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Weak solutions for one-dimensional aggregation equations

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Abstract

Existence and uniqueness of global in time measure solution for a one dimensional non-linear aggregation equation is considered. Such a system can be written as a conservation law with a velocity field computed through a selfconsistent interaction potential. Blow up of regular solutions is now well established for such system. In Carrillo et al. (Duke Math J (2011)) [14], a theory of existence and uniqueness based on the geometric approach of gradient flows on Wasserstein space has been developed. We propose in this work to establish the link between this approach and duality solutions. This latter concept of solutions allows in particular to define a flow associated to the velocity field. Then an existence and uniqueness theory for duality solutions is developed in the spirit of James and Vauchelet (NoDEA (2013)) [21]. However, since duality solutions are only known in one dimension, we restrict our study to the one dimensional case.

Keywords: duality solutions, nonlocal conservation equations, measure-valued solutions, gradient flow, optimal transport.

2010 AMS subject classifications: 35B40, 35D30, 35L60, 35Q92, 49K20.

1 Introduction

Aggregation phenomena of a population of particles interacting under a continuous interaction potential are modelled by a nonlocal nonlinear conservation equation. Letting ρ denote the density of cells, the so-called aggregation equation in N space dimension writes

$$\partial_t \rho + \nabla_x \cdot (a(\nabla_x W * \rho)\rho) = 0, \quad t > 0, \quad x \in \mathbb{R}^N, \quad (1.1)$$

and is complemented with the initial condition $\rho(0, x) = \rho^{ini}$. Here $W : \mathbb{R}^N \rightarrow \mathbb{R}$ is the interaction potential, and $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a smooth given function which depends on the actual model under consideration.

This equation is involved in many applications in physics and biology. In the framework of granular media [3, 15, 25], a is the identity function, and interaction potentials are in the form $W(x) = -|x|^\alpha$ with $\alpha > 1$. In plasma physics, the context is the high field limit of a kinetic equation describing the dynamics of electrically charged Brownian particles interacting with a thermal bath. This leads to consider potentials in the form $W(x) = -|x|$, and $a = \text{id}$ as well (see e.g. [27]). Also, continuum mathematical models have been widely proposed to model collective behaviour of individuals. Then the potential W is typically the fundamental solution of some elliptic equation, and a depends on the microscopic behaviour of the individuals. We refer for instance to the well-known Patlak-Keller-Segel model where aggregation of cells is described by a macroscopic non-local interaction equation with linear diffusion [24, 30]. More precisely, the swarming of cells can be described by aggregation equations where the typical interaction potential is the attractive Morse potential $W(x) = \frac{1}{2}e^{-|x|}$ [13, 26, 29]. Such potentials also appear when considering the hydrodynamic limit of kinetic model describing chemotaxis of bacteria [16, 17, 20].

All the potentials mentioned above have a singularity at the origin, they fall into the context of “pointy potentials” (see a precise definition below), and it is well-known that in that case concentration phenomena induce the blow-up in finite time of weak L^p solutions [4, 5, 6]. Thus the notion of solution breaks down at the blow-up time and weak measure-valued solutions for the aggregation equation have to be considered. Carrillo et al. [14] have studied the multidimensional aggregation equation when $a = \text{id}$ in the framework of gradient flow solutions. Namely, equation (1.1) is interpreted as a differential equation in time, the right-hand side being the gradient of some interaction energy defined through the potential W . This idea, known as the Otto calculus (see [28, 33]), requires the choice of a convenient space of probability measures endowed with a Riemannian structure. Then, following [2], gradient flow solutions are interpreted as curves of maximal slopes in this space. The authors obtain existence and uniqueness of weak solutions for (1.1) in \mathbb{R}^N , $N \geq 1$ when $a = \text{id}$, the main problem being now to connect these solutions to distributional solutions.

An alternative notion of weak solutions has been obtained by completely different means in the framework of positive chemotaxis in [21]. Here equation (1.1) with $W(x) = \frac{1}{2}e^{-|x|}$ can be obtained thanks to some hydrodynamic limit of the kinetic Othmer-Dunbar-Alt system [16]. The key idea is to use the notion of duality solutions, introduced in [9] for linear conservation equations with discontinuous velocities, where measure-valued solutions arise also. In that case, this allows to give a convenient meaning to the product of the velocity by the density, so that existence and uniqueness can be proved. When applying this strategy to the nonlinear case, it turns out that uniqueness is not ensured, unless the nonlinear product is given a very precise

signification, see for instance [10, 19]. In the case of chemotaxis, it is provided by the limit of the flux in the kinetic model. Once this is done, existence and uniqueness can be obtained. An important consequence of this approach is that it allows to define a flow associated to this system. Then the dynamics of the aggregates (i.e. combinations of Dirac masses) can be established, giving rise to an implementation of a particle method and numerical simulations of the dynamics of cells density after blow-up (see also [22] for a numerical approach using a discretization on a fixed grid). The principal drawback of this method is its limitation to the one-dimensional case, mainly because duality solutions are not properly defined yet in higher space dimension. Thus the theory developed in [14] is, up to our knowledge, the only one allowing to get existence of global in time weak measure solution for (1.1) in dimension higher than 2. Another possibility could be using the notion of Filippov flow [18], together with the stability results in [7], to obtain a convenient notion of solution to (1.1), thus following [31].

This work is devoted to the study of the links between these two notions of weak solutions for equation (1.1), in the one-dimensional setting. As we shall see there is no equivalence strictly speaking for a general potential and a nonlinear function a . More precisely we first consider the same situation as in [14], that is a pointy potential W , and $a = \text{id}$. We adapt the proof of [21] to define duality solutions in this context, and choose a convenient space of measures to be compatible with the gradient flows. Then we prove that duality solutions and gradient flow solutions are identical (Theorem 4.1 below), thus answering the questions raised by Remark 2.16 of [14].

Next, we investigate the nonlinear case, that is $a \neq \text{id}$. Notice that additional monotonicity properties are required to ensure the attractivity of the dynamics. The results of [14] cannot be applied as they stand, the key problem is to define a new energy for which weak solutions of (1.1) are gradient flows. However, we are able to find such an energy only in the particular case $W = -\frac{1}{2}|x|$. On the contrary existence of duality solutions for (1.1) with a nonlinear function a can be obtained for more general potentials W , even if we cannot reach the complete generality of [14].

The outline of this paper is as follows. In the next Section, we introduce notations and recall the main results obtained in [14] in the case $a = \text{id}$. A sketch of their proof is proposed. Section 3 is devoted to the duality solutions, and starts by recalling their original definition and main properties. Next, we turn to the nonlinear setting, and define precisely the velocities and fluxes that allow to state the existence and uniqueness results both for $a = \text{id}$ and $a \neq \text{id}$. The case $a = \text{id}$ is treated in Section 4: existence and uniqueness for duality solutions are proved, together with equivalence between gradient flows and duality solutions. Finally in Section 5 we investigate the case $a \neq \text{id}$, where general equivalence results no longer hold.

2 Gradient flow solutions

2.1 Notations and definitions

Let $C_0(Y, Z)$ be the set of continuous functions from Y to Z that vanish at infinity and $C_c(Y, Z)$ the set of continuous functions with compact support from Y to Z . All along the paper, we denote $\mathcal{M}_{loc}(\mathbb{R}^N)$ the space of local Borel measures on \mathbb{R}^N . For $\rho \in \mathcal{M}_{loc}(\mathbb{R}^N)$ we denote by $|\rho|(\mathbb{R}^N)$ its total variation. We will denote $\mathcal{M}_b(\mathbb{R}^N)$ the space of measures in $\mathcal{M}_{loc}(\mathbb{R}^N)$ with finite total variation. From now on, the space of measure-valued function $\mathcal{M}_b(\mathbb{R}^N)$ is always

endowed with the weak topology $\sigma(\mathcal{M}_b, C_0)$. We denote $\mathcal{S}_{\mathcal{M}} := C([0, T]; \mathcal{M}_b(\mathbb{R}^N) - \sigma(\mathcal{M}_b, C_0))$. We recall that if a sequence of measure $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_b(\mathbb{R}^N)$ satisfies $\sup_{n \in \mathbb{N}} |\mu_n|(\mathbb{R}^N) < +\infty$, then we can extract a subsequence that converges for the weak topology $\sigma(\mathcal{M}_b, C_0)$.

Since we focus on scalar conservation laws, we can assume without loss of generality that the total mass of the system is scaled to 1 and thus we will work in some space of probability measures, namely the Wasserstein space of order $q \geq 1$, which is the space of probability measures with finite order q moment:

$$\mathcal{P}_q(\mathbb{R}^N) = \left\{ \mu \text{ nonnegative Borel measure, } \mu(\mathbb{R}^N) = 1, \int |x|^q \mu(dx) < \infty \right\}.$$

This space is endowed with the Wasserstein distance defined by (see e.g. [33, 34])

$$d_{W_q}(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \left\{ \int |y - x|^q \gamma(dx, dy) \right\}^{1/q} \quad (2.1)$$

where $\Gamma(\mu, \nu)$ is the set of measures on $\mathbb{R}^N \times \mathbb{R}^N$ with marginals μ and ν , i.e.

$$\Gamma(\mu, \nu) = \left\{ \gamma \in \mathcal{P}_q(\mathbb{R}^N \times \mathbb{R}^N); \forall \xi \in C_0(\mathbb{R}^N), \int \xi(y_0) \gamma(dy_0, dy_1) = \int \xi(y_0) \mu(dy_0), \right. \\ \left. \int \xi(y_1) \gamma(dy_0, dy_1) = \int \xi(y_1) \nu(dy_1) \right\}.$$

From the Kantorovich theorem, we know that in the definition of d_{W_q} the infimum is actually a minimum. A map that realizes the minimum in the definition (2.1) of d_{W_q} is called an optimal map, the set of which is denoted by $\Gamma_0(\mu, \nu)$.

A fundamental breakthrough in the use of the geometric approach to solve PDE is the work of F. Otto [28], which is the basis of the so-called *Otto Calculus* [33]. Let X be a Riemannian manifold endowed with the Riemannian metric $g_x(\cdot, \cdot)$ (a positive quadratic form on the tangent space at X in x denoted $T_x X$). Let $\mathcal{W} : X \rightarrow \mathbb{R}$ be differentiable. The gradient of \mathcal{W} at $x \in X$ is defined by: for all $v \in T_x X$, let $\gamma(t)$ be a regular curve on X such that $\gamma(0) = x$ and $\gamma'(0) = v$, then

$$\frac{d}{dt} \Big|_{t=0} \mathcal{W}(\gamma)(t) = g_x(\nabla_x \mathcal{W}, v), \quad \nabla_x \mathcal{W} \in T_x X.$$

The gradient flow associated to \mathcal{W} is the solution $\rho : [0, +\infty) \rightarrow X$ of the differential equation :

$$\frac{d\rho}{dt} = -\nabla_{\rho} \mathcal{W}.$$

A fundamental result due to Ambrosio et al. [2] states that gradient flows are equivalent to curves of maximal slope. Therefore, solving a PDE model of gradient type boils down to prove the existence of a curve of maximal slope.

Let us be more precise. In the following, we will mainly focus on the case $q = 2$ and we will shortly denote d_W instead of d_{W_2} . In the formalism of [2], we say that a curve μ is absolutely continuous, and we denote $\mu \in AC^2((0, +\infty), \mathcal{P}_2(\mathbb{R}^N))$, if there exists $m \in L^2(0, +\infty)$, such that $d_W(\mu(s), \mu(t)) \leq \int_s^t m(r) dr$, for $0 < s \leq t < +\infty$. Then we can define the metric derivative

$$|\mu'| (t) := \limsup_{s \rightarrow t} \frac{d_W(\mu(s), \mu(t))}{|s - t|}.$$

The tangent space to a measure $\mu \in \mathcal{P}_2(\mathbb{R}^N)$ is defined by the closed vector subspace of $L^2(\mu)$

$$\text{Tan}_\mu \mathcal{P}_2(\mathbb{R}^N) := \overline{\{\nabla \phi : \phi \in C_c^\infty(\mathbb{R}^N)\}}^{L^2(\mu)}.$$

We recall the result of Theorem 8.3.1 of [2]: if $\mu \in AC^2((0, +\infty), \mathcal{P}_2(\mathbb{R}^N))$, then there exists a Borel vector field $v(t) \in L^2(\mu(t))$ such that

$$\partial_t \mu + \nabla \cdot (v\mu) = 0, \text{ in the distributional sense on } (0, +\infty) \times \mathbb{R}^N. \quad (2.2)$$

Conversely, if μ solves a continuity equation for some Borel velocity $v \in L^1((0, +\infty); L^2(\mu))$ then μ is an absolutely continuous curve and $|\mu'| (t) \leq \|v(t)\|_{L^2(\mu)}$. As a consequence, we have

$$|\mu'| (t) = \min \{ \|v\|_{L^2(\mu(t))} : v(t) \text{ satisfies (2.2)} \}.$$

Definition 2.1 (Gradient flows) *We say that a map $\mu \in AC_{loc}^2((0, +\infty); \mathcal{P}_2(\mathbb{R}^N))$ is a solution of a gradient flow equation associated to the functional \mathcal{W} if there exists a Borel vector field v such that $v(t) \in \text{Tan}_{\mu(t)} \mathcal{P}_2(\mathbb{R}^N)$ for a.e. $t > 0$, $\|v(t)\|_{L^2(\mu)} \in L_{loc}^2(0, +\infty)$, the continuity equation*

$$\partial_t \mu + \nabla \cdot (v\mu) = 0,$$

holds in the sense of distributions and $v(t) \in -\partial \mathcal{W}(\rho(t))$ for a.e. $t > 0$, where $\partial \mathcal{W}(\rho)$ is the subdifferential of \mathcal{W} at the point ρ .

We recall that the subdifferential of a functional \mathcal{W} is defined by

$$w \in \partial \mathcal{W}(\mu) \iff \mathcal{W}(\nu) - \mathcal{W}(\mu) \geq \int_{\mathbb{R}} w(x)(\theta(x) - x) \mu(dx) + o(d_W(\mu, \nu)),$$

where θ is an optimal transport map such that $\nu = \theta_\# \mu$. Next, the slope $|\partial \mathcal{W}|$ is defined by

$$|\partial \mathcal{W}|(\mu) = \limsup_{\nu \rightarrow \mu} \frac{(\mathcal{W}(\mu) - \mathcal{W}(\nu))_+}{d_W(\mu, \nu)}, \quad (2.3)$$

where $u_+ = \max\{u, 0\}$. We have the property ([2], Lemma 10.1.5)

$$|\partial \mathcal{W}|(\mu) = \min \{ \|w\|_{L^2(\mu)} : w \in \partial \mathcal{W}(\mu) \}. \quad (2.4)$$

Moreover, there exists a unique $w \in \mathcal{W}(\mu)$ which attains the minimum in (2.4). It is denoted by $\partial^0 \mathcal{W}(\mu)$.

Definition 2.2 (Curve of maximal slope) *A curve $\mu \in AC_{loc}^2((0, +\infty); \mathcal{P}_2(\mathbb{R}))$ is a curve of maximal slope for the functional \mathcal{W} if $t \mapsto \mathcal{W}(\mu(t))$ is an absolutely continuous function and if for every $0 \leq s \leq t \leq T$,*

$$\frac{1}{2} \int_s^t |\mu'|^2(\tau) d\tau + \frac{1}{2} \int_s^t |\partial \mathcal{W}|^2(\mu(\tau)) d\tau \leq \mathcal{W}(\mu(s)) - \mathcal{W}(\mu(t)).$$

Finally Theorem 11.1.3 of [2] shows that curves of maximal slope with respect to $|\partial \mathcal{W}|$ are equivalent to gradient flow solutions. Moreover, the tangent vector field $v(t)$ is the unique element of minimal norm in the subdifferential of \mathcal{W} (see (2.4)):

$$v(t) = -\partial^0 \mathcal{W}(\mu(t)) \text{ for a.e. } t > 0.$$

2.2 Strategy of the proof in [14]

The idea of the work by Carrillo et al. [14] is to extend the work of [2] to an interaction energy \mathcal{W} defined through the interaction potential W in (1.1), whose derivative has a singularity in 0. More precisely, attractive “pointy potentials” are considered, which we define now.

Definition 2.3 (pointy potential) *The interaction potential W is said to be an attractive pointy potential if it satisfies the following assumptions.*

(A0) W is continuous, $W(x) = W(-x)$ and $W(0) = 0$.

(A1) W is λ -concave for some $\lambda \geq 0$, i.e. $W(x) - \frac{\lambda}{2}|x|^2$ is concave.

(A2) There exists a constant $C > 0$ such that $W(x) \geq -C(1 + |x|^2)$ for all $x \in \mathbb{R}^N$.

(A3) $W \in C^1(\mathbb{R}^N \setminus \{0\})$.

Given a continuous potential $W : \mathbb{R} \rightarrow \mathbb{R}$, we define the interaction energy in one dimension by

$$\mathcal{W}(\rho) = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} W(x - y) \rho(dx) \rho(dy). \quad (2.5)$$

The existence and uniqueness result of [14] can now be synthetized as follows.

Theorem 2.4 *(Theorem 2.12 and 2.13 of [14].) Let W satisfies assumptions (A0)–(A3) and let $a = id$. Given $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^N)$, there exists a gradient flow solution of (1.1), i.e. a curve $\rho \in AC_{loc}^2([0, \infty); \mathcal{P}_2(\mathbb{R}^N))$ satisfying*

$$\begin{aligned} \frac{\partial \rho(t)}{\partial t} + \partial_x(v(t)\rho(t)) &= 0, \quad \text{in } \mathcal{D}'([0, \infty) \times \mathbb{R}^N), \\ v(t, x) &= -\partial^0 \mathcal{W}(\rho)(t, x) = \int_{y \neq x} W'(x - y) \rho(t, dy), \end{aligned}$$

with $\rho(0) = \rho^{ini}$. Moreover, if ρ_1 and ρ_2 are such gradient flow solutions, then there exists a constant λ such that, for all $t \geq 0$

$$d_W(\rho_1(t), \rho_2(t)) \leq e^{\lambda t} d_W(\rho_1(0), \rho_2(0)).$$

Thus the gradient flow solution of (1.1) with initial data $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^N)$ is unique. Moreover, the following energy identity holds for all $0 \leq t_0 \leq t_1 < \infty$:

$$\int_{t_0}^{t_1} \int_{\mathbb{R}} |\partial^0 W * \rho|^2 \rho(t, dx) dt + \mathcal{W}(\rho(t_1)) = \mathcal{W}(\rho(t_0)). \quad (2.6)$$

Proof. We summarize here the main steps of the proof and refer the reader to [14] for more details. The first step is to compute the element of minimal norm in the subdifferential of \mathcal{W} . By extending Theorem 10.4.11 of [2], the authors prove that

$$-\partial^0 \mathcal{W}(\rho) = \partial^0 W * \rho, \quad (2.7)$$

where

$$\partial^0 W * \rho(x) = \int_{y \neq x} W'(x-y) \rho(dy).$$

The second step is based on the so-called *JKO scheme* introduced in [23] (see also [2]). It consists in the following recursive construction for curves of maximal slope. Let $\tau > 0$ be a small time step, we set $\rho_0^\tau = \rho^{ini}$ the initial data for (1.1). Next, knowing ρ_k^τ , one proves (Proposition 2.5 in [14]) that there exists ρ_{k+1}^τ such that

$$\rho_{k+1}^\tau \in \arg \min_{\rho \in \mathcal{P}_2(\mathbb{R}^N)} \left\{ \mathcal{W}(\rho) + \frac{1}{2\tau} d_W^2(\rho_k^\tau, \rho) \right\}. \quad (2.8)$$

Next, a piecewise constant interpolation ρ^τ is defined by

$$\rho^\tau(0) = \rho^{ini}; \quad \rho^\tau(t) = \rho_k^\tau \quad \text{if } t \in (k\tau, (k+1)\tau],$$

and Proposition 2.6 states the weak compactness (in the narrow topology) of the sequence ρ^τ as $\tau \rightarrow 0$. Finally Theorem 2.8 ensures that the weak narrow limit ρ is a curve of maximal slope.

The conclusion follows by applying Theorem 11.1.3 of [2], which allows therefore to get the existence of a gradient flow for the functional \mathcal{W} . By definition, the gradient flow is a solution of a continuity equation whose velocity field is the element with minimal norm of the subdifferential of \mathcal{W} . In the first step of the proof this element has been identified to be $\partial^0 W * \rho$. Thus, it is a weak solution of the problem and moreover we have the energy estimate (2.6). \square

2.3 The one-dimensional case

We gather here several remarks specific to the one-dimensional framework. First we notice that assumptions **(A1)** and **(A3)** imply that $x \mapsto W'(x) - \lambda x$ is a nonincreasing function on $\mathbb{R} \setminus \{0\}$. Therefore $\lim_{x \rightarrow 0^\pm} W'(x) = W'(0^\pm)$ exists and from **(A0)**, we deduce that $W'(0^-) = -W'(0^+)$. Moreover, for all $x > y$ in $\mathbb{R} \setminus \{0\}$ we have $W'(x) - \lambda x \leq W'(y) - \lambda y$. Thus we have the one-sided Lipschitz estimate (OSL) for W'

$$\forall x > y \in \mathbb{R} \setminus \{0\}, \quad W'(x) - W'(y) \leq \lambda(x - y). \quad (2.9)$$

Letting $y \rightarrow 0^\pm$ we deduce that for all $x > 0$, $W'(x) - \lambda x \leq W'(0^+)$ and $W'(x) - \lambda x \leq W'(0^-)$. Thus we also have the one-sided estimate

$$W'(x) \leq \lambda x, \quad \text{for all } x > 0. \quad (2.10)$$

After an integration with **(A0)**, we deduce that $W(x) \leq \lambda x^2/2$. By concavity of the function $\widetilde{W} : x \mapsto W(x) - \lambda x^2/2$, we have

$$\widetilde{W}'(x)h \geq \widetilde{W}(x+h) - \widetilde{W}(x) \geq -C(1+x^2+h^2),$$

for any $x \neq 0$. Taking $h = \max\{|x|, 1\}$, we deduce that there exist nonnegative constants C_0 and C_1 such that

$$|W'(x)| \leq C_0 + C_1|x|. \quad (2.11)$$

The one-dimensional framework also allows to simplify several proofs in Theorem 2.4. Indeed any probability measure μ on the real line \mathbb{R} can be described in term of its cumulative distribution function $F(x) = \mu((-\infty, x))$ which is a right-continuous and nondecreasing function with $F(-\infty) = 0$ and $F(+\infty) = 1$. Then we can define the generalized inverse of F by $F^{-1}(z) = \inf\{x \in \mathbb{R}/F(x) > z\}$, it is a right-continuous and nondecreasing function as well, defined on $[0, 1]$. If μ and ν belongs to $\mathcal{P}_2(\mathbb{R})$, with generalized inverses F^{-1} and G^{-1} , we have this explicit expression of the Wasserstein distance (see [34])

$$d_W(\mu, \nu)^2 = \int_0^1 |F^{-1}(z) - G^{-1}(z)|^2 dz. \quad (2.12)$$

Moreover, we have

$$(F^{-1})'(z)\mu(F^{-1}(z)) = 1.$$

In this framework, we can then rewrite the JKO scheme (2.8) in the proof above. Let us denote by F_k^τ the cumulative distribution of the measure ρ_k^τ and by $V_k^\tau := (F_k^\tau)^{-1}$ its generalized inverse. Then, in term of generalized inverses, (2.8) rewrites

$$V_{k+1}^\tau \in \arg \min_{\{V \in L^2(0,1)\}} \left\{ \widetilde{\mathcal{W}}(V) + \frac{1}{2\tau} \|V - V_k^\tau\|_{L^2(0,1)}^2 \right\},$$

where

$$\widetilde{\mathcal{W}}(V) = \int_0^1 \int_0^1 W(V(y) - V(z)) dydz.$$

Such an approach using the generalized inverse has been used in [8] for the one dimensional Patlack-Keller-Segel equation.

3 Duality solutions

We turn now to the alternative notion of weak solution we wish to investigate. It is based on the so-called duality solutions which were introduced for linear advection equations with discontinuous coefficients in [9]. Compared with the gradient flow approach, this strategy allows a more straightforward PDE formulation, including from the numerical viewpoint. The main drawback is that presently duality solutions in any space dimension are only available for pure transport equations (see [11]). Since we have to deal here with conservative balance laws, we have to restrict ourselves to one space dimension. First we give a brief account of the theory developed in [9], summarizing the main theorems we shall use, next we define duality solutions for (1.1).

3.1 Linear conservation equations

We consider here conservation equations in the form

$$\partial_t \rho + \partial_x (b(t, x)\rho) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}, \quad (3.1)$$

where b is a given bounded Borel function. Since no regularity is assumed for b , solutions to (3.1) eventually are measures in space. A convenient tool to handle this is the notion of duality

solutions, which are defined as weak solutions, the test functions being Lipschitz solutions to the backward linear transport equation

$$\partial_t p + b(t, x)\partial_x p = 0, \quad p(T, \cdot) = p^T \in \text{Lip}(\mathbb{R}). \quad (3.2)$$

In fact, a formal computation shows that $\frac{d}{dt} \left(\int_{\mathbb{R}} p(t, x)\rho(t, dx) \right) = 0$, which defines the duality solutions for suitable p 's.

It is now quite classical that to ensure existence for (3.2), it suffices for the velocity field to be compressive, in the following sense:

Definition 3.1 *We say that the function b satisfies the one-sided Lipschitz (OSL) condition if*

$$\partial_x b(t, \cdot) \leq \beta(t) \quad \text{for } \beta \in L^1(0, T) \text{ in the distributional sense.} \quad (3.3)$$

However, to have uniqueness, we need to restrict ourselves to *reversible* solutions of (3.2): let \mathcal{L} denote the set of Lipschitz continuous solutions to (3.2), and define the set \mathcal{E} of *exceptional solutions* by

$$\mathcal{E} = \left\{ p \in \mathcal{L} \text{ such that } p^T \equiv 0 \right\}.$$

The possible loss of uniqueness corresponds to the case where \mathcal{E} is not reduced to zero.

Definition 3.2 *We say that $p \in \mathcal{L}$ is a **reversible solution** to (3.2) if p is locally constant on the set*

$$\mathcal{V}_e = \left\{ (t, x) \in [0, T] \times \mathbb{R}; \exists p_e \in \mathcal{E}, p_e(t, x) \neq 0 \right\}.$$

We refer to [9] for complete statements of the characterization and properties of reversible solutions. Then, we can state the definition of duality solutions.

Definition 3.3 *We say that $\rho \in \mathcal{S}_{\mathcal{M}} := C([0, T]; \mathcal{M}_b(\mathbb{R}) - \sigma(\mathcal{M}_b, C_0))$ is a **duality solution** to (3.1) if for any $0 < \tau \leq T$, and any **reversible** solution p to (3.2) with compact support in x , the function $t \mapsto \int_{\mathbb{R}} p(t, x)\rho(t, dx)$ is constant on $[0, \tau]$.*

We summarize now some properties of duality solutions that we shall need in the following.

Theorem 3.4 *(Bouchut, James [9])*

1. *Given $\rho^\circ \in \mathcal{M}_b(\mathbb{R})$, under the assumptions (3.3), there exists a unique $\rho \in \mathcal{S}_{\mathcal{M}}$, duality solution to (3.1), such that $\rho(0, \cdot) = \rho^\circ$.
Moreover, if ρ° is nonnegative, then $\rho(t, \cdot)$ is nonnegative for a.e. $t \geq 0$. And we have the mass conservation*

$$|\rho(t, \cdot)|(\mathbb{R}) = |\rho^\circ|(\mathbb{R}), \quad \text{for a.e. } t \in]0, T[.$$

2. *Backward flow and push-forward: the duality solution satisfies*

$$\forall t \in [0, T], \forall \phi \in C_0(\mathbb{R}), \quad \int_{\mathbb{R}} \phi(x)\rho(t, dx) = \int_{\mathbb{R}} \phi(X(t, 0, x))\rho^0(dx), \quad (3.4)$$

where the **backward flow** X is defined as the unique reversible solution to

$$\partial_t X + b(t, x)\partial_x X = 0 \quad \text{in }]0, s[\times \mathbb{R}, \quad X(s, s, x) = x.$$

3. For any duality solution ρ , we define the **generalized flux** corresponding to ρ by $b_\Delta\rho = -\partial_t u$, where $u = \int^x \rho dx$.

There exists a bounded Borel function \widehat{b} , called **universal representative** of b , such that $\widehat{b} = a$ almost everywhere, and for any duality solution ρ ,

$$\partial_t \rho + \partial_x(\widehat{b}\rho) = 0 \quad \text{in the distributional sense.}$$

4. Let (b_n) be a bounded sequence in $L^\infty(]0, T[\times \mathbb{R})$, such that $b_n \rightharpoonup b$ in $L^\infty(]0, T[\times \mathbb{R}) - w*$. Assume $\partial_x b_n \leq \alpha_n(t)$, where (α_n) is bounded in $L^1(]0, T[)$, $\partial_x b \leq \alpha \in L^1(]0, T[)$. Consider a sequence $(\rho_n) \in \mathcal{S}_M$ of duality solutions to

$$\partial_t \rho_n + \partial_x(b_n \rho_n) = 0 \quad \text{in }]0, T[\times \mathbb{R},$$

such that $\rho_n(0, \cdot)$ is bounded in $\mathcal{M}_b(\mathbb{R})$, and $\rho_n(0, \cdot) \rightharpoonup \rho^\circ \in \mathcal{M}_b(\mathbb{R})$.

Then $\rho_n \rightharpoonup \rho$ in \mathcal{S}_M , where $\rho \in \mathcal{S}_M$ is the duality solution to

$$\partial_t \rho + \partial_x(b\rho) = 0 \quad \text{in }]0, T[\times \mathbb{R}, \quad \rho(0, \cdot) = \rho^\circ.$$

Moreover, $\widehat{b}_n \rho_n \rightharpoonup \widehat{b}\rho$ weakly in $\mathcal{M}_b(]0, T[\times \mathbb{R})$.

The set of duality solutions is clearly a vector space, but it has to be noted that a duality solution is not *a priori* defined as a solution in the sense of distributions. However, assuming that the coefficient b is piecewise continuous, we have the following equivalence result:

Theorem 3.5 *Let us assume that in addition to the OSL condition (3.3), b is piecewise continuous on $]0, T[\times \mathbb{R}$ where the set of discontinuity is locally finite. Then there exists a function \widehat{b} which coincides with b on the set of continuity of b .*

With this \widehat{b} , $\rho \in \mathcal{S}_M$ is a duality solution to (3.1) if and only if $\partial_t \rho + \partial_x(\widehat{b}\rho) = 0$ in $\mathcal{D}'(\mathbb{R})$. Then the generalized flux $b_\Delta\rho = \widehat{b}\rho$. In particular, \widehat{b} is a universal representative of b .

This result comes from the uniqueness of solutions to the Cauchy problem for both kinds of solutions (see Theorem 4.3.7 of [9]).

3.2 Duality solutions for aggregation

Equipped with this notion of solutions, we can now define duality solutions for the aggregation equation. The idea was introduced in [10] in the context of pressureless gases. It was next applied to chemotaxis in [21], and we shall actually follow these steps.

Definition 3.6 *We say that $\rho \in C([0, T]; \mathcal{M}_b(\mathbb{R}))$ is a duality solution to (1.1) if there exists $\widehat{a}_\rho \in L^\infty((0, T) \times \mathbb{R})$ and $\alpha \in L^1_{loc}(0, T)$ satisfying $\partial_x \widehat{a}_\rho \leq \alpha$ in $\mathcal{D}'((0, T) \times \mathbb{R})$, such that for all $0 < t_1 < t_2 < T$,*

$$\partial_t \rho + \partial_x(\widehat{a}_\rho \rho) = 0 \quad \text{in the sense of duality on } (t_1, t_2), \quad (3.5)$$

*and $\widehat{a}_\rho = a(W' * \rho)$ a.e.*

This allows at first to give a meaning to the notion of distributional solution, but it turns out that uniqueness is a crucial issue. For that, a key point is a precise definition of the product $\hat{a}_\rho \rho$, as we shall see in more details in Section 3.3 below.

We now state the main theorems about duality solutions for the aggregation equation (1.1). Existence of such solutions in a measure space has been obtained in [21] in the particular case $W(x) = \frac{1}{2}e^{-|x|}$ and a similar result is presented in [19] when $W(x) = -|x|/2$ which appears in many applications in physics or biology. We extend here these results for a general potential satisfying assumptions **(A0)**–**(A3)**. However to do so, we have, as in [14], to restrict ourself to the linear case, that is $a = \text{id}$.

Theorem 3.7 (Duality solutions, the linear case) *Let W satisfies assumptions **(A0)**–**(A3)** and let $a = \text{id}$. Assume that $\rho^{ini} \in \mathcal{P}_1(\mathbb{R})$. Then for any $T > 0$, there exists a unique duality solution $\rho \geq 0$ of equation (1.1) in the sense of Definition 3.6 with $\rho(0) = \rho^{ini}$, $\rho(t) \in \mathcal{P}_1(\mathbb{R})$ for any $t \in (0, T)$ and which satisfies (3.5) in the distributional sense with*

$$\hat{a}_\rho(t, x) := \partial^0 W * \rho(t, x) = \int_{x \neq y} W'(x - y) \rho(t, dy). \quad (3.6)$$

Then we have $\rho = X_{\#} \rho^{ini}$ where X is the backward flow corresponding to \hat{a}_ρ .

We turn now to the case $a \neq \text{id}$. In order to be in the attractive case, we assume that the function a satisfies the following

Assumption 3.8 *a is non-decreasing, with*

$$a \in C^1(\mathbb{R}), \quad 0 \leq a' \leq \alpha. \quad (3.7)$$

In this context, existence and uniqueness of duality solutions have been proved for the case of an interaction potential $W = \frac{1}{2}e^{-|x|}$ in [21]. We extend here the techniques developed in this latter work to more general potentials W . However we are not able to prove such results in the whole generality of assumptions **(A0)**–**(A3)** and need more regularity on the interaction potential.

Assumption 3.9 *Let $W \in C^1(\mathbb{R} \setminus \{0\})$, we assume that in the distributional sense*

$$W'' = -\delta_0 + w, \quad w \in C_b^0(\mathbb{R}), \quad (3.8)$$

where δ_0 is the Dirac measure in 0.

This allows a definition of the flux in (1.1) which generalizes the one in [21]. Indeed we can formally take the convolution of (3.8) by ρ , then multiply by $a(W' * \rho)$. Denoting by A the antiderivative of a such that $A(0) = 0$ and using the chain rule we obtain formally

$$-\partial_x A(W' * \rho) = -a(W' * \rho) W'' * \rho = a(W' * \rho) (\rho + w * \rho). \quad (3.9)$$

Thus a natural formulation for the flux J is given by

$$J := -\partial_x A(W' * \rho) + a(W' * \rho) w * \rho. \quad (3.10)$$

Theorem 3.10 (duality solutions, the nonlinear case) *Let us assume that ρ^{ini} is given in $\mathcal{P}_1(\mathbb{R})$. Under Assumption 3.9 on the potential W and (3.7) for the nonlinear function a , for all $T > 0$ there exists a unique duality solution ρ of (1.1) in the sense of Definition 3.6 $\rho(t) \in \mathcal{P}_1(\mathbb{R})$ for $t \in (0, T)$ and which satisfies in the distributional sense*

$$\partial_t \rho + \partial_x J = 0, \quad (3.11)$$

where J is defined by (3.10).

Theorems 3.7 and 3.10 are proved respectively in Sections 4 and 5, but before diving into the detailed proofs, we comment the main steps, which are common to both cases.

- **existence** of duality solutions is obtained by approximation. First we obtain the dynamics of aggregates (that is combinations of Dirac masses), then we proceed using the stability of duality solutions
- **uniqueness** is obtained by a contraction argument in $\mathcal{P}_1(\mathbb{R})$. No uniqueness is expected in a general space of measures. The argument is repeated in $\mathcal{P}_2(\mathbb{R})$ so that gradient flow and duality solutions can be compared.

3.3 Velocities and fluxes

When concentrations occur in conservation equations, leading to measure-valued solutions, a key point to obtain existence and uniqueness in the sense of distributions is the definition of the flux and the corresponding velocity. This was already pointed out in [10], where duality solutions are defined for pressureless gases, and partially managed through conditions on the initial data. A more satisfactory solution came out in [21], since uniqueness was completely handled by a careful definition of the flux of the equation, or in other terms, the product $\hat{a}_\rho \rho$. An analogous situation arising in plasma physics is considered in [19], around duality solutions as well. In a similar context, other definitions of the product can be found, see [27] in the one-dimensional setting, and [32] for a generalization in two space dimensions, where defect measures are used.

We explain in more details this point in our context, in order to give a meaning to both duality and gradient flow solutions in the sense of distributions. As a rule, the product of $a(W' * \rho(t))$ by $\rho(t)$ is not well-defined when $\rho(t) \in \mathcal{M}_b(\mathbb{R})$. First we compute $W' * \rho$. We write $\rho = \partial_x u$, so that $u \in BV(\mathbb{R})$. For such a function, we denote by S_u the set of $x \in \mathbb{R}$ where u does not admit an approximate limit, $|S_u| = 0$, and by $J_u \subset S_u$ the set of approximate jump points (see Remark 3.98 of [1]). We use the decomposition $\rho = \partial_x^a u + \rho^c + \rho^j$, where $\partial_x^a u \ll \mathcal{L}$ is the regular part of the derivative, $\rho^j = \sum_{y \in J_u} \zeta_y \delta_y$ the jump part, and ρ^c the so-called Cantor part. The so-called diffuse part of the derivative is $\rho^d = \partial_x^a u + \rho^c$. For $x \notin J_u$, we easily obtain

$$W' * \rho(x) = W' * \rho^d(x) + \sum_{y \in J_u} \zeta_y W'(x - y),$$

while if $x \in J_u$, the function is not defined. Indeed, letting $z \rightarrow x$, first with $z < x$, then with $z > x$, we obtain

$$W' * \rho(x^\pm) = W' * \partial_x^a u(x) + \sum_{y \in J_u, y \neq x} \zeta_y W'(x - y) + \zeta_x W'(0^\pm). \quad (3.12)$$

Removing the indetermination amounts to define a velocity for a single Dirac mass located in x or equivalently for the center of mass of the density. Obviously, formula (3.6) sets this value to 0, hence a single Dirac mass is stationary, and the product by the measure ρ is meaningful. Therefore in the linear case we can consider the flux

$$J(t, x) := \hat{a}_\rho(t, x) \rho(t, x).$$

Recall that this value is obtained by computing the element of minimal norm in the subgradient of the energy \mathcal{W} corresponding to W .

On the other hand, in the nonlinear case, with $W'' = -\delta_0 + w$, the natural quantity to be defined is the flux J , by formula (3.10). To define the corresponding velocity, and give rigorous meaning to (3.9), we use the Vol'pert calculus for BV functions [35] (see also [1]): for a BV function u , the fonction \hat{a}_V defining the chain rule $\partial_x A(u) = \hat{a}_V \partial_x u$ is constructed by

$$\hat{a}_V(x) = \int_0^1 a(tu_1(x) + (1-t)u_2(x)) dt, \quad (3.13)$$

where

$$(u_1, u_2) = \begin{cases} (u, u) & \text{if } x \in \mathbb{R} \setminus S_u, \\ (u^+, u^-) & \text{if } x \in J_u, \\ \text{arbitrary} & \text{elsewhere.} \end{cases} \quad (3.14)$$

Now we apply that to $u = W' * \rho$, and obtain, using the antiderivative A of a ,

$$\hat{a}_V(x) = \begin{cases} a(W' * \rho(x)) & \text{if } x \in \mathbb{R} \setminus S_u, \\ \frac{A(W' * \rho(x^+)) - A(W' * \rho(x^-))}{W' * \rho(x^+) - W' * \rho(x^-)} & \text{if } x \in J_u, \\ \text{arbitrary} & \text{elsewhere.} \end{cases} \quad (3.15)$$

The connection with the linear case follows since then $A(v) = v^2/2$, hence

$$\frac{A(W' * \rho(x^+)) - A(W' * \rho(x^-))}{W' * \rho(x^+) - W' * \rho(x^-)} = \frac{W' * \rho(x^+) + W' * \rho(x^-)}{2}.$$

Therefore the undetermined term in (3.12) is replaced by $(W'(0^+) + W'(0^-))/2$, which vanishes since W is even, and we recover (3.6).

4 The linear case

By linear case we mean the case where $a = \text{id}$ in (1.1). Together with assumptions of Definition 2.3, this is exactly the context of [14]. At first we prove Theorem 3.7, obtaining existence of duality solutions in Subsection 4.1, and uniqueness in Subsection 4.2. Next, in Subsection 4.3 we establish that they are equivalent to gradient flow solutions, thus answering the questions raised by Remark 2.16 of [14]. More precisely, we prove the following theorem.

Theorem 4.1 *Let $a = \text{id}$. Let us assume that W satisfies assumptions (A0)–(A3) and that $\rho^{ini} \in \mathcal{P}_2(\mathbb{R})$.*

- (i) Let ρ be the duality solution as in Theorem 3.7. Then for all $t > 0$, $\rho(t) \in \mathcal{P}_2(\mathbb{R})$, $\rho \in AC_{loc}^2((0, +\infty); \mathcal{P}_2(\mathbb{R}))$ and ρ is the gradient flow solution as in Theorem 2.4.
- (ii) If ρ is the gradient flow solution of Theorem 2.4, then it is a duality solution as in Theorem 3.7.

4.1 Existence of duality solutions

The first step is to verify that the velocity \hat{a}_ρ defined by (3.6) satisfies the OSL condition (3.3).

Lemma 4.2 *Let $\rho(t) \in \mathcal{M}_b(\mathbb{R})$ be nonnegative for all $t \geq 0$. Then under assumptions **(A0)** – **(A3)** the function $(t, x) \mapsto \hat{a}_\rho(t, x)$ defined in (3.6) satisfies the one-sided Lipschitz estimate*

$$\hat{a}_\rho(t, x) - \hat{a}_\rho(t, y) \leq \lambda(x - y)|\rho|(\mathbb{R}), \quad \text{for all } x > y, t \geq 0$$

Proof. By definition (3.6), we have

$$\hat{a}_\rho(x) - \hat{a}_\rho(y) = \int_{z \neq x, z \neq y} (W'(x - z) - W'(y - z))\rho(dz) + W'(x - y) \int_{z=\{x\} \cup \{y\}} \rho(dz),$$

where we use the oddness of W (**A0**) in the last term. Let us assume that $x > y$, from (2.9), we deduce that $W'(x - z) - W'(y - z) \leq \lambda(x - y)$ and with (2.10), we deduce $W'(x - y) \leq \lambda(x - y)$. Thus, using the nonnegativity of ρ , we deduce the one-sided Lipschitz (OSL) estimate for \hat{a}_ρ . \square

Proof of the existence result in Theorem 3.7. This proof is split in several steps.

- **Aggregates.**

Let us assume in a first step that $\rho^{ini} = \sum_{i=1}^n m_i \delta_{x_i^0}$ where $x_1^0 < x_2^0 < \dots < x_n^0$ and the m_i -s are nonnegative. We look forward a solution $\rho_n(t, x) = \sum_{i=1}^n m_i \delta_{x_i(t)}$ in the distributional sense of the equation

$$\partial_t \rho + \partial_x(\hat{a}_\rho \rho) = 0, \tag{4.16}$$

where \hat{a}_ρ is defined in (3.6). Let $u_n := \int^x \rho_n = \sum_{i=1}^n m_i H(x - x_i(t))$ where H is the Heaviside function. Then we have that

$$-\partial_t u_n = \hat{a}_{\rho_n} \rho_n = \sum_{i=1}^n m_i \sum_{j \neq i} m_j W'(x_i - x_j) \delta_{x_i}.$$

In fact,

$$\hat{a}_{\rho_n}(x) = \begin{cases} \sum_{\substack{j \neq i \\ j=1 \\ n}} m_j W'(x_i - x_j) & \text{if } x = x_i, i = 1, \dots, n \\ \sum_{j=1}^n m_j W'(x - x_j) & \text{otherwise.} \end{cases} \tag{4.17}$$

Then the sequence $(x_i)_{i=1, \dots, n}$ satisfies the ODE system

$$x_i'(t) = \sum_{j \neq i} m_j W'(x_i - x_j), \quad x_i(0) = x_i^0, \quad i = 1, \dots, n_\ell, \tag{4.18}$$

where $n_\ell \leq n$ is the number of distinct particles, i.e. $n_\ell = \#\{i \in \{1, \dots, N\}, x_i \neq x_j, \forall j\}$. Notice that (3.12) rewrites as

$$W' * \rho_n(x_i^\pm) = \sum_{j \neq i} m_j W'(x_i - x_j) + m_i W'(0^\pm).$$

We define the dynamics of aggregates as follows.

- the x_i -s are solutions of system (4.18) (where the right-hand side is zero if $n_\ell = 1$), when they are all distinct;
- at collisions, we define a sticky dynamics: if $x_i = x_j$ at time T_ℓ when for instance $i < j$, then the two aggregates collapse in a single one and we redefine system (4.18) by changing the number n_ℓ to $n_\ell - 1$, replacing the mass m_i by $m_i + m_j$ and deleting the point x_j . We denote by $0 := T_0 < T_1 < \dots < T_k < \infty$ the times of collapse, where $k < n$.

This choice for the dynamics is clearly mass-conservative. Moreover, with this choice at the collisional times, we still have that $x_i < x_j$ when $i < j$. Thus the function defining the right-hand side satisfies the one-sided Lipschitz condition and there exists a solution to this dynamical system.

In fact, the function defining the right-hand side of (4.18) is continuous and locally bounded as long as all x_i are distincts. Therefore there exists a C^1 local solution of the ODE (4.18) thanks to the Cauchy-Peano Theorem. A difficulty appears when particles collide. Thanks to the one-sided estimate on W' (2.9), we have that the right hand side defining (4.18) is one-sided Lipschitz. Hence, there exists a unique Filippov flow which is global in time thanks to estimate (2.11). Moreover this unique solution coincides with the C^1 solution obtained by the Cauchy-Peano theorem on each time interval $[T_\ell, T_{\ell+1})$. Finally we have constructed a solution $(x_i)_{i=1, \dots, n}$ of the system of ODE (4.18) which is Lipschitz on $[0, T]$ and C^1 on each interval $(T_\ell, T_{\ell+1})$.

Thus we can define $\rho_n := \sum_{i=1}^n m_i \delta_{x_i}$. It is then straightforward by contraction that ρ is a solution in the distributional sense of (4.16)–(3.6).

• **Duality solutions.**

Using Lemma 4.2 and expression (4.17), we deduce that $\hat{\rho}_n$ satisfies the OSL condition and is piecewise continuous where the set of all discontinuities is $\{x_i\}_{i=1, \dots, n}$. From Theorem 3.5, we deduce that ρ_n is a duality solution for all $n \in \mathbb{N}^*$.

• **Finite first order moment.**

By construction, we clearly have that $\rho \geq 0$. Let us denote by $j_1(t) := \sum_{i=1}^n m_i |x_i(t)|$ the first order moment. We have $j_1(0) < +\infty$. Using (4.18), we compute

$$j_1'(t) \leq \sum_{i=1}^n m_i \sum_{j \neq i} m_j |W'(x_i - x_j)| \leq C \sum_{i=1}^n m_i \sum_{j \neq i} m_j (1 + |x_i - x_j|),$$

where we use (2.11) for the last inequality. Using $\sum_i m_i = |\rho^{ini}|(\mathbb{R}) = 1$, we deduce that there exists a nonnegative constant C such that

$$j_1'(t) \leq C(1 + j_1(t)).$$

Integrating this latter inequality implies that j_1 is bounded on $[0, T]$ by a constant only depending on T and on $j_1(0)$. It remains to pass to the limit $n \rightarrow +\infty$ in the regularization.

• **Passing to the limit** $n \rightarrow +\infty$.

Using the fact that $\rho_n(t) \in \mathcal{P}_1(\mathbb{R})$ for all $t \geq 0$ and from (2.11), we deduce that \hat{a}_{ρ_n} is bounded in $L^\infty((0, T) \times \mathbb{R})$ uniformly with respect to n . Thus, from the point 4 of Theorem 3.4, we can extract a subsequence \hat{a}_n converging in $L^\infty((0, T) \times \mathbb{R})$ towards \hat{a} and the corresponding duality solutions sequence $(\rho_n)_n$ converge in \mathcal{S}_M towards ρ which is a duality solution of

$$\partial_t \rho + \partial_x(\hat{a}\rho) = 0.$$

Moreover, since $\rho_n \rightharpoonup \rho$ in \mathcal{S}_M , we have that $\hat{a}_{\rho_n} \rightarrow \hat{a}_\rho$ a.e. with the definition in (3.6). Thus $\hat{a} = \hat{a}_\rho$ a.e. and the flux

$$J_n(t, x) \rightharpoonup J(t, x) := \hat{a}_\rho \rho \quad \text{in } \mathcal{M}_b([0, T] \times \mathbb{R}) - w.$$

4.2 Uniqueness

Let ρ be a nonnegative duality solution which satisfies in the distributional sense

$$\partial_t \rho + \partial_x(\hat{a}_\rho \rho) = 0. \tag{4.19}$$

As above, we denote by F the cumulative distribution function of ρ and by F^{-1} its generalized inverse. We have then by integration of (4.19)

$$\partial_t F + \hat{a}_\rho \partial_x F = 0.$$

Then, we deduce from standard computations that

$$\partial_t F^{-1}(t, z) \rho(t, F^{-1}(t, z)) = \hat{a}_\rho(t, F^{-1}(z)) \rho(t, F^{-1}(t, z)).$$

Then the generalized inverse is a solution of the system

$$\partial_t F^{-1}(t, z) = \hat{a}_\rho(t, F^{-1}(z)). \tag{4.20}$$

Moreover thanks to a change of variables in (3.6),

$$\hat{a}_\rho(t, F^{-1}(z)) = \int_{y \neq z} W'(F^{-1}(z) - F^{-1}(y)) dy.$$

Proposition 4.3 *Let $\rho_1(t, \cdot)$ and $\rho_2(t, \cdot)$ belong to $\mathcal{P}_1(\mathbb{R})$. If ρ_1 and ρ_2 satisfy equation (4.19) in the weak sense with initial data ρ_1^{ini} and ρ_2^{ini} . Then we have, for all $t > 0$*

$$d_{W_1}(\rho_1(t, \cdot), \rho_2(t, \cdot)) \leq e^{\lambda t} d_{W_1}(\rho_1, \rho_2).$$

Proof. Let us assume that there exists two nonnegative measures ρ_1 and ρ_2 duality solutions which satisfy moreover (4.19). Let us denote respectively F_1^{-1} and F_2^{-1} their generalized inverses. From (4.20), we have

$$\partial_t(F_1^{-1} - F_2^{-1}) = \hat{a}_\rho(t, F_1^{-1}(z)) - \hat{a}_\rho(t, F_2^{-1}(z)).$$

Multiplying the latter equation by $\text{sign}(F_1^{-1}(z) - F_2^{-1}(z))$ and integrating, we get

$$\partial_t(F_1^{-1}(z) - F_2^{-1}(z))\text{sign}(F_1^{-1}(z) - F_2^{-1}(z)) = (\hat{a}_\rho(F_1^{-1}(z)) - \hat{a}_\rho(F_2^{-1}(z)))\text{sign}(F_1^{-1}(z) - F_2^{-1}(z)). \quad (4.21)$$

The one-sided Lipschitz estimate on \hat{a}_ρ in Lemma 4.2 implies that

$$(\hat{a}_\rho(F_1^{-1}(z)) - \hat{a}_\rho(F_2^{-1}(z)))\text{sign}(F_1^{-1}(z) - F_2^{-1}(z)) \leq \lambda |F_1^{-1}(z) - F_2^{-1}(z)|.$$

Hence, after an integration,

$$\frac{d}{dt} \int_0^1 |F_1^{-1} - F_2^{-1}|(z) dz \leq \lambda \int_0^1 |F_1^{-1} - F_2^{-1}|(z) dz.$$

Applying a Gronwall argument, we deduce that

$$\|(F_1^{-1} - F_2^{-1})(t)\|_{L^1(0,1)} \leq e^{\lambda t} \|(F_1^{-1} - F_2^{-1})(0)\|_{L^1(0,1)}$$

We conclude the proof by noticing that $\|(F_1^{-1} - F_2^{-1})(t)\|_{L^1(0,1)} = d_{W_1}(\rho_1, \rho_2)$ and using a Gronwall argument. \square

Proof of Theorem 3.7. The existence has been obtained in Section 4.1. Then if we have two duality solutions ρ_1 and ρ_2 as in Theorem 3.7, Proposition 4.3 implies that their generalized inverse are equals. Therefore $\rho_1 = \rho_2$. Finally, the second point of Theorem 3.4 allows to define the duality solution as a push-forward measure from the backward flow. \square

4.3 Proof of Theorem 4.1

Let us first prove that the second order moment is bounded provided $\rho^{ini} \in \mathcal{P}_2(\mathbb{R})$. We follow the idea of the proof of finite first order moment in subsection 4.1 : we consider an approximation of ρ^{ini} by $\sum_{i=1}^n m_i \delta_{x_i^0}$ and we consider then the constructed duality solution $\rho(t, x) = \sum_{i=1}^n m_i \delta_{x_i(t)}$ where the dynamics of the nodes $\{x_i\}_{i=1, \dots, n}$ is given in (4.18). Let us denote by $j_2(t) := \sum_{i=1}^n m_i x_i^2(t)$ the second order moment. We have $j_2(0) < +\infty$. Using (4.18), we compute

$$j_2'(t) = \sum_{i=1}^n 2m_i x_i \sum_{j \neq i} m_j W'(x_i - x_j) \leq C \sum_{i=1}^n m_i |x_i| \sum_{j \neq i} m_j (1 + |x_i - x_j|),$$

where we use (2.11) for the last inequality. Since $\sum_i m_i = |\rho^{ini}|(\mathbb{R})$ and from the Cauchy-Schwarz inequality, we deduce that there exists a nonnegative constant C such that

$$j_2'(t) \leq C \sum_{i=1}^n m_i (|x_i| + |x_i|^2) \leq C(1 + j_2(t)).$$

Integrating this latter inequality implies that j_2 is bounded on $[0, T]$ by a constant only depending on T and on $j_2(0)$. Then we can pass to the limit $n \rightarrow +\infty$ to obtain a bound on $\int_{\mathbb{R}} |x|^2 \rho(t, dx)$ for any $t > 0$. Moreover, using this $L^\infty((0, T); \mathcal{P}_2(\mathbb{R}))$ bound, we deduce that the velocity field \hat{a}_ρ defined in (3.6) is bounded in $L^1((0, T); L^2(\rho))$.

Then, if ρ is a duality solution as in Theorem 3.7, it satisfies in the distributional sense the scalar conservation laws

$$\partial_t \rho + \partial_x(\hat{a}_\rho \rho) = 0.$$

Using Theorem 8.3.1 of [2] and the $L^1((0, T); L^2(\rho))$ bound on the velocity \hat{a}_ρ , we deduce that $\rho \in AC_{loc}^2((0, +\infty); \mathcal{P}_2(\mathbb{R}))$. Thus ρ is a gradient flow solution (see [14] or Sections 8.3 and 8.4 of [2]). It concludes the proof of point (i) of Theorem 4.1.

Conversely, if ρ is a gradient flow solution of Theorem 2.4, then it is a weak measure solution of (3.5)–(3.6). Such solution is unique from Theorem 3.7 then ρ is a duality solution. \square

5 On the case $a \neq \text{id}$

In many applications in physics and biology (see e.g. [10, 21]), we have to consider that $a \neq \text{id}$. This situation is not so favourable as the previous one, and one has to impose restrictions on the potential W . First we recall that attractivity implies that a is non-decreasing, see (3.7). Next, we need to assume that W has the following structure, see (3.8),

$$W'' = -\delta_0 + w, \quad w \in C_b^0(\mathbb{R}).$$

With these assumptions, we are able to prove existence and uniqueness of duality solutions, Theorem 3.10, this is the aim of subsection 5.1. Next, in subsection 5.2, we turn to gradient flow, which are definitely not well suited for that case, since we have to restrict ourselves to $w = 0$ in the previous assumption on W .

5.1 Duality solutions

Here we prove Theorem 3.10, following the same strategy as in the linear case: first we prove the OSL condition, next establish the dynamics of aggregates, which leads to existence by approximation. Finally, uniqueness follows from a contraction principle in the space \mathcal{P}_1 . In addition, we prove that duality solutions are absolutely continuous in time.

5.1.1 OSL condition

The first step consists in checking the OSL property for a .

Lemma 5.1 *Assume $0 \leq \rho \in \mathcal{M}_b(\mathbb{R})$ and that (3.7) holds, If Assumption 3.9 is satisfied, then the function $x \mapsto a(W' * \rho)$ satisfies the OSL condition (3.3).*

Proof. Using (3.8), we deduce that

$$\partial_{xx} W * \rho = -\rho + w * \rho.$$

Therefore,

$$\partial_x(a(\partial_x W * \rho)) = a'(\partial_x W * \rho)(-\rho + w * \rho) \leq a'(\partial_x W * \rho)w * \rho,$$

where we use the nonnegativity of ρ in the last inequality. Then from (3.7) we get

$$\partial_x(a(\partial_x W * \rho)) \leq \alpha \|\rho\|_{L^1} \|w\|_{L^\infty}.$$

It implies the OSL condition on the velocity. □

5.1.2 Existence

Approximation with aggregates.

Following the idea in subsection 4.1, we first approximate the initial data ρ_n^{ini} by a finite sum of Dirac masses: $\rho_n^{ini} = \sum_{i=1}^n m_i \delta_{x_i^0}$ where $x_1^0 < x_2^0 < \dots < x_n^0$ and the m_i -s are nonnegative. We look for a sequence $(\rho_n)_n$ solving in the distributional sense $\partial_t \rho_n + \partial_x J_n = 0$ where the flux J_n is given by (3.10). A function $\rho_n(t, x) = \sum_{i=1}^n m_i \delta_{x_i(t)}$ is such a solution provided the function u_n defined by

$$u_n(t, x) := \int^x \rho_n dx = \sum_{i=1}^n m_i H(x - x_i(t)), \quad (5.1)$$

where H denotes the Heaviside function, is a distributional solution to

$$\partial_t u_n - \partial_x A(\partial_x W * \rho_n) + a(\partial_x W * \rho_n) w * \rho_n = 0. \quad (5.2)$$

We have

$$W' * \rho_n(t, x) = \sum_{i=1}^n m_i W'(x - x_i(t)). \quad (5.3)$$

And from (3.8), we deduce that

$$W'(x) = -H(x) + \tilde{w}(x), \quad \text{where } \tilde{w}(x) = \int^x w(y) dy. \quad (5.4)$$

Straightforward computations show that in the distributional sense

$$\partial_x A(\partial_x W * \rho_n) = a(\partial_x W * \rho_n) w * \rho_n + \sum_{i=1}^n [A(\partial_x W * \rho_n)]_{x_i} \delta_{x_i}, \quad (5.5)$$

where $[f]_{x_i} = f(x_i^+) - f(x_i^-)$ is the jump of the function f at x_i . Injecting (5.1) and (5.5) in (5.2), we find

$$-\sum_{i=1}^n m_i x_i'(t) \delta_{x_i(t)} = \sum_{i=1}^n [A(\partial_x W * \rho_n)]_{x_i} \delta_{x_i}.$$

Thus it is a solution if we have

$$m_i x_i'(t) = -[A(\partial_x W * \rho_n)]_{x_i(t)}, \quad \text{for } i = 1, \dots, n. \quad (5.6)$$

This system of ODEs is complemented by the initial data $x_i(0) = x_i^0$. Moreover, we have from (5.4)

$$W' * \rho_n(x_i^+) = -\sum_{j=1}^i m_j + \sum_{j=1}^n m_j \tilde{w}(x_i - x_j). \quad (5.7)$$

And

$$W' * \rho_n(x_i^-) = m_i + W' * \rho_n(x_i^+). \quad (5.8)$$

Thus

$$x'_i(t) = \frac{[A(W' * \rho_n)]_{x_i}}{[W' * \rho_n]_{x_i}}, \quad i = 1, \dots, n, \quad x_i(0) = x_i^0. \quad (5.9)$$

Then we define the dynamics of aggregates as in subsection 4.1 :

- When the x_i are all distinct, they are solutions of system (5.6) or equivalently (5.9) (with zero right hand side if $n_\ell = 1$).
- At collisions, we use a sticky dynamics as defined as above.

We recall that this choice of the dynamics allows the conservation of the mass. As above, we have existence of the sequence $(x_i)_i$ satisfying (5.9) on $[0, T]$ with initial condition (x_i^0) . Then we set $\rho_n(t, x) = \sum_{i=1}^n m_i \delta_{x_i(t)}(x)$. By construction, ρ_n is a solution in the sense of distribution of (3.11)-(3.10) for given initial data ρ_n^{ini} .

Existence of duality solutions

We recall the following result due to Vol'pert [35] (see also [1]): if u belongs to $BV(\mathbb{R})$ and $f \in C^1(\mathbb{R})$ with $f(0) = 0$, then $v = f \circ u$ belongs to $BV(\mathbb{R})$ and

$$\exists \bar{f}_u \text{ with } \bar{f}_u = f'(u) \text{ a.e. such that } (f \circ u)' = \bar{f}_u u'.$$

Together with the fact that A is an antiderivative of a such that $A(0) = 0$, this result implies that there exists a function \hat{a}_n such that $\hat{a}_n = a(W' * \rho_n)$ a.e. and

$$\partial_x(A(W' * \rho_n)) = \hat{a}_n \partial_{xx} W * \rho_n = -\hat{a}_n \rho_n + \hat{a}_n w * \rho_n,$$

where the last identity comes from (3.8). Therefore, we have

$$J_n := -\partial_x(A(W' * \rho_n)) + a(W' * \rho_n)w * \rho_n = \hat{a}_n \rho_n, \quad \text{and} \quad \hat{a}_n = a(W' * \rho_n) \text{ a.e.}$$

Thus ρ_n is a solution in the distributional sense of

$$\partial_t \rho_n + \partial_x(\hat{a}_n \rho_n) = 0.$$

Moreover, we deduce from (5.3) that $a(W' * \rho_n)$ is piecewise continuous with the discontinuity lines defined by $x = x_i$, $i = 1, \dots, n$. We can apply Theorem 3.5 which gives that ρ_n is a duality solution and that \hat{a}_n is a universal representative of $a(W' * \rho_n)$. Then the flux is given by $a(W' * \rho_n) \Delta \rho_n = J_n$.

General case.

Let us yet consider the case of any initial data $\rho^{ini} \in \mathcal{M}_b(\mathbb{R})$. We approximate ρ^{ini} by $\rho_n^{ini} = \sum_{i=1}^n m_i \delta_{x_i^0}$ with $\rho_n^{ini} \rightharpoonup \rho^{ini}$ in $\mathcal{M}_b(\mathbb{R})$. By the same token as above, we can construct a sequence of solutions $(\rho_n)_n$ with $\rho_n(t=0) = \rho_n^{ini} = \sum_{i=1}^n m_i \delta_{x_i^0}$, which solves in the sense of distributions

$$\partial_t \rho_n + \partial_x J_n = 0, \quad J_n = -\partial_x A(\partial_x W * \rho_n) + a(\partial_x W * \rho_n)w * \rho_n,$$

and which satisfies

$$\hat{a}_n \rho_n = J_n, \quad \hat{a}_n = a(W' * \rho_n) \text{ a.e.}$$

Moreover, since $W' * \rho_n$ is bounded in L^∞ uniformly with respect to n by construction, we can extract a subsequence of $(a(W' * \rho_n))_n$ that converges in $L^\infty - weak^*$ towards b . Since from

Lemma 5.1, $a(W' * \rho_n)$ satisfies the OSL condition, we deduce from Theorem 3.4 4) that, up to an extraction, $\rho_n \rightarrow \rho$ in $\mathcal{S}_{\mathcal{M}}$ and $\widehat{a}_n \rho_n \rightarrow \widehat{a} \rho$ weakly in $\mathcal{M}_b([0, T[\times \mathbb{R})$, ρ being a duality solution of the scalar conservation law with coefficient b . Then $J_n \rightarrow J := -\partial_x A(W' * \rho) + a(W' * \rho)w * \rho$ in $\mathcal{D}'(\mathbb{R})$ and that $a(W' * \rho_n) \rightarrow a(W' * \rho)$ a.e. By uniqueness of the weak limit, we have $b = a(W' * \rho)$. Moreover $J = \widehat{a} \rho$ a.e. and ρ satisfies then (3.11).

Finite first order moment.

As in subsection 4.1, we define $j_1(t) = \sum_{i=1}^n m_i |x_i(t)|$ and we compute

$$j_1'(t) = \sum_{i=1}^n m_i \frac{x_i}{|x_i|} \frac{[A(W' * \rho_n)]_{x_i}}{[W' * \rho_n]_{x_i}},$$

where we use (5.9). Since a is nondecreasing, A is a convex function. Moreover, we deduce from (5.4) that $|W'(x)| \leq C(1 + |x|)$. Thus, using the fact that a' is bounded (3.7), we have

$$j_1'(t) \leq C \sum_{i=1}^n m_i (1 + \sum_j m_j |x_i - x_j|) \leq C(1 + j_1(t)),$$

where C stands for a generic nonnegative constant. Applying a Gronwall inequality and passing to the limit $n \rightarrow +\infty$ allows to conclude the proof. \square

Remark 5.2 *Let us consider the case studied in the previous Section : $a = id$ and W is even. Since W' is odd, then (5.4) rewrites*

$$W'(x) = -H(x) + w_0(x), \quad \text{where } w_0(x) = \int_0^x w(y) dy + \frac{1}{2}.$$

When $a = id$, we have $A(x) = \frac{1}{2}x^2$. Then system (5.6) rewrites

$$m_i x_i'(t) = -\frac{1}{2}(W' * \rho_n(x_i^+) - W' * \rho_n(x_i^-))(W' * \rho_n(x_i^+) + W' * \rho_n(x_i^-)).$$

Then, from (5.7) and (5.8), we have

$$x_i'(t) = -\sum_{j=1}^{i-1} m_j - \frac{m_i}{2} + \sum_{j=1}^n m_j w_0(x_i - x_j) = -\sum_{j=1}^{i-1} m_j + \sum_{j \neq i} m_j w_0(x_i - x_j).$$

From the expression of W' above, we deduce that $x_i'(t) = \sum_{j \neq i} m_j W'(x_i - x_j)$ and we recover the dynamical system (4.18) of the previous Section.

Remark 5.3 *The dynamical system (5.6) defines actually the macroscopic velocity. Indeed, if we formally take the limit $n \rightarrow +\infty$ of the right hand side of (5.9), this latter term converges towards the velocity \widehat{a}_V defined by the chain rule (3.15)*

5.1.3 Uniqueness.

The proof of the uniqueness follows straightforwardly the one in subsection 4.2 since the macroscopic velocity still satisfies the OSL condition as proved in Lemma 5.1.

5.1.4 Absolute continuity.

Following subsection 4.3, we have the following result:

Proposition 5.4 *Under the assumptions of Theorem 3.10, if moreover $\rho^{ini} \in \mathcal{P}_2(\mathbb{R})$, then the duality solution of Theorem 3.10 satisfies $\rho \in AC_{loc}^2((0, +\infty); \mathcal{P}_2(\mathbb{R}))$.*

Proof. The proof of the finite second order moment follows straightforwardly the one for the first order moment in subsection 5.1.2. Then from (3.7) and (2.11), we have that

$$|a(W' * \rho)| \leq C \left(1 + \int_{\mathbb{R}} |x - y| \rho(dy) \right).$$

Applying a Cauchy-Schwarz inequality, we deduce that

$$\int_{\mathbb{R}} |a(W' * \rho)|^2 \rho(dx) \leq C \left(1 + \int_{\mathbb{R}} |x - y|^2 \rho(dx) \rho(dy) \right),$$

which is bounded since $\rho(t) \in \mathcal{P}_2(\mathbb{R})$. Therefore the velocity field \widehat{a} is bounded in $L^1((0, T); L^2(\rho))$. Moreover ρ is a solution in the distributional sense of $\partial_t \rho + \partial_x(\widehat{a}\rho) = 0$. We conclude then from Theorem 8.3.1 of [2] that $\rho \in AC_{loc}^2((0, +\infty); \mathcal{P}_2(\mathbb{R}))$. \square

5.2 Gradient flows

When $a \neq \text{id}$, equation (1.1) does not have a structure of a gradient. Moreover, we are not able to determine a conserved energy corresponding to the system. Nevertheless, in the particular case $W(x) = -\frac{1}{2}|x|$, we are able to adapt the technique of [14] to recover the existence of gradient flow solutions. We introduce for $\rho \in \mathcal{P}_2(\mathbb{R})$ the functional

$$\mathcal{W}(\rho) = - \int_{\mathbb{R}} xa(u(x))\rho(dx), \quad \text{where } u(x) = W' * \rho(x) = \int_{-\infty}^x \rho(dy) - \frac{1}{2}. \quad (5.10)$$

Recall that we denote by A the antiderivative of a which vanishes at 0. We first verify that when $a = \text{id}$ this energy is equal to the one introduced in (2.5). In fact, we have for $W(x) = -\frac{1}{2}|x|$,

$$-\frac{1}{4} \int_{\mathbb{R}^2} |x - y| \rho(dx) \rho(dy) = -\frac{1}{4} \int_{\mathbb{R}} \int_{-\infty}^x (x - y) \rho(dy) \rho(dx) + \frac{1}{4} \int_{\mathbb{R}} \int_x^{+\infty} (x - y) \rho(dy) \rho(dx). \quad (5.11)$$

Repeated use of Fubini's Theorem in the last term of the right hand side leads to

$$\begin{aligned} -\frac{1}{4} \int_{\mathbb{R}^2} |x - y| \rho(dx) \rho(dy) &= -\frac{1}{2} \int_{\mathbb{R}} \int_{-\infty}^x (x - y) \rho(dy) \rho(dx) \\ &= -\frac{1}{2} \int_{\mathbb{R}} x \int_{-\infty}^x \rho(dy) \rho(dx) + \frac{1}{2} \int_{\mathbb{R}} \int_y^{+\infty} \rho(dx) y \rho(dy). \end{aligned}$$

Since u is given by (5.10), we have $\int_x^{+\infty} \rho(dy) = \frac{1}{2} - u(x)$. Finally,

$$-\frac{1}{4} \int_{\mathbb{R}^2} |x - y| \rho(dy) \rho(dx) = - \int_{\mathbb{R}} xu(x) \rho(dx),$$

which concludes the proof.

We are able to prove in this case the following Theorem.

Theorem 5.5 *Let $W(x) = -\frac{1}{2}|x|$ and a satisfy assumption (3.7). Given $\rho^{ini} \in \mathcal{P}_2(\mathbb{R})$, there exists a unique gradient flow solution $\rho \in AC_{loc}^2([0, +\infty), \mathcal{P}_2(\mathbb{R}))$ in the sense of Definition 2.1. Therefore ρ satisfies in the distributional sense*

$$\partial_t \rho + \partial_x(a(u)\rho) = 0, \quad \text{with} \quad \rho(0) = \rho^{ini},$$

where $u(x) = W' * \rho(x) = \rho((-\infty, x)) - \frac{1}{2}$. Moreover, this solution is unique and we have the energy estimate : for all $0 \leq t_0 \leq t_1 < \infty$,

$$\int_{t_0}^{t_1} |a(u(t, x))|^2 \rho(t, dx) dt + \mathcal{W}(\rho(t_1)) = \mathcal{W}(\rho(t_0)).$$

Proof. This proof closely follows the ideas of the Section 2 of [14]. Therefore we will only gives the main steps and do not detail all the computations. As recalled in the beginning of this paper, the proof is divided into several steps.

• $-a(u)$ is the unique element of minimal $L^2(\rho)$ -norm in the subdifferential $\partial\mathcal{W}$ of \mathcal{W} .

We first show that $-a(u) \in \partial\mathcal{W}(\rho)$, where u is defined in (5.10). That means

$$\mathcal{W}(\theta_{\#}\rho) - \mathcal{W}(\rho) \geq - \int_{\mathbb{R}} a(u)(x)(\theta(x) - x) \rho(dx) + o(d_{W_2}(\theta_{\#}\rho, \rho)), \quad (5.12)$$

where θ is an optimal transportation map. From Brenier's Theorem [12], we can take for θ a nondecreasing function. By definition of the pushforward measure, we have

$$\int_{-\infty}^x d(\theta_{\#}\rho) = \rho(\theta^{-1}(-\infty, x)),$$

so that

$$\mathcal{W}(\theta_{\#}\rho) = - \int_{\mathbb{R}} xa(\rho(\theta^{-1}(-\infty, x)))\theta_{\#}\rho(dx) = - \int_{\mathbb{R}} \theta(x)a(\rho(\{y \in \mathbb{R}/\theta(y) < \theta(x)\}))\rho(dx).$$

We deduce from the monotony of the fuction t that

$$\mathcal{W}(\theta_{\#}\rho) - \mathcal{W}(\rho) = - \int_{\mathbb{R}} a(u(x))(\theta(x) - x)\rho(dx).$$

To prove that $-a(u)$ is an element of minimal norm in $\partial\mathcal{W}(\rho)$, we first notice that for $\xi \in C_b^\infty(\mathbb{R})$ and for $\varepsilon > 0$ small enough such that $(id + \varepsilon\xi)$ is increasing, we have

$$\mathcal{W}((id + \varepsilon\xi)_{\#}\rho) = - \int_{\mathbb{R}} (id + \varepsilon\xi)(x)a(u(x))\rho(dx).$$

Thus,

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{W}((id + \varepsilon\xi)_{\#}\rho) - \mathcal{W}(\rho)}{\varepsilon} = - \int_{\mathbb{R}} a(u(x))\xi(x) d\rho(x).$$

Then, from the definition of the slope (2.3), we have

$$\liminf_{\varepsilon \searrow 0} \frac{\mathcal{W}((id + \varepsilon\xi)_{\#}\rho) - \mathcal{W}(\rho)}{d_{W_2}((id + \varepsilon\xi)_{\#}\rho, \rho)} \geq -|\partial\mathcal{W}|(\rho).$$

We deduce from above that

$$\int_{\mathbb{R}} a(u(x))\xi(x) d\rho(x) \leq |\partial\mathcal{W}|(\rho) \liminf_{\varepsilon \searrow 0} \frac{d_{W_2}((id + \varepsilon\xi)_{\#}\rho, \rho)}{\varepsilon} \leq |\partial\mathcal{W}|(\rho)\|\xi\|_{L^2(\rho)},$$

where we use the inequality $d_{W_2}(S_{\#}\rho, T_{\#}\rho) \leq \|S - T\|_{L^2(\rho)}$ for the last inequality. By the same token with $-\xi$ instead of ξ , we deduce

$$\left| \int_{\mathbb{R}} a(u(x))\xi(x) d\rho(x) \right| \leq |\partial\mathcal{W}|(\rho)\|\xi\|_{L^2(\rho)}.$$

Since ξ is arbitrary, we get $\|a(u)\|_{L^2(\rho)} \leq |\partial\mathcal{W}|(\rho)$. We conclude by using (2.4).

• **Well-posedness and convergence of the JKO scheme.**

The JKO scheme is defined in (2.8) where $\tau > 0$ is a given small time step. The convergence of this scheme can be deduced by an adaptation of the argument of Subsection 2.2 of [14]. However, in this one dimensional case, we can simplify a lot the computations by using the generalized inverse, denoted v , of the function u , in the spirit of [8]. In fact, the function u is, up to a constant, the cumulative distribution of ρ . Then, we have

$$\mathcal{W}(\rho) = \widetilde{\mathcal{W}}(v) := - \int_{-1/2}^{1/2} v(t, z)a(z) dz,$$

where we recall

$$v(t, z) = u^{-1}(t, z) := \inf\{x \in \mathbb{R}/u(t, x) > z\}.$$

We define moreover $v^{ini} = (u^{ini})^{-1}$, where $u^{ini}(x) = \rho^{ini}((-\infty, x))$. Thus, using moreover (2.12), the minimization problem defined in (2.8) is equivalent to : let $v_k^\tau \in L^2(-1/2, 1/2)$, we define

$$v_{k+1}^\tau = \arg \min_{v \in L^2(-1/2, 1/2)} \left\{ \widetilde{\mathcal{W}}(v) + \frac{1}{2\tau} \|v - v_k^\tau\|^2 \right\}. \quad (5.13)$$

We recall that a is a nondecreasing function, then $a(-1/2) \leq a(z) \leq a(1/2)$ when $z \in (-1/2, 1/2)$. Then the functional defined inside the brackets is clearly lower semi-continuous, convex and coercive on $L^2(-1/2, 1/2)$. We deduce that the above minimization problem (5.13) admits a unique solution. Moreover, computing the Fréchet derivative of the functional defining (5.13) in the L^2 -metric, the Euler-Lagrange equation associated to this minimization problem implies

$$v_{k+1}^\tau(z) = v_k^\tau(z) - \tau a(z), \quad z \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

This is an implicit Euler discretization of the equation $\partial_t v(t, z) + a(z) = 0$. It is well known that the piecewise constant interpolation v^τ defined by $v^\tau(0) = v^{ini}$ and $v^\tau(t) = v_k^\tau$ if $t \in (k\tau, (k+1)\tau]$, converges in $L^2(-1/2, 1/2)$ as $\tau \rightarrow 0$ towards $v(t)$ for all $t \in [0, +\infty)$, where $v(t, z) = v^{ini} - ta(z)$. Moreover, we have the energy estimate

$$\widetilde{\mathcal{W}}(v) = \widetilde{\mathcal{W}}(v^{ini}) - \int_0^t \int_{-1/2}^{1/2} |a(z)|^2 dz. \quad (5.14)$$

We define then $\rho := \partial_x u$, where $u(t, x) := v^{-1}(t, z) = \inf\{z \in [-1/2, 1/2]/v(t, z) > x\}$. We have $\rho \in \mathcal{P}_2(\mathbb{R})$, and from the latter energy estimate, we deduce

$$\mathcal{W}(\rho) = \mathcal{W}(\rho^{ini}) - \int_0^t \int_{\mathbb{R}} |a(u(t, x))|^2 \rho(t, dx). \quad (5.15)$$

Moreover, we have from the equation $\partial_t v(t, z) + a(z) = 0$ that the generalized inverse u solves in the weak sense $\partial_t u + a(u)\partial_x u = 0$. Therefore, in the sense of distributions, we have

$$\partial_t \rho + \partial_x(a(u)\rho) = 0.$$

From Theorem 8.3.1 of [2], we deduce that $\rho \in AC_{loc}^2((0, +\infty); \mathcal{P}_2(\mathbb{R}))$. We conclude, using moreover (5.15) that ρ is a curve of maximal slope for the functional \mathcal{W} defined in (5.10).

• **Gradient flow solutions.**

This last step is a direct consequence of Theorem 11.1.3 of [2] since all assumptions of the Theorem have been verified above. Thus curves of maximal slope are gradient flow solutions. Uniqueness is obtained thanks to a contraction estimate based on a Gronwall argument as in [14]. This concludes the proof of Theorem 5.5. \square

Finally we highlight that for the potential $W(x) = -\frac{1}{2}|x|$, the problem has a special structure explaining why we are able in that particular case to apply the theory of gradient flows. In fact, in this case we have $W''(x) = -\delta_0(x)$ that is $w = 0$ in (3.8). Then denoting by $u = W' * \rho$ and taking the convolution by W' of (3.11), we deduce that u satisfies in the distributional sense

$$\partial_t u - J + w * J = 0, \quad J = -\partial_x A(u) + a(u)w * \rho.$$

Therefore, in the particular case when $w = 0$, this latter equation reduces to the scalar conservation law

$$\partial_t u + \partial_x A(u) = 0.$$

Obviously, since we recover the absolute continuity of the duality solution in Proposition 5.4, we conclude that when $W = -\frac{1}{2}|x|$ both concepts of solutions coincide.

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