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# WEIGHTED TENT SPACES WITH WHITNEY AVERAGES: FACTORIZATION, INTERPOLATION AND DUALITY

YI HUANG

ABSTRACT. In this paper, we introduce a new scale of tent spaces which covers, the (weighted) tent spaces of Coifman-Meyer-Stein and of Hofmann-Mayboroda-McIntosh, and some other tent spaces considered by Dahlberg, Kenig-Pipher and Auscher-Axelsson in studying boundary value problems for elliptic systems. The strong factorizations within our tent spaces, with applications to quasi-Banach complex interpolation and to multiplier-duality theory, are then established. This way, we unify and extend the corresponding results obtained by Coifman-Meyer-Stein, Cohn-Verbitsky and Hytönen-Rosén.

## 0. BASIC NOTATION AND ARTICLE STRUCTURE

Let  $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+ = \mathbb{R}^n \times (0, \infty)$  be the usual upper half-space in  $\mathbb{R}^{n+1}$ . Points in  $\mathbb{R}^n$  (respectively in  $\mathbb{R}_+^{n+1}$ ) will be generally denoted by the letters  $x$  or  $z$  (respectively by  $(y, t)$  or  $(z, s)$ ). For a point  $(y, t)$  in  $\mathbb{R}_+^{n+1}$ , we let  $B(y, t) = \{z \in \mathbb{R}^n \mid |z - y| < t\}$  lie in the boundary  $\mathbb{R}^n = \partial\mathbb{R}_+^{n+1}$ . Here and below, the capital letter  $B$  denotes an open ball in  $\mathbb{R}^n$ , and  $|\cdot|$  denotes the Euclidean distance on  $\mathbb{R}^n$ .

Given  $\alpha > 0$ , we shall denote the *cone* with aperture  $\alpha$  and vertex  $x \in \mathbb{R}^n$  by

$$\Gamma_\alpha(x) := \{(y, t) \in \mathbb{R}_+^{n+1} \mid |y - x| < \alpha t\} = \{(y, t) \in \mathbb{R}_+^{n+1} \mid B(y, \alpha t) \ni x\},$$

and shall denote the *tent* with aperture  $\alpha$  and base  $B \subset \mathbb{R}^n$  by

$$\widehat{B}^\alpha := \left( \bigcup_{x \in B^c} \Gamma_\alpha(x) \right)^c = \{(y, t) \in \mathbb{R}_+^{n+1} \mid B(y, \alpha t) \subset B\}.$$

If  $\alpha = 1$ , we simply write  $\Gamma(x)$  and  $\widehat{B}$ .

Given a point  $(y, t) \in \mathbb{R}_+^{n+1}$ , we construct its *Whitney box* as

$$W(y, t) := \{(z, s) \in \mathbb{R}_+^{n+1} \mid |z - y| < \alpha_1 t, \alpha_2^{-1} t < s < \alpha_2 t\}.$$

Here, the two numbers  $(\alpha_1, \alpha_2)$  with  $\alpha_1 > 0$  and  $\alpha_2 > 1$ , are called the *Whitney parameters*. They are said to be *consistent* if  $0 < \alpha_1 < \alpha_2^{-1} < 1$ .

Throughout this article, the set of Vinogradov notations  $\{\lesssim, \simeq, \gtrsim\}$  will be used. For two quantities  $a$  and  $b$ , which can be function values, set volumes, function norms or anything else, the term  $a \lesssim b$  means that there exists a constant  $C > 0$ , which depends only on parameters at hand, such that  $a \leq Cb$ . In a similar way,  $a \gtrsim b$  means  $b \lesssim a$ , and,  $a \simeq b$  means both  $a \lesssim b$  and  $a \gtrsim b$ .

This paper is organized as follows.

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- ★ Section 1. We define in Definition 1.1 our scale of tent spaces  $T_{q,\beta}^{p,r}$  systematically. At the end of this section, we will also discuss some basic function space properties, such as convexity and separability, of these new tent spaces.
- ★ Section 2. We show that the definition of our tent spaces is independent of the aperture used for cones and tents, and of the pair of Whitney parameters used for Whitney boxes. As a reward, we can see for  $0 < r = q < \infty$ , the coincidence (Theorem 2.2) of our tent spaces with the (weighted) tent spaces of Coifman-Meyer-Stein and of Hofmann-Mayboroda-McIntosh.
- ★ Section 3 and Section 6. The core *endpoint* factorization theorem (Theorem 3.2) is presented in Section 3, with its full proof postponed to Section 6. Together with a multiplication lemma, we show the *general* multiplication and factorization theorem (Theorem 3.4) as a consequence of Theorem 3.2.
- ★ Section 4 and Section 5. Under the general multiplication and factorization theorem, the quasi-Banach complex interpolation (Theorem 4.3) and the multiplier-duality results (Theorem 5.2 and Theorem 5.4) will be established in Section 4 and Section 5 respectively. There, we will also make a detailed connection with the corresponding known results on interpolation, multiplication, factorization and duality of tent spaces, which are mainly obtained by Coifman-Meyer-Stein, Cohn-Verbitsky and Hytönen-Rosén.

### 1. DEFINITIONS OF THE TENT SPACES $T_{q,\beta}^{p,r}$

Let  $r \in (0, \infty]$ . By  $L_{\text{loc}}^r(\mathbb{R}_+^{n+1}; \mathbb{C})$ , we mean the class of complex-valued measurable functions which are defined on  $\mathbb{R}_+^{n+1}$  and locally in  $L^r$ . Note that this interpretation also makes sense when  $r = \infty$ . For  $r \in (0, \infty)$  and  $f \in L_{\text{loc}}^r(\mathbb{R}_+^{n+1}; \mathbb{C})$ , denote the (unweighted)  $L^r$ -Whitney average of  $f$  on  $W(y, t)$  by

$$\mathcal{W}_r(f)(y, t) := |W(y, t)|^{-1/r} \|f\|_{L^r(W(y, t), dz ds)},$$

while for  $r = \infty$ , we take the usual essential supremum interpretation

$$\mathcal{W}_\infty(f)(y, t) := \operatorname{ess\,sup}_{(z, s) \in W(y, t)} |f(z, s)|.$$

Here and below, apart from the Euclidean distance,  $|\cdot|$  also denotes the moduli of complex values or the set volumes in  $\mathbb{R}^n$  and  $\mathbb{R}_+^{n+1}$ .

**Definition 1.1.** I) For  $0 < p, q \leq \infty$ , we first define in  $L_{\text{loc}}^q(\mathbb{R}_+^{n+1}; \mathbb{C})$  the scale of *tent spaces*  $T_q^p$  according to the following four non-overlapping categories.

A)  $0 < p, q < \infty$ . In this case, we let

$$T_q^p := \{g \mid \mathcal{A}_q(g) \in L^p(\mathbb{R}^n)\} \quad \text{and} \quad \|g\|_{T_q^p} := \|\mathcal{A}_q(g)\|_{L^p},$$

where the *conical  $q$ -functional*  $\mathcal{A}_q$  is defined as

$$\mathcal{A}_q(g)(x) := \left( \iint_{\Gamma(x)} |g(y, t)|^q \frac{dy dt}{t^{n+1}} \right)^{1/q}, \quad x \in \mathbb{R}^n.$$

B)  $0 < q < p = \infty$ . In this case, we let

$$T_q^\infty := \{g \mid \mathcal{C}_q(g) \in L^\infty(\mathbb{R}^n)\} \quad \text{and} \quad \|g\|_{T_q^\infty} := \|\mathcal{C}_q(g)\|_{L^\infty},$$

where the *Carleson  $q$ -functional*  $\mathcal{C}_q$  is defined as

$$\mathcal{C}_q(g)(x) := \sup_{B \ni x} |B|^{-1/q} \left( \iint_{\widehat{B}} |g(y, t)|^q \frac{dy dt}{t} \right)^{1/q}, \quad x \in \mathbb{R}^n.$$

C)  $0 < p < q = \infty$ . In this case, we let

$$T_\infty^p := \{g \mid \mathcal{N}(g) \in L^p(\mathbb{R}^n)\} \quad \text{and} \quad \|g\|_{T_\infty^p} := \|\mathcal{N}(g)\|_{L^p},$$

where the *non-tangential maximal functional*  $\mathcal{N}$  is defined as

$$\mathcal{N}(g)(x) := \sup_{(y, t) \in \Gamma(x)} |g(y, t)|, \quad x \in \mathbb{R}^n.$$

D)  $p = q = \infty$ . In this case, we simply let

$$T_\infty^\infty := L^\infty(\mathbb{R}_+^{n+1}).$$

Let  $\beta \in \mathbb{R}$ . We also define the scale of *weighted tent spaces*  $T_{q, \beta}^p$  by

$$T_{q, \beta}^p := \{g \mid g(y, t)t^{-\beta} \in T_q^p\} \quad \text{and} \quad \|g\|_{T_{q, \beta}^p} := \|g(y, t)t^{-\beta}\|_{T_q^p}.$$

II) Given  $0 < r \leq \infty$  and  $\beta \in \mathbb{R}$ , and assume that the pair of Whitney parameters  $(\alpha_1, \alpha_2)$  is consistent. Then corresponding to each category above, we define in  $L_{\text{loc}}^r(\mathbb{R}_+^{n+1}; \mathbb{C})$  the scale of *tent spaces with Whitney averages*  $T_q^{p, r}$  by

$$T_q^{p, r} := \{f \mid \mathcal{W}_r(f) \in T_q^p\} \quad \text{and} \quad \|f\|_{T_q^{p, r}} := \|\mathcal{W}_r(f)\|_{T_q^p},$$

and the scale of *weighted tent spaces with Whitney averages*  $T_{q, \beta}^{p, r}$  by

$$T_{q, \beta}^{p, r} := \{f \mid f(z, s)s^{-\beta} \in T_q^{p, r}\} \quad \text{and} \quad \|f\|_{T_{q, \beta}^{p, r}} := \|f(z, s)s^{-\beta}\|_{T_q^{p, r}}.$$

In the above definitions, the  $L^r$ -Whitney average and the weight  $\beta$  are required for the applications to boundary value problems of second order elliptic PDEs in [5]. In practice  $\beta$  is a regularity index, and the weight constraint  $\beta \in [-2/q, 0]$ , with the convention  $\beta = 0$  if  $q = \infty$ , is taken in [5].

**Remark 1.2.** One easily verifies that in Category C) of Type I) spaces the functions are indeed everywhere defined. In other categories, we identify two measurable functions the same if they only differ on a set with measure 0. Moreover, in Category C) of Type II), since functions are  $L_{\text{loc}}^r$ , the averages  $\mathcal{W}_r(f)$  are everywhere finite.

**Remark 1.3.** By definition  $T_{q, 0}^p = T_q^p$  and  $T_{q, 0}^{p, r} = T_q^{p, r}$ . Moreover, for  $\beta \in \mathbb{R}$ ,  $T_{q, \beta}^p$  is isometric to  $T_q^p$  and  $T_{q, \beta}^{p, r}$  is isometric to  $T_q^{p, r}$ , via the mapping

$$\iota : f \mapsto \tilde{f}, \quad \tilde{f}(z, s) = f(z, s)s^{-\beta}.$$

Observe also that since  $(z, s) \in W(y, t)$  implies  $s \simeq t$ , we have that

$$f \in T_{q, \beta}^{p, r} \iff \mathcal{W}_r(f) \in T_{q, \beta}^p.$$

**Remark 1.4.** The classical tent spaces of Coifman-Meyer-Stein in [10], where the weight  $\beta = 0$  and Category  $C$ ) is smaller<sup>1</sup>, and the weighted tent spaces of Hofmann-Mayboroda-McIntosh in [15], where only Category  $A$ ) is considered, are all included in our scale  $T_{q,\beta}^p$ . The scale  $T_{q,\beta}^{p,r}$  with Whitney averages covers the function spaces which were introduced in [12] and [23], and further investigated in [5], [17] and [27]. In this regard, see also the concluding paragraphs of Section 5 for a detailed correspondence. Note that compared to [10] we also bring in Category  $D$ ). If  $0 < r < \infty$ , we call functions in  $T_{\infty}^{\infty,r}$  the  $r$ -Whitney multipliers. In the trivial case  $p = q = r = \infty$ , it is not difficult to observe that  $T_{\infty}^{\infty,\infty} = T_{\infty}^{\infty} = L^{\infty}(\mathbb{R}_+^{n+1})$ .

We end this section with several basic properties of our tent spaces.

Convexity and completeness. Given the tent space  $T_{q,\beta}^{p,r}$ , we let  $\tau = \min(p, q, r)$ . Observe that when  $\tau \geq 1$ , the space  $T_{q,\beta}^{p,r}$  is Banach. In fact, the triangle inequality simply follows from Minkowski's integral inequality, and the completeness can be deduced from the one of  $T_q^p$ , as we have the implication

$$f \in T_{q,\beta}^{p,r} \implies \mathcal{W}_r(f(z, s)s^{-\beta}) \in T_q^p.$$

Power-space and convexification. For a quasi-Banach function space, the trick of taking the powers is particularly useful. As for our tent spaces, let

$$[T_{q,\beta}^{p,r}]^{\theta} := \{f \text{ measurable} \mid |f|^{1/\theta} \in T_{q,\beta}^{p,r}\}, \theta \in (0, 1),$$

equipped with

$$\|f\|_{[T_{q,\beta}^{p,r}]^{\theta}} := \||f|^{1/\theta}\|_{T_{q,\beta}^{p,r}}^{\theta}.$$

This way, we have the realization

$$[T_{q,\beta}^{p,r}]^{\theta} = T_{q/\theta,\beta\theta}^{p/\theta,r/\theta}, \theta \in (0, 1).$$

Now for the quasi-Banach  $T_{q,\beta}^{p,r}$ , with  $\tau < 1$ ,  $[T_{q,\beta}^{p,r}]^{\tau}$  is then a convexification of  $T_{q,\beta}^{p,r}$ .

Separability and density. Consider the covering of  $\mathbb{R}_+^{n+1}$  by rational rectangles, which are of the product form  $\prod_{i=1}^{n+1}(a_i, b_i)$ , where for  $1 \leq i \leq n+1$ ,  $a_i$  and  $b_i$  are in  $\mathbb{Q}$  and  $b_{n+1} > 0$ . Let  $E$  be the linear span on  $\mathbb{Q}$  of the characteristic functions of rational rectangles in  $\mathbb{R}_+^{n+1}$ . Given the tent space  $T_{q,\beta}^{p,r}$ , we let  $\sigma = \max(p, q, r)$ . If  $0 < \sigma < \infty$ , one can show that the countable set  $E$  is dense in  $T_{q,\beta}^{p,r}$ , thereby in this case  $T_{q,\beta}^{p,r}$  is separable. We also point out that if  $0 < \sigma < \infty$ , the  $L^r$  functions which have compact support in  $\mathbb{R}_+^{n+1}$  are dense in  $T_{q,\beta}^{p,r}$ .

## 2. COINCIDENCE AND CHANGE OF GEOMETRY

A demanding reader may ask two natural questions: i) how do the inner (local) Whitney averages  $\mathcal{W}_r$  behave under the outer (boundary-reaching)  $\mathcal{A}_q$  or  $\mathcal{C}_q$  averages? ii) is our Definition 1.1 independent of the involved geometrical parameters?

<sup>1</sup>More precisely, [10] requires the additional boundary assumption  $g \in C_{n,t}(\mathbb{R}_+^{n+1}; \mathbb{C})$ , meaning that  $g$  is a complex-valued continuous function on  $\mathbb{R}_+^{n+1}$  and also has *non-tangential convergence*:

$$\lim_{\Gamma(x) \ni (y,t) \rightarrow x} g(y, t) \text{ exists for almost every } x \in \mathbb{R}^n.$$

Aiming at the question i), we will first investigate the relation between the classical scale  $T_q^p$  and our scale  $T_q^{p,r}$  with Whitney averages. At the end of this section, we will also give an observation on the question ii).

Let us start with the following result.

**Observation 2.1** (Change of apertures). Define for  $0 < q < \infty$  and  $\alpha > 0$  the following three  $\alpha$ -apertured functionals as

$$\begin{aligned}\mathcal{A}_q^\alpha(g)(x) &:= \left( \iint_{\Gamma_\alpha(x)} |g(y,t)|^q \frac{dydt}{t^{n+1}} \right)^{1/q}, \quad x \in \mathbb{R}^n, \\ \mathcal{N}^\alpha(g)(x) &:= \sup_{(y,t) \in \Gamma_\alpha(x)} |g(y,t)|, \quad x \in \mathbb{R}^n, \\ \mathcal{C}_q^\alpha(g)(x) &:= \sup_{B \ni x} |B|^{-1/q} \left( \iint_{\widehat{B}^\alpha} |g(y,t)|^q \frac{dydt}{t} \right)^{1/q}, \quad x \in \mathbb{R}^n.\end{aligned}$$

Similar to Definition 1.1, these functionals can also result in a scale of tent spaces  ${}^\alpha T_q^p$ , where we let  ${}^\alpha T_\infty^\infty = L^\infty$  for the trivial case  $p = q = \infty$ . It is well known that we have the change of aperture equivalence  ${}^\alpha T_q^p = T_q^p$ , with

$$(1) \quad C(n, \alpha, p, q) \|g\|_{T_q^p} \leq \|g\|_{{}^\alpha T_q^p} \leq C'(n, \alpha, p, q) \|g\|_{T_q^p}, \quad 0 < p, q \leq \infty.$$

For the proof, see [13] for the simple situation  $0 < p < q = \infty$ . For the case  $q = 2$  and  $0 < p < \infty$  (hence for  $0 < p, q < \infty$  by taking the powers of  $g$  properly), see [10] for a rough, and [30] for a refined argument on estimating  $C'$  when  $\alpha > 1$ . By using the atomic decomposition and the interpolation method, the sharp determination on both  $C$  and  $C'$  when  $\alpha > 0$ , for the case  $q = 2$  and  $0 < p \leq \infty$ , is obtained recently in [4]. Note that the methods of [4] extend to the case  $q = \infty$  under minor modifications. We also remark that, the vector-valued approach in [14] and [18] can deal with the change of apertures in a very simple manner in the Banach case, and then a convexification process takes care of the quasi-Banach case.

**Theorem 2.2.** *We have the coincidence with equivalence of quasi-norms*

$$T_{q,\beta}^{p,q} = T_{q,\beta}^p, \quad 0 < p \leq \infty, \quad 0 < q < \infty, \quad \beta \in \mathbb{R}.$$

*In particular,  $T_q^{p,q} = T_q^p$ ,  $0 < p \leq \infty$ ,  $0 < q < \infty$ , showing that the classical tent spaces are included in the tent spaces with Whitney averages.*

*Proof.* By Remark 1.3, it is enough to prove

$$T_q^{p,q} = T_q^p, \quad 0 < p \leq \infty, \quad 0 < q < \infty.$$

We start with the following *Whitney box geometry*:  $\forall (z, s) \in \mathbb{R}_+^{n+1}$

$$W) \quad W_*(z, s) \subset \{(y, t) | W(y, t) \ni (z, s)\} \subset W_{**}(z, s),$$

where  $W_*$  and  $W_{**}$  are the Whitney boxes associated to the Whitney parameters  $(\alpha_1 \alpha_2^{-1}, \alpha_2)$  and  $(\alpha_1 \alpha_2, \alpha_2)$  respectively<sup>2</sup>, and  $(\alpha_1, \alpha_2)$  is the pair of Whitney parameters which defines  $W$  and was used in Definition 1.1. We only need to verify the choices of  $\alpha_1 \alpha_2^{-1}$  and  $\alpha_1 \alpha_2$ , as the determination on  $\alpha_2$  is straightforward. To see the first inclusion in  $W$ , given any  $(y, t) \in W_*(z, s)$ , we have  $|z - y| < \alpha_1 \alpha_2^{-1} s < \alpha_1 t$ , which implies  $W(y, t) \ni (z, s)$ . To see the second inclusion, given any  $(y, t)$  with

<sup>2</sup>The pair of Whitney parameters defining  $W_{**}$  is not necessarily consistent, but for the purpose here, the consistency is not needed.

$W(y, t) \ni (z, s)$ , we have  $|y - z| < \alpha_1 t < \alpha_1 \alpha_2 s$ , which implies  $(y, t) \in W_{**}(z, s)$ . This proves the Whitney box geometry  $W$ .

For the *cone geometry*, let  $\alpha_0 = \alpha_2^{-1}(1 - \alpha_1)$ . We have that:  $\forall x \in \mathbb{R}^n$

$$C_1) \quad (z, s) \in \Gamma_{\alpha_0}(x) \text{ and } (y, t) \in W_*(z, s) \implies (y, t) \in \Gamma(x).$$

Indeed, we can compute as follow

$$|y - x| \leq |y - z| + |z - x| < \alpha_1 \alpha_2^{-1} s + \alpha_2^{-1}(1 - \alpha_1)s < t.$$

Let  $\alpha_C = \alpha_2 + \alpha_1 \alpha_2$ . There also holds:  $\forall x \in \mathbb{R}^n$

$$C_2) \quad (y, t) \in \Gamma(x) \text{ and } (z, s) \in W(y, t) \implies (z, s) \in \Gamma_{\alpha_C}(x).$$

Indeed, we can compute as follow

$$|z - x| \leq |z - y| + |y - x| < \alpha_1 t + t < (\alpha_2 + \alpha_1 \alpha_2)s.$$

Now from  $W) + C_1)$ , we have:  $\forall x \in \mathbb{R}^n$

$$\chi_{\Gamma_{\alpha_0}(x)}(z, s) \chi_{W_*(z, s)}(y, t) \leq \chi_{\Gamma(x)}(y, t) \chi_{W(y, t)}(z, s),$$

and from  $W) + C_2)$ , we have:  $\forall x \in \mathbb{R}^n$

$$\chi_{\Gamma(x)}(y, t) \chi_{W(y, t)}(z, s) \leq \chi_{\Gamma_{\alpha_C}(x)}(z, s) \chi_{W_{**}(z, s)}(y, t).$$

Then it follows from an integration in  $(y, t)$  that:  $\forall x \in \mathbb{R}^n$

$$\chi_{\Gamma_{\alpha_0}(x)}(z, s) \lesssim \iint_{\mathbb{R}_+^{n+1}} \chi_{\Gamma(x)}(y, t) \frac{\chi_{W(y, t)}(z, s)}{t^{n+1}} dy dt \lesssim \chi_{\Gamma_{\alpha_C}(x)}(z, s),$$

where in dividing  $s^{n+1}$ , we use the similarity  $s \simeq t$  implicitly.

For  $0 < q < \infty$ , multiplying by  $|f(z, s)|^q$  the above inequalities and then integrating in  $(z, s)$ , we have from Fubini's theorem that

$$\mathcal{A}_q^{\alpha_0}(f)(x) \lesssim \mathcal{A}_q(\mathcal{W}_q(f))(x) \lesssim \mathcal{A}_q^{\alpha_C}(f)(x), \quad \forall x \in \mathbb{R}^n.$$

For  $0 < p < \infty$ , taking an  $L^p$  integration in  $x$  in the above two functional relations and using the change of aperture equivalence in Observation 2.1 lead us to the coincidence  $T_q^{p, q} = T_q^p$  in Category A).

For the *tent geometry*, let  $\alpha_T = \alpha_2 + \alpha_1 \alpha_2^{-1}$ . We have that:  $\forall B \subset \mathbb{R}^n$

$$T_1) \quad (z, s) \in \widehat{B^{\alpha_T}} \text{ and } (y, t) \in W_*(z, s) \implies (y, t) \in \widehat{B}.$$

Indeed, given  $B \subset \mathbb{R}^n$ ,  $(z, s) \in \widehat{B^{\alpha_T}}$  and  $(y, t) \in W_*(z, s)$ , then  $B(z, \alpha_T s) \subset B$ . Thus

$$B(y, t) \subset B(z, t + |z - y|) \subset B(z, t + \alpha_1 \alpha_2^{-1} s) \subset B(z, \alpha_T s),$$

so  $B(y, t) \subset B$ . Recall that  $\alpha_0 = \alpha_2^{-1}(1 - \alpha_1)$ . There also holds:  $\forall B \subset \mathbb{R}^n$

$$T_2) \quad (y, t) \in \widehat{B} \text{ and } (z, s) \in W(y, t) \implies (z, s) \in \widehat{B^{\alpha_0}}.$$

Indeed, given  $B \subset \mathbb{R}^n$ ,  $(y, t) \in \widehat{B}$  and  $(z, s) \in W(y, t)$ , then  $B(y, t) \subset B$ . Thus

$$B(z, \alpha_0 s) \subset B(y, \alpha_0 s + |z - y|) \subset B(y, \alpha_0 s + \alpha_1 t) \subset B(y, t),$$

so  $B(z, \alpha_0 s) \subset B$  and  $(z, s) \in \widehat{B^{\alpha_0}}$ .

Now from  $W) + T_1)$ , we have:  $\forall B \subset \mathbb{R}^n$

$$\chi_{\widehat{B^{\alpha_T}}}(z, s) \chi_{W_*(z, s)}(y, t) \leq \chi_{\widehat{B}}(y, t) \chi_{W(y, t)}(z, s),$$

and from  $W) + T_2)$ , we have:  $\forall B \subset \mathbb{R}^n$

$$\chi_{\widehat{B}}(y, t) \chi_{W(y, t)}(z, s) \leq \chi_{\widehat{B^{\alpha_0}}}(z, s) \chi_{W_{**}(z, s)}(y, t).$$

Then it follows from an integration in  $(y, t)$  that:  $\forall B \subset \mathbb{R}^n$

$$\chi_{\widehat{B^{\alpha_T}}}(z, s) \lesssim \iint_{\mathbb{R}_+^{n+1}} \chi_{\widehat{B}}(y, t) \frac{\chi_{W(y,t)}(z, s)}{t^{n+1}} dy dt \lesssim \chi_{\widehat{B^{\alpha_0}}}(z, s),$$

where in dividing  $s^{n+1}$ , we again use the similarity  $s \simeq t$  implicitly.

For  $0 < q < \infty$ , multiplying by  $|f(z, s)|^q$  the above inequalities then integrating in  $(z, s)$  and taking a supremum over  $B \ni x$ , we have from Fubini's theorem that

$$\mathcal{C}_q^{\alpha_T}(f)(x) \lesssim \mathcal{C}_q(\mathcal{W}_q(f))(x) \lesssim \mathcal{C}_q^{\alpha_0}(f)(x), \forall x \in \mathbb{R}^n.$$

Taking an  $L^\infty$  norm in the above functional relation and using Observation 2.1 lead us to the coincidence  $T_q^{\infty, q} = T_q^\infty$  in Category  $B$ ).

We can thus conclude the proof.  $\square$

**Remark 2.3.** If  $q = \infty$ , there holds for  $f$  continuous a similar functional relation

$$\mathcal{N}^{\alpha_0}(f)(x) \lesssim \mathcal{N}(\mathcal{W}_\infty(f))(x) \lesssim \mathcal{N}^{\alpha_C}(f)(x), \forall x \in \mathbb{R}^n.$$

Therefore, for the coincidence with the ‘‘classical’’ tent spaces in Category  $C$ ), we mean indeed  $T_\infty^{p, \infty} \cap C_{n.t.} = T_\infty^p \cap C_{n.t.}$ ,  $0 < p < \infty$ .

We end this section with another geometrical result, which will be needed in Section 6 for the proof of  $F_1$ ) in Theorem 3.2.

**Observation 2.4** (Change of Whitney parameters). Note that we have frozen two consistent parameters  $(\alpha_1, \alpha_2)$  in Definition 1.1. Instead of considering different apertures as in Observation 2.1, here we replace  $(\alpha_1, \alpha_2)$  by another pair of consistent Whitney parameters  $(\alpha'_1, \alpha'_2)$ , with a prescribed chain condition

$$0 < \alpha_1 < \alpha'_1 < 1/\alpha'_2 < 1/\alpha_2 < 1.$$

Following the way in Definition 1.1, we can also define a scale of tent spaces associated to  $(\alpha'_1, \alpha'_2)$ . Denoted by  ${}^{(\alpha'_1, \alpha'_2)}T_q^{p,r}$ , they should not be mistaken for the scale  ${}^\alpha T_q^p$  in Observation 2.1. We have the change of Whitney parameters equivalence

$$(2) \quad C(\alpha_1, \alpha'_1, \alpha_2, \alpha'_2) \|f\|_{T_q^{p,r}} \leq \|f\|_{{}^{(\alpha'_1, \alpha'_2)}T_q^{p,r}} \leq C'(\alpha_1, \alpha'_1, \alpha_2, \alpha'_2) \|f\|_{T_q^{p,r}},$$

where the constants  $C$  and  $C'$  also implicitly depend on  $n, p, q$  and  $r$ .

The former part of this equivalence can be inspected from the chain condition satisfied by  $(\alpha_1, \alpha_2)$  and  $(\alpha'_1, \alpha'_2)$ . We prove the right hand inequality as follows. For  $(y, t) \in \mathbb{R}_+^{n+1}$ , denote  $\widetilde{W}(y, t) = B(y, \gamma_1 t) \times (\gamma_2^{-1} t, \gamma_2 t)$ , with  $\gamma_1 \geq \alpha'_1/\alpha_1$  and  $\gamma_2 \geq \alpha'_2/\alpha_2$ . Then one can find an integer  $N = N(n, \alpha_1, \alpha_2, \alpha'_1, \alpha'_2)$  such that, for any  $(y, t) \in \mathbb{R}_+^{n+1}$ , there exist  $N$  points  $\mathcal{P}_N(y, t)$  in  $\widetilde{W}(y, t)$  with

$$\chi_{W'(y,t)}(z, s) \leq \sum_{(\bar{y}, \bar{t}) \in \mathcal{P}_N(y,t)} \chi_{W(\bar{y}, \bar{t})}(z, s),$$

where  $W'$  is the Whitney average associated to the Whitney parameters  $(\alpha'_1, \alpha'_2)$ . Now using (1) in Observation 2.1 and the geometries  $\{W, C_1, C_2, T_1, T_2\}$  in proving Theorem 2.2, there exists  $\alpha = \alpha(\alpha_1, \alpha_2, \alpha'_1, \alpha'_2)$  such that

$$\|f\|_{{}^{(\alpha'_1, \alpha'_2)}T_q^{p,r}} \lesssim \|f\|_{\alpha T_q^{p,r}} \lesssim \|f\|_{T_q^{p,r}}.$$

We leave open the sharp determination on the bounds  $C$  and  $C'$  in (2).



## 3. MULTIPLICATION AND FACTORIZATION

The main goal of this paper, is to obtain in the spirit of [9], the corresponding multiplication and factorization results for our new scale of tent spaces  $T_{q,\beta}^{p,r}$ . Some notations and definitions in function space theory are needed.

Denote by  $\Sigma$  the  $\sigma$ -finite measure space  $(\Omega, \mu)$ , and by  $L^0$  the collection of  $\mu$ -measurable complex-valued functions on  $\Omega$ . A *quasi-Banach function lattice*  $X$  on  $\Sigma$  is a non-empty subspace of  $L^0$ , which is equipped with a quasi-norm  $\|\cdot\|_X$  such that,  $(X, \|\cdot\|_X)$  is complete and  $X$  satisfies the *lattice property*:

$$\begin{aligned} \forall f \in X, \forall g \in L^0, \text{ with } |g| \leq |f| \quad \mu - \text{a.e.} \\ \implies g \in X, \text{ with } \|g\|_X \leq \|f\|_X. \end{aligned}$$

Clearly, for any  $f$  in a quasi-Banach function lattice  $X$ ,  $\|f\|_X = \| |f| \|_X$ .

**Definition 3.1.** Let  $\{X_i\}_{i=0}^n$  be a collection of quasi-Banach function lattices on  $\Sigma$ .

M) By the multiplication:  $X_0 \leftarrow X_1 \cdots X_n$ , we mean that for any  $f_i \in X_i$ ,  $1 \leq i \leq n$ , we have  $f_1 \cdots f_n \in X_0$  and

$$\|f_1 \cdots f_n\|_{X_0} \lesssim \|f_1\|_{X_1} \cdots \|f_n\|_{X_n},$$

where the implicit constant is independent of  $f_1, \dots, f_n$ .

F) By the (strong) factorization:  $X_0 \rightarrow X_1 \cdots X_n$ , we mean that for any  $f_0 \in X_0$ , there exist  $f_i \in X_i$ ,  $1 \leq i \leq n$ , such that  $|f_0| = |f_1| \cdots |f_n|$  and

$$\|f_1\|_{X_1} \cdots \|f_n\|_{X_n} \lesssim \|f_0\|_{X_0},$$

where the implicit constant does not depend on  $f_0, f_1, \dots, f_n$ .

When M) and F) are both satisfied, we write  $X_0 \leftrightarrow X_1 \cdots X_n$ .

In this paper, our central task is to prove

**Theorem 3.2.** *For any  $0 < p_0, q_0, r_0 \leq \infty$ , we have the following factorizations*

$$\begin{aligned} F_1) \quad T_{q_0}^{p_0, r_0} &\rightarrow T_{q_0}^{p_0, \infty} \cdot T_{\infty}^{\infty, r_0}, \\ F_2) \quad T_{q_0}^{p_0, r_0} &\rightarrow T_{\infty}^{p_0, \infty} \cdot T_{q_0}^{\infty, r_0}, \\ F_3) \quad T_{q_0}^{p_0, r_0} &\rightarrow T_{\infty}^{p_0, \infty} \cdot T_{q_0}^{\infty, \infty} \cdot T_{\infty}^{\infty, r_0}. \end{aligned}$$

The proof of this endpoint factorization theorem will be postponed to Section 6. Meanwhile, there holds an endpoint multiplication result.

**Lemma 3.3.** *For any  $0 < p_0, q_0, r_0 \leq \infty$ , we have the following multiplications*

$$\begin{aligned} M_1) \quad T_{q_0}^{p_0} &\leftarrow T_{\infty}^{p_0} \cdot T_{q_0}^{\infty}, \\ M_2) \quad T_{q_0}^{p_0, r_0} &\leftarrow T_{\infty}^{p_0, \infty} \cdot T_{q_0}^{\infty, \infty} \cdot T_{\infty}^{\infty, r_0}. \end{aligned}$$

*Proof of Lemma 3.3.* If  $\max(p_0, q_0) = \infty$ , there is nothing to prove for  $M_1$ ). If  $\max(p_0, q_0) < \infty$ , then the multiplication  $M_1$ ) is essentially in [9, Lemma 2.1]. The multiplication  $M_2$ ) is a consequence of Hölder's inequality and  $M_1$ ). In fact, we have

$$\begin{aligned} \|fgh\|_{T_{q_0}^{p_0, r_0}} &\leq \|\mathcal{W}_{\infty}(f)\mathcal{W}_{\infty}(g)\mathcal{W}_{r_0}(h)\|_{T_{q_0}^{p_0}} \\ &\lesssim \|\mathcal{W}_{\infty}(f)\|_{T_{\infty}^{p_0}} \|\mathcal{W}_{\infty}(g)\|_{T_{q_0}^{\infty}} \|\mathcal{W}_{r_0}(h)\|_{T_{\infty}^{\infty, r_0}} \\ &= \|f\|_{T_{\infty}^{p_0, \infty}} \|g\|_{T_{q_0}^{\infty, \infty}} \|h\|_{T_{\infty}^{\infty, r_0}}, \end{aligned}$$

where  $f, g$  and  $h$  are all measurable functions on  $\mathbb{R}_+^{n+1}$ .

Note that for  $M_1$ ), the starting point of [9, Lemma 2.1] is the following inequality for Carleson measures (see [29, p. 58–61] for example)

$$\iint_{\mathbb{R}_+^{n+1}} |f(y, t)|^p |d\mu|(y, t) \lesssim \|f\|_{T_\infty^p}^p \sup_{B \subset \mathbb{R}^n} \frac{|\mu|(\widehat{B})}{|B|},$$

which holds for any Borel measure  $d\mu$  on  $\mathbb{R}_+^{n+1}$  and any everywhere defined measurable  $f$  such that  $\mathcal{N}(f) \in L^p$ ,  $0 < p < \infty$ . Here we apply  $M_1$ ) to  $\mathcal{W}_\infty(f) \in T_\infty^{p_0}$ , since  $\mathcal{W}_\infty(f)$  is everywhere defined and measurable. This is also why we define the Category  $C$ ) tent spaces  $T_\infty^p$  without restricting them in the class  $C_{n.t.}(\mathbb{R}_+^{n+1}; \mathbb{C})$ .  $\square$

For  $0 < p_1, p_2 \leq \infty$ , define the *Hölderian triplet*  $(p_1, p_2, (p_1, p_2)_H)$  by the relation  $(p_1, p_2)_H^{-1} = p_1^{-1} + p_2^{-1}$ , where as usual, we will admit  $1/\infty = 0$ .

Combining  $F_3$ ) in Theorem 3.2 and  $M_2$ ) in Lemma 3.3, we can deduce an intermediate claim where the Hölderian triplets enter.

**Theorem 3.4.** *Suppose for  $i \in \{0, 1, 2\}$ ,  $T_{q_i, \beta_i}^{p_i, r_i}$  lies in the scale of weighted tent spaces with Whitney averages in Definition 1.1. Assume the Hölderian relation (H):*

$$p_0 = (p_1, p_2)_H, \quad q_0 = (q_1, q_2)_H, \quad r_0 = (r_1, r_2)_H \quad \text{and} \quad \beta_0 = \beta_1 + \beta_2.$$

*Then we have the multiplication and factorization*

$$T_{q_0, \beta_0}^{p_0, r_0} \leftrightarrow T_{q_1, \beta_1}^{p_1, r_1} \cdot T_{q_2, \beta_2}^{p_2, r_2}.$$

*Proof of Theorem 3.4.* By Remark 1.3 and Definition 3.1, it is enough to assume  $\beta_i = 0$ ,  $i \in \{0, 1, 2\}$ . Thus, we are only meant to show

$$T_{q_0}^{p_0, r_0} \leftrightarrow T_{q_1}^{p_1, r_1} \cdot T_{q_2}^{p_2, r_2}.$$

Call *extremal tent spaces* those  $T_q^{p, r}$  with at least two among  $p, q, r$  equal to  $\infty$ . Therefore,  $T_{q_0}^{p_0, r_0} \leftrightarrow T_{q_1}^{p_1, r_1} \cdot T_{q_2}^{p_2, r_2}$  holds trivially if  $T_{q_0}^{p_0, r_0}$  is an extremal tent space. Indeed, multiplication is just a consequence of Hölder's inequality, and factorization follows from the trick of taking powers:  $|f| = |f|^{1-\theta} |f|^\theta$ , with  $0 \leq \theta \leq 1$ .

Now the general factorization can be proved as follows. With the Hölderian relation (H) in mind, factorizing  $T_{q_0}^{p_0, r_0}$  through  $F_3$ ) in Theorem 3.2 into extremal tent spaces, using the known factorization for extremal tent spaces, and multiplying through  $M_2$ ) in Lemma 3.3, we then have

$$\begin{aligned} T_{q_0}^{p_0, r_0} &\rightarrow T_\infty^{p_0, \infty} \cdot T_{q_0}^{\infty, \infty} \cdot T_\infty^{\infty, r_0} \\ &\rightarrow T_\infty^{p_1, \infty} \cdot T_\infty^{p_2, \infty} \cdot T_{q_1}^{\infty, \infty} \cdot T_{q_2}^{\infty, \infty} \cdot T_\infty^{\infty, r_1} \cdot T_\infty^{\infty, r_2} \rightarrow T_{q_1}^{p_1, r_1} \cdot T_{q_2}^{p_2, r_2}. \end{aligned}$$

Finally, the general multiplication can be proved as follows. With the Hölderian relation (H) in mind, factorizing  $T_{q_i}^{p_i, r_i}$  ( $i = 1, 2$ ) through  $F_3$ ) in Theorem 3.2 into extremal tent spaces, using the known multiplication for extremal tent spaces, and multiplying through  $M_2$ ) in Lemma 3.3, we then have

$$\begin{aligned} T_{q_1}^{p_1, r_1} \cdot T_{q_2}^{p_2, r_2} &\rightarrow T_\infty^{p_1, \infty} \cdot T_{q_1}^{\infty, \infty} \cdot T_\infty^{\infty, r_1} \cdot T_\infty^{p_2, \infty} \cdot T_{q_2}^{\infty, \infty} \cdot T_\infty^{\infty, r_2} \\ &\rightarrow T_\infty^{p_0, \infty} \cdot T_{q_0}^{\infty, \infty} \cdot T_\infty^{\infty, r_0} \rightarrow T_{q_0}^{p_0, r_0}. \end{aligned}$$

The quasi-norm inequalities in each proof can be obtained by inspection.  $\square$

## 4. QUASI-BANACH COMPLEX INTERPOLATION

We begin with a second look at the symbol “ $\leftrightarrow$ ” for multiplication and factorization, which we formulated in last section in Definition 3.1.

**Definition 4.1.** Given two quasi-Banach function lattices  $X_1$  and  $X_2$ , we define their *Calderón’s product*  $X_1 \bullet X_2$  as the class of  $u \in L^0$  for which

$$\|u\|_{X_1 \bullet X_2} := \inf\{\|v\|_{X_1}\|w\|_{X_2} \mid |u| = |v||w|, v \in X_1, w \in X_2\} < \infty.$$

Clearly, the usual product  $X_1 \cdot X_2 = \{vw \mid v \in X_1, w \in X_2\}$  is contained in the Calderón’s product  $X_1 \bullet X_2$ . In other words,  $X_1 \bullet X_2$  is the completion of  $X_1 \cdot X_2$  under the quasi-norm  $\|\cdot\|_{X_1 \bullet X_2}$ . Moreover,  $X_0 \leftrightarrow X_1 \cdot X_2$  amounts to say  $X_0 = X_1 \bullet X_2$ , where we interpret the equality by the equivalence of quasi-norms.

This new product  $X_1 \bullet X_2$ , was first used by Calderón in [8] as an intermediate space for the complex interpolation of a couple of Banach function lattices  $(X_1, X_2)$ . For the underlying measure space  $\Sigma = (\Omega, \mu)$ , assume that  $\Omega$  is a complete separable metric space, and  $\mu$  is a  $\sigma$ -finite Borel measure on  $\Omega$ . In a (most) natural extension of Calderón’s interpolation method to the quasi-Banach setting, Kalton and Mitrea establish in [22, Section 3] (see also [20]) that, for a couple of analytically convex separable quasi-Banach function lattices  $(X_1, X_2)$  on  $\Sigma$ , there holds the generalized *Calderón’s product formula* (see [22, Theorem 3.4]) that

$$(X_1, X_2)_\theta = [X_1]^{1-\theta} \bullet [X_2]^\theta, \quad 0 < \theta < 1.$$

Here,  $X$  analytically convex (A-convex for short) means that, for any analytic<sup>3</sup> function  $\Phi : S = \{z \in \mathbb{C} \mid \operatorname{Re} z \in (0, 1)\} \rightarrow X$ , which is also continuous to the closed strip  $\bar{S} = S \cup \partial S$ , we have the *maximum modulus principle*

$$\max_{z \in S} \|\Phi(z)\|_X \lesssim \max_{z \in \partial S} \|\Phi(z)\|_X.$$

Under this A-convexity requirement,  $X_1 + X_2$  is also A-convex, and then Calderón’s method adapts to the quasi-Banach case. In the same spirit, this analytical approach to the interpolation of quasi-Banach function lattices was also considered in [7], where the ambient A-convex space is not necessarily the usual  $X_1 + X_2$ .

It was obtained in [19] that  $X$  analytically convex is equivalent to  $X$   $r$ -convex for some  $r > 0$ . Here,  $X$  (*lattice*)  $r$ -convex means that, for any  $n \in \mathbb{N}_+$  and any  $f_i \in X$ ,  $i = 1, \dots, n$ , we have the inequality

$$\left\| \left( \sum_{i=1}^n |f_i|^r \right)^{1/r} \right\|_X \leq \left( \sum_{i=1}^n \|f_i\|_X^r \right)^{1/r}.$$

This convexification/normalization process is trivial for Banach function lattice  $X$ , as we can always take  $r = 1$  in the above inequality. Thus for our purpose here, we can change A-convex to  $r$ -convex.

Now we turn to the separability issue. Recall that a Banach function lattice  $X$  is said to satisfy the *Fatou property* [26, Remark 2 on p. 30], or *maximality in  $L^0$* , if

$$\begin{aligned} \forall 0 \leq f_n \in X \text{ and } \sup_{n \in \mathbb{N}_+} \|f_n\|_X < \infty, \text{ with } f_n \uparrow f \in L^0 \text{ } \mu - \text{a.e.} \\ \implies f \in X \text{ and } \|f\|_X = \lim_{n \rightarrow \infty} \|f_n\|_X. \end{aligned}$$

<sup>3</sup>See [22, p. 3911] for the precise definitions of analyticity and A-convexity.

It was observed<sup>4</sup> in [20] that, if both  $X_1$  and  $X_2$  satisfy the Fatou property, we only need to assume for the interpolation that either  $X_1$  or  $X_2$  is separable.

For further information on the applicability of Calderón's product formula, see [22, Section 3] and [21, Section 7] directly. *Therefore, for two quasi-Banach function lattices  $X_1$  and  $X_2$ , if  $X_i$  ( $i = 1, 2$ ) is  $r_i$ -convex and has the Fatou property, and if either  $X_1$  or  $X_2$  is separable, then we have the desired interpolation realization:*

$$(X_1, X_2)_\theta = [X_1]^{1-\theta} \bullet [X_2]^\theta, \quad 0 < \theta < 1.$$

Let us apply these to tent spaces.

**Lemma 4.2.** *All the tent spaces  $T_{q,\beta}^{p,r}$  have the Fatou property.*

*Proof.* This is an easy consequence of the monotone convergence theorem and simple measure theoretic arguments.  $\square$

For  $0 < p_1, p_2 \leq \infty$  and  $\theta \in (0, 1)$ , define the  $\theta$ -Hölderian triplet  $(p_1, p_2, (p_1, p_2)_\theta)$  by the relation  $(p_1, p_2)_\theta^{-1} = (1 - \theta)/p_1 + \theta/p_2$ , where we again admit  $1/\infty = 0$ .

**Theorem 4.3.** *Let  $0 < \theta < 1$ . Suppose for  $i \in \{0, 1, 2\}$ ,  $T_{q_i, \beta_i}^{p_i, r_i}$  lies in the scale of weighted tent spaces with Whitney averages in Definition 1.1. Assume the condition  $\min(\max(p_1, q_1, r_1), \max(p_2, q_2, r_2)) < \infty$  and the  $\theta$ -Hölderian relation  $(H)_\theta$ :*

$$p_0 = (p_1, p_2)_\theta, \quad q_0 = (q_1, q_2)_\theta, \quad r_0 = (r_1, r_2)_\theta \quad \text{and} \quad \beta_0 = (1 - \theta)\beta_1 + \theta\beta_2.$$

*Then under the Kalton-Mitrea complex interpolation method, we have*

$$(T_{q_1, \beta_1}^{p_1, r_1}, T_{q_2, \beta_2}^{p_2, r_2})_\theta = T_{q_0, \beta_0}^{p_0, r_0}.$$

*Proof.* With  $(H)_\theta$  and Theorem 3.4, we have

$$T_{q_0, \beta_0}^{p_0, r_0} \leftrightarrow T_{q_1/(1-\theta), \beta_1(1-\theta)}^{p_1/(1-\theta), r_1/(1-\theta)} \cdot T_{q_2/\theta, \beta_2\theta}^{p_2/\theta, r_2/\theta},$$

which is equivalent to say

$$T_{q_0, \beta_0}^{p_0, r_0} = T_{q_1/(1-\theta), \beta_1(1-\theta)}^{p_1/(1-\theta), r_1/(1-\theta)} \bullet T_{q_2/\theta, \beta_2\theta}^{p_2/\theta, r_2/\theta}.$$

Under the (sufficient) condition  $\min(\max(p_1, q_1, r_1), \max(p_2, q_2, r_2)) < \infty$ , at least one quasi-Banach function lattice in the interpolation couple  $(T_{q_1, \beta_1}^{p_1, r_1}, T_{q_2, \beta_2}^{p_2, r_2})$  is separable. And it follows from Minkowski's inequality that, for  $i = 1, 2$ , the quasi-Banach function lattice  $T_{q_i, \beta_i}^{p_i, r_i}$  is  $\min(\tau_i, 1)$ -convex, where  $\tau_i = \min(p_i, q_i, r_i)$ . In fact, it suffices to apply

$$\|f\|_{T_{q_i, \beta_i}^{p_i, r_i}}^{\tau_i} = \| |f|^{\tau_i} \|_{T_{q_i/\tau_i, \beta_i\tau_i}^{p_i/\tau_i, r_i/\tau_i}}, \quad i = 1, 2,$$

to the criterion of  $r$ -convexity, and notice that  $T_{q_i/\tau_i, \beta_i\tau_i}^{p_i/\tau_i, r_i/\tau_i}$  ( $i = 1, 2$ ) are Banach function lattices. Using the generalized Calderón's product formula, we have

$$\begin{aligned} (T_{q_1, \beta_1}^{p_1, r_1}, T_{q_2, \beta_2}^{p_2, r_2})_\theta &= [T_{q_1, \beta_1}^{p_1, r_1}]^{1-\theta} \bullet [T_{q_2, \beta_2}^{p_2, r_2}]^\theta \\ &= T_{q_1/(1-\theta), \beta_1(1-\theta)}^{p_1/(1-\theta), r_1/(1-\theta)} \bullet T_{q_2/\theta, \beta_2\theta}^{p_2/\theta, r_2/\theta} = T_{q_0, \beta_0}^{p_0, r_0}. \end{aligned}$$

This proves the wanted complex interpolation formula.  $\square$

<sup>4</sup>In this regard, see also the second remark following Theorem 7.9 of [21], where  $X_1$  and  $X_2$  are assumed to be sequence spaces. In fact, only the Fatou property is needed in the arguments there.

The above interpolation result is new since we considered the Whitney averaged scale and brought in the extreme tent space  $T_{\infty}^{\infty, \infty}$ . For the tent spaces without Whitney averages and with  $\beta = 0$ , the quasi-Banach complex interpolation

$$(T_{q_1}^{p_1}, T_{q_2}^{p_2})_{\theta} = T_{q_0}^{p_0}, \quad 0 < \theta < 1,$$

where  $1/p_0 = (1 - \theta)/p_1 + \theta/p_2$  and  $1/q_0 = (1 - \theta)/q_1 + \theta/q_2$ ,  $0 < p_1, p_2, q_1, q_2 < \infty$ , was considered in [6] by another analytical method. For earlier results on the Banach complex interpolation, see the references in [6]. Using the Kalton-Mitrea complex interpolation method, [9] recovers the result in [6] and obtains additionally

$$(T_q^{\infty}, \tilde{T}_{\infty}^p)_{\theta} = T_{q/(1-\theta)}^{p/\theta}, \quad 0 < \theta < 1,$$

where  $0 < p, q < \infty$ . Note that  $\tilde{T}_{\infty}^p = T_{\infty}^p \cap C_{\text{n.t.}}$  is the classical tent space, which is equivalent to the closure in  $T_{\infty}^p$  of continuous functions with compact support in  $\overline{\mathbb{R}_+^{n+1}}$  (see for example [29, p. 77]), is separable. For the weighted analogue of [9], see for instance [15], where  $\beta$  can also be any real number.

Here, we have under Theorem 4.3 and the coincidence result in Theorem 2.2 that, for the non-extremal case  $0 < p_1, p_2, q_1, q_2 < \infty$ , we have

$$(T_{q_1}^{p_1}, T_{q_2}^{p_2})_{\theta} = (T_{q_1}^{p_1, q_1}, T_{q_2}^{p_2, q_2})_{\theta} = T_{q_0}^{p_0, q_0} = T_{q_0}^{p_0}, \quad 0 < \theta < 1,$$

when  $1/p_0 = (1 - \theta)/p_1 + \theta/p_2$  and  $1/q_0 = (1 - \theta)/q_1 + \theta/q_2$ . This recovers [6, Theorem 3]. Note that the condition  $\min(\max(p_1, q_1, r_1), \max(p_2, q_2, r_2)) < \infty$  is sufficient for most of our applications to operator theory on tent spaces since we usually set one space in the interpolation pair to be  $T_2^2 = T_2^{2,2}$ .

**Remark 4.4.** For the extremal case  $\min(p_1, p_2) = \infty$ , one can show that  $T_{q_1}^{\infty}$  and  $T_{q_2}^{\infty}$  are two non-separable spaces. In this situation there exist some results in a different context. For  $\alpha \in [0, 1]$  and the space of Carleson measures of order  $\alpha$

$$V^{\alpha} := \left\{ d\mu \left| \sup_{B \subset \mathbb{R}^n} \frac{|\mu|(\widehat{B})}{|B|^{\alpha}} < \infty \right. \right\},$$

the complex interpolation  $(V^0, V^1)_{\alpha}$  was identified in [3, Theorem 3-(ii)] to a space which is strictly smaller than  $V^{\alpha}$ . In this respect, see also [1, 2] for relevant results.

## 5. MULTIPLIERS AND STANDARD DUALITY

Now we turn to the *multiplier* issue, which from the *multiplication* point of view, is more straightforward than the quasi-Banach complex interpolation.

Similarly to the last section, we restrict ourselves to the setting of (Banach) function lattices, and the underlying measure space  $\Sigma = (\Omega, \mu)$  is assumed to be complete and  $\sigma$ -finite. Here, “complete” is with respect to the measure, meaning that

$$\forall E \subset \Omega, \mu(E) = 0 \implies \forall E' \subset E, \mu(E') = 0.$$

Recall that  $L^0$  is the collection of all complex-valued  $\mu$ -measurable functions on  $\Omega$ .

**Definition 5.1.** Given two Banach function lattices  $X_0$  and  $X_1$ , we say that  $w \in L^0$  is a *multiplier* from  $X_1$  to  $X_0$ , if the associated multiplication mapping

$$M_w : X_1 \rightarrow X_0, v \mapsto vw$$

satisfies

$$\|M_w\|_{X_1 \rightarrow X_0} := \sup_{v \neq 0} \frac{\|vw\|_{X_0}}{\|v\|_{X_1}} < \infty.$$

Denote all the multipliers from  $X_1$  to  $X_0$  by  $M(X_1, X_0)$ , equipped with

$$\|w\|_{M(X_1, X_0)} = \|M_w\|_{X_1 \rightarrow X_0}.$$

Before proceeding to our main results in this section, we review a cancellation result concerning Calderón's product. It was obtained in [28, Theorem 2.5 and Corollary 2.6] that for three Banach function lattices  $\{E, F, G\}$  on  $\Sigma$ , all with the Fatou property, we have the following *cancellation formula*

$$E \bullet F = E \bullet G \implies F = G.$$

There also holds (see [28, Theorem 2.8]) that

$$F = M(E, E \bullet F),$$

if both  $E$  and  $F$  have the Fatou property. In particular situations, the above multiplier representation can also be found in [11, Theorem 3.5], which served to prove the uniqueness theorem of Calderón-Lozanovskii's interpolation method. We mention that in the literature, the construction of Calderón for intermediate spaces was further investigated by Lozanovskii in a series of papers ([24], [25]).

Let us apply these to our tent spaces.

**Theorem 5.2.** *With the same assumptions as in Theorem 3.4 and  $1 \leq p_i, q_i, r_i \leq \infty$  for  $i \in \{0, 1, 2\}$ , we have the multiplier identification*

$$T_{q_2, \beta_2}^{p_2, r_2} = M(T_{q_1, \beta_1}^{p_1, r_1}, T_{q_0, \beta_0}^{p_0, r_0}).$$

*Proof.* For  $i \in \{0, 1, 2\}$ ,  $1 \leq p_i, q_i, r_i \leq \infty$  implies  $\tau_i = \min(p_i, q_i, r_i) \geq 1$ , thus  $T_{q_i, \beta_i}^{p_i, r_i}$  is a Banach function lattice. Using the multiplier representation cited above, with the Fatou property guaranteed by Lemma 4.2, we have

$$T_{q_2, \beta_2}^{p_2, r_2} = M(T_{q_1, \beta_1}^{p_1, r_1}, T_{q_1, \beta_1}^{p_1, r_1} \bullet T_{q_2, \beta_2}^{p_2, r_2}) = M(T_{q_1, \beta_1}^{p_1, r_1}, T_{q_0, \beta_0}^{p_0, r_0}),$$

where the last equality is from Theorem 3.4:  $T_{q_0, \beta_0}^{p_0, r_0} = T_{q_1, \beta_1}^{p_1, r_1} \bullet T_{q_2, \beta_2}^{p_2, r_2}$ .  $\square$

Finally, we look at the duality theory. Given  $\beta_0 \in \mathbb{R}$ , we will consider the following  $\beta_0$ -weighted pairing

$$(f, h)_{\beta_0} := \iint_{\mathbb{R}_+^{n+1}} f(y, t)h(y, t)t^{-\beta_0-1}dydt.$$

Let  $p', q'$  and  $r'$  be the dual indice of  $1 \leq p, q, r \leq \infty$ .

**Definition 5.3.** The  $\beta_0$ -weighted *Köthe dual* of the Banach  $T_{q, \beta}^{p, r}$  is defined as

$$(T_{q, \beta}^{p, r})_{\beta_0}^* := M(T_{q, \beta}^{p, r}, L^1(\mathbb{R}_+^{n+1}, t^{-\beta_0-1}dydt)) = M(T_{q, \beta}^{p, r}, T_{1, \beta_0}^{1, 1}).$$

Here, unlike the continuous functional dual  $(\cdot)'$ , "Köthe" means the dual within the class of Banach function lattices. For a general account on this aspect, see [26]. By the *standard duality*, we mean the (Köthe) dual of the Banach  $T_{q, \beta}^{p, r}$  when  $1 \leq p < \infty$ ,  $\beta \in \mathbb{R}$  and particularly  $1 \leq \min(q, r) \leq \max(q, r) < \infty$ .

**Theorem 5.4.** *Under the pairing  $(\cdot, \cdot)_{\beta_0}$ , we have the following standard duality*

$$T_{q', \beta_0 - \beta}^{p', r'} = (T_{q, \beta}^{p, r})', \quad 1 \leq p, q, r < \infty, \beta \in \mathbb{R}.$$

*Proof.* By Theorem 5.2 and the definition of  $(\cdot)_{\beta_0}^*$ , we have

$$T_{q', \beta_0 - \beta}^{p', r'} = M(T_{q, \beta}^{p, r}, T_{1, \beta_0}^{1, 1}) = (T_{q, \beta}^{p, r})_{\beta_0}^* \subset (T_{q, \beta}^{p, r})',$$

where the last inclusion follows from the straightforward identification of multipliers to continuous linear functionals, through the pairing  $(\cdot, \cdot)_{\beta_0}$ .

For the converse, suppose that we are given a continuous linear functional  $l$  on  $T_{q, \beta}^{p, r}$ . Then whenever  $K$  is a compact set in  $\mathbb{R}_+^{n+1}$ , and whenever  $f$  is supported in  $K$ , with  $f \in L^r(K)$ , then  $\mathcal{W}_r(f) \in T_{q, \beta}^p$  with

$$\|f\|_{T_{q, \beta}^{p, r}} = \|\mathcal{W}_r(f)\|_{T_{q, \beta}^p} \leq C_K \|f\|_{L^r}.$$

Here,  $C_K$  is a constant which depends on the compact set  $K$ , and also implicitly on the indices  $p, q, r$  and  $\beta$ . Thus  $l$  induces a continuous linear functional on  $L^r(K)$  and is representable by  $h^K \in L^{r'}(K)$ , as  $1 \leq r < \infty$ . Taking an increasing family of such  $K$  which exhausts  $\mathbb{R}_+^{n+1}$ , gives us an  $h \in L_{\text{loc}}^{r'}$  such that

$$l(f) = (f, h)_{\beta_0} = \iint_{\mathbb{R}_+^{n+1}} f(y, t) h(y, t) t^{-\beta_0 - 1} dy dt,$$

whenever  $f \in L^r$  and has compact support. By density arguments, this representation of  $l$  by  $h$  extends to all  $f \in T_{q, \beta}^{p, r}$ , as we further have  $1 \leq p, q < \infty$ . By the representation through  $(\cdot, h)_{\beta_0}$ , we have  $\|l\| = \|M_h\|_{T_{q, \beta}^{p, r} \rightarrow T_{1, \beta_0}^{1, 1}}$ , which means

$$(T_{q, \beta}^{p, r})' \subset M(T_{q, \beta}^{p, r}, T_{1, \beta_0}^{1, 1}) = (T_{q, \beta}^{p, r})_{\beta_0}^* = T_{q', \beta_0 - \beta}^{p', r'}.$$

This then proves the desired standard duality.  $\square$

To end this section, we deduce as corollaries some corresponding known results on multiplication, factorization and duality, mainly obtained in the articles [10, Coifman-Meyer-Stein], [9, Cohn-Verbitsky] and [17, Hytönen-Rosén].

Relation with Coifman-Meyer-Stein. For the standard duality, it was shown in [10, Theorem 1-(b) and Theorem 2] that

$$T_2^{p'} = (T_2^p)_0^* = (T_2^p)', \quad 1 \leq p < \infty,$$

which upon using Theorem 2.2 on the coincidence for  $r = q = 2$ , then corresponds to our Theorem 5.4 in the particular case

$$T_{2,0}^{p', 2} = (T_{2,0}^{p, 2})_0^* = (T_{2,0}^{p, 2})', \quad 1 \leq p < \infty.$$

By the *Carleson duality*, we mean the continuous functional dual of  $T_{q, \beta}^{p, r}$  for  $1 \leq p < \infty, \beta \in \mathbb{R}$  and particularly  $1 \leq \min(q, r) \leq \max(q, r) = \infty$ . Let  $\widehat{B} := \widehat{\widehat{B}}$  be the closed tent on base  $B$ , and denote the Carleson measures on  $\overline{\mathbb{R}_+^{n+1}}$  by

$$\overline{\mathcal{C}} := \left\{ d\mu \left| \sup_{B \subset \mathbb{R}^n} \frac{|\mu|(\widehat{B})}{|B|} < \infty \right. \right\}.$$

Let  $\mathcal{N} = T_\infty^1 \cap C_{n.t.}$ . The classical Carleson duality ([10, Proposition 1]) states that

$$\overline{\mathcal{C}} = (\mathcal{N})'.$$

Obviously, our Theorem 5.4 on standard duality can *not* cover the Carleson duality. Nevertheless, we shall mention in Remark 6.2 a consequence of our method of proof toward factorization of bounded Borel measures on  $\overline{\mathbb{R}_+^{n+1}}$  by Carleson measures.

Relation with Hytönen-Rosén. To relate their notations,  $N_{p,q}$  and  $C_{p',q'}$  in [17] for Banach cases are just the scales  $T_{\infty,0}^{p,q}$  and  $T_{1,-1}^{p',q'}$  here, and their duality claim is

$$N_{p,q} = (C_{p',q'})', \quad 1 < p < \infty, \quad 1 < q \leq \infty.$$

This *Carleson (pre-)duality*, stated in [17, Theorem 3.2], then corresponds to our Theorem 5.4 in the particular case

$$T_{\infty,0}^{p,r} = (T_{1,-1}^{p',r'})^*_{-1} = (T_{1,-1}^{p',r'})', \quad 1 < p < \infty, \quad 1 < r \leq \infty.$$

At the multiplication side, Theorem 3.1 of [17] states

$$T_{r,-1/r}^{r,r} \leftarrow T_{\infty,0}^{p,q} \cdot T_{r,-1/r}^{\tilde{p},\tilde{q}}, \quad 1 \leq r < \infty, \quad r \leq p < \infty, \quad r \leq q \leq \infty,$$

with  $r = (p, \tilde{p})_H = (q, \tilde{q})_H$ . Again, this is a particular case of our Theorem 3.4.

Relation with Cohn-Verbitsky. Under the coincidence theorem and Remark 6.3, part  $F_2$ ) in Theorem 3.2 for  $r_0 = q_0$  corresponds to Cohn-Verbitsky

$$T_{q_0}^{p_0} = T_{q_0}^{p_0,q_0} \rightarrow T_{\infty}^{p_0,\infty} \cdot T_{q_0}^{\infty,q_0} = T_{\infty}^{p_0} \cdot T_{q_0}^{\infty}.$$

Meanwhile, with the help of  $F_1$ ) to produce Whitney multipliers, our result  $F_3$ ) is a further (polarized) factorization of the tent space  $T_{q_0}^{p_0,r_0}$ . Of course, we also bring in the endpoint spaces  $T_{\infty}^{\infty}$  and  $T_{\infty}^{\infty,r_0}$ , which makes the statement broader. Moreover, we continue with a multiplier discussion basing on the factorization result, which is seemingly new even in the situation of classical tent spaces.

We also remark that the multiplication side of Theorem 3.4 covers Lemma 5.5 in [5] and Lemma 2.4.3 in [27]. To relate the notations again, the two tent spaces  $\mathcal{X}$  and  $\mathcal{E}$  in [5], originally introduced by Kenig-Pipher in [23] and by Dahlberg in [12] respectively, correspond to  $T_{\infty,0}^{2,2}$  and  $T_{2,0}^{\infty,\infty}$  here. Our full scale  $T_{q,\beta}^{p,r}$ , mainly interested by  $\mathcal{X}^p := T_{\infty,0}^{p,2}$  and  $\mathcal{Y}_{\pm}^p := T_{2,-1\pm 1}^{p,2}$  for  $p$  in some interval containing 2, will be used as *natural function spaces* in part of a continuation work of [5], where more backgrounds on boundary value problems of elliptic PDEs can be referred.

## 6. PROOF OF THEOREM 3.2 ON FACTORIZATION

To prove  $F_3$ ) it suffices to show  $F_1$ ) and  $F_2$ ) respectively. Indeed, factorizing  $T_{q_0}^{p_0,r_0}$  through  $F_1$ ) first, then using  $F_2$ ) yields  $F_3$ ) immediately. Thus to prove Theorem 3.2, we show  $F_1$ ) and  $F_2$ ) in order.

*Proof of  $F_1$ ).* Let  $W^*(y, t)$  and  $\mathcal{W}_r^*(\cdot)(y, t)$  be the Whitney box and the  $L^r$ -Whitney average associated to the point  $(y, t) \in \mathbb{R}_+^{n+1}$ , and to the Whitney parameters

$$\alpha_1^* = \alpha_1(1 + \alpha_2^{1/2})^{-1} \quad \text{and} \quad \alpha_2^* = \alpha_2^{1/2},$$

where  $(\alpha_1, \alpha_2)$  is the pair of consistent Whitney parameters we used in Definition 1.1. Similarly, let  $W^{**}$  and  $\mathcal{W}_r^{**}(\cdot)$  be the Whitney objects associated to

$$\alpha_1^{**} = \alpha_1 \left[ 2(1 + \alpha_2^{1/2})\alpha_2^{1/4} \right]^{-1} \quad \text{and} \quad \alpha_2^{**} = \alpha_2^{1/4}.$$

Note that the two resulted pairs of Whitney parameters are also consistent, with

$$0 < \alpha_1^{**} < \alpha_1^* < \alpha_1 < \alpha_2^{-1} < (\alpha_2^*)^{-1} < (\alpha_2^{**})^{-1} < 1.$$



Moreover, for any  $(y, t) \in \mathbb{R}_+^{n+1}$ , we have the geometrical relations

$$(3) \quad \bigcup_{(z,s) \in W^*(y,t)} W^*(z, s) \subset W(y, t)$$

and

$$(4) \quad \bigcap_{(z,s) \in W^{**}(y,t)} W^*(z, s) \supset W^{**}(y, t).$$

The verification on  $\alpha_2^*$  and  $\alpha_2^{**}$  is straightforward. For the first inclusion, given any  $(z, s) \in W^*(y, t)$  and any  $(z_0, s_0) \in W^*(z, s)$ , we have

$$|z_0 - y| \leq |z_0 - z| + |z - y| < \alpha_1^* s + \alpha_1^* t < \alpha_1^* (\alpha_2^* + 1)t = \alpha_1 t,$$

which implies  $(z_0, s_0) \in W(y, t)$ . For the second inclusion, given any  $(z_0, s_0) \in W^{**}(y, t)$  and any  $(z, s) \in W^*(z, s)$ , we have

$$|z_0 - z| \leq |z_0 - y| + |y - z| < 2\alpha_1^{**} t < 2\alpha_1^{**} \alpha_2^{**} s = \alpha_1^* s,$$

which implies  $(z_0, s_0) \in W^*(z, s)$ . This proves the two relations (3) and (4).

Now for any  $u \in T_{q_0}^{p_0, r_0}$ , we construct  $v = \mathcal{W}_{r_0}^*(u)$ . Then we have from (3) that

$$\sup_{(z,s) \in W^*(y,t)} \mathcal{W}_{r_0}^*(u)(z, s) \lesssim \mathcal{W}_{r_0}(u)(y, t)$$

is valid for any  $(y, t) \in \mathbb{R}_+^{n+1}$ , thus we know

$$\mathcal{W}_\infty^*(v) \lesssim \mathcal{W}_{r_0}(u) \quad \text{and} \quad \|\mathcal{W}_\infty^*(v)\|_{T_{q_0}^{p_0}} \lesssim \|u\|_{T_{q_0}^{p_0, r_0}}.$$

For  $w = u/\mathcal{W}_{r_0}^*(u)$ , we then have from (4) that

$$\inf_{(z,s) \in W^{**}(y,t)} \mathcal{W}_{r_0}^*(u)(z, s) \gtrsim \mathcal{W}_{r_0}^{**}(u)(y, t)$$

is valid for any  $(y, t) \in \mathbb{R}_+^{n+1}$ , thus we know

$$\mathcal{W}_{r_0}^{**}(w) \lesssim 1 \quad \text{and} \quad \|\mathcal{W}_{r_0}^{**}(w)\|_{T_\infty} \lesssim 1.$$

Using the change of Whitney parameters equivalence in Observation 2.4,  $u = vw$  is then the desired factorization for  $T_{q_0}^{p_0, r_0} \rightarrow T_{q_0}^{p_0, \infty} \cdot T_\infty^{\infty, r_0}$ ,  $0 < p_0, q_0, r_0 \leq \infty$ .  $\square$

*Proof of  $F_2$* ). Observe that we can suppose  $0 < \max(p_0, q_0) < \infty$ . In fact, nothing has to be done if  $p_0 = \infty$ , and the case  $q_0 = \infty$  is already included in  $F_1$ ).

We base our arguments on the constructive proof in [9]. From the consistency of Whitney parameters, we have  $0 < \alpha_1 < \alpha_2^{-1} < 1$ . Then the following relations

$$(5) \quad \bigcap_{(z,s) \in W(y,t)} B(z, s) \supset B(y, (\alpha_2^{-1} - \alpha_1)t)$$

and

$$(6) \quad \bigcup_{(z,s) \in W(y,t)} B(z, s) \subset B(y, (\alpha_2 + \alpha_1)t)$$

hold for any  $(y, t) \in \mathbb{R}_+^{n+1}$ . In fact, for the verification of the first inclusion, given any  $x \in B(y, (\alpha_2^{-1} - \alpha_1)t)$  and any  $(z, s) \in W(y, t)$ , we compute as follow

$$|x - z| \leq |x - y| + |y - z| < (\alpha_2^{-1} - \alpha_1)t + \alpha_1 t < s,$$

which implies  $x \in B(z, s)$ . Similarly, to verify the second inclusion, given any  $(z, s) \in W(y, t)$  and any  $x \in B(z, s)$ , we compute as follow

$$|x - y| \leq |x - z| + |z - y| < s + \alpha_1 t < (\alpha_2 + \alpha_1)t,$$

which implies  $x \in B(y, (\alpha_2 + \alpha_1)t)$ . This proves the two relations (5) and (6).

As  $0 < \max(p_0, q_0) < \infty$ , the tent space  $T_{q_0}^{p_0, r_0}$  lies in Category A) and can be determined by the conical functional  $\mathcal{A}_{q_0}$ . Therefore,  $\tilde{u} = \mathcal{A}_{q_0}(\mathcal{W}_{r_0}(u)) \in L^{p_0}(\mathbb{R}^n)$ . Denote by  $P_0[h](y, t)$  the average of  $h$  on  $B(y, t) \subset \mathbb{R}^n$ , and construct  $v = P_0[\tilde{u}^{\tilde{p}}]^{1/\tilde{p}}$  for some  $\tilde{p} < p_0$ . Let  $\alpha^* = \alpha_2 + \alpha_1 > 1$ , then by (6), for any  $(y, t) \in \mathbb{R}_+^{n+1}$

$$\sup_{(z, s) \in W(y, t)} v(z, s) \lesssim v(y, \alpha^* t) =: v^*(y, t).$$

Thus we have  $\mathcal{W}_\infty(v)(y, t) \lesssim v^*(y, t)$ , and there holds

$$\mathcal{N}(\mathcal{W}_\infty(v))(x) \lesssim \mathcal{N}(v^*)(x) \leq \mathcal{M}(\tilde{u}^{\tilde{p}})^{1/\tilde{p}}(x), \forall x \in \mathbb{R}^n,$$

where  $\mathcal{N}$  is the non-tangential maximal functional,  $\mathcal{M}$  is the Hardy-Littlewood maximal operator and the last estimate follows from the fact

$$\bigcap_{(y, t) \in \Gamma(x)} B(y, \alpha^* t) \ni x, \forall x \in \mathbb{R}^n.$$

As  $p_0/\tilde{p} > 1$ , then by maximal theorem, we have

$$\|v\|_{T_{q_0}^{p_0, \infty}} \lesssim \|\mathcal{M}(\tilde{u}^{\tilde{p}})^{1/\tilde{p}}\|_{L^{p_0}} \lesssim \|\tilde{u}\|_{L^{p_0}} = \|u\|_{T_{q_0}^{p_0, r_0}}.$$

Now we turn to  $w = u/v$ . Let  $\alpha_* = \alpha_2^{-1} - \alpha_1 \in (0, 1)$ , then by (5)

$$\inf_{(z, s) \in W(y, t)} v(z, s) \gtrsim v(y, \alpha_* t)$$

is valid for any  $(y, t) \in \mathbb{R}_+^{n+1}$ . By Hölder's inequality, there holds

$$(7) \quad \|h^{-1}\|_{L^q(d\nu)}^{-1} \leq \|h\|_{L^r(d\nu)}, \forall q > 0, \forall r > 0,$$

when  $d\nu$  is a probability measure on  $\mathbb{R}^n$ . Applying this estimate with  $h = \tilde{u}$ ,  $r = \tilde{p}$ ,  $q = q_0$  and  $d\nu(x) = |B(y, \alpha_* t)|^{-1} \chi_{B(y, \alpha_* t)}(x) dx$ , we have for any  $(y, t) \in \mathbb{R}_+^{n+1}$

$$\begin{aligned} \inf_{(z, s) \in W(y, t)} v(z, s) &\gtrsim P_0[\tilde{u}^{\tilde{p}}]^{1/\tilde{p}}(y, \alpha_* t) \\ &\geq P_0[\tilde{u}^{-q_0}]^{-1/q_0}(y, \alpha_* t) \gtrsim P_0[\tilde{u}^{-q_0}]^{-1/q_0}(y, t), \end{aligned}$$

where the last estimate follows from  $0 < \alpha_* < 1$  and  $-1/q_0 < 0$ . We write  $\|\cdot\|_c = \|\cdot\|_{T_{1, -1}^\infty}$  for the Carleson norm of measurable functions on  $\mathbb{R}_+^{n+1}$ , and let

$$d\mu(y, t) = \mu(y, t) dy dt = \mathcal{W}_{r_0}(u)^{q_0}(y, t) t^{-1} dy dt.$$

The above pointwise estimates on  $v$  further imply

$$\begin{aligned} \|\mathcal{W}_{r_0}(u/v)\|_{T_{q_0}^\infty} &\lesssim \|P_0[\tilde{u}^{-q_0}]^{1/q_0} \mathcal{W}_{r_0}(u)\|_{T_{q_0}^\infty} \\ &= \|P_0[\tilde{u}^{-q_0}] \mu\|_{T_{1, -1}^\infty}^{1/q_0} = \|P_0[\mathcal{A}_1(\mu(y, t)t^{-1}) \mu]\|_c^{1/q_0} \lesssim 1. \end{aligned}$$

In the last estimate, we used the lemma below.

Therefore, we can conclude the proof of  $F_2$ .  $\square$

We record down the missing part in estimating  $\|P_0[\mathcal{A}_1(\mu(y, t)t)^{-1}]\mu\|_c \lesssim 1$ . For a non-negative measure  $d\mu$  on  $\overline{\mathbb{R}_+^{n+1}}$ , denote its (free) balayage by

$$\overline{\mathcal{A}}(d\mu)(x) := \iint_{\Gamma(x)} \frac{d\mu(z, s)}{s^n}, \quad x \in \mathbb{R}^n.$$

This way, we can reconstruct from the boundary value  $\overline{\mathcal{A}}(d\mu)$  its (free) extension

$$E(d\mu)(y, t) := P_0[\overline{\mathcal{A}}(d\mu)^{-1}](y, t), \quad \forall (y, t) \in \mathbb{R}_+^{n+1}.$$

Thus in the desired estimate, with  $d\mu(y, t) = \mu(y, t)dydt$  supported in  $\mathbb{R}_+^{n+1}$ , we have

$$P_0[\mathcal{A}_1(\mu(z, s)s)^{-1}](y, t)\mu(y, t)dydt = E(d\mu)(y, t)d\mu(y, t).$$

The next lemma is very simple and can be found in [9, Lemma 2.2], or one can refer to [3] directly. For the completeness, we still provide an argument here. Recall that  $\widehat{B}$  denotes the closed tent with base  $B \subset \mathbb{R}^n$ .

**Lemma 6.1.** *For any non-negative measure  $d\mu$  on  $\overline{\mathbb{R}_+^{n+1}}$ , we have*

$$\|E(d\mu)d\mu\|_{\overline{\mathcal{C}}} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \iint_{\widehat{B}} E(d\mu)d\mu \lesssim 1.$$

*Proof.* For any ball  $B \subset \mathbb{R}^n$ , we can estimate by Fubini's theorem that

$$\begin{aligned} & \iint_{\widehat{B}} \left[ \frac{1}{|B(y, t)|} \int_{B(y, t)} \overline{\mathcal{A}}(d\mu)^{-1}(x)dx \right] d\mu(y, t) \\ & \simeq \iint_{\widehat{B}} \left[ \int_{B(y, t)} \overline{\mathcal{A}}(d\mu)^{-1}(x)dx \right] \frac{d\mu(y, t)}{t^n} \\ & = \int_{\mathbb{R}^n} \overline{\mathcal{A}}(d\mu)^{-1}(x) \left[ \iint_{\widehat{B} \cap \Gamma(x)} \frac{d\mu(y, t)}{t^n} \right] dx \\ & \leq \int_B \overline{\mathcal{A}}(d\mu)^{-1}(x) \overline{\mathcal{A}}(d\mu)(x) dx = |B|. \end{aligned}$$

Taking a supremum over balls  $B \subset \mathbb{R}^n$  then proves the Carleson estimate.  $\square$

**Remark 6.2.** Denote by  $\overline{\mathcal{V}}$  the class of bounded (signed and complex) Borel measures on  $\overline{\mathbb{R}_+^{n+1}}$ . Note that the above lemma also implies the factorization

$$\overline{\mathcal{V}} \rightarrow (T_\infty^1 \cap C_{n.t.}) \cdot \overline{\mathcal{C}},$$

while the multiplication side  $\overline{\mathcal{V}} \leftarrow (T_\infty^1 \cap C_{n.t.}) \cdot \overline{\mathcal{C}}$  is just the Carleson's inequality (see [29, p. 63] for example). Indeed, for  $d\mu$  bounded on  $\overline{\mathbb{R}_+^{n+1}}$ ,

$$|d\mu| = E(|d\mu|)^{-1} \cdot E(|d\mu|)|d\mu|$$

is then the desired factorization. First, using the lemma above, we have

$$\|E(|d\mu|)|d\mu|\|_{\overline{\mathcal{C}}} \lesssim 1.$$

And by (7), we see for any  $(y, t) \in \mathbb{R}_+^{n+1}$  that

$$E(|d\mu|)^{-1}(y, t) \leq \left( \frac{1}{|B(y, t)|} \int_{B(y, t)} \overline{\mathcal{A}}(|d\mu|)^{p_0}(x)dx \right)^{1/p_0}, \quad 0 < p_0 < 1.$$

Then for any  $x \in \mathbb{R}^n$ , we have

$$\mathcal{N}(E(|d\mu|)^{-1})(x) \leq \mathcal{M}(\overline{\mathcal{A}}(|d\mu|)^{p_0})^{1/p_0}(x),$$

and by Lebesgue's theorem  $E(|d\mu|)^{-1} \in C_{n.t.}$ . By maximal theorem, we also have  $E(|d\mu|)^{-1} \in T_\infty^1$ , with the factorization estimate

$$\|E(|d\mu|)^{-1}\|_{T_\infty^1} \lesssim \|\overline{\mathcal{A}}(|d\mu|)\|_{L^1} \simeq |\mu|(\overline{\mathbb{R}_+^{n+1}}).$$

**Remark 6.3.** In  $F_1$ ), the case  $r_0 = \infty$  is trivial. Suppose  $0 < r_0 < \infty$  and  $\mathcal{W}_{r_0}(u) \in C_{n.t.}$ . As the constructed  $v = \mathcal{W}_{r_0}^*(u)$  is continuous and satisfies  $\mathcal{W}_\infty^*(v) \lesssim \mathcal{W}_{r_0}(u)$ , we have  $\mathcal{W}_\infty^*(v) \in C_{n.t.}$  after using the fact (3)

$$\lim_{\Gamma(x) \ni (y,t) \rightarrow x} W^*(y,t) = \lim_{\Gamma(x) \ni (y,t) \rightarrow x} W(y,t) = x, \forall x \in \mathbb{R}^n,$$

and the dominated convergence theorem.

In  $F_2$ ), if  $0 < \max(p_0, q_0) < \infty$ , we can also verify that  $\mathcal{W}_\infty(v)$  is continuous in  $\mathbb{R}_+^{n+1}$  and has the property of non-tangential convergence. In fact,

$$v^{\tilde{p}}(y,t) = |B(y,t)|^{-1} \int_{B(y,t)} \tilde{u}^{\tilde{p}}(x) dx, \forall (y,t) \in \mathbb{R}_+^{n+1},$$

where  $\tilde{u} \in L^{p_0}$  and  $p_0 > \tilde{p}$ . Then  $v \in C_{n.t.}$  follows from Lebesgue's theorem. As

$$v(y, \alpha_* t) \lesssim \inf_{(z,s) \in W(y,t)} v(z,s) \leq \sup_{(z,s) \in W(y,t)} v(z,s) \lesssim v(y, \alpha^* t)$$

hold for any  $(y,t) \in \mathbb{R}_+^{n+1}$ , we then have

$$\mathcal{W}_\infty(v) = \sup_{(z,s) \in W(y,t)} v(z,s) \in C_{n.t.},$$

which is an easy consequence of the dominated convergence theorem. In all, the constructed factorization  $v$  is in  $(T_\infty^{p_0, \infty} \cap C_{n.t.}) = (T_\infty^{p_0} \cap C_{n.t.})$ .

**Notes added after submission.** Alex Amenta and Moritz Egert confirm (via personal communications) that the main results in this paper would extend to the spaces of homogeneous type in the sense of Coifman-Weis.

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UNIV. PARIS-SUD, LABORATOIRE DE MATHÉMATIQUES, UMR 8628 DU CNRS, F-91405 ORSAY

*E-mail address:* Yi.Huang@math.u-psud.fr