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# ASYMPTOTIC SHALLOW WATER MODELS WITH NON SMOOTH TOPOGRAPHIES

MATHIEU CATHALA

**ABSTRACT.** We present new models to describe shallow water flows over non smooth topographies. The water waves problem is formulated as a system of two equations on surface quantities in which the topography is involved in a Dirichlet-Neumann operator. Starting from this formulation and using the joint analyticity of this operator with respect to the surface and the bottom parametrizations, we derive a nonlocal shallow water model which only includes smoothing contributions of the bottom. Under additional small amplitude assumptions, Boussinesq-type systems are also derived. Using these alternative shallow water models as references, we finally present numerical tests to assess the precision of the classical shallow water approximations over rough bottoms. In the case of a polygonal bottom, we show numerically that our new model is consistent with the approach developed by Nachbin.

## 1. INTRODUCTION

**1.1. Water waves over a rough bottom.** Surface water waves propagation over a variable bottom has been widely studied over the past decades because of its importance in oceanography. Assuming the fluid is incompressible, homogeneous and inviscid, its motion is governed by the Euler equations with nonlinear boundary conditions at the surface. As the free surface boundary is part of the unknowns, the full problem, known as the water waves problem, is very difficult to solve both mathematically and numerically. Nonetheless, in some specific physical regimes it is possible to derive much simpler asymptotic models (see [27] for a recent review).

In shallow water conditions *i.e.* when the typical wavelength of the waves is much larger than the typical depth, the free surface problem is frequently approximated by the Saint-Venant equations. When the bottom parametrization is smooth, it is known [1, 35, 36, 25, 28, 24] that they provide a good approximation to the exact solution of the full water waves equations. However, in case the bottom is rough, there is no evidence that the Saint-Venant equations are still a relevant approximation of the water waves problem. As a matter of fact, the topography introduces singular terms in the Saint-Venant system if the bottom parametrization is not regular.

On that basis, some models have been proposed to handle rapidly varying periodic or random topographies. We cite in particular the papers of Rosales and Papanicolaou [38], Nachbin and Sølna [30], Craig *et al.* [11], Craig, Lannes and Sulem [12].

Concerning non smooth topographies, Hamilton [22] and Nachbin [29] used a conformal mapping technique to derive long wave models in the case of two-dimensional motions. In [29], a Boussinesq system is formulated to handle non smooth one-dimensional topographies. However, this technique only applies to polygonal (one-dimensional) bottom profiles.

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The purpose of the present paper is to derive alternatives to some classical shallow water models (namely Saint-Venant equations, Serre equations and Boussinesq system) which do not involve any singular term with non smooth topographies. The systems we obtain consist in modifying the topographical terms in the classical shallow water models. In case the bottom is smooth, these new systems are consistent with the former.

**1.2. Formulation of the water waves problem.** Through this paper, we denote by  $\zeta(t, x)$  the surface elevation and by  $-H_0 + b(x)$  a parametrization of the bottom, where  $H_0$  is a reference depth (see Figure 1). The time-dependent fluid domain

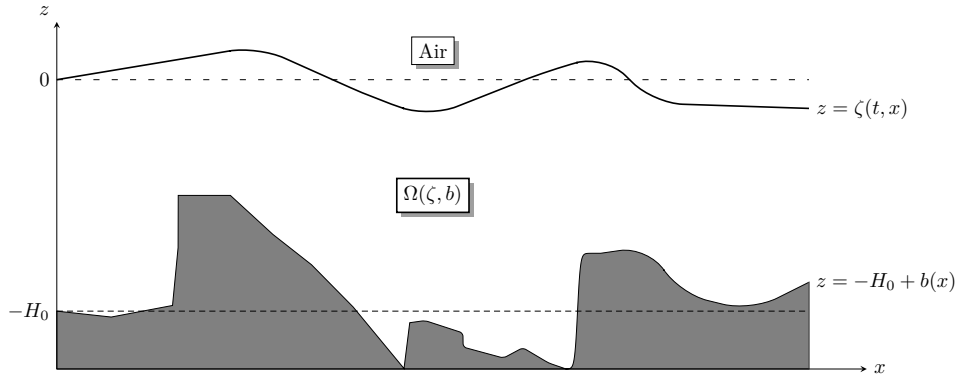


FIGURE 1. Sketch of the domain

consists of the region

$$\Omega(\zeta, b) = \{(x, z) \in \mathbb{R}^d \times \mathbb{R} ; -H_0 + b(x) < z < \zeta(t, x)\},$$

where  $d = 1, 2$  denotes the spatial dimension of the free surface (and the bottom). We further assume the flow is irrotational so that, from the incompressibility assumption, the flow field is described by an harmonic potential  $\Phi$ .

The asymptotic analysis of the water waves problem requires the use of dimensionless quantities based on characteristics of the flow. More precisely, denoting by  $\lambda$  the typical wavelength of the waves, by  $a_{\text{surf}}$  their typical amplitude and by  $a_{\text{bott}}$  the typical amplitude of the bottom variations, we define dimensionless variables and unknowns as

$$x' = \frac{x}{\lambda}, \quad z' = \frac{z}{H_0}, \quad t' = \frac{\sqrt{gH_0}}{\lambda} t,$$

and

$$\zeta' = \frac{\zeta}{a_{\text{surf}}}, \quad b' = \frac{b}{a_{\text{bott}}}, \quad \Phi' = \frac{\Phi}{a_{\text{surf}} \lambda \sqrt{g/H_0}}.$$

To simplify the notations we omit the prime symbol in the rest of the paper. From the previous physical scales we also define three independent parameters:

$$\mu = \frac{H_0^2}{\lambda^2}, \quad \varepsilon = \frac{a_{\text{surf}}}{H_0}, \quad \beta = \frac{a_{\text{bott}}}{H_0}.$$

Our analysis focuses on the shallow water regime  $\mu \ll 1$ . The parameters  $\varepsilon$  and  $\beta$  respectively account for the relative amplitude of the waves and of the bathymetry.

Zakharov [40] and Craig and Sulem [16, 15] remarked that the water waves equations can be written as a system of two scalar evolution equations on the surface elevation  $\zeta$  and on the velocity potential at the surface  $\psi = \Phi|_{z=\varepsilon\zeta}$ . The key point is that at time  $t$ , given  $\zeta(t, \cdot)$  and  $b$ , the knowledge of  $\psi(t, \cdot)$  fully determines

the velocity potential  $\Phi(t, \cdot, \cdot)$  in the fluid domain as the solution of the following non-dimensionalized elliptic problem

$$\begin{cases} \mu\Delta\Phi + \partial_z^2\Phi = 0 & \text{in } \Omega(\varepsilon\zeta, \beta b), \\ \Phi = \psi(t, \cdot) & \text{on } \{z = \varepsilon\zeta(t, x)\}, \\ \partial_{\mathbf{n}}\Phi = 0 & \text{on } \{z = -1 + \beta b(x)\}; \end{cases} \quad (1.1)$$

where  $\Delta$  denotes the Laplace operator in the horizontal variables and  $\partial_{\mathbf{n}}$  stands for the outward conormal derivative associated with the elliptic operator  $\mu\Delta + \partial_z^2$ . In particular, one may define the Dirichlet-Neumann operator as

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b] : \psi \mapsto \sqrt{1 + \varepsilon^2 |\nabla\zeta|^2} \partial_{\mathbf{n}}\Phi|_{z=\varepsilon\zeta}. \quad (1.2)$$

Using this operator the Zakharov/Craig-Sulem formulation of the water waves problem writes, in dimensionless form,

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \varepsilon\mu \frac{(\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \varepsilon\nabla\zeta \cdot \nabla\psi)^2}{2(1 + \varepsilon^2\mu|\nabla\zeta|^2)} = 0. \end{cases} \quad (1.3)$$

**1.3. Statement of results and outline of the paper.** As it appears from the above formulation, the derivation of shallow water models is governed by the asymptotic behavior of the Dirichlet-Neumann operator as  $\mu \ll 1$ . The main task consists in finding, in shallow water regime, an explicit relation between  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  and  $\nabla\psi$  through expansion of the Dirichlet-Neumann operator with respect to  $\mu$ . For smooth topographies, it is known (see Proposition 3.8 of [1]) that

$$\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\nabla \cdot ((1 + \varepsilon\zeta - \beta b)\nabla\psi) + O(\mu).$$

From the previous relation, one may deduce that, up to terms of order  $O(\mu)$ , the couple  $(\zeta, \nabla\psi)$  satisfies the classical Saint-Venant equations

$$\begin{cases} \partial_t\zeta + \nabla \cdot ((1 + \varepsilon\zeta - \beta b)\nabla\psi) = 0, \\ \partial_t\nabla\psi + \nabla\zeta + \varepsilon(\nabla\psi \cdot \nabla)\nabla\psi = 0. \end{cases} \quad (1.4)$$

Now, in the presence of non smooth topographies, the contribution of the bottom to the first equation of (1.4) may be singular whereas, as regards the full Dirichlet-Neumann operator, the topographic contribution is still infinitely smooth from the ellipticity of the potential equation (1.1).

The main result of the paper is the construction of an approximation that involves an infinitely smoothing contribution of the bottom, namely

$$\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\nabla \cdot ((1 + \varepsilon\zeta - b_\mu[\beta b])\nabla\psi) + O(\mu),$$

where  $b_\mu[\beta b]$  is a regularization operator (defined below). This construction leads to the formal derivation of a nonlocal shallow water system allowing non smooth topographies. Under additional assumptions on  $\varepsilon$ , we also derive medium and small amplitude models including dispersive effects.

The paper is organized as follows. Section 2 is devoted to the shallow water analysis of the Dirichlet-Neumann operator. Using the fact that this operator depends analytically on  $\zeta$  and  $b$ , we show that the shallow water limit of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  can be computed using explicit expressions for its shape derivatives with respect to the surface and the bottom parametrizations. This particular construction only involves smoothing contributions of the bottom. Using this asymptotic analysis, we address in section 3 the derivation of shallow water models under different

sub-regimes, depending on the size of  $\varepsilon$  (that is the wave amplitude). All these alternative models account for non smooth topographies. The numerical results we present in section 4 confirm these alternative models are consistent with the classical shallow water systems in case the bottom parametrization is smooth. Moreover these new model can be used to asses the precision of the classical systems used with rough bottoms. In the appendix, we present an additional numerical example with a polygonal bottom. In this particular case, the results obtained indicate that our new model is consistent with the approach developed by Nachbin for polygonal topographies [29].

## 2. ASYMPTOTIC ANALYSIS OF THE DIRICHLET-NEUMANN OPERATOR IN SHALLOW WATER REGIME

In this section, we focus on the asymptotic analysis of the Dirichlet-Neumann operator in shallow water regime. To handle rough bottoms, the strategy we adopt is to bring into play the shape analyticity of the Dirichlet-Neumann operator, that is to say a Taylor expansion of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  with respect to the surface and the bottom parametrizations. The shape derivatives, *i.e.* the terms of this Taylor series, can be formally calculated and only give smooth contributions of the bottom. Thus we perform a shallow water limit of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  which allows rough bottoms by analyzing the asymptotic behavior of these derivatives as  $\mu \ll 1$ . In particular, attention is payed to check that, in shallow water regime, the higher the order of the shape derivative, the higher order in  $\mu$  it contributes.

In the present section, the time variable does not play any role so that we drop the dependence on  $t$  to simplify notation.

**2.1. Shape analyticity of the Dirichlet-Neumann operator.** The analyticity of the Dirichlet-Neumann operator with respect to the surface elevation has been deeply investigated for the case of a flat bottom (see *e.g.* [6, 10, 14, 13, 23]). In case the topography is non-trivial, the shape analyticity of the Dirichlet-Neumann operator with respect to the surface and the bottom parametrizations has been more recently addressed by, among others, Nicholls and Taber [33] and Lannes [26, Theorem A.11].

**2.1.1. Taylor expansion of the Dirichlet-Neumann operator in powers of  $\zeta$ .** From the analyticity with respect to the surface, if  $\varepsilon\zeta$  lies in a small neighborhood of 0, the Dirichlet-Neumann operator can be expanded as

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = \sum_{n=0}^{+\infty} \mathcal{G}_\mu^n[\varepsilon\zeta, \beta b]\psi \quad (2.1)$$

where each mapping  $\tilde{\zeta} \mapsto \mathcal{G}_\mu^n[\tilde{\zeta}, \beta b]\psi$  is homogeneous of degree  $n^1$ .

The description of the individual terms in this Taylor series expansion has been first addressed for flat bottoms by Craig and Sulem [15] in two dimensions and a generalization to three dimensions is given in [32]. Introducing an implicit operator  $L_\mu[\beta b]$  to take into account the bottom variations, Craig *et al.* [11] showed that this description may be extended to the case of an uneven bottom.

In our non-dimensional framework, the first term of this expansion is given by

$$\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi = \sqrt{\mu} |D| \tanh(\sqrt{\mu} |D|)\psi + \sqrt{\mu} |D| L_\mu[\beta b]\psi, \quad (2.2)$$

<sup>1</sup>Denoting by  $d_\zeta^n \mathcal{G}_\mu[0, \beta b](\tilde{\zeta})\psi$  the  $n$ -th derivative of  $\zeta \mapsto \mathcal{G}_\mu[\zeta, \beta b]\psi$  at  $\zeta = 0$  in the direction  $\tilde{\zeta}$ , the  $n$ -th term in the Taylor expansion (2.1) is related to this shape derivative by  $d_\zeta^n \mathcal{G}_\mu[0, \beta b](\tilde{\zeta})\psi = n! \mathcal{G}_\mu^n[\tilde{\zeta}, \beta b]\psi$ .

where  $D = \frac{\nabla}{i}$  and where we used the Fourier multiplier notation  $f(D)u$ , defined in terms of Fourier transform by  $\widehat{f(D)u} = f\hat{u}$ . Then, adapting the computations of [11] to our dimensionless context, we get a similar recursive formula which reads, for the even terms,

$$\begin{aligned} \mathcal{G}_\mu^{2n}[\varepsilon\zeta, \beta b]\psi &= \\ & \frac{\mu^n}{(2n)!} D \cdot \{(\varepsilon\zeta)^{2n} D |D|^{2(n-1)} \mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi\} \\ & - \sum_{p=0}^{n-1} \frac{\mu^{n-p}}{(2(n-p))!} \mathcal{G}_\mu^{2p}[\varepsilon\zeta, \beta b] \{(\varepsilon\zeta)^{2(n-p)} |D|^{2(n-p)} \psi\} \\ & - \sum_{p=0}^{n-1} \frac{\mu^{n-p-1}}{(2(n-p)-1)!} \mathcal{G}_\mu^{2p+1}[\varepsilon\zeta, \beta b] \{(\varepsilon\zeta)^{2(n-p)-1} |D|^{2(n-p-1)} \mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi\}, \end{aligned} \quad (2.3)$$

and for the odd terms,

$$\begin{aligned} \mathcal{G}_\mu^{2n+1}[\varepsilon\zeta, \beta b]\psi &= \\ & \frac{\mu^{n+1}}{(2n+1)!} D \cdot \{(\varepsilon\zeta)^{2n+1} |D|^{2n} D\psi\} \\ & - \sum_{p=0}^n \frac{\mu^{n-p}}{(2(n-p)+1)!} \mathcal{G}_\mu^{2p}[\varepsilon\zeta, \beta b] \{(\varepsilon\zeta)^{2(n-p)+1} |D|^{2(n-p)} \mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi\} \\ & - \sum_{p=0}^{n-1} \frac{\mu^{n-p}}{(2(n-p))!} \mathcal{G}_\mu^{2p+1}[\varepsilon\zeta, \beta b] \{(\varepsilon\zeta)^{2(n-p)} |D|^{2(n-p)} \psi\}. \end{aligned} \quad (2.4)$$

In particular, the linear operator  $\mathcal{G}_\mu^1[\varepsilon, \beta \cdot]\psi$  is given by:

$$\mathcal{G}_\mu^1[\varepsilon\zeta, \beta b]\psi = -\mu\nabla \cdot (\varepsilon\zeta\nabla\psi) - \mathcal{G}_\mu^0[\varepsilon\zeta, \beta b](\varepsilon\zeta\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi). \quad (2.5)$$

2.1.2. *Taylor expansion of the Dirichlet-Neumann operator in powers of  $b$ .* From the analyticity of the Dirichlet-Neumann operator with respect to the bottom parametrization we know that, if  $\beta b$  lies in a sufficiently small neighborhood of 0, the operator  $|D| L_\mu[\beta b]\psi$  which stands for the contribution of the bottom in (2.2) can also be expressed as a convergent Taylor series expansion,

$$|D| L_\mu[\beta b]\psi = \sum_{n=0}^{+\infty} |D| L_\mu^n[\beta b]\psi, \quad (2.6)$$

where each mapping  $\tilde{b} \mapsto L_\mu^n[\tilde{b}]\psi$  is homogeneous<sup>2</sup> of degree  $n$ . In [11], Craig *et al.* obtained a recursion formula for the  $L_\mu^n[\beta b]\psi$ . Guyenne and Nicholls remarked in [21] that this formula involves a smoothing operator, resulting in smooth contributions of the bottom in (2.6). More precisely, starting from the recursion formula of  $|D| L_\mu^n[\beta b]\psi$  (see [11, Eq. (A 8)-(A 9)]), one can see that the individual terms in the Taylor expansion (2.6) take the form

$$\forall n \geq 1, \quad |D| L_\mu^n[\beta b]\psi = \sqrt{\mu}\nabla \cdot \{B_\mu[\beta b]F_\mu^{n-1}[\beta b] \operatorname{sech}(\sqrt{\mu}|D|)\nabla\psi\}, \quad (2.7)$$

where the smoothing operator  $B_\mu[\beta b]$  is defined by

$$B_\mu[\beta b]\mathbf{v} = \operatorname{sech}(\sqrt{\mu}|D|)(\beta b\mathbf{v}). \quad (2.8)$$

<sup>2</sup>Denoting by  $d_{\tilde{b}}^n \mathcal{G}_\mu[0, 0](\tilde{b})\psi$  the  $n$ -th derivative of  $b \mapsto \mathcal{G}_\mu[0, b]\psi$  at  $b = 0$  in the direction  $\tilde{b}$ , the  $n$ -th term in the Taylor expansion (2.6) is related to this shape derivative by  $d_{\tilde{b}}^n \mathcal{G}_\mu[0, 0](\tilde{b})\psi = n!\sqrt{\mu}|D| L_\mu^n[\tilde{b}]\psi$ .

Looking at (2.7), we see that any singular term introduced by the topography when computing  $F_\mu^n[\beta b]$  (which is made explicit below) is then regularized using  $B_\mu[\beta b]$  so much so that the topographic contribution given by (2.7) is infinitely smooth. Using (2.7), the description of the expansion (2.6) is computed from the following recursion formula for  $F_\mu^n$ :

(i) for the even terms

$$\begin{aligned} F_\mu^{2n}[\beta b]\mathbf{v} &= \frac{\mu^n}{(2n+1)!} (\beta b)^{2n} |D|^{2n} \mathbf{v} \\ &\quad - \sum_{p=1}^n \frac{\sqrt{\mu}^{2p}}{(2p)!} (\beta b)^{2p-1} |D|^{2(p-1)} D \left( D \cdot \{ \beta b F_\mu^{2(n-p)}[\beta b]\mathbf{v} \} \right) \\ &\quad + \sum_{p=0}^{n-1} \frac{\sqrt{\mu}^{2p+1}}{(2p+1)!} (\beta b)^{2p} |D|^{2p} T_\mu[\beta b] \{ F_\mu^{2(n-p)-1}[\beta b]\mathbf{v} \}, \end{aligned} \quad (2.9)$$

(ii) for the odd terms

$$\begin{aligned} F_\mu^{2n+1}[\beta b]\mathbf{v} &= - \sum_{p=1}^n \frac{\sqrt{\mu}^{2p}}{(2p)!} (\beta b)^{2p-1} |D|^{2(p-1)} D \left( D \cdot \{ \beta b F_\mu^{2(n-p)+1}[\beta b]\mathbf{v} \} \right) \\ &\quad + \sum_{p=0}^n \frac{\sqrt{\mu}^{2p+1}}{(2p+1)!} (\beta b)^{2p} |D|^{2p} T_\mu[\beta b] \{ F_\mu^{2(n-p)}[\beta b]\mathbf{v} \}, \end{aligned} \quad (2.10)$$

where  $T_\mu[\beta b]$  is defined as

$$T_\mu[\beta b]\mathbf{v} = D \left( D \cdot \frac{\tanh(\sqrt{\mu}|D|)}{|D|} \{ \beta b \mathbf{v} \} \right). \quad (2.11)$$

**Remark 2.1.** It is important to note that, even if (2.7) eventually results in infinitely differentiable bottom contributions, it is by no means obvious how to define the operators  $bF_\mu^j[\beta b]$  for general  $b \in L^\infty(\mathbb{R}^d)$ . For instance, computing the product  $bF_\mu^1[\beta b]\mathbf{v} = \sqrt{\mu} b T_\mu[\beta b]\mathbf{v}$  requires to assign meaning to  $b|D|(\mathbf{b}\mathbf{v})$ . The latter is of course well defined as soon as  $b$  belongs to  $W^{1,\infty}(\mathbb{R}^d)$  but it seems much more technical to extend its definition to  $L^\infty(\mathbb{R}^d)$  (see also Remark 4.1). More generally, it is known that the Dirichlet-Neumann operator is analytic with respect to Lipschitz deformations of the bottom (see [26, Theorem A.11]). Consequently, it is also possible to extend the definition of the higher order terms to  $W^{1,\infty}$  parametrizations of the bottom. Such extensions could be due to non-obvious cancellations in (2.9)-(2.10). That being said, in the numerical simulations of section 4, we shall also consider more general topographies for which the present approach gives promising results<sup>3</sup>.

**2.1.3. Analyticity of the Dirichlet-Neumann operator in shallow water regime.** As mentioned above, without any assumption on the shallowness parameter  $\mu$ , both Taylor expansions (2.1) and (2.6) of the Dirichlet-Neumann operator can be written for small perturbations of the surface and the bottom, that is for  $\varepsilon$  and  $\beta$  small enough. In shallow water regime  $\mu \ll 1$ , one can roughly estimate from (2.2) and the recursion formulas (2.3) and (2.4) that  $\mathcal{G}_\mu^n[\varepsilon\zeta, \beta b]\psi$  is at least of  $O(\sqrt{\mu}^{n+1})$ . Consequently the series in the right hand side of (2.1) converges (at least formally) without any further condition on  $\varepsilon$ . Similarly, because  $F_\mu^n[\beta b]$  is of  $O(\sqrt{\mu}^n)$ , the right hand side of (2.6) also converges without any further condition on  $\beta$ . For these reasons, since we study the Dirichlet-Neumann in shallow water conditions, we

<sup>3</sup>The numerical experiment presented in the Appendix also indicates that, for a step bottom, the present approach is consistent with the approach developed by Nachbin for polygonal topographies [29].

still use expansion (2.1) and (2.6) to compute  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  whereas no assumption is made on  $\varepsilon$  and  $\beta$  (we only assume that they are of  $O(1)$ ).

**2.2. Shallow water expansion of the Dirichlet-Neumann operator.** We adopt a formal procedure to derive an expansion of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  with respect to  $\mu$ . The task consists in computing the relevant contributions of each term  $\mathcal{G}_\mu^n[\varepsilon\zeta, \beta b]\psi$  from the Taylor series expansion (2.1) with respect to  $\zeta$ . This is possible thanks to the fact that the order of the contribution in  $\mu$  of  $\mathcal{G}_\mu^n[\varepsilon\zeta, \beta b]\psi$  increases with  $n$ . Concerning the contribution of the topography note that, at the formal level, very little regularity is required on  $b$  to write the recursive formulation of  $|D| L_\mu[\beta b]$ . Indeed, as mentioned above, thanks to the smoothing operator  $B_\mu[\beta b]$ , each term  $|D| L_\mu^n[\beta b]$  computed through (2.7) is well defined and gives smooth functions even for non smooth bottoms (see also remark 2.1). For this reason, allowing non smooth topographies, we use these formulas in order to estimate the shallow water contribution of the bottom *i.e.* the asymptotic behavior of  $|D| L_\mu[\beta b]\psi$  as  $\mu \rightarrow 0$ .

To begin with, let us estimate the first contribution from  $\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi$ . From (2.2),  $\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi$  can be expanded as  $\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi = \sqrt{\mu} |D| L_\mu[\beta b]\psi + O(\mu)$ . Now, using the relation (2.7) and looking at the recursion formulas (2.9) and (2.10), one sees that  $|D| L_\mu[\beta b]\psi$  gives first contributions at  $O(\sqrt{\mu})$  so that  $\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi$  gives first contributions at  $O(\mu)$ . Starting from this and using (2.3) and (2.4), one readily proves by recursion that both  $\mathcal{G}_\mu^{2n}[\varepsilon\zeta, \beta b]\psi$  and  $\mathcal{G}_\mu^{2n+1}[\varepsilon\zeta, \beta b]\psi$  first contribute at  $O(\mu^{n+1})$ . Therefore, as we restrict our asymptotic analysis of the water waves equations to  $O(\mu)$ , we only need to compute the relevant contributions from the first two terms  $\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi$  and  $\mathcal{G}_\mu^1[\varepsilon\zeta, \beta b]\psi$ . Indeed, the operator  $\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  in (1.3) can be formally expanded as

$$\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = \frac{1}{\mu} (\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi + \mathcal{G}_\mu^1[\varepsilon\zeta, \beta b]\psi) + O(\mu), \quad (2.12)$$

so that one may approximate  $\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  up to  $O(\mu)$  by expanding  $\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi$  and  $\mathcal{G}_\mu^1[\varepsilon\zeta, \beta b]\psi$  up to  $O(\mu^2)$ .

In order to determine the contributions from the term involving  $\sqrt{\mu} |D| L_\mu[\beta b]\psi$ , we use the transformation (2.7) together with the recursion formulas (2.9) and (2.10). From these formulas,  $\sqrt{\mu} |D| L_\mu[\beta b]\psi$  can be expanded as

$$\sqrt{\mu} |D| L_\mu[\beta b]\psi = \mu \nabla \cdot B_\mu[\beta b] \{ (F_\mu^0[\beta b] + F_\mu^1[\beta b]) \operatorname{sech}(\sqrt{\mu} |D|) \nabla \psi \} + O(\mu^2).$$

Setting  $b_\mu[\beta b] = B_\mu[\beta b] \circ (F_\mu^0[\beta b] + F_\mu^1[\beta b])$  that is  $b_\mu[\beta b] \mathbf{v} = B_\mu[\beta b] \{ \mathbf{v} + \sqrt{\mu} T_\mu[\beta b] \mathbf{v} \}$ , the topographical term writes

$$\sqrt{\mu} |D| L_\mu[\beta b]\psi = \mu \nabla \cdot b_\mu[\beta b] \{ \operatorname{sech}(\sqrt{\mu} |D|) \nabla \psi \} + O(\mu^2).$$

Considering the resulting approximation of  $\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi$  in (2.2) and performing a first order Taylor expansion of  $\tanh(\sqrt{\mu} |D|)\psi$  and  $\operatorname{sech}(\sqrt{\mu} |D|)\nabla\psi$  then lead to

$$\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi = -\mu \nabla \cdot ((1 - b_\mu[\beta b]) \nabla \psi) + O(\mu^2). \quad (2.13)$$

The expansion of  $\mathcal{G}_\mu^1[\varepsilon\zeta, \beta b]\psi$  in (2.5) follows from the fact that  $\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi$  is of size  $O(\mu)$ :

$$\mathcal{G}_\mu^1[\varepsilon\zeta, \beta b]\psi = -\mu \nabla \cdot (\varepsilon \zeta \nabla \psi) + O(\mu^2). \quad (2.14)$$

Gathering the last two approximations in (2.12), we finally deduce that

$$\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\nabla \cdot ((1 + \varepsilon \zeta - b_\mu[\beta b]) \nabla \psi) + O(\mu). \quad (2.15)$$

**Remark 2.2.** In (2.12), the residual is actually of order  $O(\varepsilon^2 \mu)$  so that if we consider moderate amplitude surface waves *i.e.*  $\varepsilon = O(\sqrt{\mu})$ , the resulting approximation in (2.12) is precise up to order  $O(\mu^2)$ . In that case, one may perform an asymptotic



analysis up to  $O(\mu^2)$  by expanding  $\mathcal{G}_\mu^0[\varepsilon\zeta, \beta b]\psi$  and  $\mathcal{G}_\mu^1[\varepsilon\zeta, \beta b]\psi$  up to  $O(\mu^3)$ . This can be achieved by first approximating the topographical term in (2.2) as

$$\begin{aligned} \sqrt{\mu} |D| L_\mu[\beta b]\psi = \\ \mu \nabla \cdot B_\mu[\beta b] \{ (F_\mu^0[\beta b] + F_\mu^1[\beta b] + F_\mu^2[\beta b] + F_\mu^3[\beta b]) \operatorname{sech}(\sqrt{\mu} |D|) \nabla \psi \} + O(\mu^2). \end{aligned}$$

Then, using the recursion formula (2.9)-(2.10) and following the same steps that led to (2.15), one finds that

$$\begin{aligned} \frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = & -\nabla \cdot ((1 + \varepsilon\zeta - \tilde{b}_\mu[\beta b]) \nabla \psi) - \frac{\mu}{3} \nabla \cdot \Delta \nabla \psi + \frac{\mu}{2} \nabla \cdot (b_\mu[\beta b] \Delta \nabla \psi) \\ & + \mu \beta^2 \nabla \cdot b_\mu[\beta b] \left\{ -\frac{b^2}{6} \Delta \nabla \psi + \frac{b}{2} \nabla (\nabla \cdot (b \nabla \psi)) \right\} \\ & + \mu^{3/2} \beta^2 \nabla \cdot B_\mu[\beta b] \left\{ -\frac{b^2}{6} \Delta T_\mu[\beta b] \nabla \psi + \frac{b}{2} \nabla (\nabla \cdot (b T_\mu[\beta b] \nabla \psi)) \right\} \\ & - \mu \varepsilon \nabla \cdot (1 - B_\mu[\beta b]) \{ \nabla (\zeta \nabla \cdot (1 - B_\mu[\beta b]) \nabla \psi) \} + O(\mu^2, \mu \varepsilon^2), \end{aligned} \quad (2.16)$$

where  $\tilde{b}_\mu[\beta b]$  is defined as

$$\tilde{b}_\mu[\beta b] \mathbf{v} = B_\mu[\beta b] \{ (1 + \sqrt{\mu} T_\mu[\beta b] + \mu T_\mu[\beta b]^2 + \mu^{3/2} T_\mu[\beta b]^3) \mathbf{v} \}. \quad (2.17)$$

When no assumption is made on  $\varepsilon$ , one also needs to compute the relevant contributions from  $\mathcal{G}_\mu^2[\varepsilon\zeta, \beta b]\psi$  and  $\mathcal{G}_\mu^3[\varepsilon\zeta, \beta b]\psi$  to perform an asymptotic analysis up to  $O(\mu^2)$ .

### 3. DERIVATION OF SHALLOW WATER MODELS

This section is devoted to the study of shallow water waves without any regularity assumption on the bottom parametrization. Using the shallow water expansion of the Dirichlet-Neumann operator computed in section 2.2, we derive asymptotic models that approximate, in this particular regime, the solutions of the water waves equations

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \varepsilon \mu \frac{(\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \varepsilon \nabla \zeta \cdot \nabla \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla \zeta|^2)} = 0. \end{cases} \quad (3.1)$$

In Section 3.1, a nonlinear shallow water model is obtained at first order (with respect to  $\mu$ ). Under additional assumptions on  $\varepsilon$ , asymptotic models with precision  $O(\mu^2)$  are then derived in Section 3.2 and Section 3.3.

These approximate models are written in terms of the surface elevation  $\zeta$  and the horizontal velocity at the surface  $\mathbf{v}_s = (\nabla \Phi)|_{z=\varepsilon\zeta}$ , where we recall that  $\Phi$  is the velocity potential given by (1.1). The link between  $\nabla \psi$  and  $\mathbf{v}_s$  results from the application of the chain rule which yields  $\mathbf{v}_s = \nabla \psi - \varepsilon (\partial_z \Phi)|_{z=\varepsilon\zeta} \nabla \zeta$ . Now, by definition of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]$ ,

$$(\partial_z \Phi)|_{z=\varepsilon\zeta} = \mu \frac{\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \varepsilon \nabla \psi \cdot \nabla \zeta}{1 + \mu \varepsilon^2 |\nabla \zeta|^2},$$

so that the horizontal velocity at the surface can be expressed as

$$\mathbf{v}_s = \nabla \psi - \mu \varepsilon \frac{\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \varepsilon \nabla \psi \cdot \nabla \zeta}{1 + \mu \varepsilon^2 |\nabla \zeta|^2} \nabla \zeta. \quad (3.2)$$

To achieve the formal derivation of an approximate model with precision  $O(\mu^k)$  ( $k = 1$  or  $2$ ), the strategy we adopt in the next two sections is to take the following steps:

- (1) From (3.2), find an asymptotic expansion of order  $O(\mu^k)$  of  $\nabla\psi$  in terms of the velocity  $\mathbf{v}_s$ ;
- (2) Plug this expansion into (2.15) (or (2.16)) to get an asymptotic expansion of the Dirichlet-Neumann operator in terms of  $\mathbf{v}_s$ ;
- (3) In the first equation of (3.1), replace  $-\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$  by the approximation of step (2) and neglect the terms of order  $O(\mu^k)$ ;
- (4) Take the gradient of the second equation of (3.1), insert the expansions from steps (1) and (2) and neglect the  $O(\mu^k)$  terms.

**3.1. The nonlinear shallow water equations for non smooth bottoms.** Following the steps 1–4 above, we derive a nonlocal shallow water approximation of (3.1) at order  $O(\mu)$ . To this end, let us first remark that, at first order,  $\mathbf{v}_s = \nabla\psi + O(\mu)$ . Plugging this last expansion in (2.15) we get

$$\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\nabla \cdot ((1 + \varepsilon\zeta - b_\mu[\beta b])\mathbf{v}_s) + O(\mu), \quad (3.3)$$

where we recall that the smoothing operator  $b_\mu[\beta b]$  is defined as

$$b_\mu[\beta b]\mathbf{v} = B_\mu[\beta b]\{(1 + \sqrt{\mu}T_\mu[\beta b])\mathbf{v}\}, \quad (3.4)$$

with

$$B_\mu[\beta b]\mathbf{v} = \operatorname{sech}(\sqrt{\mu}|D|)(\beta b\mathbf{v}) \quad \text{and} \quad T_\mu[\beta b]\mathbf{v} = D\left(D \cdot \frac{\tanh(\sqrt{\mu}|D|)}{|D|}\{\beta b\mathbf{v}\}\right). \quad (3.5)$$

Therefore, substituting expansion (3.3) into the first equation of (3.1), then applying  $\nabla$  to the second equation and using both  $\mathbf{v}_s = \nabla\psi + O(\mu)$  and  $\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = O(1)$ , we obtain the following nonlocal approximate equations of motion up to terms of order  $O(\mu)$

$$\begin{cases} \partial_t\zeta + \nabla \cdot ((1 + \varepsilon\zeta - b_\mu[\beta b])\mathbf{v}_s) = 0, \\ \partial_t\mathbf{v}_s + \nabla\zeta + \varepsilon(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = 0. \end{cases} \quad (3.6)$$

**Remark 3.1.** The classical shallow water approximation of (3.1) can be written

$$\begin{cases} \partial_t\zeta + \nabla \cdot ((1 + \varepsilon\zeta - \beta b)\mathbf{v}_s) = 0, \\ \partial_t\mathbf{v}_s + \nabla\zeta + \varepsilon(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = 0. \end{cases} \quad (3.7)$$

Hence in case the bottom parametrization is not regular, the alternative shallow water model (3.6) differs from the classical approximation by the presence of a regularized discharge, namely  $q_\mu = (1 + \varepsilon\zeta - b_\mu[\beta b])\mathbf{v}_s$ , instead of the classical discharge  $q = (1 + \varepsilon\zeta - \beta b)\mathbf{v}_s$ . An illustration of this regularizing effect is given in Figure 2. It is also worth mentioning that the water depth variable  $h = 1 + \varepsilon\zeta + \beta b$  has no regularized analogous in the present alternative shallow water model. Indeed, one may feel inclined to define a regularized water depth as  $h_\mu = 1 + \varepsilon\zeta - b_\mu[\beta b]$ . However, this last expression does not define a function but an operator (precisely because  $b_\mu[\beta b]$  is an operator). As a particular consequence, unlike the Saint-Venant equations which can be formulated in  $(h, q)$  variables instead of  $(\zeta, v)$ , the alternative shallow water model can neither be formulated in terms of the water depth  $h$  (which may be singular for non smooth bottoms) nor in terms of the quantity  $h_\mu$  (which is not a function).

**Remark 3.2** (Smooth bottoms). In case the bottom parametrization is regular, a Taylor expansion of  $T_\mu[\beta b]$  and  $B_\mu[\beta b]$  in (3.4) ensures that  $b_\mu[\beta b]\mathbf{v}_s = \beta b\mathbf{v}_s + O(\mu)$ .

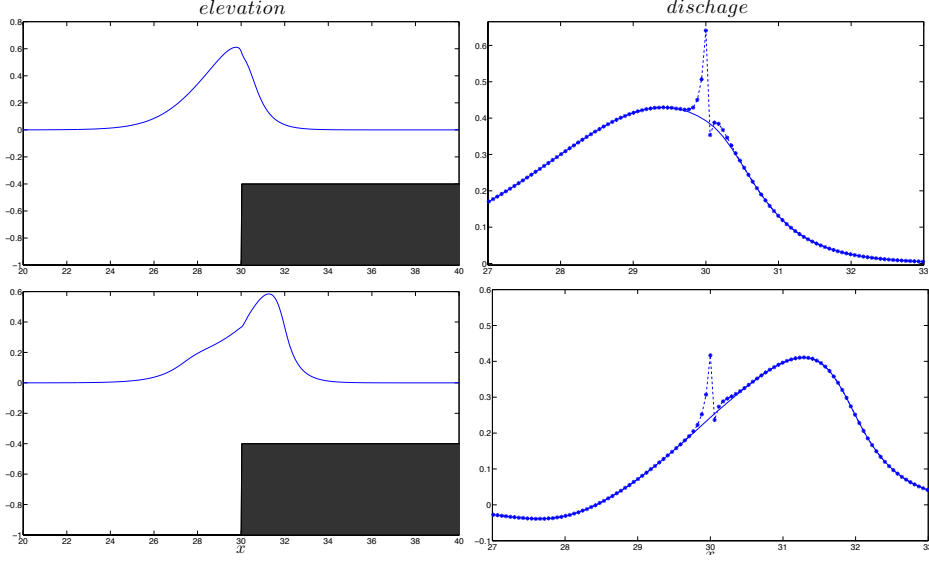


FIGURE 2. Wave passing over a step. Left: elevation. Right: comparison of the regularized discharge (—) with the classical discharge (---\*--).

Using this last approximation in (3.6), one recovers the classical Saint-Venant system (3.7) from the alternative equations (3.6).

**3.2. Medium amplitude models ( $\varepsilon = O(\sqrt{\mu})$ ) for non smooth bottoms.** In this section, besides the shallow water hypothesis, we assume that the amplitude parameter  $\varepsilon$  is of size  $O(\sqrt{\mu})$ . In case the bottom is smooth, this regime leads to the medium amplitude Green-Naghdi or Serre equations (see *e.g.* [26] for the derivation of these equations).

**3.2.1. Derivation of an approximate model with precision  $O(\mu^2)$ .** Under the previous assumption on  $\varepsilon$ , let us follow the steps (1)-(4) given above to derive an approximate model with precision  $O(\mu^2)$ . Since

$$\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\nabla \cdot (1 - B_\mu[\beta b])\mathbf{v}_s + O(\sqrt{\mu}), \quad (3.8)$$

we can see from relation (3.2) that

$$\nabla\psi = \mathbf{v}_s - \varepsilon\mu\nabla \cdot (1 - B_\mu[\beta b])\mathbf{v}_s\nabla\zeta + O(\mu^2). \quad (3.9)$$

Plugging this last approximation in (2.16), we obtain

$$\begin{aligned} \frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = & -\nabla \cdot ((1 + \varepsilon\zeta - \tilde{b}_\mu[\beta b])\mathbf{v}_s) - \frac{\mu}{3} \Delta\nabla \cdot \mathbf{v}_s + \frac{\mu}{2} S_\mu[\beta b]\mathbf{v}_s \\ & - \mu\varepsilon\nabla \cdot (1 - B_\mu[\beta b])\{\zeta\nabla(\nabla \cdot (1 - B_\mu[\beta b])\mathbf{v}_s)\} + O(\mu^2), \end{aligned}$$

where we recall that  $B_\mu[\beta b]$  and  $\tilde{b}_\mu[\beta b]$  are respectively defined in (2.8) and (2.17), while the dispersive topographical term  $S_\mu[\beta b]$  is defined as

$$\begin{aligned} S_\mu[\beta b]\mathbf{v} = & \nabla \cdot (b_\mu[\beta b]\Delta\mathbf{v}) + \beta^2\nabla \cdot b_\mu[\beta b]\left\{-\frac{b^2}{3}\Delta\mathbf{v} + b\nabla(\nabla \cdot (b\mathbf{v}))\right\} \\ & + \sqrt{\mu}\beta^2\nabla \cdot B_\mu[\beta b]\left\{-\frac{b^2}{3}\Delta T_\mu[\beta b]\mathbf{v} + b\nabla(\nabla \cdot (bT_\mu[\beta b]\mathbf{v}))\right\}. \quad (3.10) \end{aligned}$$

Inserting this last expansion in the first equation of (3.1) yields an evolution equation for the free surface up to  $O(\mu^2)$  terms:

$$\begin{aligned} \partial_t \zeta + \nabla \cdot ((1 + \varepsilon \zeta - \tilde{b}_\mu[\beta b])\mathbf{v}_s) + \frac{\mu}{3} \Delta \nabla \cdot \mathbf{v}_s - \frac{\mu}{2} S_\mu[\beta b] \mathbf{v}_s \\ + \mu \varepsilon \nabla \cdot (1 - B_\mu[\beta b]) \{ \zeta \nabla (\nabla \cdot (1 - B_\mu[\beta b])\mathbf{v}_s) \} = 0. \end{aligned} \quad (3.11)$$

Concerning the evolution of the velocity unknown we follow step (4) and take the gradient of the second equation (3.1). On using (3.8) and (3.9), the result is

$$\begin{aligned} \partial_t ((1 - \mu \varepsilon A_\mu[\zeta, \beta b])\mathbf{v}_s) + \nabla \zeta + \varepsilon (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s \\ - \frac{\mu \varepsilon}{2} \nabla (\nabla \cdot (1 - B_\mu[\beta b])\mathbf{v}_s)^2 = O(\mu^2), \end{aligned} \quad (3.12)$$

where  $A_\mu[\zeta, \beta b]$  is defined as  $A_\mu[\zeta, \beta b] \mathbf{v} = \nabla \zeta \nabla \cdot (1 - B_\mu[\beta b]) \mathbf{v}$ . Now since, from (3.11),  $\partial_t \zeta$  may be approximated as  $\partial_t \zeta = -\nabla \cdot (1 - B_\mu[\beta b])\mathbf{v}_s + O(\sqrt{\mu})$ , we get

$$\begin{aligned} \partial_t ((1 - \mu \varepsilon A_\mu[\zeta, \beta b])\mathbf{v}_s) = (1 - \mu \varepsilon A_\mu[\zeta, \beta b]) \partial_t \mathbf{v}_s \\ + \frac{\mu \varepsilon}{2} \nabla (\nabla \cdot (1 - B_\mu[\beta b])\mathbf{v}_s)^2 + O(\mu^2). \end{aligned} \quad (3.13)$$

Gathering (3.13) and (3.12) leads to

$$(1 - \mu \varepsilon A_\mu[\zeta, \beta b]) \partial_t \mathbf{v}_s + \nabla \zeta + \varepsilon (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s = O(\mu^2),$$

from which we deduce the following approximate evolution equation for the velocity up to  $O(\mu^2)$  terms

$$\partial_t \mathbf{v}_s + \nabla \zeta + \varepsilon (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s + \mu \varepsilon \nabla \zeta \nabla \cdot (1 - B_\mu[\beta b]) \nabla \zeta = 0. \quad (3.14)$$

**3.2.2. Improving the frequency dispersion of the model.** Equations (3.11) and (3.14) only differ by nonlinear terms from the Boussinesq model formulated in terms of the velocity at the surface (see *e.g.* [37, Eq. (16) and (17)]). Now it is known that the latter is linearly ill-posed (see *e.g.* [5]), and so are the former. Indeed, the existence of non-trivial solutions  $(\zeta, \psi)$  of the form  $(\zeta_0, \psi_0) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$  to the linearization of (3.11)-(3.14) around  $\zeta = 0, \nabla \psi = 0$  and for flat bottom  $b = 0$  requires the dispersion relation

$$\omega_\alpha(\mathbf{k})^2 = |\mathbf{k}|^2 - \frac{\mu}{3} |\mathbf{k}|^4,$$

and this relation does not lead to real-valued frequencies  $\omega_\alpha(\mathbf{k})$  for high wave numbers  $|\mathbf{k}|$ . To improve the linear dispersion frequencies of the model one can use the BBM "trick" [2]. The idea is to note that since  $\partial_t \zeta + \nabla \cdot ((1 + \varepsilon \zeta - b_\mu[\beta b])\mathbf{v}_s)$  is of size  $O(\mu)$ , we can introduce a real parameter  $\alpha$  by adding the quantity

$$-\mu \frac{\alpha}{3} (\Delta \partial_t \zeta + \Delta \nabla \cdot \mathbf{v}_s + \varepsilon \Delta \nabla \cdot (\zeta \mathbf{v}_s) - \Delta \nabla \cdot b_\mu[\beta b] \mathbf{v}_s) = O(\mu^2)$$

to (3.11). The resulting approximate equation, together with (3.14), yields the following asymptotic model with precision  $O(\mu^2)$  for shallow water medium amplitude waves

$$\begin{cases} (1 - \frac{\mu \alpha}{3} \Delta) \partial_t \zeta + \nabla \cdot ((1 + \varepsilon \zeta - \tilde{b}_\mu[\beta b])\mathbf{v}_s) + \frac{\mu}{3} (1 - \alpha) \Delta \nabla \cdot \mathbf{v}_s - \frac{\mu}{2} S_\mu[\beta b] \mathbf{v}_s \\ \quad + \mu \varepsilon \nabla \cdot (1 - B_\mu[\beta b]) \{ \zeta \nabla (\nabla \cdot (1 - B_\mu[\beta b])\mathbf{v}_s) \} \\ \quad + \mu \frac{\alpha}{3} \Delta \nabla \cdot b_\mu[\beta b] \mathbf{v}_s - \mu \varepsilon \frac{\alpha}{3} \Delta \nabla \cdot (\zeta \mathbf{v}_s) = 0, \\ \partial_t \mathbf{v}_s + \nabla \zeta + \varepsilon (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s + \mu \varepsilon \nabla \zeta \nabla \cdot (1 - B_\mu[\beta b]) \nabla \zeta = 0. \end{cases} \quad (3.15)$$

The dispersion relation associated to (3.15) now reads

$$\omega_\alpha(\mathbf{k})^2 = |\mathbf{k}|^2 \frac{1 + \frac{\alpha-1}{3} \mu |\mathbf{k}|^2}{1 + \frac{\alpha}{3} \mu |\mathbf{k}|^2}.$$

Consequently, the interest of the parameter  $\alpha$  is that the corresponding system (3.15) is linearly well-posed as soon as  $\alpha \geq 1$ . Moreover this parameter can be adjusted (see for instance [9, 8]) to improve the dispersive characteristics embedded in the medium amplitude model (3.15). To this end, we set  $\alpha = 1.159$  in all the numerical tests of section 4.2.2. Following [9], this value has been chosen so that the phase and group velocities associated to (3.15) stay close to the reference velocities coming from the water waves equations (3.1).

**Remark 3.3.** When the bottom is smooth, further improvements of the dispersive properties can be achieved by replacing the velocity variable  $\mathbf{v}_s$  at the surface with a different velocity variable linked to the velocity at an arbitrary elevation. The velocity at a certain depth is used in [34, 39] as dependent variable while a slightly different choice is made in [8] with the introduction of a new dependent variable (which is also related to the velocity at an arbitrary elevation as explained in [26]). In the present case, since we are dealing with rough bottoms, we decide to work with variables located at the surface where the irregularities of the bottom have the least effect.

**3.3. A Boussinesq system for non smooth bottoms.** The additional small amplitude assumption  $\varepsilon = O(\mu)$  is the traditional assumption that leads to the usual Boussinesq models. In this particular regime, neglecting in (3.15) the terms of order  $O(\varepsilon\mu) = O(\mu^2)$  yields the following Boussinesq-type approximation of the water waves equations (3.1) with precision  $O(\mu^2)$

$$\begin{cases} (1 - \frac{\mu\alpha}{3}\Delta)\partial_t\zeta + \nabla \cdot ((1 + \varepsilon\zeta - \tilde{b}_\mu[\beta b])\mathbf{v}_s) \\ \quad + \frac{\mu}{3}(1 - \alpha)\Delta\nabla \cdot \mathbf{v}_s - \frac{\mu}{2}S_\mu[\beta b]\mathbf{v}_s + \mu\frac{\alpha}{3}\Delta\nabla \cdot b_\mu[\beta b]\mathbf{v}_s = 0, \\ \partial_t\mathbf{v}_s + \nabla\zeta + \varepsilon(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = 0, \end{cases} \quad (3.16)$$

where  $S_\mu[\beta b]$  is defined as in (3.10).

**Remark 3.4** (Smooth bottoms). Using the velocity at the surface as dependent variable, the usual Boussinesq model derived by Peregrine for smooth bottoms (see [37]) can be written

$$\begin{cases} (1 - \frac{\mu\alpha}{3}\Delta)\partial_t\zeta + \nabla \cdot (h_b\mathbf{v}_s) + \frac{\mu}{3}\nabla \cdot (h_b^3\nabla(\nabla \cdot \mathbf{v}_s)) - \mu\frac{\alpha}{3}\Delta\nabla \cdot \mathbf{v}_s \\ \quad - \frac{\mu\beta}{2}\nabla \cdot (h_b^2\nabla(\nabla b \cdot \mathbf{v}_s)) - \frac{\mu\beta}{2}\nabla \cdot (h_b^2\nabla \cdot \mathbf{v}_s\nabla b) + \mu\beta\frac{\alpha}{3}\Delta\nabla \cdot (b\mathbf{v}_s) = 0, \\ \partial_t\mathbf{v}_s + \nabla\zeta + \varepsilon(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = 0, \end{cases} \quad (3.17)$$

where  $h_b = 1 - \beta b$  stands for the (nondimensional) still water depth. Assuming that the bottom is smooth, a Taylor expansion of both operators  $T_\mu[\beta b]$  and  $B_\mu[\beta b]$  in (2.17) ensures that

$$\tilde{b}_\mu[\beta b]\mathbf{v}_s = \beta b\mathbf{v}_s + \frac{\mu\beta}{2}\Delta(b\mathbf{v}_s) - \mu\beta^2 b\nabla(\nabla \cdot (b\mathbf{v}_s)) + O(\mu^2).$$

Using this last expansion together with  $b_\mu[\beta b] = \beta b\mathbf{v}_s + O(\mu)$  in the first equation of (3.16) while keeping in mind that, as the bottom is smooth,  $T_\mu[\beta b]$  gives first contributions at  $O(\sqrt{\mu})$ , one can check that this equation coincides with the first equation of (3.17) up to terms of order  $O(\mu^2)$ .

#### 4. NUMERICAL COMPUTATIONS

In this section we describe spatial discretization and time integration of the nonlocal shallow water models derived in the previous section. Then we present

some numerical simulations in order to illustrate the behavior of these asymptotic models.

**4.1. Numerical scheme.** The numerical simulations are made in the one dimensional case  $d = 1$ . In that case, the nonlocal operators  $B_\mu[\beta b]$  and  $T_\mu[\beta b]$  that occur in the definitions of  $b_\mu[\beta b]$  and  $\tilde{b}_\mu[\beta b]$  are given by

$$B_\mu[\beta b]v = \operatorname{sech}(\sqrt{\mu}|D|)(\beta bv) \quad \text{and} \quad T_\mu[\beta b]v = |D|\tanh(\sqrt{\mu}|D|)(\beta bv). \quad (4.1)$$

**4.1.1. Numerical scheme for the nonlinear shallow water equations.** In the one dimensional case, equations (3.6) reduce to

$$\begin{cases} \partial_t \zeta + \partial_x v_s + \varepsilon \partial_x (v_s \zeta) = \partial_x (b_\mu[\beta b] v_s), \\ \partial_t v_s + \partial_x \zeta + \varepsilon v_s \partial_x v_s = 0, \end{cases} \quad (4.2)$$

*Time integration.* Following the previous work of Besse and Bruneau, we use a Crank-Nicolson like scheme where the nonlinear part is avoided by doing a relaxation that is by writing the linear and the nonlinear parts to different times (see [4, 3] for a description of the method and *e.g.* [7, 19, 18] for applications to asymptotic models related to the water waves equations). More precisely, given a time step  $\Delta t$ , we consider functions  $(\zeta^n, v^n)$  which approximate  $\zeta(t^n, \cdot)$  and  $v_s(t^n, \cdot)$  at time  $t^n = n\Delta t$  and  $v^{n+\frac{1}{2}}$  which approximate  $v_s(t^{n+\frac{1}{2}}, \cdot)$  at  $t^{n+\frac{1}{2}} = (n + \frac{1}{2})\Delta t$ . Then the semi-discretized in time scheme for (3.6) reads, for all  $n \geq 1$ ,  $v^{n+\frac{1}{2}} = 2v^n - v^{n-\frac{1}{2}}$  and

$$\begin{cases} \frac{\zeta^{n+1} - \zeta^n}{\Delta t} + \varepsilon \partial_x \left( v^{n+\frac{1}{2}} \frac{\zeta^{n+1} + \zeta^n}{2} \right) = \partial_x (b_\mu[\beta b] v^{n+\frac{1}{2}}) - \partial_x v^{n+\frac{1}{2}}, \\ \frac{v^{n+1} - v^n}{\Delta t} + \partial_x \left( \frac{\zeta^{n+1} + \zeta^n}{2} \right) + \varepsilon v^{n+\frac{1}{2}} \partial_x \left( \frac{v^{n+1} + v^n}{2} \right) = 0. \end{cases}$$

*Spatial discretization.* In all the test cases, the (one-dimensional) spatial domain is  $(0, L)$ . We assume periodic boundary conditions so that the nonlocal operator  $b_\mu[\beta b]$  can be approximated using the discrete Fourier transform. This amounts to evaluating all the differential operators in (4.1) in Fourier space while performing nonlinear products in physical space. More precisely, if  $\Delta x$  is a spatial step (chosen such that  $N = \frac{L}{\Delta x}$  is an integer), the spatial domain is discretized by  $N$  equally spaced points  $x_j = j\Delta x$ ,  $j = 1, \dots, N$ , and the corresponding discrete frequencies are given by  $\mathbf{k} = \frac{2\pi}{L} \{-\frac{N}{2} + 1, \dots, \frac{N}{2}\}$ . Then, if we wish to evaluate the discrete analogue of  $B_\mu[\beta b]$  applied to a discrete function  $u = (u_j)_{1 \leq j \leq N}$ , we first multiply  $u$  by  $(b(x_j))_{1 \leq j \leq N}$ , then transform to the Fourier space (using fast Fourier transform), multiply by the diagonal operator  $\operatorname{sech}(\sqrt{\mu}\mathbf{k})$  and finally transform back to the physical space. Approximations of other such terms in (3.4) is achieved similarly which leads to a discrete approximation  $b_\mu^{\Delta x} : \mathbb{R}^N \rightarrow \mathbb{R}^N$  of the nonlocal operator  $b_\mu[\beta b]$ .

Thus considering discrete unknowns  $\zeta^n = (\zeta_j^n)_{1 \leq j \leq N}$  and  $v^n = (v_j^n)_{1 \leq j \leq N}$  at time  $t^n$  and  $v^{n+\frac{1}{2}} = (v_j^{n+\frac{1}{2}})_{1 \leq j \leq N}$  at time  $t^{n+\frac{1}{2}}$ , the fully discrete scheme reads, for all  $n \geq 1$ ,  $v^{n+\frac{1}{2}} = 2v^n - v^{n-\frac{1}{2}}$  and

$$\begin{cases} \frac{\zeta^{n+1} - \zeta^n}{\Delta t} + \varepsilon D_1 \left( v^{n+\frac{1}{2}} \frac{\zeta^{n+1} + \zeta^n}{2} \right) = D_1 (b_\mu^{\Delta x} v^{n+\frac{1}{2}}) - D_1 v^{n+\frac{1}{2}}, \\ \frac{v^{n+1} - v^n}{\Delta t} + D_1 \left( \frac{\zeta^{n+1} + \zeta^n}{2} \right) + \varepsilon v^{n+\frac{1}{2}} D_1 \left( \frac{v^{n+1} + v^n}{2} \right) = 0, \end{cases} \quad (4.3)$$

where  $D_1$  stands for the classical centered discretization of  $\partial_x$  (with periodic boundary conditions). When comparing both asymptotic models (3.6) and (3.7), we use for

the classical shallow water model (3.7) a finite difference scheme similar in principle to that described above for the alternative model.

4.1.2. *Numerical scheme for the medium and small amplitude models.* The one dimensional version of the medium amplitude model (3.15) reads

$$\left\{ \begin{array}{l} (1 - \frac{\mu\alpha}{3}\partial_x^2)\partial_t\zeta + \varepsilon(\partial_x - \frac{\mu\alpha}{3}\partial_x^3)(v_s\zeta) + \frac{\mu}{3}(1 - \alpha)\partial_x^3v_s \\ \quad = \partial_x\tilde{b}_\mu[\beta b]v_s - \partial_xv_s + \frac{\mu}{2}S_\mu[\beta b]v_s - \mu\frac{\alpha}{3}\partial_x^3b_\mu[\beta b]v_s \\ \quad \quad - \mu\varepsilon\partial_x(1 - B_\mu[\beta b])\{\zeta\partial_x^2(1 - B_\mu[\beta b])v_s\}, \\ \partial_tv_s + \partial_x\zeta + \varepsilon v_s\partial_xv_s + \mu\varepsilon\partial_x((1 - B_\mu[\beta b])\partial_x\zeta)\partial_x\zeta = 0, \end{array} \right. \quad (4.4)$$

and the dispersive topographical contribution  $S_\mu[\beta b]$  is given by

$$\begin{aligned} S_\mu[\beta b]v &= \partial_x b_\mu[\beta b]\partial_x^2v + \beta^2\partial_x b_\mu[\beta b]\left\{-\frac{b^2}{3}\partial_x^2v + b\partial_x^2(bv)\right\} \\ &\quad + \sqrt{\mu}\beta^2\partial_x B_\mu[\beta b]\left\{-\frac{b^2}{3}\partial_x^2T_\mu[\beta b]v + b\partial_x^2(bT_\mu[\beta b]v)\right\}. \end{aligned}$$

Time integration is achieved using the aforementioned Crank-Nicolson like scheme. Concerning spatial discretization, we use discrete Fourier transform as described above to approximate each nonlocal operator that appears in (4.4). Thus the fully discrete scheme reads, for all  $n \geq 1$ ,

$$\left\{ \begin{array}{l} (I - \frac{\mu\alpha}{3}D_2)\frac{\zeta^{n+1} - \zeta^n}{\Delta t} + \varepsilon(D_1 - \frac{\mu\alpha}{3}D_3)\left(v^{n+\frac{1}{2}}\frac{\zeta^{n+1} + \zeta^n}{2}\right) \\ \quad \quad \quad + \frac{\mu}{3}(1 - \alpha)D_3\left(\frac{v^{n+1} + v^n}{2}\right) \\ \quad = D_1(\tilde{b}_\mu^{\Delta x}v^{n+\frac{1}{2}}) - D_1v^{n+\frac{1}{2}} + \frac{\mu}{2}S_\mu^{\Delta x}v^{n+\frac{1}{2}} - \mu\frac{\alpha}{3}D_3(b_\mu^{\Delta x}v^{n+\frac{1}{2}}) \\ \quad \quad \quad - \mu\varepsilon D_1(1 - B_\mu^{\Delta x})\{\zeta^{n+\frac{1}{2}}D_2(1 - B_\mu^{\Delta x})v^{n+\frac{1}{2}}\}, \\ \frac{v^{n+1} - v^n}{\Delta t} + D_1\left(\frac{\zeta^{n+1} + \zeta^n}{2}\right) + \varepsilon v^{n+\frac{1}{2}}D_1\left(\frac{v^{n+1} + v^n}{2}\right) \\ \quad \quad \quad = \mu\varepsilon D_1((B_\mu^{\Delta x} - 1)D_1\zeta^{n+\frac{1}{2}})D_1\left(\frac{\zeta^{n+1} + \zeta^n}{2}\right), \end{array} \right. \quad (4.5)$$

where  $D_1$ ,  $D_2$  and  $D_3$  stand for the classical centered discretizations of  $\partial_x$ ,  $\partial_x^2$  and  $\partial_x^3$  while  $B_\mu^{\Delta x}$ ,  $b_\mu^{\Delta x}$ ,  $\tilde{b}_\mu^{\Delta x}$  and  $S_\mu^{\Delta x}$  are respectively the discrete approximations of the nonlocal operators  $B_\mu[\beta b]$ ,  $b_\mu[\beta b]$ ,  $\tilde{b}_\mu[\beta b]$  and  $S_\mu[\beta b]$ . The one-dimensional version of the Boussinesq-like system (3.16) is similarly approximated.

4.2. **Numerical results.** Our goal in the computations presented in this paper is to compare the results produced by the nonlocal shallow water systems for rough bottom derived in Section 3 with the ones obtained from the classical shallow water models.

All simulations have been performed using  $N = 1024$  points and  $\Delta t = 10^{-2}$ . In all the test cases, the initial condition  $(\zeta_0, v_0)$  consists of a unidirectional wave propagating to the right on a domain of length  $L = 60$ :

$$\zeta_0(x) = v_0(x) = a \operatorname{sech}^2\left(\frac{x-20}{2}\right), \quad 0 < x < 60, \quad (4.6)$$

where  $a$  is an arbitrary parameter. The bathymetry can be parametrized as follows

$$b(x) = \frac{1}{2}\left(\tanh\left(2\left(\frac{x-30}{\delta}\right)\right) - \tanh(x-49)\right), \quad 0 < x < 60. \quad (4.7)$$

This parametrization is regular but it involves a slope of order  $\frac{1}{\delta}$  around  $x = 30$ . Then as  $\delta \ll 1$  this slope becomes steep and the corresponding bottom becomes rough (see Figure 3).

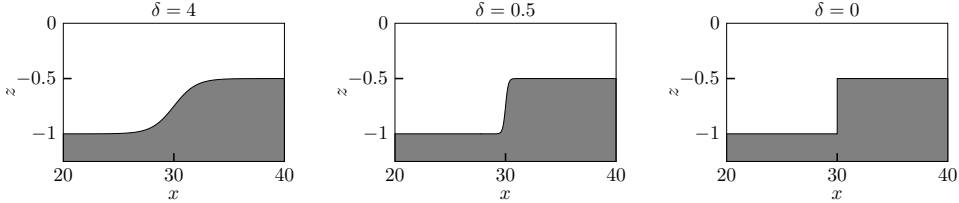


FIGURE 3. Bathymetries at  $x = 30$  for  $\delta = 4$  (left),  $\delta = 0.5$  (middle) and  $\delta = 0$  (right).

**4.2.1. Numerical results for the nonlinear shallow water equations.** We aim at evaluating the difference between both classical Saint-Venant system (3.7) and the nonlocal alternative (3.6) in terms of the shallowness parameter  $\mu$ . Since, for practical purposes, the classical Saint-Venant system is often used with non smooth topographies, the idea is to assess the price to pay when working with the classical model though the bottom is rough. In this section, the amplitude parameter is set to  $\varepsilon = 0.1$ .

*Smooth bottom.* In this test case, the topography parameter is  $\beta = 0.6$ . We set  $\delta = 4$  so that the corresponding bathymetry is smooth (see Figure 3). In this situation, we know from Remark 3.1 that the nonlocal model (3.6) reduces to the classical shallow water approximation (3.7) up to  $O(\mu)$  terms. In order to illustrate this precision, we computed the numerical solution given by (4.3) for several values of  $\mu$  and we then compared them with the numerical solution of the classical shallow water equations. For each computation and each discrete time  $t^n$ , the  $L^\infty$ -norm differences  $E_\zeta^n = \|\zeta_{\text{NL}}^n - \zeta_{\text{SV}}^n\|_\infty$  and  $E_v^n = \|v_{\text{NL}}^n - v_{\text{SV}}^n\|_\infty$  have been computed, where  $(\zeta_{\text{NL}}^n, v_{\text{NL}}^n)_n$  is the numerical solution of the nonlocal alternative system (4.3) and  $(\zeta_{\text{SV}}^n, v_{\text{SV}}^n)_n$  denotes the solution of the classical Saint-Venant scheme. Figure 4 depicts  $e_\zeta = \max_n E_\zeta^n$  as a function of  $\mu$  (the maximum is taken over a duration of 1500 time steps). The computed order of convergence is 0.95 which is consistent with the expected difference between both asymptotic models.

*Rough bottom.* In the following test cases, we focus on bottoms involving a steep slope. More precisely, the bathymetry is still given by (4.7) but the simulations have been performed for smaller values of  $\delta$ , namely  $\delta = 0.5$ ,  $\delta = 0.1$  and the limit value  $\delta = 0$ . In the latter case, the bottom parametrization has a step at  $x = 30$ :

$$b(x) = \begin{cases} 0 & \text{if } 0 \leq x < 30, \\ \frac{1}{2}(1 - \tanh(10(x - 49))) & \text{if } 30 < x \leq 60. \end{cases}$$

**Remark 4.1.** Using the alternative shallow water model with the above step parametrization raises questions as to the meaning of the term  $D \operatorname{sech}(\sqrt{\mu}|D|)bT_\mu[\beta b]v$  that appears in the definition of  $b_\mu[\beta b]v$ . Indeed, defining this term amounts to defining the term  $\mathcal{T} = D \operatorname{sech}(\sqrt{\mu}|D|)b|D|(bv)$ , which is far from obvious for general  $b \in L^\infty(\mathbb{R})$ . Let us consider the case where  $b$  is the sign function and assume that  $v$  is smooth. In this case, we can actually define  $\mathcal{T}$  as a smooth function. Indeed, note first that defining  $b|D|(bv)$  as a tempered distribution is tantamount to defining its Fourier transform  $\mathcal{F}(b|D|(bv))$ . Since, up to a multiplicative constant,  $\mathcal{F}b$  coincides with the principal value  $\text{p.v.}(\frac{1}{\xi})$ , the Fourier transform  $\mathcal{F}(b|D|(bv))$  can be formally



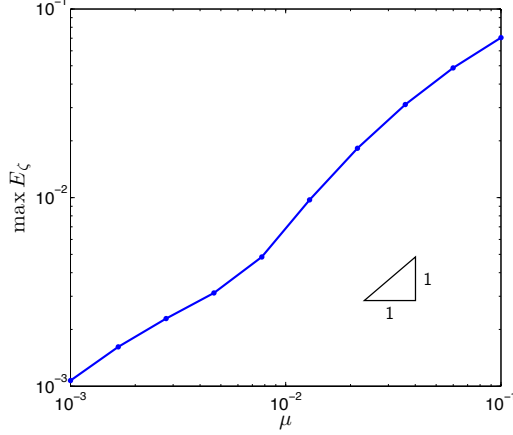


FIGURE 4. Smooth bottom: convergence between the surface elevation computed by the classical Saint-Venant model and the alternative one, as functions of the parameter  $\mu$ .

written as a convolution product of the form  $\text{p.v.}(\frac{1}{\xi}) * \mathcal{F}(|D|(bv)) = \mathcal{H}\mathcal{F}(|D|(bv))$ , where  $\mathcal{H}$  is the Hilbert transform. Now one can check that  $\mathcal{F}(|D|(bv))$  takes the form  $\mathcal{F}(|D|(bv)) = af_1 + f_2$  with  $a \in \mathbb{C}$ ,  $f_1 = v(0)\text{sign}$  and  $f_2 \in L^2(\mathbb{R})$  so that taking the Hilbert transform yields  $\mathcal{F}(b|D|(bv)) = \tilde{a}\tilde{f}_1 + \tilde{f}_2$  where  $\tilde{a} \in \mathbb{C}$ ,  $\tilde{f}_1 = v(0)\log|\cdot|$  and  $\tilde{f}_2 \in L^2(\mathbb{R})$ . Since  $\mathcal{FT}(\xi) = \text{sech}(\sqrt{\mu}|\xi|)\xi\mathcal{F}(b|D|(bv))$ , the last logarithmic singularity gives rise to a term of the form  $\text{sech}(\sqrt{\mu}|\xi|)\xi \log|\xi|$ , which is continuous and rapidly decreasing. This confirms that  $\mathcal{T}$  makes sense and is a smooth function. The same conclusion holds for piecewise continuous parametrizations of the bottom.

Figure 5 shows the comparison between the wave profiles and the velocities determined from both the classical and the nonlocal shallow water models for a flow over such a step. The shallowness parameter is set to be  $\mu = 0.01$  and the topography parameter is  $\beta = 0.6$ . As the wave passes over the step (located at  $x = 30$ ) the classical Saint-Venant model produces oscillations at the top of both main and reflected waves while the alternative model does not exhibit these oscillations. Moreover the velocity computed by the classical model has a jump discontinuity across the step. This discontinuity is "smoothed" by the nonlocal model. Note that the amplitude of the oscillations produced by the classical Saint-Venant equations decreases with decreasing the topography parameter  $\beta$ .

In order to estimate the error (in terms of the parameter  $\mu$ ) committed when using the classical Saint-Venant model, the quantity  $e_\zeta = \max_n E_\zeta^n$  has again been computed and the results are plotted in Figure 6. To study the influence of the topography parameter, this figure also presents the results obtained with  $\beta = 0.3$ . As expected the error committed by the classical model increases with  $\beta$ . The computed orders of convergence with respect to  $\mu$  for the surface elevation have been gathered in Table 1. For the case of small amplitude bottom ( $\beta = 0.3$ ), the convergence rate remains close to 1 except for the step topography ( $\delta = 0$ ). Now, for the case of large amplitude bottom ( $\beta = 0.6$ ), the order of convergence decreases with the steepness of the topography. In the limit case  $\delta = 0$ , the classical model becomes a  $O(\mu^{\frac{1}{2}})$  approximation (compared with  $O(\mu)$  for smooth topographies). In other words, with this particular kind of rough bottoms, if one decides to use the classical shallow water model, the price to pay is at most one-half order of precision.

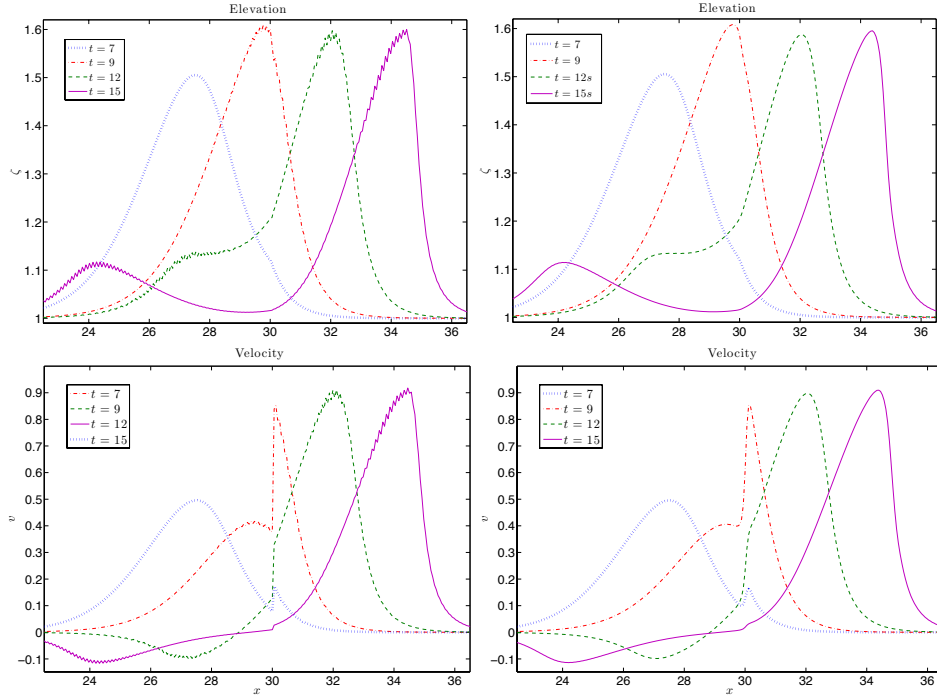


FIGURE 5. Elevation and velocity for a wave passing over a step: classical Saint-Venant model (left) and nonlocal alternative (right).

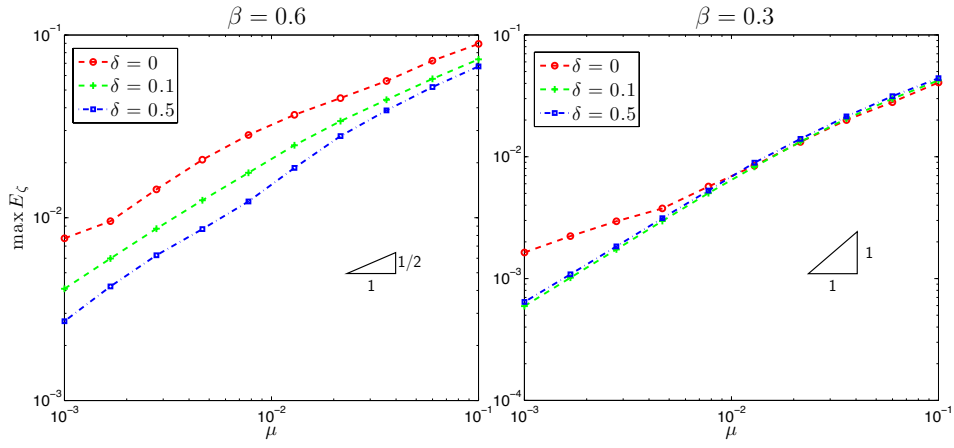


FIGURE 6. Rough bottom: convergence between the surface elevation computed by the classical Saint-Venant model and the alternative one, as functions of the parameter  $\mu$ .

$\delta$	Convergence rate	
	$\beta = 0.6$	$\beta = 0.3$
4	0.96	0.94
0.5	0.71	0.94
0.1	0.63	0.94
0	0.54	0.71

TABLE 1. Shallow water models: computed convergence rates with respect to  $\mu$ .

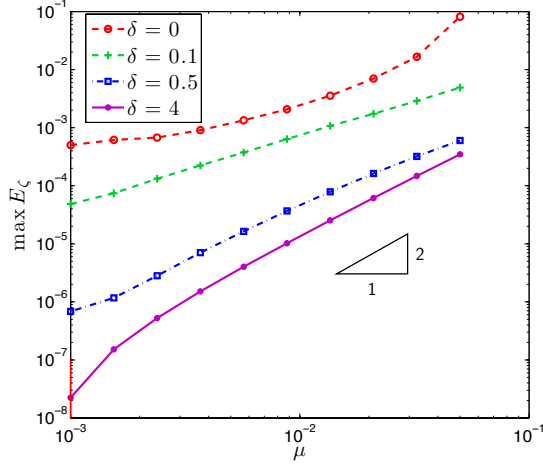


FIGURE 7. Convergence between the surface elevation computed by the classical Boussinesq model and the alternative one, as functions of the parameter  $\mu$ .

$\delta$	Convergence rate
4	2.33
0.5	1.79
0.1	1.19
0	1.18

TABLE 2. Boussinesq models: computed convergence rates with respect to  $\mu$  for the wave amplitude.

#### 4.2.2. Numerical results for the medium and small amplitude models.

*Convergence as functions of the shallowness parameter.* We consider once again the  $L^\infty$ -norm difference  $E_\zeta^n = \|\zeta_\alpha^n - \zeta_{\text{Bouss}}^n\|_\infty$ , where  $(\zeta_\alpha^n)_n$  is the elevation computed by the nonlocal alternative Boussinesq model while  $(\zeta_{\text{Bouss}}^n)_n$  denotes the numerical elevation given by the classical Boussinesq system. In the sake of evaluating the convergence between both classical Boussinesq system and the nonlocal alternative as functions of  $\mu$ , the quantity  $e_\zeta = \max_n E_\zeta^n$  has been computed over a duration of 1500 time steps and for several values of  $\delta$ . The results are depicted in Figure 7 and the computed orders of convergence are given in Table 2. As we noticed in Remark 3.4, the alternative model (3.16) reduces to the standard Boussinesq system (3.17) up to  $O(\mu^2)$  terms for smooth bottoms. The computed order of convergence of 2.33 obtained for the smooth step ( $\delta = 4$ ) is thus consistent with the expected difference between both asymptotic models. In the limit case of the step bottom ( $\delta = 0$ ) the convergence rate becomes 1.18. Consequently using the classical Boussinesq approximation costs at most about one order of precision.

*The case of a step bottom ( $\delta = 0$ ).* In this limit case, two different behaviors emerge when comparing the classical shallow water medium amplitude model with the nonlocal alternative:

- i) For the small values of the shallowness, say  $\mu < 0.01$ , the elevation and velocity computed by the classical model are close to those given by the nonlocal alternative (see Figure 8 obtained for  $\mu = 0.01$ ). In particular, as seen when comparing Figure 8 to Figure 5 both obtained for  $\mu = 0.01$  and  $\varepsilon = 0.1$ , the amplitude of the oscillations produced by the classical medium amplitude model are lower than those obtained with the classical Saint-Venant model.

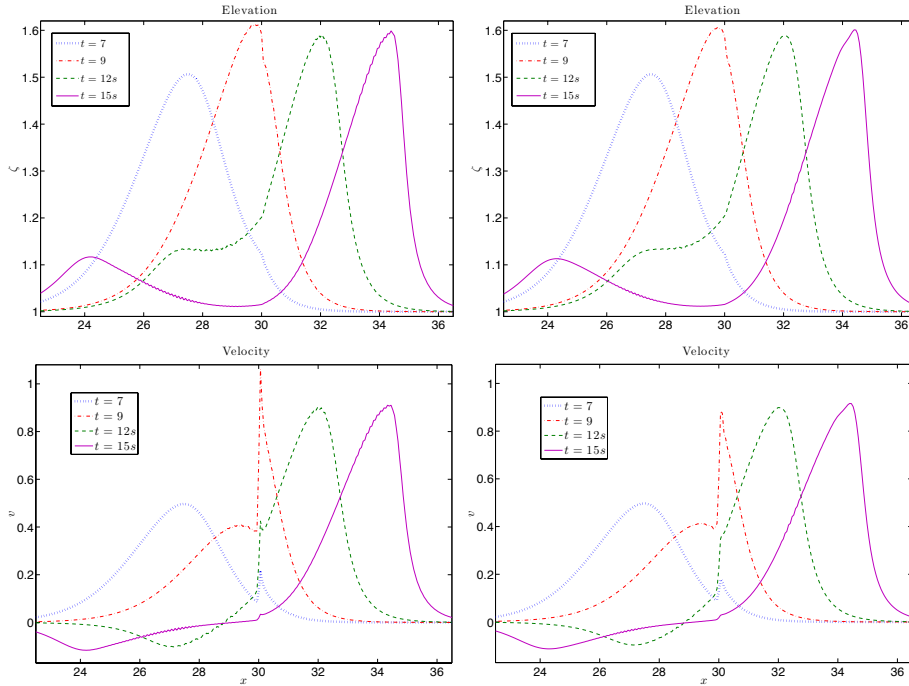


FIGURE 8. Elevation and velocity for a wave passing over a step ( $\mu = 0.01$ ,  $\varepsilon = 0.1$ ). Classical medium amplitude model (left) and nonlocal alternative (right).

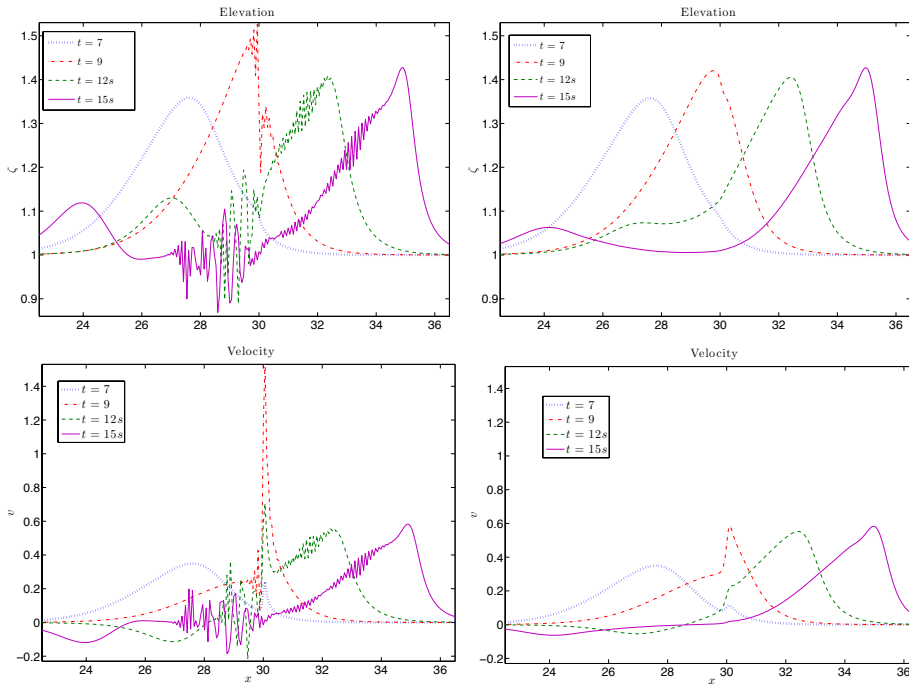


FIGURE 9. Elevation and velocity for a wave passing over a step ( $\mu = 0.04$ ,  $\varepsilon = 0.2$ ). Classical medium amplitude model (left) and nonlocal alternative (right).

- ii) For values of the shallowness parameter in the range  $0.01 < \mu < 0.05$ , some instabilities arise when using the classical medium and small amplitude models with a step bottom. This behavior is illustrated in Figure 9, which shows a comparison between the wave profiles and the velocities determined from both the classical and the nonlocal shallow water medium amplitude models. Note that these instabilities do not vanish for small values of the time step.

#### APPENDIX A. THE CASE OF POLYGONAL TOPOGRAPHIES

In the case of two dimensional motions and when the bottom has polygonal shape, Hamilton [22] and Nachbin [29] used a conformal mapping technique to derive long wave models. This conformal mapping technique can be adapted to derive shallow water models with polygonal topography. The idea is to use Schwarz-Christoffel mapping theory (see *e.g.* [31]) to find a conformal map from a strip to the fluid domain at rest (see [29, 20]). From a numerical point of view, the main interest of this technique is that such a mapping can be efficiently computed using, for instance, the Schwarz-Christoffel Toolbox [17] (see [20, Appendix A] for an application to the conformal mapping of a fluid domain with polygonal bottom). This particular conformal mapping can then be used to approximate the Dirichlet-Neumann operator. Broadly speaking, the derivation of this approximation proceeds via the following steps:

- (1) Transform the Laplace equation (1.1) into an elliptic boundary value problem defined on the flat strip.
- (2) Express the Dirichlet-Neumann operator in terms of the solution of this new problem (the so-called transformed potential).
- (3) Approximate the transformed potential using a BKW procedure.
- (4) Use this approximate solution in the expression of (2) to deduce an approximation of  $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$ .

Denoting by  $\Sigma$  the Schwarz-Christoffel mapping function and setting  $(\sigma(x), \rho(x)) = \Sigma^{-1}(x, \varepsilon\zeta(x))$  (the transformed free surface), the resulting approximation is

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\partial_x \left( \frac{1 + \rho}{\frac{d\sigma}{dx}} \partial_x \psi \right) + O(\mu),$$

giving rise to the following nonlinear shallow water system with polygonal topography

$$\begin{cases} \partial_t \zeta + \partial_x \left( \frac{1 + \rho}{\frac{d\sigma}{dx}} v_s \right) = 0, \\ \partial_t v_s + \partial_x \zeta + \varepsilon v_s \partial_x v_s = 0. \end{cases} \quad (\text{A.1})$$

To evaluate the behavior of the nonlocal shallow water model (3.6) when the bottom has polygonal shape, we compare the solutions produced by both systems (4.2) and (A.1) in the particular case of a rectangular bottom. The bathymetry is given by

$$b(x) = \begin{cases} 0 & \text{if } x < 30 \text{ or } x > 50, \\ 1 & \text{if } 30 \leq x \leq 50, \end{cases}$$

as illustrated in Figure 10. The initial condition is the unidirectional wave defined in (4.6) and the amplitude parameters are set to  $\varepsilon = 0.1$  and  $\beta = 0.6$ . Time histories of the surface elevation computed by both models are shown in Figure 11. The simulation was performed using  $N = 1024$  points and  $\Delta t = 10^{-1}$ . As the wave passes over the step, both models produce similar results.

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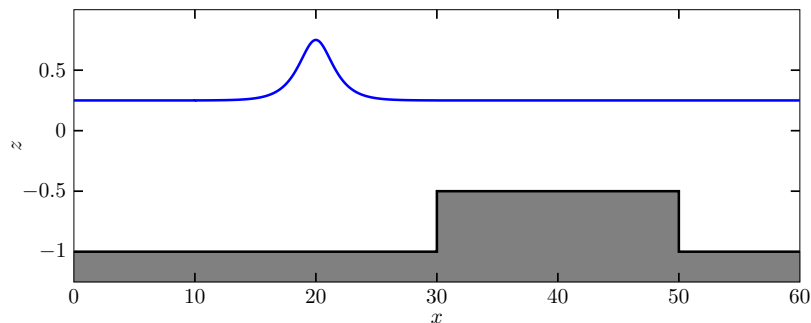


FIGURE 10. Initial condition and rectangular bottom.

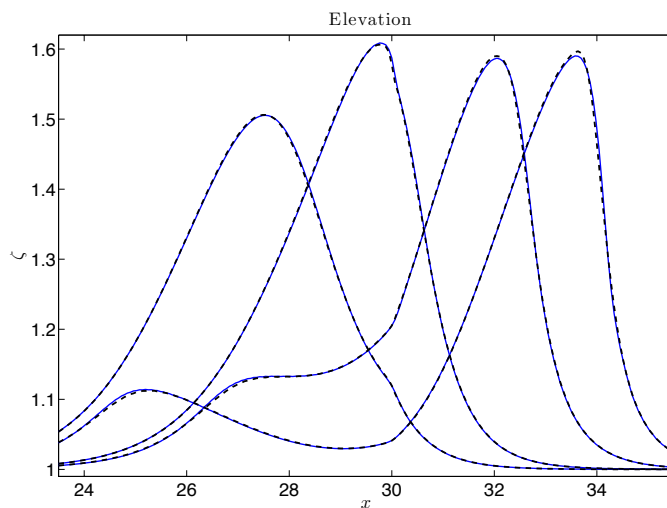


FIGURE 11. Elevation for a wave passing over a rectangular hump: conformal mapping approach (dashed line) and nonlocal model (solid line).

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