ON THE INVERSE MAPPING CLASS MONOIDS

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R. KAROUI AND V. V. VERSHININ

Abstract. Braid groups and mapping class groups have many features in common. Similarly to the notion of inverse braid monoid inverse mapping class monoid is defined. It concerns surfaces with punctures, but among given n punctures several can be omitted. This corresponds to braids where the number of strings is not fixed. In the paper we give the analogue of the Dehn-Nilsen-Baer theorem, propose a presentation of the inverse mapping class monoid for a punctured sphere and study the word problem. This shows that certain properties and objects based on mapping class groups may be extended to the inverse mapping class monoids. We also give an analogues of Artin presentation with two generators.

Contents

1. Introduction

Mapping class group is an important object in Topology, Complex Analysis, Algebraic Geometry and other domains. It is a lucky case when the method of Algebraic Topology works perfectly well, the application of the functor of fundamental group completely solves the topological problem: group of isotopy classes of homeomorphisms is described in terms of automorphisms of the fundamental group of the corresponding surface, as states the Dehn-Nilsen-Baer theorem, see [11], for example.

Let $S_{g,b,n}$ be an oriented surface of the genus g with b boundary components and a set Q_n of n fixed points. Consider the group $\mathrm{Homeo}(S_{g,b,n})$ of orientation preserving self-homeomorphisms of $S_{g,b,n}$ which fix pointwise the boundary (if it exists) and map the set Q_n into itself. Orientation reversing homeomorphisms also possible to consider, see [4], for example, but for the simplicity of exposition we restrict ourselves to orientation preserving case. Let $\mathrm{Homeo}^0(S_{g,b,n})$ be the normal subgroup of self-homeomorphisms of $S_{g,b,n}$ which are isotopic to identity. Then the *mapping class group* $\mathcal{M}_{g,b,n}$ is defined as a quotient group

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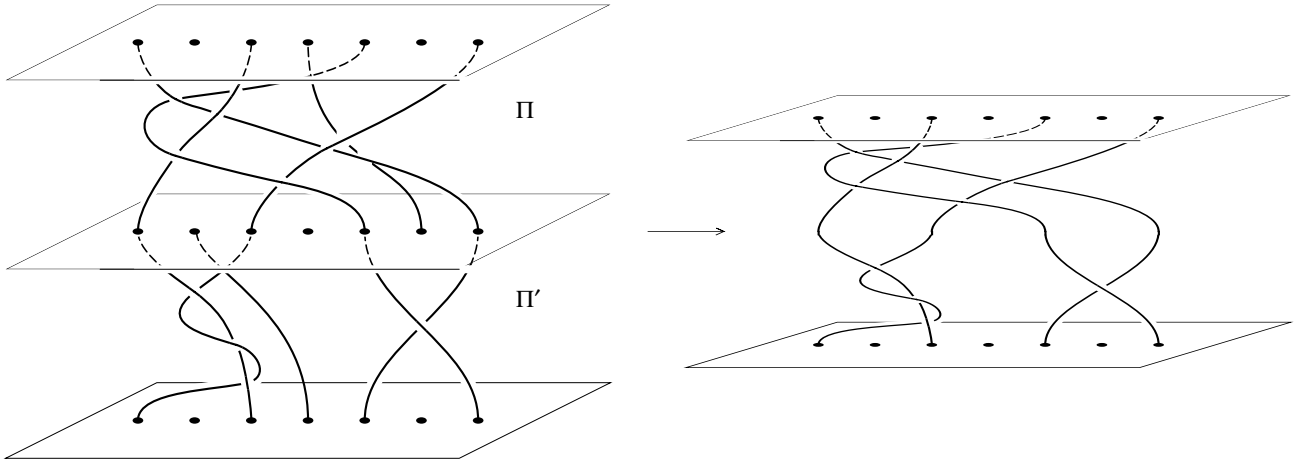

FIGURE 2.1

$$\mathcal{M}_{g,b,n} = \mathrm{Homeo}(S_{g,b,n})/\,\mathrm{Homeo}^0(S_{g,b,n})$$

These groups are connected closely with braid groups. In [16] W. Magnus interpreted the braid group as the mapping class group of a punctured disc with the fixed boundary. Braid groups have a variety of generalizations, see [25], for example. One of generalizations is the *inverse braid monoid* IB_n constructed by D. Easdown and T. G. Lavers [5].

The notion of *inverse semigroup* was introduced by V. V. Wagner in 1952 [27]. By definition it means that for any element a of a semigroup (monoid) M there exists a unique element b (which is called *inverse*) such that

$$a = aba$$

and

$$b = bab.$$

The typical example of an inverse monoid is a monoid of partial (defined on a subset) injections of a set. For a finite set this gives us the notion of a *symmetric inverse monoid* I_n which generalizes and includes the classical symmetric group Σ_n. A presentation of symmetric inverse monoid was obtained by L. M. Popova [22], see also formulas (2.1), (2.3-2.4) below.

2. Inverse braid monoids and inverse mapping class monoids

Inverse braid monoid arises from a very natural operation on braids: deleting several strands. By the application of this procedure to braids in Br_n we get *partial* braids [5]. Applying the standard procedure of multiplication of braids (concatenation) to partial braids we need to eliminate all not complete strands, as it is shown at Figure 2.1. If all strands are eliminated we get an *empty* braid. The set of all partial braids with this operation forms an inverse braid monoid IB_n.

Usually the braid group Br_n is given by the following Artin presentation [1]. It has the generators σ_i, $i = 1, ..., n-1$, and two types of relations:

$$\begin{cases} \sigma_i\sigma_j & = \sigma_j\,\sigma_i, \quad \text{if } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i & = \sigma_{i+1}\sigma_i\sigma_{i+1}\,. \end{cases} \tag{2.1}$$

There exist other presentations of the braid group. Let

$$\sigma = \sigma_1\sigma_2\ldots\sigma_{n-1},$$

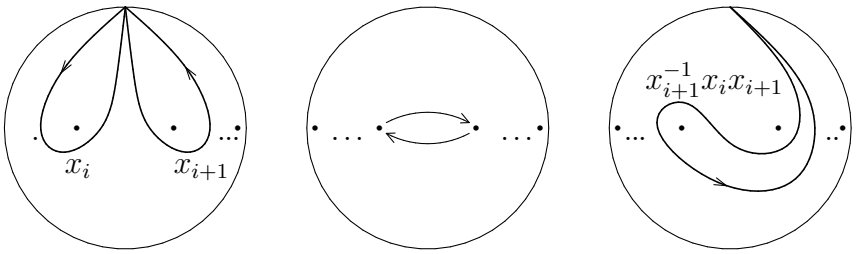

FIGURE 2.2

then the group Br_n is generated by σ_1 and σ because

$$\sigma_{i+1} = \sigma^i \sigma_1 \sigma^{-i}, \quad i = 1, \dots n-2.$$

The relations for the generators σ_1 and σ are the following

$$\begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} & = \sigma^i\sigma_1\sigma^{-i}\sigma_1 \ \text{ for } \ 2 \le i \le n/2, \\ \sigma^n & = (\sigma\sigma_1)^{n-1}. \end{cases} \tag{2.2}$$

The presentation (2.2) was given by Artin in the initial paper [1]. This presentation was also mentioned in the books by F. Klein [13] and by H. S. M. Coxeter and W. O. J. Moser [3].

Classical braid group Br_n can be defined also as the mapping class group of a disc D^2 with n points deleted (or fixed) and with its boundary fixed, or as the subgroup of the automorphism group of a free group $\operatorname{Aut} F_n$, generated by the following automorphisms:

$$\begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_{i+1}^{-1}x_ix_{i+1}, \\ x_j & \mapsto x_j, j \neq i, i+1. \end{cases}$$

Geometrically this action is depicted in Figure 2.2, where x_i correspond to the canonical loops on D^2 which form the generators of the fundamental group of the punctured disc.

This topological interpretation of the braid group was continued in [26] for the inverse braid monoid. Let Q_n the set of n fixed points of a disc D^2. The fundamental group of D^2 with these points deleted is isomorphic to F_n. Consider homeomorphisms of D^2 onto a copy of the same disc with the condition that only k points of Q_n, $k \le n$ (say $i_1, \dots, i_k$) are mapped bijectively onto the k points (say $j_1, \dots, j_k$) of the second copy of D^2, k varies from 0 to n and n is fixed. Consider the isotopy classes of such homeomorphisms and denote the set of them by $IM_n(D^2)$. Evidently it is a monoid.

Theorem 2.1. [26] *The monoids IB_n and $IM_n(D^2)$ are isomorphic.*

The following presentation for the inverse braid monoid was obtained in [5]. It has the generators σ_i, σ_i^{-1}, $i = 1, \dots, n-1$, ϵ, and relations

$$\begin{cases} \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1, \ \text{for} \ \ \text{all } i, \\ \epsilon\sigma_i = \ \sigma_i\epsilon \ \ \text{for } i \ge 2, \\ \epsilon\sigma_1\epsilon = \sigma_1\epsilon\sigma_1\epsilon = \epsilon\sigma_1\epsilon\sigma_1, \\ \epsilon = \epsilon^2 = \epsilon\sigma_1^2 = \sigma_1^2\epsilon \end{cases} \tag{2.3}$$

and the braid relations (2.1).

Geometrically the generator ϵ means that the first strand in the trivial braid is absent.

If we replace the first relation in (2.3) by the following set of relations

$$\sigma_i^2 = 1, \text{ for all } i, \tag{2.4}$$

and delete the superfluous relations

$$\epsilon = \epsilon\sigma_1^2 = \sigma_1^2\epsilon,$$

we get a presentation of the symmetric inverse monoid I_n [22] . We also can simply add the relations (2.4) if we do not worry about redundant relations. We get a canonical map [5]

$$\tau_n : IB_n \to I_n, \tag{2.5}$$

which is a natural extension of the corresponding map for the braid and symmetric groups.

More balanced relations for the inverse braid monoid were obtained in [9]. Let ϵ_i denote the trivial braid with ith strand deleted, formally:

$$\begin{cases} \epsilon_1 & = \epsilon, \\ \epsilon_{i+1} & = \sigma_i^{\pm 1}\epsilon_i\sigma_i^{\pm 1}. \end{cases}$$

The generators are: σ_i, σ_i^{-1}, $i = 1, \ldots, n-1$, ϵ_i, $i = 1, \ldots, n$, and relations are the following:

$$\begin{cases} \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1, \text{ for all } i, \\ \epsilon_j\sigma_i = \sigma_i\epsilon_j, \quad \text{for} \quad j \neq i, i+1, \\ \epsilon_i\sigma_i = \sigma_i\epsilon_{i+1}, \\ \epsilon_{i+1}\sigma_i = \sigma_i\epsilon_i, \\ \epsilon_i = \epsilon_i^2, \\ \epsilon_{i+1}\sigma_i^2 = \sigma_i^2\epsilon_{i+1} = \epsilon_{i+1}, \\ \epsilon_i\epsilon_{i+1}\sigma_i = \sigma_i\epsilon_i\epsilon_{i+1} = \epsilon_i\epsilon_{i+1}, \end{cases} \tag{2.6}$$

plus the braid relations (2.1).

If we take the presentation (2.2) for the braid group we get a presentation of the inverse braid monoid with generators $\sigma_1, \sigma_1^{-1}, \sigma, \sigma^{-1}$, ϵ, and relations [26]:

$$\begin{cases} \sigma_1\sigma_1^{-1} = \sigma_1^{-1}\sigma_1 = 1, \\ \sigma\sigma^{-1} = \sigma^{-1}\sigma = 1, \\ \epsilon\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\epsilon \quad \text{for } 1 \leq i \leq n-2, \\ \epsilon\sigma_1\epsilon = \sigma_1\epsilon\sigma_1\epsilon = \epsilon\sigma_1\epsilon\sigma_1, \\ \epsilon = \epsilon^2 = \epsilon\sigma_1^2 = \sigma_1^2\epsilon, \end{cases} \tag{2.7}$$

plus (2.2).

Similar to braids the notion of *mapping class monoid* was introduced in [26]. The definition is as follows.

It is convenient to consider the surface $S_{g,b,n}$, sometimes with n points fixed, sometimes deleted. Let f be a homeomorphism of $S_{g,b,n}$ which maps k points, $k \leq n$, from Q_n: $\{i_1, \ldots, i_k\}$ to k points $\{j_1, \ldots, j_k\}$ also from Q_n. The same way let h be a homeomorphism of $S_{g,b,n}$ which maps l points, $l \leq n$, from Q_n, say $\{s_1, \ldots, s_l\}$ to l points $\{t_1, \ldots, t_l\}$ again from Q_n. Consider the intersection of the sets $\{j_1, \ldots, j_k\}$ and $\{s_1, \ldots, s_l\}$, let it be the set of cardinality

m, $(m \leq k)$, it may be empty. Then the composition of f and h maps m points of Q_n to m points (may be different) of Q_n. For example, suppose for the simplicity of notations that

$$\{j_1, \ldots, j_k\} \cap \{s_1, \ldots, s_l\} = \{j_1, \ldots, j_m\} = \{s_1, \ldots, s_m\},$$

then the composition $h \circ f$ maps the points $\{i_1, \ldots, i_m\}$ to $\{t_1, \ldots, t_m\}$. If $m = 0$ then the composition has no relation to the set Q_n. Denote the set of isotopy classes of such maps by $\mathcal{IM}_{g,b,n}$. Composition defines a structure of monoid on $\mathcal{IM}_{g,b,n}$. It is evident that the monoid $\mathcal{IM}_{g,b,n}$ is inverse, so we call it the *inverse mapping class monoid*. If $g = 0$ and $b = 1$ we get the inverse braid monoid. In the general case of $\mathcal{IM}_{g,b,n}$ the role of the empty braid plays the mapping class group $\mathcal{M}_{g,b}$ (without fixed points). Each element $f \in \mathcal{IM}_{g,b,n}$ corresponds a (partial) bijection

$$\{i_1, \ldots, i_k\} \to \{j_1, \ldots, j_k\}.$$

This defines a canonical homomorphism to the symmetric inverse monoid:

$$\tau_{g,b,n} : \mathcal{IM}_{g,b,n} \to I_n. \tag{2.8}$$

We recall that a monoid M is *factorisable* if $M = EG$ where E is a set of idempotents of M and G is a subgroup of M.

Proposition 2.1. [26] *The monoid $\mathcal{IM}_{g,b,n}$ can be written in the form*

$$\mathcal{IM}_{g,b,n} = E\mathcal{M}_{g,b,n},$$

where E is a set of idempotents of $\mathcal{IM}_{g,b,n}$ and $\mathcal{M}_{g,b,n}$ is the corresponding mapping class group. So, this monoid is factorisable.

Similar to braids we introduce the idempotent elements $\epsilon_i \in \mathcal{IM}_{g,b,n}$ as isotopy classes of identity map

$$Id : S_{g,b} \to S_{g,b},$$

where during isotopy all points of the set Q_n are fixed with the exception of the point with number i. The element ϵ_1 is denoted by ϵ.

We call an element m of the mapping class monoid $\mathcal{IM}_{g,b,n}$ *i-Makanin* or *i-Brunnian* if it satisfies the equation:

$$\epsilon_i m = \epsilon_i. \tag{2.9}$$

Geometrically this means that if we fill the deleted point i at the surface, then a homeomorphism h lying in the class m

$$h : S_{g,b} \to S_{g,b},$$

becomes isotopical to the identity map

$$Id : S_{g,b} \setminus \{1, \ldots, \hat{i}, \ldots, n\} \to S_{g,b} \setminus \{1, \ldots, \hat{i}, \ldots, n\}.$$

The condition (2.9) is equivalent to the condition

$$m\epsilon_{\tau(m)(i)} = \epsilon_{\tau(m)(i)},$$

where τ is the canonical map to the symmetric inverse monoid (2.8). With the exception of ϵ_i itself all such elements belong to the mapping class group $\mathcal{M}_{g,b,n}$. We denote the subgroup of i-Makanin elements of the mapping class group by A_i. The subgroups A_i, $i = 1, \ldots, n$, are conjugate. The intersection of all subgroups of i-Makanin elements is the of *Makanin* or *Brunnian* subgroup of the mapping class group

$$\mathrm{Mak}_{g,b,n} = \cap_{i=1}^{n} A_i.$$

That is the same as $m \in \mathrm{Mak}_{g,b,n}$ if and only if the equation (2.9) holds for all i. Certain properties of *Makanin* subgroups of the mapping class groups are studied in the work Benson Farb, Christopher J. Leininger and Dan Margalit [7]

The purpose of this paper is to develop further the theory of inverse mapping class monoids, to demonstrate that canonical properties of mapping class groups often have there smooth continuation for the inverse mapping class monoid $\mathcal{IM}_{g,b,n}$.

3. Properties of inverse mapping class monoids

Following the ideology of considering partial symmetries instead of global ones [15] we can define the monoid of partial automorphisms of a group as follows. For a group G consider a set of partial isomorphisms

$$f : H \to K,$$

where H, K are subgroups of G. The composition of f with the isomorphism

$$g : L \to M,$$

(L, M are subgroups of G) is a superposition of f and g which is defined on $(K \cap L)f^{-1}$:

$$fg : (K \cap L)f^{-1} \to (K \cap L)g,$$

The set of all such partial isomorphisms with this operation form a monoid which is evidently inverse and which we call the *inverse partial automorphism monoid of a group* G and denote by $IPA(G)$.

In the case when the group G is a finitely generated free group F_n the following submonoid EF_n of $IPA(F_n)$ was defined in [26]. Let a be an element of the symmetric inverse monoid I_n, $a \in I_n$, so a is a partial injection with the elements $i_1, \ldots, i_k$ as its domain of the definition. Denote by $J_k = \{j_1, \ldots, j_k\}$ the image of a. The monoid EF_n consists of isomorphisms

$$< x_{i_1}, \ldots, x_{i_k} > \to < x_{j_1}, \ldots, x_{j_k} >$$

expressed by

$$f_a : x_i \mapsto w_i^{-1} x_{a(i)} w_i,$$

if i is among $i_1, \ldots, i_k$ and not defined otherwise and w_i is a word on $x_{j_1}, \ldots, x_{j_k}$. The composition of f_a and g_b, $a, b \in I_n$, is defined for x_i belonging to the domain of $a \circ b$. We put $x_{j_m} = 1$ in a word w_i if x_{j_m} does not belong to the domain of definition of g. We define a map ϕ_n from IB_n to EF_n expanding the canonical inclusion

$$Br_n \to \mathrm{Aut}\, F_n$$

by the condition that $\phi_n(\epsilon)$ as a partial isomorphism of F_n is given by the formula

$$\phi_n(\epsilon)(x_i) = \begin{cases} x_i \text{ if } i \geq 2, \\ \text{undefined, if } i = 1. \end{cases}$$

Using the presentation (2.3) we see that ϕ_n is a well defined homomorphism of monoids

$$\phi_n : IB_n \to EF_n.$$

The following statement was proved in [26].

Theorem 3.1. *The homomorphism ϕ_n is a monomorphism.*

As usual, we define the (g, b, n)-*surface group* as a group with the presentation

$$\pi_{g,b,n} =< a_1, c_1, \ldots, a_g, c_g, v_1, \ldots, v_b, u_1, \ldots, u_n \mid \prod_{i=1}^{n} u_i \prod_{l=1}^{b} v_l \prod_{m=1}^{g} [a_m, c_m] > .$$

It is the fundamental group of a surface of genus g with b components of a boundary and n punctures. The generators a_i, c_i correspond to the genus of a surface, the generators v_j correspond to boundary components and the generators u_k correspond to punctures.

In our construction of the inverse mapping class monoid $\mathcal{IM}_{g,b,n}$ we are passing from maps which are fixing a set of n points to maps which are fixing the set with a smaller number of points. If we consider these points deleted it means we are filling the holes. On the level of fundamental groups this means passing to quotient group.

Let H be a quotient group of $\pi_{g,b,n}$, defined by the conditions

$$u_i = 1 \text{ for all } \ i \notin \{i_1, \ldots, i_k\},$$

and let K be a quotient group of $\pi_{g,b,n}$, defined by the conditions

$$u_j = 1 \text{ for all } \ j \notin \{j_1, \ldots, j_k\}.$$

Let t be an element of the symmetric inverse monoid I_n, $t \in I_n$, so t is an injection with elements $i_1, \ldots, i_k$ constituting the domain of the definition of t. Let $J_k = \{j_1, \ldots, j_k\}$ be the image of t. Let w_i be a word on letters $a_1, c_1, \ldots, a_g, c_g, v_1, \ldots, v_b, u_{j_1}, \ldots, u_{j_k}$. Let $IAut\,\pi_{g,b,n}$ be the monoid consisting of isomorphisms

$$f_t : H \to K,$$

such that

$$\begin{cases} f_t(v_m) = v_m \text{ for } m = 1, \ldots, b, \\ f_t(u_i) = w_i^{-1} u_{(i)t} w_i, \text{ if } i \text{ is among } i_1, \ldots, i_k, \end{cases}$$

for all subgroups H and K of the type defined above, for all $k = 0, 1, \ldots, n$. Here the index $(i)t$ of the element $u_{(i)t}$ means the image of the element i by the action of the injection t, so $(i)t \in J_k$.

The composition of f_t and g_s, $t, s \in I_n$, is defined for u_i belonging to the domain of $t \circ s$. We put $u_{j_m} = 1$ in a word w_i if u_{j_m} does not belong to the domain of definition of g_s.

By our construction to each element of $f_t \in IAut\,\pi_{g,b,n}$ the element $t \in I_n$ is associated. This gives a canonical homomorphism

$$\tau_{g,b,n} : IAut\,\pi_{g,b,n} \to I_n. \tag{3.1}$$

For the simplicity of exposition we restrict ourselves to the case of empty boundary which was described in the book of W. Magnus, A. Karrass and D. Solitar [18, Sec. 3.7], and we denote $\pi_{g,0,n}$ by $\pi_{g,n}$. Let us define an equivalence relation in $IAut\,\pi_{g,n}$: for f_1 and f_2

$$f_1, f_2 : H \to K.$$

Such isomorphisms f_1 and f_2 are equivalent if they differ by an inner automorphism of the group H, their domain of definition, i.e. if there exists an element $q \in H$ such that

$$f_1(q^{-1} x q) = f_2(x).$$

This equivalence is in fact a congruence on the monoid $IAut\,\pi_{g,n}$ and we denote the corresponding quotient monoid by $IOut\,\pi_{g,n}$. This is similar to the classical case of groups when a

homomorphism is defined from a mapping class group of a surface to the group of outer automorphisms of the fundamental group of the surface. The homomorphism $\tau_{g,b,n} : IAut\,\pi_{g,b,n} \to I_n$ of (3.1) factors through $IOut\,\pi_{g,n}$ and we have a homomorphism

$$\tau_{g,b,n} : IOut\,\pi_{g,b,n} \to I_n. \tag{3.2}$$

Let us take an arbitrary element η of $\mathcal{IM}_{g,n}$. It is represented by a homeomorphism of a surface with k points deleted (among our n fixed points) onto another copy of the surface with (may be) different k points deleted

$$h : S_{g,n} \setminus \{i_1, \dots, i_k\} \to S_{g,n} \setminus \{j_1, \dots, j_k\}.$$

It defines the bijection $\hat{h}$ between the conjugacy classes of $\pi_1(S_{g,n} \setminus \{i_1, \dots, i_k\})$ and $\pi_1(S_{g,n} \setminus \{j_1, \dots, j_k\})$. We define $\hat{h}$ as an algebraic image of η in $IOut\,\pi_{g,n}$. Formally a homomorphism of monoids

$$\psi_{g,n} : \mathcal{IM}_{g,n} \to IOut\,\pi_{g,n}.$$

is defined by the formula

$$\psi_{g,n}(\eta) = \hat{h}.$$

This homomorphism is compatible with homomorphisms $\tau_{g,b,n}$ of (2.8) and (3.2).

Theorem 3.2. *The homomorphism ψ_n is an isomorphism of monoids.*

Proof. The monoid $\mathcal{IM}_{g,n}$ as a set is a disjoint union of copies of mapping class groups $\mathcal{M}_{g,k}$, for $k = 0, \dots n$, namely, C_n^k copies of $\mathcal{M}_{g,k}$ for each k. Our constructions are done so that for each copy of $\mathcal{M}_{g,k}$ the map $\psi_{g,n}$ is a bijection because of the Dehn-Nilsen-Baer theorem. □

4. Inverse mapping class monoids for punctured sphere

The following presentation for the mapping class group of a punctured sphere $\mathcal{M}_{0,n}$ was obtained by W. Magnus [16], see also[18]. Let $\sigma_1, \dots, \sigma_{n-1}$, denote the classes of homeomorphisms such that σ_i locally interchanges, say clockwise, the punctures with numbers i and $i+1$. Then the presentation has generators $\sigma_1, \dots, \sigma_{n-1}$, which satisfy the braid relations (2.1), the sphere relation

$$\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \dots \sigma_2\sigma_1 = 1 \tag{4.1}$$

and the following relation

$$(\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1})^n = 1. \tag{4.2}$$

Let Δ be the Garside's fundamental word in the braid group Br_n [8]. It can be in particular defined by the formula:

$$\Delta = \sigma_1 \dots \sigma_{n-1}\sigma_1 \dots \sigma_{n-2} \dots \sigma_1\sigma_2\sigma_1.$$

If we use Garside's notation $\Pi_t \equiv \sigma_1 \dots \sigma_t$, then $\Delta \equiv \Pi_{n-1} \dots \Pi_1$. If the generators σ_1, σ_2, …, σ_{n-2}, σ_{n-1}, are subject to the braid relations (2.1), then the condition (4.2) is equivalent to the following

$$\Delta^2 = 1.$$

If we consider the presentation of the braid group with two generators the sphere relation has the form

$$\sigma^n(\sigma_1^{-1}\sigma)^{1-n} = 1,$$

and the sphere mapping class relation is

$$\sigma^n = 1.$$

So, the mapping class group $\mathcal{M}_{0,n}$ in two generators is described by the braid relations and the following two relations

$$\begin{cases} \sigma^n = 1, \\ (\sigma_1^{-1}\sigma)^{n-1} = 1. \end{cases} \tag{4.3}$$

Theorem 4.1. *We get a presentation of the inverse mapping class monoid for punctured sphere $\mathcal{IM}_{0,n}$ if we take the generators σ_1, σ_2, …, σ_{n-2}, σ_{n-1}, σ_1^{-1}, σ_2^{-1}, …, σ_{n-2}^{-1}, σ_{n-1}^{-1}, ϵ (or ϵ_1, ϵ_2, …, ϵ_{n-1}, ϵ_n instead of one ϵ) subject to the sphere braid relations (2.1), (4.1), the sphere mapping class relation (4.2) and the inverse braid relations (2.3) (or (2.6)).*

Proof. We use the fact that the mapping class group $\mathcal{M}_{0,0,n}$ is a quotient group of the n-strand braid group for sphere and the ideas of the proof of the presentation (2.3) for the inverse braid monoid IB_n in the work [5].

Denote temporarily by P_n the monoid defined by the presentation by the generators σ_1, σ_2, …, σ_{n-2}, σ_{n-1}, σ_1^{-1}, σ_2^{-1}, …, σ_{n-2}^{-1}, σ_{n-1}^{-1}, ϵ the sphere braid relations (2.1), (4.1), the sphere mapping class relation (4.2) and the inverse braid relations (2.3). To define a map

$$\Theta : P_n \to \mathcal{M}_{0,n}$$

we associate to each word in the alphabet on letters σ_1, σ_2, …, σ_{n-2}, σ_{n-1}, σ_1^{-1}, σ_2^{-1}, …, σ_{n-2}^{-1}, σ_{n-1}^{-1}, ϵ the corresponding composition of (classes of) homeomorphisms. To juxtaposition of words there evidently corresponds the composition of (classes of) homeomorphisms. The fact that Θ is well defined (respects the relations) for some relations are classical facts, for others it is evident, probably, with the exception of the third relation in (2.3). The equalities of this relation follow from the fact that all three given classes (containing identity) ignore the first two points.

Let $\epsilon_{k+1,n}$ denote the the isotopy class of of a homeomorphism that fixes the first k points and does not care about the remaining $n-k$ points. On the level of braids (on a sphere) it corresponds to the partial braid with the trivial first k strands and the absent rest $n-k$ strands. The element $\epsilon_{k+1,n}$ can be expressed using the generator ϵ or the generators ϵ_i as follows

$$\epsilon_{k+1,n} = \epsilon\sigma_{n-1}\dots\sigma_{k+1}\epsilon\sigma_{n-1}\dots\sigma_{k+2}\epsilon\dots\epsilon\sigma_{n-1}\sigma_{n-2}\epsilon\sigma_{n-1}\epsilon, \tag{4.4}$$

$$\epsilon_{k+1,n} = \epsilon_{k+1}\epsilon_{k+2}\dots\epsilon_n,$$

From our construction of the inverse mapping class monoid it follows that every element of $\mathcal{IM}_{0,n}$ represented by a homeomorphism h of $S_{g,b,n}$ which maps k points, $k \le n$, from Q_n: $\{i_1, \dots, i_k\}$ to k points $\{j_1, \dots, j_k\}$ from Q_n can be expressed in the form

$$(\sigma_{i_1}\dots\sigma_1)\dots(\sigma_{i_k}\dots\sigma_k)\,\epsilon_{k+1,n}\,x\,\epsilon_{k+1,n}\,(\sigma_k\dots\sigma_{j_k})\dots(\sigma_1\dots\sigma_{j_1}), \tag{4.5}$$

$$k \in \{0,\dots,n\},\ x \in \mathcal{M}_{0,k},\ 0 \le i_1 < \dots < i_k \le n-1 \text{ and } 0 \le j_1 < \dots < j_k \le n-1.$$

Geometrically this means that we first send the points $\{i_1, \dots, i_k\}$ to the points $\{1, \dots, k\}$, then apply a homeomorphism from the mapping class *group* $\mathcal{M}_{0,k}$, and then send the points $\{1, \dots, k\}$ to the points $\{j_1, \dots, j_k\}$.

Note that in the formula (4.5) we can remove one of the $\epsilon_{k+1,n}$, but we shall use the form (4.5) because of convenience: two symbols $\epsilon_{k+1,n}$ serve as markers to distinguish the elements of $\mathcal{M}_{0,k}$.

The element x belongs to the mapping class group $\mathcal{M}_{0,k}$, so it can be expressed as a word in the letters σ_1, σ_2, …, σ_{k-1}, σ_1^{-1}, σ_2^{-1}, …, σ_{k-1}^{-1}. This proves that homomorphism Θ is onto.

Let us prove that Θ is a monomorphism. Suppose that for two words $W_1, W_2 \in P_n$ we have

$$\Theta(W_1) = \Theta(W_2).$$

It means that the corresponding homeomorphisms both map the set of points $\{i_1, \dots, i_k\}$ onto the set of points $\{j_1, \dots, j_k\}$ and they are isotopic in the class of homeomorphisms mapping $\{i_1, \dots, i_k\}$ onto $\{j_1, \dots, j_k\}$. Using relations (2.1) and (2.3) the same way as in [5] transform the words W_1, W_2 into the form (4.5)

$$\sigma(i_1, \dots i_k; k)\epsilon_{k+1,n} x \epsilon_{k+1,n}\sigma(k; j_1, \dots j_k).$$

Then the corresponding fragments $\sigma(i_1, \dots i_k; k)$ and $\sigma(k, j_1, \dots j_k; k)$ for W_1 and W_2 coincide. The elements x_1 of W_1 and x_2 of W_2 presented in the form 4.5 are the words in $\sigma_1, \dots, \sigma_k$ and $\sigma_1^{-1}, \dots, \sigma_k^{-1}$, and they correspond after Θ to isotopic homeomorphisms

$$h_1, h_2 : S_{0,k} \to S_{0,k}.$$

Hence x_1 can be transformed into x_2 using relations for the mapping class group $\mathcal{M}_{0,k}$. So, the words W_1 and W_2 represent the same element in P_n. □

Proposition 4.1. *There is a presentation of the inverse mapping class monoid for punctured sphere $\mathcal{IM}_{0,n}$ with the generators σ_1, σ, σ_1^{-1}, σ^{-1}, ϵ subject to the relations (2.2), (2.7) and (4.3) .*

□

The generators ϵ_i commute with Δ in the following way [26]:

$$\epsilon_i \Delta = \Delta \epsilon_{n+1-i}.$$

Let $\mathcal{E}$ be the monoid generated by one idempotent generator ϵ.

Proposition 4.2. *The abelianization of $\mathcal{IM}_{0,n}$, $n \geq 2$, is an abelian monoid $\mathcal{AM}$ generated (as an abelian monoid) by elements ϵ and $\bar{\alpha}$ subject to the following relations*

$$\begin{cases} 2\epsilon = \epsilon, \\ \epsilon + \bar{\alpha} = \epsilon \\ m\bar{\alpha} = 0, \text{ where } m = 2(n-1), \text{ if } n \text{ is even and } m = (n-1), \text{ if } n \text{ is odd.} \end{cases}$$

So, $\mathcal{AM}$ is isomorphic to the quotient-monoid of $\mathcal{E} \oplus \mathbb{Z}/m\mathbb{Z}$ by the relation $\epsilon + \bar{1} = \epsilon$. The canonical map of abelianization

$$a : \mathcal{IM}_{0,n} \to \mathcal{AM}$$

is given by the formula:

$$\begin{cases} a(\epsilon_i) = \epsilon, \\ a(\sigma_i) = \bar{\alpha}. \end{cases}$$

Proof. According to the presentation of the monoid $\mathcal{IM}_{0,n}$, given in Theorem 4.1 the homomorphism a is well defined. Now let $f : \mathcal{IM}_{0,n} \to B$ be an arbitrary homomorphism from $\mathcal{IM}_{0,n}$ to an abelian monoid B. Then the braid relations give that $f(\sigma_i) = f(\sigma_j)$ for any i, j. Denote this element $f(\sigma_i)$ by b and the image of ϵ by ϵ. Then the sphere braid relation and the sphere mapping class relation give that $b^m = 1$. Two last relations in (2.3) give that

$$\epsilon = \epsilon^2 = \epsilon b^2 = \epsilon b.$$

This defines a unique map

$$f_a : \mathcal{AM} \to B,$$

such that $f = f_a a$ and so $\mathcal{AM}$ is an abelianization of $\mathcal{IM}_{0,n}$. □

Corollary 4.1. *The monoid $\mathcal{AM}$ as a set consists of the following elements*

$$\mathcal{AM} = \{\bar{0}, \epsilon, \bar{\alpha}, 2\bar{\alpha}, \dots, (m-1)\bar{\alpha}\}.$$

□

Proposition 4.3. *The center of $\mathcal{IM}_{0,n}$ for $n \geq 1$ is isomorphic to $\mathcal{E}$:*

$$Z(\mathcal{IM}_{0,n}) = \{1\} \amalg \mathcal{M}_{0,0} \cong \mathcal{E},$$

i.e. the only non-unit element of $Z(\mathcal{IM}_{0,n})$ is the isotopy class of the identity map of a sphere such that isotopy does not respect punctures.

Proof. If $n = 1$, then the monoid $\mathcal{IM}_{0,1}$ consists of two elements, unit 1 and the idempotent ϵ (Example 2 below) and so it is a commutative monoid isomorphic to $\mathcal{E}$. Let $n \geq 2$. Denote by Φ the only element of $\mathcal{M}_{0,0} \hookrightarrow \mathcal{IM}_{0,n}$:

$$\Phi \in \mathcal{M}_{0,0} \hookrightarrow \mathcal{IM}_{0,n}.$$

For any element x of $\mathcal{IM}_{0,n}$ we have

$$\Phi x = x\Phi = \Phi.$$

So, Φ lies in the center. Let c be a non-unit element in the center different from Φ. It does not belong to the mapping class group $\mathcal{M}_{0,n}$, because this group is centerless for $n \geq 3$ [10] and for $n = 2$ (Example 3 below) c can not be the generator σ_1 of $\mathcal{M}_{0,2} = \Sigma_2$, since we have

$$\sigma_1\epsilon_1 \neq \epsilon_1\sigma_1.$$

So c is a homeomorphism of S^2 which maps the points $I_k = \{i_1, \dots, i_k\}$ to the points $J_k = \{j_1, \dots, j_k\}$, $0 < k < n$. Take a (class of) homeomorphism x that maps one point in the complement of J_k to a point in I_k without restrictions to other points. Then cx is equal to Φ, while xc is not. □

5. The word problem

R. Gillette and J. Van Buskirk in [10] obtained an analogue of the Markov normal form [19] for the sphere braid groups $Br_n(S^2)$ and for the punctured sphere mapping class group $\mathcal{M}_{0,n}$. This form can be obtained algorithmically, so, it gives a solution of the word problem. These ideas can be applied to the cases of inverse braid and mapping class monoids.

Let us recall the main points of the Gillette-Van Buskirk variant of Markov normal form. Define the elements $s_{i,j}$, $1 \leq i < j \leq m$, of the classical braid group Br_m by the formula:

$$s_{i,j} = \sigma_{j-1} \dots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}.$$

These elements satisfy the following Burau relations:

$$\begin{cases} s_{i,j}s_{k,l} = s_{k,l}s_{i,j} \text{ for } i < j < k < l \text{ and } i < k < l < j, \\ s_{i,j}s_{i,k}s_{j,k} = s_{i,k}s_{j,k}s_{i,j} \text{ for } i < j < k, \\ s_{i,k}s_{j,k}s_{i,j} = s_{j,k}s_{i,j}s_{i,k} \text{ for } i < j < k, \\ s_{i,k}s_{j,k}s_{j,l}s_{j,k}^{-1} = s_{j,k}s_{j,l}s_{j,k}^{-1}s_{i,k} \text{ for } i < j < k < l. \end{cases} \tag{5.1}$$

The elements $s_{i,j}$ with the relations (5.1) give a presentation of the pure braid group P_m [19].

Let us define the elements $\sigma_{k,l}$, $1 \leq k \leq l \leq m$ by the formulas

$$\sigma_{k,k} = e,$$

$$\sigma_{k,l} = \sigma_k^{-1} ... \sigma_{l-1}^{-1}.$$

We denote by the same symbols the images of the defined elements in the braid group of a sphere. Let $w(x_1, ..., x_m)$ be a word with possible entries of x_i^{δ}, where x_i are some letters and δ may be ± 1.

Theorem 5.1. [10] *Every element of the group $Br_n(S^2)$ can be uniquely written in the form*

$$\sigma_{i_n,n} ... \sigma_{i_j,j} ... \sigma_{i_2,2} w_1(s_{1,2}, ..., s_{1,n-1}) ... w_j(s_{j,j+1}, ..., s_{j,n-1}) ... w_{n-3}(s_{n-3,n-2}, s_{n-2,n-1}) \Delta^{2\delta},$$

where elements $s_{j,j+1}, ..., s_{j,n-1}$ generate a free group and δ may be 0 or 1. There exists an algorithm of obtaining the normal form for any word on letters $\sigma_1, \ldots, \sigma_{n-1}$, which gives a solution of the word problem for $Br_n(S^2)$.

Theorem 5.2. [10] *Every element of the group $\mathcal{M}_{0,n}$ can be uniquely written in the form*

$$\sigma_{i_n,n} ... \sigma_{i_j,j} ... \sigma_{i_2,2} w_1(s_{1,2}, ..., s_{1,n-1}) ... w_j(s_{j,j+1}, ..., s_{j,n-1}) ... w_{n-3}(s_{n-3,n-2}, s_{n-2,n-1}),$$

where elements $s_{j,j+1}, ..., s_{j,n-1}$ generate a free group. There exists an algorithm of obtaining the normal form for any word on letters $\sigma_1, \ldots, \sigma_{n-1}$, which gives a solution of the word problem for $\mathcal{M}_{0,n}$.

Theorem 5.3. *Every element w in $IB_n(S^2)$ can be uniquely written in the form*

$$\sigma_{i_1} \ldots \sigma_1 \ldots \sigma_{i_k} \ldots \sigma_k \epsilon_{k+1,n} x \epsilon_{k+1,n} \sigma_k \ldots \sigma_{j_k} \ldots \sigma_1 \ldots \sigma_{j_1}, \tag{5.2}$$

$$k \in \{0, \ldots, n\}, 0 \leq i_1 < \cdots < i_k \leq n-1, 0 \leq j_1 < \cdots < j_k \leq n-1, \text{ and } x \in Br_k(S^2)$$

is written in the Gillette-Van Buskirk normal form for $Br_k(S^2)$. If w is expressed in terms of σ_1, ..., σ_{n-1}, σ_1^{-1}, ..., σ_{n-1}^{-1}, ϵ, there is an effective algorithm for putting it in this normal form.

Proof. Given an element w of $IB_n(S^2)$, expressed as a word in the generators, the elements σ_{i_1}, ..., σ_1, ..., σ_{i_k}, ..., σ_k and σ_k, ..., σ_{j_k}, ..., σ_1, ..., σ_{j_1} of (5.2) are determined uniquely and algorithmically [5] by a given element of $IB_n(S^2)$. Geometrically they are determined by the fact which strands are absent and which ones are present in the braid w. The same way the element $x \in Br_k(S^2)$ is determined as a word in the generators of $Br_k(S^2)$. We apply Gillette-Van Buskirk algorithm to it to put it into Gillette-Van Buskirk normal form which is also unique. □

Theorem 5.4. *Every element w in $\mathcal{IM}_{0,n}$ can be uniquely written in the form*

$$\sigma_{i_1} \ldots \sigma_1 \ldots \sigma_{i_k} \ldots \sigma_k \epsilon_{k+1,n} x \epsilon_{k+1,n} \sigma_k \ldots \sigma_{j_k} \ldots \sigma_1 \ldots \sigma_{j_1},$$

$$k \in \{0, \ldots, n\}, \ x \in \mathcal{M}_{0,n}, \ 0 \leq i_1 < \ldots < i_k \leq n-1, \ 0 \leq j_1 < \cdots < j_k \leq n-1,$$

where x is written in the Gillette-Van Buskirk normal form for $\mathcal{M}_{0,k}$. If w is expressed in terms of σ_1, ..., σ_{n-1}, σ_1^{-1}, ..., σ_{n-1}^{-1}, ϵ, there is an effective algorithm for putting it in this normal form.

Proof. It is essentially the same as the proof of Theorem 5.3. Given an element w of $\mathcal{IM}_{0,n}$, expressed in its generators, the elements $\sigma_{i_1}, \ldots, \sigma_1, \ldots, \sigma_{i_k}, \ldots, \sigma_k$ and $\sigma_k, \ldots, \sigma_{j_k}, \ldots, \sigma_1, \ldots, \sigma_{j_1}$ are determined uniquely and algorithmically [5] by a given element of $\mathcal{IM}_{0,n}$. Geometrically they are determined by the fact which points participate in the description of a map of the class w and which ones are omitted. The same way the element $x \in \mathcal{M}_{0,k}$ is determined as a word in generators of $\mathcal{M}_{0,k}$. We apply Gillette-Van Buskirk algorithm for $\mathcal{M}_{0,k}$ to it to put it into Gillette-Van Buskirk normal form which is also unique. □

6. Examples

1. The monoid $\mathcal{IM}_{0,0}$ consists of one element of the identical map of a sphere

$$Id : S_{0,0} \to S_{0,0},$$

2. The monoid $\mathcal{IM}_{0,1}$ consists of two elements, say, unit 1 and the idempotent ϵ.

3. The mapping class group $\mathcal{M}_{0,2}$ is isomorphic to the symmetric group Σ_2, so the inverse mapping class monoid $\mathcal{IM}_{0,2}$ is isomorphic to the symmetric inverse monoid I_2.

4. The mapping class group $\mathcal{M}_{0,3}$ is isomorphic to the symmetric group Σ_3, so the inverse mapping class monoid $\mathcal{IM}_{0,3}$ is isomorphic to the symmetric inverse monoid I_3.

5. The mapping class group of a once-punctured torus $\mathcal{M}_{1,1}$ is isomorphic to the general linear group $GL_2\mathbb{Z}$ (unimodular group $SL_2\mathbb{Z}$ if we want to restrict ourselves to the orientation preserving homeomorphisms of a torus) as well as the mapping class group of a torus (without punctures) is isomorphic to $GL_2\mathbb{Z}$ (again it is isomorphic to $SL_2\mathbb{Z}$ if we want to restrict ourselves to the orientation preserving case). Then the inverse mapping class monoid $\mathcal{IM}_{1,1}$ as a set is a disjoint union of two copies of $GL_2\mathbb{Z}$:

$$\mathcal{IM}_{1,1} = GL_2\mathbb{Z} \amalg GL_2\mathbb{Z}.$$

If we denote by $\mathcal{IM}^+_{1,1}$ the inverse mapping class monoid of orientation preserving homeomorphisms then we have

$$\mathcal{IM}^+_{1,1} = SL_2\mathbb{Z} \amalg SL_2\mathbb{Z}.$$

Let the classes of maps of the first copy of $GL_2\mathbb{Z}$ respect the puncture and of the second copy are free. Then the multiplication in each copy is standard and multiplying a punctured element by a free one we get a standard product in $GL_2\mathbb{Z}$ and put it in the free copy.

This example gives a good illustration for Theorem 3.2 if we recall the classical result of Nilsen that the group of outer automorphisms of the free group F_2 (fundamental group of punctured torus) is isomorphic to $GL_2\mathbb{Z}$, see, for instance [18, Corollary N4, p.169].

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