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# ASYMPTOTIC BEHAVIOR AND CONTROL OF COUPLED DYNAMICAL SYSTEMS WITH MULTISCALE ASPECTS.

HEDY ATTOUCH AND MARC-OLIVIER CZARNECKI

**Abstract.** We study the asymptotic behaviour, as time variable  $t$  goes to  $+\infty$ , of nonautonomous dynamical systems involving multiscale features. As a benchmark case, given  $\mathcal{H}$  a general Hilbert space,  $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $\Psi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$  two closed convex functions, and  $\beta$  a function of  $t$  which tends to  $+\infty$  as  $t$  goes to  $+\infty$ , we consider the differential inclusion

$$\dot{x}(t) + \partial\Phi(x(t)) + \beta(t)\partial\Psi(x(t)) \ni 0.$$

This contains the case of nonautonomous coupled dynamical systems

$$\begin{cases} \dot{x}_1(t) + \partial f_1(x_1(t)) + \beta(t)L_1^t(L_1x_1(t) - L_2x_2(t)) \ni 0 \\ \dot{x}_2(t) + \partial f_2(x_2(t)) + \beta(t)L_2^t(L_2x_2(t) - L_1x_1(t)) \ni 0 \end{cases}$$

with  $\mathcal{H} = \mathcal{X}_1 \times \mathcal{X}_2$ ,  $x = (x_1, x_2)$ ,  $\Phi(x) = f_1(x_1) + f_2(x_2)$  and  $\Psi(x) = \frac{1}{2}\|L_1x_1 - L_2x_2\|^2$ , where  $L_1$  and  $L_2$  are two linear continuous operators acting respectively from  $\mathcal{X}_1$  and  $\mathcal{X}_2$  into a third Hilbert space  $\mathcal{Z}$ .

This system models the emergence of various collective behaviors in game theory, as well as the asymptotic control of coupled systems. Its discretized versions bear a natural link with Passty's theorem and numerical splitting methods. We show several results ranging from weak ergodic to strong convergence of the trajectories. As a key ingredient we assume that, for every  $p$  belonging to the range of  $N_C$

$$\int_0^{+\infty} \beta(t) \left[ \Psi^* \left( \frac{p}{\beta(t)} \right) - \sigma_C \left( \frac{p}{\beta(t)} \right) \right] dt < +\infty$$

where  $\Psi^*$  is the Fenchel conjugate of  $\Psi$ ,  $\sigma_C$  is the support function of  $C = \operatorname{argmin} \Psi$  and  $N_C(x)$  is the normal cone to  $C$  at  $x$ . As a by-product, we revisit the sytem

$$\dot{x}(t) + \epsilon(t)\partial\Phi(x(t)) + \partial\Psi(x(t)) \ni 0$$

where  $\epsilon(t)$  tends to zero as  $t$  goes to  $+\infty$  and  $\int_0^{+\infty} \epsilon(t)dt = +\infty$ , whose asymptotic behaviour can be derived from the preceding one by time rescaling. Applications are given in game theory, optimal control, variational problems and PDE's.

**Key words:** nonautonomous gradient-like systems; monotone inclusions; asymptotic behaviour; time multiscaling; convex minimization; hierarchical optimization; asymptotic control; slow control; potential games; best response; splitting methods; domain decomposition for PDE's.

**AMS subject classification.** 37N40, 46N10, 49M30, 65K05, 65K10 90B50, 90C25.

## 1. INTRODUCTION

$\mathcal{H}$  is a real Hilbert space, we write  $\|x\|^2 = \langle x, x \rangle$  for  $x \in \mathcal{H}$ . We denote by  $\Gamma_0(\mathcal{H})$  the class of closed (lower semi-continuous) convex proper functions from  $\mathcal{H}$  to  $\mathbb{R} \cup \{+\infty\}$  which are not identically equal to  $+\infty$ . The subdifferential of  $f \in \Gamma_0(\mathcal{H})$  is the maximal monotone operator

$$(1) \quad \begin{aligned} \partial f: \mathcal{H} &\rightarrow 2^{\mathcal{H}} \\ x &\mapsto \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \ f(y) \geq f(x) + \langle u, y - x \rangle\}. \end{aligned}$$

## 1.1. Problem statement.

- $\Phi: \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$  is a closed convex proper function.
- $\Psi: \mathcal{H} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  is a closed convex proper function,  $C = \operatorname{argmin} \Psi = \Psi^{-1}(0) \neq \emptyset$ .
- $\beta: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a function of  $t$  which tends to  $+\infty$  as  $t$  goes to  $+\infty$ .

We study the asymptotic behavior of the trajectories of the nonautonomous multiscaled differential inclusion

$$(2) \quad (MAG) \quad \dot{x}(t) + \partial\Phi(x(t)) + \beta(t)\partial\Psi(x(t)) \ni 0$$

where  $\partial\Phi$  and  $\partial\Psi$  are the subdifferentials of  $\Phi$  and  $\Psi$ .

Let us observe that  $\partial\Phi + \beta(t)\partial\Psi \subset \partial(\Phi + \beta(t)\Psi)$  (equality holds under some general qualification assumption). Hence, each trajectory of  $(MAG)$  satisfies

$$\dot{x}(t) + \partial(\Phi + \beta(t)\Psi)(x(t)) \ni 0.$$

On the other hand,  $\Phi + \beta(t)\Psi \uparrow \Phi + \delta_C$  as  $t \rightarrow +\infty$  where  $\delta_C$  is the indicator function of the set  $C$  ( $\delta_C(x) = 0$  for  $x \in C$ ,  $+\infty$  outwards). Monotone convergence is a variational convergence, ([1], theorem 3.20). As a consequence, the corresponding subdifferential operators converge in the sense of graphs (equivalently in the sense of resolvents) as  $t \rightarrow +\infty$  ([1], theorem 3.66)

$$\partial(\Phi + \beta(t)\Psi) \rightarrow \partial(\Phi + \delta_C).$$

From the asymptotical point of view, this suggests strong analogies between  $(MAG)$  and the steepest descent dynamical system associated to the closed convex proper function  $\Phi + \delta_C \in \Gamma_0(\mathcal{H})$

$$(3) \quad \dot{x}(t) + \partial(\Phi + \delta_C)(x(t)) \ni 0.$$

In our main result, theorem 3.1, we prove that the two systems (2) and (3) share similar asymptotical properties, whence the terminology  $(MAG)$  = “Multiscale Asymptotic Gradient” system.

More precisely, under general assumptions, we prove that each trajectory of  $(MAG)$  weakly converges in  $\mathcal{H}$ , with its limit belonging to  $\operatorname{argmin}_C \Phi$

$$(4) \quad x(t) \rightharpoonup x_\infty \in \operatorname{argmin}_C \Phi \text{ as } t \rightarrow +\infty.$$

This result can be seen as an extension of Bruck’s theorem [17] to multiscaled nonautonomous gradient systems.

**1.2. Notion of solution.** We consider strong solutions in the sense of Brezis ([13], definition 3.1). Such a solution  $x(\cdot)$  is continuous on  $[0, +\infty)$  and absolutely continuous on any bounded interval  $[0, T]$  with  $T < +\infty$ . Being absolutely continuous,  $x(\cdot)$  is almost everywhere differentiable, and it is assumed that the equation holds almost everywhere. Equivalently  $x(\cdot)$  is a solution of  $(MAG)$  if there exist two functions  $\xi(\cdot)$  and  $\eta(\cdot)$  with

$$(5) \quad \xi(t) \in \partial\Phi(x(t)) \text{ and } \eta(t) \in \partial\Psi(x(t)) \text{ for almost every } t > 0$$

such that

$$(6) \quad \dot{x}(t) + \xi(t) + \beta(t)\eta(t) = 0.$$

In particular,  $x(t) \in \operatorname{dom}(\partial\Phi) \cap \operatorname{dom}(\partial\Psi)$  for almost every  $t > 0$ .

Existence of strong solutions of nonautonomous monotone differential inclusions is a nontrivial topic. This question is not examined in this paper. We take for granted the existence of such trajectories. The interested reader can consult Brézis [13], Attouch-Damlamian [3], Kenmochi [24] for precise conditions insuring the existence of such solutions. In this paper, we shall be concerned only with the asymptotic behavior of the trajectories of the above systems.

**1.3. Key assumption.** We shall prove the convergence property (4) under the assumption

$$(7) \quad (\mathcal{H}_1) \quad \forall p \in R(N_C) \quad \int_0^{+\infty} \beta(t) \left[ \Psi^*\left(\frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right) \right] dt < +\infty.$$

In (7) we use classical notions and notations from convex analysis:  $\Psi^*$  is the Fenchel conjugate of  $\Psi$ ,

$$\forall y \in \mathcal{H} \quad \Psi^*(y) = \sup_{x \in \mathcal{H}} \{ \langle y, x \rangle - \Psi(x) \},$$

and  $\sigma_C$  is the support function of  $C = \operatorname{argmin} \Psi$

$$\forall y \in \mathcal{H} \quad \sigma_C(y) = \sup_{x \in C} \langle y, x \rangle.$$

Note that  $\sigma_C$  is equal to the Fenchel conjugate of  $\delta_C$ , where  $\delta_C$  is the indicator function of  $C$ .  $N_C(x)$  is the (outwards) normal cone to  $C$  at  $x$ . We denote by  $R(N_C)$  the range of  $N_C$ , i.e.,  $p \in R(N_C)$  iff  $p \in N_C(x)$  for some  $x \in C$ .

*Analysis of the condition  $(\mathcal{H}_1)$ :*

- a) Note that  $\Psi$  enters in  $(MAG)$  only via its subdifferential. Thus it is not a restriction to assume  $\min_{\mathcal{H}} \Psi = 0$ . For a function  $\Psi$  whose minimum is not equal to zero, one should replace in  $(\mathcal{H}_1)$  and in the corresponding statements  $\Psi$  by  $\Psi - \min_{\mathcal{H}} \Psi$ .  
From  $\Psi \leq \delta_C$  we get  $\Psi^* \geq (\delta_C)^* = \sigma_C$  and  $\Psi^* - \sigma_C \geq 0$ .  $(\mathcal{H}_1)$  means that the nonnegative function

$$t \mapsto \beta(t) \left[ \Psi^*\left(\frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right) \right]$$

is integrable on  $(0, +\infty)$ . It is a growth condition on  $\beta(\cdot)$  at infinity which depends only on  $\Psi$ .

- b) As an illustration, consider the model situation:  $\Psi(z) = \frac{1}{2} \operatorname{dist}^2(z, C) = \frac{1}{2} \|\cdot\|^2 +_e \delta_C$ , where  $+_e$  denotes the epigraphical sum (also called inf-convolution). From general properties of Fenchel transform

$$\Psi^*(z) = \frac{1}{2} \|z\|^2 + \sigma_C(z) \text{ and } \Psi^*(z) - \sigma_C(z) = \frac{1}{2} \|z\|^2.$$

Hence, in this situation

$$(\mathcal{H}_1) \iff \int_0^{+\infty} \frac{1}{\beta(t)} dt < +\infty$$

which is satisfied for example with  $\beta(t) = (1+t)^p$ ,  $p > 1$ .

**1.4. Contents.** Several results concerning the asymptotic convergence analysis hold true in respect of the more general differential inclusion

$$(8) \quad \dot{x}(t) + A(x(t)) + \beta(t) \partial \Psi(x(t)) \ni 0$$

with  $A$  a maximal monotone operator (in particular one may consider  $A = \partial \Phi$ ), and without any regularity assumption on function  $\beta(\cdot)$  (it may present oscillations, discontinuities...). In section 2, we prove an ergodic convergence result (theorem 2.1) which holds for (8), and which extends Baillon-Brezis theorem [10] to a nonautonomous multiscale setting. In section 3, we return to the  $(MAG)$  system with  $A = \partial \Phi$  and prove a weak convergence asymptotic convergence result (theorem 3.1), which in turn can be seen as an extension of Bruck theorem ([17]). In section 4, we revisit the asymptotic analysis of the system

$$(9) \quad \dot{x}(t) + \epsilon(t) \partial \Phi(x(t)) + \partial \Psi(x(t)) \ni 0$$

where  $\epsilon(t)$  tends to zero as  $t$  goes to  $+\infty$  and  $\int_0^{+\infty} \epsilon(t) dt = +\infty$ . Indeed (9) can be derived from  $(MAG)$  by time rescaling.

In last section 5, applications are given in game theory, optimal control, variational problems and PDE's.

## 2. WITH A MAXIMAL MONOTONE OPERATOR: ERGODIC CONVERGENCE RESULTS

In this section, we consider the differential inclusion (8) with a maximal monotone operator  $A$ . We call it  $(MAMI)$ :

$$(10) \quad (MAMI) \quad \dot{x}(t) + A(x(t)) + \beta(t) \partial \Psi(x(t)) \ni 0.$$

This terminology stands for “Multiscale Asymptotic Monotone Inclusion” with a justification similar to  $(MAG)$ : one expects that the trajectories of the nonautonomous multiscale differential inclusion  $(MAMI)$  enjoy asymptotic properties similar to the autonomous monotone inclusion

$$(11) \quad \dot{x}(t) + (A + N_C)(x(t)) \ni 0.$$

We denote by  $S$  the set of equilibria  $S := (A + N_C)^{-1}(0)$ . They are solutions of the monotone variational inequality

$$A(x) + N_C(x) \ni 0.$$

We first examine  $(MAMI)$  with a general maximal monotone operator  $A$ , in which case we prove an ergodic convergence result. Then, in the particular case of a strongly monotone operator  $A$ , we prove strong convergence of the trajectories towards the unique equilibrium.

**Theorem 2.1.** *Let*

- $A : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  *be a general maximal monotone operator.*
- $\Psi : \mathcal{H} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  *be a closed convex proper function, such that  $C = \operatorname{argmin} \Psi = \Psi^{-1}(0) \neq \emptyset$ .*

*Let us assume that,*

- $(\mathcal{H}_0)$   $A + N_C$  *is a maximal monotone operator and  $S := (A + N_C)^{-1}(0)$  is non empty.*
- $(\mathcal{H}_1)$   $\forall p \in R(N_C), \int_0^{+\infty} \beta(t) \left[ \Psi^* \left( \frac{p}{\beta(t)} \right) - \sigma_C \left( \frac{p}{\beta(t)} \right) \right] dt < +\infty$ .

*Then, for every strong solution trajectory  $x(\cdot)$  of the differential inclusion  $(MAMI)$ :*

- (i) *(weak ergodic convergence)*  $\exists x_\infty \in S$  *such that*  $w - \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x(s) ds = x_\infty$ ;
- (ii)  $\forall z \in S, \lim_{t \rightarrow +\infty} \|x(t) - z\|$  *exists*;
- (iii) *(estimation)*  $\int_0^{+\infty} \beta(t) \Psi(x(t)) dt < +\infty$ .

By taking  $\Psi = 0$  in Theorem 2.1, we recover the Baillon-Brézis result on the ergodic convergence of semi-groups of contractions in Hilbert spaces (generated by maximal monotone operators).

**Corollary 2.1.** [10, Baillon-Brézis] *Let  $A : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be a maximal monotone operator such that  $A^{-1}0 \neq \emptyset$ . Let  $x$  be a strong solution of*

$$\dot{x}(t) + A(x(t)) \ni 0.$$

*Then,*

$$w - \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x(s) ds = x_\infty \text{ exists with } x_\infty \in A^{-1}0.$$

**Remark 2.1.** An elementary example (take  $A$  equal to the rotation of angle  $\frac{\pi}{2}$  in  $\mathbb{R}^2$ ) shows that ergodic convergence can happen without convergence. Clearly, the same holds true for  $(MAMI)$ .

**2.1. Proof of Theorem 2.1.** It relies on an Opial type argument. Owing to the length of the proof, we decompose it in several lemmas.

**Lemma 2.1.** *For every  $z \in S = (A + N_C)^{-1}(0)$ ,  $\lim_{t \rightarrow +\infty} \|x(t) - z\|$  exists.*

**Proof of Lemma 2.1.** Set

$$h_z(t) = \frac{1}{2} \|x(t) - z\|^2$$

and compute the time derivative of  $h_z(t)$ . For almost every  $t > 0$

$$\begin{aligned} \dot{h}_z(t) &= \langle x(t) - z, \dot{x}(t) \rangle \\ &= \langle x(t) - z, -\xi(t) - \beta(t)\eta(t) \rangle, \end{aligned}$$

where, by definition of  $x(\cdot)$  solution of  $(MAMI)$ ,

$$\dot{x}(t) + \xi(t) + \beta(t)\eta(t) = 0$$

with

$$\xi(t) \in A(x(t)) \text{ and } \eta(t) \in \partial\Psi(x(t)).$$

Equivalently,

$$(12) \quad \dot{h}_z(t) + \langle \xi(t), x(t) - z \rangle + \beta(t) \langle \eta(t), x(t) - z \rangle = 0.$$

Since  $z \in S \subset C$  and  $\eta(t) \in \partial\Psi(x(t))$ , we have

$$0 = \Psi(z) \geq \Psi(x(t)) + \langle \eta(t), z - x(t) \rangle,$$

that is,

$$(13) \quad \langle \eta(t), x(t) - z \rangle \geq \Psi(x(t)).$$

Since  $z \in S$ , we have  $Az + N_C(z) \ni 0$  i.e., there exists some  $p \in N_C(z)$  such that  $-p \in Az$ . By monotonicity of  $A$  and  $\xi(t) \in A(x(t))$

$$(14) \quad \langle \xi(t), x(t) - z \rangle \geq \langle -p, x(t) - z \rangle.$$

In view of (12), (13) and (14), and since  $\beta(t) > 0$  we obtain

$$(15) \quad \dot{h}_z(t) \leq \langle p, x(t) - z \rangle - \beta(t)\Psi(x(t)).$$

Let us rewrite this inequality as

$$(16) \quad \dot{h}_z(t) \leq \beta(t) \left[ \left\langle \frac{p}{\beta(t)}, x(t) - z \right\rangle - \Psi(x(t)) \right].$$

Since we have no prior information on  $x(t)$ , let us take the supremum of this last expression with respect to  $x$

$$\dot{h}_z(t) \leq \beta(t) \left[ \sup_{x \in \mathcal{H}} \left\{ \left\langle \frac{p}{\beta(t)}, x \right\rangle - \Psi(x) \right\} - \left\langle \frac{p}{\beta(t)}, z \right\rangle \right],$$

which makes appear the Fenchel conjugate of  $\Psi$ , namely  $\Psi^*$ , and gives

$$(17) \quad \dot{h}_z(t) \leq \beta(t) \left[ \Psi^* \left( \frac{p}{\beta(t)} \right) - \left\langle \frac{p}{\beta(t)}, z \right\rangle \right].$$

Let us now examine the term  $\langle \frac{p}{\beta(t)}, z \rangle$  in (17). Since  $z \in C$  and  $p \in N_C(z)$  we have

$$\forall x \in C \quad \langle p, x - z \rangle \leq 0$$

which implies

$$\langle p, z \rangle = \sup_{x \in C} \langle p, x \rangle = \sigma_C(p).$$

By positive homogeneity of  $\sigma_C$  and  $\beta(t) > 0$ , we obtain

$$(18) \quad \left\langle \frac{p}{\beta(t)}, z \right\rangle = \sigma_C \left( \frac{p}{\beta(t)} \right).$$

Collecting (17, 18) we finally obtain

$$(19) \quad \dot{h}_z(t) \leq \beta(t) \left[ \Psi^* \left( \frac{p}{\beta(t)} \right) - \sigma_C \left( \frac{p}{\beta(t)} \right) \right].$$

Note that  $\Psi \leq \delta_C$ , hence  $\Psi^* \geq \sigma_C$  and  $\Psi^* - \sigma_C \geq 0$ .

It follows from (19) and assumption  $(\mathcal{H}_1)$  that  $(\dot{h}_z)_+ \in L^1(0, +\infty)$ , which classically implies that  $\lim_{t \rightarrow +\infty} h_z(t)$  exists in  $\mathbb{R}$ .  $\square$

**Lemma 2.2.** *For each  $t > 0$  set*

$$X(t) = \frac{1}{t} \int_0^t x(s) ds.$$

*Every weak limit point of  $X(\cdot)$  belongs to  $S$ .*

**Proof of Lemma 2.2.** Let  $t_n \rightarrow +\infty$  and suppose  $X(t_n) \rightharpoonup X_\infty$  (weak convergence in  $\mathcal{H}$ ). Take an arbitrary  $z \in C \cap \text{dom} A$ , and  $y \in (A + N_C)(z)$ . Consider again the function  $h_z(t) = \frac{1}{2} \|x(t) - z\|^2$ . Recall (12):

$$\dot{h}_z(t) + \langle \xi(t), x(t) - z \rangle + \beta(t) \langle \eta(t), x(t) - z \rangle = 0.$$

For some  $p \in N_C(z)$ ,  $y - p \in A(z)$ . By monotonicity of  $A$  and  $\xi(t) \in A(x(t))$

$$\langle \xi(t), x(t) - z \rangle \geq \langle y - p, x(t) - z \rangle.$$

Recall (13)

$$\langle \eta(t), x(t) - z \rangle \geq \Psi(x(t))$$

and we obtain

$$\dot{h}_z(t) + \langle y, x(t) - z \rangle \leq \beta(t) \left[ \left\langle \frac{p}{\beta(t)}, x(t) - z \right\rangle - \Psi(x(t)) \right].$$

Hence

$$(20) \quad \dot{h}_z(t) + \langle y, x(t) - z \rangle \leq \beta(t) \left[ \Psi^* \left( \frac{p}{\beta(t)} \right) - \sigma_C \left( \frac{p}{\beta(t)} \right) \right].$$

Let us integrate (20) from 0 to  $t$

$$(21) \quad h_z(t) + \left\langle y, \int_0^t x(s) ds - tz \right\rangle \leq h_z(0) + \int_0^t \beta(s) \left[ \Psi^* \left( \frac{p}{\beta(s)} \right) - \sigma_C \left( \frac{p}{\beta(s)} \right) \right] ds.$$

After division by  $t$ , and taking account of  $h_z \geq 0$ , one obtains

$$\begin{aligned} \langle y, X(t) - z \rangle &\leq \frac{1}{t} h_z(0) + \frac{1}{t} \int_0^t \beta(s) \left[ \Psi^* \left( \frac{p}{\beta(s)} \right) - \sigma_C \left( \frac{p}{\beta(s)} \right) \right] ds \\ &\leq \frac{1}{t} h_z(0) + \frac{1}{t} \int_0^{+\infty} \beta(s) \left[ \Psi^* \left( \frac{p}{\beta(s)} \right) - \sigma_C \left( \frac{p}{\beta(s)} \right) \right] ds \\ &\leq \frac{c}{t} \end{aligned}$$

where

$$c := h_z(0) + \int_0^{+\infty} \beta(s) \left[ \Psi^* \left( \frac{p}{\beta(s)} \right) - \sigma_C \left( \frac{p}{\beta(s)} \right) \right] ds,$$

is, by assumption  $(\mathcal{H}_1)$ , a finite positive number.

Recall that  $t_n \rightarrow +\infty$  and  $X(t_n) \rightharpoonup X_\infty$  (weak convergence in  $\mathcal{H}$ ). Passing to the limit as  $n \rightarrow +\infty$  on

$$\langle y, X(t_n) - z \rangle \leq \frac{c}{t_n}$$

we finally obtain

$$\langle y, X_\infty - z \rangle \leq 0.$$

An equivalent formulation is

$$(22) \quad \langle 0 - y, X_\infty - z \rangle \geq 0.$$

The above inequality (22) being true for any  $y \in \text{dom}(A + N_C)$  and any  $y \in (A + N_C)z$ , by maximal monotonicity of the operator  $A + N_C$  (Assumption  $(\mathcal{H}_0)$ ), we obtain  $0 \in (A + N_C)(X_\infty)$ , that is  $X_\infty \in S$ .  $\square$

Just like in Passty [26], we conclude to the ergodic convergence of the trajectories thanks to Lemmas 2.1 and 2.2 and the following ergodic variant of Opial's lemma.

**Lemma 2.3.** *Let  $\mathcal{H}$  be a Hilbert space,  $S$  a non empty subset of  $\mathcal{H}$  and  $x : [0, +\infty) \rightarrow \mathcal{H}$  a map. Set*

$$X(t) = \frac{1}{t} \int_0^t x(s) ds$$

and assume that

- (i) *for every  $z \in S$ ,  $\lim_{t \rightarrow +\infty} \|x(t) - z\|$  exists;*
- (ii) *every weak limit point of the map  $X$  belongs to  $S$ .*

Then

$$w - \lim_{t \rightarrow +\infty} X(t) = X_\infty \text{ for some element } X_\infty \in S.$$

**Proof of Lemma 2.3.** First note that  $x$  is bounded (from (i)), thus  $X$  also and it is sufficient to prove uniqueness of weak limit points. Let  $X(t_{n_1}) \rightharpoonup X_{\infty,1}$  and  $X(t_{n_2}) \rightharpoonup X_{\infty,1}$  be two weak converging subsequences.

Let us prove that  $X_{\infty,1} = X_{\infty,2}$ . By (ii),  $X_{\infty,1}$  and  $X_{\infty,2}$  belong to  $S$ . Hence, by (i),  $\lim_{t \rightarrow +\infty} \|x(t) - X_{\infty,1}\|^2$  and  $\lim_{t \rightarrow +\infty} \|x(t) - X_{\infty,2}\|^2$  exist. As a consequence, the following limit exist

$$\lim_{t \rightarrow +\infty} [\|x(t) - X_{\infty,1}\|^2 - \|x(t) - X_{\infty,2}\|^2].$$

After simplification

$$\lim_{t \rightarrow +\infty} \langle x(t), X_{\infty,2} - X_{\infty,1} \rangle \text{ exists.}$$

As a general classical result (Cesaro), convergence implies ergodic convergence. Hence

$$\lim_{t \rightarrow +\infty} \langle X(t), X_{\infty,2} - X_{\infty,1} \rangle \text{ exists.}$$

In particular,

$$\lim_{t_{n_1} \rightarrow +\infty} \langle X(t_{n_1}), X_{\infty,2} - X_{\infty,1} \rangle = \lim_{t_{n_2} \rightarrow +\infty} \langle X(t_{n_2}), X_{\infty,2} - X_{\infty,1} \rangle,$$

that is

$$\langle X_{\infty,1}, X_{\infty,2} - X_{\infty,1} \rangle = \langle X_{\infty,2}, X_{\infty,2} - X_{\infty,1} \rangle$$

and  $\|X_{\infty,2} - X_{\infty,1}\|^2 = 0$ , which ends the proof of the lemma.  $\square$

*End of the proof of theorem 2.1*

Let us complete the proof of theorem 2.1 and prove the estimation

$$\int_0^{+\infty} \beta(t) \Psi(x(t)) dt < +\infty.$$

Let us return to (16)

$$\dot{h}_z(t) \leq \beta(t) \left[ \left\langle \frac{p}{\beta(t)}, x(t) - z \right\rangle - \Psi(x(t)) \right]$$

which can be written in a splitted form as

$$\dot{h}_z(t) + \frac{\beta(t)}{2} \Psi(x(t)) \leq \frac{\beta(t)}{2} \left[ \left\langle \frac{2p}{\beta(t)}, x(t) - z \right\rangle - \Psi(x(t)) \right].$$

By using a device similar to the proof of lemma 2.1

$$\begin{aligned} \dot{h}_z(t) + \frac{\beta(t)}{2} \Psi(x(t)) &\leq \frac{\beta(t)}{2} \left[ \sup_{x \in \mathcal{H}} \left\{ \left\langle \frac{2p}{\beta(t)}, x \right\rangle - \Psi(x) \right\} - \left\langle \frac{2p}{\beta(t)}, z \right\rangle \right] \\ &\leq \frac{\beta(t)}{2} \left[ \Psi^* \left( \frac{2p}{\beta(t)} \right) - \sigma_C \left( \frac{2p}{\beta(t)} \right) \right]. \end{aligned}$$

Let us integrate this last inequality from 0 to  $\tau$

$$(23) \quad h_z(\tau) - h_z(0) + \frac{1}{2} \int_0^\tau \beta(t) \Psi(x(t)) dt \leq \frac{1}{2} \int_0^{+\infty} \beta(t) \left[ \Psi^* \left( \frac{2p}{\beta(t)} \right) - \sigma_C \left( \frac{2p}{\beta(t)} \right) \right] dt.$$

By assumption  $(\mathcal{H}_1)$ , the second member of (23) is a finite quantity (note that  $p \in R(N_C)$  implies  $2p \in R(N_C)$ ).

This being true for any  $\tau > 0$ , we finally obtain

$$\int_0^{+\infty} \beta(t) \Psi(x(t)) dt < +\infty.$$

**2.2. The case  $A$  strongly monotone.** The argument developed in the preceding section allows to conclude to convergence of the trajectories when the operator  $A : \mathcal{H} \rightarrow \mathcal{H}$  satisfies a strong monotonicity property. We recall that  $A$  is said to be strongly monotone if there exists some  $\alpha > 0$  such that for any  $x \in \text{dom} A$ ,  $y \in \text{dom} A$  and any  $\xi \in Ax$ ,  $\eta \in Ay$ ,

$$\langle \xi - \eta, x - y \rangle \geq \alpha \|x - y\|^2.$$

**Theorem 2.2.** *Let us assume that,*

- $(\mathcal{H}_0)$   $A + N_C$  is a maximal monotone operator and  $S := (A + N_C)^{-1}(0)$  is non empty.
- $(\mathcal{H}_1)$   $\forall p \in R(N_C) \quad \int_0^{+\infty} \beta(t) \left[ \Psi^* \left( \frac{p}{\beta(t)} \right) - \sigma_C \left( \frac{p}{\beta(t)} \right) \right] dt < +\infty.$
- $(\mathcal{H}_2)$   $A$  is a strongly monotone operator.

*Then, there exists a unique equilibrium  $\bar{x}$ , i.e.,  $S = \{\bar{x}\}$  with  $A\bar{x} + N_C(\bar{x}) \ni 0$  and any strong solution trajectory  $x(\cdot)$  of the differential inclusion (MA MI) strongly converges to  $\bar{x}$  as  $t \rightarrow +\infty$ :*

$$s - \lim_{t \rightarrow +\infty} x(t) = \bar{x}.$$



**Proof of Theorem 2.2.** By strong monotonicity of  $A$ , the operator  $A + N_C$  is also strongly monotone. As a consequence, there exists a unique solution  $\bar{x}$  to the inclusion

$$A\bar{x} + N_C(\bar{x}) \ni 0$$

and there exists some  $p \in N_C(\bar{x})$  such that  $A\bar{x} \ni -p$ .

Let us return to (12), with  $z = \bar{x}$  and  $h_{\bar{x}}(t) = \frac{1}{2} \|x(t) - \bar{x}\|^2$ ,

$$(24) \quad \dot{h}_{\bar{x}}(t) + \langle \xi(t), x(t) - \bar{x} \rangle + \beta(t) \langle \eta(t), x(t) - \bar{x} \rangle = 0.$$

Let us rewrite (24) as

$$\dot{h}_{\bar{x}}(t) + \langle \xi(t) - (-p), x(t) - \bar{x} \rangle - \langle p, x(t) - \bar{x} \rangle + \beta(t) \langle \eta(t), x(t) - \bar{x} \rangle = 0,$$

and use the strong monotonicity of  $A$  together with  $\xi(t) \in A(x(t))$  to obtain

$$\dot{h}_{\bar{x}}(t) + \alpha \|x(t) - \bar{x}\|^2 - \langle p, x(t) - \bar{x} \rangle + \beta(t) \langle \eta(t), x(t) - \bar{x} \rangle \leq 0.$$

By using the convex subdifferential inequality

$$0 = \Psi(\bar{x}) \geq \Psi(x(t)) + \langle \eta(t), \bar{x} - x(t) \rangle$$

we deduce

$$\dot{h}_{\bar{x}}(t) + \alpha \|x(t) - \bar{x}\|^2 - \langle p, x(t) - \bar{x} \rangle + \beta(t) \Psi(x(t)) \leq 0.$$

Hence,

$$\dot{h}_{\bar{x}}(t) + \alpha \|x(t) - \bar{x}\|^2 \leq \beta(t) \left[ \left\langle \frac{p}{\beta(t)}, x(t) - \bar{x} \right\rangle - \Psi(x(t)) \right].$$

By using a device similar to theorem 2.1

$$\dot{h}_{\bar{x}}(t) + \alpha \|x(t) - \bar{x}\|^2 \leq \beta(t) \left[ \Psi^* \left( \frac{p}{\beta(t)} \right) - \sigma_C \left( \frac{p}{\beta(t)} \right) \right].$$

After integration of this inequality from 0 to  $t$ ,

$$h_{\bar{x}}(t) + \alpha \int_0^t \|x(s) - \bar{x}\|^2 ds \leq h_{\bar{x}}(0) + \int_0^{+\infty} \beta(s) \left[ \Psi^* \left( \frac{p}{\beta(s)} \right) - \sigma_C \left( \frac{p}{\beta(s)} \right) \right] ds,$$

by using assumption  $(\mathcal{H}_1)$ , one obtains

$$\int_0^{+\infty} \|x(t) - \bar{x}\|^2 dt < +\infty.$$

On the other hand, by theorem 2.1,  $\lim_{t \rightarrow +\infty} \|x(t) - \bar{x}\|$  exists. As a consequence, this limit is equal to zero, i.e.  $\lim_{t \rightarrow +\infty} \|x(t) - \bar{x}\| = 0$ , which ends the proof of theorem 2.2.  $\square$

### 3. THE SUBDIFFERENTIAL CASE: WEAK CONVERGENCE RESULTS

In this section, we consider the dynamical system

$$(MAG) \quad \dot{x}(t) + \partial\Phi(x(t)) + \beta(t)\partial\Psi(x(t)) \ni 0.$$

When  $\Psi = 0$ ,  $(MAG)$  boils down to the classical steepest descent differential inclusion

$$\dot{x}(t) + \partial\Phi(x(t)) \ni 0,$$

studied by Brézis [13] [14], Bruck [17] and Baillon [9]. In accordance with these studies, in our main result, we are going to show that each trajectory of  $(MAG)$  weakly converges to a minimizer of  $\Psi$ , which also minimizes  $\Phi$  over all minima of  $\Psi$ .

Before stating our result precisely, let us specify the notion of solution. We recall that a solution  $x(\cdot)$  of  $(MAG)$  is continuous on  $[0, +\infty)$ , absolutely continuous on any bounded interval  $[0, T]$  with  $T < +\infty$ , and it is assumed that the equation holds almost everywhere. Moreover, in this section, we will assume that there exist two functions  $\xi(\cdot)$  and  $\eta(\cdot)$  which are integrable on any bounded interval  $[0, T]$  and such that

$$(25) \quad \dot{x}(t) + \xi(t) + \beta(t)\eta(t) = 0$$

with

$$(26) \quad \xi(t) \in \partial\Phi(x(t)) \text{ and } \eta(t) \in \partial\Psi(x(t)) \text{ for almost every } t > 0.$$

In several cases, for example if  $\Psi$  is differentiable, with a Lipschitz gradient  $\nabla\Psi$  on the bounded sets, or if the subdifferentials  $\partial\Phi$  and  $\partial\Psi$  satisfy an angle condition, the assumption that the two functions  $\xi(\cdot)$  and  $\eta(\cdot)$  are locally integrable is automatically fulfilled, provided that the trajectory  $x$  is absolutely continuous. However, in general, it is a nontrivial issue for which we refer to Attouch-Damlamian [3]. We will denote by  $S = \operatorname{argmin}\{\Phi|\operatorname{argmin}\Psi\}$  the set of equilibria, which is a dense notation for  $S = \{z \in \operatorname{argmin}\Psi : \Phi(z) \leq \Phi(x) \text{ for all } x \in \operatorname{argmin}\Psi\}$ .

**Theorem 3.1.** *Let*

- $\Psi : \mathcal{H} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  *be a closed convex proper function, such that*  $C = \operatorname{argmin}\Psi = \Psi^{-1}(0) \neq \emptyset$ .
- $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$  *be a closed convex proper function, such that*  $S = \operatorname{argmin}\{\Phi|\operatorname{argmin}\Psi\} \neq \emptyset$ .

*Let us assume that,*

- $(\mathcal{H}_1)$   $\forall p \in R(N_C), \int_0^{+\infty} \beta(t) \left[ \Psi^* \left( \frac{p}{\beta(t)} \right) - \sigma_C \left( \frac{p}{\beta(t)} \right) \right] dt < +\infty$ .
- $(\mathcal{H}_2)$   $\beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  *is a function of class*  $C^1$ , *such that*  $\lim_{t \rightarrow +\infty} \beta(t) = +\infty$ , *and for some*  $k \geq 0$  *and*  $t_0 \geq 0$   $0 \leq \dot{\beta}(t) \leq k\beta(t)$  *for all*  $t \geq t_0$ .

*Let*  $x$  *be a strong solution of*  $(MAG)$ . *Then:*

- (i) *weak convergence*  $\exists x_\infty \in S = \operatorname{argmin}\{\Phi|\operatorname{argmin}\Psi\}, \quad w - \lim_{t \rightarrow +\infty} x(t) = x_\infty;$
- (ii) *minimizing properties*  $\lim_{t \rightarrow +\infty} \Psi(x(t)) = 0;$   
 $\lim_{t \rightarrow +\infty} \Phi(x(t)) = \min \Phi|_{\operatorname{argmin}\Psi};$
- (iii)  $\forall z \in S \lim_{t \rightarrow +\infty} \|x(t) - z\| \text{ exists };$
- (iv) *estimations*  $\lim_{t \rightarrow +\infty} \beta(t)\Psi(x(t)) = 0;$   
 $\int_0^{+\infty} \beta(t)\Psi(x(t))dt < +\infty;$   
 $\limsup_{\tau \rightarrow +\infty} \int_0^\tau \Phi(x(t)) - \min \Phi|_{\operatorname{argmin}\Psi} dt < +\infty.$

By taking  $\Psi = 0$  in Theorem 3.1, we recover the convergence result of Bruck on the steepest descent method.

**Corollary 3.1.** [17, Bruck, Theorem 4] *Let*  $\Phi : \mathcal{H} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  *be a closed convex proper function, such that*  $\operatorname{argmin}\Phi \neq \emptyset$ . *Let*  $x$  *be a strong solution of*

$$(SD) \quad \dot{x}(t) + \partial\Phi(x(t)) \ni 0.$$

*Then*  $x$  *weakly converges to a point in*  $\operatorname{argmin}\Phi$ .

*Remark 3.1.* The counterexample of Baillon [9] shows that one may not have strong convergence for  $(SD)$ . Of course, the same holds for  $(MAG)$ .

**Proof of Theorem 3.1.** As in Bruck [17], the weak convergence is consequence of Opial's lemma, after showing the convergence of  $\|x(\cdot) - z\|$  for every  $z \in S$ , and that every weak limit point of  $x$  belongs to  $S$ . The proof is not short, and we decompose it in several lemmas. For an element  $z \in S = \operatorname{argmin}\{\Phi|\operatorname{argmin}\Psi\}$ , we define the function  $h_z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  by

$$h_z(t) = \frac{1}{2} \|x(t) - z\|^2.$$

We give the estimations on the function  $h_z$ , that we will use in the proof. Compute the derivative of  $h_z$  and use the system equation  $(MAG)$ , (25) and (26), for a.e.  $t$ :

$$\begin{aligned} \dot{h}_z(t) &= \langle \dot{x}(t), x(t) - z \rangle \\ &= \langle -\xi(t) - \beta(t)\eta(t), x(t) - z \rangle, \end{aligned}$$

with  $\xi(t) \in \partial\Phi(x(t))$  and  $\eta(t) \in \partial\Psi(x(t))$ . Thus, using  $z \in \operatorname{argmin}\Psi$ ,

$$\begin{aligned} \Phi(z) &\geq \Phi(x(t)) + \langle \xi(t), z - x(t) \rangle; \\ 0 = \Psi(z) &\geq \Psi(x(t)) + \langle \eta(t), z - x(t) \rangle. \end{aligned}$$

We deduce (note that  $\beta(t) > 0$ )

$$(27) \quad \dot{h}_z(t) + \Phi(x(t)) - \Phi(z) + \beta(t)\Psi(x(t)) \leq 0.$$

Since  $z \in S = \operatorname{argmin}\{\Phi|\operatorname{argmin}\Psi\}$ , write the first order necessary condition (with  $C = \operatorname{argmin}\Psi$ )

$$0 \in \partial\Phi(z) + N_C(z),$$

and there exists  $p \in N_C(z)$  such that  $-p \in \partial\Phi(z)$ . Thus

$$\Phi(x(t)) \geq \Phi(z) + \langle -p, x(t) - z \rangle$$

and

$$(28) \quad \dot{h}_z(t) + \beta(t)\Psi(x(t)) + \langle -p, x(t) - z \rangle \leq \dot{h}_z(t) + \Phi(x(t)) - \Phi(z) + \beta(t)\Psi(x(t)) \leq 0.$$

**Lemma 3.1.** *For every  $z \in S$ , as  $t \rightarrow +\infty$ ,*

- (i)  $\|x(t) - z\|$  converges in  $\mathbb{R}$ ;
- (ii)  $t \mapsto \int_0^t \Phi(x(s)) - \Phi(z) + \beta(s)\Psi(x(s))ds$  converges in  $\mathbb{R}$ ;
- (iii)  $\int_0^t \langle p, x(s) - z \rangle ds$  converges in  $\mathbb{R}$ ;
- (iv) Moreover,  $\int_0^{+\infty} \beta(t)\Psi(x(t))dt < +\infty$ .

**Proof of Lemma 3.1.** Parts (i) and (iv) were already proved in Section 2. However, they are obtained easily with the others, which permits an easier reading. Recall that, by definition of the Fenchel conjugate  $\Psi^*$  of  $\Psi$ ,

$$\Psi^*\left(\frac{p}{\beta(t)}\right) \geq \left\langle \frac{p}{\beta(t)}, x(t) \right\rangle - \Psi(x(t)),$$

and that  $z \in C$  and  $p \in N_C(z)$  imply

$$\sigma_C\left(\frac{p}{\beta(t)}\right) = \left\langle \frac{p}{\beta(t)}, z \right\rangle,$$

thus, in view of (28)

$$\dot{h}_z(t) + \beta(t) \left( -\Psi^*\left(\frac{p}{\beta(t)}\right) + \sigma_C\left(\frac{p}{\beta(t)}\right) \right) \leq \dot{h}_z(t) + \beta(t)\Psi(x(t)) + \langle -p, x(t) - z \rangle \leq 0.$$

From Assumption  $(\mathcal{H}_1)$ , by integrating the above equation between two large enough real numbers, we first deduce that the function  $h_z$  satisfies the Cauchy criterion, and is bounded, thus converges in  $\mathbb{R}$ , that's (i). Then we deduce the convergence of the function

$$t \mapsto \int_0^t \beta(s)\Psi(x(s)) + \langle -p, x(s) - z \rangle ds,$$

and, in view of (28), of the function

$$t \mapsto \int_0^t \Phi(x(s)) - \Phi(z) + \beta(s)\Psi(x(s))ds,$$

that's (ii). Since  $\Psi \geq 0$ , and in view of (28),

$$(29) \quad \dot{h}_z(t) + \frac{\beta(t)}{2}\Psi(x(t)) + \langle -p, x(t) - z \rangle \leq \dot{h}_z(t) + \beta(t)\Psi(x(t)) + \langle -p, x(t) - z \rangle \leq 0.$$

As  $\frac{2p}{\beta(t)} \in N_C(z)$ ,

$$\dot{h}_z(t) + \frac{\beta(t)}{2} \left( -\Psi^*\left(\frac{2p}{\beta(t)}\right) + \sigma_C\left(\frac{2p}{\beta(t)}\right) \right) \leq \dot{h}_z(t) + \frac{\beta(t)}{2}\Psi(x(t)) + \langle -p, x(t) - z \rangle \leq 0,$$

which, in view of  $(\mathcal{H}_1)$ , implies the convergence of

$$t \mapsto \int_0^t \frac{\beta(s)}{2}\Psi(x(s)) + \langle -p, x(s) - z \rangle ds.$$

From (ii) we deduce that

$$\int_0^{+\infty} \beta(t)\Psi(x(t))dt < +\infty,$$

and that  $t \mapsto \int_0^t \langle p, x(s) - z \rangle ds$  converges in  $\mathbb{R}$ , these are respectively (iv) and (iii).

Let us recall and translate a lemma from Brézis [13], in the special case of the subdifferential of a convex function:

**Lemma 3.2.** [13, Lemme 4, p73] *Let  $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a closed convex proper function. Let  $x \in L^2(0, T; H)$  be such that  $\dot{x} \in L^2(0, T; H)$  and  $x(t) \in \text{Dom}(\partial\Phi)$  a.e.  $t$ . Assume that there exists  $\xi \in L^2(0, T; H)$  such that  $\xi(t) \in \partial\Phi(x(t))$  for a.e.  $t$ . Then the function  $t \mapsto \Phi(x(t))$  is absolutely continuous and for every  $t$  such that  $x(t) \in \text{Dom}(\partial\Phi)$ ,  $x$  and  $\Phi(x)$  are differentiable at  $t$ , we have*

$$\forall h \in \partial\Phi(x(t)), \quad \frac{d}{dt}\Phi(x(t)) = \langle \dot{x}(t), h \rangle.$$

**Lemma 3.3.** *Let*

$$E_1(t) := \frac{\Phi(x(t))}{\beta(t)} + \Psi(x(t)).$$

*Then the function  $E_1$  is absolutely continuous, and, for a.e.  $t$ ,*

$$\dot{E}_1(t) = -\frac{\dot{\beta}(t)}{\beta(t)^2}\Phi(x(t)) - \frac{|\dot{x}(t)|^2}{\beta(t)}.$$

*The function  $E_1$  converges to zero, and  $\lim_{t \rightarrow +\infty} \Psi(x(t)) = 0$ .*

**Proof of Lemma 3.3.**

From Lemma 3.2, the functions  $t \mapsto \Phi(x(t))$  and  $t \mapsto \Psi(x(t))$  are absolutely continuous, and for a.e.  $t$ ,

$$\begin{aligned} \frac{d}{dt}\Phi(x(t)) &= \langle \dot{x}(t), \xi(t) \rangle \\ \frac{d}{dt}\Psi(x(t)) &= \langle \dot{x}(t), \eta(t) \rangle. \end{aligned}$$

Thus the function  $E_1$  is absolutely continuous (recall that  $\beta$  is of class  $C^1$ ) and, by using (25), for a.e.  $t$ ,

$$\begin{aligned} \dot{E}_1(t) &= \frac{1}{\beta(t)}\langle \dot{x}(t), \xi(t) \rangle - \frac{\dot{\beta}(t)}{\beta(t)^2}\Phi(x(t)) + \langle \dot{x}(t), \eta(t) \rangle \\ &= -\frac{\dot{\beta}(t)}{\beta(t)^2}\Phi(x(t)) - \frac{|\dot{x}(t)|^2}{\beta(t)}. \end{aligned}$$

By Lemma 3.1, the trajectory  $x$  is bounded. Write, for example,

$$\Phi(x(t)) \geq \Phi(z) + \langle -p, x - z \rangle,$$

to see that  $\Phi(x(t))$  is bounded from below. By Assumption  $(\mathcal{H}_2)$ , the function  $\beta$  is non decreasing. Hence

$$\dot{E}_1(t) \leq -\frac{\dot{\beta}(t)}{\beta(t)^2} \inf_{s \geq 0} \Phi(x(s)) - \frac{|\dot{x}(t)|^2}{\beta(t)}$$

which implies the convergence of the function  $E_1$ , recalling that  $\lim_{t \rightarrow +\infty} \beta(t) = +\infty$ . From Lemma 3.1, (ii), we have

$$\liminf_{t \rightarrow +\infty} \Phi(x(t)) - \Phi(z) + \beta(t)\Psi(x(t)) \leq 0.$$

Using that  $\Phi(x(t))$  is bounded from below and that  $\Psi(x(t)) \geq 0$ , we see that the above lower limit belongs to  $\mathbb{R}$ . As a consequence, we can take a sequence  $(t_n)$  which tends to  $+\infty$  such that  $\Phi(x(t_n)) - \Phi(z) + \beta(t_n)\Psi(x(t_n))$  converges, and obtain

$$\lim_{t \rightarrow +\infty} E_1(t) = \lim_{n \rightarrow +\infty} E_1(t_n) = \lim_{n \rightarrow +\infty} \frac{1}{\beta(t_n)} (\Phi(x(t_n)) - \Phi(z) + \beta(t_n)\Psi(x(t_n))) + \frac{1}{\beta(t_n)} \Phi(z) = 0.$$

Write

$$0 \leq \Psi(x(t)) \leq \Psi(x(t)) + \frac{1}{\beta(t)} \left( \Phi(x(t)) - \inf_{s \geq 0} \Phi(x(s)) \right) = E_1(t) - \frac{1}{\beta(t)} \inf_{s \geq 0} \Phi(x(s))$$

to deduce

$$\lim_{t \rightarrow +\infty} \Psi(x(t)) = 0.$$

**Lemma 3.4.**

$$\begin{aligned} \liminf_{t \rightarrow +\infty} \langle -p, x(t) - z \rangle &= 0; \\ \liminf_{t \rightarrow +\infty} \Phi(x(t)) &\geq \Phi(z). \end{aligned}$$

**Proof of Lemma 3.4.** Since  $t \mapsto \int_0^t \langle p, x(s) - z \rangle ds$  converges in  $\mathbb{R}$  (Lemma 3.1),

$$\liminf_{t \rightarrow +\infty} \langle -p, x(t) - z \rangle \leq 0.$$

Now take a sequence  $(t_n)$ ,  $t_n \rightarrow +\infty$ , such that

$$\lim_{n \rightarrow +\infty} \langle -p, x(t_n) - z \rangle = l$$

for some  $l \in \mathbb{R}$ . Since the trajectory  $x$  is bounded, and up to a subsequence,

$$x(t_n) \rightharpoonup x_\infty$$

for some  $x_\infty \in H$ . Then

$$\lim_{n \rightarrow +\infty} \langle p, x(t_n) - z \rangle = \langle p, x_\infty - z \rangle.$$

By weak lower semicontinuity of the function  $\Psi$

$$\Psi(x_\infty) \leq \liminf_{n \rightarrow +\infty} \Psi(x(t_n)) = 0$$

thus

$$x_\infty \in \operatorname{argmin} \Psi.$$

Since  $p \in N_{\operatorname{argmin} \Psi}(z)$ ,

$$\langle -p, x_\infty - z \rangle \geq 0.$$

Thus, every limit point of  $\langle -p, x(t) - z \rangle$  is nonnegative, that is

$$\liminf_{t \rightarrow +\infty} \langle -p, x(t) - z \rangle \geq 0.$$

Now, for every  $t$ , since  $-p \in \partial \Phi(z)$ ,

$$\Phi(x(t)) \geq \Phi(z) + \langle -p, x(t) - z \rangle,$$

hence

$$\liminf_{t \rightarrow +\infty} \Phi(x(t)) \geq \Phi(z).$$

**Lemma 3.5.** *Let*

$$E_2(t) = \Phi(x(t)) + \beta(t)\Psi(x(t)).$$

a) *The function  $E_2$  is absolutely continuous, and, for a.e.  $t$ ,*

$$\dot{E}_2(t) = -|\dot{x}(t)|^2 + \dot{\beta}(t)\Psi(x(t)).$$

b) *The function  $E_2$  converges,  $\lim_{t \rightarrow +\infty} \Phi(x(t)) = \Phi(z)$ , and  $\lim_{t \rightarrow +\infty} \beta(t)\Psi(x(t)) = 0$ .*

**Proof of Lemma 3.5.** See the proof of Lemma 3.3 for the proof of part a). Since  $\int_0^{+\infty} \beta(t)\Psi(x(t))dt < +\infty$  (Lemma 3.1) and  $\dot{\beta} \leq k\beta$  (Assumption  $(\mathcal{H}_2)$ ), then  $\int_0^{+\infty} \dot{\beta}(t)\Psi(x(t))dt < +\infty$ , which implies the convergence of the function  $E_2$ . By Lemma 3.1, (ii),

$$\lim_{t \rightarrow +\infty} \int_0^t \Phi(x(s)) - \Phi(z) + \beta(s)\Psi(x(s))ds \text{ exists in } \mathbb{R},$$

thus

$$\lim_{t \rightarrow +\infty} E_2(t) = \lim_{t \rightarrow +\infty} \Phi(x(t)) + \beta(t)\Psi(x(t)) = \Phi(z).$$

Hence

$$\limsup_{t \rightarrow +\infty} \Phi(x(t)) \leq \lim_{t \rightarrow +\infty} \Phi(x(t)) + \beta(t)\Psi(x(t)) \leq \Phi(z).$$

Since (Lemma 3.4)

$$\liminf_{t \rightarrow +\infty} \Phi(x(t)) \geq \Phi(z),$$

we deduce

$$\lim_{t \rightarrow +\infty} \Phi(x(t)) = \Phi(z),$$

and

$$\lim_{t \rightarrow +\infty} \beta(t)\Psi(x(t)) = 0.$$

In view of Opial's lemma, since we already proved the convergence of  $\|x(t) - z\|$  for every  $z \in S = \operatorname{argmin} \Phi$ , the proof of Theorem 3.1 is finished with the following lemma.

**Lemma 3.6.** *If  $x(t_n) \rightharpoonup x_\infty$ , then  $x_\infty \in S = \operatorname{argmin}\{\Phi | \operatorname{argmin} \Psi\}$ .*

**Proof of Lemma 3.6.** By weak lower semicontinuity of the functions  $\Phi$  and  $\Psi$

$$\Psi(x_\infty) \leq \liminf_{n \rightarrow +\infty} \Psi(x(t_n)) = 0 \text{ thus } x_\infty \in \operatorname{argmin} \Phi.$$

$$\Phi(x_\infty) \leq \liminf_{n \rightarrow +\infty} \Phi(x(t_n)) = \Phi(z) \text{ thus } x_\infty \in \operatorname{argmin}\{\Phi | \operatorname{argmin} \Psi\}.$$

#### 4. MULTISCALE ASPECTS

In this section, we show that the system (where  $\beta(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ )

$$(MAG) \quad \dot{x}(t) + \partial \Phi(x(t)) + \beta(t)\partial \Psi(x(t)) \ni 0,$$

after time rescaling, can be equivalently rewritten as

$$(MAG)_\varepsilon \quad \dot{x}(t) + \partial \Psi(x(t)) + \varepsilon(t)\partial \Phi(x(t)) \ni 0,$$

with a positive control  $t \mapsto \varepsilon(t)$  that converges to 0 as  $t \rightarrow \infty$ .

By taking  $\Phi(x) = \|x\|^2/2$ ,  $(MAG)_\varepsilon$  system amounts to the  $(SDC)_\varepsilon$  system (steepest descent with control)

$$(SDC)_\varepsilon \quad \dot{x}(t) + \partial \Psi(x(t)) + \varepsilon(t)x(t) \ni 0.$$

The  $(SDC)_\varepsilon$  system plays an important role in optimization and game theory as well as in asymptotic control theory and the study of ill-posed problems. It can be viewed as a Tikhonov-like dynamical system. In the particular

setting of  $(SDC)_\varepsilon$ , the asymptotic convergence properties of the trajectories depend on whether  $\varepsilon(\cdot)$  is in  $L^1(0, \infty)$  or not.

For the case  $\varepsilon(\cdot) \notin L^1$ , the first general convergence results go back to [28] (based on previous work by [16]) and require in addition  $\varepsilon(\cdot)$  to be non-increasing. Under these conditions, each trajectory of  $(SDC)_\varepsilon$  converges strongly to  $\bar{x}$ , the point of minimal norm in  $\operatorname{argmin} \Psi$ . This case is often referred to as the slow parametrization case (slow convergence of  $\varepsilon(\cdot)$  to zero). In a recent contribution to this subject [20], it is proved that this convergence result still holds without assuming  $\varepsilon(\cdot)$  to be non-increasing.

By contrast, for the case  $\varepsilon(\cdot) \in L^1$ , each trajectory of  $(SDC)_\varepsilon$  weakly converges to some point in  $\operatorname{argmin} \Psi$  (which depends on the trajectory), a result which is in the line of Bruck theorem. This is usually referred to as the fast parametrization case.

Among the many papers devoted to  $(SDC)_\varepsilon$  and related systems, let us mention [6], [27], [23], [20]. In [7] the authors show similar properties concerning the second order system

$$(HBFC) \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Psi(x(t)) + \varepsilon(t)x(t) = 0$$

with  $\gamma > 0$ .

Let us now return to the connection with  $(MAG)$ . As we shall see, the  $(MAG)$  system studied in this paper leads to an equivalent  $(MAG)_\varepsilon$  system with a corresponding control  $\varepsilon(\cdot)$  which automatically satisfies

$$\varepsilon(\cdot) \notin L^1.$$

The strong convergence of the trajectories of  $(MAG)$  with

$$\Phi(x) = \frac{1}{2} \|x\|^2$$

and hence of  $(SDC)_\varepsilon$ , is a consequence of the strongly monotone case given by Theorem 2.2. When  $\Phi$  is not strongly monotone, the techniques of the preceding papers remain useless! Indeed, the study of  $(MAG)$  makes the situation clearer, and, for general  $\Phi$  and  $\Psi$ , allows to find conditions permitting to obtain (weak) convergence to a point in  $S = \operatorname{argmin}\{\Phi | \operatorname{argmin} \Psi\}$ . This is exactly condition  $(\mathcal{H}_1)$ , which, in terms of the function  $\varepsilon(\cdot)$  for the system  $(MAG)_\varepsilon$ , in classical cases, can be seen as an  $L^2$  integrability condition. For example, if  $\Psi(\cdot) = \frac{1}{2} \operatorname{dist}^2(\cdot, C)$ , the condition will turn to be exactly

$$\varepsilon(\cdot) \in L^2 \text{ and } \varepsilon(\cdot) \notin L^1.$$

Let us now state precisely the equivalence between the formulations  $(MAG)$  and  $(MAG)_\varepsilon$ , and also between  $(MAMI)$  and  $(MAMI)_\varepsilon$ .

**Lemma 4.1** (dictionary). *Let  $T_\beta$  and  $T_\varepsilon$  be two elements in  $(\mathbb{R}_+ \setminus \{0\}) \cup +\infty$ . Take two functions of class  $C^1$*

$$\begin{aligned} \beta : [0, T_\beta) &\rightarrow \mathbb{R}_+ \setminus \{0\}; \\ \varepsilon : [0, T_\varepsilon) &\rightarrow \mathbb{R}_+ \setminus \{0\}. \end{aligned}$$

*Define  $t_\beta : [0, T_\varepsilon) \rightarrow [0, T_\beta)$  and  $t_\varepsilon : [0, T_\beta) \rightarrow [0, T_\varepsilon)$  by*

$$\int_0^{t_\beta(t)} \beta(s) ds = t \text{ and } \int_0^{t_\varepsilon(t)} \varepsilon(s) ds = t.$$

*Assume that, for every  $t$ ,*

$$\varepsilon(t)\beta(t_\beta(t)) = 1.$$

*Then*

$$\begin{aligned} t_\varepsilon \circ t_\beta &= \operatorname{id}_{[0, T_\varepsilon)} \quad \text{and} \quad T_\varepsilon = \int_0^{T_\beta} \beta; \\ t_\beta \circ t_\varepsilon &= \operatorname{id}_{[0, T_\beta)} \quad \text{and} \quad T_\beta = \int_0^{T_\varepsilon} \varepsilon; \end{aligned}$$

*if  $x$  is a strong solution of*

$$(MAG) \quad \dot{x}(t) + \partial \Phi(x(t)) + \beta(t) \partial \Psi(x(t)) \ni 0,$$

*then  $x \circ t_\beta$  is a strong solution of*

$$(MAG)_\varepsilon \quad \dot{w}(t) + \varepsilon(t) \partial \Phi(w(t)) + \partial \Psi(w(t)) \ni 0;$$

*conversely, if  $w$  is a strong solution of  $(MAG)_\varepsilon$ , then  $w \circ t_\varepsilon$  is a strong solution of  $(MAG)$ .*

*Now, if  $x$  is a strong solution of*

$$(MAMI) \quad \dot{x}(t) + A(x(t)) + \beta(t) \partial \Psi(x(t)) \ni 0,$$

*then  $x \circ t_\beta$  is a strong solution of*

$$(MAMI)_\varepsilon \quad \dot{w}(t) + \varepsilon(t) A(w(t)) + \partial \Psi(w(t)) \ni 0;$$

*conversely, if  $w$  is a strong solution of  $(MAMI)_\varepsilon$ , then  $w \circ t_\varepsilon$  is a strong solution of  $(MAMI)$ .*

**Proof of Lemma 4.1.** Since  $\int_0^{t_\beta(t)} \beta(s)ds = t$ , then  $t_\beta(\cdot)$  is of class  $C^1$  and  $\dot{t}_\beta(t)\beta(t_\beta(t)) = 1$ . Thus  $\dot{t}_\beta(t) = \varepsilon(t)$ , and  $t_\beta(t_\varepsilon(t)) = \int_0^{t_\varepsilon(t)} \varepsilon(s)ds = t$ . We are now prepared to make the change of variable associated with function  $t_\beta(\cdot)$ . Let  $x(\cdot)$  be a strong solution of (MAG) and write the system (MAG) at the point  $t_\beta(t)$ :

$$\dot{x}(t_\beta(t)) + \partial\Phi(x(t_\beta(t))) + \beta(t_\beta(t))\partial\Psi(x(t_\beta(t))) \ni 0.$$

After multiplication by  $\dot{t}_\beta(t)$  we get

$$\dot{t}_\beta(t)\dot{x}(t_\beta(t)) + \dot{t}_\beta(t)\partial\Phi(x(t_\beta(t))) + \dot{t}_\beta(t)\beta(t_\beta(t))\partial\Psi(x(t_\beta(t))) \ni 0.$$

Set  $w = x \circ t_\beta$ . The map  $w = x \circ t_\beta$  is absolutely continuous, and according to  $\dot{w}(t) = \dot{t}_\beta(t)\dot{x}(t_\beta(t))$ ,  $\dot{t}_\beta(t) = \varepsilon(t)$  and  $\dot{t}_\beta(t)\beta(t_\beta(t)) = 1$ , we obtain

$$\dot{w}(t) + \varepsilon(t)\partial\Phi(w(t)) + \partial\Psi(w(t)) \ni 0,$$

that's  $(MAG)_\varepsilon$ . The notion of strong solution, as given in Section 3, remains valid. Similar arguments work with (MAMI).  $\square$

Accordingly, all our results can be written for the systems  $(MAG)_\varepsilon$  and  $(MAMI)_\varepsilon$ . Before doing so, let us analyse the corresponding assumption  $(\mathcal{H}_1)$ .

*Remark 4.1.* About Assumption  $(\mathcal{H}_1)$ . Take  $\beta(\cdot)$  and  $\varepsilon(\cdot)$  as in Lemma 4.1, with  $T_\beta = T_\varepsilon = +\infty$ . Then, by making the change of variable  $t = t_\beta(s)$  in the following integral, and by using  $\dot{t}_\beta(t)\beta(t_\beta(t)) = 1$ , we obtain

$$\begin{aligned} \int_0^{+\infty} \beta(t) \left( \Psi^*\left(\frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right) \right) dt &= \int_0^{+\infty} \dot{t}_\beta(s)\beta(t_\beta(s)) \left( \Psi^*\left(\frac{p}{\beta(t_\beta(s))}\right) - \sigma_C\left(\frac{p}{\beta(t_\beta(s))}\right) \right) ds \\ &= \int_0^{+\infty} \Psi^*(\varepsilon(s)p) - \sigma_C(\varepsilon(s)p) ds. \end{aligned}$$

Thus condition  $(\mathcal{H}_1)$  becomes

$$\forall p \in R(N_C) \quad \int_0^{+\infty} \Psi^*(\varepsilon(t)p) - \sigma_C(\varepsilon(t)p) dt < +\infty.$$

In particular, when  $\Psi(x) = \frac{1}{2} \text{dist}^2(x, C)$ , then  $\Psi^*(x) = \frac{1}{2} \|x\|^2 + \sigma_C(x)$  and

$$(\mathcal{H}_1) \iff \int_0^{+\infty} \frac{1}{\beta(t)} dt < +\infty \iff \int_0^{+\infty} \varepsilon(t)^2 dt < +\infty. \quad \square$$

**Theorem 4.1.** Let  $\Phi$  and  $\Psi$  satisfy the assumptions of Theorem 3.1. Let us assume that,

- $(\mathcal{H}_1)_\varepsilon \quad \forall p \in R(N_C), \int_0^{+\infty} \Psi^*(\varepsilon(t)p) - \sigma_C(\varepsilon(t)p) dt < +\infty.$
- $(\mathcal{H}_2)_\varepsilon \quad \varepsilon(\cdot)$  is a non increasing function of class  $C^1$ , such that  $\lim_{t \rightarrow +\infty} \varepsilon(t) = 0$ ,  $\int_0^{+\infty} \varepsilon(t) dt = +\infty$ , and for some  $k \geq 0$ ,  $-k\varepsilon^2 \leq \dot{\varepsilon}$ .

Let  $x(\cdot)$  be a strong solution of  $(MAG)_\varepsilon$ . Then:

- (i) weak convergence  $\exists x_\infty \in S = \text{argmin}\{\Phi | \text{argmin}\Psi\}, \quad w - \lim_{t \rightarrow +\infty} x(t) = x_\infty;$
- (ii) minimizing properties  $\lim_{t \rightarrow +\infty} \Psi(x(t)) = 0;$   
 $\lim_{t \rightarrow +\infty} \Phi(x(t)) = \min \Phi |_{\text{argmin}\Psi};$
- (iii)  $\forall z \in S \quad \lim_{t \rightarrow +\infty} \|x(t) - z\| \text{ exists };$
- (iv) estimations  $\lim_{t \rightarrow +\infty} \frac{1}{\varepsilon(t)} \Psi(x(t)) = 0;$   
 $\int_0^{+\infty} \Psi(x(t)) dt < +\infty;$   
 $\limsup_{\tau \rightarrow +\infty} \int_0^\tau \varepsilon(t) \left( \Phi(x(t)) - \min \Phi |_{\text{argmin}\Psi} \right) dt < +\infty.$

**Proof of Theorem 4.1** Equivalence between Theorem 4.1 and Theorem 3.1 is a consequence of Lemma 4.1, Remark 4.1 and the equivalent formulation of condition  $(\mathcal{H}_2)$  as given below:

Write  $(\mathcal{H}_2)$  at the point  $t_\beta(t)$

$$\dot{\beta}(t_\beta(t)) \leq k\beta(t_\beta(t))$$

and multiply by  $\dot{t}_\beta(t)$  (which is nonnegative)

$$\dot{\beta}(t_\beta(t))\dot{t}_\beta(t) \leq k\beta(t_\beta(t))\dot{t}_\beta(t).$$

Owing to  $\beta(t_\beta(t))\dot{t}_\beta(t) = 1$ , we get

$$\frac{d}{dt}\beta(t_\beta(t)) \leq k,$$

which, by using  $\varepsilon(t)\beta(t_\beta(t)) = 1$ , finally yields

$$\frac{d}{dt}\left(\frac{1}{\varepsilon(t)}\right) = -\frac{\dot{\varepsilon}(t)}{\varepsilon^2(t)} \leq k$$

that's  $(\mathcal{H}_2)_\varepsilon$ .  $\square$

## 5. FURTHER CONVERGENCE RESULTS, WITHOUT THE GROWTH CONDITION $\dot{\beta} \leq k\beta$

It worth noticing that, in the proof of weak convergence theorem 3.1, the growth condition  $\dot{\beta} \leq k\beta$  is only used ultimately, in lemma 3.5. It allows to develop an energetic argument involving  $E_2$ . As a byproduct of this technic, one obtains too that the trajectories have finite kinetic energy:  $\int_0^{+\infty} |\dot{x}(t)|^2 dt < +\infty$ . It is a natural question to ask whether theorem 3.1 still holds without this growth condition. As a positive answer, when  $\mathcal{H}$  is a finite dimensional space, we are going to prove that convergence of the trajectories holds true without this growth condition. Indeed, having in view applications to possibly infinite dimensional problems (PDE's, control), we consider the more general situation with functionals  $\Phi$  or  $\Psi$  which are supposed to be inf-compact. Let us recall that a function  $\phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$  is said to be inf-compact if, for every  $R > 0$  and  $l \in \mathbb{R}$  the lower level set

$$\{x \in \mathcal{H} : |x| \leq R, \phi(x) \leq l\} \text{ is relatively compact in } \mathcal{H}.$$

Our proof is close to Baillon-Cominetti argument developed in [11] theorem 2.1. It relies mainly on topological arguments. It turns out that it is more convenient to work with the  $(MAG)_\varepsilon$  version of our dynamics (see section 4)

$$(MAG)_\varepsilon \quad \dot{x}(t) + \partial\Psi(x(t)) + \varepsilon(t)\partial\Phi(x(t)) \ni 0,$$

with a positive control  $t \mapsto \varepsilon(t)$  that converges to 0 as  $t \rightarrow \infty$ . Then, the result can be easily converted in terms of  $(MAG)$ .

**Theorem 5.1.** *Let  $\Phi$  and  $\Psi$  satisfy the assumptions of Theorem 3.1. Let us assume that,*

- $(\mathcal{H}_1)_\varepsilon$   $\forall p \in R(N_C), \int_0^{+\infty} \Psi^*(\varepsilon(t)p) - \sigma_C(\varepsilon(t)p) dt < +\infty$ .
- $(\mathcal{H}_2)_\varepsilon$   $\varepsilon(\cdot)$  is a non increasing function of class  $C^1$ , such that  $\lim_{t \rightarrow +\infty} \varepsilon(t) = 0$ ,  $\int_0^{+\infty} \varepsilon(t) dt = +\infty$ .
- $(\mathcal{H}_3)$   $\Phi$  or  $\Psi$  is inf-compact and  $S$  is bounded.

*Let  $x(\cdot)$  be a trajectory solution of  $(MAG)_\varepsilon$ . Then the following convergence result holds*

$$\exists x_\infty \in S = \operatorname{argmin}\{\Phi | \operatorname{argmin}\Psi\}, \quad \lim_{t \rightarrow +\infty} x(t) = x_\infty$$

*the convergence being taken in the strong sense when  $\Psi$  is inf-compact and in the weak sense when  $\Phi$  is inf-compact.*

**Proof of Theorem 5.1** In view of Lemma 4.1, one can easily check that the assumptions of Theorem 3.1 are satisfied, except for the growth condition  $-k\varepsilon^2 \leq \dot{\varepsilon} \quad / \quad \dot{\beta} \leq k\beta$ . So all the results in the proof of Theorem 3.1 hold, except for Lemma 3.5.

*Weak convergence of the trajectories.* We know that, for every  $z$  in  $S = \operatorname{argmin}\{\Phi | \operatorname{argmin}\Psi\}$ ,  $\lim_{t \rightarrow +\infty} \|x(t) - z\|$  exists (Lemma 3.1). In order to obtain weak convergence of the trajectory we use Opial lemma. Thus, we just need to prove that every weak-limit point of  $x(\cdot)$  belongs to  $S$ . In turn, this will be a straight consequence of

$$(30) \quad \operatorname{dist}(x(t), S) \rightarrow 0 \text{ as } t \rightarrow +\infty$$

and of the weak lower semicontinuity of the convex continuous function  $\operatorname{dist}(\cdot, S)$ . In order to prove (30) let us introduce  $h(t) := \frac{1}{2} \operatorname{dist}(x(t), S)^2$ , and estimate  $\dot{h}(t)$ . Set  $D_S(x) = \frac{1}{2} \operatorname{dist}(x, S)^2$ , which is convex and differentiable with  $\nabla D_S(x) = x - P_S(x)$ , where  $P_S$  is the projection of  $x$  onto  $S$ . Then for almost every  $t$ ,

$$(31) \quad \dot{h}(t) = \langle \dot{x}(t), x(t) - P_S x(t) \rangle.$$

Let us rewrite  $(MAG)_\varepsilon$  as

$$(32) \quad \dot{x}(t) + \varepsilon(t)\partial\phi_t(x(t)) \ni 0$$

with

$$\phi_t(x) := \Phi(x) + \frac{1}{\varepsilon(t)}\Psi(x).$$



For almost every  $t$ , by using (31), (32) and convexity of  $\phi_t$

$$\begin{aligned}\dot{h}(t) &= \varepsilon(t) \langle \partial \phi_t(x(t)), P_S x(t) - x(t) \rangle \\ &\leq \varepsilon(t) (\phi_t(P_S x(t)) - \phi_t(x(t))).\end{aligned}$$

Let us notice that  $\phi_t(P_S x(t))$  is independent of  $t$  (recall that  $\Psi = 0$  on  $S$ ). It is equal to the optimal value of the limit equilibrium problem, we set

$$v_{opt} := \Phi(z) \text{ for all } z \in S = \operatorname{argmin}\{\Phi | \operatorname{argmin}\Psi\}.$$

Hence,

$$(33) \quad \dot{h}(t) + \varepsilon(t) (\phi_t(x(t)) - v_{opt}) \leq 0.$$

Integrating (33), and using that the function  $h$  is bounded, we get

$$\int_0^{+\infty} \varepsilon(t) (\phi_t(x(t)) - v_{opt}) dt < +\infty,$$

and since  $\int_0^{+\infty} \varepsilon(t) dt = +\infty$  we deduce that

$$\liminf_{t \rightarrow +\infty} (\phi_t(x(t)) - v_{opt}) \leq 0.$$

By definition of  $\liminf$ , this implies the existence of a sequence  $t_k \rightarrow +\infty$  such that  $\lim_{k \rightarrow +\infty} \phi_{t_k}(x(t_k)) \leq v_{opt}$ . From boundedness of the trajectory  $x(\cdot)$  and inf-compactness assumption  $(\mathcal{H}_3)$ , we deduce that the sequence  $(x(t_k))$  is relatively compact in  $\mathcal{H}$ . On the other hand, as  $t \rightarrow +\infty$ , the sequence of functions  $\phi_t$  converges increasingly to  $\Phi + \delta_C$  where  $C = \operatorname{argmin}\Psi$ . By a classical result, monotone convergence implies epiconvergence ( $\Gamma$ -convergence) (see [1] theorem 2.40) with its accompanying variational properties. Still denoting  $x(t_k)$  a subsequence which converges to some  $x_\infty$ , we obtain that  $x_\infty \in \operatorname{argmin}\{\Phi + \delta_C\} = S$ . From

$$h(t_k) = \frac{1}{2} \operatorname{dist}(x(t_k), S)^2 \leq \frac{1}{2} |x(t_k) - x_\infty|^2,$$

we obtain the convergence of  $h(t_k)$  to zero. Thus, we have established the existence of a sequence  $t_k \rightarrow +\infty$  such that  $h(t_k)$  tends to zero. The proof will be completed by proving that  $\lim_{t \rightarrow +\infty} h(t)$  exists. Let us return to (33), introduce the set

$$S_t := \{x : \phi_t(x) \geq v_{opt}\}$$

and observe that

$$x(t) \in S_t \Rightarrow \dot{h}(t) \leq 0.$$

Set

$$(34) \quad b(t) := \sup\{D_S(x) : \phi_t(x) \leq v_{opt}\}.$$

Thus,

$$D_S(x) > b(t) \Rightarrow x \in S_t$$

and

$$h(t) = D_S(x(t)) > b(t) \Rightarrow x(t) \in S_t \Rightarrow \dot{h}(t) \leq 0.$$

Let us show that  $\lim_{t \rightarrow +\infty} b(t) = 0$ , with  $b(t)$  defined in (34). Let us argue by contradiction and suppose that there exists  $\varepsilon > 0$ ,  $t_k \rightarrow +\infty$ , and  $x_k \in \mathcal{H}$  with  $\phi_{t_k}(x_k) \leq v_{opt}$  and  $\operatorname{dist}(x_k, S) > \varepsilon$ . By convex combination of  $x_k$  with  $P_S(x_k)$  we can assume, without loss of generality, that  $\operatorname{dist}(x_k, S) = \varepsilon$ . The set  $S$  has been assumed to be bounded. It follows that the sequence  $(x_k)$  is bounded. From the inf-compactness assumption  $(\mathcal{H}_3)$  and  $\phi_{t_k}(x_k) \leq v_{opt}$  we deduce that the sequence  $(x_k)$  is relatively compact. Thus, passing to a subsequence, we can assume that  $x_k$  strongly converges to some  $\bar{x}$ . From  $\phi_{t_k}(x_k) \leq v_{opt}$  and the preceding epi-convergent argument we get  $\bar{x} \in S$ . This clearly contradicts the fact that  $\operatorname{dist}(x_k, S) = \varepsilon$  and  $x_k$  strongly converges to  $\bar{x}$ .

Collecting the preceding results, we are in position to apply the following lemma to obtain that  $\lim_{t \rightarrow +\infty} h(t)$  exists, and hence conclude to the weak convergence. We give the proof of the lemma for the convenience of the reader.

**Lemma 5.1.** ([11] lemma 2.2) *Let  $h(\cdot)$  and  $b(\cdot)$  be two real-valued functions defined on  $(0, +\infty)$  with  $h(\cdot)$  absolutely continuous and nonnegative. Let us assume*

- (i)  $b(t) \rightarrow 0$  as  $t \rightarrow +\infty$ ;
- (ii) *there exist a set  $N \subset (0, +\infty)$  of zero Lebesgue measure such that  $h(t) > b(t) \Rightarrow \dot{h}(t) \leq 0$  for all  $t \notin N$ .*

*Then  $h(t)$  has a limit for  $t \rightarrow +\infty$ .*

**Proof of Lemma 5.1.** Replacing  $b(t)$  by  $\sup\{b(s) : s \geq t\}$ , we can assume, without loss of generality, that  $b(\cdot)$  is nonincreasing. Since by assumption  $h(\cdot)$  is nonincreasing when above  $b(\cdot)$ , it follows easily that, if  $h(s) \leq b(s)$  for a given  $s \geq 0$ , then  $h(t) \leq b(s)$  for all  $t \geq s$ .

Let us examine the two situations: Either there exists some  $\bar{s}$  with  $h(t) > b(t)$  for all  $t \geq \bar{s}$ , then  $h(\cdot)$  is nonincreasing over  $[\bar{s}, +\infty)$  and therefore it converges when  $t \rightarrow +\infty$ . Or there exists a sequence  $s_k \rightarrow +\infty$  such that  $h(s_k) \leq b(s_k)$

for all  $k \in \mathbb{N}$ . By using the above remark, we deduce  $0 \leq h(t) \leq b(s_k)$  for all  $t \geq s_k$ . Since  $b(s_k) \rightarrow 0$ , we conclude that  $h(t) \rightarrow 0$  as  $t \rightarrow +\infty$ .  $\square$

*Strong convergence when  $\Psi$  is inf-compact.* Since the trajectory  $x(\cdot)$  is bounded, and since  $\Psi(x(t)) \rightarrow 0$ , we conclude that the trajectory  $x(\cdot)$  is relatively compact. It weakly converges, thus strongly converges.  $\square$

## 6. APPLICATIONS

### 6.1. Coupled gradient systems.

- $\mathcal{H} = \mathcal{X}_1 \times \mathcal{X}_2$  the cartesian product of two Hilbert spaces,  $x = (x_1, x_2)$ .
- $\Phi(x) = f_1(x_1) + f_2(x_2) + \phi(x_1, x_2)$ ,  $f_1 \in \Gamma_0(\mathcal{X}_1)$ ,  $f_2 \in \Gamma_0(\mathcal{X}_2)$  are closed convex functions,  $\phi : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathbb{R}$  is a smooth convex coupling function.
- $\Psi(x) = \frac{1}{2} \|L_1 x_1 - L_2 x_2\|^2$ ,  $L_1 \in L(\mathcal{X}_1, \mathcal{Z})$  and  $L_2 \in L(\mathcal{X}_2, \mathcal{Z})$  are linear continuous operators acting respectively from  $\mathcal{X}_1$  and  $\mathcal{X}_2$  into a third Hilbert space  $\mathcal{Z}$ .

$$(MAG) \quad \dot{x}(t) + \partial\Phi(x(t)) + \beta(t)\partial\Psi(x(t)) \ni 0$$

$$\Updownarrow$$

$$\begin{cases} \dot{x}_1(t) + \partial f_1(x_1(t)) + \nabla_{x_1} \phi(x_1(t), x_2(t)) + \beta(t)L_1^t(L_1 x_1(t) - L_2 x_2(t)) \ni 0 \\ \dot{x}_2(t) + \partial f_2(x_2(t)) + \nabla_{x_2} \phi(x_1(t), x_2(t)) + \beta(t)L_2^t(L_1 x_1(t) - L_2 x_2(t)) \ni 0 \end{cases}$$

*Claim:* Under ad hoc conditions on the convergence of  $\beta(t)$  to  $+\infty$  as  $t$  goes to  $+\infty$ ,  $x(t) = (x_1(t), x_2(t)) \rightarrow x_\infty = (x_{1,\infty}, x_{2,\infty})$  where  $(x_{1,\infty}, x_{2,\infty})$  is a solution of

$$\min \{f_1(x_1) + f_2(x_2) + \phi(x_1, x_2) : L_1 x_1 - L_2 x_2 = 0\}.$$

The dynamical approach to such equilibria is a central question in various domains:

- In game theory (see subsection 5.3 for further details), such solution is a Nash equilibrium of the potential game (here team game) where we have two players 1 and 2 with respective static loss functions:

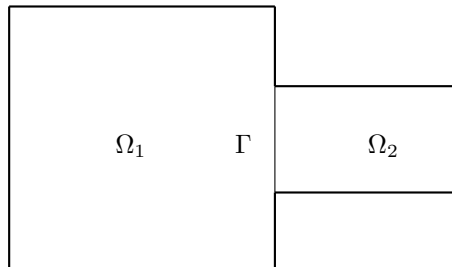
$$\begin{cases} F_1 : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow F_1(x_1, x_2) = f_1(x_1) + \phi(x_1, x_2) \text{ if } L_1 x_1 - L_2 x_2 = 0, +\infty \text{ elsewhere} \\ F_2 : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow F_2(x_1, x_2) = f_2(x_2) + \phi(x_1, x_2) \text{ if } L_1 x_1 - L_2 x_2 = 0, +\infty \text{ elsewhere} \end{cases}$$

The  $f_i(\cdot)$  represent the individual payoff of the agents,  $\phi(\cdot, \cdot)$  is their joint payoff and  $L_1 x_1 - L_2 x_2 = 0$  is a normalized form of a constraint expressing in a coupled way, for example some limitation of the resources.

In that case, a discrete version of our dynamical system provides a best response dynamical approach to such equilibria.

- In optimal control theory, the constraint  $L_1 x_1 = L_2 x_2$  is the state equation which relates the state variable  $x_1$  to the corresponding control  $x_2$ . In that case, the criteria which is to minimize naturally splits into the sum of two costs: the cost to be far from a desired state and the cost of the control.
- Indeed, most applications deal with some modelling and algorithmic aspects and are directly connected to the corresponding discrete dynamical systems. In that case, the continuous version offers a flexible tool allowing a deeper understanding of the mathematical properties. Moreover it may suggest extensions and connections with other domains. We now consider various applications including domain decomposition methods in PDE's, best response dynamics in potential games and optimal control.

### 6.2. Domain decomposition for PDE's.



*Dirichlet problem* on  $\Omega$ :  $h \in L^2(\Omega)$  given, find  $u : \Omega \rightarrow \mathbb{R}$  solution of

$$\begin{cases} -\Delta u = h & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*Variational formulation*:

$$\min \left\{ \frac{1}{2} \int_{\Omega_1} |\nabla u_1|^2 + \frac{1}{2} \int_{\Omega_2} |\nabla u_2|^2 - \int_{\Omega} h u : u_1 \in \mathcal{X}_1, u_2 \in \mathcal{X}_2, [u] = 0 \text{ on } \Gamma \right\}.$$

$$\min \{ f_1(u_1) + f_2(u_2) : u_1 \in \mathcal{X}_1, u_2 \in \mathcal{X}_2, L_1(u_1) - L_2(u_2) = 0 \}.$$

- $\mathcal{X}_i = \{u \in H^1(\Omega_i), u = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}, u = u_i \text{ on } \Omega_i, i = 1, 2.$
- $f_i(u_i) = \frac{1}{2} \int_{\Omega_i} |\nabla u_i|^2 - \int_{\Omega_i} h u_i.$
- $L_i : H^1(\Omega_i) \rightarrow \mathcal{Z} = L^2(\Gamma)$  is the trace operator,  $i = 1, 2.$
- $[u]$  = jump of  $u$  through the interface  $\Gamma$ .

$$\begin{cases} -\Delta \frac{\partial u_1}{\partial t} - \Delta u_1 = h_1 & \text{on } \Omega_1 \\ -\Delta \frac{\partial u_2}{\partial t} - \Delta u_2 = h_2 & \text{on } \Omega_2 \\ \frac{\partial u_1(t)}{\partial \nu_1} + \frac{\partial u_1}{\partial \nu_1}(t) - \beta(t)[u(t)] = 0 & \text{on } \Gamma \\ \frac{\partial u_2(t)}{\partial \nu_2} + \frac{\partial u_2}{\partial \nu_2}(t) + \beta(t)[u(t)] = 0 & \text{on } \Gamma \end{cases}$$

Discrete version: Alternating Algorithm with Dirichlet-Neumann transmission conditions:

$$(u_{1,k}, u_{2,k}) \rightarrow (u_{1,k+1}, u_{2,k}) \rightarrow (u_{1,k+1}, u_{2,k+1}) \text{ with } \beta_k \rightarrow +\infty.$$

$$\begin{cases} -(1+\alpha)\Delta u_{1,k+1} = h_1 - \alpha\Delta u_{1,k} & \text{on } \Omega_1 \\ (1+\alpha)\frac{\partial u_{1,k+1}}{\partial \nu_1} + \beta_k u_{1,k+1} = \beta_k u_{2,k} + \alpha\frac{\partial u_{1,k}}{\partial \nu_1} & \text{on } \Gamma \\ u_{1,k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ -(1+\alpha)\Delta u_{2,k+1} = h_2 - \alpha\Delta u_{2,k} & \text{on } \Omega_2 \\ (1+\alpha)\frac{\partial u_{2,k+1}}{\partial \nu_2} + \beta_k u_{2,k+1} = \beta_k u_{1,k+1} + \alpha\frac{\partial u_{2,k}}{\partial \nu_2} & \text{on } \Gamma \\ u_{2,k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \end{cases}$$

**6.3. Best response dynamics for potential games.** Consider the potential game (here team game) where we have two players 1 and 2 with respective static loss functions:

$$\begin{cases} F_1 : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow F_1(x_1, x_2) = f_1(x_1) + \phi(x_1, x_2) & \text{if } L_1 x_1 - L_2 x_2 = 0, +\infty \text{ elsewhere} \\ F_2 : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow F_2(x_1, x_2) = f_2(x_2) + \phi(x_1, x_2) & \text{if } L_1 x_1 - L_2 x_2 = 0, +\infty \text{ elsewhere} \end{cases}$$

The Nash equilibria are the solutions of the convex optimization problem:

$$\min \{ f_1(x_1) + f_2(x_2) + \phi(x_1, x_2) : L_1 x_1 - L_2 x_2 = 0 \}.$$

The (MAG) dynamical approach to such equilibria leads to consider the following nonautonomous coupled dynamical system:

$$\begin{cases} \dot{x}_1(t) + \partial f_1(x_1(t)) + \nabla_{x_1} \phi(x_1(t), x_2(t)) + \beta(t) L_1^t (L_1 x_1(t) - L_2 x_2(t)) \ni 0 \\ \dot{x}_2(t) + \partial f_2(x_2(t)) + \nabla_{x_2} \phi(x_1(t), x_2(t)) + \beta(t) L_2^t (L_1 x_2(t) - L_2 x_1(t)) \ni 0 \end{cases}$$

Discretization of this continuous dynamics leads to the following “Best reply dynamic with cost to change”, (players 1 and 2 play alternatively):

$$(x_{1,k}, x_{2,k}) \rightarrow (x_{1,k+1}, x_{2,k}) \rightarrow (x_{1,k+1}, x_{2,k+1}) \text{ with } \beta_k \rightarrow +\infty.$$

$$\begin{cases} x_{1,k+1} = \operatorname{argmin}\{f_1(\xi) + \phi(\xi, x_{2,k}) + \frac{\beta_k}{2} \|L_1 \xi - L_2 x_{2,k}\|^2 + \frac{\alpha}{2} \|\xi - x_{1,k}\|_{\mathcal{X}}^2 : \xi \in \mathcal{X}\} \\ x_{2,k+1} = \operatorname{argmin}\{f_2(\eta) + \phi(x_{1,k+1}, \eta) + \frac{\beta_k}{2} \|L_1 x_{1,k+1} - L_2 \eta\|^2 + \frac{\nu}{2} \|\eta - x_{2,k}\|_{\mathcal{Y}}^2 : \eta \in \mathcal{Y}\} \end{cases}$$

For the cognitive and psychological interpretation of the costs to move term, consult [4], [5], [8]. This discrete dynamic will be thoroughly studied in a further study.

#### 6.4. Optimal control.

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