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Abstract

Existence and uniqueness of global in time measure solution for a one dimensional nonlinear aggregation equation is considered. Such a system can be written as a conservation law with a velocity field computed through a self-consistent interaction potential. Blow up of regular solutions is now well established for such system. In Carrillo et al. (Duke Math J (2011)) [18], a theory of existence and uniqueness based on the geometric approach of gradient flows on Wasserstein space has been developed. We propose in this work to establish the link between this approach and duality solutions. This latter concept of solutions allows in particular to define a flow associated to the velocity field. Then an existence and uniqueness theory for duality solutions is developed in the spirit of James and Vauchelet (NoDEA (2013)) [26]. However, since duality solutions are only known in one dimension, we restrict our study to the one dimensional case.

Keywords: duality solutions, aggregation equation, nonlocal conservation equations, measure-valued solutions, gradient flow, optimal transport.

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1 Introduction

Aggregation phenomena in a population of particles interacting under a continuous interaction potential are modelled by a nonlocal nonlinear conservation equation. Letting ρ denote the density of cells, the so-called aggregation equation in N space dimension writes

$$\partial_t \rho + \nabla_x \cdot \big(a(\nabla_x W * \rho)\rho\big) = 0, \qquad t > 0, \quad x \in \mathbb{R}^N, \tag{1.1}$$

and is complemented with the inital condition $\rho(0, x) = \rho^{ini}$. Here $W : \mathbb{R}^N \to \mathbb{R}$ is the interaction potential, and $a : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth given function which depends on the actual model under consideration. In this paper, we only focus on the strongly attractive case and consider attractive pointy and Lipschitz potentials W (see Definition 2.3 below) and nondecreasing smooth function a.

This equation is involved in many applications in physics and biology. In the framework of granular media [4, 19, 31], a is the identity function, and interaction potentials are in the form $W(x) = -|x|^\alpha$ with $\alpha > 1$. In plasma physics, the context is the high field limit of a kinetic equation describing the dynamics of electrically charged Brownian particles interacting with a thermal bath. This leads to consider potentials in the form $W(x) = -|x|$, and $a = \mathrm{id}$ as well (see e.g. [33]). Also, continuum mathematical models have been widely proposed to model collective behaviour of individuals. Then the potential W is typically the fundamental solution of some elliptic equation, and a depends on the microscopic behaviour of the individuals. In the context of pedestrian motion nonlinear functions a are considered but with smooth potential W (see [20] and later references with generalizations to systems in [21]). The well-known Patlak-Keller-Segel model describes aggregation of cells by a macroscopic non-local interaction equation with linear diffusion [30, 36]. More precisely, the swarming of cells can be described by aggregation equations where the typical interaction potential is the attractive Morse potential $W(x) = \frac{1}{2}e^{-|x|}$ [17, 32, 35]. Such potentials also appear when considering the hydrodynamic limit of kinetic model describing chemotaxis of bacteria [22, 23, 26].

Most of the potentials mentioned above have a singularity at the origin, they fall into the context of "pointy potentials" (see a precise definition below), and it is well-known that in that case concentration phenomena induce the blow-up in finite time of weak L^p solutions [5, 6, 7]. Thus the notion of solution breaks down at the blow-up time and weak measure-valued solutions for the aggregation equation have to be considered [3, 8, 9]. Carrillo et al. [18] have studied the multidimensional aggregation equation when $a = \mathrm{id}$ in the framework of gradient flow solutions. Namely, equation (1.1) is interpreted as a differential equation in time, the right-hand side being the gradient of some interaction energy defined through the potential W. This idea, known as the Otto calculus (see [34, 40]), requires the choice of a convenient space of probability measures endowed with a Riemannian structure. Then, following [2], gradient flow solutions are interpreted as curves of maximal slopes in this space. The authors obtain existence and uniqueness of weak solutions for (1.1) in $\mathbb{R}^N$, $N \geq 1$ when $a = \mathrm{id}$, the main problem being now to connect these solutions to distributional solutions.

An alternative notion of weak solutions has been obtained by completely different means in the framework of positive chemotaxis in [26]. Here equation (1.1) with $W(x) = \frac{1}{2}e^{-|x|}$ can be obtained thanks to some hydrodynamic limit of the kinetic Othmer-Dunbar-Alt system [22]. The key idea is to use the notion of duality solutions, introduced in [13] for linear conservation equations with discontinuous velocities, where measure-valued solutions also arise. In that case, this allows to give a convenient meaning to the product of the velocity by the density, so that existence and uniqueness can be proved. When applying this strategy to the nonlinear case, it turns out that uniqueness is not ensured, unless the nonlinear product is given a very precise signification, see for instance [14, 25]. In the case of chemotaxis, it is provided by the limit of the flux in the kinetic model. Once this is done, existence and uniqueness can be obtained. An important consequence of this approach is that it allows to define a flow associated to this system. Then the dynamics of the aggregates (i.e. combinations of Dirac masses) can be established, giving rise to an implementation of a particle method and numerical simulations of the dynamics of cells density after blow-up (see also [27] for a numerical approach using a discretization on a fixed grid). The principal drawback of this method is its limitation to the one-dimensional case, mainly because duality solutions are not properly defined yet in higher space dimension. Thus the theory developed in [18] is, up to our knowledge, the only one allowing to get existence of global in time weak measure solution for (1.1) in dimension higher than 2. Another possibility could be using the notion of Filippov flow [24], together with the stability results in [10], to obtain a convenient notion of solution to (1.1), thus following [37].

This work is devoted to the study of the links between these two notions of weak solutions for equation (1.1), in the one-dimensional setting. As we shall see there is no equivalence strictly speaking for a general potential and a nonlinear function a. More precisely we first consider the same situation as in [18], that is a pointy potential W, and $a = \mathrm{id}$. We adapt the proof of [26] to define duality solutions in this context, and choose a convenient space of measures to be compatible with the gradient flows. Then we prove that duality solutions and gradient flow solutions are identical (Theorem 4.1 below), thus answering the questions raised by Remark 2.16 of [18].

Next, we investigate the nonlinear case, that is $a \neq \mathrm{id}$. Notice that additional monotonicity properties are required to ensure the attractivity of the dynamics. The results of [18] cannot be applied as they stand, the key problem is to define a new energy for which weak solutions of (1.1) are gradient flows. However, we are able to find such an energy only in the particular case $W = -\frac{1}{2}|x|$. On the contrary existence of duality solutions for (1.1) with a nonlinear function a can be obtained for more general potentials W, even if we cannot reach the complete generality of [18]. As in the linear case, for this specific choice of potential, we have equivalence of duality and gradient flows solutions. Moreover, in this case, this solution can be seen as the derivative of the entropy solution of a scalar conservation law (Theorem 5.7 below).

The outline of this paper is as follows. In the next Section, we introduce notations and recall the main results obtained in [18] in the case $a = \mathrm{id}$. A sketch of their proof is proposed. Section 3 is devoted to the duality solutions, and starts by recalling their original definition and main properties. Next, we turn to the nonlinear setting, and define precisely the velocities and fluxes that allow to state the existence and uniqueness results both for $a = \mathrm{id}$ and $a \neq \mathrm{id}$. The case $a = \mathrm{id}$ is treated in Section 4: existence and uniqueness for duality solutions are proved, together with equivalence between gradient flows and duality solutions. Finally in Section 5 we investigate the case $a \neq \mathrm{id}$, where general equivalence results no longer hold. For the completeness of the paper, a technical Lemma is given in Appendix.

2 Gradient flow solutions

2.1 Notations and definitions

Let $C_0(Y,Z)$ be the set of continuous functions from Y to Z that vanish at infinity and $C_c(Y,Z)$ the set of continuous functions with compact support from Y to Z, where Y and Z are metric spaces. All along the paper, we denote $\mathcal{M}_{loc}(\mathbb{R}^N)$ the space of local Borel measures on $\mathbb{R}^N$. For $\rho \in \mathcal{M}_{loc}(\mathbb{R}^N)$ we denote by $|\rho|(\mathbb{R}^N)$ its total variation. We will denote $\mathcal{M}_b(\mathbb{R}^N)$ the space of measures in $\mathcal{M}_{loc}(\mathbb{R}^N)$ with finite total variation. From now on, the space of measures $\mathcal{M}_b(\mathbb{R}^N)$ is always endowed with the weak topology $\sigma(\mathcal{M}_b, C_0)$. We denote $\mathcal{S}_\mathcal{M} := C([0,T]; \mathcal{M}_b(\mathbb{R}^N) - \sigma(\mathcal{M}_b, C_0))$. We recall that if a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_b(\mathbb{R}^N)$ satisfies $\sup_{n\in\mathbb{N}} |\mu_n|(\mathbb{R}^N) < +\infty$, then we can extract a subsequence that converges for the weak topology $\sigma(\mathcal{M}_b, C_0)$.

Since we focus on scalar conservation laws, we can assume without loss of generality that the total mass of the system is scaled to 1 and thus we will work in some space of probability measures, namely the Wasserstein space of order $q \geq 1$, which is the space of probability measures with finite order q moment:

$$\mathcal{P}_q(\mathbb{R}^N) = \left\{ \mu \text{ nonnegative Borel measure}, \mu(\mathbb{R}^N) = 1, \int |x|^q \mu(dx) < \infty \right\}.$$

This space is endowed with the Wasserstein distance defined by (see e.g. [40, 41])

$$d_{Wq}(\mu,\nu) = \inf_{\gamma\in\Gamma(\mu,\nu)} \left\{ \int |y-x|^q \, \gamma(dx,dy) \right\}^{1/q} \tag{2.1}$$

where $\Gamma(\mu,\nu)$ is the set of measures on $\mathbb{R}^N \times \mathbb{R}^N$ with marginals μ and ν, i.e.

$$\Gamma(\mu,\nu) = \left\{ \gamma \in \mathcal{P}_q(\mathbb{R}^N \times \mathbb{R}^N);\ \forall \xi \in C_0(\mathbb{R}^N), \int \xi(y_0)\gamma(dy_0,dy_1) = \int \xi(y_0)\mu(dy_0), \right.$$
$$\left. \int \xi(y_1)\gamma(dy_0,dy_1) = \int \xi(y_1)\nu(dy_1) \right\}.$$

From a simple minimization argument, we know that in the definition of d_{Wq} the infimum is actually a minimum. A map that realizes the minimum in the definition (2.1) of d_{Wq} is called an optimal map, the set of which is denoted by $\Gamma_0(\mu,\nu)$.

A fundamental breakthrough in the use of the geometric approach to solve PDE is the work of F. Otto [34], which is the basis of the so-called *Otto Calculus* [40]. Let X be a Riemannian manifold endowed with the Riemannian metric $g_x(\cdot,\cdot)$ (a positive quadratic form on the tangent space at X in x denoted T_xX). Let $\mathcal{W} : X \to \mathbb{R}$ be differentiable. The gradient of $\mathcal{W}$ at $x \in X$ is defined as follows : for all $v \in T_xX$, let $\gamma(t)$ be a regular curve on X such that $\gamma(0) = x$ and $\gamma'(0) = v$, then

$$\frac{d}{dt}|_{t=0} \mathcal{W}(\gamma)(t) = g_x(\nabla_x \mathcal{W}, v), \qquad \nabla_x \mathcal{W} \in T_xX.$$

The gradient flow associated to $\mathcal{W}$ is the solution $\rho : [0,+\infty) \to X$ of the differential equation :

$$\frac{d\rho}{dt} = -\nabla_\rho \mathcal{W}.$$

A fundamental result due to Ambrosio et al. [2] states that gradient flows are equivalent to curves of maximal slope. Therefore, solving a PDE model of gradient type boils down to prove the existence of a curve of maximal slope.

Let us be more precise. In the following, we will mainly focus on the case $q = 2$ and we will shortly denote d_W instead of d_{W2}. In the formalism of [2], we say that a curve μ is absolutely continuous, and we denote $\mu \in AC^2((0,+\infty), \mathcal{P}_2(\mathbb{R}^N))$, if there exists $m \in L^2(0,+\infty)$, such that $d_W(\mu(s),\mu(t)) \leq \int_s^t m(r)dr$, for $0 < s \leq t < +\infty$. Then we can define the metric derivative

$$|\mu'|(t) := \limsup_{s\to t} \frac{d_W(\mu(s),\mu(t))}{|s-t|}.$$

The tangent space to a measure $\mu \in \mathcal{P}_2(\mathbb{R}^N)$ is defined by the closed vector subspace of $L^2(\mu)$

$$Tan_\mu \mathcal{P}_2(\mathbb{R}^N) := \overline{\{\nabla\phi : \phi \in C_c^\infty(\mathbb{R}^N)\}}^{L^2(\mu)}.$$

We recall the result of Theorem 8.3.1 of [2]: if $\mu \in AC^2((0,+\infty), \mathcal{P}_2(\mathbb{R}^N))$, then there exists a Borel vector field $v(t) \in L^2(\mu(t))$ such that

$$\partial_t \mu + \nabla\cdot(v\mu) = 0, \text{ in the distributional sense on } (0,+\infty)\times\mathbb{R}^N. \tag{2.2}$$

Conversely, if μ solves a continuity equation for some Borel velocity $v \in L^1((0,+\infty); L^2(\mu))$ then μ is an absolutely continuous curve and $|\mu'|(t) \leq \|v(t)\|_{L^2(\mu)}$. As a consequence, we have

$$|\mu'|(t) = \min\left\{\|v\|_{L^2(\mu(t))} : v(t) \text{ satisfies (2.2)}\right\}.$$

Let $\mathcal{W}$ be a functional on $\mathcal{P}_2(\mathbb{R}^N)$. We denote by $\partial\mathcal{W}$ its subdifferential. The general definition of subdifferential on $\mathcal{P}_2(\mathbb{R}^d)$ being pretty involved, we refer the interested reader to [2, Definition 10.3.1]. In principle, the element of $\partial\mathcal{W}$ are plans. In the case at hand, such plans are concentrated on the graph of a vector field, which allows to reduce the general definition of subdifferential to the following one: a vector field $w \in L^2(\mu)$ is said to be an element of the subdifferential of $\mathcal{W}$ at μ if

$$\mathcal{W}(\mu) - \mathcal{W}(\nu) \geq \inf_{\gamma\in\Gamma_0(\mu,\nu)} \int_{\mathbb{R}^N\times\mathbb{R}^N} w(x)\cdot(y-x)d\gamma(x,y) + o(d_{W2}(\mu,\nu)).$$

Next, the slope $|\partial\mathcal{W}|$ is defined by

$$|\partial\mathcal{W}|(\mu) = \limsup_{\nu\to\mu} \frac{(\mathcal{W}(\mu)-\mathcal{W}(\nu))_+}{d_W(\mu,\nu)}, \tag{2.3}$$

where $u_+ = \max\{u, 0\}$. We have the property ([2], Lemma 10.1.5)

$$|\partial\mathcal{W}|(\mu) = \min\{\|w\|_{L^2(\mu)} \ : \ w \in \partial\mathcal{W}(\mu)\}. \tag{2.4}$$

Moreover, there exists a unique $w \in \mathcal{W}(\mu)$ which attains the minimum in (2.4). It is denoted by $\partial^0\mathcal{W}(\mu)$.

Definition 2.1 (Gradient flows) *We say that a map* $\mu \in AC^2_{loc}((0,+\infty);\mathcal{P}_2(\mathbb{R}^N))$ *is a solution of a gradient flow equation associated to the functional* $\mathcal{W}$ *if there exists a Borel vector field* v *such that* $v(t) \in Tan_{\mu(t)}\mathcal{P}_2(\mathbb{R}^N)$ *for a.e.* $t > 0$, $\|v(t)\|_{L^2(\mu)} \in L^2_{loc}(0,+\infty)$, *the continuity equation*

$$\partial_t \mu + \nabla \cdot (v\mu) = 0,$$

holds in the sense of distributions and $v(t) \in -\partial\mathcal{W}(\rho(t))$ *for a.e.* $t > 0$, *where* $\partial\mathcal{W}(\rho)$ *is the subdifferential of* $\mathcal{W}$ *at the point* ρ.

Definition 2.2 (Curve of maximal slope) *A curve* $\mu \in AC^2_{loc}((0,+\infty);\mathcal{P}_2(\mathbb{R}))$ *is a curve of maximal slope for the functional* $\mathcal{W}$ *if* $t \mapsto \mathcal{W}(\mu(t))$ *is an absolutely continuous function and if for every* $0 \le s \le t \le T$,

$$\frac{1}{2}\int_s^t |\mu'|^2(\tau)\,d\tau + \frac{1}{2}\int_s^t |\partial\mathcal{W}|^2(\mu(\tau))d\tau \le \mathcal{W}(\mu(s)) - \mathcal{W}(\mu(t)).$$

Finally Theorem 11.1.3 of [2] shows that curves of maximal slope with respect to $|\partial\mathcal{W}|$ are equivalent to gradient flow solutions. Moreover, the tangent vector field $v(t)$ is the unique element of minimal norm in the subdifferential of $\mathcal{W}$ (see (2.4)):

$$v(t) = -\partial^0\mathcal{W}(\mu(t)) \text{ for a.e. } t > 0.$$

2.2 Strategy of the proof in [18]

The idea of the work by Carrillo et al. [18] is to extend the work of [2] to an interaction energy $\mathcal{W}$ defined through the interaction potential W in (1.1), whose derivative has a singularity in 0. More precisely, attractive "pointy potentials" are considered, which we define now.

Definition 2.3 (pointy potential) *The interaction potential* W *is said to be an attractive pointy potential if it satisfies the following assumptions.*

(A0) W *is continuous,* $W(x) = W(-x)$ *and* $W(0) = 0$.

(A1) W *is* λ*-concave for some* $\lambda \ge 0$, *i.e.* $W(x) - \frac{\lambda}{2}|x|^2$ *is concave.*

(A2) *There exists a constant* $C > 0$ *such that* $W(x) \ge -C(1+|x|^2)$ *for all* $x \in \mathbb{R}^N$.

(A3) $W \in C^1(\mathbb{R}^N \setminus \{0\})$.

Given a continuous potential $W : \mathbb{R} \to \mathbb{R}$, we define the interaction energy in one dimension by

$$\mathcal{W}(\rho) = -\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} W(x-y)\,\rho(dx)\rho(dy). \tag{2.5}$$

The existence and uniqueness result of [18] can now be synthetized as follows.

Theorem 2.4 *([18, Theorems 2.12 and 2.13]) Let W satisfies assumptions* **(A0)–(A3)** *and let $a = id$. Given $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^N)$, there exists a gradient flow solution of* (1.1)*, i.e. a curve $\rho \in AC^2_{loc}([0,\infty); \mathcal{P}_2(\mathbb{R}^N))$ satisfying*

$$\frac{\partial \rho(t)}{\partial t} + \partial_x(v(t)\rho(t)) = 0, \qquad \text{in } \mathcal{D}'([0,\infty) \times \mathbb{R}^N),$$
$$v(t,x) = -\partial^0 \mathcal{W}(\rho)(t,x) = \int_{y \neq x} \nabla W(x-y)\, \rho(t,dy),$$

with $\rho(0) = \rho^{ini}$. Moreover, if ρ_1 and ρ_2 are such gradient flow solutions, then there exists a constant λ such that, for all $t \geq 0$

$$d_W(\rho_1(t), \rho_2(t)) \leq e^{\lambda t} d_W(\rho_1(0), \rho_2(0)).$$

Thus the gradient flow solution of (1.1) *with initial data $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^N)$ is unique. Moreover, the following energy identity holds for all $0 \leq t_0 \leq t_1 < \infty$:*

$$\int_{t_0}^{t_1} \int_{\mathbb{R}} |\partial^0 W * \rho|^2 \rho(t,dx)dt + \mathcal{W}(\rho(t_1)) = \mathcal{W}(\rho(t_0)). \tag{2.6}$$

Proof. We summarize here the main steps of the proof and refer the reader to [18] for more details. The first step is to compute the element of minimal norm in the subdifferential of $\mathcal{W}$. By extending Theorem 10.4.11 of [2], the authors prove [18, Proposition 2.6] that

$$-\partial^0 \mathcal{W}(\rho) = \partial^0 W * \rho, \tag{2.7}$$

where

$$\partial^0 W * \rho(x) = \int_{y \neq x} W'(x-y)\, \rho(dy).$$

The second step is based on the so-called *JKO scheme* introduced in [29] (see also [2]). It consists in the following recursive construction for curves of maximal slope. Let $\tau > 0$ be a small time step, we set $\rho_0^\tau = \rho^{ini}$ the initial data for (1.1). Next, knowing ρ_k^τ, one proves [18, Proposition 2.5] that there exists ρ_{k+1}^τ such that

$$\rho_{k+1}^\tau \in \underset{\rho \in \mathcal{P}_2(\mathbb{R}^N)}{\arg\min} \left\{ \mathcal{W}(\rho) + \frac{1}{2\tau} d_W^2(\rho_k^\tau, \rho) \right\}. \tag{2.8}$$

Next, a piecewise constant interpolation ρ^τ is defined by

$$\rho^\tau(0) = \rho^{ini}\ ; \qquad \rho^\tau(t) = \rho_k^\tau \quad \text{if} \quad t \in (k\tau, (k+1)\tau],$$

and Proposition 2.6 states the weak compactness (in the narrow topology) of the sequence ρ^τ as $\tau \to 0$. Finally Theorem 2.8 ensures that the weak narrow limit ρ is a curve of maximal slope.

The conclusion follows by applying Theorem 11.1.3 of [2], which allows therefore to get the existence of a gradient flow for the functional $\mathcal{W}$. By definition, the gradient flow is a solution of a continuity equation whose velocity field is the element with minimal norm of the subdifferential of $\mathcal{W}$. In the first step of the proof this element has been identified to be $\partial^0 W * \rho$. Thus, it is a weak solution of the problem and moreover we have the energy estimate (2.6). $\square$

2.3 The one-dimensional case

We gather here several remarks specific to the one-dimensional framework. First we notice that assumptions **(A1)** and **(A3)** imply that $x \mapsto W'(x) - \lambda x$ is a nonincreasing function on $\mathbb{R} \setminus \{0\}$. Therefore $\lim_{x \to 0^\pm} W'(x) = W'(0^\pm)$ exists and from **(A0)**, we deduce that $W'(0^-) = -W'(0^+)$. Moreover, for all $x > y$ in $\mathbb{R} \setminus \{0\}$ we have $W'(x) - \lambda x \leq W'(y) - \lambda y$. Thus we have the one-sided Lipschitz estimate (OSL) for W'

$$\forall\, x > y \in \mathbb{R} \setminus \{0\}, \qquad W'(x) - W'(y) \leq \lambda(x - y). \tag{2.9}$$

Letting $y \to 0^\pm$ we deduce that for all $x > 0$, $W'(x) - \lambda x \leq W'(0^+)$ and $W'(x) - \lambda x \leq W'(0^-)$. Thus we also have the one-sided estimate

$$W'(x) \leq \lambda x, \qquad \text{for all } x > 0. \tag{2.10}$$

In the following, we will assume that the potential W is Lipschitz in order to avoid possible linear growth at infinity of the velocity field :

(A4) There exists a nonnegative constant C_0 such that $|W'(x)| \leq C_0$ for all $x \in \mathbb{R}^*$.

The one-dimensional framework also allows to simplify several proofs in Theorem 2.4. Indeed any probability measure μ on the real line $\mathbb{R}$ can be described in term of its cumulative distribution function $M(x) = \mu((-\infty, x))$ which is a right-continuous and nondecreasing function with $M(-\infty) = 0$ and $M(+\infty) = 1$. Then we can define the generalized inverse F_μ of M (or monotone rearrangement of μ) by $F_\mu(z) = M^{-1}(z) := \inf\{x \in \mathbb{R}/M(x) > z\}$, it is a right-continuous and nondecreasing function as well, defined on $[0, 1]$. We have for every nonnegative Borel map ξ,

$$\int_{\mathbb{R}} \xi(x)\rho(dx) = \int_0^1 \xi(F_\mu(z))\, dz.$$

In particular, $\mu \in \mathcal{P}_2(\mathbb{R})$ if and only if $F_\mu \in L^2(0, 1)$. Moreover, in the one-dimensional setting, there exists a unique optimal map realizing the minimum in (2.1). More precisely, if μ and ν belongs to $\mathcal{P}_2(\mathbb{R})$, with monotone rearrangement F_μ and F_ν, then $\Gamma_0(\mu, \nu) = \{(F_\mu, F_\nu)_\# \mathbb{L}_{(0,1)}\}$ where $\mathbb{L}_{(0,1)}$ is the restriction of the Lebesgue measure on $(0, 1)$. Then we have the explicit expression of the Wasserstein distance (see [39, 41])

$$d_W(\mu, \nu)^2 = \int_0^1 |F_\mu(z) - F_\nu(z)|^2\, dz, \tag{2.11}$$

and the map $\mu \mapsto F_\mu$ is an isometry between $\mathcal{P}_2(\mathbb{R})$ and the convex subset of (essentially) nondecreasing functions of $L^2(0, 1)$.

In this framework, we can then rewrite the JKO scheme (2.8) in the proof above. Let us denote by M_k^τ the cumulative distribution of the measure ρ_k^τ and by $V_k^\tau := F_k^\tau$ its generalized inverse. Then, in term of generalized inverses, (2.8) rewrites

$$V_{k+1}^\tau \in \underset{\{V \in L^2(0,1), \partial_x V \geq 0\}}{\arg\min} \left\{ \widetilde{\mathcal{W}}(V) + \frac{1}{2\tau} \|V - V_k^\tau\|^2_{L^2(0,1)} \right\},$$

where

$$\widetilde{\mathcal{W}}(V) = \frac{1}{2} \int_0^1 \int_0^1 W(V(y) - V(z))\, dy dz.$$

Such an approach using the generalized inverse has been used in [11] for the one dimensional Patlak-Keller-Segel equation.

3 Duality solutions

We turn now to the alternative notion of weak solution we wish to investigate. It is based on the so-called duality solutions which were introduced for linear advection equations with discontinuous coefficients in [13]. Compared with the gradient flow approach, this strategy allows a more straightforward PDE formulation. In particular from the numerical viewpoint, classical finite volume approach strongly relying on this formulation is proposed in [28]. The main drawback is that presently duality solutions in any space dimension are only available for pure transport equations (see [15]). Since we have to deal here with conservative balance laws, we have to restrict ourselves to one space dimension. First we give a brief account of the theory developed in [13], summarizing the main theorems we shall use, next we define duality solutions for (1.1).

3.1 Linear conservation equations

We consider here conservation equations in the form

$$\partial_t \rho + \partial_x \big(b(t,x)\rho\big) = 0, \qquad (t,x) \in (0,T) \times \mathbb{R}, \tag{3.1}$$

where b is a given bounded Borel function. Since no regularity is assumed for b, solutions to (3.1) eventually are measures in space. A convenient tool to handle this is the notion of duality solutions, which are defined as weak solutions, the test functions being Lipschitz solutions to the backward linear transport equation

$$\partial_t p + b(t,x)\partial_x p = 0, \tag{3.2}$$

$$p(T,.) = p^T \in \mathrm{Lip}(\mathbb{R}). \tag{3.3}$$

In fact, a formal computation shows that $\frac{d}{dt}\left(\int_{\mathbb{R}} p(t,x)\rho(t,dx)\right) = 0$, which defines the duality solutions for suitable p's.

It is quite classical that a sufficient condition to ensure existence for (3.2) is that the velocity field to be compressive, in the following sense:

Definition 3.1 *We say that the function b satisfies the one-sided Lipschitz (OSL) condition if*

$$\partial_x b(t,.) \le \beta(t) \qquad \text{for } \beta \in L^1(0,T) \text{ in the distributional sense.} \tag{3.4}$$

However, to have uniqueness, we need to restrict ourselves to *reversible* solutions of (3.2): let $\mathcal{L}$ denote the set of Lipschitz continuous solutions to (3.2), and define the set $\mathcal{E}$ of *exceptional solutions* by

$$\mathcal{E} = \Big\{ p \in \mathcal{L} \text{ such that } p^T \equiv 0 \Big\}.$$

The possible loss of uniqueness corresponds to the case where $\mathcal{E}$ is not reduced to zero.

Definition 3.2 *We say that $p \in \mathcal{L}$ is a* **reversible solution** *to (3.2) if p is locally constant on the set*

$$\mathcal{V}_e = \Big\{ (t,x) \in [0,T] \times \mathbb{R};\ \exists\, p_e \in \mathcal{E},\ p_e(t,x) \neq 0 \Big\}.$$

We refer to [13] for complete statements of the characterization and properties of reversible solutions. Then, we can state the definition of duality solutions.

Definition 3.3 *We say that $\rho \in \mathcal{S}_{\mathcal{M}} := C([0,T];\mathcal{M}_b(\mathbb{R}) - \sigma(\mathcal{M}_b, C_0))$ is a* **duality solution** *to (3.1) if for any $0 < \tau \le T$, and any* **reversible** *solution p to (3.2) with compact support in x, the function $t \mapsto \displaystyle\int_{\mathbb{R}} p(t,x)\rho(t,dx)$ is constant on $[0,\tau]$.*

We summarize now some properties of duality solutions that we shall need in the following.

Theorem 3.4 *(Bouchut, James [13])*

1. *Given $\rho^\circ \in \mathcal{M}_b(\mathbb{R})$, under the assumptions (3.4), there exists a unique $\rho \in \mathcal{S}_{\mathcal{M}}$, duality solution to (3.1), such that $\rho(0,.) = \rho^\circ$.*
   Moreover, if ρ° is nonnegative, then $\rho(t,\cdot)$ is nonnegative for a.e. $t \ge 0$. And we have the mass conservation
   $$|\rho(t,\cdot)|(\mathbb{R}) = |\rho^\circ|(\mathbb{R}), \qquad \text{for a.e. } t \in]0,T[.$$

2. *Backward flow and push-forward: the duality solution satisfies*
   $$\forall\, t \in [0,T], \forall\, \phi \in C_0(\mathbb{R}), \quad \int_{\mathbb{R}} \phi(x)\rho(t,dx) = \int_{\mathbb{R}} \phi(X(t,0,x))\rho^0(dx), \tag{3.5}$$
   where the **backward flow** *X is defined as the unique reversible solution to*
   $$\partial_t X + b(t,x)\partial_x X = 0 \quad \text{in }]0,s[\times\mathbb{R}, \qquad X(s,s,x) = x.$$

3. *For any duality solution ρ, we define the* **generalized flux** *corresponding to ρ by $b_\Delta \rho = -\partial_t u$, where $u = \int^x \rho\, dx$.*

   There exists a bounded Borel function $\widehat{b}$, called **universal representative** *of b, such that $\widehat{b} = b$ almost everywhere, $b_\Delta \rho = \widehat{b}\rho$ and for any duality solution ρ,*
   $$\partial_t \rho + \partial_x(\widehat{b}\rho) = 0 \qquad \text{in the distributional sense.}$$

4. *Let (b_n) be a bounded sequence in $L^\infty(]0,T[\times\mathbb{R})$, such that $b_n \rightharpoonup b$ in $L^\infty(]0,T[\times\mathbb{R}) - w\star$. Assume $\partial_x b_n \le \beta_n(t)$, where (β_n) is bounded in $L^1(]0,T[)$, $\partial_x b \le \beta \in L^1(]0,T[)$. Consider a sequence $(\rho_n) \in \mathcal{S}_{\mathcal{M}}$ of duality solutions to*
   $$\partial_t \rho_n + \partial_x(b_n \rho_n) = 0 \quad \text{in} \quad]0,T[\times\mathbb{R},$$
   such that $\rho_n(0,.)$ is bounded in $\mathcal{M}_b(\mathbb{R})$, and $\rho_n(0,.) \rightharpoonup \rho^\circ \in \mathcal{M}_b(\mathbb{R})$.

   Then $\rho_n \rightharpoonup \rho$ in $\mathcal{S}_{\mathcal{M}}$, where $\rho \in \mathcal{S}_{\mathcal{M}}$ is the duality solution to
   $$\partial_t \rho + \partial_x(b\rho) = 0 \quad \text{in} \quad]0,T[\times\mathbb{R}, \qquad \rho(0,.) = \rho^\circ.$$
   Moreover, $\widehat{b}_n \rho_n \rightharpoonup \widehat{b}\rho$ weakly in $\mathcal{M}_b(]0,T[\times\mathbb{R})$.

The set of duality solutions is clearly a vector space, but it has to be noted that a duality solution is not *a priori* defined as a solution in the sense of distributions. However, assuming that the coefficient b is piecewise continuous, we have the following equivalence result:

Theorem 3.5 *Let us assume that in addition to the OSL condition* (3.4)*, b is piecewise continuous on $]0,T[\times\mathbb{R}$ where the set of discontinuity is locally finite. Then there exists a function $\widehat{b}$ which coincides with b on the set of continuity of b.*

With this $\widehat{b}$, $\rho \in \mathcal{S}_{\mathcal{M}}$ is a duality solution to (3.1) *if and only if $\partial_t\rho + \partial_x(\widehat{b}\rho) = 0$ in $\mathcal{D}'(\mathbb{R})$. Then the generalized flux $b_\Delta\rho = \widehat{b}\rho$. In particular, $\widehat{b}$ is a universal representative of b.*

This result comes from the uniqueness of solutions to the Cauchy problem for both kinds of solutions, see [13, Theorem 4.3.7].

3.2 Duality solutions for aggregation

Equipped with this notion of solutions, we can now define duality solutions for the aggregation equation. The idea was introduced in [14] in the context of pressureless gases. It was next applied to chemotaxis in [26], and we shall actually follow these steps.

Definition 3.6 *We say that $\rho \in \mathcal{S}_{\mathcal{M}}$ is a duality solution to* (1.1) *if there exists $\widehat{a}_\rho \in L^\infty((0,T)\times\mathbb{R})$ and $\beta \in L^1_{loc}(0,T)$ satisfying $\partial_x\widehat{a}_\rho \leq \beta$ in $\mathcal{D}'((0,T)\times\mathbb{R})$, such that for all $0 < t_1 < t_2 < T$,*

$$\partial_t\rho + \partial_x(\widehat{a}_\rho\rho) = 0 \tag{3.6}$$

in the sense of duality on (t_1,t_2), and $\widehat{a}_\rho = a(W' * \rho)$ a.e. We emphasize that it means that the final datum for (3.2) *should be at t_2 instead of T.*

This allows at first to give a meaning to the notion of distributional solution, but it turns out that uniqueness is a crucial issue. For that, a key point is a precise definition of the product $\widehat{a}_\rho\rho$, as we shall see in more details in Section 3.3 below.

We now state the main theorems about duality solutions for the aggregation equation (1.1). Existence of such solutions in a measure space has been obtained in [26] in the particular case $W(x) = \frac{1}{2}e^{-|x|}$ and a similar result is presented in [25] when $W(x) = -|x|/2$ which appears in many applications in physics or biology. We extend here these results for a general potential satisfying assumptions **(A0)–(A4)**. However to do so, we have, as in [18], to restrict ourselves to the linear case, that is $a = \mathrm{id}$.

Theorem 3.7 (Duality solutions, linear case) *Let W satisfy assumptions* **(A0)–(A4)** *and $a = id$. Assume that $\rho^{ini} \in \mathcal{P}_1(\mathbb{R})$. Then for any $T > 0$, there exists a unique $\rho \in \mathcal{S}_{\mathcal{M}}$ such that $\rho(0) = \rho^{ini}$, $\rho(t) \in \mathcal{P}_1(\mathbb{R})$ for any $t \in (0,T)$, and ρ is a duality solution to equation* (1.1) *with universal representative $\widehat{a}_\rho$ in* (3.6) *defined by*

$$\widehat{a}_\rho(t,x) := \partial^0 W * \rho(t,x) = \int_{x\neq y} W'(x-y)\rho(t,dy). \tag{3.7}$$

Moreover we have $\rho = X_\#\rho^{ini}$ where X is the backward flow corresponding to $\widehat{a}_\rho$.

We turn now to the case $a \neq \mathrm{id}$. In order to be in the attractive case, we assume that the following

Assumption 3.8 *The potential W is Lipschitz and pointy, i.e. satisfies* **(A0)–(A4)**. *The function a is non-decreasing, with*

$$a \in C^1(\mathbb{R}), \quad 0 \le a' \le \alpha, \quad \alpha > 0. \tag{3.8}$$

In this context, existence and uniqueness of duality solutions have been proved for the case of an interaction potential $W = \frac{1}{2}e^{-|x|}$ in [26]. We extend here the techniques developed in this latter work to more general potentials W. However we are not able to prove such results in the whole generality of assumptions **(A0)–(A4)** and need more regularity on the interaction potential, as follows

Assumption 3.9 *We assume that $W \in C^1(\mathbb{R} \setminus \{0\})$ and that in the distributional sense*

$$W'' = -\delta_0 + w, \quad w \in Lip \cap L^\infty(\mathbb{R}), \tag{3.9}$$

where δ_0 is the Dirac measure in 0.

This allows a definition of the flux in (1.1) which generalizes the one in [26]. Indeed we can formally take the convolution of (3.9) by ρ, then multiply by $a(W' * \rho)$. Denoting by A the antiderivative of a such that $A(0) = 0$ and using the chain rule we obtain formally

$$-\partial_x(A(W' * \rho)) = -a(W' * \rho)W'' * \rho = a(W' * \rho)(\rho - w * \rho). \tag{3.10}$$

Thus a natural formulation for the flux J is given by

$$J := -\partial_x\big(A(W' * \rho)\big) + a(W' * \rho)w * \rho. \tag{3.11}$$

The product $a(W' * \rho)w * \rho$ is well defined since $w * \rho$ is Lipschitz. Then J is defined in the sense of measures.

Theorem 3.10 (duality solutions, nonlinear case) *Let be given $\rho^{ini} \in \mathcal{P}_1(\mathbb{R})$. Under Assumptions 3.8 and 3.9 on the potential W and the nonlinear function a, for all $T > 0$ there exists a unique duality solution ρ of* (1.1) *in the sense of Definition 3.6 $\rho(t) \in \mathcal{P}_1(\mathbb{R})$ for $t \in (0, T)$ and which satisfies in the distributional sense*

$$\partial_t\rho + \partial_x J = 0, \tag{3.12}$$

where J is defined by (3.11).

Theorems 3.7 and 3.10 are proved respectively in Sections 4 and 5, but before diving into the detailed proofs, we comment the main steps, which are common to both cases.

- **existence** of duality solutions is obtained by approximation. First we obtain the dynamics of aggregates (that is combinations of Dirac masses), then we proceed using the stability of duality solutions

- **uniqueness** is obtained by a contraction argument in $\mathcal{P}_1(\mathbb{R})$. No uniqueness is expected in a general space of measures. The argument is repeated in $\mathcal{P}_2(\mathbb{R})$ so that gradient flow and duality solutions can be compared. In the nonlinear case, the contraction argument relies on an entropy inequality.

3.3 Velocities and fluxes

When concentrations occur in conservation equations, leading to measure-valued solutions, a key point to obtain existence and uniqueness in the sense of distributions is the definition of the flux and the corresponding velocity. This was already pointed out in [14], where duality solutions are defined for pressureless gases, and partially managed through conditions on the initial data. A more satisfactory solution came out in [26], since uniqueness was completely handled by a careful definition of the flux of the equation, or in other terms, the product $\widehat{a}_\rho\rho$. An analogous situation arising in plasma physics is considered in [25], around duality solutions as well. In a similar context, other definitions of the product can be found, see [33] in the one-dimensional setting, and [38] for a generalization in two space dimensions, where defect measures are used.

We explain in more details this point in the present context, in order to give a meaning to both duality and gradient flow solutions in the sense of distributions. As a rule, the product of $a(W' * \rho(t))$ by $\rho(t)$ is not well-defined when $\rho(t) \in \mathcal{M}_b(\mathbb{R})$. First we compute $W' * \rho$. We write $\rho = -\partial_x u$, so that $u \in BV(\mathbb{R})$. For such a function, we denote by S_u the set of $x \in \mathbb{R}$ where u does not admit an approximate limit, $|S_u| = 0$, and by $J_u \subset S_u$ the set of approximate jump points (see [1, Proposition 3.69]). We use the decomposition $\rho = -\partial_x^a u + \rho^c + \rho^j$, where $\partial_x^a u \ll \mathcal{L}$ is the regular part of the derivative, $\rho^j = \sum_{y\in J_u} \zeta_y \delta_y$ the jump part, and ρ^c the so-called Cantor part. The diffuse part of the derivative is defined as $\rho^d = -\partial_x^a u + \rho^c$. For $x \notin J_u$, we easily obtain

$$W' * \rho(x) = W' * \rho^d(x) + \sum_{y\in J_u} \zeta_y W'(x-y),$$

while if $x \in J_u$, the function is not defined. Indeed, letting $z \to x$, first with $z < x$, then with $z > x$, we obtain

$$W' * \rho(x^\pm) = -W' * \partial_x^a u(x) + \sum_{y\in J_u, y\neq x} \zeta_y W'(x-y) + \zeta_x W'(0^\pm). \tag{3.13}$$

Removing the indetermination amounts to define a velocity for a single Dirac mass located in x or equivalently for the center of mass of the density. Obviously, formula (3.7) sets this value to 0, hence a single Dirac mass is stationary, and the product by the measure ρ is meaningful. Therefore in the linear case we can consider the flux

$$J(t,x) := \widehat{a}_\rho(t,x)\,\rho(t,x).$$

Recall that this value is obtained by computing the element of minimal norm in the subgradient of the energy $\mathcal{W}$ corresponding to W.

On the other hand, in the nonlinear case, with $W'' = -\delta_0 + w$, the natural quantity to be defined is the flux J, by formula (3.11). To define the corresponding velocity, and give rigorous meaning to (3.10), we use the Vol'pert calculus for BV functions [42] (see also [1, Remark 3.98]): for a BV function u, the fonction $\widehat{a}_u$ defining the chain rule $\partial_x(A(u)) = \widehat{a}_u \partial_x u$ is constructed by

$$\widehat{a}_u(x) = \int_0^1 a(u_1(x) + (1-t)u_2(x))\,dt, \tag{3.14}$$

where

$$(u_1, u_2) = \begin{cases} (u, u) & \text{if } x \in \mathbb{R} \setminus S_u, \\ (u^+, u^-) & \text{if } x \in J_u, \\ \text{arbitrary} & \text{elsewhere.} \end{cases} \tag{3.15}$$

Now we apply that to $u = W' * \rho$, and obtain, using the antiderivative A of a,

$$\widehat{a}_u(x) = \begin{cases} a(W' * \rho(x)) & \text{if } x \in \mathbb{R} \setminus S_u, \\ \dfrac{A(W' * \rho(x^+)) - A(W' * \rho(x^-))}{W' * \rho(x^+) - W' * \rho(x^-)} & \text{if } x \in J_u, \\ \text{arbitrary} & \text{elsewhere.} \end{cases} \tag{3.16}$$

The connection with the linear case follows since then $A(v) = v^2/2$, hence

$$\frac{A(W' * \rho(x^+)) - A(W' * \rho(x^-))}{W' * \rho(x^+) - W' * \rho(x^-)} = \frac{W' * \rho(x^+) + W' * \rho(x^-)}{2}.$$

Therefore the undetermined term in (3.13) is replaced by $(W'(0^+) + W'(0^-))/2$, which vanishes since W is even, and we recover (3.7).

4 The linear case

By linear case we mean the case where $a = \mathrm{id}$ in (1.1). Together with assumptions of Definition 2.3, this is exactly the context of [18]. At first we prove Theorem 3.7, obtaining existence of duality solutions in Subsection 4.1, and uniqueness in Subsection 4.2. Next, in Subsection 4.3 we establish that they are equivalent to gradient flow solutions, thus answering the questions raised by Remark 2.16 of [18]. More precisely, we prove the following theorem.

Theorem 4.1 *Let $a = id$. Let us assume that W satisfies assumptions* **(A0)–(A4)** *and that $\rho^{ini} \in \mathcal{P}_2(\mathbb{R})$.*

(i) Let ρ be the duality solution as in Theorem 3.7. Then for all $t > 0$, $\rho(t) \in \mathcal{P}_2(\mathbb{R})$, $\rho \in AC^2_{loc}((0, +\infty); \mathcal{P}_2(\mathbb{R}))$ and ρ is the gradient flow solution as in Theorem 2.4.

(ii) If ρ is the gradient flow solution of Theorem 2.4, then it is a duality solution as in Theorem 3.7.

4.1 Existence of duality solutions

The first step is to verify that the velocity $\widehat{a}_\rho$ defined by (3.7) satisfies the OSL condition (3.4).

Lemma 4.2 *Let $\rho(t) \in \mathcal{M}_b(\mathbb{R})$ be nonnegative for all $t \geq 0$. Then under assumptions* **(A0) – (A4)** *the function $(t, x) \mapsto \widehat{a}_\rho(t, x)$ defined in* (3.7) *satisfies the one-sided Lipschitz estimate*

$$\widehat{a}_\rho(t, x) - \widehat{a}_\rho(t, y) \leq \lambda(x - y)|\rho|(\mathbb{R}), \quad \textit{for all } \ x > y, \ t \geq 0$$

Proof. By definition (3.7), we have

$$\widehat{a}_\rho(x) - \widehat{a}_\rho(y) = \int_{z\neq x, z\neq y} (W'(x-z) - W'(y-z))\rho(dz) + W'(x-y)\int_{z\in\{x\}\cup\{y\}} \rho(dz),$$

where we use the oddness of W' **(A0)** in the last term. Let us assume that $x > y$, from (2.9), we deduce that $W'(x-z) - W'(y-z) \le \lambda(x-y)$ and with (2.10), we deduce $W'(x-y) \le \lambda(x-y)$. Thus, using the nonnegativity of ρ, we deduce the one-sided Lipschitz (OSL) estimate for $\widehat{a}_\rho$. $\square$

Proof of the existence result in Theorem 3.7. This proof is split in several steps.

• **Aggregates**.

Consider first $\rho_n^{ini} = \sum_{i=1}^n m_i \delta_{x_i^0}$ where $x_1^0 < x_2^0 < \cdots < x_n^0$ and the m_i-s are nonnegative. We assume that $\sum_{i=1}^n m_i = 1$ and that $\sum_{i=1}^n m_i |x_i^0| < +\infty$ such that $\rho_n^{ini} \in \mathcal{P}_1(\mathbb{R})$. We look forward a solution $\rho_n(t,x) = \sum_{i=1}^n m_i \delta_{x_i(t)}$ in the distributional sense of the equation

$$\partial_t \rho + \partial_x(\widehat{a}_\rho \rho) = 0, \tag{4.17}$$

where $\widehat{a}_\rho$ is defined in (3.7). Let $u_n := \int^x \rho_n = \sum_{i=1}^n m_i H(x - x_i(t))$ where H is the Heaviside function. Then we have

$$-\partial_t u_n = \widehat{a}_{\rho_n}\rho_n = \sum_{i=1}^n m_i \sum_{j\neq i} m_j W'(x_i - x_j)\delta_{x_i}.$$

In fact,

$$\widehat{a}_{\rho_n}(x) = \begin{cases} \displaystyle\sum_{j\neq i} m_j W'(x_i - x_j) & \text{if } x = x_i,\ i = 1,\dots,n \\ \displaystyle\sum_{j=1}^n m_j W'(x - x_j) & \text{otherwise.} \end{cases} \tag{4.18}$$

From Lemma 4.2 and expression (4.18), we deduce that $\widehat{a}_{\rho_n}$ satisfies the OSL condition. Hence, there exists a unique Filippov flow [24] which is global in time thanks to assumption **(A4)**. Then the sequence $(x_i)_{i=1,\dots,n}$ satisfies the ODE system

$$x_i'(t) = \sum_{j\neq i} m_j W'(x_i - x_j), \qquad x_i(0) = x_i^0, \qquad i = 1,\dots,n_\ell, \tag{4.19}$$

where $n_\ell \le n$ is the number of distinct particles, i.e. $n_\ell = \#\{i \in \{1,\dots,N\}, x_i \neq x_j, \forall j\}$. Equation (4.19) should be understood as looking for absolute continuous solutions to the integral problem

$$x_i(t) = x_i^0 + \sum_{j\neq i}\int_0^t m_j W'(x_i(s) - x_j(s))\,ds. \tag{4.20}$$

Notice that (3.13) rewrites as

$$W' * \rho_n(x_i^\pm) = \sum_{j\neq i} m_j W'(x_i - x_j) + m_i W'(0^\pm).$$

We define the dynamics of aggregates as follows.

- the x_i-s are solutions of system (4.19) (where the right-hand side is zero if $n_\ell = 1$), when they are all distinct;

- at collisions, we define a sticky dynamics: if $x_i = x_j$ at time T_ℓ when for instance $i < j$, then the two aggregates collapse in a single one and we redefine system (4.19) by changing the number n_ℓ to $n_\ell - 1$, replacing the mass m_i by $m_i + m_j$ and deleting the point x_j. We denote by $0 := T_0 < T_1 < \dots T_k < \infty$ the times of collapse, where $k < n$.

This choice for the dynamics is clearly mass-conservative. Moreover, with this choice at the collisional times, we still have $x_i < x_j$ when $i < j$.

Thus we can define $\rho_n := \sum_{i=1}^{n_\ell} m_i \delta_{x_i}$. It is then straightforward by construction that ρ is a solution in the distributional sense of (4.17)–(3.7).

• **Finite first order moment**.

By construction, we clearly have that $\rho \geq 0$. Let us denote by $j_1^n(t) := \sum_{i=1}^{n_\ell} m_i |x_i(t)|$ the first order moment. We have $j_1(0) < +\infty$. Using (4.19), we compute

$$\frac{d}{dt} j_1^n(t) \leq \sum_{i=1}^{n_\ell} m_i \sum_{j \neq i} m_j |W'(x_i - x_j)| \leq C,$$

where we use **(A4)** and $\sum_i m_i = |\rho_n^{ini}|(\mathbb{R}) = 1$ for the last inequality. We deduce that there exists a nonnegative constant C such that for all $t \in [0, T]$,

$$j_1^n(t) \leq CT + j_1^n(0). \tag{4.21}$$

• **Duality solutions**.

From Lemma 4.2 and expression (4.18), we deduce that $\widehat{a}_{\rho_n}$ satisfies the OSL condition and is piecewise continuous outside the set of discontinuities $\{x_i\}_{i=1,\dots,n_\ell}$. Theorem 3.5 implies then that ρ_n is a duality solution for all $n \in \mathbb{N}^*$.

• **Passing to the limit** $n \to +\infty$.

Using the fact that $\rho_n(t) \in \mathcal{P}_1(\mathbb{R})$ for all $t \geq 0$ and from **(A4)**, we deduce that $\widehat{a}_{\rho_n}$ is bounded in $L^\infty((0,T)\times\mathbb{R})$ uniformly with respect to n. Thus, from point 4 of Theorem 3.4, we can extract a subsequence $\widehat{a}_n$ converging in $L^\infty((0,T)\times\mathbb{R}) - w\star$ towards $\widehat{a}$ and the corresponding sequence of duality solutions $(\rho_n)_n$ converges in $\mathcal{S}_\mathcal{M}$ towards ρ which is a duality solution to $\partial_t \rho + \partial_x(\widehat{a}\rho) = 0$. Moreover, since $\rho_n \rightharpoonup \rho$ in $\mathcal{S}_\mathcal{M}$, the formula (3.7) defining $\widehat{a}_\rho$ implies $\widehat{a}_{\rho_n} \to \widehat{a}_\rho$ a.e. (a proof of such result is postponed in Lemma A.1 in Appendix). Thus $\widehat{a} = \widehat{a}_\rho$ a.e., the flux $J_n(t,x) \rightharpoonup J(t,x) := \widehat{a}_\rho \rho$ in $\mathcal{M}_b(]0,T[\times\mathbb{R}) - w$ and the conservation equation (3.6) holds both in the duality and distributional sense. Finally passing to the limit $n \to +\infty$ in (4.21), we deduce that ρ has a bounded first order moment.

4.2 Uniqueness

Let ρ be a nonnegative duality solution which satisfies (3.6) in the distributional sense. As above, we denote by F the cumulative distribution function of ρ and by F^{-1} its generalized inverse. We have then by integration of (3.6)

$$\partial_t F + \widehat{a}_\rho \partial_x F = 0,$$

so that the generalized inverse is a solution to

$$\partial_t F^{-1}(t,z) = \widehat{a}_\rho(t, F^{-1}(z)). \tag{4.22}$$

Moreover thanks to a change of variables in (3.7),

$$\widehat{a}_\rho(t, F^{-1}(z)) = \int_{y\neq z} W'(F^{-1}(z) - F^{-1}(y))\, dy.$$

Proposition 4.3 *Assume $\rho_1(t,\cdot), \rho_2(t,\cdot) \in \mathcal{P}_1(\mathbb{R})$ satisfy (3.6) in the sense of distributions, with $\widehat{a}_{\rho_i}$ given by (3.7), and initial data ρ_1^{ini} and ρ_2^{ini}. Then we have, for all $t > 0$*

$$d_{W1}(\rho_1(t,\cdot), \rho_2(t,\cdot)) \leq e^{2\lambda t} d_{W1}(\rho_1^{ini}, \rho_2^{ini}).$$

Proof. Let F_i^{-1} denote the generalized inverse of ρ_i, $i = 1, 2$. From (4.22), we have

$$\begin{aligned}
\partial_t(F_1^{-1} - F_2^{-1}) &= \widehat{a}_{\rho_1}(t, F_1^{-1}(z)) - \widehat{a}_{\rho_2}(t, F_2^{-1}(z)) \\
&= \int_{y\neq z} \big(W'(F_1^{-1}(z) - F_1^{-1}(y)) - W'(F_2^{-1}(z) - F_2^{-1}(y))\, dy.
\end{aligned}$$

Multiplying the latter equation by $sign(F_1^{-1}(z) - F_2^{-1}(z))$ and integrating, we get

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \big|F_1^{-1}(z) - F_2^{-1}(z)\big|\, dz = \\
&\iint_{\{y\neq z\}} \big(W'(F_1^{-1}(z) - F_1^{-1}(y)) - W'(F_2^{-1}(z) - F_2^{-1}(y))\big)\mathrm{sign}\big(F_1^{-1}(z) - F_2^{-1}(z)\big)\, dzdy.
\end{aligned}$$

Using the oddness of W' and exchanging the role of y and z in the integral above, we also have

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \big|F_1^{-1}(z) - F_2^{-1}(z)\big|\, dz = \\
&-\iint_{\{y\neq z\}} \big(W'(F_1^{-1}(z) - F_1^{-1}(y)) - W'(F_2^{-1}(z) - F_2^{-1}(y))\big)\mathrm{sign}\big(F_1^{-1}(y) - F_2^{-1}(y)\big)\, dzdy.
\end{aligned}$$

Then we deduce

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \big|F_1^{-1}(z) - F_2^{-1}(z)\big|\, dz = \ &\frac{1}{2}\iint_{\{y\neq z\}} \big(W'(F_1^{-1}(z) - F_1^{-1}(y)) - W'(F_2^{-1}(z) - F_2^{-1}(y))\big)\times \\
&\big(\mathrm{sign}\big(F_1^{-1}(z) - F_2^{-1}(z)\big) - \mathrm{sign}\big(F_1^{-1}(y) - F_2^{-1}(y)\big)\big)\, dzdy.
\end{aligned} \tag{4.23}$$

The one-sided Lipschitz estimate for W' in (2.9) implies that the integrand in the right-hand side is bounded by

$$2\lambda\big|F_1^{-1}(z) - F_1^{-1}(y) - F_2^{-1}(z) + F_2^{-1}(y)\big|.$$

Hence, after an integration, we deduce

$$\frac{d}{dt}\int_0^1 |F_1^{-1} - F_2^{-1}|(z)\, dz \leq 2\lambda \int_0^1 |F_1^{-1} - F_2^{-1}|(z)\, dz.$$

Since $\|(F_1^{-1} - F_2^{-1})(t)\|_{L^1(0,1)} = d_{W1}(\rho_1, \rho_2)$, we conclude the proof by a Gronwall argument. $\square$

Proof of Theorem 3.7. The existence has been obtained in Section 4.1. Then if we have two duality solutions ρ_1 and ρ_2 as in Theorem 3.7, Proposition 4.3 implies that their generalized inverse are equal. Therefore $\rho_1 = \rho_2$. Finally, the second point of Theorem 3.4 allows to define the duality solution as the push-forward of ρ^{ini} by the backward flow. $\square$

4.3 Proof of Theorem 4.1

To cope with gradient flow solutions, we need first to prove that the second order moment is bounded provided $\rho^{ini} \in \mathcal{P}_2(\mathbb{R})$. We follow the idea of the proof of finite first order moment in subsection 4.1: we consider an approximation of ρ^{ini} by $\sum_{i=1}^{n} m_i \delta_{x_i^0}$ and build the corresponding duality solution $\rho_n(t,x) = \sum_{i=1}^{n_\ell} m_i \delta_{x_i(t)}$ where the dynamics of the nodes $\{x_i\}_{i=1,\dots,n_\ell}$ is given in (4.19). Let us denote by $j_2^n(t) := \sum_{i=1}^{n_\ell} m_i x_i^2(t)$ the second order moment. We have $j_2^n(0) < +\infty$. Using (4.19), we compute

$$\frac{d}{dt} j_2^n(t) = \sum_{i=1}^{n_\ell} 2 m_i x_i \sum_{j \neq i} m_j W'(x_i - x_j) \leq C \sum_{i=1}^{n_\ell} m_i |x_i| = C j_1^n(t).$$

where we use **(A4)** for the last inequality. Since j_1^n is uniformly bounded (see (4.21)), we deduce that j_2^n is uniformly bounded on $[0,T]$ by a constant only depending on T and on $j_2^n(0)$. Then we can pass to the limit $n \to +\infty$ to obtain a bound on $\int_{\mathbb{R}} |x|^2 \rho(t,dx)$ for any $t > 0$. Moreover, we deduce that the velocity field $\widehat{a}_\rho$ defined in (3.7) is bounded in $L^1((0,T); L^2(\rho))$.

Now, if ρ is a duality solution as in Theorem 3.7, then it satisfies (3.6) in the distributional sense. Using [2, Theorem 8.3.1] and the $L^1((0,T); L^2(\rho))$ bound on the velocity $\widehat{a}_\rho$, we deduce that $\rho \in AC^2_{loc}((0,+\infty); \mathcal{P}_2(\mathbb{R}))$. Thus ρ is a gradient flow solution (see [18] or [2, Sections 8.3 and 8.4]). This concludes the proof of point (i) of Theorem 4.1.

Conversely, if ρ is a gradient flow solution of Theorem 2.4, by uniqueness of both duality solutions and gradient flow solutions, we deduce that ρ is also a duality solution. $\square$

5 On the case $a \neq \mathrm{id}$

The situation $a \neq \mathrm{id}$ is not so favourable as the previous one, and one has to impose restrictions on the potential W. First we recall that attractivity implies that a is non-decreasing, see (3.8). Next, we need to assume that W has the following structure, see (3.9),

$$W'' = -\delta_0 + w, \qquad w \in Lip \cap L^\infty(\mathbb{R}).$$

With these assumptions, we are able to prove existence and uniqueness of duality solutions, Theorem 3.10, this is the aim of subsection 5.1. Next, in subsection 5.2, we turn to gradient flow, which are definitely not well suited for that case, since we have to restrict ourselves to $w = 0$ in the previous assumption on W.

5.1 Duality solutions

Here we prove Theorem 3.10, following the same strategy as in the linear case: first we prove the OSL condition, next establish the dynamics of aggregates, which leads to existence by approximation. Finally, uniqueness follows from a contraction principle in the space $\mathcal{P}_1$. In addition, we prove that duality solutions are absolutely continuous in time.

5.1.1 OSL condition

The first step consists in checking the OSL property for a.

Lemma 5.1 *Assume $0 \leq \rho \in \mathcal{M}_b(\mathbb{R})$ and that (3.8) holds, If Assumption 3.9 is satisfied, then the function $x \mapsto a(W' * \rho)$ satisfies the OSL condition (3.4).*

Proof. Using (3.9), we deduce that

$$\partial_{xx} W * \rho = -\rho + w * \rho.$$

Therefore,

$$\partial_x(a(\partial_x W * \rho)) = a'(\partial_x W * \rho)(-\rho + w * \rho) \leq a'(\partial_x W * \rho) w * \rho,$$

where we use the nonnegativity of ρ in the last inequality. Then from (3.8) we get

$$\partial_x(a(\partial_x W * \rho)) \leq \alpha \|\rho\|_{L^1} \|w\|_{L^\infty}.$$

It implies the OSL condition on the velocity. $\square$

5.1.2 Proof of the existence result in Theorem 3.10

Approximation by aggregates.

Following the idea in subsection 4.1, we first approximate the initial data ρ^{ini} by a finite sum of Dirac masses: $\rho_n^{ini} = \sum_{i=1}^n m_i \delta_{x_i^0}$ where $x_1^0 < x_2^0 < \cdots < x_n^0$ and the m_i-s are nonnegative. We assume that $\sum_{i=1}^n m_i = 1$ and $\sum_{i=1}^n m_i |x_i^0| < +\infty$, i.e. $\rho_n^{ini} \in \mathcal{P}_1(\mathbb{R})$. We look for a sequence $(\rho_n)_n$ solving in the distributional sense $\partial_t \rho_n + \partial_x J_n = 0$ where the flux J_n is given by (3.11). A function $\rho_n(t,x) = \sum_{i=1}^n m_i \delta_{x_i(t)}$ is such a solution provided the function u_n defined by

$$u_n(t,x) := \int^x \rho_n \, dx = \sum_{i=1}^n m_i H(x - x_i(t)), \tag{5.1}$$

where H denotes the Heaviside function, is a distributional solution to

$$\partial_t u_n - \partial_x \big(A(\partial_x W * \rho_n) \big) + a(\partial_x W * \rho_n) w * \rho_n = 0. \tag{5.2}$$

From (3.9), we deduce that

$$W'(x) = -H(x) + \widetilde{w}(x), \qquad \text{where } \widetilde{w}(x) = \int_0^x w(y)\, dy + \frac{1}{2}. \tag{5.3}$$

Then, we have

$$W' * \rho_n(x_i^+) = -\sum_{j=1}^{i} m_j + \sum_{j=1}^{n} m_j \widetilde{w}(x_i - x_j). \tag{5.4}$$

And

$$W' * \rho_n(x_i^-) = m_i + W' * \rho_n(x_i^+). \tag{5.5}$$

From these identities together with (3.16), straightforward computations show that in the distributional sense

$$\partial_x\big(A(W' * \rho_n)\big) = a(W' * \rho_n) w * \rho_n + \sum_{i=1}^{n} [A(W' * \rho_n)]_{x_i} \delta_{x_i}, \tag{5.6}$$

where $[f]_{x_i} = f(x_i^+) - f(x_i^-)$ is the jump of the function f at x_i. Injecting (5.1) and (5.6) in (5.2), we find

$$-\sum_{i=1}^{n} m_i x_i'(t) \delta_{x_i(t)} = \sum_{i=1}^{n} [A(W' * \rho_n)]_{x_i} \delta_{x_i}.$$

Thus it is a solution if we have

$$m_i x_i'(t) = -[A(W' * \rho_n)]_{x_i(t)}, \qquad \text{for } i = 1, \dots, n. \tag{5.7}$$

This system of ODEs is complemented by the initial data $x_i(0) = x_i^0$. Thus we are looking for absolute continuous solutions to the integral problem

$$x_i(t) = x_i^0 + \int_0^t \frac{[A(W' * \rho_n(s))]_{x_i(s)}}{[W' * \rho_n(s)]_{x_i(s)}} \, ds, \qquad i = 1, \dots n. \tag{5.8}$$

Then we define the dynamics of aggregates as in subsection 4.1 :

- When the x_i are all distinct, they are solutions of system (5.7) or equivalently (5.8) (with zero right hand side if $n_\ell = 1$, where we recall $n_\ell(t)$ is the number of distinct particles at time t).
- At collisions, we use we use the same sticky dynamics as above.

We recall that this choice of the dynamics implies mass conservation. As above, we have existence of the sequence $(x_i)_i$ satisfying (5.8) on $[0, T]$ with initial condition (x_i^0). Then we set $\rho_n(t, x) = \sum_{i=1}^{n_\ell} m_i \delta_{x_i(t)}(x)$. By construction, ρ_n is a solution in the sense of distribution of (3.12)-(3.11) for given initial data ρ_n^{ini}.

Finite first order moment.

As in subsection 4.1, we define $j_1^n(t) = \sum_{i=1}^{n_\ell} m_i |x_i(t)|$ and we compute

$$\frac{d}{dt} j_1^n(t) = \sum_{i=1}^{n_\ell} m_i \frac{x_i}{|x_i|} \frac{[A(W' * \rho_n)]_{x_i}}{[W' * \rho_n]_{x_i}},$$

where we use (5.8). From **(A4)** and the fact that a' is bounded (3.8), we deduce that $a(W' * \rho_n)$ is uniformly bounded. Moreover, since a is nondecreasing, A is a convex function, therefore the quantity $\dfrac{[A(W' * \rho_n)]_{x_i}}{[W' * \rho_n]_{x_i}}$ is uniformly bounded. Then we have

$$j_1^n(t) \le CT + j_1^n(0), \qquad \forall t \in [0, T], \tag{5.9}$$

where C stands for a generic nonnegative constant.

Existence of duality solutions

By the Vol'pert calculus recalled in Section 3.3, we have

$$J_n := -\partial_x(A(W' * \rho_n)) + a(W' * \rho_n)w * \rho_n = \widehat{a}_n \rho_n, \qquad \text{and} \qquad \widehat{a}_n = a(W' * \rho_n) \text{ a.e.}$$

Then ρ_n is a solution in the distributional sense of

$$\partial_t \rho_n + \partial_x(\widehat{a}_n \rho_n) = 0.$$

Moreover, by definition $a(W' * \rho_n)$ is piecewise continuous with the discontinuity lines defined by $x = x_i$, $i = 1, \ldots, n$, and by assumption 3.8 it is bounded in L^∞. We can apply Theorem 3.5 which gives that ρ_n is a duality solution and that $\widehat{a}_n$ is a universal representative of $a(W' * \rho_n)$. Then the flux is given by $a(W' * \rho_n)_\Delta \rho_n = J_n$.

General case.

Let us yet consider the case of any initial data $\rho^{ini} \in \mathcal{P}_1(\mathbb{R})$. We approximate ρ^{ini} by $\rho_n^{ini} = \sum_{i=1}^n m_i \delta_{x_i^0}$, $\rho_n^{ini} \in \mathcal{P}_1(\mathbb{R})$ with $\rho_n^{ini} \rightharpoonup \rho^{ini}$ in $\mathcal{M}_b(\mathbb{R})$. By the same token as above, we can construct a sequence of solutions $(\rho_n)_n$ with $\rho_n(t=0) = \rho_n^{ini} = \sum_{i=1}^n m_i \delta_{x_i^0}$, which solves in the sense of distributions

$$\partial_t \rho_n + \partial_x J_n = 0, \quad J_n = -\partial_x\big(A(\partial_x W * \rho_n)\big) + a(\partial_x W * \rho_n)w * \rho_n,$$

and which satisfies

$$\widehat{a}_n \rho_n = J_n, \quad \widehat{a}_n = a(W' * \rho_n) \text{ a.e.}$$

Moreover, since $W' * \rho_n$ is bounded in L^∞ uniformly with respect to n by construction and assumption **(A4)**, we can extract a subsequence of $(a(W' * \rho_n))_n$ that converges in $L^\infty - weak*$ towards b. Since from Lemma 5.1, $a(W' * \rho_n)$ satisfies the OSL condition, we deduce from Theorem 3.4 4) that, up to an extraction, $\rho_n \rightharpoonup \rho$ in $\mathcal{S}_\mathcal{M}$ and $\widehat{a}_n \rho_n \rightharpoonup \widehat{a}\rho$ weakly in $\mathcal{M}_b(]0, T[\times\mathbb{R})$, ρ being a duality solution of the scalar conservation law with coefficient b. Then $J_n \to J := -\partial_x(A(W' * \rho)) + a(W' * \rho)w * \rho$ in $\mathcal{D}'(\mathbb{R})$ and that $a(W' * \rho_n) \to a(W' * \rho)$ a.e. By uniqueness of the weak limit, we have $b = a(W' * \rho)$. Moreover $J = \widehat{a}\rho$ a.e. and ρ satisfies then (3.12). Finally, we recover the bound on the first order moment by passing to the limit $n \to +\infty$ in the estimate (5.9). $\square$

Remark 5.2 *Let us consider the case studied in the previous Section : $a = id$ and W is even. Since W' is odd, then* (5.3) *rewrites*

$$W'(x) = -H(x) + w_0(x), \qquad \textit{where} \;\; w_0(x) = \int_0^x w(y)\,dy + \frac{1}{2}.$$

When $a = id$, we have $A(x) = \frac{1}{2}x^2$. Then system (5.7) *rewrites*

$$m_i x_i'(t) = -\frac{1}{2}(W' * \rho_n(x_i^+) - W' * \rho_n(x_i^-))(W' * \rho_n(x_i^+) + W' * \rho_n(x_i^-)).$$

Then, from (5.4) *and* (5.5), *we have*

$$x_i'(t) = -\sum_{j=1}^{i-1} m_j - \frac{m_i}{2} + \sum_{j=1}^{n} m_j w_0(x_i - x_j) = -\sum_{j=1}^{i-1} m_j + \sum_{j\neq i} m_j w_0(x_i - x_j).$$

From the expression of W' above, we deduce that $x_i'(t) = \sum_{j\neq i} m_j W'(x_i - x_j)$ and we recover the dynamical system (4.19) *of the previous Section.*

Remark 5.3 *The dynamical system* (5.7) *defines actually the macroscopic velocity. Indeed, if we formally take the limit $n \to +\infty$ of the right-hand side of* (5.8)*, this latter term converges towards the velocity $\widehat{a}_u$ defined by the chain rule* (3.16)

5.1.3 Uniqueness.

We first notice that the strategy used in subsection 4.2 cannot be used here, since it strongly relies on the linearity of a. Then we have to use the approach proposed in [26] which uses an entropy estimate. The key point is to observe that the quantity $W' * \rho$ solves a scalar conservation laws with source term.

Proposition 5.4 (Entropy estimate) *Let us assume that Assumptions 3.9 and* (3.8) *hold. For $T > 0$, let $\rho \in C([0,T], \mathcal{P}_1(\mathbb{R}))$ satisfying in the distribution sense* (3.12)-(3.11)*. Then $u := W' * \rho$ is a weak solution of*

$$\partial_t u + \partial_x A(u) = a(u) w * \rho + \partial_x (w * A(u)) - w * (a(u) w * \rho). \tag{5.10}$$

Moreover, if we assume that the entropy condition

$$\partial_x u \leq w * \rho \tag{5.11}$$

holds, then for any twice continuously differentiable convex function η, we have

$$\partial_t \eta(u) + \partial_x q(u) - \eta'(u) a(u) w * \rho + \eta'(u)\big(\partial_x (w * A(u)) - w * (a(u) w * \rho)\big) \leq 0, \tag{5.12}$$

where the entropy flux is given by $q(x) = \displaystyle\int_0^x \eta'(y) a(y)\, dy$.

Proof. Equation (5.10) is obtained by taking the convolution product of (3.12) with W'. The entropy inequality is then a straightforward adaptation of the proof of Lemma 4.5 of [26]. $\square$

We turn now to the proof of the uniqueness. Once again, we use the idea developed in [26] and extend it to the case at hand. Consider two solutions ρ_1 and ρ_2 such as in Theorem 3.10. We denote $u_1 := W' * \rho_1$ and $u_2 = W' * \rho_2$. Starting from the entropy inequality (5.12) with the family of Kružkov entropies $\eta_\kappa(u) = |u - \kappa|$ and using the doubling of variable technique developed by Kružkov, we obtain as in the proof of Theorem 5.1 of [26]

$$\frac{d}{dt}\int_{\mathbb{R}} \big|u_1 - u_2\big| \leq \|w\|_{Lip} \int_{\mathbb{R}} \big|A(u_1) - A(u_2)\big|\, dx + \big(1 + \|w\|_\infty\big) \int_{\mathbb{R}} \big|a(u_1) w * \rho_1 - a(u_2) w * \rho_2\big|\, dx.$$

From **(A4)** and the bound of $\rho(t)$ in $\mathcal{P}_1(\mathbb{R})$ for all t, we deduce that u_i, $i = 1, 2$ are bounded in $L^\infty_{t,x}$. Then we get

$$\frac{d}{dt}\int_{\mathbb{R}} \big|u_1 - u_2\big| \leq C\Big(\int_{\mathbb{R}} \big|u_1 - u_2\big|\, dx + \int_{\mathbb{R}} \big|w * \rho_1 - w * \rho_2\big|\, dx\Big), \tag{5.13}$$

where we use moreover (3.8). Taking the convolution with w of equation (3.12) we deduce

$$\partial_t w * \rho_i - \partial_x\big(w * A(u_i)\big) + w * \big(a(u_i) w * \rho_i\big) = 0, \qquad i = 1, 2.$$

We deduce from (3.8) and the Lipschitz bound of w that

$$\frac{d}{dt}\int_{\mathbb{R}} \big|w * \rho_1 - w * \rho_2\big| \leq C\Big(\int_{\mathbb{R}} \big|u_1 - u_2\big|\,dx + \int_{\mathbb{R}} \big|w * \rho_1 - w * \rho_2\big|\,dx\Big). \tag{5.14}$$

Adding (5.13) and (5.14), we deduce applying a Gronwall lemma that $u_1 = u_2$ and $w*\rho_1 = w*\rho_2$, which implies $\rho_1 = \rho_2$. □

Remark 5.5 *We point out that the entropy condition* (5.11) *is equivalent to* $\rho \geq 0$*, which is required since* ρ *is supposed to be a density. As a consequence, if we allow* ρ *to be nonpositive, uniqueness of solutions is not guaranteed (see Section 5.3 of [26] for a counter-example of uniqueness in the case* $W(x) = \frac{1}{2}e^{-|x|}$*).*

5.1.4 Absolute continuity.

Following subsection 4.3, we have the following result:

Proposition 5.6 *Under the assumptions of Theorem 3.10, if moreover* $\rho^{ini} \in \mathcal{P}_2(\mathbb{R})$*, then the duality solution of Theorem 3.10 satisfies* $\rho \in AC^2_{loc}((0, +\infty); \mathcal{P}_2(\mathbb{R}))$*.*

Proof. The proof of the finite second order moment follows straightforwardly the one for the first order moment in subsection 5.1.2. Then from (3.8) and **(A4)**, we have that $a(W' * \rho)$ is uniformly bounded in L^∞. Therefore the velocity field $\widehat{a}$ is bounded in $L^1((0,T); L^2(\rho))$. Moreover ρ is a solution in the distributional sense of $\partial_t \rho + \partial_x(\widehat{a}\rho) = 0$. We conclude then from Theorem 8.3.1 of [2] that $\rho \in AC^2_{loc}((0, +\infty); \mathcal{P}_2(\mathbb{R}))$. □

5.2 Gradient flows

When $a \neq \mathrm{id}$, equation (1.1) does not have a structure of a gradient. Moreover, we are not able to determine a conserved energy corresponding to the system. Nevertheless, in the particular case $W(x) = -\frac{1}{2}|x|$, we are able to adapt the technique of [18] to recover the existence of gradient flow solutions. Before inroducing the energy functional $\mathcal{W}$ for this case, let us first recall the observation of Section 3.3. Denoting A an antiderivative of a, we have from the Vol'pert calculus

$$\partial_x(A(u)) = \widehat{a}_u(x)\partial_x u, \quad \text{where } u(x) = \int_{y\neq x} W'(x-y)\rho(dy) = \frac{1}{2}\big(\rho(x,+\infty) - \rho(-\infty, x)\big). \tag{5.15}$$

The function $\widehat{a}_u$ is defined in (3.16) and we recall that when $a = id$, $\widehat{a}_u = u$. Then, we define for $\rho \in \mathcal{P}_2(\mathbb{R})$ the functional

$$\mathcal{W}(\rho) = -\int_{\mathbb{R}} x\widehat{a}_u(x)\rho(dx), \qquad u = W' * \rho. \tag{5.16}$$

We first verify that when $a = \text{id}$ this energy is equal to the one introduced in (2.5). In fact, we have for $W(x) = -\frac{1}{2}|x|$,

$$\frac{1}{4}\int_{\mathbb{R}^2} |x-y|\rho(dx)\rho(dy) = \frac{1}{4}\int_{\mathbb{R}}\int_{-\infty}^{x}(x-y)\rho(dy)\rho(dx) - \frac{1}{4}\int_{\mathbb{R}}\int_{x}^{+\infty}(x-y)\rho(dy)\rho(dx). \quad (5.17)$$

Repeated use of Fubini's Theorem in the last term of the right hand side leads to

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{R}^2} |x-y|\rho(dx)\rho(dy) &= \frac{1}{2}\int_{\mathbb{R}}\int_{-\infty}^{x}(x-y)\rho(dy)\rho(dx) \\ &= \frac{1}{2}\int_{\mathbb{R}} x\int_{-\infty}^{x}\rho(dy)\rho(dx) - \frac{1}{2}\int_{\mathbb{R}}\int_{y}^{+\infty}\rho(dx)y\rho(dy).\end{aligned}$$

With the definition of u in (5.15), we have

$$\frac{1}{4}\int_{\mathbb{R}^2}|x-y|\rho(dy)\rho(dx) = -\int_{\mathbb{R}} xu(x)\rho(dx),$$

which concludes the proof.

We are able to prove in this case the following Theorem.

Theorem 5.7 *Let $W(x) = -\frac{1}{2}|x|$ and a satisfy assumption* (3.8)*. Let $\rho^{ini} \in \mathcal{P}_2(\mathbb{R})$ be given.*

(i) There exists a unique gradient flow solution $\rho \in AC^2_{loc}([0,+\infty),\mathcal{P}_2(\mathbb{R}))$ in the sense of Definition 2.1. Therefore ρ satisfies in the distributional sense

$$\partial_t\rho + \partial_x(\widehat{a}_u\rho) = 0, \qquad \text{with} \qquad \rho(0) = \rho^{ini},$$

where $u(x) = W' * \rho(x) = \frac{1}{2}(\rho(x,+\infty) - \rho(-\infty,x))$ and $\widehat{a}_u$ is defined in (3.16)*. Moreover, this solution is unique and we have the energy estimate : for all $0 \le t_0 \le t_1 < \infty$,*

$$\int_{t_0}^{t_1} |\widehat{a}_u(t,x))|^2\,\rho(t,dx)\,dt + \mathcal{W}(\rho(t_1)) = \mathcal{W}(\rho(t_0)).$$

(ii) The duality solution of Theorem 3.10 satisfies $\rho(t) \in \mathcal{P}_2(\mathbb{R})$ for all $t \ge 0$ and coincides with the gradient flow solution of the first item. Moreover we have $\rho = -\partial_x u$ where u is the unique entropy solution of the scalar conservation law

$$\partial_t u + \partial_x A(u) = 0, \qquad u(0,x) = W' * \rho^{ini},$$

where A is an antiderivative of a.

Notice that the equivalence between entropy solutions and gradient flow solutions (item (ii)) has been observed independently in [12] in the linear case $a = Id$ and for $W(x) = \pm|x|$ (including then a repulsive case).

Proof. (i) We use the ideas of Section 2 of [18] recalled in the beginning of this paper, Section 2.2. The proof is divided into several steps.

We first notice that due to the one dimensional framework, we can simplify the computations by working in the Hilbert space $L^2(-\frac{1}{2},\frac{1}{2})$. In fact, the function $u = W' * \rho$ is, up to a constant, the cumulative distribution of $-\rho$, since $\rho = -\partial_x u$. Using the definition (3.16), since $\widehat{a}_u$ is arbitrary on $S_u \setminus J_u$, we have :

$$\int_{\mathbb{R}} x\widehat{a}_u \rho(dx) = \int_{\mathbb{R}\setminus J_u} x a(u(x))\rho(dx) + \sum_{x\in J_u} x \frac{A(u(x^+)) - A(u(x^-))}{u(x^+) - u(x^-)}(u(x^+) - u(x^-)).$$

We introduce the generalized inverse v of u whose definition is

$$v(t,z) = u^{-1}(t,z) := \inf\{x \in \mathbb{R}/u(t,x) > z\}.$$

Since u is nonincreasing, v is nonincreasing. For $x \in J_u$, we denote $z_x^- = u(x^-)$ and $z_x^+ = u(x^+)$. We deduce after a change of variable that

$$\mathcal{W}(\rho) = -\int_{(-\frac{1}{2},\frac{1}{2})\setminus u(J_u)} v(t,z)a(z)dz - \sum_{x\in J_u} v(t,z)(A(z_x^+) - A(z_x^-)).$$

By definition, the function v is constant on the set $u(J_u)$. We deduce, recalling that A is an antiderivative of a,

$$\sum_{x\in J_u} v(t,z)(A(z_x^+) - A(z_x^-)) = \sum_{x\in J_u} \int_{z_x^-}^{z_x^+} v(t,z)a(z)dz.$$

Thus, we can rewrite the functional $\mathcal{W}$ as

$$\mathcal{W}(\rho) = \widetilde{\mathcal{W}}(v) := -\int_{-1/2}^{1/2} v(t,z)a(z)\,dz. \tag{5.18}$$

We define moreover $v^{ini} = (u^{ini})^{-1}$, where $u^{ini} = \frac{1}{2}\big(\rho^{ini}(x,+\infty) - \rho^{ini}(-\infty,x)\big)$.

• **$-\widehat{a}_u$ is the unique element of minimal $L^2(\rho)$-norm in the subdifferential $\partial\mathcal{W}$ of $\mathcal{W}$.**

Let $\rho \in \mathcal{P}_2(\mathbb{R})$ and $u = \frac{1}{2}(\rho(x,+\infty) - \rho(-\infty,x))$. As above, we denote v the pseudo-inverse of u. We first show that $-\widehat{a}_u \in \partial\mathcal{W}(\rho)$, where $\widehat{a}_u$ is defined in (3.16). From [2, Definition 10.3.1] (see also equation (10.3.12) of the same book), it means that for all μ in $\mathcal{P}_2(\mathbb{R})$, we have

$$\mathcal{W}(\mu) - \mathcal{W}(\rho) \geq \inf_{\gamma\in\Gamma_0(\rho,\mu)} \int_{\mathbb{R}} -\widehat{a}_u(x)(y-x)\,\gamma(dx,dy) + o(d_{W2}(\mu,\rho)). \tag{5.19}$$

For $\mu \in \mathcal{P}_2(\mathbb{R})$ we denote $u_\mu = W' * \mu$ and v_μ its pseudo-inverse. As for (5.18) with μ instead of ρ, we deduce :

$$\mathcal{W}(\mu) - \mathcal{W}(\rho) = -\int_{-1/2}^{1/2} a(z)\big(v_\mu(z) - v(z)\big)\,dz. \tag{5.20}$$

We have recalled in Subsection 2.3 that in one dimension, the set of optimal map is given by $\Gamma_0(\rho,\mu) = \{(v,v_\mu)_\#\mathbb{L}_{(-1/2,1/2)}\}$. Therefore we have that for $\gamma \in \Gamma_0(\rho,\mu)$,

$$\int_{\mathbb{R}} -\widehat{a}_u(x)(y-x)\,\gamma(dx,dy) = -\int_{-1/2}^{1/2} \widehat{a}_u(v(z))\big(v_\mu(z) - v(z)\big)\,dz.$$

Let us consider then the quantity

$$R := \int_{-1/2}^{1/2} \big(\widehat{a}_u(v(z)) - a(z)\big)\big(v_\mu(z) - v(z)\big)\, dz.$$

We have from (5.20)

$$\mathcal{W}(\mu) - \mathcal{W}(\rho) = -\int_{-1/2}^{1/2} \widehat{a}_u(v(z))\big(v_\mu(z) - v(z)\big)\, dz + R. \tag{5.21}$$

Using definition (3.16), we have that on $(-\frac{1}{2}, \frac{1}{2}) \setminus u(J_u)$, $\widehat{a}_u(v(z)) = a(z)$. For $x \in J_u$, we denote $z_x^- = u(x^-)$ and $z_x^+ = u(x^+)$, we have

$$\forall\, z \in (z_x^-, z_x^+), \quad \widehat{a}_u(v(z)) = \frac{A(z_x^+) - A(z_x^-)}{z_x^+ - z_x^-} = \frac{1}{z_x^+ - z_x^-} \int_{z_x^-}^{z_x^+} a(y)\, dy.$$

Therefore, we can rewrite

$$R = \sum_{x \in J_u} \frac{1}{z_x^+ - z_x^-} \int_{z_x^-}^{z_x^+} \int_{z_x^-}^{z_x^+} \big(a(y) - a(z)\big)\big(v_\mu(z) - v(z)\big)\, dydz.$$

Hence, if $v_\mu - v$ is piecewise constant on (z_x^-, z_x^+) for each $x \in J_u$, we have that $R = 0$. By definition, for each $x \in J_u$, we have that v is constant on (z_x^-, z_x^+). Let $(v_\mu^n)_{n\in\mathbb{N}}$ be a sequence of approximation of v_μ such that v_μ^n is piecewise constant on (z_x^-, z_x^+) for all $x \in J_u$ and converge in $L^2(-\frac{1}{2}, \frac{1}{2})$ and a.e. towards v_μ. Moreover v_μ^n is the monotone rearrangement of $\mu_n = -\partial_x u_\mu^n$ where $u_\mu^n(x) = \sup\{z \in (-\frac{1}{2}, \frac{1}{2}) / v_\mu^n(z) < x\}$. We deduce from the above discussion and from (5.21) that

$$\mathcal{W}(\mu_n) - \mathcal{W}(\rho) = -\int_{-1/2}^{1/2} \widehat{a}_u(v(z))\big(v_\mu^n(z) - v(z)\big)\, dz.$$

Using the Fatou Lemma, we conclude by letting $n \to +\infty$

$$\mathcal{W}(\mu) - \mathcal{W}(\rho) \geq -\int_{-1/2}^{1/2} \widehat{a}_u(v(z))\big(v_\mu(z) - v(z)\big)\, dz$$

To prove that $-\widehat{a}_u$ is an element of minimal norm in $\partial\mathcal{W}(\rho)$, we consider $\xi \in C^\infty \cap Lip(\mathbb{R})$ and for $\varepsilon > 0$ small enough such that $(id + \varepsilon\xi)$ is increasing. We have

$$\mathcal{W}((id + \varepsilon\xi)_\# \rho) = -\int_{\mathbb{R}} (id + \varepsilon\xi)(x)\widehat{a}_{u_\varepsilon}(x + \varepsilon\xi(x))\rho(dx),$$

where $u_\varepsilon = W' * ((id + \varepsilon\xi)_\#\rho)$. We notice that for any increasing function θ, we have

$$\begin{aligned} W' * (\theta_\#\rho)(\theta(x)) &= \frac{1}{2}\Big(\rho\big(\theta^{-1}(\theta(x), +\infty)\big) - \rho\big(\theta^{-1}(-\infty, \theta(x))\big)\Big) \\ &= \frac{1}{2}\Big(\rho\big(\{y \in \mathbb{R}/\theta(y) > \theta(x)\}\big) - \rho\big(\{y \in \mathbb{R}/\theta(y) < \theta(x)\}\big)\Big). \end{aligned}$$

Due to the monotonicity of the function θ, we have $\{y \in \mathbb{R}/\theta(y) < \theta(x)\} = \{y \le x\}$. Hence $W' * (\theta_\# \rho)(\theta(x)) = W' * \rho(x)$. We deduce

$$\mathcal{W}((id + \varepsilon\xi)_\#\rho) = -\int_{\mathbb{R}} (id + \varepsilon\xi)(x)\widehat{a}_u(x)\rho(dx).$$

Thus,

$$\lim_{\varepsilon\to 0} \frac{\mathcal{W}((id+\varepsilon\xi)_\#\rho) - \mathcal{W}(\rho)}{\varepsilon} = -\int_{\mathbb{R}} \widehat{a}_u(x)\xi(x)\, d\rho(x).$$

Then, from the definition of the slope (2.3), we have

$$\liminf_{\varepsilon\searrow 0} \frac{\mathcal{W}((id+\varepsilon\xi)_\#\rho) - \mathcal{W}(\rho)}{d_{W2}((id+\varepsilon\xi)_\#\rho, \rho)} \ge -|\partial\mathcal{W}|(\rho).$$

We deduce from above that

$$\int_{\mathbb{R}} \widehat{a}_u(x)\xi(x)\, d\rho(x) \le |\partial\mathcal{W}|(\rho) \liminf_{\varepsilon\searrow 0} \frac{d_{W2}((id+\varepsilon\xi)_\#\rho, \rho)}{\varepsilon} \le |\partial\mathcal{W}|(\rho)\|\xi\|_{L^2(\rho)},$$

where we use the inequality $d_{W2}(S_\#\rho, T_\#\rho) \le \|S - T\|_{L^2(\rho)}$ for the last inequality. By the same token with $-\xi$ instead of ξ, we deduce

$$\Big| \int_{\mathbb{R}} \widehat{a}_u(x)\xi(x)\, d\rho(x)\Big| \le |\partial\mathcal{W}|(\rho)\|\xi\|_{L^2(\rho)}.$$

Since ξ is arbitrary, we get $\|\widehat{a}_u\|_{L^2(\rho)} \le |\partial\mathcal{W}|(\rho)$. We conclude by using (2.4).

• **Well-posedness and convergence of the JKO scheme.**

The JKO scheme is defined in (2.8) where $\tau > 0$ is a given small time step. Thus, using (2.11) and (5.18), the minimization problem defined in (2.8) is equivalent to : let $v_k^\tau \in L^2(-\frac{1}{2}, \frac{1}{2})$, we define

$$v_{k+1}^\tau = \operatorname*{arg\,min}_{v\in L^2(-1/2,1/2), \partial_z v \le 0} \Big\{ \widetilde{\mathcal{W}}(v) + \frac{1}{2\tau}\|v - v_k^\tau\|^2 \Big\}. \tag{5.22}$$

We recall that a is a nondecreasing function, then $a(-\frac{1}{2}) \le a(z) \le a(\frac{1}{2})$ when $z \in (-\frac{1}{2}, \frac{1}{2})$. Then the functional defined inside the brackets is clearly lower semi-continuous, convex and coercive on $L^2(-\frac{1}{2}, \frac{1}{2})$. We deduce that the above minimization problem (5.22) admits an unique solution. Moreover, computing the Fréchet derivative of the functional defining (5.22) in the L^2-metric, the Euler-Lagrange equation associated to this minimization problem implies

$$v_{k+1}^\tau(z) = v_k^\tau(z) - \tau a(z), \qquad z \in \big(-\frac{1}{2}, \frac{1}{2}\big).$$

This is an implicit Euler discretization of the equation $\partial_t v(t, z) + a(z) = 0$. The function a being nondecreasing, it is clear that if v_k^τ is nonincreasing, then v_{k+1}^τ is nonincreasing. It is well known that the piecewise constant interpolation v^τ defined by $v^\tau(0) = v^{ini}$ and $v^\tau(t) = v_k^\tau$ if $t \in (k\tau, (k+1)\tau]$, converges in $L^2(-\frac{1}{2}, \frac{1}{2})$ as $\tau \to 0$ towards $v(t)$ for all $t \in [0, +\infty)$, where $v(t, z) = v^{ini} - ta(z)$. Moreover, we have the energy estimate

$$\widetilde{\mathcal{W}}(v) = \widetilde{\mathcal{W}}(v^{ini}) - \int_0^t \int_{-1/2}^{1/2} |a(z)|^2\, dzds.$$

We define then $\rho := -\partial_x u$, where $u(t,x) := v^{-1}(t,z) = \sup\{z \in [-\frac{1}{2}, \frac{1}{2}] / \ v(t,z) < x\}$. We have $\rho \in \mathcal{P}_2(\mathbb{R})$, and decomposing the integral between the regular part and the jump part as above, we state that for $s \in (0,t)$,

$$\int_{-1/2}^{1/2} |a(z)|^2 \, dz = \int_{\mathbb{R}} |\widehat{a}_u(s,x))|^2 \rho(s,dx).$$

We deduce from the latter energy estimate,

$$\mathcal{W}(\rho) = \mathcal{W}(\rho^{ini}) - \int_0^t \int_{\mathbb{R}} |\widehat{a}_u(s,x))|^2 \rho(s,dx). \tag{5.23}$$

Moreover, we have from the equation $\partial_t v(t,z) + a(z) = 0$ that the generalized inverse u solves in the weak sense $\partial_t u + \widehat{a}_u \partial_x u = 0$. Therefore, in the sense of distributions, we have

$$\partial_t \rho + \partial_x(\widehat{a}_u \rho) = 0.$$

From Theorem 8.3.1 of [2], we deduce that $\rho \in AC^2_{loc}((0,+\infty); \mathcal{P}_2(\mathbb{R}))$. We conclude, using moreover (5.23) that ρ is a curve of maximal slope for the functional $\mathcal{W}$ defined in (5.16).

• **Gradient flow solutions.**

This last step is a direct consequence of Theorem 11.1.3 of [2] since all assumptions of the Theorem have been verified above. Thus curves of maximal slope are gradient flow solutions. Uniqueness is obtained thanks to a contraction estimate based on a Gronwall argument as in [18]. This concludes the proof of the first item of Theorem 5.7.

(ii) From Proposition 5.6, we deduce that the duality solution of Theorem 3.10 belongs to $AC^2_{loc}((0,+\infty); \mathcal{P}_2(\mathbb{R}))$. Then by uniqueness of duality solutions and gradient flow solutions, we deduce that both notion of solutions coincide. Moreover, from Proposition 5.4 with $w = 0$, equation (5.10) reduces to

$$\partial_t u + \partial_x A(u) = 0.$$

It is well-known that this scalar conservation law admits an unique nonincreasing solution which is the entropy solution (see e.g. [14, Lemma 3.3]). Then $\rho = -\partial_x u$. $\square$

Appendix

Technical Lemma

In this appendix we state a technical lemma which is used in the paper.

Lemma A.1 *Let us assume that W satisfies assumptions* **(A0)–(A4)**. *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of measures such that $\rho_n \rightharpoonup \rho$ weakly in $\mathcal{M}_b(\mathbb{R})$. Then*

$$\lim_{n\to+\infty}\int_{x\neq y} W'(x-y)\rho_n(dy) = \int_{x\neq y} W'(x-y)\rho(dy), \qquad \textit{for a.e. } x\in\mathbb{R}.$$

Proof. We consider a regularization of W by W_ε such that for all $\varepsilon > 0$, $W_\varepsilon \in C^1(\mathbb{R}^d)$, $W_\varepsilon(-x) = W_\varepsilon(x)$, W_ε and W'_ε uniformly bounded with respect to ε, and

$$\sup_{x\in\mathbb{R}\setminus(-\varepsilon,\varepsilon)} |W'_\varepsilon(x) - W'(x)| \leq \varepsilon. \tag{A.1}$$

By definition of the weak convergence, we have

$$\lim_{n\to+\infty}\int_{x\neq y} W'_\varepsilon(x-y)\rho_n(dy) = \int_{x\neq y} W'_\varepsilon(x-y)\rho(dy), \qquad \text{for a.e. } x\in\mathbb{R}. \tag{A.2}$$

In fact, we can remove the point $y = x$ in the integral since by construction W'_ε is odd, then $W'_\varepsilon(0) = 0$. Moreover for all $n\in\mathbb{N}$, we have that

$$\begin{aligned}\Big|\int_{x\neq y}(W'_\varepsilon - W')(x-y)\rho_n(dy)\Big| = \ & \Big|\int_{(x-\varepsilon,x+\varepsilon)\setminus\{x\}}(W'_\varepsilon - W')(x-y)\rho_n(dy)\Big| \\ & + \Big|\int_{\mathbb{R}\setminus(x-\varepsilon,x+\varepsilon)}(W'_\varepsilon - W')(x-y)\rho_n(dy)\Big|.\end{aligned}$$

We can bound the first term of the right hand side by $C\rho_n((x-\varepsilon, x+\varepsilon)\setminus\{x\}) \to 0$ when $\varepsilon \to 0$, where C stand for a nonnegative constant. For the second term, we use the property (A.1) to state the convergence towards 0 as $\varepsilon \to 0$. We deduce that

$$\lim_{\varepsilon\to 0}\int_{x\neq y} W'_\varepsilon(x-y)\rho_n(dy) = \int_{x\neq y} W'(x-y)\rho_n(dy),$$

and by the same token with ρ instead of ρ_n, we have

$$\lim_{\varepsilon\to 0}\int_{x\neq y} W'_\varepsilon(x-y)\rho(dy) = \int_{x\neq y} W'(x-y)\rho(dy).$$

We conclude by passing into the limit $\varepsilon \to 0$ in (A.2). □

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