser. I,* **346**, 53–58.

[16] Pöschel, J. & Trubowitz, E. (1987) *Inverse spectral theory.* Boston, MA, Academic Press Inc.,

[17] Ramdani, K., Tucsnak, M. & Weiss, G. (2010) Recovering the initial state of an infinite-dimensional system using observers. *Automatica,* **46**, 1616–1625.

[18] Vinokurov, V.A., Sadovnichii, V.A. (2000) Asymptotics of any order for the eigenvalues and eigenfunctions of the Sturm–Liouville boundary value problem on a segment with a summable potential, *Izvestia: Mathematics*, **64**, 47–108.

[19] Tenenbaum, G. (1995) *Introduction to analytic and probabilistic number theory.* Cambridge, Cambridge University Press.