Generalised canonical basic sets for Ariki-Koike algebras

Thomas Gerber

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Abstract

Let $\mathscr{H}$ be a non semi-simple Ariki-Koike algebra. According to [18] and [14], each "cyclotomic" specialisation induces a natural order on Specht modules. In some cases, Geck and Jacon have proved that this order makes unitriangular the decomposition matrix of $\mathscr{H}$. The algebra $\mathscr{H}$ is then said to admit a "canonical basic set". We fully classify which cyclotomic specialisations afford a canonical basic set for $\mathscr{H}$ and which do not. When this is the case, we describe these sets in terms of "twisted Uglov" or "twisted Kleshchev" multipartitions.

1 Introduction

Over a field of characteristic 0, the representation theory of the symmetric group $\mathfrak{S}_n$ is well-known. In particular, Maschke's semi-simplicity criterion ensures that it is sufficient to understand its irreducible representations. In fact, they are parametrised by partitions of n, via an explicit bijection. It is then possible to study the representations of $\mathfrak{S}_n$ in a combinatorial manner (using the notion of Young tableaux), and deduce for instance the classic "hook-length formula" to compute the dimension of any irreducible representation.

In the so-called "modular" case, that is when the ground field is of prime characteristic e, one loses many convenient properties, notably the semi-simplicity of the theory. One can however gather some information. For instance, the irreducible representations are known to be parametrised by "e-regular" partitions of n (that is, the partitions of n with at most $e-1$ equal parts). Also, the study of the "decomposition matrix", which measures the default of semi-simplicity, is of vivid interest in modular representation theory. This matrix is known to have a unitriangular shape with respect to the dominance order on the set of e-regular partitions, which parametrise its columns. It is natural to try to extend this property to the more general groups $G(l,1,n) = (\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n$, and to their quantizations, which are known as "Ariki-Koike algebras".

Inspired by this typical example, Geck and Rouquier introduced in [9] and [16] the notion of "canonical basic set" for a non semi-simple Hecke algebra $\mathcal{H}$. This approach formalises the fact that the decomposition matrix D of $\mathcal{H}$ is unitriangular, with respect to a certain ordering of its columns. Besides, it yields a bijection between $\mathrm{Irr}(\mathcal{H})$ and a subset $\mathscr{B}$ of $\mathrm{Irr}(\mathbf{H})$, where $\mathbf{H}$ is the semi-simple generic Hecke algebra which specialises to $\mathcal{H}$. The columns of D are then parametrised by $\mathscr{B}$, which is endowed with a natural order induced by Lusztig's $\mathbf{a}$-function, the order on the so-called "$\mathbf{a}$-invariants".

In this paper, we focus on Ariki-Koike algebras, that is, Hecke algebras $\mathbf{H}_{k,n}$ of the complex reflection groups $G(l,1,n)$. When $\mathbf{H}_{k,n}$ is non semi-simple, it is regarded as a specialisation, parametrised by a pair $(e,\mathbf{r})$ (where $e \in \mathbb{Z}_{>0} \cup \{\infty\}$ and $\mathbf{r} \in \mathbb{Z}^l$) , of a generic Ariki-Koike algebra $\mathbf{H}_n$. The irreducible representations of $\mathbf{H}_n$ are known to be parametrised by l-partitions of n. Using Broué and Malle's "cyclotomic Hecke algebras", see [5], Jacon defined in [18] $\mathbf{a}$-invariants for Ariki-Koike algebras, depending on a parameter $\mathbf{m} \in \mathbb{Q}^l$, which induce an order on the set of l-partitions of n. In [14], Geck and Jacon showed compatibility between this order, which has several geometric interpretations, and a combinatorial order $\ll_{\mathbf{m}}$ defined using Lusztig's "symbols". It turns out that the order $\ll_{\mathbf{m}}$ naturally arises when it comes to the study of $G(l,1,n)$, for instance in Kazhdan-Lusztig theory in type B_n (that is when $l=2$), see [12], or via the representations of Cherednik algebras, as studied in [6] or [26].

Ariki's proof in [2] of the LLT conjecture ([23]) enables us to compute the decomposition numbers of $\mathbf{H}_{k,n}$, when $\mathrm{char}(k)=0$, via the canonical basis of the Fock space $\mathcal{F}_{\mathbf{r}}$ (Theorem

3.6). With this approach, Geck and Jacon showed (see [14, Theorem 6.7.2]) that it is always possible to find a canonical basic set $\mathscr{B}$ for $\mathbf{H}_{k,n}$ for an appropriate choice of the parameter $\mathbf{m}$. In fact, they showed that $\mathscr{B}$ is the set $\Phi_{\mathbf{r}}(n)$ of Uglov l-partitions of rank n associated to $\mathbf{r}$.

We study here the canonical basic set for $\mathbf{H}_{k,n}$ with respect to $\ll_{\mathbf{m}}$ in full generality: given an element $\mathbf{m} \in \mathbb{Q}^l$, is there a canonical basic set for $\mathbf{H}_{k,n}$ with respect to $\ll_{\mathbf{m}}$? After defining a certain finite set of hyperplanes $\mathscr{P}^* \in \mathbb{Q}^l$, we establish the following classification (see Theorem 6.10):

- If $\mathbf{m} \notin \mathscr{P}^*$, then there exists a canonical basic set, which we explicitely determine. It can be regarded as a generalisation of the canonical basic set of Uglov multipartitions determined by Geck and Jacon.
- If $\mathbf{m} \in \mathscr{P}^*$, then there is no canonical basic set for $\mathbf{H}_{k,n}$.

The paper is structured as follows. In Section 2 we recall some background on the representation theory of Ariki-Koike algebras in the non semi-simple case. We also introduce cyclotomic Ariki-Koike algebras, and the order $\ll_{\mathbf{m}}$ we use throughout this article, and we formulate the precise question we are interested in. Section 3 summarizes the results of Geck Jacon in [14, Chapters 5 and 6] which are relevant for our purpose. We introduce the Fock space, which is a $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module, and its highest weight submodule $V(\mathbf{r})$ which appears in Ariki's theorem. In Section 4 we show, using Theorem 3.8, that any so-called "regular" element $\mathbf{m}$ affords a canonical basic set, which we describe in terms of "twisted Uglov multipartitions". Section 5 is devoted to the "asymptotic" case, which roughly speaking corresponds to the case where the difference between two arbitrary components of $\mathbf{m}$ is large. After explaining the particularity of this setting, we establish the existence of a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set of some "twisted Kleshchev multipartitions". In the final Section 6, we prove that when $\mathbf{m}$ is "singular", that is when $\mathbf{m} \in \mathscr{P}^*$, there is no canonical basic set with respect to $\ll_{\mathbf{m}}$. We sum up these results in Theorem 6.10.

2 Preliminaries

2.1 General notations

We start with some notations about partitions and multipartitions.

Let $n \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}_{>0}$. A *partition* of n is a decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\sum_i \lambda_i = n$. We write $|\lambda| = n$, the *rank* of n. The elements λ_i are called the *parts* of λ. We consider that a partition λ has an infinite number of parts $\lambda_i = 0$. We denote by $\Pi(n)$ the set of partitions of n, and we write $\lambda \vdash n$ if $\lambda \in \Pi(n)$. A *l-partition* of n (also referred to as a *multipartition*) is a sequence of partitions $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \dots, \lambda^l)$ such that $|\lambda^1| + \cdots + |\lambda^l| = n$. We define $\Pi_l(n)$ the set of l-partitions of n, and we write $\lambda \vdash_l n$ if $\lambda \in \Pi_l(n)$. The integer $|\boldsymbol{\lambda}| = n$ is called the rank of $\boldsymbol{\lambda}$.

Let $\boldsymbol{\lambda} \vdash_l n$. The *Young diagram* of $\boldsymbol{\lambda}$ is the set

$$[\boldsymbol{\lambda}] := \{(a, b, c)\,;\, a \geq 1, c \in [\![1, l]\!], \text{and } 1 \leq b \leq \lambda^c_a\}.$$

The elements of $[\boldsymbol{\lambda}]$ are called the nodes of $\boldsymbol{\lambda}$. For the sake of simplicity, we sometimes identify a multipartition with its Young diagram. A node γ of $[\boldsymbol{\lambda}]$ is called a *removable node* if the element $\boldsymbol{\mu}$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \backslash \{\gamma\}$ is still a multipartition (of rank $n-1$). In this case, γ is also called an *addable node* of $[\boldsymbol{\mu}]$.

A *multicharge* is an element $\mathbf{r} = (r_1, \dots, r_l) \in \mathbb{Z}^l$. Given $e \in \mathbb{Z}_{>1}$, a multicharge $\mathbf{r}$ and a multipartition $\boldsymbol{\lambda} \vdash_l n$, we can associate to each node $\gamma = (a, b, c)$ of $[\boldsymbol{\lambda}]$ its *residue modulo e* $\mathrm{r}_e(\gamma) := r_c + b - a \mod e$. For $i \in [\![0, e-1]\!]$, we call γ an *i-node* if $\mathrm{r}_e(\gamma) = i$.

We recall the classic *dominance* order on $\Pi(n)$. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ be two partitions of n. We say that that λ *dominates* μ, and we write $\lambda \trianglerighteq \mu$, if $\sum_{1 \leq i \leq d} \lambda_i \geq \sum_{1 \leq i \leq d} \mu_i$ for all $d \geq 1$. More generally, we introduce a dominance order on the the set of sequences of rational numbers in the same manner. Precisely, if $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ are such sequences, we define $\alpha \trianglerighteq \beta$ by $\sum_{1 \leq i \leq d} \alpha_i \geq \sum_{1 \leq i \leq d} \beta_i$ for all $d \geq 1$. We write $\alpha \triangleright \beta$ if $\alpha \trianglerighteq \beta$ and $\alpha \neq \beta$.

These are partial orders. In the following sections, we shall also consider other orders on the set $\Pi_l(n)$ of multipartitions.

We also need to introduce the *extended affine symmetric group*.

Let $l \in \mathbb{Z}_{>0}$. Denote by $\mathfrak{S}_l$ the symmetric group on $[\![1, l]\!]$, and by σ_i, $i \in [\![1, l-1]\!]$ its generators in its Coxeter presentation. Let $\{y_1, \dots, y_l\}$ be the standard basis of $\mathbb{Z}^l$. The extended affine symmetric group $\widehat{\mathfrak{S}}_l$ is the group with the following presentation:

- Generators: σ_i, $i \in [\![1, l-1]\!]$ and y_i, $i \in [\![1, l]\!]$.
- Relations :
  - $\sigma_i^2 = 1$ for all $i \in [\![1, l-1]\!]$,
  - $\sigma_i\sigma_{i+1}\sigma = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $i \in [\![1, l-2]\!]$,
  - $\sigma_i\sigma_j = \sigma_j\sigma_i$ whenever $i - j \neq 1 \mod l$,
  - $y_iy_j = y_jy_i$ for all $i, j \in [\![1, l]\!]$,
  - $\sigma_i y_j = y_j\sigma_i$ for all $i \in [\![1, l-1]\!]$ and $j \in [\![1, l]\!]$ such that $j \neq i, i+1 \mod l$,
  - $\sigma_i y_i \sigma_i = y_{i+1}$ for all $i \in [\![1, l-1]\!]$.

Note that $\widehat{\mathfrak{S}}_l$ is not a Coxeter group. Also, this group can be seen as the semi-direct product $\mathbb{Z}^l \rtimes \mathfrak{S}_l$.

Given $e \in \mathbb{Z}_{>1}$, there is an action of $\widehat{\mathfrak{S}}_l$ on $\mathbb{Z}^l$, via the formulas:

- $\sigma_i\mathbf{r} = (r_1, \dots, r_{i-1}, r_{i+1}, r_i, \dots, r_l)$ for all $i \in [\![1, l-1]\!]$, and
- $y_i\mathbf{r} = (r_1, \dots, r_{i-1}, r_i + e, r_{i+1}, \dots, r_l)$ for all $i \in [\![1, l]\!]$,

where $\mathbf{r} = (r_1, \dots, r_l) \in \mathbb{Z}^l$.

The set $\{\mathbf{r} \in \mathbb{Z}^l \mid 1 \leq r_1 \leq \cdots \leq r_l \leq e\}$ is a fundamental domain for this action.

2.2 Ariki-Koike algebras and decomposition maps

Let $l \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{>1}$. Let R be a subring of $\mathbb{C}$, $u, V_1, \dots, V_l$ be independent indeterminates, and set $A = R[u^{\pm 1}, V_1, \dots, V_l]$.

Definition 2.1. The generic Ariki-Koike algebra $\mathbf{H}_n = \mathbf{H}_{A,n}(u, V_1, \dots, V_l)$ is the unital associative A-algebra with generators T_i, $i = 0, \dots, n-1$, and relations

- $(T_i - u)(T_i + 1) = 0$ for all $i \in [\![1, n-1]\!]$.
- $(T_0 - V_1)\dots(T_0 - V_l) = 0$
- $T_0T_1T_0T_1 = T_1T_0T_1T_0$
- $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for all $i \in [\![1, n-2]\!]$
- $T_iT_j = T_jT_i$ whenever $|i - j| > 1$

Let K be the field of fractions of A. We set $\mathbf{H}_{K,n} = K \otimes_A \mathbf{H}_n$. We denote by $\mathrm{Irr}(\mathbf{H}_{K,n})$ the set of irreducible $\mathbf{H}_{K,n}$-modules.

Theorem 2.2 (Ariki, [4])**.** *The algebra $\mathbf{H}_{K,n}$ is split semi-simple.*

As a consequence, if R contains $\xi_l = \exp(2i\pi/l)$, Tits' deformation theorem implies that the irreducible representations of $\mathbf{H}_{K,n}$ are in one-to-one correspondence with the irreducible representations of the complex reflection group $W_n = G(l, 1, n)$ over $\mathbb{K} = \mathrm{Frac}(R)$. It is known that these representations are in one-to-one correspondence with $\Pi_l(n)$, the set of l-partitions of n. Therefore, we can write

$$\mathrm{Irr}(\mathbf{H}_{K,n}) = \{E^{\boldsymbol{\lambda}}; \boldsymbol{\lambda} \vdash_l n\}.$$

The representations $E^{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} \vdash_l n$, are called the Specht representations. The correspondence $\Pi_l(n) \to \mathrm{Irr}(\mathbf{H}_{K,n})$, $\boldsymbol{\lambda} \mapsto E^{\boldsymbol{\lambda}}$ is explicitely described in [4, Section 3].

2.2.1 Specialisations

We have the following criterion of semi-simplicity for specialised Ariki-Koike algebras.

Theorem 2.3 (Ariki, [1]). *Let $\theta : A \longrightarrow k$ be a specialisation, with $k = Frac(\theta(A))$. Denote $\eta = \theta(u)$ and $\eta_i = \theta(V_i)$, for all $i \in [\![1, l]\!]$. Then the specialised algebra $\mathbf{H}_{k,n}(\eta, \eta_1, \dots, \eta_l) = k \otimes_A \mathbf{H}_n$ is (split) semi-simple if and only if*

$$\Big(\prod_{-n<d<n}\prod_{1\leq i<j\leq l}(\eta^d\eta_i - \eta_j)\Big)\Big(\prod_{1\leq i\leq n}(1+\eta+\cdots+\eta^{i-1})\Big) \neq 0.$$

Moreover, we have this result by Dipper and Mathas:

Theorem 2.4 (Dipper and Mathas, [7]). *Let $\mathcal{E} = \{\eta_1, \dots, \eta_l\}$ and suppose that there exists a partition $\mathcal{E} = \mathcal{E}_1 \sqcup \cdots \sqcup \mathcal{E}_s$ such that*

$$\prod_{1\leq\alpha<\beta\leq s}\prod_{(V_i,V_j)\in\mathcal{E}_\alpha\times\mathcal{E}_\beta}\prod_{-n<a<n}(\eta^a\eta_i - \eta_j) \neq 0.$$

Then $\mathbf{H}_{k,n}(\eta, \eta_1, \dots, \eta_l)$ is Morita equivalent to the algebra

$$\bigoplus_{\substack{n_1+\cdots+n_s=n\\ n_1,\dots,n_s\geq 0}} \mathbf{H}_{k,n_1}(\eta, \mathcal{E}_1) \otimes_k \cdots \otimes_k \mathbf{H}_{k,n_s}(\eta, \mathcal{E}_s).$$

As a consequence, in order to study non semi-simple Ariki-Koike algebras, it is sufficient to consider the specialisations $\mathbf{H}_{k,n}(\eta, \eta_1, \dots, \eta_l)$ of $\mathbf{H}_n$ defined via

$$\begin{array}{rccc} \theta: & A & \longrightarrow & k \\ & u & \longmapsto & \zeta \\ & V_i & \longmapsto & \zeta^{r_i}, \end{array}$$

where ζ is a primitive root of unity of order e (possibly infinite), and $r_i \in \mathbb{Z}$ for all $i \in [\![1, l]\!]$. Hence each specialisation considered in the sequel will be characterised by a pair $(e, \mathbf{r})$ where e is the multiplicative order of $\zeta = \theta(u)$, and $\mathbf{r} = (r_1, \dots, r_l) \in \mathbb{Z}^l$. We will denote equally $\mathbf{H}_{k,n}(\zeta, \zeta^{r_1}, \dots, \zeta^{r_j}) = \mathbf{H}_{k,n}^{(e,\mathbf{r})}$ the specialised Ariki-Koike algebra corresponding to $(e, \mathbf{r})$. We also denote $\theta_{(e,\mathbf{r})}$ the associated specialisation map.

Consider the specialisation $\theta_{(e,\mathbf{r})} : A \longrightarrow k$ with $k = \mathrm{Frac}(\theta_{(e,\mathbf{r})}(A))$. In accordance with [15], there is an associated decomposition map $d_{\theta_{(e,\mathbf{r})}} : R_0(\mathbf{H}_{K,n}) \longrightarrow R_0(\mathbf{H}_{k,n}^{(e,\mathbf{r})})$, and we can write

$$d_{\theta_{(e,\mathbf{r})}}([E^\lambda]) = \sum_{M\in \mathrm{Irr}(\mathbf{H}_{k,n}^{(e,\mathbf{r})})} d_{\lambda,M}[M].$$

The *decomposition matrix* of $\mathbf{H}_{k,n}^{(e,\mathbf{r})}$ is the matrix

$$D_{\theta_{(e,\mathbf{r})}} = (d_{\lambda,M})_{\substack{\lambda\vdash_l n \\ M\in \mathrm{Irr}(\mathbf{H}_{k,n}^{(e,\mathbf{r})})}}.$$

The elements $d_{\lambda,M}$ are called the *decomposition numbers* of $\mathbf{H}_{k,n}^{(e,\mathbf{r})}$. If the specialised algebra is semi-simple, the decomposition map is trivial and $D_{\theta_{(e,\mathbf{r})}}$ is the identity matrix. In general, this matrix has a rectangular shape, since $|\mathrm{Irr}(\mathbf{H}_{k,n}^{(e,\mathbf{r})})| \leq |\{E^\lambda; \boldsymbol{\lambda} \vdash_l n\}|$.

Terminology: For simplicity, we say that $\boldsymbol{\lambda}$ *appears* in the column C indexed by M if $d_{\lambda,M} \neq 0$. We also say that C *contains* $\boldsymbol{\lambda}$.

Actually, we can recover this matrix $D_{\theta_{(e,\mathbf{r})}}$ from any specialisation $\mathbf{H}_{k,n}^{(e,\mathbf{s})}$ where $\mathbf{s}$ is in the class of $\mathbf{r}$ modulo $\widehat{\mathfrak{S}}_l$. It is important to understand the consequences of choosing another multicharge to get the decomposition matrix.

Denote by $C(\mathbf{r})$ (or simply C) the class of $\mathbf{r}$ modulo $\widehat{\mathfrak{S}}_l$, and by $C_e(\mathbf{r})$ (or simply C_e) the class of $\mathbf{r}$ modulo the subgroup $\langle y_1, ..., y_l\rangle$ of $\widehat{\mathfrak{S}}_l$ (i.e. the set of all $\mathbf{s} = (s_1, ..., s_l) \in \mathbb{Z}^l$ such that $\forall 1 \leq i \leq l, s_i = r_i \mod e$).

- If $\mathbf{s} \in C_e$, then it is clear that $\mathbf{H}_{k,n}^{(e,\mathbf{s})} = \mathbf{H}_{k,n}^{(e,\mathbf{r})}$. Besides, the decompositions maps are the same. Therefore, the decomposition matrices are strictly equal.

- If $\mathbf{s} \in C$, one still has $\mathbf{H}_{k,n}^{(e,\mathbf{s})} = \mathbf{H}_{k,n}^{(e,\mathbf{r})}$. We therefore denote $\mathbf{H}_{k,n}$ this algebra. However, the decomposition maps do not necessarily coincide. In fact, if we have $\mathbf{s} = \sigma(\mathbf{r})$, for some $\sigma \in \mathfrak{S}_l$, denote $\theta := \theta_{(e,\mathbf{r})}$ and $\theta_\sigma := \theta_{(e,\mathbf{s})}$. Then we can write $d_\theta([E^\lambda]) = \sum_{M \in \mathrm{Irr}(\mathbf{H}_{k,n})} d_{\lambda,M}[M]$ and $d_{\theta_\sigma}([E^\lambda]) = \sum_{M \in \mathrm{Irr}(\mathbf{H}_{k,n})} d^\sigma_{\lambda,M}[M]$.
  Now since $\theta_\sigma(V_i) = \zeta^{s_i} = \zeta^{r_{\sigma(i)}} = \theta(V_{\sigma(i)})$, we have

$$d^\sigma_{\lambda,M} = d_{\lambda^\sigma,M}, \quad \forall \boldsymbol{\lambda} \vdash_l n\,,\ \forall M \in \mathrm{Irr}(\mathbf{H}_{k,n})$$

  where $\boldsymbol{\lambda}^\sigma = (\lambda^{\sigma(1)}, \dots \lambda^{\sigma(l)})$.
  This means that the decomposition matrices are equal up to a permutation of the rows. Equivalently, they are strictly equal (denoted by D) provided the parametrisation of the rows is changed: the row of D labeled by $\boldsymbol{\lambda}$ with respect to the parametrisation afforded by $\mathbf{r}$ is labeled by $\boldsymbol{\lambda}^\sigma$ with respect to the parametrisation afforded by $\mathbf{s} = \sigma(\mathbf{r})$.

To sum up, the specialised Ariki-Koike algebra only depends on C. We denote it by $\mathbf{H}_{k,n}$. Hence we consider that for any $\mathbf{s} \in C$, we obtain one genuine matrix, that we denote dy D, but that each element $\mathbf{s} \in C$ affords a different (in general) parametrisation of the rows of D.

In our purpose to study canonical basic sets, introduced in the next section, it is crucial to understand which parametrisation we use. In fact, we will fix a multicharge $\mathbf{r}$ once and for all, and this will fix a parametrisation of the rows of D.

In the sequel, we will be interested with the shape of this decomposition matrix. One of the classic problems is to find an indexation of the simple $\mathbf{H}_{k,n}$-modules so that D is upper unitriangular, that is

$$D = \overbrace{\begin{pmatrix} 1 & 0 & \dots & 0 \\ \star & 1 & \dots & 0 \\ \vdots & \star & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & & \star \\ \vdots & \vdots & \dots & \vdots \end{pmatrix}}^{\mathrm{Irr}(\mathbf{H}_{k,n})} \Bigg\} \Pi_l(n)$$

More precisely, we ask for an order $\leq$ on $\mathrm{Irr}(\mathbf{H}_{k,n})$ such that D has the above shape with respect to this order, that is $i > j \Rightarrow M_i \leq M_j$, if $M_i \in \mathrm{Irr}(\mathbf{H}_{k,n})$ parametrises the i-th column of D. These problems have been solved in [14], using the theory of canonical basic sets. This approach enables us find a bijection between $\mathrm{Irr}(\mathbf{H}_{k,n})$ and a subset of $\Pi_l(n)$, and therefore to label the simple $\mathbf{H}_{k,n}$-modules by certain l-partitions, the *Uglov multipartitions*.

In this paper, we address the "converse" question: given a certain natural order on the set of multipartitions (coming from some *cyclotomic* Hecke Algebra), is it possible to find a parametrisation of $\mathrm{Irr}(\mathbf{H}_{k,n})$ by a subset of $\Pi_l(n)$ such that D is upper unitriangular with respect to this order? We first need to precise which specific orders we are interested in, and some background about canonical basic sets.

2.3 Cyclotomic specialisations and canonical basic sets

We introduce a new kind of specialisation of $\mathbf{H}_n$, independent of the previous non semi-simple specialisation. This so-called *cyclotomic* specialisation yields a one-parameter algebra, which enables us to construct an "algebraic" order on the simple modules of $\mathbf{H}_{K,n}$.

Precisely, a cyclotomic specialisation is any specialisation

$$\begin{array}{rccl} \theta_y : & A & \longrightarrow & \mathbb{K}(y) \\ & V_i & \longmapsto & \eta_i = y^{t_i}\xi_l^{i-1} \text{ for } i \in [\![1, l]\!] \\ & u & \longmapsto & \eta = y^t, \end{array}$$

where y is an indeterminate, $t \in \mathbb{Z}_{>0}$ and $(t_1, \dots, t_l) \in \mathbb{Z}^l$. We recall that $\xi_l = \exp(2i\pi/l)$.

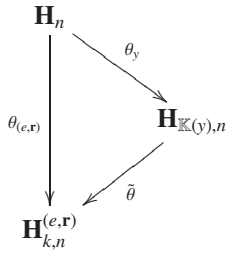
We denote by $\mathbf{H}_{\mathbb{K}(y),n}$ the corresponding specialised algebra, which we call *cyclotomic Ariki-Koike algebra*. Since

$$\Big(\prod_{-n<d<n} \prod_{1 \leq i < j \leq l} (\eta^d \eta_i - \eta_j)\Big)\Big(\prod_{1 \leq i \leq n} (1 + \eta + \dots + \eta^{i-1})\Big) \neq 0,$$

the algebra $\mathbf{H}_{\mathbb{K}(y),n}$ is (split) semi-simple by Theorem 2.3. Hence, the irreducible representations of $\mathbf{H}_{\mathbb{K}(y),n}$ are parametrised by l-partitions. We denote $\mathrm{Irr}(\mathbf{H}_{\mathbb{K}(y),n}) = \{E_y^{\boldsymbol{\lambda}}; \boldsymbol{\lambda} \vdash_l n\}$. These representations can be seen as reductions of the elements $E^{\boldsymbol{\lambda}}$ of $\mathrm{Irr}(\mathbf{H}_{k,n})$ via θ_y.

For such a cyclotomic specialisation, we also set $m_i = t_i/t$ for all $i \in [\![1, l]\!]$, and $\mathbf{m} = (m_1, \ldots, m_l) \in \mathbb{Q}^l$. This element $\mathbf{m}$ is called a *weight sequence*.

Note that any (non semi-simple) algebra $\mathbf{H}_{k,n}^{(e,\mathbf{r})} = \mathbf{H}_{k,n}$ can be obtained by specialising a certain cyclotomic algebra. Denote $\theta_{(e,\mathbf{r})}(u) = \zeta = \exp(2ik\pi/e)$ with $\gcd(k, e) = 1$. In fact, if $t \in \mathbb{Z}_{>0}$ is arbitrary and $m_i = r_i - e(i-1)/kl$, we have a cyclotomic algebra $\mathbf{H}_{\mathbb{K}(y),n}$ depending on a parameter y, which can be specialised via $y \mapsto \zeta^{1/t} = \exp(2ik\pi/et)$. One can check that the resulting algebra is indeed $\mathbf{H}_{k,n}$. In other terms, the following diagram commutes:



$$\text{where } \theta_y : \begin{array}{lcl} A & \longrightarrow & \mathbb{K}(y) \\ u & \longmapsto & y^t \quad \text{with } t \text{ such that } t(r_i - \frac{e(i-1)}{kl}) \in \mathbb{Z}, \\ V_i & \longmapsto & y^{t(r_i - \frac{e(i-1)}{kl})} \xi_l^{i-1} \quad \text{for } i \in [\![1, l]\!], \end{array}$$

$$\text{and } \tilde{\theta} : \begin{array}{lcl} \mathbb{K}(y) & \longrightarrow & k \quad \text{such that } \zeta^{1/t} \in k \\ y & \longmapsto & \zeta^{1/t}. \end{array}$$

Let $H_{\mathbb{K}(y),n}$ be a cyclotomic Ariki-Koike algebra with parameter $t \in \mathbb{Z}_{>0}$ and weight sequence $\mathbf{m}$. We can compute the Schur elements $\mathbf{c}_y^{\boldsymbol{\lambda}}$ of the simple $H_{\mathbb{K}(y),n}$-modules $E_y^{\boldsymbol{\lambda}}$, by specialising (via θ_y) the Schur elements $\mathbf{c}^{\boldsymbol{\lambda}}$ of $E^{\boldsymbol{\lambda}}$. Explicit formulas for $\mathbf{c}^{\boldsymbol{\lambda}}$ have been given independently in [13] and [28]. We can then set

$$\mathbf{a}^{(\mathbf{m},t)}(\boldsymbol{\lambda}) = \min\{s \in \mathbb{Z} \,|\, y^s \mathbf{c}_y^{\boldsymbol{\lambda}} \in R[y]\}.$$

These elements are called the $\mathbf{a}$-invariants for $E_y^{\boldsymbol{\lambda}}$. In particular, when $l = 1$, the map $E^{\boldsymbol{\lambda}} \mapsto \mathbf{a}^{(\mathbf{m},t)}(\boldsymbol{\lambda}) = \mathbf{a}(\boldsymbol{\lambda})$ coincides with Lusztig's $\mathbf{a}$-function, in the sense of [27]. The $\mathbf{a}$-invariants induce an order on Specht modules, namely $E^{\boldsymbol{\lambda}} \sqsubseteq E^{\boldsymbol{\mu}} \Leftrightarrow [\boldsymbol{\lambda} = \boldsymbol{\mu} \text{ or } \mathbf{a}^{(\mathbf{m},t)}(\boldsymbol{\lambda}) < \mathbf{a}^{(\mathbf{m},t)}(\boldsymbol{\mu})]$. The general notion of canonical basic sets requires an order on the Specht modules. In the case of Ariki-Koike algebras, it is natural to use this algebraic order. In fact, we will use a combinatorial order $\ll_{\mathbf{m}}$ which contains the order $\sqsubseteq$ above. This is the order on shifted $\mathbf{m}$-symbols defined in [14].

Recall that the shifted $\mathbf{m}$-symbol of $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^l) \vdash_l n$ of size $p \in \mathbb{Z}$ is the l-tuple $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda}) = (\mathfrak{B}_{\mathbf{m}}^1(\boldsymbol{\lambda}), \ldots, \mathfrak{B}_{\mathbf{m}}^l(\boldsymbol{\lambda}))$, where $\mathfrak{B}_{\mathbf{m}}^j(\boldsymbol{\lambda}) = (\mathfrak{B}_{p+\lfloor m_j \rfloor}^j(\boldsymbol{\lambda}), \ldots, \mathfrak{B}_1^j(\boldsymbol{\lambda}))$, with $\mathfrak{B}_i^j(\boldsymbol{\lambda}) = \lambda_i^j - i + p + m_j$, for all $j \in [\![1, l]\!]$ and $i \in [\![1, p + \lfloor m_j \rfloor]\!]$. Note that p must be sufficiently large, so that $p + \lfloor m_j \rfloor \geq 1$ for all $j \in [\![1, l]\!]$. This ensures that each $\mathfrak{B}_{\mathbf{m}}^j(\boldsymbol{\lambda})$ is well defined. As usual, we consider that each partition λ^j of $\boldsymbol{\lambda}$ has an infinite number of parts $\lambda_i^j = 0$.

The shifted $\mathbf{m}$-symbol $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$ is pictured by an array whose j-th line (numbered from bottom to top) corresponds to $\mathfrak{B}_{\mathbf{m}}^j(\boldsymbol{\lambda})$.

Example 2.5. Let $\mathbf{m} = (1/2, 2, -1)$ and $\boldsymbol{\lambda} = (1.1, \emptyset, 2) \vdash_3 4$. We choose $p = 3$. Then

$$\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 3 & & & \\ 0 & 1 & 2 & 3 & 4 \\ 1/2 & 5/2 & 7/2 & & \end{pmatrix}.$$

Note that this symbol can easily be obtained from the shifted $\mathbf{m}$-symbol of the empty l-partition, by adding the parts of λ^i to the i-th row (numbered from bottom to top) of $\mathfrak{B}_{\mathbf{m}}(\emptyset)$, from right to left.

The symbol $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$ has $h = lp + \sum_{1 \leq j \leq l} \lfloor m_j \rfloor$ elements.

Write $\mathfrak{b}_{\mathbf{m}}(\boldsymbol{\lambda}) = (\mathfrak{b}^1_{\mathbf{m}}(\boldsymbol{\lambda}), \mathfrak{b}^2_{\mathbf{m}}(\boldsymbol{\lambda}), \dots, \mathfrak{b}^h_{\mathbf{m}}(\boldsymbol{\lambda}))$ the sequence of elements in $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$, in decreasing order. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Pi_l(n)$, we define the order $\ll_{\mathbf{m}}$ by

$$\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu} \quad \overset{\text{def}}{\iff} \quad \boldsymbol{\lambda} = \boldsymbol{\mu} \text{ or } \mathfrak{b}_{\mathbf{m}}(\lambda) \rhd \mathfrak{b}_{\mathbf{m}}(\boldsymbol{\mu}),$$

in the general sense of dominance order on sequences of rational numbers defined in section 2.1.

Set also $n_{\mathbf{m}}(\boldsymbol{\lambda}) = \sum_{1 \leq i \leq h} (i-1)\mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda})$. By [14, Proposition 5.5.11], we can compute the $\mathbf{a}$-invariant of $\boldsymbol{\lambda}$ using symbols, namely $\mathbf{a}^{(\mathbf{m},t)}(\boldsymbol{\lambda}) = t(n_{\mathbf{m}}(\boldsymbol{\lambda}) - n_{\mathbf{m}}(\emptyset))$. As a direct consequence, we have the following compatibility property ([14, Proposition 5.7.7]):

$$[\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu} \text{ and } \boldsymbol{\lambda} \neq \boldsymbol{\mu}] \quad \Rightarrow \quad \mathbf{a}^{(\mathbf{m},t)}(\boldsymbol{\lambda}) < \mathbf{a}^{(\mathbf{m},t)}(\boldsymbol{\mu}). \tag{1}$$

This order on symbols has the advantage of being easier to handle, since it is purely combinatorial. Besides, it naturally appears in the representation theory of the complex reflection groups of type $G(l, 1, n)$. For instance, when $l = 2$, Geck and Iancu showed in [12, Theorem 7.11] that this order is compatible with an order $\leq_L$, defined (in [11]) on $\mathrm{Irr}(G(l, 1, n))$ using Lusztig's families (and related to the order $\leq_{\mathcal{LR}}$ defining Kazhdan-Lusztig cells). In fact, they showed that in general, one has $\boldsymbol{\lambda} \leq_L \boldsymbol{\mu} \Rightarrow \boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}$, and that in some particular cases, both orders are equivalent. Note that the version of the order $\ll_{\mathbf{m}}$ defined in [12, Section 3] is slightly different from the one we just defined. In another vein, Chlouveraki, Gordon and Griffeth have used the compatibility property (1) above to deduce information on the decomposition of standard modules of Cherednik algebras, [6, Theorem 5.7]. Also, Liboz showed in [26] that the order $\ll_{\mathbf{m}}$ contains the order induced by the "$\mathbf{c}$-function" on $\mathrm{Irr}(G(l, 1, n))$ used in the representation theory of Cherednik algebras.

We can now state the definition of a canonical basic set in the sense of [10], using the order $\ll_{\mathbf{m}}$ instead of the order induced by the $\mathbf{a}$-invariants.

Fix $\mathbf{r} \in \mathbb{Z}^l$ and $e \in \mathbb{Z}_{>1} \cup \{\infty\}$. Consider the specialised algebra $\mathbf{H}^{(e,\mathbf{r})}_{k,n} = \mathbf{H}_{k,n}$. For $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$, set $\mathscr{S}(M) = \{\boldsymbol{\lambda} \vdash_l n \,|\, d_{\boldsymbol{\lambda},M} \neq 0\}$. Note that this set strongly depends on the choice of $\mathbf{r}$ (which is fixed once and for all), as explained on page 4.

Definition 2.6. Assume that the following conditions hold:

1. For $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$, there exists a unique element $\boldsymbol{\lambda}_M \in \mathscr{S}(M)$ such that for all $\boldsymbol{\mu} \in \mathscr{S}(M)$, $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}$.
2. The map $\mathrm{Irr}(\mathbf{H}_{k,n}) \to \Pi_l(n)$, $M \mapsto \boldsymbol{\lambda}_M$ is injective.
3. We have $d_{\boldsymbol{\lambda}_M,M} = 1$, for all $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$.

Then the set $\mathscr{B} := \{\boldsymbol{\lambda}_M; M \in \mathrm{Irr}(\mathbf{H}_{k,n})\} \subseteq \Pi_l(n)$ is in one-to-one correspondence with $\mathrm{Irr}(\mathbf{H}_{k,n})$. It is called a *canonical basic set for* $(\mathbf{H}_{k,n}, \mathbf{r})$ *with respect to* $\ll_{\mathbf{m}}$.

Remark 2.7. As a direct consequence, if there exists a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$, it is unique. Moreover, the three conditions of Definition 2.6 encode the fact that D is upper unitriangular with respect to $\ll_{\mathbf{m}}$.

Remark 2.8. Just like decomposition matrices, it is important to understand how the notion of canonical basic set depends on $\mathbf{r}$. Indeed, this multicharge determines a parametrisation of the rows of D. This parametrisation being invariant in the class C_e, we are ensured that for $\mathbf{s} \in C_e$, if $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set $\mathscr{B}$ (with respect to $\ll_{\mathbf{m}}$), then $\mathscr{B}$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ (with respect to $\ll_{\mathbf{m}}$).

However, this is not true for general $\mathbf{s} \in C$. For such a multicharge, it is sometimes possible to find a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$, even if $(\mathbf{H}_{k,n}, \mathbf{r})$ does not admit a canonical basic set and both algebras are equal, see Example 6.9. Also, if $\mathscr{B}$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$, it is sometimes possible to find $\mathbf{s} \in C$ such that $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set $\mathscr{B}'$ and $\mathscr{B}' \neq \mathscr{B}$, see Example 4.3.

Fix $\mathbf{r} \in \mathbb{Z}^l, n, l \in \mathbb{Z}_{>1}, e \in \mathbb{Z}_{>1} \cup \{\infty\}$. We want to review which values of $\mathbf{m} \in \mathbb{Q}^l$ afford a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$, and which do not. We will see that unless $\mathbf{m}$ belongs to some hyperplanes of $\mathbb{Q}^l$, the algebra $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, which we can explicitely describe.

Remark 2.9. We could have adressed a slightly different question. Since we can recover the decomposition matrix from any $\mathbf{s} \in C$ (up to a change of parametrisation of the rows), we could also ask which weight sequences $\mathbf{m}$ afford a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$, for some $\mathbf{s} \in C$. Note that solving the first question automatically solves this weaker question, by taking the reunion over $\mathbf{s} \in C$ of all weight sequences $\mathbf{m}$ that afford a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$.

First, let us recall what particular values of $\mathbf{m}$ are known to afford canonical basic sets for $(\mathbf{H}_{k,n}, \mathbf{r})$.

3 Existence of canonical basic sets for appropriate parameters

Consider the specialised Ariki-Koike algebra $\mathbf{H}_{k,n} = \mathbf{H}_{k,n}^{(e,\mathbf{r})}$ where $e \in \mathbb{Z}_{>1} \cup \{\infty\}$, and $\mathbf{r} = (r_1, \dots, r_l) \in \mathbb{Z}^l$. We want to find a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$, in the sense of Definition 2.6. The following results prove that it is always possible to find $\mathbf{m} \in \mathbb{Q}^l$ such that $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$. Besides, they can be explicitely described, either "directly" (FLOTW l-partitions) or recursively (Uglov l-partitions). These results can be found in [14].

3.1 FLOTW multipartitions as canonical basic sets

Let $\mathscr{S}_e^l = \{\mathbf{r} = (r_1, \dots, r_l) \in \mathbb{Z}^l \,|\, 0 \leq r_j - r_i < e \text{ for all } i < j\}$. In this section, we assume that $\mathbf{r} \in \mathscr{S}_e^l$.

The following definition is due to Foda, Leclerc, Okado, Thibon and Welsh, see [8].

Definition 3.1. Let $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l) \vdash_l n$ and $\mathbf{r} \in \mathscr{S}_e^l$. Then $\boldsymbol{\lambda}$ is called a *FLOTW l-partition* if:

1. For all $j \in [\![1, l-1]\!]$, $\lambda_i^j \geq \lambda^{j+1}_{i+r_{j+1}-r_j}, \forall i \geq 1$; and $\lambda_i^l \geq \lambda^1_{i+e+s_1-s_l}, \forall i \geq 1$.
2. The residues of the rightmost nodes of the length p rows (for all $p > 0$) of $\boldsymbol{\lambda}$ do not cover $[\![0, e-1]\!]$.

Denote by $\Psi_{(\mathbf{r},e)}$ the set of FLOTW l-partitions associated to $\mathbf{r} \in \mathscr{S}_e^l$, and by $\Psi_{(\mathbf{r},e)}(n) \subset \Psi_{(\mathbf{r},e)}$ the ones of rank n.

Remark 3.2. If $l = 1$, the FLOTW l-partitions are exactly the e-regular partitions.

We have the following result by Geck and Jacon.

Theorem 3.3 ([14, Theorem 5.8.2])**.** *Let $\mathbf{r} \in \mathscr{S}_e^l$, $\mathbf{v} = (v_1, \dots, v_l) \in \mathbb{Q}^l$ such that $i < j \Rightarrow 0 < v_j - v_i < e$, and set $\mathbf{m} = \mathbf{r} - \mathbf{v} = (r_1 - v_1, \dots, r_l - v_l)$. Then $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set $\Psi_{(\mathbf{r},e)}(n)$.*

Despite the restriction that $\mathbf{r}$ must be in $\mathscr{S}_e^l$, this is however relevant since $\mathscr{S}_e^l$ contains a fundamental domain for the action of $\widehat{\mathfrak{S}}_l$ on $\mathbb{Z}^l$ (the one described in section 2.1), and since $\mathbf{H}_{k,n}$ depends only on the class C of $\mathbf{r}$ modulo $\widehat{\mathfrak{S}}_l$. Besides, this theorem holds regardless of the characteristic of the field k, and $\Psi_{(\mathbf{r},e)}(n)$ has the advantage of being directly computable.

3.2 Ariki's theorem and Uglov multipartitions as canonical basic sets

We now want to find a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ for an arbitrary value of $\mathbf{r}$. In the sequel, we will need to assume that $\mathrm{char}(k) = 0$. Indeed, this is an essential condition for Ariki's theorem to apply. This result links the theory of canonical bases (in the sense of Kashiwara, or Lusztig) of quantum groups with the modular representation of Ariki-Koike algebras.

We do not recall here the theory of quantum groups. We refer to [3] and [17] for detailed background on $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$ in particular. We denote by $e_i, f_i, t_i, t_i^{-1}, \mathfrak{d}, \mathfrak{d}^{-1}$ the generators of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$, and by ω_i, $i \in [\![0, e-1]\!]$, the fundamental weights of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$.

We redefine the Fock space and its properties. Let $\mathbf{r} \in \mathbb{Z}^l$, q be an indeterminate. The Fock space $\mathcal{F}_{\mathbf{r}}$ is the $\mathbb{Q}(q)$-vector space with formal basis $|\boldsymbol{\lambda}, \mathbf{r}\rangle$ where $\boldsymbol{\lambda} \vdash_l n$, i.e.

$$\mathcal{F}_{\mathbf{r}} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{\boldsymbol{\lambda} \vdash_l n} \mathbb{Q}(q)|\boldsymbol{\lambda}, \mathbf{r}\rangle.$$

We define an order on the set of addable or removable i-node of a l-partition $\boldsymbol{\lambda}$. Let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be two removable or addable i-nodes of $\boldsymbol{\lambda} \vdash_l n$. We write

$$\gamma \prec_{(\mathbf{r},e)} \gamma' \text{ if } \begin{cases} b - a + s_c < b' - a' + s_{c'} \text{ or} \\ b - a + s_c = b' - a' + s_{c'} \quad \text{and} \quad c > c'. \end{cases}$$

Let $\boldsymbol{\lambda} \vdash_l n$ and $\boldsymbol{\mu} \vdash_l n+1$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$ where γ is an i-node. We set

$$\begin{aligned} N_i^{\prec}(\boldsymbol{\lambda}, \boldsymbol{\mu}) =& \sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \prec_{(\mathbf{s},e)} \gamma\} - \\ & \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \prec_{(\mathbf{s},e)} \gamma\}, \end{aligned}$$

$$\begin{aligned} N_i^{\succ}(\boldsymbol{\lambda}, \boldsymbol{\mu}) =& \sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \succ_{(\mathbf{s},e)} \gamma\} - \\ & \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \succ_{(\mathbf{s},e)} \gamma\}, \end{aligned}$$

$$N_i(\boldsymbol{\lambda}) = \sharp\{\text{addable } i\text{-nodes of } \boldsymbol{\lambda}\} - \sharp\{\text{removable } i\text{-nodes of } \boldsymbol{\lambda}\},$$

$$\text{and} \qquad N_{\mathfrak{b}}(\boldsymbol{\lambda}) = \sharp\{0\text{-nodes of } \boldsymbol{\lambda}\}.$$

The following result is due to Jimbo, Misra, Miwa and Okado.

Theorem 3.4 ([21])**.** *The formulas*

$$e_i\, |\boldsymbol{\lambda}, \mathbf{r}\rangle = \sum_{\mathrm{r}_e([\boldsymbol{\lambda}]\setminus[\boldsymbol{\mu}])=i} q^{-N_i^{\prec}(\boldsymbol{\mu},\boldsymbol{\lambda})}\, |\boldsymbol{\mu}, \mathbf{r}\rangle\,,$$

$$f_i\, |\boldsymbol{\lambda}, \mathbf{r}\rangle = \sum_{\mathrm{r}_e([\boldsymbol{\mu}]\setminus[\boldsymbol{\lambda}])=i} q^{-N_i^{\succ}(\boldsymbol{\lambda},\boldsymbol{\mu})}\, |\boldsymbol{\mu}, \mathbf{r}\rangle\,,$$

$$t_i|\boldsymbol{\lambda}, \mathbf{r}\rangle = q^{N_i(\boldsymbol{\lambda})}|\boldsymbol{\lambda}, \mathbf{r}\rangle \quad \textit{and}$$

$$\mathfrak{d}|\boldsymbol{\lambda}, \mathbf{r}\rangle = -(\Delta(\mathbf{r}) + N_{\mathfrak{b}}(\boldsymbol{\lambda}))|\boldsymbol{\lambda}, \mathbf{r}\rangle, \textit{ for all } i \in [\![0, e-1]\!]$$

endow $\mathcal{F}_{\mathbf{r}}$ with the structure of an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module. Here, $\Delta(\mathbf{r})$ is rational number defined in [29].

The element $|\emptyset, \mathbf{r}\rangle \in \mathcal{F}_{\mathbf{r}}$ is a highest weight vector, of highest weight $\Lambda_{\mathbf{r}} := \omega_{r_1 \bmod e} + \cdots + \omega_{r_l \bmod e}$. We denote by $V(\mathbf{r}) \subset \mathcal{F}_{\mathbf{r}}$ the irreducible highest weight $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module spanned by $|\emptyset, \mathbf{r}\rangle$. This module $V(\mathbf{r})$ is endowed with a crystal basis, a crystal graph $\mathscr{G}_{\mathbf{r}}$, and a canonical (or global) basis, in the sense of [22]. Define the *Uglov l-partitions* to be the elements labelling the vertices of $\mathscr{G}_{\mathbf{r}}$. In order to determine these so-called Uglov multipartitions, we need to recall the definition of *good addable* and *good removable* i-nodes. Denote by $\Phi_{\mathbf{r}}$ the set of Uglov l-partitions, labelling the vertices of $V(\mathbf{r})$, and by $\Phi_{\mathbf{r}}(n)$ the ones of rank n.

Let $\boldsymbol{\lambda} \vdash_l n$. Consider the set of its addable and removable i-nodes, ordered with respect to $\prec_{(\mathbf{r},e)}$. Encode each addable (resp. removable) i-node with the letter A (resp. R). This yields a word of the form $A^{\alpha_1}R^{\beta_1}\ldots A^{\alpha_p}R^{\beta_p}$. Delete recursively all the occurences of type RA in this word. We get a word of the form $A^{\alpha}R^{\beta}$. Denote it by $w_i(\boldsymbol{\lambda})$. Let γ be the rightmost addable (resp. leftmost removable) i-node in $w_i(\boldsymbol{\lambda})$. Then γ is called the *good addable* (resp. *good removable*) i-node of $\boldsymbol{\lambda}$.

Theorem 3.8 in [21] gives a way to describe the crystal graph of the whole Fock space $\mathcal{F}_{\mathbf{r}}$, which also is an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module. Taking the connected component of this graph containing $|\emptyset, \mathbf{r}\rangle$, we get $\mathscr{G}_{\mathbf{r}}$. Precisely, we have the following result:

Theorem 3.5 ([14, Proposition 6.2.14])**.** *The set $\Phi_{\mathbf{r}}$ is recursively characterised by:*

- $\emptyset \in \Phi_{\mathbf{r}}$,
- *If $\boldsymbol{\mu} \in \Phi_{\mathbf{r}}$, then any $\boldsymbol{\lambda}$ obtained from $\boldsymbol{\mu}$ by adding a good addable node is also in $\Phi_{\mathbf{r}}$.*

Precisely, $\mathscr{G}_{\mathbf{r}}$ consists of vertices $|\boldsymbol{\lambda}, \mathbf{r}\rangle$ with $\boldsymbol{\lambda} \in \Phi_{\mathbf{r}}$ and arrows $|\boldsymbol{\mu}, \mathbf{r}\rangle \xrightarrow{\ i\ } |\boldsymbol{\lambda}, \mathbf{r}\rangle$ if and only if $[\boldsymbol{\lambda}] = [\boldsymbol{\mu}] \cup \{\gamma\}$ where γ is the good removable i-node of $\boldsymbol{\mu}$.

We can now state Ariki's theorem, proved in [2].

Consider the canonical basis $\mathcal{B}_{\mathbf{r}}$ of $V(\mathbf{r})$. It is indexed by Uglov l-partitions. We write $\mathcal{B}_{\mathbf{r}} = \{G(\boldsymbol{\mu}, \mathbf{r})\,;\, \boldsymbol{\mu} \in \Phi_{\mathbf{r}}\}$. Each element of $\mathcal{B}_{\mathbf{r}}$ decomposes on the basis of l-partitions. Write $G(\boldsymbol{\mu}, \mathbf{r}) = \sum_{\boldsymbol{\lambda} \vdash_l n} c_{\boldsymbol{\lambda},\boldsymbol{\mu}}(q)|\boldsymbol{\lambda}, \mathbf{r}\rangle$.

Let $\mathcal{B}^1_{\mathbf{r}}$ be the specialisation of $\mathcal{B}_{\mathbf{r}}$ at $q = 1$, that is

$$\mathcal{B}^1_{\mathbf{r}} = \{\sum_{\boldsymbol{\lambda} \vdash_l n} c_{\boldsymbol{\lambda},\boldsymbol{\mu}}(1)|\boldsymbol{\lambda}, \mathbf{r}\rangle\,;\, \boldsymbol{\mu} \in \Phi_{\mathbf{r}}\}.$$

Recall that the elements $d_{\boldsymbol{\lambda},M} \in \mathbb{Z}_{>0}$, $\boldsymbol{\lambda} \vdash_l n$, are the decomposition numbers associated to $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$. Define

$$B(M, \mathbf{r}) = \sum_{\boldsymbol{\lambda} \vdash_l n} d_{\boldsymbol{\lambda},M}|\boldsymbol{\lambda}, \mathbf{r}\rangle.$$

Theorem 3.6 ([3, Theorem 12.5])**.** *Suppose that char$(k) = 0$. Then*

$$\mathcal{B}^1_{\mathbf{r}} = \{B(M, \mathbf{r})\,;\, M \in Irr(\mathbf{H}_{k,n}), n \in \mathbb{Z}_{\geq 0}\}.$$

Hence we have the following result concerning the decomposition matrix D of $\mathbf{H}_{k,n}$:

Corollary 3.7. *Set $C = (c_{\boldsymbol{\lambda},\boldsymbol{\mu}}(1))_{\boldsymbol{\lambda} \vdash_l n, \boldsymbol{\mu} \in \Phi_{\mathbf{r}}}$. Then $C = D$ up to a reordering of the columns.*

In other words, if $\mathrm{char}(k) = 0$, it is sufficient to compute the canonical basis of the irreducible highest weight $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module $V(\mathbf{r})$ in order to recover the decomposition matrix D.

Uglov, in [29], determined the canonical basis $\mathcal{B}_{\mathbf{r}}$ of $V(\mathbf{r})$, generalising the work of Leclerc and Thibon in [25]. Another good reference is the thesis of Yvonne, [30]. This requires some theory about the *affine Hecke algebra of type A* and *q-wedge products*. This approach permits us to establish the existence of a "canonical" basis of the whole Fock space, and also to compute $\mathcal{B}_{\mathbf{r}}$.

We introduce one more notation. Let $\mathbf{s} = (s_1, \ldots, s_l) \in C$. For $\mathbf{m} = (m_1, \ldots, m_l) \in \mathbb{Q}^l$, we set $\mathbf{v} = (v_1, \ldots, v_l) = (s_1 - m_1, \ldots, s_l - v_l)$, and we define

$$\mathscr{D}_{\mathbf{s}} = \{\mathbf{m} \in \mathbb{Q}^l \mid i < j \Rightarrow 0 < v_j - v_i < e\}.$$

Using Uglov's canonical basis of the Fock space, Geck and Jacon have proved the following result about canonical basic sets ([14],):

Theorem 3.8 ([14, Theorem 6.7.2])**.** *Suppose that char$(k) = 0$.*

Let $\mathbf{r} \in \mathbb{Z}^l$ and $\mathbf{m} \in \mathbb{Q}^l$. If $\mathbf{m} \in \mathscr{D}_{\mathbf{r}}$, then $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set $\Phi_{\mathbf{r}}(n)$.

In the sequel, we will assume that $\mathrm{char}(k) = 0$, so that Ariki's theorem (and hence Theorem 3.8) holds.

We now wish to review the existence or non-existence of a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$, depending on the values of $\mathbf{m}$, and explicitely describe these sets when they exist.

In the following sections, we will denote by $\mathscr{P}$ the following subset of $\mathbb{Q}^l$:

$$\mathscr{P} = \{m \in \mathbb{Q}^l \mid \exists\, i \neq j \text{ such that } (r_i - m_i) - (r_j - m_j) \in e\mathbb{Z}\}$$

Precisely, $\mathscr{P}$ consists in the union of the hyperplanes

$$\begin{aligned}\mathcal{P}_{i,j}(\mathbf{s}) &= \{\mathbf{m} \in \mathbb{Q}^l \mid (s_i - m_i) - (s_j - m_j) = 0\} \\ &= \{\mathbf{m} \in \mathbb{Q}^l \mid v_i - v_j = 0\} \quad (\text{where } v_i := s_i - m_i\ \forall i \in [\![1, l]\!])\end{aligned}$$

over all $\mathbf{s} \in C_e$ and all $1 \leq i < j \leq l$.

Indeed, for $k \in \mathbb{Z}$, we have

$$\begin{aligned}(r_i - m_i) - (r_j - m_j) = ke &\Leftrightarrow (r_i - m_i) - (r_j + ke - m_j) = 0 \\ &\Leftrightarrow (s_i - m_i) - (s_j - m_j) = 0 \\ &\Leftrightarrow v_i - v_j = 0\end{aligned}$$

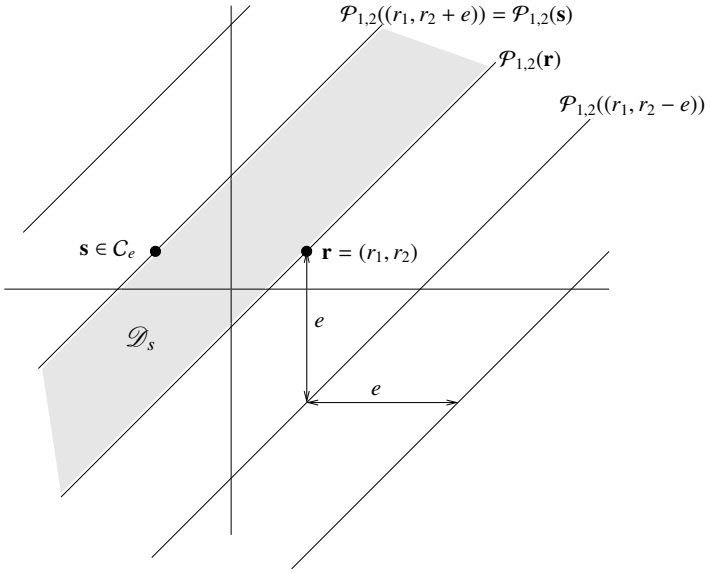


Figure 1: The set $\mathscr{P}$ and the domains $\mathscr{D}_{\mathbf{s}}$ in level 2.

with $\mathbf{s} = (s_1, \ldots, s_i, \ldots, s_j, \ldots, s_l) = (r_1, \ldots, r_i, \ldots, r_j + ke, \ldots, r_l)$.

Clearly, this is not a disjoint union, since $\mathcal{P}_{i,j}(\mathbf{s}) = \mathcal{P}_{i,j}(\tilde{\mathbf{s}})$ whenever $\tilde{s}_i = s_i + pe$ and $\tilde{s}_j = s_j + pe$ for some $p \in \mathbb{Z}$. Also, when $l > 2$, the hyperplanes $\mathcal{P}_{i,j}(\mathbf{s})$ and $\mathcal{P}_{i',j'}(\mathbf{s})$ always intersect, even when $(i, j) \neq (i', j')$.

4 Canonical basic sets for regular m

If $\mathbf{m} \in \mathbb{Q}^l \backslash \mathscr{P}$, we say that $\mathbf{m}$ is regular. In this section, we show that any regular $\mathbf{m}$ defines an order $\ll_{\mathbf{m}}$ with respect to which $\mathbf{H}_{k,n}$ admits a canonical basic set. We use the fact that, for $\mathbf{s} \in C_e$, if $\mathscr{B}$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$, then $\mathscr{B}$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$ (see Remark 2.8). We first study the case $l = 2$, and then the general case.

4.1 $l = 2$

Here we have $\mathbf{r} = (r_1, r_2) \in \mathbb{Z}^2$, and $\mathscr{P}$ is just a collection of parallel lines, namely the lines passing through $(s_1, s_2) \in C_e$ with slope 1. The set $\mathscr{D}_{\mathbf{s}}$ is the domain strictly between the lines passing through (s_1, s_2) and $(s_1, s_2 - e)$, see Figure 1.

Notation: For $\mathbf{s} = (s_1, s_2) \in C$, we denote $\tilde{\mathbf{s}} = (s_1, s_2 + e)$.

Proposition 4.1. *Let $\mathbf{m} = (m_1, m_2) \in \mathbb{Q}^2 \backslash \mathscr{P}$. Then $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely either $\Phi_{\mathbf{s}}(n)$ or $\Phi_{\tilde{\mathbf{s}}}(n)$, where $\mathbf{s} \in C_e$ is explicitely determined.*

Proof. The idea is to show that any such $\mathbf{m}$ belongs to a certain $\mathscr{D}_{\hat{\mathbf{s}}}$, for $\hat{\mathbf{s}} \in C_e$.

In fact, for such an $\mathbf{m}$, we can find an element $\mathbf{s} \in C_e$ which is "close" to $\mathbf{m}$ in the following sense: there exists $\mathbf{s} \in C_e$ such that $0 \leq |s_i - m_i| \leq \frac{e}{2}$, $i = 1, 2$. We set $v_i = s_i - m_i$.

In other terms, there exists $\mathbf{s} \in C_e$ such that $\mathbf{m}$ is in the closed ball (with respect to the infinity norm) of radius $\frac{e}{2}$ centered at $\mathbf{s}$. This is clear since these balls form a cover of $\mathbb{R}^2$. Hence we have $0 \leq |v_1 - v_2| \leq e$.

Moreover the fact that $\mathbf{m}$ is not located on a hyperplane of $\mathscr{P}$ (and that $\mathbf{s}$ is obtained from $\mathbf{r}$ after translation of each coordinate by an element of $e\mathbb{Z}$) ensures that these inequalities are strict, i.e. that $0 < |v_1 - v_2| < e$.

- If $0 < v_2 - v_1 < e$, then $\mathbf{m} \in \mathscr{D}_{\mathbf{s}}$. By Theorem 3.8, the set $\Phi_{\mathbf{s}}(n)$ is the canonical basic set for the algebra $\mathbf{H}_{k,n}^{(e,\mathbf{s})}$ with respect to the order $\ll_{\mathbf{m}}$. Therefore, by Remark 2.8, $\Phi_{\mathbf{s}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$.

- If $0 < v_1 - v_2 < e$, then $0 < (v_2 + e) - v_1 < e$. Hence $\mathbf{m} \in \mathscr{D}_{\tilde{\mathbf{s}}}$ (where we recall that $\tilde{\mathbf{s}} = (s_1, s_2 + e)$), so that by Theorem 3.8, the set $\Phi_{\tilde{\mathbf{s}}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \tilde{\mathbf{s}})$ with respect to the order $\ll_{\mathbf{m}}$. Since $\tilde{\mathbf{s}} \in C_e$, using Remark 2.8, $\Phi_{\tilde{\mathbf{s}}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$.

□

Remark 4.2. In level 2, the domains $\mathscr{D}_{\mathbf{s}}$, $\mathbf{s} \in C_e$, actually tile $\mathbb{Q}^2 \backslash \mathscr{P}$. In higher level this does not hold anymore, and we need to find other canonical basic sets than Uglov multipartitions.

Note that we can sometimes find different canonical basic sets for $(\mathbf{H}_{k,n}, \mathbf{r})$ and $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to the same order $\ll_{\mathbf{m}}$ if $\mathbf{s} \in C \backslash C_e$, as mentioned in Remark 2.8. This is what the following example shows.

Example 4.3. Let $l = 2$, $e = 4$, $\mathbf{r} = (1, 0)$ and $\mathbf{s} = \mathbf{r}^\sigma = (0, 1)$ (where $\sigma = (12)$). Take $\mathbf{m} = (1, -1)$. Then $\mathbf{s} \notin C_e$.

Then $\mathbf{m} \in \mathscr{D}_{\mathbf{r}}$ since $\mathbf{r} - \mathbf{m} = (0, 1)$. Hence $\Phi_{\mathbf{r}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$. Besides, $\mathbf{m} \in \mathscr{D}_{\mathbf{s}}$ since $\mathbf{s} - \mathbf{m} = (-1, 2)$. Hence $\Phi_{\mathbf{s}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$, but $\Phi_{\mathbf{s}}(n) \neq \Phi_{\mathbf{r}}(n)$ for $n \geq 2$ (which one can easily check).

4.2 $l > 2$

In the sequel we use the following notations. For $\boldsymbol{\alpha} = (\alpha_1, \dots \alpha_l) \in \mathbb{Q}^l$ and $\sigma \in \mathfrak{S}_l$, we denote $\boldsymbol{\alpha}^\sigma = (\alpha_{\sigma(1)}, \dots \alpha_{\sigma(l)})$. Similarly, for $\boldsymbol{\lambda} = (\lambda^1, ..., \lambda^l) \vdash_l n$, we write $\boldsymbol{\lambda}^\sigma = (\lambda^{\sigma(1)}, \dots, \lambda^{\sigma(l)})$.

Let $\mathbf{m} = (m_1, ..., m_l)$ be an element of $\mathbb{Q}^l \backslash \mathscr{P}$.

Proposition 4.4. *Let $\mathbf{s} \in C$ and $\sigma \in \mathfrak{S}_l$. If $\Phi_{\mathbf{s}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$, then the set*

$$\sigma(\Phi_{\mathbf{s}}(n)) := \{\boldsymbol{\lambda}^\sigma \,;\, \boldsymbol{\lambda} \in \Phi_{\mathbf{s}}(n)\}$$

of σ-twisted Uglov l-partitions is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s}^\sigma)$ with respect to $\ll_{\mathbf{m}^\sigma}$.

Proof. In order to prove this result, we need to define a *twisted Fock space* $\mathcal{F}^\sigma_{\mathbf{s}^\sigma}$, which, as a vector space, is the Fock space $\mathcal{F}_{\mathbf{s}^\sigma}$, but has a σ-twisted $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-action.

Recall that the action of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$ on the Fock space $\mathcal{F}_{\mathbf{s}}$ is derived from an order on the i-nodes of l-partitions. We define a twisted order on the removable and addable i-nodes of a multipartition in the following way : let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be two removable or addable i-nodes of $\boldsymbol{\lambda} \vdash_l n$. We write

$$\gamma \prec^\sigma_{(\mathbf{s}^\sigma, e)} \gamma' \text{ if } \begin{cases} b - a + s_{\sigma(c)} < b' - a' + s_{\sigma(c')} \text{ or} \\ b - a + s_{\sigma(c)} = b' - a' + s_{\sigma(c')} \quad \text{and} \quad \sigma(c) > \sigma(c'). \end{cases}$$

Now, if $\gamma = (a, b, c)$ is a removable (resp. addable) i-node of $\boldsymbol{\lambda} = (\lambda^1, ..., \lambda^l)$, then $\gamma^\sigma := (a, b, \sigma^{-1}(c))$ is a removable (resp. addable) i-node of $\boldsymbol{\lambda}^\sigma := (\lambda^{\sigma(1)}, ..., \lambda^{\sigma(l)})$, so that we have

$$\gamma^\sigma \prec^\sigma_{(\mathbf{s}^\sigma, e)} \gamma'^\sigma \Leftrightarrow \gamma \prec_{(\mathbf{s}, e)} \gamma'.$$

This order enables us to define the numbers $N_i^{\prec^\sigma}$ and $N_i^{\succ^\sigma}$. Let $\boldsymbol{\lambda} \vdash_l n$ and $\boldsymbol{\mu} \vdash_l n + 1$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$ where γ is an i-node. Then set

$$\begin{aligned} N_i^{\prec^\sigma}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = &\sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \prec^\sigma_{(\mathbf{s}^\sigma, e)} \gamma\} - \\ &\sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \prec^\sigma_{(\mathbf{s}^\sigma, e)} \gamma\} \end{aligned}$$

and

$$\begin{aligned} N_i^{\succ^\sigma}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = &\sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \succ^\sigma_{(\mathbf{s}^\sigma, e)} \gamma\} - \\ &\sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \succ^\sigma_{(\mathbf{s}^\sigma, e)} \gamma\} \end{aligned}$$

We abuse the notation by denoting σ the isomorphism of vector spaces

$$\sigma : \begin{array}{ccc} \mathcal{F}_{\mathbf{s}} & \longrightarrow & \mathcal{F}_{\mathbf{s}^\sigma} \\ |\boldsymbol{\lambda}, \mathbf{s}\rangle & \longmapsto & |\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma\rangle \end{array}$$

Now we want do define a twisted action of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ on $\mathcal{F}_{\mathbf{s}^\sigma}$.

The action of e_i and f_i, denoted by $e_i^\sigma.\,|\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma\rangle$ and $f_i^\sigma.\,|\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma\rangle$, are defined as follows:

$$e_i^\sigma.\,|\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma\rangle = \sum_{\mathrm{r}_e([\boldsymbol{\lambda}^\sigma]\setminus[\boldsymbol{\mu}^\sigma])=i} q^{-N_i^{\lhd^\sigma}(\boldsymbol{\mu}^\sigma,\boldsymbol{\lambda}^\sigma)}\,|\boldsymbol{\mu}^\sigma, \mathbf{s}^\sigma\rangle\,.$$

Then we have

$$\begin{aligned} e_i^\sigma.\,|\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma\rangle &= \sum_{\mathrm{r}_e([\boldsymbol{\lambda}]\setminus[\boldsymbol{\mu}])=i} q^{-N_i^{\lhd}(\boldsymbol{\mu},\boldsymbol{\lambda})}\sigma(\,|\boldsymbol{\mu}, \mathbf{s}\rangle) \\ &= \sigma\Big(\sum_{\mathrm{r}_e([\boldsymbol{\lambda}]\setminus[\boldsymbol{\mu}])=i} q^{-N_i^{\lhd}(\boldsymbol{\mu},\boldsymbol{\lambda})}\,|\boldsymbol{\mu}, \mathbf{s}\rangle\Big) \end{aligned}$$

that is e_i^σ acts as $\sigma e_i \sigma^{-1}$.

Similarly, if we set

$$f_i^\sigma.\,|\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma\rangle = \sum_{\mathrm{r}_e([\boldsymbol{\mu}^\sigma]\setminus[\boldsymbol{\lambda}^\sigma])=i} q^{-N_i^{\rhd^\sigma}(\boldsymbol{\lambda}^\sigma,\boldsymbol{\mu}^\sigma)}\,|\boldsymbol{\mu}^\sigma, \mathbf{s}^\sigma\rangle\,,$$

we have

$$\begin{aligned} f_i^\sigma.\,|\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma\rangle &= \sum_{\mathrm{r}_e([\boldsymbol{\mu}]\setminus[\boldsymbol{\lambda}])=i} q^{-N_i^{\rhd}(\boldsymbol{\lambda},\boldsymbol{\mu})}\sigma(\,|\boldsymbol{\mu}, \mathbf{s}\rangle) \\ &= \sigma\Big(\sum_{\mathrm{r}_e([\boldsymbol{\mu}]\setminus[\boldsymbol{\lambda}])=i} q^{-N_i^{\lhd}(\boldsymbol{\lambda},\boldsymbol{\mu})}\,|\boldsymbol{\mu}, \mathbf{s}\rangle\Big) \end{aligned}$$

that is f_i^σ acts as $\sigma f_i \sigma^{-1}$.

Hence by Theorem 3.4, these new formulas, combined with the formulas

$$t_i.\,|\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma\rangle = q^{N_i(\boldsymbol{\lambda})}|\boldsymbol{\lambda}, \mathbf{s}\rangle$$

and

$$\mathfrak{d}.\,|\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma\rangle = -(\Delta(\mathbf{s}) + N_{\mathfrak{d}}(\boldsymbol{\lambda}))|\boldsymbol{\lambda}, \mathbf{s}\rangle$$

endow $\mathcal{F}_{\mathbf{s}^\sigma}$ with the structure of an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module, that we denote by $\mathcal{F}^\sigma_{\mathbf{s}^\sigma}$.

We continue the construction as in the non-twisted case. Denote by $V(\mathbf{s}^\sigma)^\sigma$ the submodule of $\mathcal{F}^\sigma_{\mathbf{s}^\sigma}$ generated by the empty l-partition $|\boldsymbol{\emptyset}, \mathbf{s}^\sigma\rangle$. This is an irreducible highest weight $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module for this twisted action, and the crystal basis of $V(\mathbf{s})$ is mapped to the one of $V(\mathbf{s}^\sigma)^\sigma$ by the isomorphism σ. In particular the vertices of the crystal graph of $V(\mathbf{s}^\sigma)^\sigma$ are the *σ-twisted Uglov l-partitions*:

$$\sigma(\Phi_{\mathbf{s}}(n)) := \{(\lambda^{\sigma(1)}, ..., \lambda^{\sigma(l)})\,;\,(\lambda^1, ..., \lambda^l) \in \Phi_{\mathbf{s}}(n)\}.$$

It is an indexing set for the global basis of $V(\mathbf{s}^\sigma)^\sigma$. Now this basis is also obtained from the global basis of $V(\mathbf{s})$ by applying σ. That is, if we denote by $G^\sigma(\boldsymbol{\lambda}, \mathbf{s}^\sigma)$ ($\boldsymbol{\lambda} \in \sigma(\Phi_{\mathbf{s}}(n))$) the elements of the canonical basis of $V(\mathbf{s}^\sigma)^\sigma$, we have:

$$\sigma(G(\boldsymbol{\lambda}, \mathbf{s})) = G^\sigma(\boldsymbol{\lambda}^\sigma, \mathbf{s}^\sigma). \tag{2}$$

Write $G^\sigma(\boldsymbol{\lambda}, \mathbf{s}^\sigma) = \sum_{\boldsymbol{\mu}^\sigma \vdash_l n} d_{\boldsymbol{\mu}^\sigma,\boldsymbol{\lambda}^\sigma}(q)\,|\boldsymbol{\mu}^\sigma, \mathbf{s}^\sigma\rangle$ the decomposition of $G^\sigma(\boldsymbol{\lambda}, \mathbf{s}^\sigma)$ on the basis of all l-partitions. By (2), we have

$$\begin{aligned} \sigma\Big(\sum_{\boldsymbol{\mu}\vdash_l n} d_{\boldsymbol{\mu},\boldsymbol{\lambda}}(q)\,|\boldsymbol{\mu}, \mathbf{s}\rangle\Big) &= \sum_{\boldsymbol{\mu}^\sigma\vdash_l n} d_{\boldsymbol{\mu}^\sigma,\boldsymbol{\lambda}^\sigma}(q)\,|\boldsymbol{\mu}^\sigma, \mathbf{s}^\sigma\rangle \\ \text{i.e. } \sum_{\boldsymbol{\mu}\vdash_l n} d_{\boldsymbol{\mu},\boldsymbol{\lambda}}(q)\,|\boldsymbol{\mu}^\sigma, \mathbf{s}^\sigma\rangle &= \sum_{\boldsymbol{\mu}\vdash_l n} d_{\boldsymbol{\mu}^\sigma,\boldsymbol{\lambda}^\sigma}(q)\,|\boldsymbol{\mu}^\sigma, \mathbf{s}^\sigma\rangle\,. \end{aligned}$$

Hence $\forall \boldsymbol{\lambda} \in \Phi_{\mathbf{s}}(n)$, $\forall \boldsymbol{\mu} \vdash_l n$, we have

$$d_{\boldsymbol{\mu}^\sigma,\boldsymbol{\lambda}^\sigma}(q) = d_{\boldsymbol{\mu},\boldsymbol{\lambda}}(q). \tag{3}$$

In particular, this is true at $q = 1$.

By Ariki's theorem (Theorem 3.6), which holds for any realisation of the highest weight $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module $V(\mathbf{r})$, the matrix $(d_{\mu,\lambda(1)})_{\mu\vdash_l n,\lambda\in\Phi_{\mathbf{s}}(n)}$ is the decomposition matrix D of $\mathbf{H}_{k,n}$. Hence by (3), one can also parametrise the irreducible modules of $\mathbf{H}_{k,n}$ by the elements of $\sigma(\Phi_{\mathbf{s}}(n))$ and recover the same matrix by labelling the i-th column by $\boldsymbol{\lambda}_i^\sigma \in \sigma(\Phi_{\mathbf{s}}(n))$ and the j-th line by $\boldsymbol{\mu}_j^\sigma \vdash_l n$.

Moreover, the fact that $\Phi_{\mathbf{s}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$ means that D is upper unitriangular with respect to $\ll_{\mathbf{m}}$. Since we have

$$\boldsymbol{\lambda}_i \ll_{\mathbf{m}} \boldsymbol{\lambda}_j \Leftrightarrow \boldsymbol{\lambda}_i^\sigma \ll_{\mathbf{m}^\sigma} \boldsymbol{\lambda}_j^\sigma,$$

the matrix D (with columns indexed by $\sigma(\Phi_{\mathbf{s}}(n))$) is upper unitriangular with respect to $\ll_{\mathbf{m}^\sigma}$, i.e. $\sigma(\Phi_{\mathbf{s}}(n)))$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s}^\sigma)$.

□

We are now ready to prove the following general result.

Proposition 4.5. *Let* $\mathbf{m} = (m_1, ..., m_l) \in \mathbb{Q}^l \backslash \mathscr{P}$. *Then* $(\mathbf{H}_{k,n}, \mathbf{r})$ *admits a canonical basic set with respect to* $\ll_{\mathbf{m}}$, *namely* $\sigma(\Phi_{\mathbf{s}^{\sigma^{-1}}}(n))$, *where* $\mathbf{s} \in C_e$ *and* $\sigma \in \mathfrak{S}_l$ *are explicitely determined. We then say that* $\mathbf{s}$ *is the* $\mathbf{m}$*-adapted multicharge, and* σ *the* $\mathbf{m}$*-adapted permutation.*

Proof. As in the level 2 case, we start by finding a $\mathbf{s} \in C_e$ which is "close" to $\mathbf{m}$.

Since the closed balls (with respect to the infinity norm) of radius $\frac{e}{2}$ centered at $\mathbf{s}$, $\mathbf{s} \in C_e$, form a cover of $\mathbb{R}^l$, $\mathbf{m}$ is in one of these balls. That is there exists $\mathbf{s} = (s_1, ..., s_l) \in C_e$ such that for all i, $0 \leq |s_i - m_i| \leq \frac{e}{2}$. We set $v_i = s_i - m_i$.

Now if $\mathbf{m}$ is not on the boundary of this ball, such an $\mathbf{s}$ is uniquely determined. If $\mathbf{m}$ is on the boundary, there exists $i \in [\![1, l]\!]$ such that $|v_i| = \frac{e}{2}$. Then we choose the $\mathbf{s}$ such that $v_i = \frac{e}{2}$ if $i > 1$ and such that $v_i = -\frac{e}{2}$ if $i = 1$ (to be in accordance with what has been done in level 2), which determines uniquely $\mathbf{s}$. Note that since $\mathbf{m} \notin \mathscr{P}$, at most one coordinate v_i can verify $|v_i| = \frac{e}{2}$. Moreover one can never have $|v_i - v_j| = 0$ for $i \neq j$.

Hence, for this choice of $\mathbf{s}$, we have that $i \neq j \Rightarrow 0 < |v_i - v_j| < e$, which implies that

$$\exists \tau \in \mathfrak{S}_l \text{ such that } i < j \Rightarrow 0 < v_{\tau(j)} - v_{\tau(i)} < e. \tag{4}$$

Since $\mathbf{m} = \mathbf{s} - \mathbf{v}$, we have $\mathbf{m}^\tau = \mathbf{s}^\tau - \mathbf{v}^\tau$. Because of (4), we see that $\mathbf{m}^\tau \in \mathscr{D}_{\mathbf{s}^\tau}$, hence by Theorem 3.8, $\Phi_{\mathbf{s}^\tau}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s}^\tau)$. Since $\mathbf{s} \in C_e$, $\mathbf{s}^\tau \in C_e(\mathbf{r}^\tau)$ and therefore (using Remark 2.8 again), $\Phi_{\mathbf{s}^\tau}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r}^\tau)$ with respect to $\ll_{\mathbf{m}^\tau}$.

Thus by Proposition 4.4, $\tau^{-1}(\Phi_{\mathbf{s}^\tau}(n))$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$. Setting $\sigma = \tau^{-1}$, we get the result.

□

Remark 4.6. When $l = 2$, the $\mathbf{m}$-adapted multicharge coincides with the multicharge $\mathbf{s}$ of Proposition 4.1.

In the particular level 2 case, we thus have two different approaches which yield canonical basic sets. Let $l = 2$. Let $\mathbf{m} \in \mathbb{Q}^2$, and take $\mathbf{s}$ the $\mathbf{m}$-adapted multicharge. Denote $\sigma = (12)$ (in particular, $\sigma = \sigma^{-1}$). Suppose that $\mathbf{m} \notin \mathscr{D}_{\mathbf{s}}$. On the one hand, by Proposition 4.5, $\sigma(\Phi_{\mathbf{s}^\sigma}(n))$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$. On the other hand, we also have $\mathbf{m} \in \mathscr{D}_{\tilde{\mathbf{s}}}$, so that $\Phi_{\tilde{\mathbf{s}}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ (this is precisely Proposition 4.1).

Hence one must have $\Phi_{\tilde{\mathbf{s}}}(n) = \sigma(\Phi_{\mathbf{s}^\sigma}(n))$. In other terms,

$$\Phi_{(s_1, s_2+e)}(n) = \{(\lambda^2, \lambda^1)\,;\, (\lambda^1, \lambda^2) \in \Phi_{(s_2, s_1)}(n)\}.$$

We recover a result by Jacon, namely Proposition 3.1 of [19]. However, in level $l > 2$, the application $\boldsymbol{\lambda} \longmapsto \boldsymbol{\lambda}^\sigma$ is not a crystal isomorphism. Consequently, the canonical basic set $\sigma(\Phi_{\mathbf{s}^{\sigma^{-1}}}(n))$ is not a priori a set of Uglov l-partitions. Let us detail which of these applications are indeed isomorphisms between sets of some Uglov multipartitions.

The crystal isomorphisms between the different sets of Uglov multipartitions (associated to $\mathbf{s} \in C$) have been described by Jacon and Lecouvey in [20]. In particular, Proposition 5.2.1 of [20] claims that

$$\Phi_{(s_1, ..., s_{l-1}, s_l+e)}(n) = \{(\lambda^2, \ldots, \lambda^l, \lambda^1)\,;\, (\lambda^1, \ldots, \lambda^l) \in \Phi_{(s_l, s_1, ..., s_{l-1})}(n)\}.$$

This proves that the application

$$\begin{array}{ccc} \Phi_{\mathbf{s}^{\sigma^{-1}}}(n) & \longrightarrow & \Phi_{\tilde{\mathbf{s}}}(n) \\ \boldsymbol{\lambda} & \longmapsto & \boldsymbol{\lambda}^{\sigma_0} \end{array} \quad \text{with } \sigma_0 = (1\ 2\ \ldots\ l),$$

where $\tilde{\mathbf{s}} = (s_1, \ldots, s_{l-1}, s_l + e)$, is a crystal isomorphism. Applying several times ρ (which is of order l), we obtain $l-1$ different crystal isomorphisms $\boldsymbol{\lambda} \longmapsto \boldsymbol{\lambda}^{{\sigma_0}^k}$.

Example 4.7. Take $l = 3$. Then $\sigma_0 = (1\ 2\ 3)$ and ${\sigma_0}^2 = (1\ 3\ 2)$. Then the following applications are crystal isomorphisms:

$$\begin{array}{ccc} \Phi_{(s_3,s_1,s_2)}(n) & \longrightarrow & \Phi_{(s_1,s_2,s_3+e)}(n) \\ (\lambda^1, \lambda^2, \lambda^3) & \longmapsto & (\lambda^2, \lambda^3, \lambda^1) \end{array},$$

$$\begin{array}{ccc} \Phi_{(s_2,s_3,s_1)}(n) & \longrightarrow & \Phi_{(s_3,s_1,s_2+e)}(n) \\ (\lambda^1, \lambda^2, \lambda^3) & \longmapsto & (\lambda^2, \lambda^3, \lambda^1) \end{array}.$$

As in the level 2 case, it is possible to recover these results by looking at the domains $\mathscr{D}_{\mathbf{s}}$, $\mathbf{s} \in C_e$. Indeed, even though these domains do not tile $\mathbb{Q}^l$ (as already mentioned in Remark 4.2) some weight sequences $\mathbf{m}$ whose adapted permutation σ verify $\sigma \neq \mathrm{Id}$ can also lie in a domain $\mathscr{D}_{\hat{\mathbf{s}}}$, for some $\hat{\mathbf{s}} \in C_e$. In that case we have two different constructions of the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$, which must therefore coincide. That is, for some values of $\sigma \in \mathfrak{S}_l$, the set $\sigma(\Phi_{\mathbf{s}^{\sigma^{-1}}}(n))$ is necessarily a set of Uglov multipartitions $\Phi_{\hat{\mathbf{s}}}(n)$, for some $\hat{\mathbf{s}} \in C_e$.

Of course, it gets difficult to visualise the domains $\mathscr{D}_{\mathbf{s}}$ when $l \geq 3$. Moreover, this argument does not hold whenever $\boldsymbol{\lambda} \longmapsto \boldsymbol{\lambda}^{\sigma}$ is not a crystal isomorphism between Uglov multipartitions. In fact, these σ-twisted Uglov multipartitions yield in general new canonical basic sets for $(\mathbf{H}_{k,n}, \mathbf{r})$.

5 Canonical basic sets for asymptotic m

5.1 Kleshchev multipartitions and asymptotic setting

Let $\mathbf{s} \in C_e$. We show that when the difference between the values of $\mathbf{s}$ is large, the set of Uglov multipartitions stabilises, and coincides with the set of Kleshchev multipartitions. This is what we call the *asymptotic* case, and such an l-tuple $\mathbf{s}$ will be called asymptotic, see Definition 5.9.

Let us recall in detail the relation between Uglov l-partitions and Kleshchev l-partitions. The Kleshchev l-partitions are defined in the same manner as the Uglov l-partitions, except the order on i-nodes used to define an action of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$ on the Fock space is different. Indeed, let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be two removable or addable i-nodes of the same l-partition of n. We define

$$\gamma \prec_{\mathcal{K}} \gamma' \Leftrightarrow \begin{cases} c' < c & \text{or} \\ c' = c & \text{and} \quad a' < a. \end{cases}$$

Note that this order only depends on the class C_e, not on some particular $\mathbf{s} \in C_e$ anymore.

This permits us to give $\mathcal{F}_{\mathbf{s}}$ the structure of integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module via the same formulas used with $\prec_{(\mathbf{s},e)}$ in Theorem 3.4. We can then construct the crystal graph of the highest weight submodule spanned by $|\boldsymbol{\emptyset}, \mathbf{s}\rangle$, in the same way as the Uglov multipartions (Theorem 3.5). Its vertices are labeled by what we call the *Kleshchev l-partitions*. We denote by $\mathcal{K}_{C_e}(n)$ the set of Kleshchev l-partitions of rank n.

Note that with this realisation as an $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module, $\mathcal{F}_{\mathbf{s}}$ is actually a tensor product of Fock spaces of level 1, see [29].

The following proposition connects both orders for certain values of $\mathbf{s}$.

Proposition 5.1. *Let $\mathbf{s} \in C_e$ such that $i < j \Rightarrow s_i - s_j \geq n - e$ [1]. Then for all $m \leq n$, $\Phi_{\mathbf{s}}(m) = \mathcal{K}_{C_e}(m)$. In particular , $\Phi_{\mathbf{s}}(n) = \mathcal{K}_{C_e}(n)$.*

[1] Of course, this is equivalent to $s_i - s_{i+1} \geq n - e$ for all $i \in [\![1, l-1]\!]$.

Proof. It is sufficient to show that in this case, both orders on i-nodes are equivalent, i.e. $\gamma \prec_{(\mathbf{s},e)} \gamma' \Leftrightarrow \gamma \prec_{\mathcal{K}} \gamma'$, where $\gamma = (a,b,c)$ and $\gamma' = (a',b',c')$ are two removable or addable i-nodes of $\boldsymbol{\lambda} \vdash_l m$.

Note that $-n+1 \leq b'-a'-(b-a) \leq n-1$. Indeed, the difference between $b'-a'$ and $b-a$ is minimal if and only if ($\lambda^{c'} = \emptyset$ and $\lambda^c = (n)$) or ($\lambda^{c'} = (1^n)$ and $\lambda^c = \emptyset$); and is maximal if and only if ($\lambda^{c'} = (n)$ and $\lambda^c = \emptyset$) or ($\lambda^{c'} = \emptyset$ and $\lambda^c = (1^n)$).

First assume that $\gamma \prec_{\mathcal{K}} \gamma'$. Then:

- If $c' < c$, then $s_{c'} - s_c \geq n-e$, hence $b'-a'+s_{c'}-(b-a+s_c) \geq -n+1+n-e = -e+1$. Since γ and γ' have the same residue, this implies that $b'-a'+s_{c'}$ and $b-a+s_c$ are congruent modulo e, thus $b'-a'+s_{c'}-(b-a+s_c) \geq 0$, and therefore $\gamma \prec_{(\mathbf{s},e)} \gamma'$.
- If $c' = c$ and $a' < a$. Then $b < b'$ since $\lambda^{c'} = \lambda^c$ is a partition and γ and γ' are on the border of λ^c. Hence $b-a < b'-a'$, and $b-a+s_c < b'-a'+s_{c'}$, hence $\gamma \prec_{(\mathbf{s},e)} \gamma'$.

Conversely, assume that $\gamma \prec_{(\mathbf{s},e)} \gamma'$. Then:

- If $b-a+s_c < b'-a'+s_{c'}$ then suppose $c' > c$. Then $s_c - s_{c'} \geq n-e$. Since γ and γ' have the same residue, we have $b'-a'+s_{c'}-(b-a+s_c) \geq e$, and thus $b'-a'-(b-a) \geq e+n-e = n$, whence a contradiction. Hence $c' \leq c$. If $c' < c$ then $\gamma \prec_{\mathcal{K}} \gamma'$, and if $c' = c$ then $b'-a' > b-a$ thus $a' < a$ for the same reason as before, and $\gamma \prec_{\mathcal{K}} \gamma'$.
- If $b-a+s_c = b'-a'+s_{c'}$ and $c' < c$ then it is straightforward that $\gamma \prec_{\mathcal{K}} \gamma'$.

Now since the rest of the constructions of the Uglov l-partitions on the one hand, and of the Kleshchev l-partitions on the other hand are exactly the same, it is clear that both sets coincide.

□

From this Proposition, we directly deduce:

Corollary 5.2. *Suppose $\mathbf{s} \in C_e$. When the difference $s_i - s_j$, for all $i < j$, is sufficiently large, the set of Uglov multipartitions $\Phi_{\mathbf{s}}(n)$ stabilises, and is equal to $\mathcal{K}_{C_e}(n)$.*

Remark 5.3. Note that the bound $n-e$ is not necessarily sharp (even though it is optimal for the orders on i-nodes to coincide), it is a priori possible for Uglov multipartitions to stabilise at a weaker condition on $\mathbf{s}$.

Actually, the set of Uglov multipartitions stabilises in other directions, that is under other conditions of $\mathbf{s}$. Precisely, we will show that they stabilise whenever the difference between any arbitrary coordinates of $\mathbf{s}$ (without the condition $i < j$) is "large enough".

In order to describe this phenomenon, we introduce the set of *twisted Kleshchev multipartitions*. Let $\pi \in \mathfrak{S}_l$. We define the *π-twisted Kleshchev order* on i-nodes as follows: Let $\gamma = (a,b,c)$ and $\gamma' = (a',b',c')$ be two removable or addable i-nodes of the same l-partition of n. We set

$$\gamma \prec_{\mathcal{K}}^{\pi} \gamma' \Leftrightarrow \begin{cases} \pi(c') < \pi(c) & \text{or} \\ \pi(c') = \pi(c) & \text{and} \quad a' < a \end{cases}$$

This just means that the lexicographic convention on the coordinates of the l-partition is twisted by π. The *π-twisted Kleshchev l-partitions* are then defined as in the non-twisted case (and as the in the "Uglov" case): they label the vertices of the crystal graph of the same highest weight module defined via the action of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$ derived from this order $\prec_{\mathcal{K}}^{\pi}$. We denote them by $\mathcal{K}_{C_e}^{\pi}(n)$.

Remark 5.4. Note that it is equivalent to either build the set of π-twisted Kleshchev multipartitions associated to C_e, or to twist via π the set of Kleshchev multipartitions associated to $C_e(\mathbf{r}^{\pi^{-1}})$, i.e.

$$\mathcal{K}_{C_e}^{\pi}(n) = \pi(\mathcal{K}_{C_e^{\pi^{-1}}}(n)),$$

where $C_e^{\pi^{-1}} := C_e(\mathbf{r}^{\pi^{-1}})$.

We have the following "asymptotic" property:

Proposition 5.5. *Let* $\mathbf{s} \in C_e$ *such that there exists* $\pi \in \mathfrak{S}_l$ *verifying* $\pi(i) < \pi(j) \Rightarrow s_i - s_j \geq n$. *Then* $\Phi_{\mathbf{s}}(m) = \mathcal{K}^{\pi}_{C_e}(m)$ *for all* $m \leq n$. *In particular,* $\Phi_{\mathbf{s}}(n) = \mathcal{K}^{\pi}_{C_e}(n)$.

Proof. It is very similar to the one of Proposition 5.1. Indeed, we show that for $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ two removable or addable i-nodes of $\boldsymbol{\lambda} \vdash_l m$, $\gamma \prec_{(\mathbf{s},e)} \gamma' \Leftrightarrow \gamma \prec^{\pi}_{\mathcal{K}} \gamma'$.

Assume that $\gamma \prec^{\pi}_{\mathcal{K}} \gamma'$. Then:

- If $\pi(c') < \pi(c)$, then $s_{c'} - s_c \geq n$, hence $b' - a' + s_{c'} - (b - a + s_c) \geq -n + 1 + n = 1 > 0$. Hence $b' - a' + s_{c'} > (b - a + s_c)$ and $\gamma \prec_{(\mathbf{s},e)} \gamma'$.
- If $\pi(c') = \pi(c)$ and $a' < a$. Then $c' = c$ since π is a permutation. Thus $b < b'$ since $\lambda^{c'} = \lambda^c$ is a partition and γ and γ' are on the border of λ^c. Hence $b - a < b' - a'$, and $b - a + s_c < b' - a' + s_{c'}$, hence $\gamma \prec_{(\mathbf{s},e)} \gamma'$.

Conversely, assume that $\gamma \prec_{(\mathbf{s},e)} \gamma'$. Then:

- If $b - a + s_c < b' - a' + s_{c'}$ then suppose $\pi(c') > \pi(c)$. Then $s_c - s_{c'} \geq n$, and thus $b' - a' - (b - a) > n$, whence a contradiction. Hence $\pi(c') \leq \pi(c)$. If $\pi(c') < \pi(c)$ then $\gamma \prec^{\pi}_{\mathcal{K}} \gamma'$, and if $\pi(c') = \pi(c)$ then $c' = c$ and $b' - a' > b - a$ thus $a' < a$, and $\gamma \prec^{\pi}_{\mathcal{K}} \gamma'$.
- If $b - a + s_c = b' - a' + s_{c'}$ and $\pi(c') < \pi(c)$ then $\gamma \prec^{\pi}_{\mathcal{K}} \gamma'$.

Now since the rest of the constructions of the Uglov l-partitions on the one hand, and of the π-twisted Kleshchev l-partitions on the other hand are the same, it is clear that both sets coincide.

□

Hence, we directly deduce the following stabilisation property, whenever the difference between two arbitrary coordinates of $\mathbf{s}$ is large:

Corollary 5.6. *Let* $\mathbf{s} \in C_e$ *and let* $\pi \in \mathfrak{S}_l$. *When the difference* $s_i - s_j$, *for all* $\pi(i) < \pi(j)$, *is sufficiently large, then the set of Uglov l-partitions* $\Phi_{\mathbf{s}}(n)$ *stabilises, and is equal to* $\mathcal{K}^{\pi}_{C_e}(n)$.

Remark 5.7. As in Remark 5.3, note that the bound is not necessarily sharp, and that Uglov multipartitions are likely to stabilise under weaker conditions. In fact, when $\pi = \mathrm{Id}$, Proposition 5.5 gives a bound (namely n) on each $s_i - s_j$ beyond which $\Phi_{\mathbf{s}}(n) = \mathcal{K}_{C_e}(n)$, but which is less precise than the one given in Proposition 5.1 (namely $n - e$).

Remark 5.8. Let $\mathbf{s}$ be as in Corollary 5.6, i.e. $\Phi_{\mathbf{s}}(n) = \mathcal{K}^{\pi}_{C_e}(n)$.

It is important to notice that for all $\sigma \in \mathfrak{S}_l$,

$$\Phi_{\mathbf{s}^\sigma}(n) = \sigma(\mathcal{K}^{\pi}_{C_e}(n)). \tag{5}$$

Indeed, this directly follows from the definition of the Kleshchev order on i-nodes. Since in this case $\Phi_{\mathbf{s}}(n)$ is a set of (π-twisted) Kleshchev multipartitions, it is equivalent to either

- twist the multicharge via $\mathbf{s} \mapsto \mathbf{s}^\sigma$ and build the corresponding Uglov crystal, or
- twist via $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda}^\sigma$ these π-twisted Kleshchev l-partitions.

In other terms, replacing σ by σ^{-1}, (5) is equivalent to:

$$\sigma(\Phi_{\mathbf{s}^{\sigma^{-1}}}(n)) = \mathcal{K}^{\pi}_{C_e}(n).$$

In particular, this shows that the canonical basic set $\sigma(\Phi_{\mathbf{s}^{\sigma^{-1}}}(n))$ of Proposition 4.5 is always equal to $\mathcal{K}^{\pi}_{C_e}(n)$, for any value of $\sigma \in \mathfrak{S}_l$.

We can now define *asymptotic* multicharges and weight sequences.

Definition 5.9.

1. Let $\mathbf{s} \in C_e$. We say that $\mathbf{s}$ is asymptotic if $\Phi_{\mathbf{s}}(n) = \mathcal{K}^{\pi}_{C_e}(n)$ for some $\pi \in \mathfrak{S}_l$.
2. Let $\mathbf{m} \in \mathbb{Q}^l$. We say that $\mathbf{m}$ is asymptotic if either:
   - there exists an asymptotic multicharge $\mathbf{s}$ and $1 \leq i < j \leq l$ such that $\mathbf{m} \in \mathcal{P}_{i,j}(\mathbf{s})$, or

- **m** is regular and the **m**-adapted multicharge (see Proposition 4.5) is asymptotic.

Remark 5.10. According to Remark 5.8, **s** is asymptotic if and only if for all $\sigma \in \mathfrak{S}_l$, $\sigma(\Phi_{\mathbf{s}^{\sigma^{-1}}}(n)) = \mathcal{K}^{\pi}_{C_e}(n)$.

Let us now focus on the question of the existence of canonical basic sets, given an asymptotic weight sequence **m**. In the case where $\mathbf{m} \notin \mathscr{P}$, **m** is regular, and we have already shown in Proposition 4.5 that $\mathbf{H}_{k,n}$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set $\sigma(\Phi_{\mathbf{s}^{\sigma^{-1}}}(n))$ where **s** is the **m**-adapted multicharge and σ the **m** adapted permuation. In virtue of Remark 5.8, these sets of l-partitions are all equal to $\mathcal{K}^{\pi}_{C_e}(n)$. We will show that in the remaining asymptotic cases, that is when **m** belongs to some hyperplane $\mathcal{P}_{i,j}(\mathbf{s})$ with **s** asymptotic, the order $\ll_{\mathbf{m}}$ affords a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ which is also a set of twisted Kleshchev multipartitions.

5.2 Kleshchev multiparitions as canonical basic sets

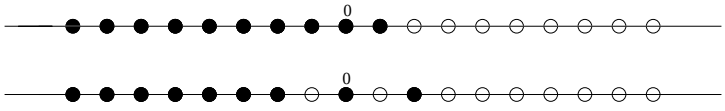
Fix $\mathbf{m} \in \mathscr{P}$ such that **m** is asymptotic, i.e. $\mathbf{m} = (m_1, \dots, m_l) = (s_1 - v_1, \dots, s_l - v_l)$ where $v_i = v_j$ for some i and j, and where **s** is asymptotic.

In order to understand the phenomenon that appears, let us recall some results. In [29], Uglov defined a combinatorial order $\leq_{\mathcal{K}}$ to study the matrix of the *canonical basis of* $\mathcal{F}_{\mathbf{s}}$. First, he showed that the Fock space can be endowed with a so-called (positive) canonical basis, which generalises the notion of canonical bases for integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-modules. The elements of this basis are indexed by l-partitions. Therefore, there is a transition matrix between this basis and the basis of l-partitions Δ, whose rows and columns are indexed by $\Pi_l(n)$. The coefficients appearing are elements of $\mathbb{N}[q]$, we denote them by $\delta_{\mu,\lambda}(q)$ $(\boldsymbol{\mu}, \boldsymbol{\lambda} \vdash_l n)$.

Besides, one can recover the decomposition matrix D of the Ariki-Koike algebra $\mathbf{H}_{k,n}$ by specialising Δ at $q = 1$, and by keeping only the columns indexed by Uglov multipartitions.

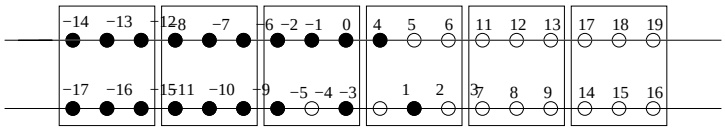
Let us recall briefly how the order $\leq_{\mathcal{K}}$ is constructed. To each ordered pair $(\boldsymbol{\lambda}, \mathbf{s})$ where $\boldsymbol{\lambda} \vdash_l n$ and **s** is a multicharge, we associate uniquely an ordered pair $(\tau(\boldsymbol{\lambda}), \tau(\mathbf{s}))$ where $\tau(\boldsymbol{\lambda})$ is a partition of some $n' \in \mathbb{N}$, and $\tau(\mathbf{s}) = s_1 + \cdots + s_l$. Note that this integer n' is not necessarily the same for all $\boldsymbol{\lambda} \vdash_l n$. This is done by using the representation of $(\boldsymbol{\lambda}, \mathbf{s})$ by an l-abacus (see [14] or [30]). The map τ is explicitely described in [29] and [30]. We just give an example to illustrate this correspondence.

Example 5.11. Let $l = 2$ and $e = 3$, let $\mathbf{s} = (0, 1)$ and $n = 3$. Let $\boldsymbol{\lambda} = (2.1, \emptyset) \vdash_2 3$. Then the associated l-abacus $\mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$ is the following:

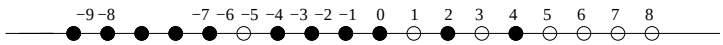


The i-th row of the abacus (numbered from bottom to top) represents the partition λ^i: the parts of λ^i are obtained by counting the numbers of white beads appearing on the left of each black bead. The multicharge **s** is recovered by moving the black beads to the left as much as possible, and then determining the position of the rightmost bead with respect to the origin 0.

We can then divide the abacus into rectangles of height l and length e and index the beads as follows:



We then construct a 1-abacus by ordering the numbered beads on one single row. By definition of τ, this is precisely $\mathcal{A}^1_{\tau(\mathbf{s})}(\tau(\boldsymbol{\lambda}))$, the 1-abacus associated to $\tau(\boldsymbol{\lambda})$. Here, we get the following 1-abacus:



From this abacus, we get $\tau(\boldsymbol{\lambda})$ and $\tau(s)$ by the same procedure as previously explained. We find $\tau(\boldsymbol{\lambda}) = (3.2.1.1.1.1) = (3.2.1^4) \vdash 9$ and $\tau(\mathbf{s}) = 1$. Note that we have indeed $\tau(\mathbf{s}) = s_1 + s_2$.

Now that we have constructed this map τ, we can set a new definition.

Definition 5.12. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$. We write $\boldsymbol{\lambda} \leq_{\mathcal{K}} \boldsymbol{\mu}$ if $\tau(\boldsymbol{\lambda}) \trianglelefteq \tau(\boldsymbol{\mu})$, in the usual sense of dominance on partitions.

Remark 5.13. Obviously, this is a partial order on the set of l-partitions of n, since $\trianglelefteq$ is a partial order on the set of partitions. Besides, in order to write $\boldsymbol{\lambda} \leq_{\mathcal{K}} \boldsymbol{\mu}$, $\tau(\boldsymbol{\lambda})$ and $\tau(\boldsymbol{\mu})$ must be partitions of the same integer m.

For this order $\leq_{\mathcal{K}}$, Uglov showed that the following unitriangularity property holds:

Proposition 5.14. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$. Then $\delta_{\mu,\lambda}(q) \neq 0 \Rightarrow \boldsymbol{\mu} \leq_{\mathcal{K}} \boldsymbol{\lambda}$. Moreover $\delta_{\lambda,\lambda}(q) = 1$. In other terms, Δ is lower unitriangular with respect to (a total order on $\Pi_l(n)$ refining) $\leq_{\mathcal{K}}$.*

Corollary 5.15. *The decomposition matrix of $\mathbf{H}_{k,n}$ is lower unitriangular with respect to $\leq_{\mathcal{K}}$.*

Now, when the difference between the values of $\mathbf{s}$ is large, say there exists $\pi \in \mathfrak{S}_l$ such that $[\pi(i) < \pi(j) \Rightarrow s_i - s_j > 2n]$, one can see that the order $\ll_{\mathbf{m}}$ is a refinement of $\leq_{\mathcal{K}}$, that is, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$,

$$\boldsymbol{\mu} \leq_{\mathcal{K}} \boldsymbol{\lambda} \Rightarrow \boldsymbol{\mu} \ll_{\mathbf{m}} \boldsymbol{\lambda} \tag{6}$$

for $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$.

Indeed, in this setting, $\mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$ can be partitioned into l sections, delimited vertically, each of which encoding one of the parts λ^i of $\boldsymbol{\lambda}$. This partitioning of $\mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$ is preserved by the application $\boldsymbol{\lambda} \longmapsto \tau(\boldsymbol{\lambda})$, so that one can still recover the parts of $\boldsymbol{\lambda}$ simply by looking at $\mathcal{A}_{\tau(\mathbf{s})}(\tau(\boldsymbol{\lambda}))$. Consequently, one can use the so-called *dominance order on* $\Pi_l(n)$ instead of $\leq_{\mathcal{K}}$: precisely, if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are comparable with respect to $\leq_{\mathcal{K}}$, then $\boldsymbol{\mu} \leq_{\mathcal{K}} \boldsymbol{\lambda} \Leftrightarrow \boldsymbol{\mu} \trianglelefteq^{\tau} \boldsymbol{\lambda}$,

where

$$\begin{aligned} \boldsymbol{\mu} \trianglelefteq^{\tau} \boldsymbol{\lambda} &\overset{\text{def}}{\Longleftrightarrow} \boldsymbol{\mu}^{\tau} \trianglelefteq \boldsymbol{\lambda}^{\tau} \\ &\overset{\text{def}}{\Longleftrightarrow} \sum_{1 \leq j \leq k-1} |\mu^{\tau(j)}| + \sum_{1 \leq i \leq d} \mu_i^{\tau(k)} \leq \sum_{1 \leq j \leq k-1} |\lambda^{\tau(j)}| + \sum_{1 \leq i \leq d} \lambda_i^{\tau(k)}, \end{aligned}$$

and where $\tau = \pi^{-1}$.

For the same reasons, one can also check that the order $\ll_{\mathbf{m}}$ is also compatible with this dominance order, so that $\leq_{\mathcal{K}}$ and $\ll_{\mathbf{m}}$ are compatible in the sense of (6).

Since with respect to this construction, one labels the columns of D with the Uglov l-partitions, which coincide with the π-twisted Kleshchev l-partitions, we have proved:

Proposition 5.16. *Suppose that $\mathbf{m} \in \mathcal{P}_{i,j}(\mathbf{s})$ for some $i, j \in [\![1, l]\!]$, with $\mathbf{s}$ verifying $[\pi(i) < \pi(j) \Rightarrow s_i - s_j > 2n]$ for some $\pi \in \mathfrak{S}_l$. In particular, $\mathbf{s}$ and $\mathbf{m}$ are asymptotic.*

Then $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set $\mathcal{K}^{\pi}_{C_e}(n)$.

Of course, this particular setting does not cover all the asymptotic cases. This just means that when $\mathbf{s}$ is "asymptotic enough", any $\mathbf{m} \in \mathcal{P}_{i,j}$ (for all $i, j \in [\![1, l]\!]$, $i \neq j$) affords a canonical basic set, namely the set of π-twisted Kleshchev l-partitions, which is also the canonical basic set with respect to $\ll_{\tilde{\mathbf{m}}}$ whenever $\tilde{\mathbf{m}}$ is a regular weight sequence "close" to $\mathbf{m}$, see Proposition 4.5 and Remark 5.8.

Hence it is natural to try to extend this property whenever the canonical basic sets $\sigma(\Phi_{\mathbf{s}^{\sigma^{-1}}}(n))$ of Proposition 4.5 stabilise to $\mathcal{K}^{\pi}_{C_e}(n)$. In fact, we have the following result:

Proposition 5.17. *Let $\mathbf{m} \in \mathscr{P}$, say $\mathbf{m} \in \mathcal{P}_{i,j}(\mathbf{s})$, be an asymptotic weight sequence. Then $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set $\mathcal{K}^{\pi}_{C_e}(n)$, where $\pi \in \mathfrak{S}_l$ is such that $[\pi(i) < \pi(j) \Rightarrow s_i > s_j]$.*

In order to prove this, we need the following technical lemma. We introduce the following notation. Given a weight sequence $\mathbf{m}$, $\varepsilon > 0$ and $\rho \in \mathfrak{S}_l$, we define a new weight sequence by

$$\mathbf{m}^{[\varepsilon,\rho]} := (m_1 + \varepsilon/\rho(1), \ldots, m_l + \varepsilon/\rho(l)).$$

Lemma 5.18. *Let* $\mathbf{m}$ *be an arbitrary weight sequence. Let* $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$, $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. *Then there exists* $\alpha_{\lambda,\mu} > 0$ *such that for all* $\varepsilon \in]0, \alpha_{\lambda,\mu}[$,

1.
$$\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu} \quad \Rightarrow \quad \left[\begin{array}{c} \boldsymbol{\lambda} \ll_{\mathbf{m}^{[\varepsilon,\rho]}} \boldsymbol{\mu} \quad \textit{or} \quad \boldsymbol{\lambda} \textit{ and } \boldsymbol{\mu} \textit{ are not} \\ \textit{comparable with respect to } \ll_{\mathbf{m}^{[\varepsilon,\rho]}},\ \forall \rho \in \mathfrak{S}_l \end{array} \right]$$

and

2.
$$\left[\begin{array}{c} \boldsymbol{\lambda} \textit{ and } \boldsymbol{\mu} \textit{ are not comparable} \\ \textit{with respect to } \ll_{\mathbf{m}} \end{array} \right] \Rightarrow \left[\begin{array}{c} \boldsymbol{\lambda} \textit{ and } \boldsymbol{\mu} \textit{ are not comparable with} \\ \textit{respect to } \ll_{\mathbf{m}^{[\varepsilon,\rho]}},\ \forall \rho \in \mathfrak{S}_l. \end{array} \right],$$

This means that for a small perturbation of $\mathbf{m}$, the order $\ll_{\mathbf{m}}$ never reverses: at worst, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ become uncomparable. Moreover, one can never gain comparability between multipartitions uncomparable with respect to $\ll_{\mathbf{m}}$ when slightly perturbing $\mathbf{m}$.

Proof. First, note that it is sufficient to prove these properties for the perturbations $\mathbf{m}^{[\varepsilon,k]}$ of $\mathbf{m}$ defined by $\mathbf{m}^{[\varepsilon,k]} = (m_1, \dots, m_{k-1}, m_k + \varepsilon, m_{k+1}, \dots, m_l)$, for all $k \in [\![1, l]\!]$. Indeed, the result then follows by induction, since $\mathbf{m}^{[\varepsilon,\rho]}$ is constructed by iterating this procedure.

Recall that, by definition (Section 2.3), $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$ means that $\mathfrak{b}_{\mathbf{m}}(\boldsymbol{\lambda}) \rhd \mathfrak{b}_{\mathbf{m}}(\boldsymbol{\mu})$, where $\mathfrak{b}_{\mathbf{m}}(\lambda) = (\mathfrak{b}^1_{\mathbf{m}}(\boldsymbol{\lambda}), \dots, \mathfrak{b}^h_{\mathbf{m}}(\boldsymbol{\lambda}))$ is the decreasing sequence consisting of the elements of $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$.

Let us first prove Assertion 1.

Let $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}$. For $k \in [\![1, l]\!]$, consider the order $\ll_{\mathbf{m}^{[\varepsilon,k]}}$. It is obtained from $\ll_{\mathbf{m}}$ simply by translating the k-th row of the symbols by ε, and taking the dominance order on the decreasing sequences of these new elements. Informally, when we choose ε to be "small", one cannot have $\boldsymbol{\mu} \ll_{\mathbf{m}^{[\varepsilon,k]}} \boldsymbol{\lambda}$. Indeed, the fact that $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}$ creates a gap at some point between $\sum_i \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\mu})$ and $\sum_i \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda})$, which cannot be recovered if ε is small enough.

Let us prove this properly. We also use a running example to illustrate the different points of the coming proof. For simplicity, since $\mathbf{m} \in \mathcal{P}_{i,j}(\mathbf{s})$ for some i, j and some $\mathbf{s} \in \mathbb{Z}^l$, we can assume without restriction that m_i and m_j are integers.

Example: Take $l = e = 2$, $n = 38$, $\mathbf{r} = (1, 0, 0)$, $\mathbf{m} = (3^1/_3, 7, 5) \in \mathcal{P}_{1,2}(1, 2, 0)$. Let $\boldsymbol{\lambda} = (4.1, 4^2.3.2^3.1, 4^3.2.1) \vdash_l n$ and $\boldsymbol{\mu} = (4.2, 4.3.2^5, 5.4^2.1^2) \vdash_l n$. The shifted $\mathbf{m}$-symbols of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ of size 1 are the following:

$$\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 2 & 4 & 7 & 8 & 9 & & \\ 0 & 2 & 4 & 5 & 6 & 8 & 10 & 11 \\ 0^1/_3 & 1^1/_3 & 3^1/_3 & 7^1/_3 & & & & \end{pmatrix}$$

and

$$\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\mu}) = \begin{pmatrix} 0 & 2 & 3 & 7 & 8 & 10 & & \\ 0 & 3 & 4 & 5 & 6 & 7 & 9 & 11 \\ 0^1/_3 & 1^1/_3 & 4^1/_3 & 7^1/_3 & & & & \end{pmatrix}.$$

The corresponding sequences $\mathfrak{b}_{\mathbf{m}}$ are

$$\mathfrak{b}_{\mathbf{m}}(\boldsymbol{\lambda}) = (11, 10, 9, 8, 8, 7^1/_3, 7, 6, 5, 4, 4, 3^1/_3, 2, 2, 1^1/_3, 0^1/_3, 0, 0)$$

and

$$\mathfrak{b}_{\mathbf{m}}(\boldsymbol{\mu}) = (11, 10, 9, 8, 7^1/_3, 7, 7, 6, 5, 4^1/_3, 4, 3, 3, 2, 1^1/_3, 0^1/_3, 0, 0).$$

Since $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$, there exists a smallest integer p such that $\mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\lambda}) > \mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\mu})$. Denote $\delta = \mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\lambda}) - \mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\mu})$.

In our example, $p = 5$ and $\delta = 2/3$, since $\mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda}) = \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\mu})\, \forall i < 5$ and $\mathfrak{b}^5_{\mathbf{m}}(\boldsymbol{\lambda}) = 8$ and $\mathfrak{b}^5_{\mathbf{m}}(\boldsymbol{\mu}) = 7^1/_3$.

Now for all i, denote $\{m_i\} = m_i - \lfloor m_i \rfloor$ the fractional part of m_i, whenever $m_i \notin \mathbb{Z}$. Set $\beta_i = \min(\{m_i\}, 1 - \{m_i\})$ (for all $m_i \notin \mathbb{Z}$), and $\beta = \min_i \beta_i$. If $m_i \in \mathbb{Z}$ for all i, then set $\beta = 1$. In particular $\beta \leq \delta$. In the example, $\beta = 1/3$.

Hence, set $0 < \varepsilon < \beta$. Now, for all $k \in [\![1, l]\!]$, consider the $\mathbf{m}^{[\varepsilon,k]}$-symbols of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. In our example, for $k = 1$ for instance, we get

$$\mathfrak{B}_{\mathbf{m}^{[1]}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0+\varepsilon & 2+\varepsilon & 4+\varepsilon & 7+\varepsilon & 8+\varepsilon & 9+\varepsilon & & \\ 0 & 2 & 4 & 5 & 6 & 8 & 10 & 11 \\ 0^{1}/_{3} & 1^{1}/_{3} & 3^{1}/_{3} & 7^{1}/_{3} & & & & \end{pmatrix}$$

and

$$\mathfrak{B}_{\mathbf{m}^{[1]}}(\boldsymbol{\mu}) = \begin{pmatrix} 0+\varepsilon & 2+\varepsilon & 3+\varepsilon & 7+\varepsilon & 8+\varepsilon & 10+\varepsilon & & \\ 0 & 3 & 4 & 5 & 6 & 7 & 9 & 11 \\ 0^{1}/_{3} & 1^{1}/_{3} & 4^{1}/_{3} & 7^{1}/_{3} & & & & \end{pmatrix}.$$

Since $\varepsilon < \beta$, the "perturbed" elements (of $\mathfrak{B}_{\mathbf{m}^{[\varepsilon,k]}}$) are ordered in the same way as the original ones (those of $\mathfrak{B}_{\mathbf{m}}$). Precisely, for all i, we either have

$$\mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) = \begin{cases} \mathfrak{b}^{i}_{\mathbf{m}}(\boldsymbol{\lambda}) & \text{or} \\ \mathfrak{b}^{i}_{\mathbf{m}}(\boldsymbol{\lambda}) + \varepsilon, \end{cases} \tag{7}$$

and similarly for $\boldsymbol{\mu}$.

Now, let $\alpha_{\lambda,\mu} = \min(\beta, 1/p)$ and take $0 < \varepsilon < \alpha_{\lambda,\mu}$. One can then compute $\sum_{i=1}^{s} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda})$ and $\sum_{i=1}^{s} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu})$for all $s < p$. Clearly, it is possible to have $\sum_{i=1}^{s} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) < \sum_{i=1}^{s} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu})$. This is the case in the example, for $k = 1$, since if we take $s = 2$, we have $\mathfrak{b}^{1}_{\mathbf{m}^{[1]}}(\boldsymbol{\lambda}) + \mathfrak{b}^{2}_{\mathbf{m}^{[1]}}(\boldsymbol{\lambda}) = 11 + 10 < 11 + 10 + \varepsilon = \mathfrak{b}^{1}_{\mathbf{m}^{[1]}}(\boldsymbol{\mu}) + \mathfrak{b}^{2}_{\mathbf{m}^{[1]}}(\boldsymbol{\mu})$. Hence, one can have $\boldsymbol{\lambda} \not\ll_{\mathbf{m}^{[\varepsilon,k]}} \boldsymbol{\mu}$.

However, we necessarily have:

- $\displaystyle\sum_{i=1}^{p-1} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) - \sum_{i=1}^{p-1} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) \leq (p-1)\varepsilon$, and
- $\mathfrak{b}^{p}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) - \mathfrak{b}^{p}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) > 1 - \varepsilon$ since $\mathfrak{b}^{p}_{\mathbf{m}}(\boldsymbol{\lambda}) - \mathfrak{b}^{p}_{\mathbf{m}}(\boldsymbol{\mu}) = \delta \geq 1$.

Thus,

$$\begin{aligned} \sum_{i=1}^{p} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) - \sum_{i=1}^{p} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) &> -(p-1)\varepsilon + 1 - \varepsilon \\ &= -p\varepsilon + 1 \\ &> -p\frac{1}{p} + 1 \quad \text{since} \quad \varepsilon < \frac{1}{p} \\ &= 0 \end{aligned}$$

Hence one can never have $\boldsymbol{\mu} \ll_{\mathbf{m}^{[\varepsilon,k]}} \boldsymbol{\lambda}$, which proves the first point.

The proof of Assertion 2. is completely similar: one shows that there exists p_1 and p_2 such that

$$\sum_{i=1}^{p_1} \mathfrak{b}^{i}_{\mathbf{m}}(\boldsymbol{\lambda}) > \sum_{i=1}^{p_1} \mathfrak{b}^{i}_{\mathbf{m}}(\boldsymbol{\mu}) \quad \text{and} \quad \sum_{i=1}^{p_2} \mathfrak{b}^{i}_{\mathbf{m}}(\boldsymbol{\lambda}) < \sum_{i=1}^{p_2} \mathfrak{b}^{i}_{\mathbf{m}}(\boldsymbol{\mu}),$$

and such that for an appropriate $\varepsilon > 0$, one has

$$\sum_{i=1}^{p_1} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) > \sum_{i=1}^{p_1} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) \quad \text{and} \quad \sum_{i=1}^{p_2} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) < \sum_{i=1}^{p_2} \mathfrak{b}^{i}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}),$$

for all $k \in [\![1, l]\!]$.

As mentioned at the beginning of the proof, both assertions thus still hold for the perturbations $\mathbf{m}^{[\varepsilon,\rho]}$ of $\mathbf{m}$. Note that this also relies on the fact that whenever $\varepsilon < \alpha_{\lambda,\mu}$, one has $\varepsilon/\rho(k) \leq \varepsilon < \alpha_{\lambda,\mu}$ for all $\rho \in \mathfrak{S}_l$ and for all $k \in [\![1, l]\!]$.

□

Proof of Proposition 5.17. Since $\mathbf{s}$ is asymptotic, we have $s_i \neq s_j$ for all $i \neq j$. Let π be the element of $\mathfrak{S}_l$ verifying $[\pi(i) < \pi(j) \Rightarrow s_i > s_j]$.

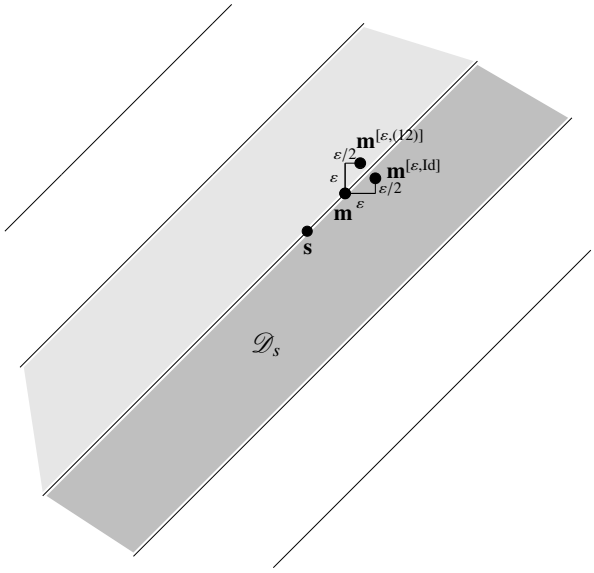


Figure 2: The perturbations $\mathbf{m}^{[\varepsilon,\mathrm{Id}]}$ and $\mathbf{m}^{[\varepsilon,(12)]}$ of $\mathbf{m}$ in level 2.

Now, let $\beta = \min_{m_i \neq m_j} |m_i - m_j|$ and $\alpha = \min_{\boldsymbol{\mu} \in \mathscr{S}(M)} (\alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}})$, where the elements $\alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ are defined in Lemma 5.18, and take $0 < \varepsilon < \min(\alpha, \beta)$. Since $\varepsilon < \beta$, the weight sequences $\mathbf{m}^{[\varepsilon,\rho]}$ no longer belong to $\mathscr{P}$ for all $\rho \in \mathfrak{S}_l$. Therefore, by Proposition 4.5, $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}^{[\varepsilon,\rho]}}$ (for all $\rho \in \mathfrak{S}_l$), namely a set of twisted Uglov l-partitions, which is always equal to the set $\mathcal{K}^{\pi}_{C_e}(n)$ (see Remark 5.8 for instance). Denote $\mathscr{B} = \mathcal{K}^{\pi}_{C_e}(n)$.

For $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$, write $\mathscr{S}(M) = \{\boldsymbol{\mu} \vdash_l n \mid d_{\boldsymbol{\mu},M} \neq 0\}$, as in Definition 2.6. Since $\mathscr{B}$ is the canonical basic set with respect to $\ll_{\mathbf{m}^{[\varepsilon,\rho]}}$ (for all $\rho \in \mathfrak{S}_l$), there exists $\boldsymbol{\lambda} \in \mathscr{B}$ verifying:

$$\text{for all } \rho \in \mathfrak{S}_l, \quad \boldsymbol{\lambda} \ll_{\mathbf{m}^{[\varepsilon,\rho]}} \boldsymbol{\mu} \quad \forall \boldsymbol{\mu} \in \mathscr{S}(M). \tag{8}$$

Suppose that $\mathscr{B}$ is not the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$. Then either:

1. there exists $\boldsymbol{\mu} \in \mathscr{S}(M)$ such that $\boldsymbol{\mu} \ll_{\mathbf{m}} \boldsymbol{\lambda}$. Because $\varepsilon < \alpha$, Lemma 5.18 applies. In particular, Assertion 1. ensures that one can never find $\rho \in \mathfrak{S}_l$ such that $\boldsymbol{\lambda} \ll_{\mathbf{m}^{[\varepsilon,\rho]}} \boldsymbol{\mu}$, which contradicts (8).
2. there exists $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}$ such that $\boldsymbol{\lambda}'$ is also minimal in $\mathscr{S}(M)$ with respect to $\ll_{\mathbf{m}}$. In this case, by Point 2. of Lemma 5.18, $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are not comparable with respect to $\ll_{\mathbf{m}^{[\varepsilon,\rho]}}$ (for all $\rho \in \mathfrak{S}_l$). Therefore, $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are both minimal with respect to $\ll_{\mathbf{m}^{[\varepsilon,\rho]}}$, which, again, contradicts (8).

6 Canonical basic sets for singular m

Denote by $\mathscr{P}^*$ the set of all $\mathbf{m}$ in $\mathscr{P}$ such that $\mathbf{m}$ is not asymptotic. If $\mathbf{m} \in \mathscr{P}^*$, we say that $\mathbf{m}$ is singular. We want to show that $(\mathbf{H}_{k,n}, \mathbf{r})$ does not admit a canonical basic set with respect to $\ll_{\mathbf{m}}$ whenever $\mathbf{m}$ belongs to $\mathscr{P}^*$.

We start by a particular case.

6.1 $l = 2$ and $e = \infty$

In this section, we fix $l = 2$ and take $e = +\infty$, i.e. ζ is not a root of unity.

Here, we have $C_e = C_\infty = \{\mathbf{r}\}$ where $r \in \mathbb{Z}^2$. We write $\mathbf{r} = \mathbf{s}$.

Since $e = \infty$, $\mathscr{P}$ consits of just one hyperplane, namely the line passing through $\mathbf{s} = (s_1, s_2) = (r_1, r_2)$ with slope one, and $\mathscr{P}^* = \mathscr{P}$, see figure 3. This is what makes it easier to study.

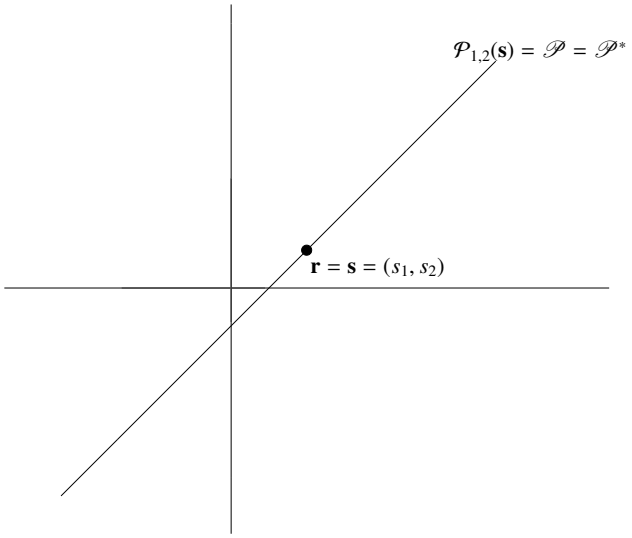


Figure 3: The set $\mathscr{P}$ in level 2 with $e = \infty$.

In this level 2 case, we introduce a combinatorial tool: the algebraic $\mathbf{s}$-symbol of an bipartition $\boldsymbol{\lambda}$, which we denote by $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})$. It can be seen as the shifted $\mathbf{s}$-symbol $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$, of infinite size. On the other hand, it will permit us to read some algebraic properties of $|\boldsymbol{\lambda}, \mathbf{s}\rangle \in \mathcal{F}_{\mathbf{s}}$.

When $s_1 \leq s_2$, the *empty algebraic s-symbol* is the following symbol:

$$\begin{pmatrix} \dots & -j & -(j-1) & \dots & -1 & 0 & \dots & \overset{s_2-s_1}{\longleftrightarrow} & & & s_2-1 & s_2 \\ \dots & -j & -(j-1) & \dots & -1 & 0 & \dots & s_1-1 & s_1 & & & \end{pmatrix}$$

When $s_1 \geq s_2$, then the second row will be shorter than the first one (numbered as usual from bottom to top).

Now let $\boldsymbol{\lambda} = (\lambda^1, \lambda^2) \vdash_2 n$. Denote by λ^i_k the parts of λ^i ($i = 1, 2$). Similarly to the shifted symbols (see section 2.3), the *algebraic* $\mathbf{s}$*-symbol* $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$ is obtained from the empty algebraic $\mathbf{s}$-symbol and by adding the parts of λ^i to the row i, from right to left:

$$\begin{pmatrix} \dots & -j & -(j-1) & \dots & \dots & \dots & \overset{s_2-s_1}{\longleftrightarrow} & s_2-1+\lambda^2_2 & s_2+\lambda^2_1 \\ \dots & -j & -(j-1) & \dots & s_1-1+\lambda^1_2 & s_1+\lambda^1_1 & & & \end{pmatrix}$$

Example 6.1. Let $\mathbf{s} = (1, 0)$ and $\boldsymbol{\lambda} = (2, 1.1.1) \vdash_2 5$. We have

$$\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} \dots & -3 & -1 & 0 & 1 & \\ \dots & -3 & -2 & -1 & 0 & 3 \end{pmatrix}$$

Remark 6.2. This data is equivalent to the 2-abacus associated to $\boldsymbol{\lambda}$ and $\mathbf{s}$ up to another convention for the indexation of the beads, in accordance with [14, Section 6.5].

The following proposition explains the algebraic meaning of $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})$. Recall that we denote by ω_j, $j \in \mathbb{Z}$, the fundamental weights of $U_q(\mathfrak{sl}_\infty)$. Set $\varepsilon_j = \omega_{j+1} - \omega_j$.

Proposition 6.3. *Denote by* $wt(\boldsymbol{\lambda})$ *the weight of* $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ *as an element of* $\mathcal{F}_{\mathbf{s}}$. *Then* $wt(\boldsymbol{\lambda})$ *can be read on the algebraic* $\mathbf{s}$*-symbol of* $\boldsymbol{\lambda}$. *Precisely, the contribution of each* j *appearing in* $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})$ *is equal to* ε_j.

Proof. The multipartition $\boldsymbol{\lambda}$ is a vertex in the connected component of the crystal graph of $\mathcal{F}_{\mathbf{s}}$ containing $|\boldsymbol{\emptyset}, \mathbf{s}\rangle$, which is a highest weight vector of weight $\Lambda = \omega_{s_1} + \dots + \omega_{s_l}$. The result follows from the fact that each addition of an j-node decreases the weight by $\alpha_j := \varepsilon_{j-1} - \varepsilon_j = -\omega_{j-1} + 2\omega_j - \omega_{j+1}$.

□

Proposition 6.4. *Let $\boldsymbol{\lambda} \in \Phi_{\mathbf{s}}(n)$, let $G_\infty(\boldsymbol{\lambda}, \mathbf{s})$ be the corresponding element of the global basis of $V(\mathbf{s})$. Write the decomposition of $G_\infty(\boldsymbol{\lambda}, \mathbf{s})$ on the basis of all bipartitions:*

$$G_\infty(\boldsymbol{\lambda}, \mathbf{s}) = \boldsymbol{\lambda} + \sum_{\boldsymbol{\mu} \vdash_2 n} d_{\mu,\lambda}(q) |\boldsymbol{\mu}, \mathbf{s}\rangle.$$

Then $d_{\mu,\lambda}(q) \neq 0$ only if $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})$ and $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\mu})$ coincide up to some permutation of their elements.

Proof. In the expansion of $G_\infty(\boldsymbol{\lambda}, \mathbf{s})$, the elements $|\boldsymbol{\mu}, \mathbf{s}\rangle$ appearing all have the same weight. In other terms,

$$d_{\mu,\lambda}(q) \neq 0 \Rightarrow \mathrm{wt}(\boldsymbol{\mu}) = \mathrm{wt}(\boldsymbol{\lambda})$$

By proposition 6.3, $\mathrm{wt}(\boldsymbol{\mu}) = \mathrm{wt}(\boldsymbol{\lambda})$ if and only if $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\mu})$ and $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})$ have the same elements, i.e. there is a permutation σ of the elements of $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})$ such that $\sigma(\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})) = \mathfrak{A}_{\mathbf{s}}(\boldsymbol{\mu})$.

□

Remark 6.5. Note that in this case ($l = 2$ and $e = \infty$), we know explicit formulas for computing the elements of the canonical basis (see [24]). These formulas directly show that the elements appearing in the decomposition of $G_\infty(\boldsymbol{\lambda}, \mathbf{s})$ have the same algebraic symbol up to some permutation of their elements.

Here we have used a more general argument, which is sufficient to prove the following corollary, and which also holds in higher level.

Corollary 6.6. *Let $\mathbf{m}$ be a singular element of $\mathbb{Q}^2$, i.e. $\mathbf{m} \in \mathscr{P}$. If $|\boldsymbol{\mu}, \mathbf{s}\rangle$ and $|\boldsymbol{\nu}, \mathbf{s}\rangle$ appear in the expansion of $G_\infty(\boldsymbol{\lambda}, \mathbf{s})$, then $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are not comparable with respect to $\ll_{\mathbf{m}}$.*

Proof. This relies on the fact that, in the singular case, the shifted $\mathbf{m}$-symbol $\mathfrak{B}_{\mathbf{m}}$ and the algebraic $\mathbf{s}$-symbol $\mathfrak{A}_{\mathbf{s}}$ carry the same information.

Indeed, for $\boldsymbol{\lambda} \vdash_2 n$, one can first recover its shifted $\mathbf{s}$-symbol of size p (for p sufficiently large) $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ from its algebraic $\mathbf{s}$-symbol $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})$ by:

1. truncating $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\lambda})$ in such a way that each line has the same number of elements as $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$,
2. translating this truncated symbol by $p - 1$, i.e. adding $p - 1$ to each element.

Now $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$ is obtained from $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ by translating the line i by $-v_i$ (where $v_i = s_i - m_i$), but since $\mathbf{m}$ is singular, $v_1 = v_2$. Hence $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$ is obtained by translating the whole $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ by $-v_1$.

Consequently, if $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\mu})$ and $\mathfrak{A}_{\mathbf{s}}(\boldsymbol{\nu})$ have the same elements, then so do $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\mu})$ and $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\nu})$. Hence by Proposition 6.4, if $|\boldsymbol{\mu}, \mathbf{s}\rangle$ and $|\boldsymbol{\nu}, \mathbf{s}\rangle$ appear in the expansion of $G_\infty(\boldsymbol{\lambda}, \mathbf{s})$, then $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\mu})$ and $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\nu})$ have the same elements, and thus are not comparable with respect to $\ll_{\mathbf{m}}$.

□

Let us get back to our initial problem, and the determination of the existence of a canonical basic set with respect to $\ll_{\mathbf{m}}$ for a singular $\mathbf{m}$. When $e = \infty$, by Theorem 2.3, we know that the algebra $\mathbf{H}_{k,n}$ split semi-simple if and only if $s_2 - s_1 \geq n$ or $s_2 - s_1 \leq -n$. Hence the interesting case is when $s_2 - s_1 \in [\![-(n-1), n-1]\!]$, which is what we will assume in the sequel.

Actually, when $l = 2$ and $e = \infty$, we have already noticed that explicit formulas for computing the expansion of the element of the canonical basis $G_\infty(\boldsymbol{\lambda})$ are known ([24]). Using these formulas, it is straightforward that if $s_2 - s_1 \geq n$ or $s_2 - s_1 \leq -n$, then it is not possible to find an element of the canonical basis whose expansion contains several bipartitions, therefore the decomposition matrix is trivial.

Proposition 6.7. *Assume $\mathbf{H}_{k,n}$ is non semi-simple. Let $\mathbf{m} \in \mathscr{P}$. Then $(\mathbf{H}_{k,n}, \mathbf{r})$ does not admit a canonical basic set with respect to $\ll_{\mathbf{m}}$.*

Proof. We assumed that $\mathbf{H}_{k,n}$ was non semi-simple. In this case, there exists a non-trivial column of the decomposition matrix D_∞, i.e. there exists $\boldsymbol{\lambda}$ such that $G_\infty(\boldsymbol{\lambda}, \mathbf{s}) = |\boldsymbol{\lambda}, \mathbf{s}\rangle + \sum_{\boldsymbol{\mu} \vdash_2 n} d_{\boldsymbol{\mu},\boldsymbol{\lambda}}(q)|\boldsymbol{\mu}, \mathbf{s}\rangle$ with at least one element $\boldsymbol{\mu}$ such that $d_{\boldsymbol{\mu},\boldsymbol{\lambda}}(q) \neq 0$.

Now by Corollary 6.6, all elements appearing in the expansion of $G_\infty(\boldsymbol{\lambda}, \mathbf{s})$ are pairwise uncomparable with respect to $\ll_{\mathbf{m}}$. Hence there cannot be a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$. □

6.2 General singular case

In order to solve the general case, we need once again lemma 5.18.

Proposition 6.8. *Let* $\mathbf{m} \in \mathscr{P}^*$. *Then* $(\mathbf{H}_{k,n}, \mathbf{r})$ *does not admit a canonical basic set with respect to* $\ll_{\mathbf{m}}$.

Proof. Suppose that there exists a canonical basic set $\mathscr{B}$ for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$.

For $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$, recall that we have denoted $\mathscr{S}(M) = \{\boldsymbol{\mu} \vdash_l n \,|\, d_{\boldsymbol{\mu},M} \neq 0\}$. By definition, there exists a unique element $\boldsymbol{\lambda} \in \mathscr{S}(M)$ such that $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}$ for all $\boldsymbol{\mu} \in \mathscr{S}(M)$.

Since $\mathbf{m} \in \mathscr{P}^*$, there exists $\mathbf{s} \in C_e$ non-asymptotic and $i, j \in [\![1, l]\!]$, $i < j$ such that $\mathbf{m} \in \mathcal{P}_{i,j}(\mathbf{s})$. As inthe proof of Proposition 5.17, let $\beta = \min_{m_i \neq m_j} |m_i - m_j|$ and $\alpha = \min_{\boldsymbol{\mu} \in \mathscr{S}(M)} (\alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}})$, and set $0 < \varepsilon < \min(\alpha, \beta)$. Then all $\mathbf{m}^{[\varepsilon,\rho]}$ are regular because $\varepsilon < \beta$. By Proposition 4.5, there exists a canonical basic set $\mathscr{B}^{[\rho]}$ for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}^{[\varepsilon,\rho]}}$, namely the set of some twisted Uglov l-partitions. Since $\mathbf{m}$ (or, equivalently, $\mathbf{s}$) is not asymptotic, Remark 5.10 implies that there exists ρ_1 and ρ_2 such that

$$\mathscr{B}^{[\rho_1]} \neq \mathscr{B}^{[\rho_2]}. \tag{9}$$

Since $\mathscr{B}^{[\rho_1]}$ is the canonical basic set with respect to $\ll_{\mathbf{m}^{[\varepsilon,\rho_1]}}$, there exists a unique element $\boldsymbol{\lambda}^{[1]}$ such that for all $\boldsymbol{\mu} \in \mathscr{S}(M)$, $\boldsymbol{\lambda}^{[1]} \ll_{\mathbf{m}^{[\varepsilon,\rho_1]}} \boldsymbol{\mu}$. Similarly, there exists a unique element $\boldsymbol{\lambda}^{[2]}$ such that for all $\boldsymbol{\mu} \in \mathscr{S}(M)$, $\boldsymbol{\lambda}^{[2]} \ll_{\mathbf{m}^{[\varepsilon,\rho_2]}} \boldsymbol{\mu}$.

Now by (9), there exists a particular $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$ such that

$$\boldsymbol{\lambda}^{[1]} \neq \boldsymbol{\lambda}^{[2]}. \tag{10}$$

Thus, we have:

- $\boldsymbol{\lambda}^{[1]} \ll_{\mathbf{m}^{[\varepsilon,\rho_1]}} \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\lambda}^{[1]}$. But by Lemma 5.18 (which applies since $\varepsilon < \alpha$), this not possible if $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}^{[1]}$. Hence $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{[1]}$.
- $\boldsymbol{\lambda}^{[2]} \ll_{\mathbf{m}^{[\varepsilon,\rho_2]}} \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\lambda}^{[2]}$. Again, by Lemma 5.18, this not possible if $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}^{[2]}$. Hence $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{[2]}$.

Hence, $\boldsymbol{\lambda}^{[1]} = \boldsymbol{\lambda}^{[2]}$, which contradicts (10). □

As previously mentioned in Remark 2.8, a singular weight sequence $\mathbf{m}$ can however afford a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$, but with $\mathbf{s} \in C \backslash C_e$ (i.e. with a different parametrisation of the rows of D). This is illustrated in the following example.

Example 6.9. Let $l = 2$, $e = 3$, $n \geq 4$, $\mathbf{r} = (1, 0)$ (in particular $\mathbf{r}$ is not asymptotic), and $\mathbf{s} = \mathbf{r}^\sigma = (0, 1)$ (where $\sigma = (12)$). Take $m = (0, -1)$.

Then $\mathbf{m} \in \mathcal{P}_{1,2}(\mathbf{r})$ since $\mathbf{r} - \mathbf{m} = (1, 1)$, and by Proposition 6.8, $(\mathbf{H}_{k,n}, \mathbf{r})$ does not admit a canonical basic set with respect to $\ll_{\mathbf{m}}$. However, $\mathbf{s} - \mathbf{m} = (0, 2)$, so that $\mathbf{m} \in \mathscr{D}_{\mathbf{s}}$. By Proposition 4.1, $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set $\Phi_{\mathbf{s}}(n)$.

Putting together Propositions 4.5, 5.17 and 6.8, we have proved:

Theorem 6.10. *Given a multicharge* $\mathbf{r} \in \mathbb{Z}^l$ *and a weight sequence* $\mathbf{m} \in \mathbb{Q}^l$, *we have the following exhaustive classification:*

- *If* $\mathbf{m}$ *is regular, then* $(\mathbf{H}_{k,n}, \mathbf{r})$ *admits a canonical basic set with respect to* $\ll_{\mathbf{m}}$*, namely the set of* σ*-twisted Uglov l-partitions* $\sigma(\Phi_{\mathbf{s}}(n))$ *where* σ *is the* $\mathbf{m}$*-adapted permutation and* $\mathbf{s}$ *is the* $\mathbf{m}$*-adapted multicharge (see Proposition 4.5).*
- *If* $\mathbf{m}$ *is asymptotic, then* $(\mathbf{H}_{k,n}, \mathbf{r})$ *admits a canonical basic set with respect to* $\ll_{\mathbf{m}}$*, namely the set of* π*-twisted Kleshchev l-partitions fo rank* $\mathcal{K}^{\pi}_{C_e}(n)$*, where* π *is described in Proposition 5.17.*
- *If* $\mathbf{m}$ *is singular, then* $(\mathbf{H}_{k,n}, \mathbf{r})$ *does not admit a canonical basic set with respect to* $\ll_{\mathbf{m}}$*.*

Remark 6.11. Note that a weight sequence $\mathbf{m}$ can be simultaneously regular and asymptotic. In this case, one must have $\sigma(\Phi_{\mathbf{s}}(n)) = \mathcal{K}^{\pi}_{C_e}(n)$, which is precisely what is stated in Remark 5.8.

References

[1] Susumu Ariki. On the semi-simplicity of the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$. *J. Algebra*, 177:164–185, 1994.

[2] Susumu Ariki. On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$. *J. Math. Kyoto Univ.*, 36(4):789–808, 1996.

[3] Susumu Ariki. *Representations of quantum algebras and combinatorics of Young tableaux*, volume 26 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2002. Translated from the 2000 Japanese edition and revised by the author.

[4] Susumu Ariki and Kazuhiko Koike. A Hecke Algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and Construction of Its Irreducible Representations. *Advances in Mathematics*, 106(2):216 – 243, 1994.

[5] Michel Broué and Gunter Malle. Zyklotomische Heckealgebren. *Astérisque*, 212:119–189, 1993.

[6] Maria Chlouveraki, Iain Gordon, and Stephen Griffeth. Cell modules and canonical basic sets for hecke algebras from cherednik algebras. April 2011.

[7] Richard Dipper and Andrew Mathas. Morita equivalences of Ariki-Koike algebras. 2002.

[8] Omar Foda, Bernard Leclerc, Masato Okado, Jean-Yves Thibon, and Trevor Welsh. Branching functions of $A^{(1)}_{n-1}$ and Jantzen-Seitz problem for Ariki-Koike algebras. 2007.

[9] Meinolf Geck. Kazhdan-Lusztig cells and decomposition numbers. *Represent. Theory*, 2:264–277, 1998.

[10] Meinolf Geck. Modular representations of Hecke algebras. 2007.

[11] Meinolf Geck. On the Kazhdan–Lusztig order on cells and families. October 2009.

[12] Meinolf Geck and Lacrimioara Iancu. Ordering Lusztig's families in type B_n. May 2012.

[13] Meinolf Geck, Lacrimioara Iancu, and Gunter Malle. Weights of Markov traces and generic degrees. *Indagationes Mathematicae*, 2000.

[14] Meinolf Geck and Nicolas Jacon. *Representations of Hecke Algebras at Roots of Unity*. Springer, 2011.

[15] Meinolf Geck and Götz Pfeiffer. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*. Oxford University Press, 2000.

[16] Meinolf Geck and Raphael Rouquier. Filtrations on Projective Modules for Iwahori-Hecke Algebras. 2001.

[17] Jin Hong and Seok-Jin Kang. *Introduction to Quantum Groups and Crystal Bases*. American Mathematical Society, 2002.

[18] Nicolas Jacon. On the parametrization of the simple modules for Ariki-Koike algebras at roots of unity. 2004.

[19] Nicolas Jacon. Crystal graphs of irreducible $U_v(\hat{sl}_e)$-modules of level two and Uglov bipartitions. 2007.

[20] Nicolas Jacon and Cédric Lecouvey. Crystal isomorphisms for irreducible highest weight $U_v(\hat{sl}_e)$-modules of higher level. *Algebras and Representation Theory*, 13:467–489, 2007.

[21] Michio Jimbo, Kailash C. Misra, Tetsuji Miwa, and Masato Okado. Combinatorics of representations of $U_q(\widehat{sl(n)})$ at $q = 0$. *Comm. Math. Phys.*, 136(3):543–566, 1991.

[22] Masaki Kashiwara. On crystal bases of the q-analogue of universal enveloping algebras. 1991.

[23] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Hecke algebras at roots of unity and crystal bases of quantum affine algebras. 1995.

[24] Bernard Leclerc and Hyohe Miyachi. Constructible characters and canonical bases. 2007.

[25] Bernard Leclerc and Jean-Yves Thibon. Canonical Bases of q-Deformed Fock Spaces. *Int. Math. Res. Notices*, 9:447–456, 1996.

[26] Emilie Liboz. *Algèbres de Cherednik et ordres sur les blocs de Calogero-Moser des groupes imprimitifs*. PhD thesis, Université de Franche-Comté, 2012.

[27] George Lusztig. *Characters of reductive groups over a finite field*, volume 107 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1984.

[28] Andrew Mathas. Matrix units and generic degrees for the Ariki–Koike algebras. 2007.

[29] Denis Uglov. Canonical bases of higher-level q-deformed Fock spaces and Kazhdan-Lusztig polynomials. 1999.

[30] Xavier Yvonne. *Bases canoniques d'espaces de Fock de niveau supérieur*. PhD thesis, Université de Caen, 2005.