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A SECOND-ORDER DIFFERENTIAL SYSTEM WITH HESSIAN-DRIVEN DAMPING; APPLICATION TO NON-ELASTIC SHOCK LAWS

Hedy ATTOUCH [1], Paul-Emile MAINGÉ[2] and Patrick REDONT [3]

Dedicated to J.I. Diaz on the occasion of his 60th birthday

Abstract. We consider the second-order differential system with Hessian-driven damping $\ddot{u}+\alpha\dot{u}+\beta\nabla^2\Phi(u)\dot{u}+\nabla\Phi(u)+\nabla\Psi(u)=0$, where $\mathcal{H}$ is a real Hilbert space, $\Phi,\Psi:\mathcal{H}\to\mathbb{R}$ are scalar potentials, and α,β are positive parameters. An interesting property of this system is that, after introduction of an auxiliary variable y, it can be equivalently written as a first-order system involving only the time derivatives $\dot{u}$, $\dot{y}$ and the gradient operators $\nabla\Phi$, $\nabla\Psi$. This allows to extend our analysis to the case of a convex lower semicontinuous function $\Phi:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$, and so to introduce constraints in our model. When $\Phi=\delta_K$ is the indicator function of a closed convex set $K\subseteq\mathcal{H}$, the subdifferential operator $\partial\Phi$ takes account of the contact forces, while $\nabla\Psi$ takes account of the driving forces. In this setting, by playing with the geometrical damping parameter β, we can describe nonelastic shock laws with restitution coefficient. Taking advantage of the infinite dimensional framework, we introduce a nonlinear hyperbolic PDE describing a damped oscillating system with obstacle. The first-order system is dissipative; each trajectory weakly converges to a minimizer of $\Phi+\Psi$, provided that Φ and $\Phi+\Psi$ are convex functions. Exponential stabilization is obtained under strong convexity assumptions.

Key words: Asymptotic stabilization; convex variational analysis; dissipative dynamical systems; exponential stabilization; gradient-like systems; Hessian-driven damping; impact dynamics; nonelastic shocks; nonsmooth potentials; restitution coefficient; second-order nonlinear differential equations; unilateral mechanics; viscoelastic membrane.

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1 Introduction

Let $\Phi,\Psi:\mathcal{H}\to\mathbb{R}$ be smooth real-valued functions operating on a real Hilbert space $\mathcal{H}$ (we will consider later the case Φ nonsmooth), and let α, β be positive real parameters. The function Φ is assumed to be convex. We consider the second-order differential system

$$\ddot{u}(t)+\alpha\dot{u}(t)+\beta\nabla^2\Phi(u(t))\dot{u}(t)+\nabla\Phi(u(t))+\nabla\Psi(u(t))=0, \tag{1.1}$$

[1] Institut de Mathématiques et de Modélisation de Montpellier, UMR CNRS 5149, CC 51, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France. (attouch@math.univ-montp2.fr). Supported by French ANR grant ANR-08-BLAN-0294-03.

[2] Université des Antilles-Guyane, D.S.I., LAMIA, Campus de Schoelcher, 97233 Cedex, Martinique, F.W.I. (Paul-Emile.Mainge@martinique.univ-ag.fr).

[3] Institut de Mathématiques et de Modélisation de Montpellier, UMR CNRS 5149, CC 51, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France. (redont@math.univ-montp2.fr). Supported by French ANR grant ANR-08-BLAN-0294-03.

where $\nabla\Phi(u)$ and $\nabla\Psi(u)$ denote the respective gradient operators of Φ and Ψ at u, $\nabla^2\Phi(u)$ is the Hessian of Φ at u, while $\dot{u}(t)$ and $\ddot{u}(t)$ respectively denote the first (velocity) and second derivative (acceleration) of the solution u at time $t \geq 0$.
We aim at showing that system (1.1) and its nonsmooth version provide a flexible mathematical model for nonelastic shocks with perfect contact in unilateral mechanics (like a ball bouncing on the floor, or a vibrating membrane over an obstacle).

Equation (1.1) is the Newton equation of Mechanics for an inertial system (the mass matrix has been normalized) subject to the following forces:

The *driving* forces have been split into the sum $\nabla\Phi + \nabla\Psi$, where $\nabla\Psi$ stands for classical smooth driving forces (like gravity), and $\nabla\Phi$ takes account of the contact forces. This will lead us naturally to consider nonsmooth potentials Φ, with, as a typical situation, $\Phi = \delta_K$, the indicator function of a closed convex constraint set $K \subseteq \mathcal{H}$ ($\delta_K(v) = 0$ if $v \in K$, $+\infty$ elsewhere).

The *damping* forces are the most original part of our model: the geometrical damping term $\nabla^2\Phi(u)\dot{u}$ only involves the (contact) potential Φ, which makes functions Φ and Ψ play asymmetric roles; we shall see that playing both with this term and its coefficient β allows to control the energy dissipation due to shocks, and so to obtain nonelastic shock laws with restitution coefficient. The viscous damping term $\alpha\dot{u}$ is not involved in the analysis of shocks, it plays a role in the asymptotic stabilization analysis.

As we already stressed, in order to model contact forces, it is important to consider system (1.1) with a nonsmooth potential Φ (like δ_K). At first glance this looks difficult, because a meaning has to be given to $\nabla^2\Phi$, the derivative of a noncontinuous vector field! A simple but central remark is that $\nabla\Phi(u(t))$ comes with its time derivative $\nabla^2\Phi(u(t))\dot{u}(t)$ in (1.1), which suggests that system (1.1) can be simplified by performing some time integration. Indeed, as a key property of our approach, we show (Section 2, Theorem 2.1) that (1.1) can be equivalently written as a first-order in time differential system

$$\begin{cases} \dot{u}(t) + \beta\nabla\Phi(u(t)) + au(t) + by(t) = 0, \\ \dot{y}(t) - \beta\nabla\Psi(u(t)) + au(t) + by(t) = 0, \end{cases} \tag{1.2}$$

where a and b are real numbers such that $a + b = \alpha$ and $\beta b = 1$. More precisely, the set of solutions of (1.1) is made of functions u verifying (1.2) (for some auxiliary function y). It is important to note that (1.2), an alternative formulation of (1.1) with no occurrence of the Hessian of Φ, makes sense for possibly nonsmooth convex functions ($\nabla\Phi$ is then replaced by $\partial\Phi$, the subdifferential of Φ in the sense of convex analysis).

Our program becomes clear. We first study (1.1) and (1.2) in the smooth case (Section 3) with the help of classical tools (Cauchy-Lipschitz theorem, energy estimates).

Then in Section 4 we consider a potential $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ which is just assumed to be convex and lower semicontinuous (lsc for short); from now on we write ϕ for a nonsmooth potential, and Φ for a smooth one. We show that the Cauchy problem for the differential inclusion system

$$\begin{cases} \dot{u}(t) + \partial\phi(u(t)) + au(t) + by(t) \ni 0, \text{ for a.e. } t > 0, \\ \dot{y}(t) - \nabla\Psi(u(t)) + au(t) + by(t) = 0, \quad \text{for } t > 0 \end{cases} \tag{1.3}$$

has a unique strong global solution $(u,y) \in \mathcal{C}\left([0,+\infty[,\mathcal{H}^2\right)$, which is absolutely continuous on all compact subsets of $]0,+\infty[$ (Theorem 4.1). Our analysis is quite similar to that followed by Brézis in the study of semigroups of contractions generated by subdifferentials of convex lsc functions (see [15–17]). It consists in taking the Moreau-Yosida approximation of ϕ (a smooth approximation), getting uniform estimates on the trajectories of the approximate system, and passing to the limit. Relying on the laziness property of the solutions of (1.3), by taking $\phi = \delta_K$ equal to the indicator function of a closed convex set $K \subseteq \mathcal{H}$, we show that the solutions of the corresponding system exhibit a completely inelastic shock law. To our knowledge, this is one of the first existence and uniqueness results for trajectories of an inertial system evolving in a general convex subset of a Hilbert space and satisfying a completely inelastic shock law; see Moreau [28], Paoli-Schatzman [29–32], and Cabot and Paoli [18] for a recent account on this subject.

The Moreau-Yosida approximation plays an important theoretical role, but it may be difficult to compute, and hence not easy to handle practically. This leads us to consider an abstract variational approximation scheme based on Mosco-epiconvergence (Theorem 4.2) which covers various practical approximations schemes like exterior penalization, or barrier methods.

Sections 5 and 6 provide some illustrations of our approach to impact dynamics:
- in Section 5, we take ϕ equal to the Dirichlet energy functional with obstacle constraint, and treat it as a lower semicontinuous functional on $\mathcal{H} = L^2(\Omega)$;
- in Section 6, we take advantage of the fact that the exterior penalization of a convex constraint by the square of the distance provides a $C^{1,1}$ function ((1.2) becomes a classical differential system). By playing both with this approximation and the damping parameter β we obtain an application to impact dynamics with restitution coefficient; our dynamics in this case is closely related to that introduced by Paoli-Schatzman in [29–32] (and hopefully simpler!).

In the last Section 7, we analyze the asymptotic behaviour (as time $t \to +\infty$) of trajectories of systems (1.1) and (1.2). Under the sole extra assumption that $\Phi + \Psi$ is convex with a nonempty set S of minimizers, we show in Theorem 7.1 that any solution trajectory u of (1.1) or (1.2) weakly converges to a minimizer (S may be a continuum). Note that S is also the set of equilibria of (1.1). Exponential decay is obtained under the assumption of strong convexity of $\Phi + \Psi$ (Theorem 7.3). This asymptotic stabilization property bears natural link with the analysis and control of infinite dimensional systems.
Actually, systems (1.1) and (1.2) combine the convergence properties of the following two systems, which appear as particular cases:
- if $\Phi = 0$, (1.1) reduces to the so-called heavy ball with friction system (see Alvarez [3], Attouch-Goudou-Redont [10]):

$$\ddot{u}(t) + \alpha\dot{u}(t) + \nabla\Psi(u(t)) = 0.$$

A nonsmooth potential Ψ gives rise to a differential inclusion modelling shocks: elastic (Schatzman [34], Attouch-Cabot-Redont [9]), or nonelastic ([18]).
- if $\Psi = 0$, (1.1) reduces to the so-called dynamical inertial Newton system (see

Alvarez-Attouch-Bolte-Redont [4]):

$$\ddot{u}(t) + \alpha\dot{u}(t) + \beta\nabla^2\Phi(u(t))\dot{u}(t) + \nabla\Phi(u(t)) = 0.$$

Indeed it can be viewed as a (hyperbolic) regularization of the ill-posed continuous dynamical Newton method in optimization $\nabla^2\Phi(u(t))\dot{u}(t) + \nabla\Phi(u(t)) = 0$.

From the point of view of optimization, (1.1) and (1.2) offer greater flexibility by allowing to consider two potentials, one for the objective function (Ψ) and the other for the constraint (Φ). One can consult [2–4, 7, 12] for related work concerning Newton dynamics and optimization.

The natural link between continuous dissipative dynamical systems and optimization algorithms (by time discretization) paves the way for new numerical methods. Let us mention for instance that the first-order in time system treated in [4] has been used for solving nonsmooth convex minimization problems [7], general monotone inclusions [25, 27], fixed-point problems [24], and also for minimizing the nonsmooth extended difference of convex functions [26]. In this respect, the present work offers new perspective concerning splitting algorithms for constrained minimization, and for minimizing a sum of convex functions $\Phi+\Psi$ (interestingly, in (1.2) the sum $\Phi+\Psi$ has been split in a form suitable for parallel computing).

All these interesting questions regarding impact dynamics, asymptotic control, and optimization algorithms require further theoretical and applied studies which go beyond the scope of the present article. We try to give an overview on these issues and show the remarkable mathematical and modelling properties of system (1.1) and its companion (1.2).

Throughout this work, $\mathcal{H}$ is a real Hilbert space; its scalar product is denoted by $\langle\cdot,\cdot\rangle$ and its associated norm by $|\cdot|$.

2 Equivalence of systems (1.1), (1.2)

Consider systems (1.1), (1.2) and suppose that constants a, b, α, β verify

$$\beta \neq 0, \quad b = 1/\beta, \quad a = \alpha - 1/\beta. \tag{2.1}$$

Theorem 2.1 *Suppose Φ twice-differentiable and Ψ differentiable on $\mathcal{H}$. Suppose that* (2.1) *holds. Let $(u_0, v_0, y_0) \in \mathcal{H}^3$ verify $v_0 + \beta\nabla\Phi(u_0) + au_0 + by_0 = 0$. Then the following assertions are equivalent:*

(i1) *u is a twice-differentiable solution of* (1.1) *with initial conditions $u(0) = u_0$, $\dot{u}(0) = v_0$;*

(i2) *there exists a function y such that (u, y) is a differentiable solution of* (1.2), *with initial conditions $u(0) = u_0$, $y(0) = y_0$.*

Proof: (i1 $\Rightarrow$ i2) Consider the function y defined by the first equation of (1.2)

$$\dot{u} + \beta\nabla\Phi(u) + \left(\alpha - \frac{1}{\beta}\right) u + \frac{1}{\beta}\, y = 0\,, \tag{2.2}$$

and note that y is differentiable with $y(0) = y_0$. Differentiating, we obtain

$$\ddot{u} + \beta\nabla^2\Phi(u)\dot{u} + \left(\alpha - \frac{1}{\beta}\right) \dot{u} + \frac{1}{\beta}\, \dot{y} = 0.$$

Whence, with (1.1), we deduce

$$-\beta(\nabla\Phi(u) + \nabla\Psi(u)) - \dot{u} + \dot{y} = 0.$$

Adding this equality and (2.2) yields the second equation of (1.2).

(i2 $\Rightarrow$ i1) Conversely, suppose that (u, y) is a differentiable solution of (1.2). Note that $\dot{u}(0) = v_0$. Subtract the first equation of (1.2) from the second one to obtain

$$\dot{y} = \dot{u} + \beta(\nabla\Phi(u) + \nabla\Psi(u)).$$

The first equation of (1.2) shows that $\dot{u}$ is differentiable and $\ddot{u}$ satisfies

$$\ddot{u} + \beta\nabla^2\Phi(u)\dot{u} + \left(\alpha - \frac{1}{\beta}\right)\dot{u} + \frac{1}{\beta}\dot{y} = 0.$$

Eliminating $\dot{y}$ in the two equalities above gives (1.1) . •

Remark 2.1 Equivalently, by taking $-y$ (instead of y) as an auxiliary variable in (1.2), and setting $a = \left(\alpha - \frac{1}{\beta}\right)$, $b = \frac{1}{\beta}$, we obtain the following first-order system

$$\begin{cases} \dot{u}(t) + \beta\nabla\Phi(u(t)) + au(t) - by(t) = 0, \\ \dot{y}(t) + \beta\nabla\Psi(u(t)) - au(t) + by(t) = 0. \end{cases}$$

The auxiliary variable y can be expressed as $y(t) = \frac{1}{b}\left(\dot{u}(t) + \beta\nabla\Phi(u(t)) + au(t)\right)$. It involves both $\dot{u}(t)$ and $\beta\nabla\Phi(u(t))$. Thus, despite some similarities, it is different from the classical transformation which consists in taking $y = \dot{u}$ as in classical mechanics, when passing from second-order Newtonian equation to first-order Hamiltonian system.

3 Dynamics with smooth potentials

This section is devoted to the study of existence and uniqueness results for systems (1.1) and (1.2). Without loss of generality we essentially focus our attention on the special case of (1.2) with $\beta = 1$, that is the first-order system

$$\begin{cases} \dot{u}(t) + \nabla\Phi(u(t)) + au(t) + by(t) = 0, \\ \dot{y}(t) - \nabla\Psi(u(t)) + au(t) + by(t) = 0, \\ u(0) = u_0, \quad y(0) = y_0, \end{cases} \tag{3.1}$$

where $(u_0, y_0) \in \mathcal{H}^2$ and parameters a and b are real numbers satisfying

(CP) $b \geq 0$, $a + b \geq 0$.

Functions Φ and Ψ are required to fulfill the following assumptions:

(A1) $\Psi : \mathcal{H} \to \mathbb{R}$ is differentiable, its gradient $\nabla\Psi$ is Lipschitz continuous on bounded sets;

(A2) $\Phi : \mathcal{H} \to \mathbb{R}$ is convex and differentiable, its gradient $\nabla\Phi$ is Lipschitz continuous on bounded sets;

(A3) $\Theta := \Phi + \Psi$ is bounded from below on $\mathcal{H}$.

Remark 3.1 By difference between the first and second equations of (3.1), and recalling $\Theta = \Psi + \Phi$, we obtain the following useful formula

$$\dot{u} - \dot{y} + \nabla\Theta(u) = 0. \tag{3.2}$$

We are interested in establishing global existence and uniqueness results for classical solutions of the first-order system (3.1) under conditions (A1)-(A3) and (CP). Note that, in this section, no convexity assumption is made on functions Ψ and $\Phi + \Psi$.

With any classical solution (u, y) of (3.1), we associate the energy-like function E defined for $t \geq 0$ by

$$E(t) = b\Theta(u(t)) + \frac{1}{2}|\dot{u}(t)|^2. \tag{3.3}$$

The global existence property relies on the decreasing property of the energy E.

Lemma 3.1 *Suppose that conditions* (A1)-(A2) (CP) *are satisfied. Suppose also that* (u, y) *is a classical solution of* (3.1) *defined on some nonempty interval* $[0, T_m[$. *Then,* E *is a nonincreasing function; more precisely, for all* s, t *with* $0 \leq s \leq t < T_m$, *we have*

$$\frac{1}{2}|\dot{u}(t)|^2 + b\Theta(u(t)) + (a+b)\int_s^t |\dot{u}(\tau)|^2 d\tau \leq \frac{1}{2}|\dot{u}(s)|^2 + b\Theta(u(s)). \tag{3.4}$$

Proof: For any $0 < T < T_m$, u and y are Lipschitz continuous over $[0, T]$, as functions of class C^1. Let h be a positive real number, $0 < h < T_m - T$. For any mapping $f : [0, T] \to \mathcal{H}$ and for any $t \in [0, T]$, set $d_h f(t) = f(t+h) - f(t)$. From the first equation of (3.1), owing to the autonomous property of the system, we have for $0 \leq t \leq T$

$$d_h\dot{u}(t) + a d_h u(t) + b d_h y(t) + d_h((\nabla\Phi) \circ u)(t) = 0.$$

By taking the scalar product in $\mathcal{H}$ with $d_h u(t)$ we obtain

$$\langle d_h\dot{u}(t), d_h u(t)\rangle + a|d_h u(t)|^2 + b\langle d_h y(t), d_h u(t)\rangle + \langle d_h((\nabla\Phi)\circ u)(t), d_h u(t)\rangle = 0. \tag{3.5}$$

The first term in the left side of (3.5) can be equivalently written as

$$\langle d_h\dot{u}(t), d_h u(t)\rangle = \frac{1}{2}\frac{d}{dt}|d_h u(t)|^2. \tag{3.6}$$

Concerning the last term in the left side of (3.5), by convexity of Φ, and thus by monotonicity of its gradient $\nabla\Phi$ (see, e.g., [15]), we additionally have

$$\langle d_h((\nabla\Phi) \circ u)(t), d_h u(t)\rangle = \langle \nabla\Phi(u(t+h)) - \nabla\Phi(u(t)), u(t+h) - u(t)\rangle \geq 0. \tag{3.7}$$

Combining these last three results we obtain

$$\frac{1}{2}\frac{d}{dt}|d_h u(t)|^2 + a|d_h u(t)|^2 + b\langle d_h y(t), d_h u(t)\rangle \leq 0. \tag{3.8}$$

Given two real values s and t such that $0 \leq s < t \leq T$, after integrating (3.8) on $[s,t]$ and dividing by h^2, we obtain

$$\frac{1}{2}\left|\frac{d_h u(t)}{h}\right|^2 - \frac{1}{2}\left|\frac{d_h u(s)}{h}\right|^2 + a\int_s^t \left|\frac{d_h u(r)}{h}\right|^2 dr + b\int_s^t \left\langle \frac{d_h y(r)}{h}, \frac{d_h u(r)}{h}\right\rangle dr \leq 0.$$

One can easily deduce from the first and second equations of (3.1) that $\dot{u}$ and $\dot{y}$ are Lipschitz continuous on $[0,T]$. This is a consequence of the Lipschitz continuity of u and y on $[0,T]$, and of the Lipschitz continuity on bounded sets of $\nabla\Phi$ and $\nabla\Psi$ (by (A1) and (A2)). It is then classically deduced that $\frac{d_h u}{h}$ and $\frac{d_h y}{h}$ converge uniformly to $\dot{u}$ and $\dot{y}$, respectively, on $[s,t]$ (as $h \to 0$). Passing to the limit as $h \to 0$ in the above inequality gives

$$\frac{1}{2}|\dot{u}(t)|^2 - \frac{1}{2}|\dot{u}(s)|^2 + a\int_s^t |\dot{u}(r)|^2 dr + b\int_s^t \langle \dot{y}(r), \dot{u}(r)\rangle dr \leq 0. \tag{3.9}$$

On the other hand, from (3.2) we deduce $\langle \dot{y}, \dot{u}\rangle = |\dot{u}|^2 + \frac{d}{dt}\Theta(u)$, which, together with (3.9), yields (3.4). •

We can now formulate the existence and uniqueness theorem.

Theorem 3.1 *Suppose that conditions* (A1)-(A3) *and* (CP) *are satisfied. Then, problem* (3.1) *has a unique global classical solution* (u,y) *defined on* $[0,+\infty[$, *and which satisfies*

- **(r1)** $u, y \in C^1([0,+\infty);\mathcal{H})$;
- **(r2)** $\dot{u} \in L^\infty(0,+\infty;\mathcal{H})$ *; if moreover* $a+b>0$ *then* $\dot{u} \in L^2(0,+\infty;\mathcal{H})$;
- **(r3)** *the energy* $E(t) = b\Theta(u(t)) + (1/2)|\dot{u}(t)|^2$ *is a decreasing function of* t.

Proof: System (3.1) can be equivalently rewritten as

$$\dot{Z} + \mathcal{D}(Z) = 0, \tag{3.10}$$

with $Z(t) = (u(t), y(t)) \in \mathcal{H}^2$, and $\mathcal{D} : \mathcal{H}^2 \to \mathcal{H}^2$ defined for (x_1,x_2) in $\mathcal{H}^2$ by

$$\mathcal{D}(x_1,x_2) = \begin{pmatrix} \nabla\Phi(x_1) + ax_1 + bx_2 \\ -\nabla\Psi(x_1) + ax_1 + bx_2 \end{pmatrix}^T. \tag{3.11}$$

Applying the Cauchy-Lipschitz theorem, under conditions (A1)-(A3), we deduce that there exists a maximal time T_m (finite or infinite) such that existence and uniqueness of a classical local solution (u,y) to (3.1) in $C^1([0,T_m[;\mathcal{H}^2)$ hold.
To extend this existence and uniqueness result to the whole interval $[0,+\infty[$, we follow a standard argument and proceed by contradiction. Suppose that T_m is finite. Then, from (3.4) and using (A3), we deduce that $|\dot{u}|$, and further $|u|$, is bounded on $[0,T_m[$. This result and the Lipschitz continuity of $\nabla\Theta = \nabla\Phi + \nabla\Psi$ on bounded sets show that $\dot{y} = \dot{u} + \nabla\Theta(u)$ is also bounded on $[0,T_m[$. Then it can be checked that $u(t)$ and $y(t)$ converge as $t \to T_m^-$, which contradicts the maximality of T_m. Consequently, $T_m = +\infty$ and there exists a unique global solution (u,y) to

(3.1) which satisfies $u \in C^1([0,+\infty[;\mathcal{H})$ and $y \in C^1([0,+\infty[;\mathcal{H})$, so that (r1) holds. Items (r2) and (r3) are then readily deduced from Lemma 3.1 . •

Returning to the equivalent second-order equation (1.1), we can reformulate the preceding results as follows.

Corollary 3.1 *Let the parameters in equation* (1.1) *be such that* $\alpha \geq 0$ *and* $\beta > 0$, *and suppose, besides conditions* (A1)-(A3)*, that* $\Phi : \mathcal{H} \to \mathbb{R}$ *is twice-differentiable. Then, for any* $(u_0, v_0) \in \mathcal{H}^2$*, problem (1.1) with Cauchy data* $u(0) = u_0$, $\dot{u}(0) = v_0$, *has a unique global twice-differentiable solution* u *defined on* $[0,+\infty[$. *Moreover, the following properties hold:*
- $\dot{u} \in L^\infty(0,+\infty;\mathcal{H})$*; if* $\alpha > 0$ *then* $\dot{u} \in L^2(0,+\infty;\mathcal{H})$*;*
- *if* $\Phi \in \mathcal{C}^2(\mathcal{H})$ *then* $u \in \mathcal{C}^2([0,+\infty);\mathcal{H})$.

Proof: According to Theorem 2.1, u is a twice-differentiable solution of (1.1), with initial data $u(0) = u_0$ and $\dot{u}(0) = v_0$, if and only if, for some function y, (u, y) is a differentiable solution of (3.1) with initial data $u(0) = u_0$, $y(0) = -(1/b)(v_0 + \beta\nabla\Phi(u_0) + au_0)$ (with Φ and Ψ replaced by $\beta\Phi$ and $\beta\Psi$, also with $b = 1/\beta$, $a = \alpha - 1/\beta$). In view of the continuity of $\nabla\Phi$ and $\nabla\Psi$, (u, y) is necessarily a classical (*i. e.* $\mathcal{C}^1$) solution. Noticing that $a + b = \alpha \geq 0$, and $b > 0$, we are precisely in the situation examined in Theorem 3.1. •

4 Dynamics with a nonsmooth damping potential

Replacing the smooth potential $\Phi : \mathcal{H} \to \mathbb{R}$ by a nonsmooth potential $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is motivated by applications to unilateral mechanics, PDE's, and optimization (this last algorithmic aspect is not considered in this paper). A direct approach which would consist in studying the singular differential inclusion

$$\ddot{u}(t) + \alpha\dot{u}(t) + \beta\partial^2\phi(u(t))\dot{u}(t) + \partial\phi(u(t)) + \nabla\Psi(u(t)) \ni 0, \tag{4.1}$$

is out of reach, a major obstacle being to give a meaning to $\partial^2\phi$. By contrast, our approach, which relies on the study of the first-order system (3.1), still makes sense when considering a convex lower semicontinuous potential $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$. This naturally suggests to apply it to unilateral mechanics (take ϕ equal to the indicator function of a closed convex set $K \subseteq \mathcal{H}$), or to nonlinear hyperbolic PDE's (take ϕ equal to a Dirichlet type energy functional with an obstacle constraint).

4.1 Notion of strong solution and Moreau-Yosida approximation

Instead of working with the singular system (4.1), we consider the following system

$$\begin{cases} (u, y) \in \mathcal{C}\left([0,+\infty[,\mathcal{H}^2\right), \\ \quad \text{absolutely continuous on all compact subset of }]0,+\infty[, \\ \dot{u}(t) + \partial\phi(u(t)) + au(t) + by(t) \ni 0, \text{ for a.e. } t > 0, \\ \dot{y}(t) - \nabla\Psi(u(t)) + au(t) + by(t) = 0, \quad \text{for } t > 0, \\ u(0) = u_0, \quad y(0) = y_0, \end{cases} \tag{4.2}$$

when $(u_0, y_0) \in \mathcal{H}^2$ and the following conditions hold:

(CP) a and b are real numbers such that $b \geq 0$ and $a + b \geq 0$;

(A1) $\Psi : \mathcal{H} \to \mathbb{R}$ is a differentiable function with $\nabla\Psi$ Lipschitz continuous on bounded sets;

($\tilde{\mathbf{B}}$1) $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a convex l.s.c. (lower semi-continuous) function;

($\tilde{\mathbf{B}}$2) ϕ and Ψ are bounded from below over $\mathcal{H}$;

($\tilde{\mathbf{B}}$3) $(u_0, y_0) \in \overline{\operatorname{dom} \partial\phi} \times \mathcal{H}$.

Clearly (4.2) is nothing but the formulation of system (3.1) with a nonsmooth potential ϕ (instead of Φ). In order to investigate (4.2), we give an equivalent formulation.

Setting $Z(t) = (u(t), y(t)) \in \mathcal{H}^2$, system (4.2) can be equivalently written as

$$\begin{cases} Z \in \mathcal{C}\left([0, +\infty[, \mathcal{H}^2\right), \\ \quad \text{absolutely continuous on all compact subset of }]0, +\infty[, \\ \dot{Z}(t) + \partial\widetilde{\phi}(Z(t)) + \mathcal{D}(Z(t)) \ni 0, \text{ for a.e. } t > 0, \\ Z(0) = (u_0, y_0), \end{cases} \tag{4.3}$$

with $\widetilde{\phi} : \mathcal{H}^2 \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{D} : \mathcal{H}^2 \to \mathcal{H}^2$ defined for any (v_1, v_2) in $\mathcal{H}^2$ by

$$\widetilde{\phi}(v_1, v_2) = \phi(v_1) \quad \text{and} \quad \mathcal{D}(v_1, v_2) = \begin{pmatrix} av_1 + bv_2 \\ -\nabla\Psi(v_1) + av_1 + bv_2 \end{pmatrix}^T.$$

Note that system (4.3) is governed by the sum of the convex subdifferential operator $\partial\widetilde{\phi}$ and the locally Lipschitz operator $\mathcal{D}$.

Remark 4.1 If $\nabla\Psi$ was assumed to be globally Lipschitz continuous, we could apply Proposition 3.12 of [15] to obtain the global existence and uniqueness of a strong solution of (4.3). But in many instances this assumption is too strong. We then adopt another strategy. Following the classical proof of the nonlinear Hille-Yosida theorem, we use the Yosida regularization of the nonsmooth operator $\partial\phi$. Existence and uniqueness of a global classical solution for the approximate problem is a direct consequence of the results of Section 3. Next we pass to the limit on the approximated equations to finally obtain the solution of system (4.2). As a byproduct, this approach provides estimations which will be useful for the asymptotic analysis.

Before we introduce our approximate problem, we recall some classical definitions and facts from convex analysis (see [15, ch. 2] for a reference).

Definition 4.1 *The Yosida regularization $(\partial\phi)_\lambda$ of index $\lambda > 0$ of the maximal monotone operator $\partial\phi$ is defined by $(\partial\phi)_\lambda = \nabla\phi_\lambda$, where ϕ_λ (which is $\mathcal{C}^1$ indeed) is the Moreau-Yosida regularization of index $\lambda > 0$ of ϕ, that is*

$$\forall v \in \mathcal{H}, \ \phi_\lambda(v) = \inf_{w \in \mathcal{H}} \left\{ \phi(w) + \frac{1}{2\lambda}|v - w|^2 \right\}. \tag{4.4}$$

Definition 4.2 *The resolvent of index $\lambda > 0$ of the operator $\partial\phi$ is the single-valued operator defined by $J_\lambda = (I + \lambda\partial\phi)^{-1}$.*

Remark 4.2 The following properties hold:

(d1) The Yosida regularization $(\partial\phi)_\lambda = \nabla\phi_\lambda$ is $(1/\lambda)$-Lipschitz continuous and

$$\nabla\phi_\lambda(v) = (1/\lambda)\,(v - J_\lambda(v))\,. \tag{4.5}$$

(d2) The operator J_λ is a contraction, and $J_\lambda(v)$ is the unique point where the minimum in (4.4) is achieved, namely $\phi_\lambda(v) = \phi(J_\lambda(v)) + \frac{1}{2\lambda}|v - J_\lambda(v)|^2$. So, it follows immediately that

$$\forall v \in \mathcal{H},\ \nabla\phi_\lambda(v) \in \partial\phi(J_\lambda(v)). \tag{4.6}$$

Remark 4.3 The following inequalities and convergence properties are satisfied:

$$\forall v \in \mathcal{H},\ \inf\phi_\lambda = \inf\phi \le \phi_\lambda(v) \le \phi(v), \tag{4.7}$$

$$\forall v \in \operatorname{dom}\phi,\ |\nabla\phi_\lambda(v)| \nearrow |\partial\phi(v)^\circ| \text{ and } \nabla\phi_\lambda(v) \to \partial\phi(v)^\circ, \lambda \to 0, \tag{4.8}$$

where $\partial\phi(v)^\circ$ denotes the element of minimum norm of $\partial\phi(v)$.

We now formulate a well-suited approximate formulation of (4.2). Given $\lambda > 0$, we associate with (4.2) the following approximate system

$$\begin{cases} \dot u_\lambda(t) + \nabla\phi_\lambda(u_\lambda(t)) + au_\lambda(t) + by_\lambda(t) = 0, \text{ for } t > 0, \\ \dot y_\lambda(t) - \nabla\Psi(u_\lambda(t)) + au_\lambda(t) + by_\lambda(t) = 0, \text{ for } t > 0, \\ u_\lambda(0) = u_0, \quad y_\lambda(0) = y_0. \end{cases} \tag{4.9}$$

Lemma 4.1 *Under assumptions* (CP)-(A1)-($\tilde{\text{B}}$1)-($\tilde{\text{B}}$2)-($\tilde{\text{B}}$3)*, system* (4.9) *admits a unique global solution* $(u_\lambda, y_\lambda) : [0, +\infty[\to \mathcal{H}\times\mathcal{H}$*.*

Proof: The functions ϕ_λ and Ψ are both differentiable, and their gradients $\nabla\phi_\lambda$ and $\nabla\Psi$ are Lipschitz continuous and Lipschitz continuous on bounded sets, respectively. Moreover, in light of (4.7), the function $\phi_\lambda + \Psi$ satisfies

$$\inf(\phi_\lambda + \Psi) \ge \inf\phi_\lambda + \inf\Psi \ge \inf\phi + \inf\Psi > -\infty.$$

Thus, all the conditions of Theorem 3.1 are fulfilled. •

4.2 Lazy solutions and inelastic shock law

Let us claim the main result of this section.

Theorem 4.1 *Under assumptions* (CP)-(A1)-($\tilde{\text{B}}$1)-($\tilde{\text{B}}$2)-($\tilde{\text{B}}$3)*, system* (4.2) *admits a unique solution* (u, y)*. Moreover, for each* $T > 0$*,* (u, y) *is the uniform limit on* $[0, T]$ *of the sequence* (u_λ, y_λ)*, where* (u_λ, y_λ) *is the solution of problem (4.9), and it also holds that*

$$\dot u_\lambda \rightharpoonup \dot u \textit{ weakly in } L^2(0, T; \mathcal{H}). \tag{4.10}$$

Proof: The proof closely follows that of [15, Theorem 3.1] for solving the system $\dot{u}+Au \ni 0$, where A is a maximal monotone operator, and consists mainly in showing that (u_λ, y_λ), the solution of (4.9), is Cauchy.

(Uniqueness): Let us switch to the equivalent system (4.3). If Z_1, Z_2 are two solutions then, by the monotonicity of $\widetilde{\partial\phi}$, we have

$$0 \leq -\langle \dot{Z}_2(t) + \mathcal{D}(Z_2(t)) - \dot{Z}_1(t) - \mathcal{D}(Z_1(t)), Z_2(t) - Z_1(t)\rangle, \quad \text{for a.e. } t > 0.$$

Fix $T > 0$. Z_1 and Z_2 are bounded on $t \in [0,T]$. Since $\mathcal{D}$ is Lipschitz continuous on bounded sets, there exists some positive constant L such that

$$t \in [0,T] \Rightarrow \langle \mathcal{D}(Z_2(t)) - \mathcal{D}(Z_1(t)), Z_2(t) - Z_1(t)\rangle \leq L|Z_2(t) - Z_1(t)|^2.$$

Hence

$$0 \leq -\langle \dot{Z}_2(t) - \dot{Z}_1(t), Z_2(t) - Z_1(t)\rangle + L|Z_2(t) - Z_1(t)|^2, \text{ for a.e. } t > 0.$$

Now fix some S with $0 < S < T$. The function $t \mapsto \frac{1}{2}|Z_2(t) - Z_1(t)|^2 \exp(-2Lt)$ is absolutely continuous on $[S,T]$ with an almost everywhere nonpositive derivative; hence it is nonincreasing on $[S,T]$, also on $[0,T]$ thanks to its continuity and the arbitrariness of S; so we have

$$t \in [0,T] \Rightarrow \frac{1}{2}|Z_2(t) - Z_1(t)|^2 \exp(-2Lt) \leq \frac{1}{2}|Z_2(0) - Z_1(0)|^2. \tag{4.11}$$

Hence $Z_2(t) = Z_1(t)$ for all $t > 0$ (due to the arbitrariness of T) whenever $Z_2(0) = Z_1(0)$.

(Existence): We begin with showing the existence of a strong solution on $[0,T]$ for any fixed value $T > 0$. This part of the proof is divided into two cases (a) and (b) regarding the initial data (i.e., $(u_0, y_0) \in \operatorname{dom}\partial\phi \times \mathcal{H}$ and $(u_0, y_0) \in \overline{\operatorname{dom}\partial\phi} \times \mathcal{H}$):

(a) We assume first that $(u_0, y_0) \in \operatorname{dom}\partial\phi \times \mathcal{H}$ and we give a proof in three steps.
Let $(u_\lambda, y_\lambda) \in \mathcal{C}\left([0,+\infty[, \mathcal{H}^2\right)$ be the solution of (4.9).

(a1) Let us state some bounds. With Lemma 3.1, the energy $E_\lambda := \frac{1}{2}|\dot{u}_\lambda|^2 + b(\phi_\lambda + \Psi) \circ u_\lambda$ is nonincreasing, so that $E_\lambda(t) \leq E_\lambda(0)$, for all $t \geq 0$. Hence, in view of ($\tilde{\text{B}}$2) and (4.7)

$$\begin{aligned}\frac{1}{2}|\dot{u}_\lambda(t)|^2 &\leq E_\lambda(0) - b(\phi_\lambda(u_\lambda(t)) + \Psi(u_\lambda(t))) \\ &\leq E_\lambda(0) - b(\inf\phi + \inf\Psi).\end{aligned} \tag{4.12}$$

Moreover, with (4.9) and (4.8), we have $\dot{u}_\lambda(0) = -(\nabla\phi_\lambda(u_0) + au_0 + by_0)$, and so $|\dot{u}_\lambda(0)| \leq |\partial\phi(u_0)^\circ| + |au_0 + by_0|$. Hence with (4.7), and for any $\lambda > 0$, we also have

$$\begin{aligned}E_\lambda(0) &= \frac{1}{2}|\dot{u}_\lambda(0)|^2 + b(\phi_\lambda(u_0) + \Psi(u_0)) \\ &\leq \frac{1}{2}\left(|\partial\phi(u_0)^\circ| + |au_0 + by_0|\right)^2 + b(\phi(u_0) + \Psi(u_0)).\end{aligned} \tag{4.13}$$

This shows, with (4.12), that $|\dot{u}_\lambda(t)|$ is uniformly bounded with respect to $\lambda > 0$ and $t \geq 0$. As a consequence $|u_\lambda(t)|$ is uniformly bounded with respect to $\lambda > 0$ and $t \in [0, T]$.

Now, the second equation in (4.9) admits a solution in closed form

$$y_\lambda(t) = y_0 e^{-bt} + \int_0^t (\nabla\Psi(u_\lambda(s)) - a u_\lambda(s)) e^{-b(t-s)} ds, \tag{4.14}$$

which shows that $|y_\lambda(t)|$ is uniformly bounded for $\lambda > 0$ and $t \in [0, T]$, since $\nabla\Psi$ is Lipschitz continuous on bounded subsets. Consequently (return to the second equation in (4.9)), $|\dot{y}_\lambda(t)|$ is also uniformly bounded for $\lambda > 0$ and $t \in [0, T]$.

Finally, the first equation in (4.9) shows that $|\nabla\phi_\lambda(u_\lambda(t))|$ is uniformly bounded for $\lambda > 0$ and $t \in [0, T]$.

Summing up, we have the following upper bounds for some constant M

$$\sup_{\lambda>0,\, t\geq 0} |\dot{u}_\lambda(t)| \leq M, \quad \sup_{\lambda>0,\, 0\leq t\leq T} (|u_\lambda(t)|, |y_\lambda(t)|, |\dot{y}_\lambda(t)|, |\nabla\phi_\lambda(u_\lambda(t))|) \leq M. \tag{4.15}$$

(a2) Let us show that $(u_\lambda, y_\lambda)_{\lambda>0}$ is Cauchy. We consider two solutions (u_λ, y_λ) and (u_μ, y_μ) of (4.9) corresponding to two parameters λ and μ. We drop variable $t \in [0, T]$, for the sake of simplicity, and we compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u_\mu - u_\lambda|^2 &= \langle \dot{u}_\mu - \dot{u}_\lambda, u_\mu - u_\lambda \rangle \\
&= -\langle \nabla\phi_\mu(u_\mu) - \nabla\phi_\lambda(u_\lambda), u_\mu - u_\lambda \rangle \\
&\quad -a|u_\mu - u_\lambda|^2 - b\langle y_\mu - y_\lambda, u_\mu - u_\lambda \rangle.
\end{aligned} \tag{4.16}$$

Successively invoking (4.5), (4.6), the monotonicity of $\nabla\phi_\lambda$ and (4.15), we have

$$\begin{aligned}
&\langle \nabla\phi_\mu(u_\mu) - \nabla\phi_\lambda(u_\lambda), u_\mu - u_\lambda \rangle \\
&\quad = \langle \nabla\phi_\mu(u_\mu) - \nabla\phi_\lambda(u_\lambda), J_\mu(u_\mu) - J_\lambda(u_\lambda) \rangle \\
&\qquad + \langle \nabla\phi_\mu(u_\mu) - \nabla\phi_\lambda(u_\lambda), \mu\nabla\phi_\mu(u_\mu) - \lambda\nabla\phi_\lambda(u_\lambda) \rangle \\
&\quad \geq \langle \nabla\phi_\mu(u_\mu) - \nabla\phi_\lambda(u_\lambda), \mu\nabla\phi_\mu(u_\mu) - \lambda\nabla\phi_\lambda(u_\lambda) \rangle \\
&\quad = \mu \left\{ \left| \nabla\phi_\mu(u_\mu) - \frac{1}{2}\nabla\phi_\lambda(u_\lambda) \right|^2 - \frac{1}{4}|\nabla\phi_\lambda(u_\lambda)|^2 \right\} \\
&\qquad + \lambda \left\{ \left| \nabla\phi_\lambda(u_\lambda) - \frac{1}{2}\nabla\phi_\mu(u_\mu) \right|^2 - \frac{1}{4}|\nabla\phi_\mu(u_\mu)|^2 \right\} \\
&\quad \geq -\frac{\mu}{4}|\nabla\phi_\lambda(u_\lambda)|^2 - \frac{\lambda}{4}|\nabla\phi_\mu(u_\mu)|^2 \\
&\quad \geq -\frac{M^2}{4}(\lambda + \mu).
\end{aligned} \tag{4.17}$$

Let us write M instead of $M^2/4$, by abuse of notation; we then have combining (4.16) and (4.17)

$$\frac{1}{2}\frac{d}{dt}|u_\mu - u_\lambda|^2 \leq M(\lambda + \mu) - a|u_\mu - u_\lambda|^2 - b\langle y_\mu - y_\lambda, u_\mu - u_\lambda \rangle. \tag{4.18}$$

Concerning y_λ and y_μ, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|y_\mu - y_\lambda|^2 \\
&\quad = \langle \dot{y}_\mu - \dot{y}_\lambda, y_\mu - y_\lambda\rangle \\
&\quad = \langle \nabla\Psi(u_\mu) - \nabla\Psi(u_\lambda), y_\mu - y_\lambda\rangle - a\langle u_\mu - u_\lambda, y_\mu - y_\lambda\rangle - b|y_\mu - y_\lambda|^2, \\
&\quad \leq L|u_\mu - u_\lambda||y_\mu - y_\lambda| - a\langle u_\mu - u_\lambda, y_\mu - y_\lambda\rangle - b|y_\mu - y_\lambda|^2,
\end{aligned} \tag{4.19}$$

where L is the Lipschitz constant of $\nabla\Psi$ on some bounded set containing $u_\nu(t)$ for $\nu > 0$ and for $t \in [0, T]$ (recall (4.15)).
Collecting (4.18) and (4.19), we further have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|u_\mu - u_\lambda|^2 + |y_\mu - y_\lambda|^2\right) \\
&\qquad \leq \quad M(\lambda+\mu) - a|u_\mu - u_\lambda|^2 - b|y_\mu - y_\lambda|^2 \\
&\qquad\qquad -(a+b)\langle y_\mu - y_\lambda, u_\mu - u_\lambda\rangle + L|u_\mu - u_\lambda||y_\mu - y_\lambda|.
\end{aligned}$$

Whence we may infer the existence of some constant M' such that

$$\frac{1}{2}\frac{d}{dt}\left(|u_\mu - u_\lambda|^2 + |y_\mu - y_\lambda|^2\right) \leq M(\lambda+\mu) + \frac{M'}{2}\left(|u_\mu - u_\lambda|^2 + |y_\mu - y_\lambda|^2\right).$$

For simplicity of notation, let M denote the greater of the two constants M and M'. Then, integrating the above inequality on $[0, t]$ readily yields (recall $0 < t < T$)

$$|u_\mu(t) - u_\lambda(t)|^2 + |y_\mu(t) - y_\lambda(t)|^2 \leq 2(\lambda+\mu)(e^{Mt} - 1) \leq 2(\lambda+\mu)(e^{MT} - 1),$$

which shows that $(u_\lambda, y_\lambda)_{\lambda>0}$ is Cauchy in $\mathcal{C}([0,T], \mathcal{H}^2)$.

(a3) Next we derive our results by passing to the limit. As $\lambda \to 0$, u_λ and y_λ converge to some u and y in $C([0,T], \mathcal{H})$, hence in $L^2([0,T], \mathcal{H})$. Since $\dot{u}_\lambda$ and $\dot{y}_\lambda$ are bounded in $C([0,T], \mathcal{H})$, hence in $L^2([0,T], \mathcal{H})$, u and y are absolutely continuous, and $\dot{u}_\lambda$ and $\dot{y}_\lambda$ converge weakly to $\dot{u}$ and $\dot{y}$ in $L^2([0,T], \mathcal{H})$.

To deal with y, we note that we may pass to the limit in (4.14) in view of the uniform convergence of u_λ and of the Lipschitz continuity on bounded sets of $\nabla\Psi$; so we get

$$y(t) = y_0 e^{-bt} + \int_0^t (\nabla\Psi(u(s)) - au(s))e^{-b(t-s)} ds.$$

Differentiating yields the second equation in (4.2).

Now, the first equation in (4.9) also reads (recall (4.6))

$$-(\dot{u}_\lambda(t) + au_\lambda(t) + by_\lambda(t)) \in \partial\phi(J_\lambda u_\lambda(t)), \quad 0 \leq t \leq T. \tag{4.20}$$

Let $\mathcal{A}$ be the maximal monotone operator extension of $\partial\phi$ to $L^2([0,T], \mathcal{H})$ ([15, ex. 2.3.3, p. 25]). We have

$$-(\dot{u}_\lambda + au_\lambda + by_\lambda) \in \mathcal{A}(J_\lambda u_\lambda), \text{ in } L^2([0,T], \mathcal{H}).$$

On the one hand $\dot{u}_\lambda + au_\lambda + by_\lambda$ converges weakly to $\dot{u} + au + by$ in $L^2([0,T], \mathcal{H})$. On the other hand $J_\lambda u_\lambda$ converges strongly to u in $L^2([0,T], \mathcal{H})$; indeed: $|u_\lambda(t) - J_\lambda u_\lambda(t)| =$

$\lambda|\nabla\phi_\lambda(u_\lambda(t))| \leq \lambda M$ for $t \in [0,T]$ (recall (4.5, 4.15)). Taking into account the fact that the graph of $\mathcal{A}$ is sequentially strong-weak closed ([15, prop. 2.5, p. 27]), we deduce that

$$-(\dot{u} + au + by) \in \mathcal{A}u, \text{ in } L^2([0,T],\mathcal{H});$$

explicitly

$$-(\dot{u}(t) + au(t) + by(t)) \in \partial\phi(u(t)), \text{ for a.e. } t > 0.$$

(b) Case $u_0 \in \overline{\mathrm{dom}\,\partial\phi} = \overline{\mathrm{dom}\,\phi}$ and $y_0 \in \mathcal{H}$. Let $u_{0,n} \in \mathrm{dom}\,\partial\phi$ be some sequence converging to u_0, and let us switch again to formalism (4.3). For each initial condition $Z_{0,n} = (u_{0,n}, y_0)$ system (4.3) admits a solution Z_n. For $(p,n) \in \mathbb{N}^2$ we may derive from inequality (4.11) that $|Z_p(t) - Z_n(t)| \leq |u_{0,p} - u_{0,n}|e^{LT}$ for $t \in [0,T]$, which shows that (Z_n) is Cauchy in $\mathcal{C}([0,T],\mathcal{H}^2)$ and that its limit $\bar{Z}$ only depends on u_0. Set $f_n = -\mathcal{D}(Z_n)$, $\bar{f} = -\mathcal{D}(\bar{Z})$. The following properties hold:
- Z_n is a strong solution (in the sense of [15, def. 3.1, p. 64]) of $Z + \partial\widetilde{\phi}(Z) \ni f_n$, $Z(0) = Z_{0,n}$;
- $f_n \to \bar{f}$ in $\mathcal{C}([0,T],\mathcal{H}^2)$, hence in $\mathcal{L}^1([0,T],\mathcal{H}^2)$;
- $Z_n \to \bar{Z}$ in $\mathcal{C}([0,T],\mathcal{H}^2)$.

Consequently, $\bar{Z}$ is a weak solution ([15, def. 3.1, p. 64]) of $Z + \partial\widetilde{\phi}(Z) \ni \bar{f}$, with $Z(0) = (u_0, y_0)$. But ([15, th. 3.6, p. 76]) due to the special form of (4.3) (namely the maximal monotone operator involved is a subdifferential), $\bar{Z}$ is a strong solution of $Z + \partial\widetilde{\phi}(Z) \ni \bar{f}$; that is, $\bar{Z}$ is a solution of (4.2).

The existence of u on $[0,\infty[$ definitely follows from the arbitrariness of T . •

We next make precise some properties of the solution of (4.2). We denote by P_K the projection operator onto the closed convex set $K \subseteq \mathcal{H}$ and by $T_K(v)$ the closed convex tangent cone to K at point $v \in K$.

Proposition 4.1 *The solution (u,y) of (4.2) enjoys the following properties:*
a. *if $u_0 \in \overline{\mathrm{dom}\,\partial\phi} = \overline{\mathrm{dom}\,\phi}$, then $u(t) \in \mathrm{dom}\,\partial\phi$ for all $t > 0$;*
b. *$\dot{u}^+(t)$, the right derivative of u, exists for all $t > 0$ and we have*

$$\dot{u}^+(t) = -(au(t) + by(t) + \partial\phi(u(t)))^\circ = -au(t) - by(t) - P_{\partial\phi(u(t))}(-au(t) - by(t)).$$

c. *In particular, if ϕ is the indicator function δ_K of the closed convex set $K \subseteq \mathcal{H}$*

$$\dot{u}^+(t) = P_{T_K(u(t))}(-au(t) - by(t)).$$

Furthermore, for any $t > 0$ such that there exists $\epsilon > 0$ such that $t - \epsilon < s < t \Rightarrow u(s) \in$ int K, then $\dot{u}^-(t)$, the left derivative of u, exists and the following bounce law holds

$$\dot{u}^+(t) = P_{T_K(u(t))}(\dot{u}^-(t)).$$

Proof. a. Let us first suppose that $u_0 \in \mathrm{dom}\,\partial\phi$. Recall equation (4.20). As $\lambda \to 0$, we can extract from $-(\dot{u}_\lambda(t) + au_\lambda(t) + by_\lambda(t))$ a subsequence that converges weakly to some $\xi \in \mathcal{H}$. Besides $J_\lambda u_\lambda(t)$ converges strongly to $u(t)$. Taking into account the fact that the graph of $\partial\phi$ is sequentially strong-weak closed, we have $\xi(t) \in \partial\phi(u(t))$. When $u_0 \in \overline{\mathrm{dom}\,\partial\phi}$, we can notice that $t \mapsto au(t) + by(t)$ is absolutely continuous on

each interval of the form (δ, T) with $0 < \delta < T < \infty$. The conclusion then follows from [15, Th. 3.7, p. 76].
b. This point is a direct application of [15, Th. 3.5, p. 66].
c. Equality $\dot{u}^+(t) = P_{T_K(u(t))}(-au(t) - by(t))$ is a specialization of the previous point.
Let $0 < \eta < \epsilon$. Function u verifies $\dot{u}(s) + \partial\delta_K(u(s)) + au(s) + by(s) = 0$ almost everywhere on $]t-\epsilon, t[$. Since $u(s) \in \operatorname{int} K$, the cone tangent to K at $u(s)$ equals H [4]; then $\partial\delta_K(u(s))$, which is equal to the normal cone to K at $u(s)$, reduces to $\{0\}$. Hence, u being absolutely continuous

$$u(t) - u(t-\eta) = \int_{t-\eta}^{t} \dot{u}(s)ds = -\int_{t-\eta}^{t} (au(s) + by(s))ds$$

Dividing by η and letting $\eta \to 0$ yields the existence of $\dot{u}^-(t) = -(au(t) + by(t))$; hence the bounce law. . •

4.3 Dissipative properties

Let us extend to the nonsmooth case the decreasing property of the energy.

Proposition 4.2 *For $u_0 \in \operatorname{dom}(\partial\phi)$, the following results hold:*

$$\dot{u_\lambda} \to \dot{u} \;\; in \;\; L^2(0, T; \mathcal{H}) \;\; for\ all\ T > 0; \tag{4.21}$$

$$\phi_\lambda(u_\lambda(t)) \to \phi(u(t)) \;\; for\ almost\ every \;\; t > 0. \tag{4.22}$$

Moreover, the mapping $t \to E(t) = \frac{1}{2}|\dot{u}(t)|^2 + b\Theta(u(t))$ is essentially nonincreasing for any solution u of (4.2).

Proof: Let us first prove (4.21) and (4.22). Let us recall (Theorem 4.1) that the sequence (u_λ, y_λ) converges uniformly on each interval $[0, T]$ to (u, y), which is the unique solution of system (4.2). From (4.9) we have $\dot{u}_\lambda + \nabla\phi_\lambda(u_\lambda) + au_\lambda + by_\lambda = 0$. By taking the scalar product with $\dot{u_\lambda}$ in this last expression, and integrating on $[0, t]$, we obtain

$$\int_0^t |\dot{u_\lambda}(\tau)|^2 d\tau + \phi_\lambda(u_\lambda(t)) - \phi_\lambda(u_0) + \int_0^t \langle au_\lambda(\tau) + by_\lambda(\tau), \dot{u_\lambda}(\tau)\rangle d\tau = 0. \tag{4.23}$$

We now use an elementary argument ([6, Lemma 1.18]) which tells us that if, given a finite number of filtered sequences of real numbers $(d_{i,\lambda})_\lambda$, $i = 1, 2, ..., k$, one has the following conditions (c1)-(c3) satisfied:
(c1) $\sum_{i=1}^k d_{i,\lambda} = 0 \quad \forall\lambda > 0$,
(c2) $d_i \leq \liminf_{\lambda\to 0} d_{i,\lambda}$, $i = 1, 2, ..., k$,
(c3) $\sum_{i=1}^k d_i = 0$
then $d_i = \lim_{\lambda\to 0} d_{i,\lambda}$ for all $i = 1, 2, ..., k$.
Let us verify that conditions (c1)-(c3) are satisfied in our situation: Clearly, (4.23) can be rewritten as $\sum_{k=1}^4 d_{k,\lambda} = 0$, with $(d_k)_{k=1,..,4}$ defined by

[4] this is the important point, and it may hold even if $\operatorname{int} K = \emptyset$; *e. g.* $H = L^2(0, 1; \mathbb{R})$, $K = \{u \in H; u \geq 0\}$, for any $u \in K$ such that $\operatorname{ess-sup} u > 0$ the tangent cone to K at u is H.

$d_{1,\lambda} = \int_0^t |\dot{u_\lambda}(\tau)|^2 d\tau$, $d_{2,\lambda} = \phi_\lambda(u_\lambda(t))$,
$d_{3,\lambda} = -\phi_\lambda(u_0)$, and $d_{4,\lambda} = \int_0^t \langle au_\lambda(\tau) + by_\lambda(\tau), \dot{u_\lambda}(\tau)\rangle d\tau$. That is (c1).
Let us now prove that (c2) is satisfied with the following quantities
$d_1 = \int_0^t |\dot{u}(\tau)|^2 d\tau$, $d_2 = \phi(u(t))$,
$d_3 = -\phi(u_0)$, and $d_4 = \int_0^t \langle au(\tau) + by(\tau), \dot{u}(\tau)\rangle d\tau$.
¿From (4.10) we know that $\dot{u}_\lambda \rightharpoonup \dot{u}$ weakly in $L^2(0,t;\mathcal{H})$. From the weak lower semicontinuity of the mapping $v \to \int_0^t |v|^2 d\tau$ on $L^2(0,t;\mathcal{H})$, we immediately deduce that $\liminf_{\lambda\to 0} \int_0^t |\dot{u}_\lambda|^2 d\tau \geq \int_0^t |\dot{u}|^2 d\tau$, that is $\liminf_{\lambda\to 0} d_{1,\lambda} \geq d_1$.
We know that ϕ_λ increases as λ decreases to 0^+, hence, for $0 < \lambda < \lambda_0$ (where λ_0 is any positive value) we have $\phi_\lambda(u_\lambda(t)) \geq \phi_{\lambda_0}(u_\lambda(t))$. Then, for any $\lambda_0 > 0$, by continuity of ϕ_{λ_0} over $\mathcal{H}$ and using the fact that $u_\lambda(t) \to u$ strongly in $\mathcal{H}$ as $\lambda \to 0$ (see Theorem 4.1), we deduce that

$$\liminf_{\lambda\to 0} \phi_\lambda(u_\lambda(t)) \geq \liminf_{\lambda\to 0} \phi_{\lambda_0}(u_\lambda(t)) = \phi_{\lambda_0}(u(t)) \ .$$

Hence, recalling that $\phi = \sup_{\lambda>0} \phi_\lambda$, we get

$$\liminf_{\lambda\to 0} \phi_\lambda(u_\lambda(t)) \geq \sup_{\lambda_0>0} \phi_{\lambda_0}(u(t)) = \phi(u(t)),$$

that is $\liminf_{\lambda\to 0} d_{2,\lambda} \geq d_2$.
Since $u_0 \in \mathrm{dom}(\phi)$ we have $\lim_{\lambda\to 0} \phi_\lambda(u_0) = \phi(u_0)$, that is $\lim_{\lambda\to 0} d_{3,\lambda} = d_3$.
¿From Theorem 4.1 we know that $au_\lambda + by_\lambda \to au + by$ strongly in $L^2(0,t;\mathcal{H})$ and that $\dot{u}_\lambda \to \dot{u}$ weakly in $L^2(0,t;\mathcal{H})$ (as $\lambda \to 0$). It is then readily seen that $\lim_{\lambda\to 0^+} \int_0^t \langle au_\lambda + by_\lambda, \dot{u}_\lambda\rangle d\tau = \int_0^t \langle au + by, \dot{u}\rangle d\tau$, that is $\lim_{\lambda\to 0} d_{4,\lambda} = d_4$.
It remains to prove (c3). Indeed from (4.2), we know that, for a.e. $t > 0$, there exists some $\xi(t) \in \partial\phi(u(t))$ such that

$$\dot{u}(t) + \xi(t) + au(t) + by(t) = 0. \tag{4.24}$$

It is also clear that $\xi(.)$ belongs to $L^2(0,T;\mathcal{H})$ (by $\dot{u} \in L^2(0,T;\mathcal{H})$ and by u and y in $C([0,T];\mathcal{H})$). By taking the scalar product with $\dot{u}(t)$, (4.24) entails for a.e. $t \geq 0$,

$$|\dot{u}(t)|^2 + \langle \xi(t), \dot{u}(t)\rangle + \langle au(t) + by(t), \dot{u}(t)\rangle = 0. \tag{4.25}$$

Since $\xi \in L^2(0,T;\mathcal{H})$ and $\xi(t) \in \partial\phi(u(t))$ a.e. $t \geq 0$, according to [15, Lemma 3.3], the function $\phi(u)$ is absolutely continuous, and the derivation formula $(\phi(u))'(t) = \langle \xi(t), \dot{u}(t)\rangle$ holds. Therefore, by integrating (4.25) on $[0,t]$, $t > 0$, we obtain

$$\int_0^t |\dot{u}|^2 d\tau + \phi(u(t)) - \phi(u_0) + \int_0^t \langle au + by, \dot{u}\rangle d\tau = 0,$$

which is the desired result $d_1 + d_2 + d_3 + d_4 = 0$.

Consequently, for all $t > 0$, we deduce (4.22) as well as $\lim_{\lambda\to 0} \int_0^t |\dot{u}_\lambda|^2 d\tau = \int_0^t |\dot{u}|^2 d\tau$. The latter result together with (4.10) shows that, in the Hilbert space $L^2(0,T;\mathcal{H})$, we both have (as $\lambda \to 0$) weak convergence of $\dot{u}_\lambda$ to $\dot{u}$, and convergence of the norm of $\dot{u}_\lambda$ to the norm of $\dot{u}$. By a classical result we conclude that $\dot{u}_\lambda \to \dot{u}$ strongly in $L^2(0,T;\mathcal{H})$, that is (4.21).

Let us now prove that the energy-like functional E is essentially nonincreasing. The function ϕ_λ being smooth, by (3.4), for any $\lambda > 0$ and any $t \geq s \geq 0$ we have

$$\begin{aligned}\tfrac{1}{2}|\dot{u_\lambda}(t)|^2 + b(\phi_\lambda + \Psi)(u_\lambda(t)) + (a+b)\textstyle\int_s^t |\dot{u_\lambda}(\tau)|^2 d\tau \\ \leq \tfrac{1}{2}|\dot{u_\lambda}(s)|^2 + b(\phi_\lambda + \Psi)(u_\lambda(s)).\end{aligned} \tag{4.26}$$

Moreover by (4.21) we know that there exists a subsequence (that we still denote u_λ) such that $\dot{u}_\lambda(t) \to \dot{u}(t)$, for a.e. $t > 0$, strongly in $\mathcal{H}$. Using also (4.22), we can pass to the limit in the above inequality. We obtain that, for a.e. t and s verifying $t \geq s > 0$,

$$\begin{aligned}\tfrac{1}{2}|\dot{u}(t)|^2 + b(\phi + \Psi)(u(t)) + (a+b)\textstyle\int_s^t |\dot{u}(\tau)|^2 d\tau \\ \leq \tfrac{1}{2}|\dot{u}(s)|^2 + b(\phi + \Psi)(u(s)),\end{aligned}$$

which yields the desired result . •

Remark 4.4 Writing (4.26) for $s = 0$ and $t \geq 0$, we obtain

$$\frac{1}{2}|\dot{u}_\lambda(t)|^2 + b(\phi_\lambda + \Psi)(u_\lambda(t)) + (a+b)\int_0^t |\dot{u}_\lambda(\tau)|^2 d\tau \leq E_\lambda(0).$$

Arguing as in the end of the proof of Proposition 4.2 (recall $u_0 \in \operatorname{dom} \partial\phi$ and (4.13)) we deduce for almost every $t \geq 0$

$$\frac{1}{2}|\dot{u}(t)|^2 + b(\phi + \Psi)(u(t)) \leq \frac{1}{2}\left(|\partial\phi(u_0)^\circ| + |au_0 + by_0|\right)^2 + b(\phi(u_0) + \Psi(u_0)).$$

4.4 Epigraphical approximations

The Moreau-Yosida approximation enjoys nice theoretical properties, but it may be difficult to compute, and hence not convenient for numerical purpose. Instead one can use a sequence of (smooth) potentials ϕ_n which *Mosco-epiconverges* to ϕ, a notion which covers a large number of approximation methods, like exterior penalization, barrier methods, viscosity methods, Galerkin method (see [5]). This fact is illustrated in the next section.

Let us recall some classical facts concerning variational convergences for sequences of functions and operators (see [5], [6] for further details).

A sequence of convex lower semicontinuous functions $\phi_n : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is said to Mosco-epiconverge to a convex lower semicontinuous function $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ if the following two convergence properties hold in $\mathbb{R} \cup \{+\infty\}$: for any $u \in \mathcal{H}$

$$\begin{cases} \exists (u_n),\ u_n \to u \text{ strongly in } \mathcal{H},\ \text{such that } \phi_n(u_n) \to \phi(u) \\ \forall (v_n) \text{ such that } v_n \rightharpoonup u \text{ weakly in } \mathcal{H},\ \text{one has } \phi(u) \leq \liminf \phi_n(v_n). \end{cases} \tag{4.27}$$

As shown in [5, Theorem 3.66], up to a normalization, this is equivalent to the convergence of the subdifferential operators $\partial\phi_n \to \partial\phi$ in the graph sense (or equivalently in the sense of resolvents). In particular, for any sequence $(u_n, z_n) \in \mathcal{H}^2$

$$z_n \in \partial\phi_n(u_n),\ u_n \to u \text{ strongly in } \mathcal{H},\ z_n \to z \text{ weakly in } \mathcal{H}\ \Rightarrow z \in \partial\phi(u). \tag{4.28}$$

Let us stress the fact that monotone convergence implies Mosco-epiconvergence, see [5, Theorem 3.20].

Let us now fix our assumptions concerning the Mosco-approximation scheme of system (4.2):

(M1) $\phi_n : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ and $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ are convex proper lower semicontinuous functions such that the sequence (ϕ_n) Mosco-epiconverges to ϕ;

(M2) there exists some $m \in \mathbb{R}$ such that $\Psi \geq m$, $\phi \geq m$, $\phi_n \geq m$ for all $n \in \mathbb{N}$;

(M3) the sequence $(\phi_n)_{n\in\mathbb{N}}$ satisfies the following inf-compactness property: $\sup_n |v_n| < \infty$ and $\sup_n \phi_n(v_n) < \infty$ imply that the sequence $(v_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{H}$;

(M4) there exists a sequence $(u_{0n}, \xi_n) \in \mathcal{H}^2$ such that: $\xi_n \in \partial\phi_n(u_{0n})$ (hence $u_{0n} \in \operatorname{dom} \partial\phi_n$), $u_{0n} \to u_0$, the sequences (ξ_n), $(\phi_n(u_{0n}))$ are bounded.

In view of Theorem (4.1), system (4.2) admits a unique solution $(u, y) \in \mathcal{C}\left([0, +\infty[, \mathcal{H}^2\right)$ under the additional assumptions

(CP') a and b are real numbers such that $b > 0$ and $a + b \geq 0$;

(A1) $\Psi : \mathcal{H} \to \mathbb{R}$ is a differentiable function with $\nabla\Psi$ Lipschitz continuous on bounded sets.

Likewise, the approximate system

$$\begin{cases} \dot{u}_n(t) + \partial\phi_n(u_n) + au_n + by_n \ni 0, \\ \dot{y}_n(t) - \nabla\Psi(u_n) + au_n + by_n = 0, \\ u_n(0) = u_{0n}, \quad y_n(0) = y_0. \end{cases} \tag{4.29}$$

admits a unique solution $(u_n, y_n) \in \mathcal{C}\left([0, +\infty[, \mathcal{H}^2\right)$.

We will also need the following technical assumption

(H) $\mathcal{H}$ is separable.

Theorem 4.2 *Under assumptions* (M1-M2-M3-M4, CP', A1, H)*, for any* $T > 0$*, the solution* (u_n, y_n) *of the approximate problem* (4.29) *converges uniformly on* $[0, T]$ *to* (u, y)*, the solution of system* (4.2)*. Moreover*

$$\dot{u}_n \to \dot{u} \textit{ strongly in } L^2([0, T], \mathcal{H}). \tag{4.30}$$

Proof: Let (u_n, y_n) be the solution of (4.29). In view of remark (4.4) we have

$$\frac{1}{2}|\dot{u}_n(t)|^2 + b(\phi_n + \Psi)(u_n(t)) \leq \frac{1}{2}\left(|\partial\phi_n(u_{0n})^\circ| + |au_{0n} + by_0|\right)^2 + b(\phi_n + \Psi)(u_{0n}).$$

¿From assumption (M4) (notice in particular $|\partial\phi_n(u_{0n})^\circ| \leq |\xi_n|$) we deduce that there exists a constant $C \in \mathbb{R}^+$ such that, for all $n \in \mathbb{N}$ and almost all $t \geq 0$

$$\frac{1}{2}|\dot{u}_n(t)|^2 + b(\phi_n + \Psi)(u_n(t)) \leq C. \tag{4.31}$$

¿From (4.31), and the uniform minorization assumption (M2), we obtain that the sequences $(\dot{u}_n)$, (u_n) are uniformly bounded on each bounded interval $[0, T]$. By the second equation of (4.29) we infer the uniform boundedness of (y_n) and $(\dot{y}_n)$ on $[0, T]$. Let us now argue with $T > 0$ fixed. Returning to (4.31), from (M2) and (CP'), we obtain the existence of some $C_1 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$

$$\phi_n(u_n(t)) \leq C_1 \quad \forall t \in [0, T]. \tag{4.32}$$

¿From the inf-compactness assumption (M3) we deduce that the sequence (u_n) satisfies the conditions of the Ascoli-Arzela theorem on $[0, T]$. Hence there exist $u \in \mathcal{C}([0, T], \mathcal{H})$ and a subsequence u_{n_k} such that

$$u_{n_k} \to u \text{ uniformly on } [0, T]. \tag{4.33}$$

¿From the second equation of (4.29) we immediately obtain

$$y_{n_k} \to y \text{ uniformly on } [0, T]. \tag{4.34}$$

Further, from the uniform boundedness of $(\dot{u}_n, \dot{y}_n)$ and from the absolute continuity of (u_n, y_n) on $[0, T]$, we deduce the absolute continuity of (u, y) on $[0, T]$ and $(\dot{u}_{n_k}, \dot{y}_{n_k}) \rightharpoonup (\dot{u}, \dot{y})$ in $L^2([0, T], \mathcal{H}^2)$.

For simplicity of notation, we write u_n and y_n (instead of u_{n_k} and y_{n_k}). We now argue in the space $L^2([0, T], \mathcal{H})$ and consider $\mathcal{A}_n$, $\mathcal{A}$ the maximal monotone operator extensions of $\partial\phi_n$, $\partial\phi$ to $L^2([0, T], \mathcal{H})$ (as in the proof of Theorem 4.1). Actually $\mathcal{A}_n$ and $\mathcal{A}$ are convex subdifferential operators; namely: $\mathcal{A}_n = \partial\mathcal{F}_n$, $\mathcal{A} = \partial\mathcal{F}$ (see [15, Proposition 2.16]) where

$$\begin{array}{c} \mathcal{F}_n(v) = \int_0^T \phi_n(v(s))ds, \text{ if } \phi_n(v) \in L^1([0, T], \mathcal{H}), \ +\infty \text{ elsewhere,} \\ \mathcal{F}(v) = \int_0^T \phi(v(s))ds, \text{ if } \phi(v) \in L^1([0, T], \mathcal{H}), \ +\infty \text{ elsewhere.} \end{array} \tag{4.35}$$

Thus the first inclusion in (4.29) also reads

$$-(\dot{u}_n + au_n + by_n) \in \partial\mathcal{F}_n(u_n) \quad \text{in } L^2([0, T], \mathcal{H}).$$

Now, the sequence $\mathcal{F}_n$ Mosco-epiconverges to $\mathcal{F}$ in $L^2([0, T], \mathcal{H})$ (see [6, Corollaire 1.17], where assumption (H) is used). On the one hand $-(\dot{u}_n + au_n + by_n)$ converges weakly to $-(\dot{u} + au + by)$ in $L^2([0, T], \mathcal{H})$. On the other hand u_n converges strongly to u in $L^2([0, T], \mathcal{H})$. From the weak-strong closedness property (4.28) we deduce

$$-(\dot{u} + au + by) \in \partial\mathcal{F}(u) \quad \text{in } L^2([0, T], \mathcal{H});$$

explicitly

$$\dot{u}(t) + \partial\phi(u(t)) + au(t) + by(t) \ni 0, \text{ for a.e. } t > 0.$$

Passing to the limit on the second equation of (4.29) is immediate, because $\nabla\Psi$ is continuous on $\mathcal{H}$. Hence (u, y) is the solution of system (4.2) with Cauchy data $u(0) = u_0$ (recall (M4)) and $y(0) = y_0$. By uniqueness of the solution of system (4.2) (Theorem 4.1), the whole sequence (u_n, y_n) has a unique limit point, and we conclude by a standard compactness argument that (u_n, y_n) converges uniformly to (u, y) on $[0, T]$, where (u, y) is the solution of system (4.2). Following the same argument as in the proof of Proposition 4.2, we then pass from the weak to the strong convergence of sequence $(\dot{u}_n)$ in $L^2([0, T], \mathcal{H})$. •

5 A nonlinear hyperbolic model for viscoelastic material with unilateral constraint

We consider the vibration problem for a viscoelastic membrane clamped on its boundary and with unilateral conditions (obstacle problem). We wish to study this problem in light of the results of Section 4.
Let us specify the notations and the functional setting. Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is locally C^1. Take $\mathcal{H}$ equal to the Hilbert space $L^2(\Omega)$ of the Lebesgue square integrable functions on Ω. Let $\chi : \Omega \to \mathbb{R}$ be an obstacle function, which is supposed to be regular, say $\chi \in H^1_0(\Omega) \cap H^2(\Omega)$. Let

$$K = \Big\{ v \in H^1_0(\Omega) : \ v(x) \geq \chi \ \ a.e. \ x \in \Omega \Big\} \tag{5.1}$$

be the constraint set of admissible deplacements. Clearly, $K \neq \emptyset$, since $\chi \in K$.
Let us define the internal energy functional $\phi : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi(v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla v(x)|^2 dx & \text{if } v \in K \\ +\infty & \text{if } v \in L^2(\Omega), \ v \notin K. \end{cases}$$

One can easily verify that ϕ is a convex l.s.c. (lower semi-continuous) function on $L^2(\Omega)$. Indeed, for any $\lambda \in \mathbb{R}$, by coerciveness of ϕ on $H^1_0(\Omega)$ (which itself is a consequence of Poincaré inequality) we have that the sublevel set $\{v \in L^2(\Omega) : \ \phi(v) \leq \lambda\}$ is bounded in $H^1_0(\Omega)$ and hence weakly relatively compact in $H^1_0(\Omega)$. The conclusion follows from the lower semicontinuity of ϕ for the weak topology of $H^1_0(\Omega)$ (note that K is a closed convex subset of $H^1_0(\Omega)$).
Let us compute the subdifferential operator $\partial\phi$. Making a translation in $H^1_0(\Omega)$, we need only consider the case $\chi = 0$. Set $u = \widetilde{u} + \chi$. An elementary computation shows that

$$z \in \partial\phi(u) \Leftrightarrow z + \Delta\chi \in \partial\widetilde{\phi}(\widetilde{u}) \tag{5.2}$$

where

$$\widetilde{\phi}(v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla v(x)|^2 dx & \text{if } v \geq 0 \quad \text{on } \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $\widetilde{\phi} = \phi_1 + \phi_2$, where ϕ_1 is the Dirichlet energy and ϕ_2 is the indicator function of the positive cone in $L^2(\Omega)$. As a key property, we use that contractions operate with respect to the Dirichlet energy ([8] Theorem 5.8.2), in particular

$$\|v \vee 0\|_{H^1} \leq \|v\|_{H^1} \quad \forall v \in H^1(\Omega), \tag{5.3}$$

where $v \vee 0$ is the positive part of v. Noticing that, for any $\lambda > 0$, $(I + \lambda\partial\phi_2)^{-1} v = v \vee 0$, we deduce that, for any $v \in \mathrm{dom}\phi_1 = H^1_0(\Omega)$, for any $\lambda > 0$

$$\phi_1 \left((I + \lambda\partial\phi_2)^{-1} v \right) \leq \phi_1(v). \tag{5.4}$$

Following [15, Prop. 2.17] (see also [16, Thm. 9]), we can apply the additivity rule for the subdifferential of the sum of two convex lsc. functions, namely $\partial\widetilde{\phi} = \partial\phi_1 + \partial\phi_2$.

We use the description of the subdifferential of ϕ_1 given in [17] (which makes use of the regularity assumption on $\partial\Omega$, and Agmon-Douglis-Nirenberg regularity result for the Poisson equation on Ω) to obtain
$z \in \partial\phi(u) \Leftrightarrow \ \tilde{u} \in H^2(\Omega) \cap H_0^1(\Omega), \ \ \tilde{u} \geq 0$ on Ω, and there exists $\mu \in L^2(\Omega)$ such that

$$\begin{cases} z + \Delta\chi = -\Delta\tilde{u} - \mu \quad \text{on } \Omega, \\ \mu \geq 0 \quad \text{on } \Omega \\ \mu\tilde{u} = 0 \quad \text{on } \Omega. \end{cases} \tag{5.5}$$

Equivalently $z \in \partial\phi(u) \Leftrightarrow \ u \in H^2(\Omega) \cap H_0^1(\Omega), \ \ u \geq \chi$ on Ω, and there exists $\mu \in L^2(\Omega)$ such that

$$\begin{cases} z = -\Delta u - \mu \quad \text{on } \Omega, \\ \mu \geq 0 \quad \text{on } \Omega \\ \mu(u - \chi) = 0 \quad \text{on } \Omega, \end{cases} \tag{5.6}$$

which provides the description of the subdifferential operator $\partial\phi$ in the space $L^2(\Omega)$. Let us now consider the external energy potential $\Psi : L^2(\Omega) \to \mathbb{R}$, which is prescribed to be convex and differentiable. For example, take $f \in L^2(\Omega)$ and

$$\Psi(v) = \int_\Omega j(v(x))dx \ - \int_\Omega f(x)v(x)dx, \tag{5.7}$$

with $j : r \in \mathbb{R} \mapsto j(r) \in \mathbb{R}$ a convex function, whose derivative $p = j'$ satisfies the following growth condition: there exists some constant $k \in \mathbb{R}^+$ such that, for any $r, s \in \mathbb{R}$

$$|p(r) - p(s)| \leq k|r - s|. \tag{5.8}$$

(In contrast with the finite dimensional setting, in infinite dimensional spaces the local Lipschitz property of the gradient does not give much more modelling possibilities than the corresponding global Lipschitz property). The gradient of Ψ is described as follows

$$z = \nabla\Psi(u) \Leftrightarrow z(x) = p(u(x)) - f(x) \ \ a.e. \ x \in \Omega. \tag{5.9}$$

We are ready to apply the results of Section 4. Given parameters $\alpha \geq 0$ and $\beta > 0$, we consider the system

$$\begin{cases} \dot{u}(t) + \beta\partial\phi(u(t)) + \left(\alpha - \dfrac{1}{\beta}\right)u(t) + \dfrac{1}{\beta}y(t) \ni 0, \\ \dot{y}(t) - \beta\nabla\Psi(u(t)) + \left(\alpha - \dfrac{1}{\beta}\right)u(t) + \dfrac{1}{\beta}y(t) = 0. \end{cases} \tag{5.10}$$

By Theorem 4.1, there exists a unique global strong solution (u, y) of system (5.10). Let us interpret (5.10) by using description (5.6) of $\partial\phi$. By making the same transformation as in Section 2 (equivalence of the two systems), we obtain at least formally (some derivations have to be justified)

$$\begin{cases} \ddot{u}(t) - \beta\Delta\dot{u}(t) - \beta\dot{\mu}(t) + \alpha\dot{u}(t) + p(u(t)) + (-\Delta u(t) - f(t) - \mu(t)) \ni 0 \quad \text{on } \Omega, \\ \mu(t) \geq 0 \quad \text{on } \Omega \\ u(t) \geq \chi \quad \text{on } \Omega \\ \mu(t)(u(t) - \chi) = 0 \quad \text{on } \Omega. \end{cases} \tag{5.11}$$

This system must be interpreted rigorously as (5.10). Moreover the following shock law holds (lazy system): Set $X(t) = -\left(\alpha - \frac{1}{\beta}\right) u(t) - \frac{1}{\beta} y(t)$, by Proposition 4.1,

$$\dot{u}^+(t) = X(t) - P_{\partial\phi(u(t))}(X(t)). \tag{5.12}$$

By (5.6), and after computation of a projection in $L^2(\Omega)$ (which indeed can be localized) we obtain

$$\dot{u}^+(t,x) = \begin{cases} X(t) + \Delta u(t,x), & \text{if } u(t,x) > \chi(x) \\ \sup\{X(t,x) + \Delta u(t,x); 0\} & \text{if } u(t,x) = \chi(x). \end{cases}$$

Numerically, we can compute an approximate trajectory by using the approximation of system (5.10) obtained by replacing ϕ by the penalization

$$\phi_n(v) = \frac{1}{2}\int_\Omega |\nabla v(x)|^2 dx \ + \frac{n}{2}\int_\Omega ((v(x) - \chi(x))^-)^2 dx. \tag{5.13}$$

The assumptions of Theorem 4.2 are satified: monotone convergence implies Mosco-epiconvergence, and the inf-compactness property is satisfied (thanks to the Rellich-Kondrachov theorem). By contrast with the Moreau-Yosida approximation, the above penalization approach is convenient for computation. It yields the following system

$$\begin{cases} \dot{u}(t) - \beta\Delta u(t) + n\,(u(t) - \chi)^- + \left(\alpha - \dfrac{1}{\beta}\right) u(t) + \dfrac{1}{\beta} y(t) = 0, \\ \dot{y}(t) - \beta\nabla\Psi(u(t)) + \left(\alpha - \dfrac{1}{\beta}\right) u(t) + \dfrac{1}{\beta} y(t) = 0. \end{cases} \tag{5.14}$$

In [33], Petrov-Schatzman use a different method (Fourier analysis) and contact law, in the study of a viscoelastic monodimensional bar with unilateral conditions.

6 Impact dynamics with restitution coefficient

Let us give an illustration of our approach in a finite dimentional setting $\mathcal{H} = \mathbb{R}^n$. Specifically, we consider the following differential inclusion problem which arises in the modelling of nonelastic shocks:

$$\begin{cases} \ddot{u}(t) + \alpha\dot{u}(t) + \partial\delta_K(u(t)) + \nabla\Psi(u(t)) \ni 0, \\ \dot{u}(t^+) = -e\dot{u}_N(t^-) + \dot{u}_T(t^-) \quad \text{for any } t \text{ such that } u(t) \in \partial K. \end{cases} \tag{6.1}$$

In (6.1), K is a closed convex subset of $\mathcal{H}$, ∂K denotes the boundary of K, $\partial\delta_K$ is the Fenchel subdifferential of the indicator function δ_K of K, ($\partial\delta_K(u)$ is the outward normal cone to K at point u). The parameter $e \in (0,1)$ is a restitution coefficient (of the normal velocity). This latter system models the evolution of a mechanical system with inertia, whose state $u(t)$ is forced to remain in K. The system is subject to different forces: the inertia force (involving the acceleration, for simplicity the inertia matrix has been taken equal to the identity), a potential driving force $-\nabla\Psi(u)$, and a viscous friction force $f = -\alpha\dot{u}$. When the system hits the boundary at $u(t)$, it is subject to a reaction force (which belongs to $-\partial\delta_K(u(t))$), with a shock law which can be described as follows: $\dot{u}_T(t^-)$, the tangential velocity of $\dot{u}(t^-)$, is preserved, while $\dot{u}_N(t^-)$, the normal velocity of $\dot{u}(t^-)$, is reversed and multiplied by a restitution coefficient $e \in (0,1)$. One can consult [1,14,19,28] for an extended presentation of the theory of impact dynamics.

6.1 The Paoli-Schatzman model

In [29], Paoli and Schatzman considered differential inclusion problems of the form

$$\begin{cases} \ddot{u}(t) + \partial\delta_K(u(t)) \ni f(t, u(t), \dot{u}(t)), \\ \dot{u}(t^+) = -e\dot{u}_N(t^-) + \dot{u}_T(t^-) \quad \text{for any } t \text{ such that } u(t) \in \partial K. \end{cases} \tag{6.2}$$

As a key ingredient of their approach, they introduced an approximate version of (6.2) given by the following second-order differential equation

$$\ddot{u}_\lambda(t) + \frac{2\epsilon}{\sqrt{\lambda}} G((I - P_K)u_\lambda(t), \dot{u}_\lambda(t)) + \frac{(I - P_K)u_\lambda(t)}{\lambda} = f(t, u_\lambda(t), \dot{u}_\lambda(t)). \tag{6.3}$$

In (6.3), λ is a positive (penalization) parameter, I is the identity mapping, P_K is the metric projection from $\mathcal{H}$ onto K, the parameter ϵ is linked with e by

$$\epsilon = -(\log e)/\sqrt{\pi^2 + (\log e)^2}, \tag{6.4}$$

and G is the mapping defined for any $(v, w) \in \mathcal{H}^2$ by

$$G(v, w) = \frac{(v, w)v}{|v|^2} \quad \text{if } v \neq 0, \quad G(v, w) = 0 \quad \text{otherwise.} \tag{6.5}$$

It is proved in [29] (when $\mathcal{H}$ is finite dimensional) that the solution u_λ of (6.3) admits a sub-sequence that converges (in some sense) as $\lambda \to 0^+$ to a solution of (6.2).
In the one-dimensional setting one can perform an explicit computation of the trajectories of system (6.3), which shows the relevancy of this approach: Take $\mathcal{H} = \mathrm{I\!R}, K = \mathrm{I\!R}^+, f = 0$; set $\mathrm{sgn}^-(v) = 0$ if $v \geq 0$, 1 otherwise. Equation (6.3) becomes

$$\ddot{u}_\lambda(t) + \frac{2\epsilon}{\sqrt{\lambda}} \dot{u}_\lambda(t)\mathrm{sgn}^-(u_\lambda(t)) + \frac{1}{\lambda} u_\lambda(t)\mathrm{sgn}^-(u_\lambda(t)) = 0. \tag{6.6}$$

For given Cauchy data $u_\lambda(0) = u_0 > 0$, $\dot{u}_\lambda(0) = \dot{u}_0 < 0$, the analytic solution of equation (6.6) is given by

$$u_\lambda(t) = \begin{cases} u_0 + t\dot{u}_0 \quad \text{if } t \in [0, \tau] \\ \frac{\dot{u}_0\sqrt{\lambda}}{\sqrt{1-\epsilon^2}} e^{-\epsilon\frac{t-\tau}{\sqrt{\lambda}}} \sin\left(\frac{\sqrt{1-\epsilon^2}}{\sqrt{\lambda}}(t - \tau)\right) \quad \text{if } t \in [\tau, \tau_\lambda] \\ -\dot{u}_0(t - \tau_\lambda) \exp\left(-\frac{\pi\epsilon}{\sqrt{1-\epsilon^2}}\right) \quad \text{if } t \in [\tau_\lambda, +\infty]. \end{cases} \tag{6.7}$$

where $\tau = -\frac{u_0}{\dot{u}_0}$, $\tau_\lambda = \tau + \frac{\pi\sqrt{\lambda}}{\sqrt{1-\epsilon^2}}$.
As $\lambda \to 0$, we have $\tau_\lambda \to \tau$, $u_\lambda \to u$, and $\dot{u}(\tau^+) = -e\dot{u}(\tau^-)$ with e and ϵ tied by (6.4).

6.2 A model based on the Hessian-damping

Relying on this type of idea, we wish to propose an approximating process of practical interest regarding the evolution problem (6.1). To that end, we denote by $\Phi_{K,\lambda}$ the Moreau-Yosida approximation of δ_K (the indicator function of $K \subset \mathcal{H}$) defined for any $v \in \mathcal{H}$ and any $\lambda > 0$ by $\Phi_{K,\lambda}(v) = (1/2\lambda)\mathrm{dist}^2(v,K)$. It is a classical result that $\Phi_{K,\lambda}$ is a C^1 function whose gradient is given by $\nabla\Phi_{K,\lambda}(v) = (1/\lambda)\,(v - P_K v)$. A main observation which was done in [4] is that, if K is a half space

$$\frac{1}{\sqrt{\lambda}} G((I - P_K)u_\lambda(t), \dot{u}_\lambda(t)) = \sqrt{\lambda}\nabla^2\Phi_{K,\lambda}(u_\lambda(t))\dot{u}_\lambda(t), \tag{6.8}$$

whenever the Hessian $\nabla^2\Phi_{K,\lambda}(.)$ is defined at $u_\lambda(t)$. Thus, we propose to consider the following system

$$\ddot{u}_\lambda(t) + \alpha\dot{u}_\lambda(t) + 2\epsilon\sqrt{\lambda}\nabla^2\Phi_{K,\lambda}(u_\lambda(t))\dot{u}_\lambda(t) + \nabla\Phi_{K,\lambda}(u_\lambda(t)) + \nabla\Psi(u_\lambda(t)) = 0 \tag{6.9}$$

as a hopefully simpler approach to (6.1).

It is worth recalling that (6.9) was discussed in [9], only in the special instance $\Psi = 0$, $\epsilon = 0$. Thanks to Theorem 2.1, we can equivalently formulate equation (6.9) as a first order system. Setting $\beta = 2\epsilon\sqrt{\lambda}$, $b = 1/\beta$ and $a = \alpha - 1/\beta$, we obtain

$$\begin{cases} \dot{u}_\lambda(t) + 2\epsilon\sqrt{\lambda}\nabla\Phi_{K,\lambda}(u_\lambda(t)) + \left(\alpha - \dfrac{1}{2\epsilon\sqrt{\lambda}}\right) u_\lambda(t) + \dfrac{1}{2\epsilon\sqrt{\lambda}} y_\lambda(t) = 0, \\ \dot{y}_\lambda(t) - 2\epsilon\sqrt{\lambda}\nabla\Psi(u_\lambda(t)) + \left(\alpha - \dfrac{1}{2\epsilon\sqrt{\lambda}}\right) u_\lambda(t) + \dfrac{1}{2\epsilon\sqrt{\lambda}} y_\lambda(t) = 0. \end{cases} \tag{6.10}$$

Clearly, (6.9) does not make sense when $\nabla^2\Phi_{K,\lambda}(.)$ is not defined at $u_\lambda(t)$, which precisely occurs when $u_\lambda(t)$ hits the boundary of K! By contrast, system (6.10) can always be defined. This naturally suggests using (6.10) (with no occurrence of the Hessian of Φ_λ) in order to approximate the differential inclusion (6.1). From a theoretical viewpoint, it would be of great interest to establish rigourously the link (by making $\lambda \to 0^+$) between trajectories of (6.10) and trajectories of (6.1). This requires further studies which are out of the scope of the present article

We conclude this section by showing some numerical experiments (Figures 1, 2). Recall that $\Phi_{K,\lambda} \equiv 0$ on K, so that $\nabla\Phi_{K,\lambda} \equiv 0$ and $\nabla^2\Phi_{K,\lambda} \equiv 0$ on $\mathrm{int}(K)$ (the interior of K). Hence, as long as $u_\lambda(t)$ lies in $\mathrm{int}(K)$, for some positive time t, it satisfies (see (6.9)) $\ddot{u}_\lambda(t) + \alpha\dot{u}_\lambda(t) + \nabla\Psi(u_\lambda(t)) = 0$. It is then clear that the trajectories of (6.10) are independent of the parameters ϵ and λ as long as the state $u_\lambda(t)$ remains in $\mathrm{int}(K)$. These trajectories are influenced by ϵ, λ and the term $\nabla^2\Phi_{K,\lambda}(u_\lambda(t))$ only when $u_\lambda(t)$ is located outside $\mathrm{int}(K)$.

Impacts in a disk: Figures 1, 2 illustrate some trajectories $u_\lambda = (u_1, u_2)$ verifying system (6.10) when $\mathcal{H} = \mathbb{R}^2$, K is the disk of center $(0,0)$ and radius 1, and Ψ is the convex function defined for any $(u_1,u_2) \in \mathbb{R}^2$ by $\Psi(u_1,u_2) = 0.0025 * ((u_1 - u_{01})^2 + (u_2 - u_{02})^2)$ for several data $(u_{01}, u_{02}) \in \mathbb{R}^2$. The initial conditions are $u_\lambda(0) = (0.5, 0)$ and $y_\lambda(0) = -2\epsilon\sqrt{\lambda}\left(\frac{2\epsilon\sqrt{\lambda}}{2\alpha\epsilon\sqrt{\lambda}-1} + (0, 0.1)\right)$, which ensures that $\dot{u}_\lambda(0) = (0, 0.1)$.

In Figure 1, $(u_{01}, u_{02}) = (0,0)$, the minimum of Ψ over K is attained at $(0,0)$ which belongs to the set K. After a finite number of shocks the constraint K is no more active and the trajectories converge to $(0,0)$. It can be noticed that the normal velocity of u_λ, up to a shock, increases with respect to the restitution coefficient e.

In Figure 2, $(u_{01}, u_{02}) = (-1,-1)$, the minimum of Ψ over $\mathcal{H} = \mathbb{R}^2$ is attained outside K. In that case, the constraint is active at the optimum which is equal to $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. We observe many shocks (whose number increases as the shocks tend to be elastic) which are located close to the constrained optimum.

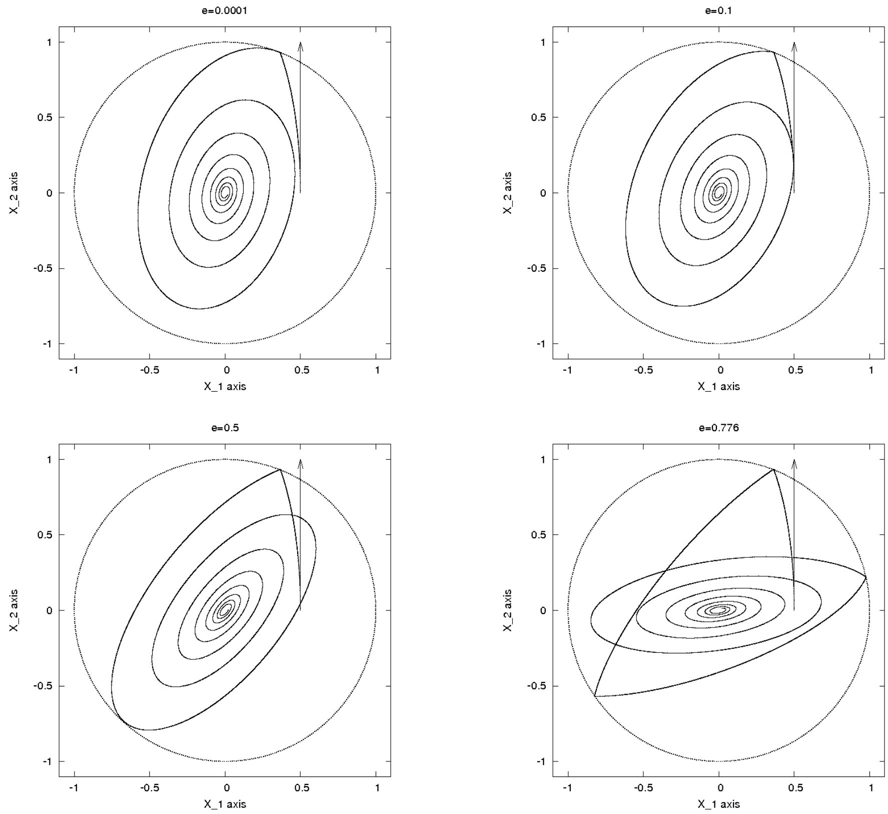

Figure 1: Evolution of the profile of $u_\lambda(t) = (\mathrm{u}_1(t), \mathrm{u}_2(t))$ for several values of the restitution coefficient e. The other parameters are $\lambda = 0.001$ and $\alpha = 0.01$.

7 Convergence and asymptotic stabilization

In this section we investigate the asymptotic behaviour of the trajectories of sytem (3.1), and to keep the exposition within reasonable length the potential Φ is supposed smooth. Some of the results established here require the following additional conditions:

(A4) $\Theta := \Phi + \Psi$ is a convex function;

(CPP) a and b are real numbers such that $b > 0$ and $a + b > 0$.

¿From now on we make the standing assumptions **(A1)-(A4)** and **(CPP)**. Note that Φ and $\Psi + \Phi$ are assumed to be convex while Ψ may be nonconvex.

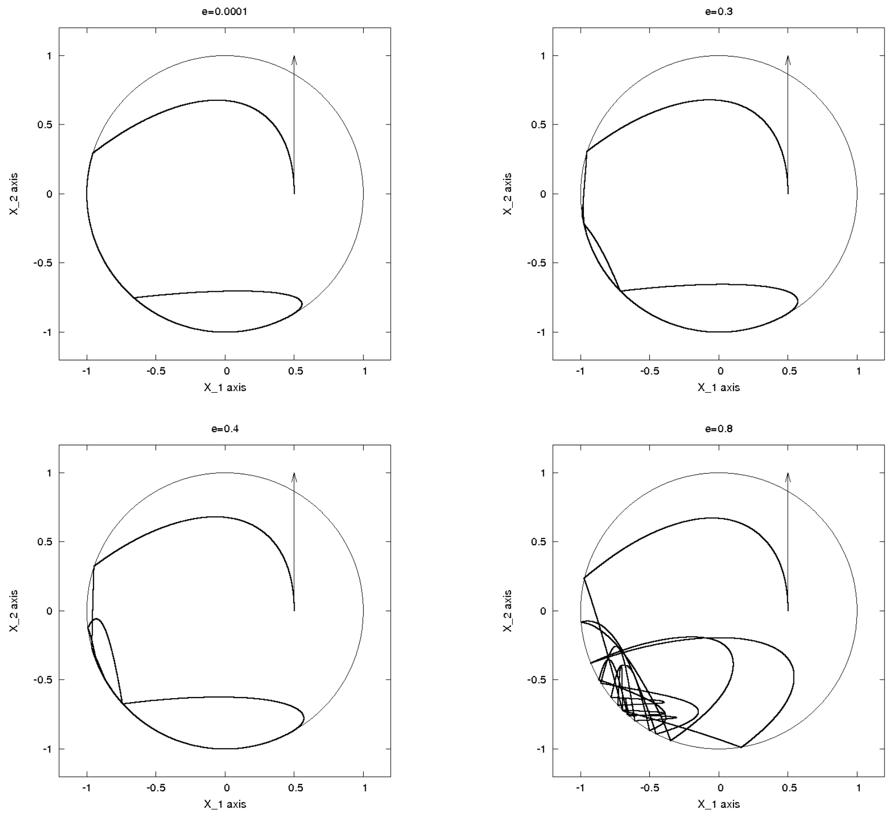

Figure 2: Evolution of the profile of $u_\lambda(t) = (\mathrm{u}_1(t), \mathrm{u}_2(t))$ for several values of e (with $\lambda = 0.001$ and $\alpha = 0.01$). On each sub-figure, $u_\lambda(t)$ converges as $t \to +\infty$ to $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, the minimizer of Ψ over K.

7.1 Weak convergence

Our analysis is mainly based upon Lyapunov techniques. These approaches are widely used for investigating asymptotic properties of dissipative systems, one may consult [3, 11, 13, 20–23] for related studies.

With any element $q \in \mathcal{H}$ and any classical solution (u, y) of (3.1), we associate the real-valued function F defined for $t \geq 0$ by

$$F(t) = \langle q - u(t), au(t) + by(t)\rangle + \frac{(a+b)}{2}|u(t) - q|^2 - \Phi(u(t)) - \int_0^t |\dot{u}(s)|^2 ds. \quad (7.11)$$

In order to set up our main convergence result, we first establish a lemma underlining the dissipative nature of system (3.1).

Lemma 7.1 *For any $t \geq 0$, the time derivative of F is given by*

$$\dot{F}(t) = -b\langle \nabla\Theta(u(t)), u(t) - q\rangle, \quad (7.12)$$

where $\Theta := \Phi + \Psi$. Hence, F is nonincreasing provided that $q \in S := (\nabla\Theta)^{-1}(0)$.

Proof: Derivating F we obtain (we drop variable t for simplicity)

$$\begin{aligned}\dot F(t) &= -\langle \dot u, au+by\rangle + \langle q-u, a\dot u + b\dot y\rangle + (a+b)\langle u-q,\dot u\rangle + \langle \dot u + au + by, \dot u\rangle - |\dot u|^2\\ &= -\langle u-q, b(\dot y - \dot u)\rangle\\ &= -b\langle u-q, \nabla\Theta(u)\rangle \quad \text{(according to (3.2))}.\end{aligned}$$

From the convexity of Θ (hence the monotonicity of $\nabla\Theta$), we deduce that $F(.)$ is nonincreasing whenever $q \in S$. •

Let us state our main convergence result.

Theorem 7.1 *Suppose that conditions* **(A1)-(A4)** *and* **(CPP)** *are satisfied. Then, the unique classical global solution* (u, y) *of problem* (3.1) *satisfies:*

(r1) Convergence of the energy $E = b\Theta(u) + (1/2)|\dot u|^2$:

$$\lim_{t\to+\infty} E(t) = b\inf_{\mathcal{H}} \Theta, \qquad \lim_{t\to+\infty} \Theta(u(t)) = \inf_{\mathcal{H}} \Theta, \qquad \lim_{t\to+\infty} |\dot u(t)| = 0.$$

If, in addition, the set S *of minimizers of* $\Theta := \Phi + \Psi$ *is nonempty, then we have:*

(r2) $u, y, \dot y \in L^\infty([0,+\infty), \mathcal{H})$*;*

(r3) Estimates on the energy decay:

$$E(.) - b\inf_{\mathcal{H}}\Theta \in L^1(0,+\infty), \quad \Theta(u) - \inf_{\mathcal{H}}\Theta \in L^1(0,+\infty);$$

$$\lim_{t\to+\infty} t(E(t) - b\inf_{\mathcal{H}}\Theta) = \lim_{t\to+\infty} t(\Theta(u(t)) - \inf_{\mathcal{H}}\Theta) = \lim_{t\to+\infty} t|\dot u(t)|^2 = 0;$$

(r4) Convergence of the trajectory:
$u(t)$ *weakly converges to some element* u_∞ *in* S *as* t *goes to infinity, provided that one of the following additional conditions is satisfied:*
(i1) S *is a singleton;*
(i2) $\nabla\Phi$ *or* $\nabla\Psi$ *is weakly (sequentially) closed;* i. e. $\nabla\Phi$ *or* $\nabla\Psi$ *is sequentially continuous from* $\mathcal{H}$ *endowed with its weak topology to* $\mathcal{H}$ *endowed with its weak topology.*
(i3) Ψ *is convex.*

Proof: **(r1)** By convexity of Θ and Φ, given any $(z,q) \in \mathcal{H}^2$, we have

$$\begin{aligned}\Theta(q) &\geq \Theta(z) + \langle \nabla\Theta(z), q - z\rangle\\ \Phi(q) &\geq \Phi(z) + \langle \nabla\Phi(z), q - z\rangle.\end{aligned} \tag{7.13}$$

Moreover, by the first equation of (3.1), the quantity $F(t)$ given by (7.11) can be rewritten as (for simplicity we often ignore the dependency of u, $\dot u \dots$ on t)

$$F(t) = -\langle q-u, \nabla\Phi(u)\rangle - \langle q-u,\dot u\rangle + \frac{(a+b)}{2}|u-q|^2 - \Phi(u) - \int_0^t |\dot u(s)|^2 ds. \tag{7.14}$$

On the one hand, from the second inequality in (7.13), we deduce that

$$F(t) \geq -\Phi(q) + \langle u-q,\dot u\rangle + \frac{(a+b)}{2}|u-q|^2 - \int_0^t |\dot u(s)|^2 ds. \tag{7.15}$$

Recalling $\dot{u} \in L^2(0,+\infty;\mathcal{H})$ (Theorem 3.1-r2) and setting $C = -\Phi(q) - \int_0^\infty |\dot{u}(s)|^2 ds$, we obtain

$$F(t) \geq C + \frac{d}{dt}\left(\frac{1}{2}|u-q|^2\right). \tag{7.16}$$

On the other hand the energy mapping $E(.) = b\Theta(u(.)) + (1/2)|\dot{u}(.)|^2$ is nonincreasing and bounded below (Theorem 3.1-r3 and (A3)); hence $E_\infty := \lim_{t\to+\infty} E(t)$ exists, so we have $\Theta(u) \geq (1/b)E_\infty - (1/2b)|\dot{u}|^2$. This inequality, combined with the first one in (7.13), yields

$$b\Theta(q) - E_\infty \geq -(1/2)|\dot{u}|^2 + b\langle \nabla\Theta(u), q-u\rangle.$$

Integrating between 0 and t, and using (7.12), we obtain

$$\begin{aligned} t(b\Theta(q) - E_\infty) &\geq -(1/2)\int_0^t |\dot{u}(s)|^2 ds + b\int_0^t \langle \nabla\Theta(u(s)), q-u(s)\rangle ds \\ &\geq -(1/2)\int_0^\infty |\dot{u}(s)|^2 ds + F(t) - F(0). \end{aligned} \tag{7.17}$$

Combining (7.16) with (7.17), and setting $C_1 = C - (1/2)\int_0^\infty |\dot{u}(s)|^2 ds - F(0)$, we deduce $t(b\Theta(q) - E_\infty) \geq C_1 + \frac{d}{dt}\left(\frac{1}{2}|u-q|^2\right)$. By integrating this inequality from 0 to t, we obtain

$$\frac{t^2}{2}(b\Theta(q) - E_\infty) \geq C_1 t + (1/2)|u(t)-q|^2 - (1/2)|u_0 - q|^2,$$

which readily entails $b\Theta(q) - E_\infty \geq \frac{2C_1}{t} - \frac{1}{t^2}|u_0 - q|^2$. Whence we deduce $b\inf\Theta \geq E_\infty$ (in view of the arbitrariness of $t > 0$ and $q \in \mathcal{H}$). Joining this last estimate to the inequality $b\Theta(u) \leq E$ gives us the chain of inequalities

$$\limsup_{t\to+\infty} b\Theta(u(t)) \leq \limsup_{t\to+\infty} E(t) = E_\infty \leq b\inf\Theta \leq \liminf_{t\to+\infty} b\Theta(u(t)), \tag{7.18}$$

which allows to conclude $\lim_{t\to\infty} b\Theta(u(t)) = \lim_{t\to\infty} E(t) = b\inf\Theta$. It follows immediately from the definition of E that $\lim_{t\to+\infty} |\dot{u}(t)| = 0$.

(r2) The functional F is nonincreasing if $q \in S$ (by Lemma 7.1). Therefore we deduce from (7.15)

$$F(0) \geq C - |u-q| \times |\dot{u}| + (1/2)(a+b)|u-q|^2.$$

We know from Theorem 3.1-r2 that $\dot{u} \in L^\infty(0,+\infty;\mathcal{H})$. Therefore the inequality above shows that $|u-q|$ is bounded on $[0,+\infty[$, and so is $|u|$. Let us now turn to y. The second equation in (3.1) can be solved in closed form

$$y(t) = y_0 e^{-bt} + \int_0^t (\nabla\Psi(u(s)) - au(s))e^{-b(t-s)} ds. \tag{7.19}$$

The boundedness of u and assumption (A1) entail the boundedness of y. It follows from the second equation of (3.1) that $\dot{y}$ also is bounded.

(r3) Fix $q \in S$. The functional F is bounded below (use (7.16) and (r2)), besides it is nonincreasing; hence $= \lim_{t\to+\infty} F(t)$ exists. Thus, the nonnegative mapping

$t \to \langle \nabla\Theta(u(t)), u(t)-q\rangle$ belongs to $L^1(0,+\infty)$ (in light of Lemma 7.1). Moreover, by (7.13), and recalling $\Theta(q) = \inf\Theta$, we have $0 \le \Theta(u) - \inf\Theta \le \langle \nabla\Theta(u), u-q\rangle$. This clearly shows that $\Theta(u) - \inf\Theta$ belongs to $L^1(0,+\infty)$, and so does $E - b\inf\Theta$ (as $E - b\inf\Theta = (1/2)|\dot{u}|^2 + b(\Theta(u) - \inf\Theta)$, with $\dot{u} \in L^2(0,+\infty;\mathcal{H})$). It turns out that the function $E - b\inf\Theta$ is nonnegative, nonincreasing and belongs to $L^1(0,+\infty)$. It follows immediately that

$$\int_t^{2t} (E(s) - b\inf\Theta)ds \ge t(E(2t) - b\inf\Theta) \ge 0.$$

Noticing that the above integral vanishes as $t \to +\infty$, we deduce that so does the quantity $t(E(2t) - b\inf\Theta)$.
Now write $t(E(t) - b\inf\Theta) = (t/2)|\dot{u}(t)|^2 + bt(\Theta(u(t)) - \inf\Theta)$ and observe that each term in the sum is nonnegative to conclude that each of them vanishes, as $t \to +\infty$.

(r4) The set of weak cluster-points of $(u(t)_{t\ge 0})$ in $\mathcal{H}$ is nonempty, because $u \in L^\infty(0,+\infty;\mathcal{H})$ (by (r2)). Let us prove that any of these weak cluster-points belongs to S. Let $t_n \to +\infty$ and $\overline{u} \in \mathcal{H}$ be such that $u(t_n) \rightharpoonup \overline{u}$ weakly in $\mathcal{H}$ (as $n \to +\infty$). From $\Theta(u(t_n)) \to \inf_{\mathcal{H}}\Theta$ (by (r1)) and by the weak lower semi-continuity of Θ, we have $\inf_{\mathcal{H}}\Theta = \liminf_{n\to+\infty}\Theta(u(t_n)) \ge \Theta(\overline{u}) \ge \inf_{\mathcal{H}}\Theta$. Hence $\Theta(\overline{u}) = \inf_{\mathcal{H}}\Theta$, *i. e.* $\overline{u} \in S$.
Consequently, to obtain the weak convergence of $u(t)$ (as $t \to +\infty$) to some element in S, we just need to prove the uniqueness of a weak cluster-point under each of the additional conditions (i1)-(i3). This already settles point (i1).

Let us turn to points (i2), (i3). Set $\gamma = \frac{a+b}{2}$ and for any q and z in $\mathcal{H}$ define

$$A(z,q) = \gamma|z-q|^2 + N(z,q), \text{ where } N(z,q) = \langle \nabla\Phi(z), z-q\rangle - \Phi(z) + \Phi(q).$$

Clearly, from (7.14), $F(t)$ can be alternatively expressed as

$$F(t) = A(u(t),q) - \Phi(q) - \langle q - u(t), \dot{u}(t)\rangle - \int_0^t |\dot{u}(s)|^2 ds. \tag{7.20}$$

Taking q in S, and recalling $u \in L^\infty(0,\infty;\mathcal{H})$, $\lim_{t\to+\infty}|\dot{u}(t)| = 0$, $\dot{u} \in L^2(0,\infty;\mathcal{H})$ and that $\lim_{t\to+\infty} F(t)$ exists, we immediately deduce

$$\lim_{t\to+\infty} A(u(t),q) \text{ exists.} \tag{7.21}$$

Note also

$$\langle \nabla\Theta(u(t)), u(t) - q\rangle \to 0 \text{ as } t \to +\infty. \tag{7.22}$$

Indeed, this function is integrable and globally Lipschitz continuous (as a consequence of u and $\dot{u}$ being bounded together with $\nabla\Theta$ being Lipschitz continuous on bounded sets).

Now let $\bar{v}$ and $\bar{w}$ be two weak cluster-points of $(u(t))_{t\ge 0}$. That is, there exist two positive sequences (s_n) and (t_n) such that $s_n \to +\infty$, $u(s_n) \rightharpoonup \bar{v}$ and $t_n \to +\infty$, $u(t_n) \rightharpoonup \bar{w}$ (as $n \to +\infty$). Define

$$R_n = A(u(t_n),\bar{w}) - A(u(s_n),\bar{w}) - A(u(t_n),\bar{v}) + A(u(s_n),\bar{v}) + 2\gamma\langle u(t_n) - u(s_n), \bar{w} - \bar{v}\rangle \tag{7.23}$$

Taking (7.21) into account and passing to the limit in this equality readily yields

$$\lim_{n\to+\infty} R_n = 2\gamma|\bar{v}-\bar{w}|^2. \tag{7.24}$$

Purely algebraic computations show that R_n also admits the following expressions

$$\begin{aligned}
R_n &= -\langle \nabla\Phi(u(t_n)) - \nabla\Phi(u(s_n)), \bar{w}-\bar{v}\rangle & (7.25)\\
R_n &= \langle \nabla\Psi(u(t_n)) - \nabla\Psi(u(s_n)), \bar{w}-\bar{v}\rangle & \\
&\quad +\langle \nabla\Theta(u(t_n)), u(t_n)-\bar{w}\rangle - \langle \nabla\Theta(u(t_n)), u(t_n)-\bar{v}\rangle & (7.26)\\
&\quad -\langle \nabla\Theta(u(s_n)), u(s_n)-\bar{w}\rangle + \langle \nabla\Theta(u(s_n)), u(s_n)-\bar{v}\rangle & \\
R_n &= N(u(t_n),\bar{w}) - N(u(s_n),\bar{w}) - N(u(t_n),\bar{v}) + N(u(s_n),\bar{v}) & (7.27)
\end{aligned}$$

Now we are in a position to conclude.

(i2) If $\nabla\Phi$ is weakly sequentially continuous, then from (7.24) and (7.25) we deduce $2\gamma|\bar{v}-\bar{w}|^2 = -\langle \nabla\Phi(\bar{w}) - \nabla\Phi(\bar{v}), \bar{w}-\bar{v}\rangle$; hence $\bar{w}=\bar{v}$, since Φ is convex.

If $\nabla\Psi$ is weakly sequentially continuous, then from (7.26) and (7.22) we deduce $\lim_{n\to+\infty} R_n = \langle \nabla\Psi(\bar{w}) - \nabla\Psi(\bar{v}), \bar{w}-\bar{v}\rangle$. But the latter quantity is equal to $-\langle \nabla\Phi(\bar{w}) - \nabla\Phi(\bar{v}), \bar{w}-\bar{v}\rangle$ in view of $\nabla\Theta(\bar{w}) = \nabla\Theta(\bar{v}) = 0$. We conclude as above .

(i3) For any q in S, (7.22) can be rewritten as

$$\lim_{t\to+\infty} (\langle \nabla\Phi(u(t)) - \nabla\Phi(q), u(t)-q\rangle + \langle \nabla\Psi(u(t)) - \nabla\Psi(q), u(t)-q\rangle) = 0.$$

Since Ψ and Φ are assumed to be convex, each term in the sum above is nonnegative; hence

$$\lim_{t\to+\infty} \langle \nabla\Phi(u(t)) - \nabla\Phi(q), u(t)-q\rangle = 0. \tag{7.28}$$

For any z and q in $\mathcal{H}$, a simple computation (using the convexity of Φ) yields $0 \le N(z,q) \le \langle \nabla\Phi(z) - \nabla\Phi(q), z-q\rangle$. In particular, when q belongs to S, with (7.28) we deduce $\lim_{t\to+\infty} N(u(t),q) = 0$. Noticing that $\bar{v}$ and $\bar{w}$ belong to S, by using the expression of R_n given in (7.27) we readily obtain $\lim_{n\to+\infty} R_n = 0$, which in light of (7.24) yields $\bar{v}=\bar{w}$. •

Remark 7.1 Condition (i2) of Theorem 7.1 always holds when $\mathcal{H}$ is finite dimensional.

Remark 7.2 Specifically, when Φ is twice-differentiable, Theorem 7.1 allows to cover the asymptotic behaviour of the second-order equation (1.1). To that end, take $\beta > 0$, $\alpha > 0$, initial data $u(0) = u_0$ and $\dot{u}(0) = v_0$ in $\mathcal{H}$, and consider system (3.1) with Φ and Ψ replaced by $\beta\Phi$ and $\beta\Psi$, respectively, $b = 1/\beta$, $a = \alpha - 1/\beta$, and the initial data $u(0) = u_0$ and $y(0) = -(1/b)(v_0 + \beta\nabla\Phi(u_0) + au_0)$. As a direct consequence of Theorems 7.1 and 2.1, the classical solution of (1.1) enjoys the properties stated for u in Theorem 7.1.

The following result provides additional information concerning the asymptotic convergence properties of $\Phi(u(.))$ and $\Psi(u(.))$ when Φ and Ψ are convex.

Proposition 7.1 *Suppose in addition to conditions* **(A1)-(A4)** *and* **(CPP)** *that* Ψ *is convex and that* S*, the set of minimizers of* $\Theta := \Phi + \Psi$*, is nonempty. Let* (u, y) *be a classical global solution of system* (3.1) *such that* $u(t)$ *weakly converges to some element* u_∞ *in* S *as* t *goes to infinity. Then, the following holds:*

$$\lim_{t\to+\infty} \Phi(u(t)) = \Phi(u_\infty) \quad \textit{and} \quad \lim_{t\to+\infty} \Psi(u(t)) = \Psi(u_\infty).$$

Proof: By the weak lower semi-continuity of Φ and Ψ, we have $\Phi(u_\infty) \leq \liminf_{t\to+\infty} \Phi(u(t))$ and $\Psi(u_\infty) \leq \liminf_{t\to+\infty} \Psi(u(t))$. These inequalities, together with $\inf_{\mathcal{H}} \Theta = \Theta(u_\infty) = \Phi(u_\infty) + \Psi(u_\infty)$ (theorem 7.1-r1), imply

$$\begin{aligned}
\limsup_{t\to+\infty} \Phi(u(t)) &= \limsup_{t\to+\infty} (\Phi(u(t)) + \Psi(u(t)) - \Psi(u(t))) \\
&\leq \limsup_{t\to+\infty} (\Phi(u(t)) + \Psi(u(t))) - \liminf_{t\to+\infty} \Psi(u(t)) \\
&\leq \Phi(u_\infty) + \Psi(u_\infty) - \Psi(u_\infty) = \Phi(u_\infty)
\end{aligned}$$

Clearly, we deduce $\lim_{t\to+\infty} \Phi(u(t)) = \Phi(u_\infty)$, and it is then a simple matter to obtain $\lim_{t\to+\infty} \Psi(u(t)) = \Psi(u_\infty)$, which is our claim . •

7.2 Strong convergence

Let us now reinforce the assumptions on potentials Φ and Ψ in order to obtain strong convergence results. The following classical notions will be helpful:

A function $\phi : \mathcal{H} \to \mathbb{R}$ is said to be boundedly inf-compact if for any $R > 0$ and any $\lambda \in \mathbb{R}$, the set $\{x \in \mathcal{H} : \phi(x) \leq \lambda; |x| \leq R\}$ is relatively compact for the strong topology of $\mathcal{H}$.

A function $\phi : \mathcal{H} \to \mathbb{R}$ is said to be uniformly convex at $q \in \mathcal{H}$ if $\nabla\phi$ is uniformly monotone at q, i.e., there exists some $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ which is nondecreasing, and vanishes only at 0, such that for any $v \in \mathcal{H}$

$$\langle \nabla\phi(v), v - q\rangle \geq \omega(|v - q|). \tag{7.29}$$

Clearly q is the unique possible minimum point of ϕ. This property is satisfied if ϕ is strongly convex, i.e., there exists some positive constant c such that $\phi - c|.|^2$ is convex.

Theorem 7.2 *Suppose that conditions* **(A1)-(A4)** *and* **(CPP)** *are satisfied and that* S*, the set of minimizers of* $\Theta := \Phi + \Psi$*, is nonempty. Suppose, in addition, that one of the following assumptions is satisfied:*

i) $\Theta : \mathcal{H} \to \mathbb{R}$ *is boundedly inf-compact;*

ii) $\Theta : \mathcal{H} \to \mathbb{R}$ *is uniformly convex at some* $q \in \mathcal{H}$*.*

Then, the unique classical solution (u, y) *of system* (3.1)*, is such that* $u(t)$ *strongly converges to some element* u_∞ *in* S *as* t *goes to infinity. Moreover, the auxiliary variable* $y(t)$ *strongly converges in* $\mathcal{H}$*, as* $t \to +\infty$*, to* $y_\infty = -\frac{1}{b}(\nabla\Phi(u_\infty) + au_\infty)$*.*

Proof: i) By Theorem 7.1-r1-r2, u is a bounded minimizing trajectory for Θ. Since Θ is boundedly inf-compact, $u(t)$ admits strong cluster points as $t \to \infty$, which are also minimizers of Θ (same reasoning as in Theorem 7.1-r4). Now, using (7.24), (7.25), the strong continuity and the monotonicity of $\nabla\Phi$ (as in the proof of Theorem 7.1-r4-i2), one easily obtains that there exists a unique limit point of $t \mapsto u(t)$ (for the strong topology of $\mathcal{H}$). As a consequence, strong convergence holds.

ii) The assumption $S \neq \emptyset$ and the uniform convexity of Θ at q yield $S = \{q\}$. Then (7.22) and (7.29) (with Θ in place of ϕ) entail $\omega(|u(t) - q|) \to 0$ as $t \to \infty$, which forces $|u(t) - q| \to 0$.

Let us consider the first equation of (3.1), $\dot{u}(t)+\nabla\Phi(u(t))+au(t)+by(t) = 0$, and use that $u(t)$ strongly converges to some u_∞, while $\dot{u}(t)$ strongly converges to zero. Hence, by using the continuity property of $\nabla\Phi$, we obtain that the auxiliary variable $y(\cdot)$ strongly converges in $\mathcal{H}$ at $t \to +\infty$ to some y_∞ which satisfies $\nabla\Phi(u_\infty)+au_\infty+by_\infty = 0$, which is our claim. •

Remark 7.3 In the assumptions of Theorem 7.2, functions Φ and Ψ play symmetric roles. It is worth noticing that it is enough that one of the two functions is boundedly inf-compact to ensure that the sum is boundedly inf-compact (just argue by using a continuous affine minorant of the other function).
Similarly, if one of the two functions is strongly convex, so is their sum. Indeed in the next section, in the study of the exponential stabilization, we shall see that the strong convexity assumption can be formulated in a weaker local way.

7.3 Exponential stabilization

The following assumption will be needed in this section.

(A5) S, the set of minimizers of Θ, is a singleton $\{q\}$, and Θ satisfies

There exist $R > 0$ and $\sigma > 0$ such that

$$|v - q| \leq R \Rightarrow \langle \nabla\Theta(v), v - q\rangle \geq \sigma|v - q|^2. \tag{7.30}$$

Note that assumption **(A5)** holds if $\Theta(.) - \frac{\sigma}{2}|.|^2$ is convex, *i.e.* Θ is strongly convex.

Lemma 7.2 *Suppose the condition* (A5) *is satisfied. Then, for any positive real C, there exists some $\gamma_C > 0$ such that, for any $v \in \mathcal{H}$ satisfying $|v - q| \leq C$,*

$$\langle \nabla\Theta(v), v - q\rangle \geq \gamma_C|v - q|^2. \tag{7.31}$$

Precisely, one can take $\gamma_C = \sigma\rho$ where $\rho \in (0, \min\{1, \frac{R}{C}\})$. Function Θ satisfies further, for $|v - q| < C$,

$$\Theta(v) - \Theta(q) \geq \frac{\gamma_C}{2}|v - q|^2. \tag{7.32}$$

Proof: Let $v \in \mathcal{H}$ be such that $|v - q| \leq C$ and set $z = \rho v + (1 - \rho)q$ with some positive value $\rho \in (0, \min\{1, \frac{R}{C}\})$. Clearly, besides $\rho \in (0, 1)$, we have $\rho \leq R/C$, and so $|z - q| = \rho|v - q| \leq R$ (as $|v - q| \leq C$). Furthermore, from the convexity of Θ (hence $\langle \nabla\Theta(v), v - z\rangle \geq \langle \nabla\Theta(z), v - z\rangle$) , we have

$$\begin{aligned}
\langle \nabla\Theta(v), v - q\rangle &= \langle \nabla\Theta(v), v - z\rangle + \langle \nabla\Theta(v), z - q\rangle \\
&\geq \langle \nabla\Theta(z), v - z\rangle + \langle \nabla\Theta(v), z - q\rangle \\
&= \langle \nabla\Theta(z), v - q\rangle + \langle \nabla\Theta(z), q - z\rangle + \langle \nabla\Theta(v), z - q\rangle.
\end{aligned}$$

Observing that $z - q = \rho(v - q)$, we equivalently obtain

$$\begin{aligned}\langle \nabla\Theta(v), v - q\rangle &\geq (1/\rho)\langle \nabla\Theta(z), z - q\rangle - \langle \nabla\Theta(z), z - q\rangle + \rho\langle \nabla\Theta(v), v - q\rangle \\ &= (1/\rho - 1)\langle \nabla\Theta(z), z - q\rangle + \rho\langle \nabla\Theta(v), v - q\rangle.\end{aligned}$$

Since $\rho \in (0, 1)$, we can obviously rewrite this latter inequality as

$$\langle \nabla\Theta(v), v - q\rangle \geq (1/\rho)\langle \nabla\Theta(z), z - q\rangle.$$

Therefore, recalling that $|z - q| \leq R$, and using (A5), we deduce that

$$\langle \nabla\Theta(v), v - q\rangle \geq \sigma(1/\rho)|z - q|^2,$$

hence, by $|z - q| = \rho|v - q|$ we are led to (7.31). As a consequence, for $|v - q| < C$, we additionally obtain

$$\begin{aligned}\Theta(v) - \Theta(q) &= \textstyle\int_0^1 \langle \nabla\Theta(q + t(v - q)), v - q\rangle dt \\ &\geq \textstyle\int_0^1 \gamma t|v - q|^2 dt = \frac{\gamma}{2}|v - q|^2,\end{aligned}$$

which completes the proof. •

Theorem 7.3 *Under assumptions (A1) (A2) (A3) (A4) (A5) (CPP), the solution* (u, y) *to system (3.1) is such that* $u(t)$ *converges exponentially to* q *as* $t \to \infty$. *Moreover the energy* $E(t)$ *and* $\Theta(u(t))$ *converge exponentially to* $\inf\Theta$, *and* $|\dot{u}(t)|$ *converges exponentially to zero as* $t \to \infty$.

Proof: Let u be the first component of the solution to system (3.1).We recall that the trajectory $t \mapsto u(t)$ is bounded, that is (for any $t \geq 0$) $|u(t) - q| \leq C$ for some positive constant C. Set $\gamma_C = \sigma\rho$, with $\rho \in (0, \min\{1, \frac{R}{C}\})$, where σ and R are given by condition (A5). Hence from Lemma 7.2 for $t \geq 0$,

$$\langle \nabla\Theta(u(t)), u(t) - q\rangle \geq \gamma|u(t) - q|^2, \tag{7.33}$$

and

$$\Theta(u(t)) - \Theta(q) \geq \frac{\gamma}{2}|u(t) - q|^2. \tag{7.34}$$

¿From now on, for sake of simplicity of notations, we take $\Theta(q) = 0$ and we will often omit argument $t \in [0, +\infty[$ of function u.
Function u is bounded ((r2) Thm. 7.1) and $\nabla\Phi$ is Lipschitz continuous on bounded subsets, hence, with the first equation of system (3.1), $\dot{u}$ is absolutely continuous. The energy $E(t) = b\Theta(u) + \frac{1}{2}|\dot{u}(t)|^2$ is also absolutely continuous, time differentiable almost everywhere, and satisfies in view of (3.4)

$$\dot{E}(t) + (a + b)|\dot{u}(t)|^2 \leq 0 \ \ a.e. \ \ t \in [0, +\infty[.$$

Set

$$\begin{aligned}G(t) &= F(t) + \int_0^t |\dot{u}|^2 + \Phi(q) \qquad (cf.\ (7.11)) \\ &= \frac{a+b}{2}|u - q|^2 + \langle u - q, \dot{u}\rangle + [\langle \nabla\Phi(u), u - q\rangle - \Phi(u) + \Phi(q)] \quad (cf.\ (7.14))\end{aligned}$$

Function G is time differentiable with derivative $\dot{G}$ satisfying (recall (7.12))

$$\dot{G}(t) + b\langle \nabla\Theta(u(t)), u(t) - q\rangle = |\dot{u}(t)|^2, \ \forall t \in [0, +\infty[.$$

Combining derivatives $\dot{E}$ and $\dot{G}$ yields

$$\dot{E} + \frac{a+b}{2}\dot{G} + \frac{a+b}{2}\Big[|\dot{u}|^2 + b\langle \nabla\Theta(u), u - q\rangle\Big] \leq 0, \quad a.e. \text{ on } [0, +\infty[. \tag{7.35}$$

Our goal is to show $|\dot{u}|^2 + b\langle \nabla\Theta(u), u - q\rangle \geq \delta[E + \frac{a+b}{2}G]$ for some $\delta > 0$, to derive from (7.35) a differential inequation for $E + \frac{a+b}{2}G$ and further for function $\frac{1}{2}|u-q|^2$ showing that the latter has exponential decay.
In view of the convexity of Θ (hence $\langle \nabla\Theta(u), u - q\rangle \geq \Theta(u)$) and from (7.33) we first derive for $t \in (0, +\infty[$

$$\begin{aligned}
|\dot{u}|^2 + b\langle \nabla\Theta(u), u - q\rangle &= \left(\frac{1}{4}|\dot{u}|^2 + \frac{b}{2}\langle \nabla\Theta(u), u - q\rangle\right) \\
&\quad + \left(\frac{3}{4}|\dot{u}|^2 + \frac{b}{2}\langle \nabla\Theta(u), u - q\rangle\right) \\
&\geq \left(\frac{1}{4}|\dot{u}|^2 + \frac{b}{2}\Theta(u)\right) + \left(\frac{3}{4}|\dot{u}|^2 + \frac{b\gamma}{2}|u - q|^2\right) \\
&\geq \frac{1}{2}\left[E + m\left(|\dot{u}|^2 + |u - q|^2\right)\right], \qquad (7.36)
\end{aligned}$$

with $m = \min\left(\frac{3}{2}, b\gamma\right)$.

We now turn to G. Let L be a Lipschitz constant of $\nabla\Phi$ on some bounded set containing u; we have

$$\begin{aligned}
0 \leq \Phi(q) - \Phi(u) + \langle \nabla\Phi(u), u - q\rangle &= \int_0^1 \langle \nabla\Phi(u + s(q-u)) - \nabla\Phi(u), q - u\rangle ds \\
&\leq \int_0^1 Ls|q-u|^2 ds = \frac{L}{2}|q-u|^2.
\end{aligned}$$

Hence for any constant $M \geq \frac{a+b+L+1}{2}$, we have

$$G \leq \frac{a+b}{2}|u-q|^2 + \frac{1}{2}|u-q|^2 + \frac{1}{2}|\dot{u}|^2 + \frac{L}{2}|q-u|^2 \leq M\left(|u-q|^2 + |\dot{u}|^2\right).$$

Combining the inequation above with (7.36) yields

$$|\dot{u}|^2 + b\langle \nabla\Theta(u), u - q\rangle \geq \frac{1}{2}\left[E + \frac{m}{M}G\right] = \frac{m}{M(a+b)}\left[\frac{M(a+b)}{2m}E + \frac{a+b}{2}G\right].$$

Choosing now $M = \max(\frac{a+b+L+1}{2}, \frac{2m}{a+b})$, so that $\frac{M(a+b)}{2m}E \geq E$, since $\frac{M(a+b)}{2m} \geq 1$ and $E \geq 0$, we obtain

$$|\dot{u}|^2 + b\langle \nabla\Theta(u), u - q\rangle \geq \frac{m}{M(a+b)}\left[E + \frac{a+b}{2}G\right].$$

Finally, we deduce from (7.35), with $\delta = \frac{m}{2M}$

$$\dot{E} + \frac{a+b}{2}\dot{G} + \delta\left[E + \frac{a+b}{2}G\right] \leq 0.$$

Let us integrate this inequation between 0 and t. Since E is and G are absolutely continuous, we obtain

$$E(t) + \frac{a+b}{2}G(t) \le e^{-\delta t}\left(E(0) + \frac{a+b}{2}G(0)\right), \forall t \in [0, +\infty[. \tag{7.37}$$

Owing to the nonnegativity of E and $\langle \nabla\Theta(u), u-q\rangle - \Phi(u) + \Phi(q)$, we have

$$\frac{a+b}{2}\left[\frac{a+b}{2}|u-q|^2 + \langle u-q, \dot{u}\rangle\right] \le E + \frac{a+b}{2}G;$$

hence, with $C = \frac{2}{a+b}E(0) + G(0)$

$$(a+b)\frac{1}{2}|u(t)-q|^2 + \langle u(t)-q, \dot{u}(t)\rangle \le Ce^{-\delta t}.$$

Integrating between 0 and t readily yields

$$\frac{1}{2}|u(t)-q|^2 \le \frac{1}{2}|u_0-q|^2 e^{-(a+b)t} + \frac{C}{a+b-\delta}\left(e^{-\delta t} - e^{-(a+b)t}\right).$$

Noticing that $\delta \le \frac{a+b}{4}$, in view of $M \ge \frac{2m}{a+b}$, we obtain the simpler bound

$$\frac{1}{2}|u(t)-q|^2 \le \frac{1}{2}|u_0-q|^2 e^{-(a+b)t} + \frac{4C}{3(a+b)}e^{-\delta t} = O(e^{-\delta t}). \bullet$$

Let us now return to (7.37) which gives the exponential decay of $E(t) + \frac{a+b}{2}G(t)$. Using the formulation of G, the nonnegativity of $[\langle \nabla\Phi(u), u-q\rangle - \Phi(u) + \Phi(q)]$, and the exponential decay of $|u(t)-q|$ which has just been obtained, we finally obtain that the energy $E(t)$ and $\Theta(u(t))$ converge exponentially to $\Theta(q) = \inf\Theta$, and $|\dot{u}(t)|$ converges exponentially to zero as $t \to \infty$. $\bullet$

References

[1] P. Alart, O. Maisonneuve, R.T. Rockafellar, Nonsmooth Mechanics and Analysis, *Advances in Mechanics and Mathematics*, **12**, Springer, 2006.

[2] F. Alvarez, J.M. Pérez, A dynamical system associated with Newton's method for parametric approximations of convex minimization problems, *Applied Mathematics and Optimization*, **38**, (1998), pp. 193-217.

[3] F. Alvarez, On the minimizing property of a second-order dissipative system in Hilbert space, *SIAM J. Control Optim.* **38**, (2000), pp. 1102–1119.

[4] F. Alvarez, H. Attouch, J. Bolte, P. Redont, A second-order gradient-like dissipative dynamical system with Hessian-driven damping. Application to optimization and mechanics, *J. Math. Pures Appl.*, **81**, (2002), pp. 747–779.

[5] H. Attouch, Variational convergence for functions and operators, Pitman Advanced Publishing Program, Applicable Mathematics Series, 1984.

[6] H. Attouch, Convergences de fonctionnelles convexes, Journées d'Analyse Non Linéaire, Besançon, *Lecture Notes in Mathematics*, Springer-Verlag **665**, (1977), pp. 1-40.

[7] H. Attouch, J. Bolte, P. Redont, Optimizing properties of an inertial dynamical system with geometric damping, *Control and Cybernetics*, **31**, (2002), pp. 643–657.

[8] H. Attouch, G. Buttazzo, and G. Michaille, *Variational Analysis in Sobolev and BV Spaces,* MPS-SIAM Series on Optimization, **6**, 2006.

[9] H. Attouch, A. Cabot, P. Redont, The dynamics of elastic shocks via epigraphical regularization of a differential inclusion, *Adv. Math. Sci. Appl.*, **12** (1), (2002), pp. 273–306.

[10] H. Attouch, X. Goudou, P. Redont, The heavy ball with friction method, *Commun. Contemp. Math.*, **2** (1), (2000), pp. 1–34.

[11] H. Attouch, P.E. Maingé, Asymptotic behavior of second order dissipative evolution equations combining potential with non-potential effects, *ESAIM: Control, Optimisation and Calculus of Variations*, Published online: 19 janvier 2011.

[12] H. Attouch, B. F. Svaiter, A continuous dynamical Newton-Like approach to solving monotone inclusions, *SIAM J. Control Optim.*, **49** (2), (2011), pp. 574-598.

[13] B. Baji, A. Cabot, J.I. Diaz, Asymptotics for some nonlinear damped wave equation: finite time convergence versus exponential decay results, *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, **24, No 6**, (2007), pp. 1009–1028.

[14] P. Ballard, The dynamics of discrete mechanical systems with perfect unilateral constraints, *Arch. Rational Mech. Anal.*, **154**, (2000), pp. 199–274.

[15] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland, Mathematical Studies, 1973.

[16] H. Brézis, Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, in: *Contributions to Nonlinear Functional Analysis*, E. Zarantonello, ed., Academic Press, New York, 1971.

[17] H. Brézis, Problèmes unilatéraux, *J. Math. Pures et Appl.*, **51**, (1972), pp. 1-168.

[18] A. Cabot, L. Paoli, Asymptotics for some vibro-impact problems with a linear dissipation term, *J. Math. Pures Appl.*, **87**, (2007), pp. 291–323.

[19] S. Faik, H. Witteman, Modeling of impact dynamics: A literature survey, - 2000 International ADAMS User Conference, 2000.

[20] J.K. Hale, Asymptotic behavior of dissipative systems, Math. surveys Monogr., **25**, AMS, Providence, RI, 1987.

[21] A. Haraux, Systèmes dynamiques dissipatifs et applications, RMA, **17**, Masson, 1991.

[22] A. Haraux, M. Jendoubi, Convergence of solutions of second-order gradient-like systems with analytic nonlinearities, *J. Differential Equations*, **144**(2) (1998), pp. 313–320.

[23] C. Lageman, Pointwise convergence of gradient-like systems, *Math. Nachr.*, **280**, (13-14), (2007), pp. 1543-1558.

[24] P.E. Maingé, Fixed-point iterations coupled with relaxation factors and inertial effects, *Nonlinear Analysis: Theory, Methods, Applications*, **72**, (2), (2010), pp. 720-733.

[25] P.E. Maingé, N. Merabet, A new inertial-type hybrid projection- proximal algorithm for monotone inclusions, *Applied Mathematics and Computation*, (2009), doi:10.1016/j.amc.2009.10.26.

[26] P.E. Maingé, A. Moudafi, Convergence of new inertial proximal methods for DC programming, *SIAM J. Optimization*, **19**,(1) (2008), pp. 397–413.

[27] P.E. Maingé, A. Moudafi, A proximal method for maximal monotone operators via discretization of a first order dissipative dynamical system, *J. Convex Analysis*, **14**, (4) (2007), pp. 869–878.

[28] J.J. Moreau, Standard inelastic shocks and the dynamics of unilateral constraints: in Unilateral problems in structural analysis, CISM Courses, **Vol 288**, Springer Verlag, 1985, 173-221.

[29] L. Paoli, M. Schatzman, Mouvement à un nombre fini de degrés de liberté avec contraintes unilatérales: cas avec perte d'énergie, *Mod. Math. Anal. Num.*, **27** (1993), pp. 673–717.

[30] L. Paoli, M. Schatzman, Penalty approximation for nonsmooth constraints in vibro-impact, *J. Differential Equations*, **177** (2001), pp. 375–418.

[31] L. Paoli, M. Schatzman, A numerical scheme for impact problems I and II, *SIAM J. Numer. Anal.*, **40** (2) (2002), pp. 702–733, 734–768.

[32] L. Paoli, M. Schatzman, Penalty approximation for dynamical systems submitted to multiple nonsmooth constraints, *Multibody System Dynamics*, **8** (2003), pp. 347–366.

[33] A. Petrov, M. Schatzman, One-dimensional viscoelastodynamics with Signorini boundary conditions, *CRAS Paris* **334, No 11** (2002), pp. 983–988.

[34] M. Schatzman, A class of nonlinear differential equations of second order in time, *Nonlinear Anal.*, **2** (1978), pp. 355–373.