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# Vibrational stabilization of the upright statically unstable position of a double pendulum

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## A B S T R A C T

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The phenomenon of stabilization by parametric excitation of an unstable, elastically restrained double inverted pendulum under its own weight is addressed. The solution is pursued by the Multiple Scale Method, as a perturbation of a critical Hamiltonian system, possessing a zero and a real frequency. Several asymptotic expansions are carried out, which are able to capture the long term behavior of the system, for generic (non resonant) values of the excitation frequency, and some special (resonant) values of excitation to natural frequency ratio. It is shown that a proper ordering of the control parameters must be performed, and proper use of integer or fractional power expansions must be made, according to the resonance under study. In particular, a non standard application of the Multiple Scale Method is illustrated for the 1:1 resonant case, requiring fractional powers and accounting for the 'arbitrary constants', generally omitted in regular cases. A comprehensive scenario of the stabilization regions is given in which lower bound as well as upper bound curves are evaluated, thus integrating results that recently appeared in the literature.

## 1. Introduction

The problem of stabilization of an otherwise unstable, autonomous, mechanical system is a fascinating task, both from a theoretical and practical point of view. There exists a huge literature concerning the stabilization of spacecrafts, satellites, under water vehicles and floating bodies, mostly devoted to the analysis of the effects of internal moving masses, and acting as actuators for retro action control laws. However, the simplest form of control for regaining stability is expected to be a mere parametric excitation, in which a mass, a stiffness, or a characteristic length is periodically varied in time, causing the system to become non autonomous and, possibly, stable. An example of theoretical and experimental studies concerning the stabilization of canoes and kayaks via parametric excitation, is given in [1].

There are many scientific works on the stabilization of statically unstable systems by parametric excitation. Important contributions were made by the works of Stephenson [2], Kapitza [3], Chelomei [4], Bogolyubov and Mitropol'skii [5], Acheson [6], Acheson and Mullin [7], Blekhman [8], Champneys and Fraser [9], Mullin et al. [10], Thomsen [11,15], Seyranian and Seyranian [13], Yabuno and Tsumoto [14], Shishkina et al. [16] and many others. Most of these works deal with high frequency stabilization problems. However, in the recent paper by Seyranian and Seyranian [17] it was shown that a statically unstable elastic beam, compressed by an axial periodic force, can be stabilized in its horizontal position by

the excitation frequencies of the order of the main frequency of transverse vibrations of the beam. In the paper by Mailybaev and Seyranian [18] the stabilization conditions of statically unstable systems close to the critical point were obtained for a wide range of excitation frequencies by analyzing the Floquet multipliers. An inverted double pendulum was one of the simplest examples. It was shown that stabilization of the upper vertical unstable position is possible for low, medium and high excitation frequencies. A lower bound for the stabilization region was found.

In this paper we consider again the inverted double pendulum, and investigate the stability regions by looking also for the upper bound. This is the main motive of the paper. Use is made of the Multiple Scale Method, which is able to highlight the mechanism of stabilization, thus giving more insights in the physical problem. The application of the method, however, is not straightforward when a 1:1 resonant case has to be analyzed. An adapted version of the algorithm, which is believed to be new, is implemented here.

## 2. Mechanical model

An undamped inverted double pendulum is considered (Fig. 1), consisting of two masses  $m_1$  and  $m_2$ , connected by rigid weightless rods of length  $l_1$  and  $l_2$  with elastic joints of stiffnesses  $c_1$  and  $c_2$ , respectively. In the gravitational field the kinetic and potential energies for the system have the form

$$T = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2),$$

$$P = \frac{1}{2}c_1\theta_1^2 + \frac{1}{2}c_2(\theta_2 - \theta_1)^2 + (m_1 + m_2)gl_1 \cos \theta_1 + m_2gl_2 \cos \theta_2, \quad (1)$$

where  $\theta_1, \theta_2$  are the rotation angles of the rods and the dot denotes differentiation with respect to the time  $t$ .

Inserting the Lagrange function  $L = T - P$  in the Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = 0, \quad i = 1, 2, \quad (2)$$

we obtain nonlinear equations of motion of the system as

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2l_1l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2l_1l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + (c_1 + c_2)\theta_1 - c_2\theta_2 - (m_1 + m_2)gl_1 \sin\theta_1 = 0,$$

$$m_2l_1l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + m_2l_2^2\ddot{\theta}_2 - m_2l_1l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 - c_2\theta_1 + c_2\theta_2 - m_2gl_2 \sin\theta_2 = 0. \quad (3)$$

Let us assume that the support undergoes some vertical harmonic excitation  $z = a \cos \Omega t$ , where  $a$  is small with respect to the rods' lengths. According to the d'Alembert principle, the relevant equations of motion have the previous form, with  $g \rightarrow g - a\Omega^2 \cos \Omega t$  instead of  $g$ .

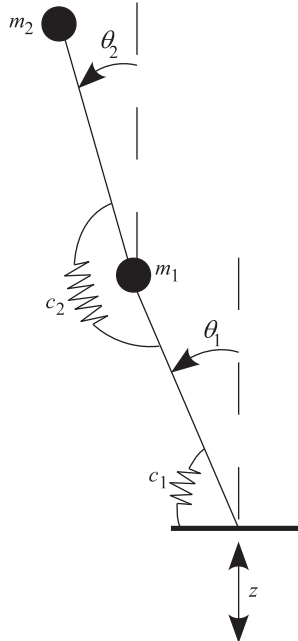


Fig. 1. Inverted double pendulum.

We are interested in studying the stability of equilibrium of the system in the vertical position under excitation. For this purpose we consider small oscillations of the pendulum around the vertical position  $\theta_1 = \theta_2 = 0$  governed by the following linear equations:

$$(m_1 + m_2)l_1^2\dot{\theta}_1 + m_2l_1l_2\dot{\theta}_2 + (c_1 + c_2)(m_1 + m_2)(g - a\Omega^2 \cos \Omega t)l_1\theta_1 - c_2\theta_2 = 0,$$

$$m_2l_1l_2\dot{\theta}_1 + m_2l_2^2\dot{\theta}_2 - c_2\theta_1 + (c_2 - m_2(g - a\Omega^2 \cos \Omega t)l_2)\theta_2 = 0. \quad (4)$$

According to Lyapunov's theorem [20], if the linearized system (4) is unstable with Floquet multipliers greater than 1 by their absolute values, then the equilibrium  $\theta_1 = \theta_2 = 0$  of the initial nonlinear system is unstable.

We consider the case of a double pendulum with equal lengths  $l_1 = l_2 = l$  and stiffnesses  $c_1 = c_2 = c$ . Then the linear system of differential equations in the dimensionless matrix form will be as follows [18]:

$$\mathbf{M}\dot{\boldsymbol{\theta}} + (\mathbf{C}(p) + \delta\tilde{\Omega}^2 \cos \tilde{\Omega}\tilde{t}\mathbf{B})\boldsymbol{\theta} = 0 \quad (5)$$

with the matrices

$$\mathbf{M} = \begin{pmatrix} 1+\eta & \eta \\ \eta & \eta \end{pmatrix}, \quad \mathbf{C}(p) = \begin{pmatrix} 2p & \eta & 1 & p \\ p & & p & \eta \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1+\eta & 0 \\ 0 & \eta \end{pmatrix} \quad (6)$$

and  $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$ . Here the dot denotes differentiation with respect to the dimensionless time  $\tilde{t} = \Omega^*t$ , where  $\Omega^* = \sqrt{g/l}$ , and the dimensionless parameters are defined as follows:

$$\delta = \frac{a}{l}, \quad p = \frac{c}{m_1gl}, \quad \tilde{\Omega} = \frac{\Omega}{\Omega^*}, \quad \eta = \frac{m_2}{m_1}. \quad (7)$$

Tildes will be omitted ahead. Eq. (5) is the governing equation of a linear, parametrically excited system.

Let  $p = p_0$  be the critical value of the nondimensional negative stiffness, such that when  $\delta = 0$ , the system (5) is stable at  $p < p_0$  and unstable at  $p > p_0$ . This value was found in [18] to be

$$p_0 = \frac{3\eta + 1 + \sqrt{5\eta^2 + 2\eta + 1}}{2}. \quad (8)$$

Let us consider parameter  $p$  close to the critical stability value  $p_0$

$$p = p_0 + \Delta p, \quad \Delta p \ll 1. \quad (9)$$

Consequently, the stiffness matrix  $\mathbf{C}(p)$  can be written as

$$\mathbf{C}(p) = \mathbf{C}_0 + \Delta p\mathbf{C}_1, \quad (10)$$

where  $\mathbf{C}_0 = \mathbf{C}(p_0)$  and

$$\mathbf{C}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \quad (11)$$

At  $p = p_0$  and  $\delta = 0$  the eigenfrequencies and corresponding eigenmodes for (5) are [18]

$$\omega_1 = 0, \quad \mathbf{w}_1 = \begin{pmatrix} p_0 \\ 2p_0 + \eta + 1 \end{pmatrix},$$

$$\omega_2 \equiv \omega = \sqrt{\left(5 + \frac{1}{\eta}\right)p_0 - 2(\eta + 1)}, \quad \mathbf{w}_2 = \begin{pmatrix} \eta + \omega^2\eta + p_0 \\ \omega^2\eta + p_0 \end{pmatrix}. \quad (12)$$

By introducing the linear transformation of coordinates,  $(\theta_1, \theta_2)^T = \mathbf{W}(X, Y)^T$ , where  $\mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2)$  is the modal matrix, and  $(X, Y)^T$  the modal coordinate vector, and pre multiplying Eq. (5) by  $\mathbf{W}^T$ , the equations of motion, after rearrangement, read

$$X + \Delta p(c_{11}X + c_{12}Y) + \frac{\delta\Omega^2}{2}(b_{11}X + b_{12}Y)(e^{i\Omega t} + e^{-i\Omega t}) = 0,$$

$$Y + \omega^2Y + \Delta p(c_{21}X + c_{22}Y) + \frac{\delta\Omega^2}{2}(b_{21}X + b_{22}Y)(e^{i\Omega t} + e^{-i\Omega t}) = 0, \quad (13)$$

where  $c_{ij}$  are coefficients defined by the matrix  $\mathbf{C}_1$  and the harmonic excitation has been expressed in complex form. Note that the two equations (13) uncouple when  $\Delta p = 0$  and  $\delta = 0$ .

### 3. The Multiple Scale Method: integer power expansion

To introduce a perturbation parameter  $\varepsilon \ll 1$  in the equations, we perform the rescaling  $\delta \rightarrow \varepsilon\delta$  and  $\Delta p \rightarrow \varepsilon^2\Delta p$ , according to [18]. As it will appear clearer later, this ordering sets the parameter  $\Delta p$  at the same order at which the first meaningful

information is supplied by the perturbation algorithm, as usually done in perturbation methods. According to the Multiple Scale Method [19], we assume that the system's dynamics depends on several independent time scales, namely

$$t_0 = t, \quad t_1 = \varepsilon t, \quad t_2 = \varepsilon^2 t, \quad \dots, \quad (14)$$

so that  $X(t) = X(t_0(t), t_1(t), \dots)$ ,  $Y(t) = Y(t_0(t), t_1(t), \dots)$ . Consequently, by the chain rule, it follows that:

$$\frac{d}{dt} = \sum_{k=0}^{\infty} \varepsilon^k d_k, \quad \frac{d^2}{dt^2} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \varepsilon^{k+j} d_k d_j, \quad (15)$$

where  $d_k = \partial/\partial t_k$ . Then, we expand the configuration variables in series of integer powers of the perturbation parameter:

$$\begin{aligned} X(t, \varepsilon) &= X_0(t_0, t_1, t_2, \dots) + \varepsilon X_1(t_0, t_1, t_2, \dots) + \varepsilon^2 X_2(t_0, t_1, t_2, \dots) + \dots, \\ Y(t, \varepsilon) &= Y_0(t_0, t_1, t_2, \dots) + \varepsilon Y_1(t_0, t_1, t_2, \dots) + \varepsilon^2 Y_2(t_0, t_1, t_2, \dots) + \dots \end{aligned} \quad (16)$$

By substituting all these expressions in the equations of motion (13), and separately equating to zero terms with the same power of  $\varepsilon$ , we obtain a chain of perturbation in the unknowns  $X_k$ ,  $Y_k$ , namely

$$\begin{aligned} \varepsilon^0 : & \begin{cases} d_0^2 X_0 = 0, \\ d_0^2 Y_0 + \omega^2 Y_0 = 0, \end{cases} \\ \varepsilon^1 : & \begin{cases} d_0^2 X_1 = 2d_0 d_1 X_0 - \frac{\delta \Omega^2}{2} (b_{11} X_0 + b_{12} Y_0) (e^{i\Omega t_0} + e^{-i\Omega t_0}), \\ d_0^2 Y_1 + \omega^2 Y_1 = 2d_0 d_1 Y_0 - \frac{\delta \Omega^2}{2} (b_{21} X_0 + b_{22} Y_0) (e^{i\Omega t_0} + e^{-i\Omega t_0}), \end{cases} \\ \varepsilon^2 : & \begin{cases} d_0^2 X_2 = 2d_0 d_1 X_1 - 2d_0 d_2 X_0 - d_1^2 X_0 - \Delta p (c_{11} X_0 + c_{12} Y_0) - \frac{\delta \Omega^2}{2} (b_{11} X_1 + b_{12} Y_1) (e^{i\Omega t_0} + e^{-i\Omega t_0}), \\ d_0^2 Y_1 + \omega^2 Y_1 = 2d_0 d_1 Y_1 - 2d_0 d_2 Y_0 - d_1^2 Y_0 - \Delta p (c_{21} X_0 + c_{22} Y_0) - \frac{\delta \Omega^2}{2} (b_{21} X_1 + b_{22} Y_1) (e^{i\Omega t_0} + e^{-i\Omega t_0}), \end{cases} \\ & \vdots \end{aligned} \quad (17)$$

The first two of Eqs. (17) are the generating equations of the perturbation process. They govern the dynamics of the unexcited system ( $\delta = 0$ ) at the divergence bifurcation point ( $\Delta p = 0$ ), i.e. a marginally stable Hamiltonian system, in which a couple of eigenvalues is zero, while the remaining two are purely imaginary. By disregarding the divergent solution, proportional to  $t_0$ , the generating solution reads

$$\begin{cases} X_0 = A(t_1, t_2, \dots), \\ Y_0 = B(t_1, t_2, \dots) e^{i\omega t_0} + c.c., \end{cases} \quad (18)$$

where  $i$  is the imaginary unit and  $c.c.$  denotes the complex conjugate of the preceding terms. In it,  $A(t_1, t_2, \dots)$  and  $B(t_1, t_2, \dots)$  are real and complex unknown amplitudes, respectively, whose modulation on the slower time scales must be determined at the higher orders, as dependent on incremental parameter  $\Delta p$ , excitation amplitude  $\delta$  and frequency  $\Omega$ . These amplitudes are responsible for long time behavior (hence stability) of the first and second modes, respectively. Substitution of Eqs. (18) in the  $\varepsilon^1$  order perturbation equations (17) leads to

$$\begin{cases} d_0^2 X_1 = \frac{\delta \Omega^2}{2} b_{12} B e^{i(\Omega + \omega)t_0} - \frac{\delta \Omega^2}{2} b_{12} \bar{B} e^{i(\Omega - \omega)t_0} - \frac{\delta \Omega^2}{2} b_{11} A e^{i\Omega t_0} + c.c., \\ d_0^2 Y_1 + \omega^2 Y_1 = 2i\omega d_1 B e^{i\omega t_0} - \frac{\delta \Omega^2}{2} b_{22} B e^{i(\Omega + \omega)t_0} - \frac{\delta \Omega^2}{2} b_{22} \bar{B} e^{i(\Omega - \omega)t_0} - \frac{\delta \Omega^2}{2} b_{21} A e^{i\Omega t_0} + c.c. \end{cases} \quad (19)$$

where the overbar denotes complex conjugate.

Eqs. (19) suggest the following general discussion of the solution. In these equations, exponential terms appear on the right members, of the type  $\exp(ivt_0)$ , in which  $v = (0, \Omega, \Omega + \omega, \Omega - \omega)$  has the meaning of a forcing frequency, generated by combinations of the natural and excitation frequencies. Therefore, the cases  $\Omega = \omega$  and  $\Omega = 2\omega$  are detected as (first order) resonance conditions (in which excitation frequency  $\Omega - \omega$  equates to 0 or  $\omega$ , respectively). However, if the perturbation solution is carried out at the  $\varepsilon^2$  order, new harmonics like  $2\Omega + \omega$ ,  $2\Omega - \omega$  are expected to appear in the 'known' term, and therefore a new (second order) resonance can be foreseen, namely  $\Omega = \omega/2$ . In all cases, in order for the series expansions (16) to be uniformly valid, resonant terms of frequencies 0 and  $\omega$  that would lead to divergent solutions on the  $t_0$  scale (secular terms), must be eliminated from the right hand side of the perturbation equations, thus leading to the desired amplitude modulation equations. The perturbation scheme highlights the mechanism leading to stabilization: combinations of the frequencies producing a zero frequency, if of suitable sign, compensate for the lack of stiffness of the system. As a drawback, the (originally) stable mode could be made unstable by the parametric excitation and that would limit the

range in which stabilization works. In the next sub sections three different cases will be analyzed: (a) the non resonant case  $\Omega \neq \omega, 2\omega, \omega/2$ ; (b) the first order resonant case  $\Omega = 2\omega$ ; (c) the second order resonant case  $\Omega = \omega/2$ . In contrast the case  $\Omega = \omega$ , which calls for a special treatment, will be analyzed in the next section.

### 3.1. Non resonant case

Here  $\Omega \neq \omega, 2\omega, \omega/2$ . The resonant forcing term of frequency  $\omega$  would cause secular, type  $t_0 e^{i\omega t_0}$  terms to appear in  $Y_1$ . To remove such secular terms we must enforce

$$d_1 B = 0. \quad (20)$$

Note that, at this order, the dependence of  $A$  on the  $t_1$  scale is still undetermined. The solution for  $X_1, Y_1$ , by neglecting the unessential complementary part (which repeats the generating solution), is

$$\begin{cases} X_1 = \frac{\delta\Omega^2}{2} b_{12} \frac{B}{(\Omega+\omega)^2} e^{i(\Omega+\omega)t_0} + \frac{\delta\Omega^2}{2} b_{12} \frac{\bar{B}}{(\Omega-\omega)^2} e^{i(\Omega-\omega)t_0} + \frac{\delta A}{2} b_{11} e^{i\Omega t_0} + \text{c.c.}, \\ Y_1 = \frac{\delta\Omega^2}{2} b_{22} \frac{B}{\omega^2 (\Omega+\omega)^2} e^{i(\Omega+\omega)t_0} - \frac{\delta\Omega^2}{2} b_{22} \frac{\bar{B}}{\omega^2 (\Omega-\omega)^2} e^{i(\Omega-\omega)t_0} - \frac{\delta\Omega^2}{2} b_{21} \frac{A}{\omega^2 \Omega^2} e^{i\Omega t_0} + \text{c.c.} \end{cases} \quad (21)$$

By substituting solutions (18), (21) into the  $\varepsilon^2$  order equations (17) we obtain

$$\begin{cases} d_0^2 X_2 = d_1^2 A \Delta p c_{11} A - \frac{\delta^2 \Omega^4}{2} \left( b_{11}^2 \frac{A}{\Omega^2} - b_{12} b_{21} \frac{A}{\omega^2 \Omega^2} \right) + \text{NRT}, \\ d_0^2 Y_2 + \omega^2 Y_2 = \left( 2i\omega d_2 B - \Delta p c_{22} B - \frac{\delta^2 \Omega^4}{4} b_{12} b_{21} \left( \frac{B}{(\Omega-\omega)^2} + \frac{B}{(\Omega+\omega)^2} \right) \right. \\ \left. + \frac{\delta^2 \Omega^4}{4} b_{22}^2 \left( \frac{B}{\omega^2 (\Omega-\omega)^2} + \frac{B}{\omega^2 (\Omega+\omega)^2} \right) \right) e^{i\Omega t_0} + \text{c.c.} + \text{NRT}, \end{cases} \quad (22)$$

where NRT means non resonant terms. To remove secular terms we write

$$\begin{cases} d_1^2 A = C_1(\delta, \Omega, \Delta p) A, \\ d_2 B = iC_2(\delta, \Omega, \Delta p) B, \end{cases} \quad (23)$$

where the following quantities have been defined:

$$C_1(\delta, \Omega, \Delta p) = \Delta p c_{11} - \frac{\delta^2 \Omega^4}{2} \left( \frac{b_{11}^2}{\Omega^2} - \frac{b_{12} b_{21}}{\omega^2 \Omega^2} \right),$$

$$C_2(\delta, \Omega, \Delta p) = \frac{1}{2\omega} \left( \Delta p c_{22} + \frac{\delta^2 \Omega^4}{4} \left( b_{12} b_{21} \left( \frac{1}{(\Omega-\omega)^2} + \frac{1}{(\Omega+\omega)^2} \right) - b_{22}^2 \left( \frac{1}{\omega^2 (\Omega-\omega)^2} + \frac{1}{\omega^2 (\Omega+\omega)^2} \right) \right) \right). \quad (24)$$

Eqs (23) govern the modulation on the  $t_1$  and  $t_2$  scales, for the two amplitudes, respectively. It is possible to come back to the true time  $t$  by multiplying Eqs. (23) by  $\varepsilon^2$  and reabsorbing the perturbation parameter by backward transformations,  $\varepsilon\delta \rightarrow \delta$ ,  $\varepsilon^2 \Delta p \rightarrow \Delta p$ ,  $\varepsilon^2 d_1^2 A \rightarrow A$ ,  $\varepsilon^2 d_2 B \rightarrow \hat{B}$  thus obtaining

$$\begin{cases} A = C_1(\delta, \Omega, \Delta p) A, \\ \hat{B} = iC_2(\delta, \Omega, \Delta p) \hat{B}. \end{cases} \quad (25)$$

The stability region for the system (25) in the parameter space is defined by the inequality  $C_1 < 0$ . It turns out that  $C_1 = 0$  is the locus of the lower boundary of stability, and it coincides with the one obtained in [18], where use was made of perturbation expansions of the Floquet multipliers.

### 3.2. Resonant case $\Omega = 2\omega$

To express the closeness of  $\Omega$  to  $2\omega$  we introduce a small detuning  $\varepsilon\sigma$  and let  $\Omega = 2\omega + \varepsilon\sigma$ . With this position, Eqs. (19) now reads

$$\begin{cases} d_0^2 X_1 = \frac{\delta\Omega^2}{2} b_{12} B e^{i(\Omega+\omega)t_0} - \frac{\delta\Omega^2}{2} b_{12} \bar{B} e^{i(\Omega-\omega)t_0} - \frac{\delta\Omega^2}{2} b_{11} A e^{i\Omega t_0} + \text{c.c.}, \\ d_0^2 Y_1 + \omega^2 Y_1 = \left( 2i\omega d_1 B - \frac{\delta\Omega^2}{2} b_{22} \bar{B} e^{i\sigma t_1} \right) e^{i\omega t_0} - \frac{\delta\Omega^2}{2} b_{22} B e^{i(\Omega+\omega)t_0} - \frac{\delta\Omega^2}{2} b_{21} A e^{i\Omega t_0} + \text{c.c.} \end{cases} \quad (26)$$

By removing secular terms we shall have

$$2i\omega d_1 B = \frac{\delta\Omega^2}{2} b_{22} \bar{B} e^{i\sigma t_1}, \quad (27)$$

while, again, modulation of  $A$  is undetermined at this order. Truncation of the analysis at this level already furnishes useful information. By using the inverse transformations  $\varepsilon\delta \rightarrow \delta$ ,  $\varepsilon\sigma \rightarrow \sigma$ ,  $t_1 \rightarrow \varepsilon t$ ,  $\varepsilon d_1 B \rightarrow \dot{B}$  we shall obtain an equation in the original quantities:

$$2i\omega \dot{B} = \frac{\delta\Omega^2}{2} b_{22} \bar{B} e^{i\sigma t}. \quad (28)$$

To transform Eq. (28) in an autonomous form, let  $B = R e^{i\sigma t/2}$  and, to pass to real quantities, let  $R = u + i v$ . Then we can write the system of equations for the new variables:

$$\begin{cases} \dot{u} = \left( \frac{\sigma}{2} + \frac{\delta\Omega^2}{4\omega} b_{22} \right) v, \\ \dot{v} = \left( \frac{\sigma}{2} + \frac{\delta\Omega^2}{4\omega} b_{22} \right) u \end{cases} \quad (29)$$

from which

$$u = C(\delta, \Omega) u, \quad v = C(\delta, \Omega) v, \quad (30)$$

where  $C(\delta, \Omega) = ((\delta\Omega^2/4\omega)b_{22})^2 - (\sigma/2)^2$ . The condition  $C(\delta, \Omega) = 0$  determines the upper boundary of stability near the principal parametric resonance  $\Omega = 2\omega$ . Note that  $C(\delta, \Omega)$  is, of course, independent of  $\Delta p$ , which has been scaled at the  $\varepsilon^2$  order. However, it would be impossible to use the rescaling  $\Delta p \rightarrow \varepsilon \Delta p$ , since this would entail the appearance of a resonant term,  $\Delta p c_{11} X_0$ , in the first equation, that could not be removed, due to the fact that  $d_0 d_1 X_0 = 0$ . To investigate the role played by  $\Delta p$  and to evaluate the dynamics of  $A$ , we have to resort to the  $\varepsilon^2$  order.

After removing secular terms (27) the solution for  $X_1, Y_1$  takes the form

$$\begin{cases} X_1 = \frac{\delta\Omega^2}{2(\Omega + \omega)^2} b_{12} B e^{i(\Omega + \omega)t_0} + \frac{\delta\Omega^2}{2(\Omega - \omega)^2} b_{12} \bar{B} e^{i(\Omega - \omega)t_0} + \frac{\delta}{2} b_{11} A e^{i\Omega t_0} + \text{c.c.}, \\ Y_1 = \frac{\delta\Omega^2}{2(\omega^2 - (\Omega + \omega)^2)} b_{22} B e^{i(\Omega + \omega)t_0} - \frac{\delta\Omega^2}{2(\omega^2 - \Omega^2)} b_{21} A e^{i\Omega t_0} + \text{c.c.} \end{cases} \quad (31)$$

By substituting solution (31) into the  $\varepsilon^2$  equations (17) we obtain

$$\begin{cases} d_0^2 X_2 = d_1^2 A \Delta p c_{11} A - \frac{\delta^2 \Omega^2}{2} \left( b_{11}^2 + b_{12} b_{21} \frac{\Omega^2}{\Omega^2 - \omega^2} \right) A + \text{NRT}, \\ d_0^2 Y_2 + \omega^2 Y_2 = \left( d_1^2 B \Delta p c_{22} B - \frac{\delta^2 \Omega^4}{4} \left( b_{12} b_{21} \left( \frac{1}{(\Omega + \omega)^2} + \frac{1}{(\Omega - \omega)^2} \right) - \frac{b_{22}^2}{\omega^2 (\Omega + \omega)^2} \right) B - 2i\omega d_2 B \right) e^{i\omega t_0} + \text{c.c.} + \text{NRT}. \end{cases} \quad (32)$$

Elimination of secular terms leads to

$$\begin{cases} d_1^2 A = \Delta p c_{11} A - \frac{\delta^2 \Omega^2}{2} \left( b_{11}^2 + b_{12} b_{21} \frac{\Omega^2}{\Omega^2 - \omega^2} \right) A, \\ 2i\omega d_2 B = d_1^2 B \Delta p c_{22} B - \frac{\delta^2 \Omega^4}{4} \left( b_{12} b_{21} \left( \frac{1}{(\Omega + \omega)^2} + \frac{1}{(\Omega - \omega)^2} \right) - \frac{b_{22}^2}{\omega^2 (\Omega + \omega)^2} \right) B. \end{cases} \quad (33)$$

By using the inverse transformations  $\varepsilon\delta \rightarrow \delta$ ,  $\varepsilon^2 \Delta p \rightarrow \Delta p$ ,  $t_1 \rightarrow \varepsilon t$ ,  $\varepsilon\sigma \rightarrow \sigma$  and reconstruction

$$A = \varepsilon^2 d_1^2 A,$$

$$\dot{B} = \varepsilon d_1 B + \varepsilon^2 d_2 B \quad (34)$$

we shall obtain two independent equations in the real (not rescaled) quantities

$$\begin{cases} A = \tilde{C}_1(\delta, \Omega, \Delta p) A, \\ \dot{B} = \frac{i\delta\Omega^2}{4\omega} b_{22} \left( 1 - \frac{\sigma}{2\omega} \right) \bar{B} e^{i\sigma t} + i\tilde{C}_2(\delta, \Omega, \Delta p) B, \end{cases} \quad (35)$$

where

$$\tilde{C}_1(\delta, \Omega, \Delta p) = \Delta p c_{11} - \frac{\delta^2 \Omega^2}{2} \left( b_{11}^2 + b_{12} b_{21} \frac{\Omega^2}{\Omega^2 - \omega^2} \right),$$

$$\tilde{C}_2(\delta, \Omega, \Delta p) = \frac{1}{2\omega} \left( \Delta p c_{22} + \frac{\delta^2 \Omega^4}{4} \left( b_{12} b_{21} \left( \frac{1}{(\Omega + \omega)^2} + \frac{1}{(\Omega - \omega)^2} \right) - \frac{b_{22}^2}{\omega^2 (\Omega + \omega)^2} \right) \right). \quad (36)$$

As a first result of the higher order analysis, a new equation for amplitude  $A$ , uncoupled from that of  $B$ , has been determined. The dynamics of  $A$  is governed by coefficient  $\tilde{C}_1(\delta, \Omega, \Delta p)$  in the first of Eqs. (36); however it should be noted that this coefficient coincides with coefficient  $C_1(\delta, \Omega, \Delta p)$  defined by the first of Eqs. (24), relevant to the non resonant case. Therefore, stability of the first mode is not affected by resonance  $\Omega = 2\omega$ , and instability occurs at the lower bound previously determined. As a second result of the more refined analysis,  $\Delta p$  now appears in the modulation equation for amplitude  $B$ .

By proceeding as for Eq. (28), we obtain

$$u = \tilde{C}(\delta, \Omega, \Delta p)u, \quad v = \tilde{C}(\delta, \Omega, \Delta p)v, \quad (37)$$

where  $\tilde{C}(\delta, \Omega, \Delta p) = ((\delta \Omega^2 / 4\omega) b_{22} (1 - \sigma / 2\omega))^2 (\sigma / 2 - \tilde{C}_2)^2$ . Now the condition  $\tilde{C}(\delta, \Omega, \Delta p) = 0$  determines the upper bound of stability. Note that the  $\tilde{C}(\delta, \Omega, \Delta p)$  and  $C(\delta, \Omega)$  only differ for  $\varepsilon^2$  order quantities.

### 3.3. Resonant case $\Omega = \omega/2$

In this case we have  $\Omega = \omega/2 + \varepsilon\sigma$ . Since this is a second order resonance, the first order solution (21) still holds in this case. By substituting the solutions for  $X_0, Y_0$  and  $X_1, Y_1$  into the  $\varepsilon^2$  order equations (17), and accounting for the closeness of  $\Omega$  to  $\omega/2$ , we obtain

$$\left\{ \begin{array}{l} d_0^2 X_2 = d_1^2 A \Delta p c_{11} A - \frac{\delta^2 \Omega^4}{2} \left( \frac{b_{11}^2}{\Omega^2} - \frac{b_{12} b_{21}}{\omega^2 \Omega^2} \right) A \\ \quad - \frac{\delta^2 \Omega^4}{4} \left( \frac{b_{11} b_{12}}{(\Omega - \omega)^2} - \frac{b_{12} b_{22}}{\omega^2 (\Omega - \omega)^2} \right) (\bar{B} e^{2i\sigma t_1} + B e^{-2i\sigma t_1}) + \text{NRT}, \\ d_0^2 Y_2 + \omega^2 Y_2 = \left( 2i\omega d_2 B \Delta p c_{22} B - \frac{\delta^2 \Omega^4}{4} \left( \frac{b_{11} b_{21}}{\Omega^2} - \frac{b_{21} b_{22}}{\omega^2 \Omega^2} \right) A e^{2i\sigma t_1} \right. \\ \quad \left. - \frac{\delta^2 \Omega^4}{4} \left( \frac{b_{12} b_{21}}{(\Omega + \omega)^2} + \frac{b_{12} b_{21}}{(\Omega - \omega)^2} - \frac{b_{22}^2}{\omega^2 (\Omega + \omega)^2} - \frac{b_{22}^2}{\omega^2 (\Omega - \omega)^2} \right) B \right) e^{i\omega t_0} + \text{c.c.} + \text{NRT}. \end{array} \right. \quad (38)$$

By removing secular terms and using the inverse transformations  $\varepsilon\delta \rightarrow \delta$ ,  $\varepsilon^2 \Delta p \rightarrow \Delta p$ ,  $t_1 \rightarrow \varepsilon t$ ,  $\varepsilon\sigma \rightarrow \sigma$ ,  $\varepsilon^2 d_1^2 A \rightarrow A$ ,  $\varepsilon^2 d_2 B \rightarrow \dot{B}$  we obtain the system of equations:

$$\left\{ \begin{array}{l} A = \alpha_1 A - \alpha_2 (\bar{B} e^{2i\sigma t} + B e^{-2i\sigma t}), \\ \dot{B} = \frac{i\alpha_3}{2\omega} B + \frac{i\alpha_4}{2\omega} A e^{2i\sigma t}, \end{array} \right. \quad (39)$$

where

$$\begin{aligned} \alpha_1 &= \Delta p c_{11} + \frac{\delta^2 \Omega^4}{2} \left( \frac{b_{11}^2}{\Omega^2} - \frac{b_{12} b_{21}}{\omega^2 \Omega^2} \right), \\ \alpha_2 &= \frac{\delta^2 \Omega^4}{4} \left( \frac{b_{11} b_{12}}{(\Omega - \omega)^2} - \frac{b_{12} b_{22}}{\omega^2 (\Omega - \omega)^2} \right), \\ \alpha_3 &= \Delta p c_{22} + \frac{\delta^2 \Omega^4}{4} \left( \frac{b_{12} b_{21}}{(\Omega + \omega)^2} + \frac{b_{12} b_{21}}{(\Omega - \omega)^2} - \frac{b_{22}^2}{\omega^2 (\Omega + \omega)^2} - \frac{b_{22}^2}{\omega^2 (\Omega - \omega)^2} \right), \\ \alpha_4 &= \frac{\delta^2 \Omega^4}{4} \left( \frac{b_{11} b_{21}}{\Omega^2} - \frac{b_{21} b_{22}}{\omega^2 \Omega^2} \right). \end{aligned} \quad (40)$$

By letting  $B = R e^{2i\sigma t}$ , with  $R = u + iv$ , we can rewrite the system (39) in the matrix form  $\dot{\mathbf{x}} = \mathbf{H}\mathbf{x}$ , where vector  $\mathbf{x}$  and matrix  $\mathbf{H}$  are

$$\mathbf{x} = \begin{pmatrix} A \\ \dot{A} \\ u \\ v \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_1 & 0 & 2\alpha_2 & 0 \\ 0 & 0 & 0 & 2\Omega - \omega \\ \frac{\alpha_4}{2\omega} & 0 & 2\Omega + \omega + \frac{\alpha_3}{2\omega} & 0 \end{pmatrix}. \quad (41)$$



The characteristic equation for the matrix  $\mathbf{H}$  will have only even powers of the characteristic number  $\lambda$ :

$$\lambda^4 + C_1\lambda^2 + C_2 = 0, \quad (42)$$

where

$$C_1 = \alpha_1 + \alpha_3 + (\omega - 2\Omega)^2 + \frac{\alpha_3^2}{4\omega^2} - \frac{2\Omega\alpha_3}{\omega},$$

$$C_2 = \frac{(4\Omega\omega - 2\omega^2 - \alpha_3)(4\alpha_1\omega\Omega - 2\alpha_1\omega^2 + 2\alpha_2\alpha_4 - \alpha_1\alpha_3)}{4\omega^2} \quad (43)$$

and arguments  $\delta, \Omega, \Delta p$  of  $C_1, C_2$  have been understood.

In this case the condition for stability requires that the following inequalities hold simultaneously:

$$C_1 > 0, \quad C_2 > 0, \quad C_1^2 - 4C_2 > 0. \quad (44)$$

#### 4. The Multiple Scale Method: fractional power expansion

The resonant case  $\Omega = \omega$  is now addressed. This is a special case in which the (standard) procedure of the previous section fails. Indeed, when  $\Omega = \omega$  is substituted in Eqs. (19), a resonant term of frequency 0 appears in the  $X_1$  equation which cannot be removed from the right hand side, since no amplitude derivatives such  $d_1A$  are available in the equation. This is a consequence of the fact that  $d_0d_1X_0 = 0$  in the first equation (since the associated natural frequency is zero), while  $d_0d_1Y_0 \neq 0$  in the second equation (since the relevant natural frequency is not zero). Therefore, the drawback is just due to the static instability (divergence) manifesting itself at  $p = p_0$ . This circumstance is in strict similarity with perturbations from the defective (not semisimple) eigenvalues, encountered in algebraic problems [20,21], as well as in nonlinear dynamical system [22], where fractional power expansions, instead of integer power, must be used. Fractional powers of suitable order, indeed, introduce intermediate steps between the generating equation and the resonant equation, that permits the appearance of key terms, able to remove the resonance. As a result of this idea, we will expand  $X(t, \varepsilon), Y(t, \varepsilon)$  in series of  $\varepsilon^{1/2}$ :

$$X(t, \varepsilon) = X_0(t_0, t_1, t_2, \dots) + \varepsilon^{1/2}X_1(t_0, t_1, t_2, \dots) + \varepsilon X_2(t_0, t_1, t_2, \dots) + \varepsilon^{3/2}X_3(t_0, t_1, t_2, \dots) + \dots,$$

$$Y(t, \varepsilon) = Y_0(t_0, t_1, t_2, \dots) + \varepsilon^{1/2}Y_1(t_0, t_1, t_2, \dots) + \varepsilon Y_2(t_0, t_1, t_2, \dots) + \varepsilon^{3/2}Y_3(t_0, t_1, t_2, \dots) + \dots, \quad (45)$$

where

$$t_0 = t, \quad t_1 = \varepsilon^{1/2}t, \quad t_2 = \varepsilon t, \quad \dots \quad (46)$$

are fractional times. The chain rule now reads

$$\frac{d}{dt} = \sum_{k=0}^{\infty} \varepsilon^{k/2} d_k, \quad \frac{d^2}{dt^2} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \varepsilon^{(k+j)/2} d_k d_j \quad (47)$$

and the perturbation equations are

$$\varepsilon^0 : \begin{cases} d_0^2 X_0 = 0, \\ d_0^2 Y_0 + \omega^2 Y_0 = 0, \end{cases}$$

$$\varepsilon^{1/2} : \begin{cases} d_0^2 X_1 = 2d_0 d_1 X_0, \\ d_0^2 Y_1 + \omega^2 Y_1 = 2d_0 d_1 Y_0, \end{cases}$$

$$\varepsilon^1 : \begin{cases} d_0^2 X_2 = 2d_0 d_2 X_0 - d_1^2 X_0 - 2d_0 d_1 X_1 - \frac{\delta\Omega^2}{2}(b_{11}X_0 + b_{12}Y_0)(e^{i\Omega t_0} + e^{-i\Omega t_0}), \\ d_0^2 Y_2 + \omega^2 Y_2 = 2d_0 d_2 Y_0 - d_1^2 Y_0 - 2d_0 d_1 Y_1 - \frac{\delta\Omega^2}{2}(b_{21}X_0 + b_{22}Y_0)(e^{i\Omega t_0} + e^{-i\Omega t_0}), \end{cases}$$

$$\varepsilon^{3/2} : \begin{cases} d_0^2 X_3 = 2d_1 d_2 X_0 - 2d_0 d_3 X_0 - 2d_0 d_2 X_1 - d_1^2 X_1 - 2d_0 d_1 X_2 - \frac{\delta\Omega^2}{2}(b_{11}X_1 + b_{12}Y_1)(e^{i\Omega t_0} + e^{-i\Omega t_0}), \\ d_0^2 Y_3 + \omega^2 Y_3 = 2d_1 d_2 Y_0 - 2d_0 d_3 Y_0 - 2d_0 d_2 Y_1 - d_1^2 Y_1 - 2d_0 d_1 Y_2 - \frac{\delta\Omega^2}{2}(b_{21}X_1 + b_{22}Y_1)(e^{i\Omega t_0} + e^{-i\Omega t_0}), \end{cases}$$



we have the solution

$$\begin{cases} X_3 = \frac{i\delta}{\Omega} b_{11} d_1 A e^{i\Omega t_0} + \frac{\delta \Omega^2}{2(\Omega + \omega)^2} b_{12} B_1 e^{i(\Omega + \omega)t_0} + \text{c.c.}, \\ Y_3 = \frac{\delta \Omega^2}{2(\omega^2 - (\Omega + \omega)^2)} b_{22} B_1 e^{i(\Omega + \omega)t_0} - \frac{\delta \Omega^2}{2(\omega^2 - (\omega - \Omega)^2)} b_{22} B_1 e^{i(\omega - \Omega)t_0} + \text{c.c.} \end{cases} \quad (57)$$

The equations of the  $\varepsilon^2$  order will be treated next:

$$\begin{cases} d_0^2 X_4 = d_2^2 A - 2d_1 d_3 A - \Delta p c_{11} A - \frac{\delta^2 \Omega^2}{2} b_{11}^2 A + \text{NRT}, \\ d_0^2 Y_4 + \omega^2 Y_4 = (2i\omega(d_4 B_0 + d_3 B_1) - d_2^2 B_0 - 2d_1 d_2 B_1 + \gamma_1 B_0 + \gamma_2 B_0 e^{2i\sigma t_1}) e^{i\omega t_0} + \text{c.c.} + \text{NRT}, \end{cases} \quad (58)$$

where

$$\begin{aligned} \gamma_1 &= \delta p c_{22} + \frac{\delta^2 \Omega^2}{4} \left( \frac{b_{12} b_{21}}{(\Omega + \omega)^2} + b_{22}^2 \left( \frac{1}{\omega^2 - (\Omega + \omega)^2} + \frac{1}{\omega^2 - (\omega - \Omega)^2} \right) \right), \\ \gamma_2 &= \frac{\delta^2 \Omega^2}{4} \frac{b_{22}^2}{\omega^2 - (\omega - \Omega)^2}. \end{aligned} \quad (59)$$

Elimination of secular terms leads to

$$\begin{aligned} d_2^2 A + 2d_1 d_3 A &= \Delta p c_{11} A - \frac{\delta^2 \Omega^2}{2} b_{11}^2 A, \\ 2i\omega(d_4 B_0 + d_3 B_1) &= d_2^2 B_0 - 2d_1 d_2 B_1 + \gamma_1 B_0 + \gamma_2 B_0 e^{2i\sigma t_1}. \end{aligned} \quad (60)$$

To recombine all the previous results, and in line with the second of Eqs. (45), (49) and (51), let us introduce the total amplitude  $B = B_0 + \varepsilon^{1/2} B_1$ . Then, by using the inverse transformations  $\varepsilon \delta \rightarrow \delta$ ,  $\varepsilon^2 \Delta p \rightarrow \Delta p$ ,  $t_1 \rightarrow \varepsilon^{1/2} t$ ,  $\varepsilon^{1/2} \sigma \rightarrow \sigma$  and reconstruction

$$\begin{aligned} A &= \varepsilon d_1^2 A + \varepsilon^{3/2} 2d_1 d_2 A + \varepsilon^2 (d_2^2 A + 2d_1 d_3 A), \\ \dot{B} &= \varepsilon (d_2 B_0 + d_1 B_1) + \varepsilon^{3/2} (d_3 B_0 + d_2 B_1) + \varepsilon^2 (d_4 B_0 + d_3 B_1) \end{aligned} \quad (61)$$

we shall obtain equations in the true (not rescaled) quantities

$$\begin{aligned} A &= \alpha_1 A + \alpha_2 (B e^{-i\sigma t} + \bar{B} e^{i\sigma t}), \\ \dot{B} &= i\alpha_3 A e^{i\sigma t} + \alpha_4 \dot{A} e^{i\sigma t} + i\alpha_5 B + i\alpha_6 \bar{B} e^{2i\sigma t}, \end{aligned} \quad (62)$$

where

$$\begin{aligned} \alpha_1 &= \left( \Delta p c_{11} + \frac{\delta^2 \Omega^2}{2} b_{11}^2 \right), \quad \alpha_2 = \frac{\delta \Omega^2}{2} b_{12}, \\ \alpha_3 &= \frac{\delta \Omega^2}{4\omega} b_{21} \left( 1 - \frac{\sigma}{2\omega} + \left( \frac{\sigma}{2\omega} \right)^2 \right), \quad \alpha_4 = \frac{\delta \Omega^2}{8\omega^2} b_{21} \left( 1 - \frac{\sigma}{\omega} \right), \\ \alpha_5 &= \frac{1}{2\omega} \left( \Delta p c_{22} + \frac{\delta^2 \Omega^4}{4} \left( b_{12} b_{21} \left( \frac{1}{4\omega^2} + \frac{1}{(\Omega + \omega)^2} \right) - b_{22}^2 \left( \frac{1}{\omega^2 - (\Omega + \omega)^2} + \frac{1}{\omega^2 - (\omega - \Omega)^2} \right) \right) \right), \\ \alpha_6 &= \frac{\delta^2 \Omega^4}{8\omega} \left( \frac{b_{12} b_{21}}{4\omega^2} - \frac{b_{22}^2}{\omega^2 - (\omega - \Omega)^2} \right). \end{aligned} \quad (63)$$

By substituting  $B = R e^{i\sigma t}$ , where  $R = u + iv$ , we can recast the system (62) in matrix form, similar to the resonant case  $\Omega = \omega/2$ . Then we will have the same stability conditions

$$C_1 > 0, \quad C_2 > 0, \quad C_1^2 - 4C_2 > 0, \quad (64)$$

where the coefficients are re defined as follows:

$$\begin{aligned} C_1 &= \alpha_1 + \alpha_5^2 - \alpha_6^2 - 2\alpha_5(\Omega - \omega) + (\Omega - \omega)^2 - 2\alpha_2 \alpha_4, \\ C_2 &= (\Omega - \omega - \alpha_5 + \alpha_6)(\alpha_1(\alpha_5 + \alpha_6 - (\Omega - \omega)) - 2\alpha_2 \alpha_4). \end{aligned} \quad (65)$$

## 5. Numerical results

In this section, we will show numerical results for a sample inverted double pendulum. A selected system with mass ratio  $\eta = 5$  was considered, for which the nondimensional stiffness assumes the critical value (Eq. (8))  $p_0 = 13.831$  and the stable frequency (the third of Eqs. (12) the value  $\omega = 7.741$ . Then, a small perturbation of the stiffness  $\Delta p = 0.1$  was introduced, in order to make the system statically unstable, and the boundaries of the dynamic re stabilization region were sought in the plane of the excitation frequency  $\Omega$  and the nondimensional excitation amplitude  $\delta$ .

In order to validate the asymptotic results supplied by the Multiple Scale Method, a comparison was performed with exact numerical results furnished by Floquet theory of ordinary differential equations with periodic coefficients [20]. According to this method, a matrix  $\mathbf{X}(t)$  of independent solutions for the linear system (5), satisfying the initial conditions  $\mathbf{X}(0) = \mathbf{I}$ , was numerically computed in a period  $T = 2\pi/\Omega$  for each set of the control parameters  $\Omega, \delta$ ; then the Floquet (or monodromy) matrix  $\mathbf{F} = \mathbf{X}(T)$  was evaluated. For algorithmic purposes, a small damping was added to the system, able to change marginal into asymptotic stability. The latter was investigated by checking that all the multipliers  $\rho$  (i.e. the roots of the characteristic polynomial of  $\mathbf{F}$ ) were, in modulus, less than 1. Fig. 2 provides an overall view of the system behavior. Here, the grey zone denotes the stable region, obtained according to the Floquet method, while continuous lines represent the analytical stability boundaries furnished by the Multiple Scale Method. Agreement between the two results appears to be excellent at low excitation level, and qualitatively good at moderately high levels. It is seen that re stabilization can occur at any excitation frequencies (as found in [18]), except for frequencies close to, and less than, the resonant value  $\Omega = \omega$ , as well as for frequencies close to the resonant value  $\Omega = 2\omega$ . Low excitation frequencies require a comparatively higher level of amplitude, with respect to middle and high frequencies. This threshold represents a lower bound which is well described by the non resonant solution (Eqs. (25)), which coincides with the one obtained in [18]. As revealed by the perturbation procedure, re stabilization at the lower bound is due to the creation of a zero frequency, produced by the combination of natural and excitation frequencies, which is able to make up for the lack of stiffness of the associated static system. In contrast, the stable natural frequency plays a passive role in the stabilization phenomenon, since, although it contributes to the solution (see the coefficient  $C_1$  in the first of Eqs. (24)), the modulation equations (25) are uncoupled, and the amplitude  $B(t)$  is harmonic. Fig. 2, however, also reveals the existence of upper bounds for the excitation level, which originate from the resonant values  $\Omega = \omega/2, \omega, 2\omega$ . Here, differently from the lower bound, the phenomenon is strongly affected by the behavior of the (otherwise stable) mode. In other words, re stabilization that would occur for a single d.o.f. system with zero frequency, cannot occur, since the second mode is rendered unstable by the parametric excitation. It should be noted that a narrow island of stable solutions (already observed in [18]) does exist at the middle frequencies; however, since it occurs at high excitation levels it cannot be captured by the perturbation method.

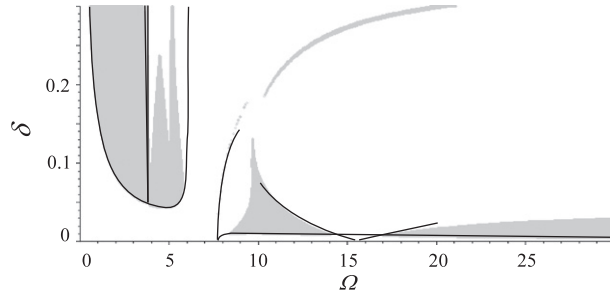


Fig. 2. Stability regions and analytical stability boundaries for sample system  $\eta = 5, \Delta p = 0.1$ .

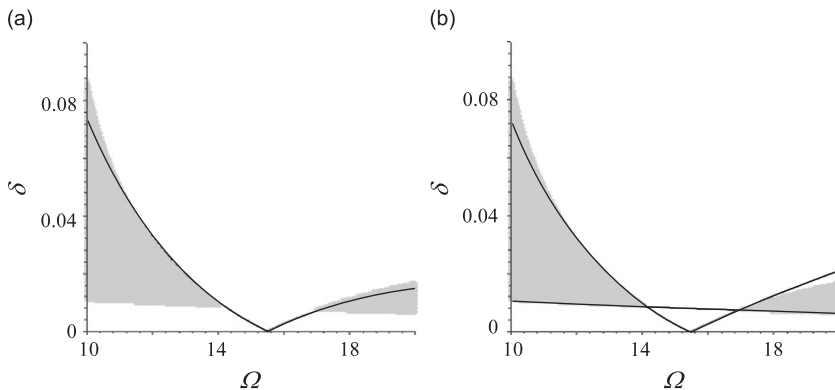


Fig. 3. Stability regions and analytical stability boundaries around  $\Omega = 2\omega$ : first-order asymptotic solution (a) and second-order asymptotic solution (b).

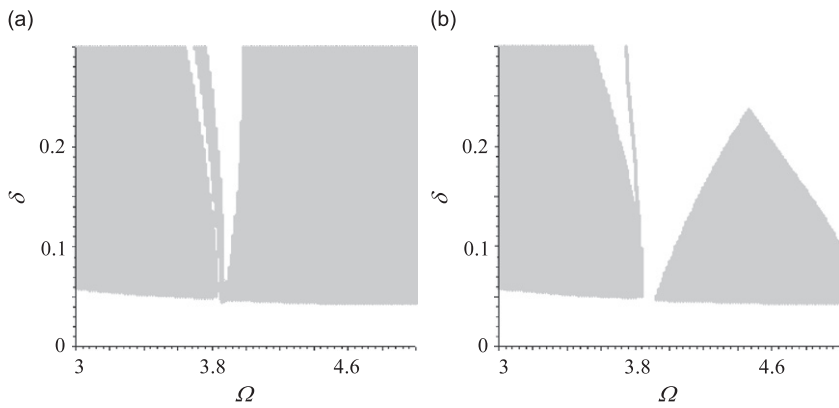


Fig. 4. Stability regions around  $\Omega = \omega/2$ : asymptotic solution (a) and Floquet solution (b).

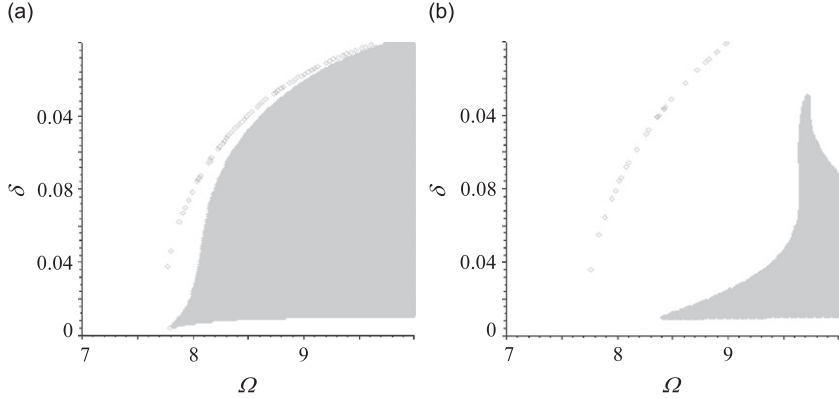


Fig. 5. Stability regions around  $\Omega = \omega$ : asymptotic solution (a) and Floquet solution (b).

Some zooms of Fig. 2 are also shown in subsequent pictures. Fig. 3(a,b) displays the analytical stability boundaries close to the resonance  $\Omega = 2\omega$ , both for the first order solution (Fig. 3(a), Eqs. (30)), and the second order solution (Fig. 3(b), Eqs. (37)). Here the agreement with exact results is excellent, already at the lower order. Second order expansion also leads to spurious solutions, as well known in the literature (see [23] for a discussion) and therefore disregarded in Fig. 3(b).

An expansion of the neighborhood of  $\Omega = \omega/2$  is also shown in Fig. 4(a,b). In particular, Fig. 4(a) shows the stable zone as obtained by inequalities (44), whereas Fig. 4(b) displays the exact solution. It appears that the two solutions are in good accordance; in particular, the existence of a narrow stability zone, not appreciable in the scale of Fig. 2, is revealed, breaking the main instability region.

Finally, a magnification of the region close to  $\Omega = \omega$  is depicted in Fig. 5(a,b), comparing results following from inequalities (64) (Fig. 5(a)) and the Floquet method (Fig. 5(b)). The perturbation solution is able to capture two interesting features of the exact solution, namely: (a) stabilization occurs on the right side only of the resonant value; (b) a very thin zone of stable solutions exists at a higher level. Note that we also have good quantitative agreement in the resonance condition, but far from the resonance at  $\Omega > 8$  we have only qualitative agreement.

## 6. Conclusions

In this paper, the phenomenon of dynamic stabilization of a (statically) unstable two d.o.f. system, already addressed in [18], was studied. The perturbation Multiple Scale Method was used here, which is able to provide both the lower and the upper bounds of the stability regions in the plane of parameters, thus completing and enriching the investigation carried out in [18], where the sensitivities of the Floquet multipliers were instead used. The following main results were achieved:

1. The Multiple Scale Method reveals the intimate essence of the re-stabilization phenomenon, which is connected to the creation of zero frequencies originated by the combination of the natural and excitation frequencies. This result is consistent with the concept of 'effective mechanical stiffness' of parametrically excited systems, discussed in the literature [12,15,16]. Depending on the parameters, the 'apparent' stiffness can be lower or higher than the elastic stiffness, this entailing a beneficial or detrimental effect of the excitation.

2. Stabilization, however, is limited by the occurrence of dynamic instability of the second mode, which occurs close to resonant values. Here, boundary curves originate, which constitute an upper bound far from resonance.
3. Non-straightforward perturbation algorithms were implemented, leading to amplitude modulation equations valid in the non-resonant and resonant cases. The method requires a proper ordering of the parameters, and a proper use of integer or fractional power expansions, accounting for the (defective) nature of the double-zero eigenvalue, similar to what happens in bifurcation theory of the eigenvalues of algebraic systems, as well as in the bifurcation of autonomous systems. Differently from the last case, however, the so-called 'arbitrary constants', rising from the complementary solution of the perturbation equations, here play a fundamental role, so that they cannot be neglected, as is usual in standard cases.
4. Accuracy of the perturbation solution, when compared with exact numerical results, was found to be excellent in some cases, and less good in others. In these circumstances, however, the qualitative aspects were correctly captured. Major differences occurred for fractional power expansions, where it is well known that the degeneracy of the individual point would call for higher-order perturbation expansions.
5. Although the method was proved here to work for a two-d.o.f. system, it is believed that it could be extended to larger-dimensional systems and applied for stability study in other nonlinear physical systems containing parameters.

These results were obtained for the undamped system, while it is well-known that the presence of damping might change the picture of instability. However, the introduction of damping will make the analysis much more complicated, so that it is a matter for future study.

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