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# Cut-generating functions 

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#### Abstract

In optimization problems such as integer programs or their relaxations, one encounters feasible regions of the form $\left\{x \in \mathbb{R}_{+}^{n}: R x \in\right.$ $S\}$ where $R$ is a general real matrix and $S \subset \mathbb{R}^{q}$ is a specific closed set with $0 \notin S$. For example, in a relaxation of integer programs introduced in [ALWW2007], $S$ is of the form $\mathbb{Z}^{q}-b$ where $b \notin \mathbb{Z}^{q}$. One would like to generate valid inequalities that cut off the infeasible solution $x=0$. Formulas for such inequalities can be obtained through cut-generating functions. This paper presents a formal theory of minimal cut-generating functions and maximal $S$-free sets which is valid independently of the particular $S$. This theory relies on tools of convex analysis.


Keywords: Integer programming; Convex analysis; Separation; Generalized gauges; $S$-free sets

## 1 Introduction

### 1.1 The separation problem, examples

This paper deals with sets of the form

$$
\begin{equation*}
X=X(R, S):=\left\{x \in \mathbb{R}_{+}^{n}: R x \in S\right\} \tag{1}
\end{equation*}
$$

where

$$
\begin{align*}
& R=\left[r_{1}, \ldots, r_{n}\right] \text { is a real } q \times n \text { matrix, } \\
& S \subset \mathbb{R}^{q} \text { is a closed set with } 0 \notin S \tag{2}
\end{align*}
$$

In other words, our set $X$ is the intersection of a closed convex cone (the nonnegative orthant) with a reverse image by a linear mapping. Since $0 \notin S$, it is not difficult to show that 0 does not lie in the closed convex hull of $X$.

We are interested in separating 0 from $X$ : we want to generate cuts, i.e. inequalities valid for $X$, which we write as

$$
\begin{equation*}
c^{\top} x \geqslant 1, \quad \text { for all } x \in X \tag{3}
\end{equation*}
$$

Geometrically, we want to generate half-spaces $H^{+}=\left\{x \in \mathbb{R}^{n}: c^{\top} x \geqslant 1\right\}$ (note: $0 \notin H^{+}$) satisfying $H^{+} \supset X$. This paper presents an overview of a formal theory of the functions that generate the coefficients $c_{j}$ of such cuts.

Let us first give some motivation for our model (1), (2), arising in mixed integer programming. Starting from a polyhedron

$$
P=\left\{(x, y) \in \mathbb{R}_{+}^{n} \times \mathbb{R}^{m}: A x+y=b\right\}
$$

(nonnegativity of the $y$-variables can also be imposed), assume that $b \notin \mathbb{Z}^{m}$. Several situations have been considered in the literature.

Example 1 (An integer linear program). Suppose first that all variables must be integers: the set of interest is $P \cap\left\{\mathbb{Z}^{n} \times \mathbb{Z}^{m}\right\}$, i.e. the set of points $(x, y=b-A x)$ such that $x \in \mathbb{Z}_{+}^{n}$ and $b-A x \in \mathbb{Z}^{m}$. Our problem has the form (1), (2) if we set

$$
q=n+m, \quad R=\left[\begin{array}{c}
I  \tag{4}\\
-A
\end{array}\right], \quad S=\mathbb{Z}^{n} \times \mathbb{Z}^{m}-\left[\begin{array}{l}
0 \\
b
\end{array}\right]
$$

Since $b \notin \mathbb{Z}^{m}$, the above $S$ is a closed set not containing the origin; (4) is the model considered by Gomory [G1969].

Example 2 ( $A$ mixed integer linear program). Consider now $P \cap\left\{\mathbb{R}^{n} \times \mathbb{Z}^{m}\right\}$ : the set of interest is the set of points $(x, y=b-A x)$ such that $x \in \mathbb{R}_{+}^{n}$ and $b-A x \in \mathbb{Z}^{m}$. Then (4) is replaced by

$$
q=m, \quad R=-A, \quad S=\mathbb{Z}^{m}-b
$$

which is the model considered by Andersen, Louveaux, Weismantel and Wolsey [ALWW2007].

We will retain from the above two examples the asymmetry between $S$ (a very particular and highly structured set) and $R$ (an arbitrary matrix). Keeping this in mind, we will consider that $(q, S)$ is given and fixed, while $(n, R)$ is instancedependent data: our cutting problem can be viewed as parametrized by $(n, R)$. A number of papers have appeared in recent years, dealing with the above problem with various special forms for $S$, see [ALWW2007], [DW2010], [BCCZ2010] and references therein.

### 1.2 Cut-generating functions and $S$-free sets

Let $(q, S)$ be given and fixed. To generate cuts in the present situation, it would be convenient to have a mapping, taking instances of (1), (2) as input, and producing cuts as output. What we need for this is a function

$$
\mathbb{R}^{q} \ni r \mapsto \rho(r) \in \mathbb{R}
$$

We will apply the function $\rho$ to the columns $r_{j}$ of $R$ (an arbitrary matrix, with an arbitrary number of columns) to produce the coefficients $c_{j}:=\rho\left(r_{j}\right)$ of a cut (3). In summary, we require that our $\rho$ satisfies, for any instance $X$ of (1),

$$
\begin{equation*}
x \in X \quad \Longrightarrow \quad \sum_{j=1}^{n} \rho\left(r_{j}\right) x_{j} \geqslant 1 \tag{5}
\end{equation*}
$$

Such a $\rho$ can be called a cut-generating function (CGF). So far, a CGF is a rather abstract object; but the (vast!) class of functions from $\mathbb{R}^{q}$ to $\mathbb{R}$ can be drastically reduced from the following observations.
(i) First consider in (3) a vector $c^{\prime}$ with $c_{j}^{\prime} \leqslant c_{j}$ for $j=1, \ldots, n$; then $c^{\prime \top} x \leqslant$ $c^{\top} x$ for any $x \geqslant 0$. If $c^{\prime}$ is a cut, it is tighter than $c$ in the sense that it cuts a bigger portion of $\mathbb{R}_{+}^{n}$. We can impose some "minimal" character to a CGF, in order to reach some "tightness" of the resulting cuts.
(ii) Next observe that changing $R$ to $t R(t>0)$ divides $X$ by $t$; the set of cuts is just multiplied by $t$. Since we seek a minimal $\rho$, we can impose without loss of generality $\rho(t r)=t \rho(r)$, for any $r \in \mathbb{R}^{q}$ and $t>0$ : only positively homogeneous CGF's are of interest.
(iii) It can be proved that the closed convex hull of a CGF $\rho$ is again a CGF. Moreover, if $\rho$ is positively homogeneous, then the closed convex hull of $\rho$ is positively homogeneous as well.

A function is sublinear if it is convex and positively homogeneous. The above observations show that the class of sublinear functions suffices to generate all relevant cuts; a fairly narrow class indeed, which is fundamental in convex analysis. Sublinear functions are in correspondence with closed convex sets and in our context, such a correspondence is based on the mapping $\rho \mapsto V$ defined by

$$
\begin{equation*}
V=V(\rho):=\left\{r \in \mathbb{R}^{q}: \rho(r) \leqslant 1\right\} \tag{6}
\end{equation*}
$$

Sublinear functions $\rho: \mathbb{R}^{q} \mapsto \mathbb{R}$ are convex, continuous and satisfy $\rho(0)=0$, which implies that $V(\rho)$ in (6) is a closed convex neighborhood of 0 in $\mathbb{R}^{q}$. The set $V$ turns out to be a cornerstone: via Theorem 1 below, (6) establishes a correspondence between the (sublinear) CGF's and the so-called $S$-free sets.

Definition 1 ( $S$-free set). Given a closed set $S \subset \mathbb{R}^{q}$ not containing the origin, a closed convex neighborhood $V$ of $0 \in \mathbb{R}^{q}$ is called $S$-free if its interior contains no point in $S: \operatorname{int}(V) \cap S=\emptyset$.

Theorem 1. Let $\rho$ be a sublinear function from $\mathbb{R}^{q}$ to $\mathbb{R}$ and $V(\rho)$ the closed convex neighborhood of $0 \in \mathbb{R}^{q}$ defined in (6). Then $\rho$ is a CGFfor (1), (2) if and only if $V(\rho)$ is $S$-free.

As a result, the cut generation problem for $X$ can alternatively be studied from a geometric point of view, involving sets $V$ instead of functions $\rho$. This situation, common in convex analysis, is often very fruitful.

Definition 2 (CGF as representation). Let $V \subset \mathbb{R}^{q}$ be a closed convex neighborhood of the origin. A representation of $V$ is a (finite-valued) sublinear function $\rho$ satisfying (6). We will say that $\rho$ represents $V$. A (sublinear) cut-generating function for (1), (2) is a representation of an $S$-free set.

A sublinear $\rho$ represents a unique $V=V(\rho)$, well-defined by (6). One easily checks

$$
\begin{equation*}
\rho \leqslant \rho^{\prime} \quad \Longrightarrow \quad V(\rho) \supset V\left(\rho^{\prime}\right) . \tag{7}
\end{equation*}
$$

Hence, minimality of $\rho$ corresponds to maximality of $V$. By contrast, the mapping $\rho \mapsto V(\rho)$ in (6) is many-to-one and therefore has no inverse. There is a difficulty here: a given neighborhood $V$ may have several representations, and we are interested in the small ones.

### 1.3 Goals and outline of the paper

The aim of this paper is to present the main points of a formal theory of minimal cut-generating functions and maximal $S$-free sets which is valid independently of the particular $S$. This theory of cut-generating functions gathers, generalizes and synthesizes some existing results (see [BCZ2011], [DW2010], [BCCZ2010] and references therein). The complete theory is presented in an extended version of this paper [CCDLM2013]; in particular, the proofs of the results are omitted here, so the reader is referred to [CCDLM2013] to see precisely how things combine.

The paper is organized as follows. We study the mapping (6) in Section 2. We show that the pre-images of a given $V$ (the representations of $V$ ) have a unique maximal element $\gamma_{V}$ and a unique minimal element $\mu_{V}$; in view of (i) above, the latter is the relevant inverse of $\rho \mapsto V(\rho)$. Then we study in Section 3 the correspondence $V \leftrightarrow \mu_{V}$. We show that different concepts of minimality come into play for $\rho$ in (i). Geometrically they correspond to different concepts of maximality for $V$. We also show that they coincide in a number of cases.

## 2 Largest and smallest representations

In this section, we study the representation operation introduced in Definition 2 and its geometric counterpart. We first recall some basic definitions of convex analysis; The monograph [HL2001] (especially its Chapter C) is suggested for an elementary introduction, while textbooks [HL1993,R1970] are more complete.

### 2.1 Basic definitions of convex analysis

The support function of a set $G \subset \mathbb{R}^{q}$ is

$$
\begin{equation*}
\sigma_{G}(r):=\sup _{d \in G} d^{\top} r \tag{8}
\end{equation*}
$$

It is seen to be sublinear, to grow when $G$ grows, but to remain unchanged if $G$ is replaced by its closed convex hull: $\sigma_{G}=\sigma_{\overline{\operatorname{Conv}}(G)}$. Conversely, any sublinear function $\rho$ is the support function of a closed convex set, unambiguously defined by

$$
G=G_{\rho}:=\left\{d \in \mathbb{R}^{n}: d^{\top} r \leqslant \rho(r) \text { for all } r \in \mathbb{R}^{q}\right\} ;
$$

we say that $\rho$ supports $G$. Note that a sublinear function $\rho$ is finite valued if and only if $\rho$ is the support function of a bounded closed convex set.

Another relevant object for our purpose is the gauge

$$
\begin{equation*}
\mathbb{R}^{q} \ni r \mapsto \gamma_{V}(r):=\inf \{\lambda>0: r \in \lambda V\} \tag{9}
\end{equation*}
$$

of our neighborhood $V$. In fact, results in convex analysis [HL2001, Theorem C.1.2.5 and Proposition C.3.2.4] show that $\gamma_{V}$

- also appears as a representation of $V$
- is the support function of the polar set of $V$ defined by

$$
\begin{equation*}
V^{\circ}:=\left\{d: d^{\top} r \leqslant 1 \text { for all } r \in V\right\}=\left\{d: \sigma_{V}(d) \leqslant 1\right\} . \tag{10}
\end{equation*}
$$

### 2.2 Prepolars and representations

From now on in this section, we are given a subset $V$ of $\mathbb{R}^{q}$, which is a closed convex neighborhood of the origin. If $G$ is such that $G^{\circ}=V$, we can say that $G$ is a prepolar of $V$, i.e. that $\sigma_{G}$ represents $V$ in the sense of Definition 2. As already mentioned, $V$ may have several representations, and there may be several $G$ 's such that $G^{\circ}=V$, that is, several $G$ 's may be prepolars of $V$. Because $\left(V^{\circ}\right)^{\circ}=V$, the standard polar $V^{\circ}$ is itself a prepolar - which is somewhat confusing - and turns out to be the largest one; or equivalently $\gamma_{V}$ turns out to be the largest representation of $V$, as shown by Theorem 2 below. This theorem states furthermore that $V$ has also a smallest prepolar, or equivalently a smallest representation; keeping (i) of Section 1 in mind, this is exactly what we want. This result is actually [BCZ2011, Theorem 1]; we give a different treatment here.

The following geometric objects turn out to be relevant:

$$
\left\{\begin{array}{l}
\tilde{V}^{\circ}:=\left\{d \in V^{\circ}: d^{\top} r=\sigma_{V}(d)=1 \text { for some } r \in V\right\},  \tag{11}\\
\widehat{V}^{\circ}:=\left\{d \in V^{\circ}: \sigma_{V}(d)=1\right\} .
\end{array}\right.
$$

For later use, we illustrate this construction with a simple example.


Fig. 1. Constructing $\tilde{V}^{\circ}$ or $\widehat{V}^{\circ}$

Example 3. With $\left[\begin{array}{l}r^{1} \\ r^{2}\end{array}\right] \in \mathbb{R}^{2}$, take for $V$ the polyhedron given by the following three inequalities (see Figure 1):

$$
r^{1} \leqslant 1, \quad r^{2} \leqslant 1, \quad r^{2} \leqslant 2+r^{1}
$$

Recalling that extreme points of $V^{\circ}$ correspond to facets of $V$, we see that $V^{\circ}$ has the three extreme points $A, B, C$ defined by the equation $d^{\top} r=1$, for $r$ respectively on the three lines making up the boundary of $V$. We obtain $A=(1,0), B=(0,1), C=\frac{1}{2}(-1,1)$.

In this example, $\widetilde{V}^{\circ}$ and $\widehat{V}^{\circ}$ are the same set, namely the union of the two segments $[A, B]$ and $[B, C]$. To obtain $V^{\circ}$, convexify them with the fourth point 0 ; if $V$ had a fourth constraint, say $r^{2} \geqslant-1$, then this fourth point would be moved down to $D=(0,-1)$ - and would be part of the sets $\widetilde{V}^{\circ}$ and $\widehat{V}^{\circ}$.

Because $0 \in \operatorname{int} V$, the definition (8) of a support function shows that $\sigma_{V}$ is positive whenever it is finite: for some $\varepsilon=\varepsilon(V)>0$,

$$
\begin{equation*}
\varepsilon\|d\| \leqslant \sigma_{V}(d) \leqslant+\infty \quad \text { for all } d \in \mathbb{R}^{q} \tag{12}
\end{equation*}
$$

The two sets in (11) are therefore bounded. Besides, the next proposition shows that they differ very little.

Proposition 1. We have $\widetilde{V}^{\circ} \subset \widehat{V}^{\circ} \subset \operatorname{cl}\left(\widetilde{V}^{\circ}\right)$. It follows that $\widehat{V}^{\circ}$ and $\widetilde{V}^{\circ}$ have the same closed convex hull.

The closed convex hull revealed by this proposition deserves a notation, as well as its support function: we set

$$
\begin{equation*}
V^{\bullet}:=\overline{\operatorname{conv}}\left(\widetilde{V}^{\circ}\right)=\overline{\operatorname{conv}}\left(\widehat{V}^{\circ}\right) \quad \text { and } \quad \mu_{V}:=\sigma_{V^{\bullet}}=\sigma_{\widetilde{V}^{\circ}}=\sigma_{\widehat{V}^{\circ}} \tag{13}
\end{equation*}
$$

(in Figure $1, V^{\bullet}$ is the triangle conv $(A, B, C)$ ). In fact, the next result shows that $\mu_{V}$ is the smallest representation we are looking for. From now on, we assume $V \neq \mathbb{R}^{q}$ (otherwise $V^{\bullet}=\emptyset, \mu_{V} \equiv-\infty$, a degenerate situation which is trivial).

Proposition 2 (Smallest representation). Any $\rho$ representing $V$ satisfies $\rho \geqslant \mu_{V}$. Geometrically, $V^{\bullet}$ is the smallest closed convex set whose support function represents $V$.

Thus, $V$ does have a smallest representation, whose supported set is $V^{\bullet}$. On the other hand, it is interesting to link it with $V^{\circ}$. The intuition suggested by Figure 1 is confirmed by the following result.

Proposition 3. Appending 0 to $V^{\bullet}$ gives the standard polar:

$$
\gamma_{V}=\max \left\{\mu_{V}, 0\right\} \quad \text { i.e. } \quad V^{\circ}=\overline{\operatorname{conv}}\left(V^{\bullet} \cup\{0\}\right)=[0,1] V^{\bullet}
$$

We actually have an equivalence.
Theorem 2 (Representations). A sublinear function $\rho$ represents $V$ if and only if it satisfies

$$
\begin{equation*}
\mu_{V} \leqslant \rho \leqslant \gamma_{V} \tag{14}
\end{equation*}
$$

Geometrically, the support function of a set $G$ represents $V$ if and only if $G$ is sandwiched between the two extreme prepolars of $V$ :

$$
G^{\circ}=V \quad \Longleftrightarrow \quad V^{\bullet} \subset \overline{\operatorname{conv}}(G) \subset V^{\circ} .
$$

## 3 Minimal CGF's and maximal $S$-free sets

### 3.1 Minimal CGF's

In our quest for small CGF's, the following definition is natural.
Definition 3 (Minimality). A CGF $\rho$ is called minimal if any CGF $\rho^{\prime} \leqslant \rho$ is $\rho$ itself.

A minimal CGF is certainly a smallest representation:

$$
\begin{equation*}
\rho \text { is a minimal CGF } \quad \Longrightarrow \quad \rho=\mu_{V(\rho)}=\sigma_{V(\rho)} \bullet \tag{15}
\end{equation*}
$$

(indeed, Theorem 2 states that $\mu_{V(\rho)}$ represents the same set $V(\rho)$ as $\rho$ - and is therefore a CGF if so is $\rho$ ).

If $\rho$ is a minimal CGF, $V(\rho)$ must of course be a special $S$-free set. Take for example $S=\{1\} \subset \mathbb{R}$ and the $S$-free set $V=[-1,+1] ; \rho(r):=|r|$ is the smallest (because unique) representation of $V$ but $\rho$ is not minimal: $\rho^{\prime}(r):=\max \{0, r\}$ is also a CGF, representing $\left.\left.V^{\prime}=\right]-\infty,+1\right]$. From (7), a smaller $\rho$ describes a larger $V$; so Definition 3 has its geometrical counterpart:

Definition 4 (Maximality). An $S$-free set $V$ of Definition 1 is called maximal if any $S$-free set $V^{\prime} \supset V$ is $V$ itself.

Actually, this "duality" is deceiving, as the two definitions do not match: the set represented by a minimal CGF need not be maximal. Here is a trivial example.

Example 4. When $\rho$ is linear, the property introduced in Definition 3 holds vacuously: no sublinear function can properly lie below a linear function. Thus, any linear CGF $\rho$ is minimal; yet, a linear $\rho$ represents a neighborhood $V(\rho)$ (a half-space) which is $S$-free but has not reason to be maximal. See Figure 2: with $n=1$, the set $V=]-\infty, 1]$ (represented by $\rho(x)=x$ ) is $\{2\}$-free but is obviously not maximal.


Fig. 2. A linear CGF is always maximal

Note that, if the half-space represented by a linear function is $S$-free, it actually separates $S$ from 0 . A simple assumption such as $0 \in \operatorname{conv} S$ will therefore rule out the above counterexample; but Example 5 below will reveal a more serious deficiency. So a subtlety is necessary, indeed the smallest representation of a maximal $V$ enjoys a stronger property than minimality.

### 3.2 Strongly minimal CGF's

Let $\rho$ be a CGF, which represents via (6) the set $V=V(\rho)$. The gauge $\gamma_{V(\rho)}$ is then a function of $\rho$ and here comes the correct substitute to Definition 3.

Definition 5 (Strongly minimal CGF). A CGF $\rho$ is called strongly minimal if any CGF $\rho^{\prime} \leqslant \gamma_{V(\rho)}$ satisfies $\rho^{\prime} \geqslant \rho$.

Needless to say, the class of strong minimality CGF's is a subclass of the class of minimal CGF's. Example 5 below will complement Example 4, showing that the restriction is a real one. At any rate, strong minimality turns out to be the appropriate definition in general:

Theorem 3 (Strongly minimal $\Leftrightarrow$ maximal). An $S$-free set $V$ is maximal if and only if its smallest representation $\mu_{V}$ of (13) is a strongly minimal CGF.

In fact, the concept of minimality involves two properties from a sublinear function:

- it must be the smallest representation of some $V$ (recall (15)),
- the neighborhood $V$ must enjoy some maximality property.

In view of the first property, a CGF can be imposed to be not only sublinear but also to support a set that is a smallest prepolar. Then Definition 3 has a geometric counterpart: minimality of $\rho=\mu_{V}=\sigma_{V} \cdot$ means

$$
\begin{array}{ccc}
G^{\prime} \subset V^{\bullet} \\
{\left[\rho^{\prime}=\sigma_{G^{\prime}} \leqslant \mu_{V}\right]}
\end{array} \begin{gathered}
\text { and }\left(G^{\prime}\right)^{\circ} \text { is } S \text {-free } \\
{\left[\rho^{\prime} \text { is a CGF }\right]}
\end{gathered} \quad \Longrightarrow \quad G^{\prime}=V^{\bullet}, \text { i.e. }\left(G^{\prime}\right)^{\circ}=V .
$$

Likewise for Definition 5: strong minimality of $\rho=\gamma_{V}=\sigma_{V^{\circ}}$ means

$$
\begin{array}{ccc}
G^{\prime} \subset V^{\circ} \text { and }\left(G^{\prime}\right)^{\circ} \text { is } S \text {-free } & \Longrightarrow & G^{\prime} \supset V^{\bullet}, \text { i.e. }\left(G^{\prime}\right)^{\circ} \subset V . \\
{\left[\rho^{\prime}=\sigma_{G^{\prime}} \leqslant \gamma_{V}\right]} & {\left[\rho^{\prime} \text { is a CGF }\right]} &
\end{array}
$$

These observations allow some more insight into the $(\cdot)^{\bullet}$ operation:
Proposition 4. Let $\rho=\mu_{V}=\sigma_{V} \bullet$ be a minimal CGF. If an $S$-free neighborhood $W$ satisfies $W^{\bullet} \subset V^{\bullet}$, then $W=V$.

Thus, the trouble necessitating strong minimality lies in (7): even though the reverse implication holds when $\rho=\gamma_{V}$, it does not hold for $\rho=\mu_{V}$ : the mapping $V \mapsto V^{\bullet}$ is not monotonic; and of course, this phenomenon is linked to the presence of the recession cone $V_{\infty}$. The following example helps for a better understanding.

Example 5. In Example 3, take for $S$ the union of the three lines given respectively by the three equations

$$
r^{1}=1, \quad r^{2}=1, \quad r^{2}=2+r^{1}
$$

so that $V$ is clearly maximal $S$-free.


Fig. 3. The mapping $V \mapsto V^{\bullet}$ is not monotonic

Now shrink $V$ to $V_{t}$ (left part of Figure 3) by moving its right vertical boundary to $r^{1} \leqslant 1-t$. Then $A$ is moved to $A_{t}=\left(\frac{1}{1-t}, 0\right)$; there is no inclusion between the new $V_{t}^{\bullet}=\operatorname{conv}\left(A_{t}, B, C\right)$ and the original $V^{\bullet}=\operatorname{conv}(A, B, C)$; this is the key to our example.

Let us show that $\mu_{V_{t}}$ is minimal, even though $V_{t}$ is not maximal. Take for this a CGF $\rho \leqslant \mu_{V_{t}}$, which represents an $S$-free set $W$; by (7), $W \supset V_{t}$. With the notation (13), we therefore have

$$
\sigma_{W}^{\bullet}=\mu_{W} \leqslant \rho \leqslant \mu_{V_{t}}=\sigma_{V_{t}^{\bullet}}, \quad \text { i.e., } \quad W^{\bullet} \subset V_{t}^{\bullet}
$$

and we proceed to show that equality does hold, i.e. the three extreme points of $V_{t}^{\bullet}$ do lie in $W^{\bullet}$.

- If $A_{t} \notin W^{\bullet}$, the right part of Figure 3 shows that $W^{\bullet}$ is included in the open upper half-space. Knowing that

$$
W=\left(W^{\bullet}\right)^{\circ}=\left\{r: d^{\top} r \leqslant 1 \text { for all } d \in W^{\bullet}\right\}
$$

(see the end of Section 2), this implies that the recession cone $W_{\infty}$ has a vector of the form $r_{A}=(\varepsilon,-1)(\varepsilon>0) ; W$ cannot be $S$-free.

- If $C \notin W^{\bullet}$, there is $r_{C} \in \mathbb{R}^{2}$ such that $C^{\top} r_{C}>\sigma_{W} \bullet\left(r_{C}\right)=\mu_{W}\left(r_{C}\right)$ (we denote also by $C$ the 2 -vector representing $C)$. For example $r_{C}=(-2,0) \in \operatorname{bd}(V)$ (see Figure 3), so that

$$
C^{\top} r_{C}=1>\sigma_{W} \bullet(-2,0)=\mu_{W}(-2,0)
$$

By continuity, $\mu_{W}(-2-\varepsilon, 0) \leqslant 1$ for $\varepsilon>0$ small enough. Because $\mu_{W}$ represents $W$, this implies that $(-2-\varepsilon, 0) \in W ; W$ (which contains $V_{t}$ ) is not $S$-free.

- By the same token, we prove that $B \in W^{\bullet}$ (the separator $r_{B}=(0,1) \in \operatorname{bd}(V)$ does the job).
We have therefore proved that $W^{\bullet}=V_{t}^{\bullet}$, i.e $\mu_{W}=\mu_{V_{t}}$, i.e. $\mu_{V_{t}}$ is minimal.
Examples 4 and 5 show that minimality does not imply strong minimality in general. On the other hand, the following theorem provides two favorable cases when this implication holds.

Theorem 4. Suppose $0 \in \hat{S}:=\overline{\operatorname{conv}} S$ and that $\mu_{V}$ is minimal. Then $\mu_{V}$ is strongly minimal under any of the following conditions:
(i) $V_{\infty} \cap \hat{S}_{\infty}=\{0\}$ (in particular $S$ bounded),
(ii) $V_{\infty} \cap \hat{S}_{\infty}=L \cap \hat{S}_{\infty}$ where $L$ stands for the lineality space of $V$, and $\hat{S}=G+\hat{S}_{\infty}$ where $G$ in any nonempty bounded set.

Theorem 4 generalizes several earlier results. The special case where $S$ is a finite set of points in $\mathbb{Z}^{q}-b$ was first considered by Johnson [J1981] and more recently by Dey and Wolsey [DW2010]. Theorem 4(ii) was proven by [DW2010] and $[\mathrm{BCCZ2010}]$ in the special case where $S=P \cap\left(\mathbb{Z}^{q}-b\right)$ for some rational polyhedron $P$.

### 3.3 Asymptotically maximal sets

Finally a natural question arises: how far from being maximal are the $S$-free sets represented by minimal CGF's? For this, we introduce one more concept, which does not seem to have arisen in the literature on cut-generating functions.

Definition 6. An $S$-free set $V$ of Definition 1 is called asymptotically maximal if any $S$-free set $V^{\prime} \supset V$ satisfies $V_{\infty}^{\prime}=V_{\infty}$.
Then we have a partial answer to the question about $S$-free sets represented by minimal CGF's.

Theorem 5 (Minimal $\Rightarrow$ asymptotically maximal). The $S$-free neighborhood represented by a minimal CGF is asymptotically maximal.

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