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QUASI-ISOMETRIES BETWEEN HYPERBOLIC METRIC SPACES, QUANTITATIVE ASPECTS

VLADIMIR SHCHUR

ABSTRACT. This is a preliminary version of my PhD thesis. In this text we discuss possible ways to give quantitative measurement for two spaces not being quasi-isometric. From this quantitative point of view, we reconsider the definition of quasi-isometries and propose a notion of "quasi-isometric distortion growth" between two metric spaces. We revise our article [30] where an optimal upper-bound for Morse Lemma is given, together with the symmetric variant which we call Anti-Morse Lemma, and their applications.

Next, we focus on lower bounds on quasi-isometric distortion growth for hyperbolic metric spaces. In this class, \mathbb{L}^p -cohomology spaces provides useful quasi-isometry invariants and Poincaré constants of balls are their quantitative incarnation. We study how Poincaré constants are transported by quasi-isometries. For this, we introduce the notion of a cross-kernel. We calculate Poincaré constants for locally homogeneous metrics of the form $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$, and give a lower bound on quasi-isometric distortion growth among such spaces.

This allows us to give examples of different quasi-isometric distortion growths, including a sublinear one (logarithmic) provided by unipotent locally homogeneous spaces.

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Key words and phrases. Hyperbolic space, quasi-isometrie, quasi-geodesic, Morse Lemma, Poincaré inequality, Poincaré constant.

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Part 1. Introduction

1. THE QUANTITATIVE QUASI-ISOMETRY PROBLEM

1.1. **General idea.** Gromov’s quasi-isometry classification problem for groups [4] has given rise to a large amount of works (for the reader’s convenience, we include a survey of the quasi-isometry classification problem in Section 4). When two groups are shown to be non-quasi-isometric, it would be desirable to give a quantitative measurement of this (we thank Itai Benjamini for bringing this issue to our attention). The aim of our research is to measure quantitatively how far two spaces are from being quasi-isometric at scale $R > 0$, and study on examples what may happen as R tends to infinity.

Let X and Y be two metric spaces not quasi-isometric to each other. Given some positive real number R , consider quasi-isometries between subsets in X and Y respectively of diameter of the order of R . These subsets are bounded spaces so there exists a (λ, c) -quasi-isometry with minimal $\lambda = \lambda(R)$. For simplicity, we shall assume that additive constants c are much less than $\lambda(R)$. We want to study how $\lambda(R)$ behaves as R goes to infinity. Later, we shall give precise (and rather cumbersome) definitions, but in this introduction, we content ourselves with a rather vague one.

1.2. **Example.** We consider the following theorem as the prototype of a quantitative result. Y. Shalom and T. Tao gave a quantitative version of Gromov’s famous theorem stating that every finitely generated group of polynomial growth is virtually nilpotent.

Theorem 1. (*Y. Shalom, T. Tao [23]*) *Let G be a group generated by a finite (symmetric) set S and suppose that one has a polynomial growth condition*

$$|B_S(R)| \leq R^d$$

for some

$$R > \exp(\exp(Cd^C))$$

for some sufficiently large absolute constant C . Then G contains a finite index subgroup H which is nilpotent of step at most C^d .

A corollary of this theorem is

Corollary 1. *Let (G, S) be a finitely generated group. Assume that G is not virtually nilpotent. Then*

$$|B_S(R)| \geq R^{\sigma(\log \log R)^\sigma}$$

for any $R > 1/\sigma$, where $\sigma > 0$ is a sufficiently small absolute constant.

This has the following consequence for our quantitative quasi-isometry problem.

Example 1. Nilpotent versus non-nilpotent groups.

Let G and H be finitely generated groups, with H virtually nilpotent and G not virtually nilpotent group. Pick finite generating systems $S \subset G$ and $S' \subset H$ and get metric spaces G_S and $H_{S'}$. If $\Theta : B_{G_S}(R) \rightarrow H_{S'}$ is a (λ, c) -quasi-isometric embedding, then $\Theta(B_{G_S}(R)) \subset B_{H_{S'}}(\lambda R + c)$. Let Λ be a $\lambda + c$ -lattice in $B_{G_S}(R)$. One can pick Λ in such a way that

$$|\Lambda| \geq \frac{|B_{G_S}(R)|}{B_{G_S}(\lambda + c)} \geq e^{-C(\lambda+c)} |B_{G_S}(R)|.$$

On the other hand, since Θ is injective on Λ ,

$$|B_{H_{S'}}(\lambda R + c)| \geq |\Lambda|.$$

Hence,

$$|B_{S'}(\lambda R + c)| \geq e^{-C(\lambda+c)} |B_S(R)|,$$

where $C = C(G, S)$.

Now as H is virtually nilpotent, $|B_{S'}(R')| \leq K(R')^d$ where d depends on H only and K depends on H and S' . So Corollary 1 implies that

$$R^{\sigma(\log \log R)^\sigma} \leq |B_S(\lambda + c)| K(\lambda R + c)^d$$

and for R big enough we conclude that

$$\lambda + c \geq C(\log \log R)^\sigma \log R,$$

where $C = C(G, S, H, S')$ is a constant depending on the groups and generating systems, but σ is universal.

The fact that G does not have polynomial growth gives a mere $\lambda(R) \geq \Omega(\log R)$. Shalom and Tao's theorem gives an extra factor of $(\log \log R)^\sigma$.

2. SUMMARY OF RESULTS

Here we will briefly discuss our results.

2.1. Morse Lemma. Roughly speaking, the Morse lemma states that in a hyperbolic metric space, a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasigeodesic (see definitions 3, 14) γ belongs to a $\lambda_1 \lambda_2 (c_1 + c_2)$ -neighborhood of every geodesic σ with the same endpoints. Our aim is to prove the optimal upper bound for the Morse lemma.

Theorem 2 (Morse lemma). *Let γ be a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic in a δ -hyperbolic space E and let σ be a geodesic segment connecting its endpoints. Then γ belongs to an H -neighborhood of σ , where*

$$H = A\lambda_1\lambda_2\left(c_1 + c_2 + \delta + 1\right),$$

and A is some universal constant.

We will prove this theorem in Section 9.2. This result is optimal, i.e., there exists an example of a quasi-geodesic such that the distance of the farthest point of γ from σ is $\lambda_1 \lambda_2 \min\{c_1, c_2\}/4$ (see Section 10).

The Morse lemma plays an important role in the geometry of hyperbolic spaces. For example, it is used to prove that hyperbolicity is invariant under quasi-isometries between geodesic spaces [1] (see Chapter 5.2, Theorem 12): let E and F be δ_1 - and δ_2 -hyperbolic geodesic spaces. If there exists a (λ, c) -quasi-isometry between these two spaces, then

$$\delta_1 \leq 8\lambda(2H + 4\delta_2 + c).$$

We expect our optimal bound in the Morse lemma to be a useful tool in the quantitative quasi-isometric embedding problem for hyperbolic metric spaces.

2.2. Anti-Morse Lemma. We give a second illustration. In certain hyperbolic metric spaces, self-quasi-isometries fixing the ideal boundary move points a bounded distance. Directly applying the Morse lemma yields a bound of $H \sim \lambda^2 c$, while the examples that we know achieve merely λc . For this problem, we can fill the gap partially. Our argument relies on the following theorem, which we call the anti-Morse lemma.

Theorem 3 (anti-Morse lemma). *Let γ be a (λ, c) -quasi-geodesic in a δ -hyperbolic metric space and σ be a geodesic connecting the endpoints of γ . Let $4\delta \ll \ln \lambda$. Then σ belongs to a H_{am} -neighborhood of γ , where $H_{am} = A_3(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)$, here A_3 is some universal constant.*

We prove Theorem 3 in Section 11.

As an example of an application of Anti-Morse Theorem we show that the center of a ball in a tree cannot be moved very far by a self-quasi-isometry.

Proposition 1. *Let O be the center of a ball of radius R in a d -regular metric tree T ($d \geq 3$). Let f be $(\lambda_1, \lambda_2, c_1, c_2)$ -self-quasi-isometry of this ball. Then*

$$d(f(O), O) \leq \min\{R, \lambda_1 H_{am} + c_1 + \lambda_1(c_1 + c_3 + 1)\}.$$

Because $\delta = 0$ for a tree, we have $d(f(O), O) \lesssim \lambda \log \lambda c$ for sufficiently large λ . We prove this proposition in Section 12.

In Section 14, we define the class of geodesically rich hyperbolic spaces (it contains all Gromov hyperbolic groups), for which we can prove the following statement.

Theorem 4. *Let X be a geodesically rich δ -hyperbolic metric space and f be a $(\lambda_1, \lambda_2, c_1, c_2)$ -self-quasi-isometry fixing the boundary ∂X . Then for any point $O \in X$, the displacement $d(O, f(O)) \leq \lambda_1(H_{am} + r_1) + 2c_1 + A_4$, where r_1 and A_4 are constants depending on the geometry of the space X .*

In Part 2, we shall first discuss the geometry of hyperbolic spaces and prove a lemma on the exponential contraction of lengths of curves with projections on geodesics. We then discuss the invariance of the Δ -length of geodesics under quasi-isometries. Using these results, we prove the quantitative version of the Morse and anti-Morse lemmas. We define the class of geodesically rich spaces; for this class, we estimate the displacement of points by self-quasi-isometries that fix the ideal boundary. Finally, we show that this class includes all Gromov hyperbolic groups.

2.3. Lower bounds for negatively curved locally homogeneous spaces. The third part is devoted to the study of the transport of Poincaré inequalities by quasi-isometries. Using these results we will give a lower bound for the (λ, c) -quasi-isometric distortion between balls of radius R in spaces of the form $Z_\mu = \mathbb{T}^n \times \mathbb{R}$ with exponential metrics $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$ and $dt^2 + \sum_i e^{2\mu'_i t} dx_i^2$, where all μ_i, μ'_i are assumed to remain bounded both from below and above. Essentially our theorem states that the quasi-isometric distortion growth function is linear.

Theorem 5. *(Rough version. For a precise statement, see Theorem 24). Every (λ, c) -quasi-isometric embedding of an R -ball in Z_μ into $Z_{\mu'}$ satisfies*

$$\lambda + c \geq \left(\frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_i}{\mu'_n} \right) R.$$

The proof of this theorem involves several results which could have an independent interest and more applications. First, we study the transport of Poincaré inequalities by quasi-isometries. For this purpose we propose to use “cross-kernels”. These objects are naturally obtained as follows. Let X and Y be two metric spaces, $f : X \rightarrow Y$ a quasi-isometry and $\psi(y_1, y_2)$ a kernel on Y . The composition relatively to the first argument $\psi(f(x), y)$ is an example of a cross-kernel. Cross-kernels help us to transport functions from Y to X and allow us to control quantitatively their Poincaré constants.

Further, we establish an upper-bound for the Poincaré constant of ball in an exponential metric $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$,

$$C_p(\mu) \leq c \left(p, \sum \mu_i \right) \left(1 + (\max_i \mu_i) R \right),$$

where $c(p, \sum \mu_i)$ is a constant depending only on p and the sum of μ_i .

2.4. Upper bounds. In Part 4, we shall give a construction of quasi-isometries between balls in hyperbolic metric spaces. We begin with the approximation (up to an additive error depending on hyperbolicity constant) of the distance between two points. Let (X, P_0) be

a hyperbolic metric space with the base points P_0 . Let $P_1, P_2 \in X$ be two points in this space, the distances to the base point are $d(P_1, P_0) = t_1$ and $d(P_2, P_0) = t_2$. Now consider the geodesics P_0P_1 and P_0P_2 , denote by $-t^\infty$ the logarithm of visual distance between the ends at infinity of this geodesics. Then up to an additive error

$$d(P_1, P_2) = t_1 + t_2 - \min\{t_1, t_2, t^\infty\}.$$

Using this formula we find quasi-isometry constants for the restriction on balls of a map Θ between X and Y which is a kind of radial extension of a homeomorphism θ between ideal boundaries. The following is a non technical statement of Theorem 25, see Section 18 for a complete statement.

Theorem 6. *Let X, Y be hyperbolic metric spaces. Let $\theta : \partial X \rightarrow \partial Y$ be a homeomorphism. We define the following function. For $R > 0$,*

$$K(R) = \sup \left\{ \left| \log \frac{d_{y_0}(\theta(\xi_1), \theta(\xi_2))}{d_{x_0}(\xi_1, \xi_2)} \right| \mid d_{x_0}(\xi_1, \xi_2) \geq e^{-R} \vee d_{x_0}(\theta(\xi_1), \theta(\xi_2)) \geq e^{-R} \right\}.$$

Here d_{x_0}, d_{y_0} denote visual metrics on ideal boundaries. Then there exists a $(K(R), K(R))$ -quasi-isometry between $B_X(x_0, R)$ and $B_Y(y_0, R)$.

For $Z_\mu, Z_{\mu'} = \mathbb{T}^n \times \mathbb{R}$ with exponential metrics we show that $K(R) = \max_i |\mu_i / \mu'_i - 1| R$. Then we give an example of non-quasi-isometric negatively curved homogeneous manifolds with $K(R) \lesssim \log R$.

3. STATEMENT OF THE QUANTITATIVE QUASI-ISOMETRY PROBLEM

3.1. Definition of quasi-isometry.

Definition 1. Two metric spaces X and Y are said to be roughly quasi-isometric if there exists two maps $f : X \rightarrow Y, g : Y \rightarrow X$ and two constants $\lambda > 0$ and $c \geq 0$ such that

- $|f(x) - f(y)| \leq \lambda|x - y| + c$ for every $x, y \in X$,
- $|g(x') - g(y')| \leq \lambda|x' - y'| + c$ for every $x', y' \in Y$,
- $|g(f(x)) - x| \leq c$ for every $x \in X$,
- $|f(g(x')) - x'| \leq c$ for every $x' \in Y$.

The word *rough* is often dropped away.

The first two conditions mean that f and g are nearly Lipschitz if we are looking from afar. The two latter conditions provide that f and g are nearly inverse of each other. It is easy to check that the composition of two quasi-isometries is also a quasi-isometry. So, quasi-isometries provide an equivalence relation on the class of metric spaces.

Remark 1. Definition 1 is invariant under taking inverse maps.

Definition 2. A map $f : E \rightarrow F$ between metric spaces is a *rough (λ, c) -quasi-isometric embedding* if for any two points x, y of E

$$\frac{1}{\lambda}(|x - y|_E - c) \leq |f(x) - f(y)|_F \leq \lambda|x - y|_E + c.$$

This definition follows from the definition for two spaces being quasi-isometric but it does not include the existence of a nearly inverse map. We can easily transform Definition 2 to make it equivalent to Definition 1 by adding the condition that f is nearly surjective. We ask that the image of E is c -dense in F : for every point y of F there exists a point x of E such that $d(y, f(x)) < c$.

3.2. Choice of a class of maps. What do we exactly mean by quasi-isometric distortion at scale R ?

We propose three different settings. Let X and Y be metric spaces. Let x_0, y_0 be base point in X and Y . Given $R > 0$, three families of maps can be considered.

- (1) Quasi-isometries of $B_X(x_0, R)$ onto $B_X(y_0, R)$.
- (2) Quasi-isometries of $B_X(x_0, R)$ onto $B_X(y_0, \rho(R))$, for some function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.
- (3) Quasi-isometric embeddings of $B_X(x_0, R)$ to Y .

Neglecting the additive constant c for a while, these families give rise to distortion functions $\lambda_1(R)$, $\lambda_2(R)$ and $\lambda_3(R)$.

λ_1 has the advantage of letting X and Y play symmetric roles. We shall see next that lower bounds on λ_1 can be obtained easily. In fact, λ_1 may tend to infinity even if X and Y are quasi-isometric. It is therefore rather surprising that non trivial upper bounds on λ_1 can be given (Theorem 6).

λ_2 seems to be appropriate in certain settings, as examples below will show.

λ_3 is non-symmetric. It is natural in the sense that it stays bounded if and only if there exists a quasi-isometric embedding of X to Y . It looks harder to estimate from below. Nevertheless, this is what is done in Theorem 5.

3.3. Example illustrating the behaviour of λ_1 . Let X and Y be two regular trees T_{d_1} and T_{d_2} respectively, suppose that $d_1 < d_2$. Consider two balls of radius R in both of these spaces, denote them by $B_{d_1}(R)$ and $B_{d_2}(R)$ respectively. What is the lower bound for the constants of quasi-isometry between them? The volume of $B_{d_1}(R)$ is d_1^R and the volume of $B_{d_2}(R)$ is d_2^R . A (λ_R, c_R) -quasi-isometry $f'_R : B_{d_1}(R) \rightarrow B_{d_2}(R)$ should preserve (in quasi-isometric sense) volumes. In our future calculations we will drop some multiplicative constants (which are bounded constants which depend only on a whole space and not on the particular radius R)

Divide $B_{d_1}(R)$ in balls of radius c_R . The image of such a ball has maximal possible radius $(\lambda_R + 1)c_R$ and the number of such balls is $Vol(B_{d_1}(R))/Vol(B_{d_1}(c_R)) = d_1^R/d_1^{c_R}$. By definition of a quasi-isometry $B_{d_2}(R)$ should be covered by images of these balls, hence $Vol(B_{d_2}(R)) \leq d_1^R/d_1^{c_R} Vol(B_{d_2}((\lambda_R + 1)c_R))$

$$d_2^R \leq d_1^R/d_1^{c_R} d_2^{(\lambda_R+1)c_R}.$$

From this relation we conclude that $\lambda_R c_R = \Omega(R)$. On the other hand, we know from [24] that two regular trees of degrees at least 4 are quasi-isometric.

3.4. Example illustrating the behaviour of λ_2 . Take a d -regular tree. Now transform it in a $d(d-1)$ -regular tree in a following way. Take an origin, drop away all its neighbours and add edges to all their ancestors (all the points of second level). Now we delete all points of third level and connect directly the points of second levels with corresponding points of fourth level. As a result we get a new tree which is evidently $(2, 1)$ -quasi-isometric to the initial one. Moreover, any ball $B_d(R)$ is $(2, 1)$ -quasi-isometric to a ball in a new tree of radius $R/2$.

3.5. Role of the additive parameter c . Quasi-isometry constants are pairs (λ, c) . Up to now, we have neglected the additive constant c . But this cannot be done with impunity, as the following examples show.

Example 2. Intervals.

Consider intervals $I_{\mathbb{R}} = [0, 1]$, $I_{\mathbb{R}}^{\lambda} = [0, \lambda]$ in \mathbb{R} and $I_{\mathbb{Z}} = [0, 1]$, $I_{\mathbb{Z}}^{\lambda} = [0, \lambda]$ in \mathbb{Z} . The λ times stretching of $I_{\mathbb{R}}$ to $I_{\mathbb{R}}^{\lambda}$ is a $(\lambda, 0)$ -quasi-isometry as inner points of $I_{\mathbb{R}}$ fill the inner points of an image. The natural embeddings of $I_{\mathbb{Z}}$ in $I_{\mathbb{R}}$ and $I_{\mathbb{Z}}^{\lambda}$ in $I_{\mathbb{R}}^{\lambda}$ are both $(1, 1)$ -quasi-isometries, though the stretching of $I_{\mathbb{Z}}$ to $I_{\mathbb{Z}}^{\lambda}$ is a $(1, \lambda)$ -quasi-isometry.

$$\begin{array}{ccc} I_{\mathbb{R}} & \xrightarrow{(\lambda, 0)} & I_{\mathbb{R}}^{\lambda} \\ (1, 1) \downarrow & & \downarrow (1, 1) \\ I_{\mathbb{Z}} & \xrightarrow{(1, \lambda)} & I_{\mathbb{Z}}^{\lambda} \end{array}$$

Example 3. Line versus plane.

Consider \mathbb{R} and \mathbb{R}^2 . Here we will describe a $(c_1 R, c_2)$ -quasi-isometry between balls in these spaces (c_1 and c_2 are two universal constants). A ball in \mathbb{R} is just an interval of length R . Stretch it R times and then fill a ball in \mathbb{R}^2 with a serpentine or a zigzag with width 1. It is easy to check that this is indeed a $(R, 1)$ -quasi-isometry. Now change \mathbb{R} by \mathbb{Z} and \mathbb{R}^2 by \mathbb{Z}^2 . Though there exist evident $(1, 1)$ -quasi-isometries between balls in \mathbb{R} and \mathbb{Z} and balls in \mathbb{R}^2 and \mathbb{Z}^2 provided by natural embeddings, there is no $(c'_1 R, c'_2)$ -quasi-isometry between \mathbb{Z} and \mathbb{Z}^2 . Moreover, the additive constant should be of order R with small constant c'_2 . The reason is that by definition, the image should be c'_2 -dense. That is the range should be covered by the balls of radius c'_2 centered in the images of points of the departure space, hence

$$\text{Vol}(B_{\mathbb{Z}^2}(R)) \leq |B_{\mathbb{Z}}(R)| \text{Vol}(B_{\mathbb{Z}^2}(c'_2)).$$

In $B_{\mathbb{Z}}(R)$ we have only R points and up to some universal multiplicative constants we get

$$R^2 \leq R(c'_2)^2$$

what leads to

$$c'_2 \geq R.$$

$$\begin{array}{ccc} B_R(\mathbb{R}) & \xrightarrow{(R,0)} & I_R(\mathbb{R}^2) \\ (1,1) \downarrow & & \downarrow (1,1) \\ I_R(\mathbb{Z}) & \xrightarrow{(1,R)} & I_R(\mathbb{Z}^2) \end{array}$$

We arrive at

Conclusion 1. In the quantitative problem both the multiplicative and the additive parameters are important.

3.6. Choice of a numerical measurement of distortion. Here we want to present a form of definition of quasi-isometries which is more convenient for quantitative problems and to study compositions of quasi-isometries. For this purpose, we shall observe that, under composition, quasi-isometry constants behave like elements of the affine group of the line. We shall introduce a natural distance on the affine group and prove that it is a function of $\lambda + c^2/\lambda + 1/\lambda$, where λ and c are quasi-isometry's constants.

Sometimes it will be useful for us to distinguish constants as follows.

Definition 3. We say that a map $f : X \rightarrow Y$ is a quasi-isometric embedding if there exist constants $\lambda_1, \lambda_2, c_1, c_2$ such that for any two points $x_1, x_2 \in X$

$$\frac{1}{\lambda_2}(d_X(x_1, x_2) - c_2) \leq d_Y(f(x_1), f(x_2)) \leq \lambda_1 d_X(x_1, x_2) + c_1.$$

We say that X and Y are quasi-isometric if the image $f(X)$ is c_3 -dense in Y for some given constant c_3 .

Study compositions of quasi-isometries. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be $(\lambda_1, c_1, \lambda_2, c_2)$ - and (μ_1, c_1, μ_2, d_2) -quasi-isometries respectively (we use 3 here as definition of quasi-isometries). x_1, x_2 are two points in X . Hence

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leq \lambda_1 d_X(x_1, x_2) + c_1, \\ d_X(f^{-1}(y_1), f^{-1}(y_2)) &\leq \lambda_2 d_Y(y_1, y_2) + c_2. \end{aligned}$$

and

$$\begin{aligned} d_Z(g(y_1), g(y_2)) &\leq \mu_1 d_Y(y_1, y_2) + d_2, \\ d_Y(g^{-1}(z_1), g^{-1}(z_2)) &\leq \mu_2 d_Z(z_1, z_2) + d_2. \end{aligned}$$

hence for $g \circ f$ we have

$$\begin{aligned} d_Z(g \circ f(x_1), g \circ f(x_2)) &\leq \lambda_1 \mu_1 d_X(x_1, x_2) + \mu_1 c_1 + d_1, \\ d_X((g \circ f)^{-1}(z_1), (g \circ f)^{-1}(z_2)) &\leq \lambda_2 \mu_2 d_Z(z_1, z_2) + \lambda_2 d_2 + c_2. \end{aligned}$$

We see that the distortion of metrics by a quasi-isometry f can be encoded into two matrices

$$F_1 = \begin{pmatrix} \lambda_1 & c_1 \\ 0 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} \lambda_2 & c_2 \\ 0 & 1 \end{pmatrix},$$

and in matrix form we can write

$$\begin{pmatrix} d_Y \\ 1 \end{pmatrix} \leq F_1 \begin{pmatrix} d_X \\ 1 \end{pmatrix}.$$

Hence the composition $h = g \circ f$ is encoded by matrices

$$G_1 F_1, F_2 G_2.$$

Let D be a left-invariant distance on $\mathbb{R} \times \mathbb{R}$. We set $D(f) = D((\lambda_1, c_1), (1, 0))$ the distance to an isometry and $\tilde{D} = \max\{D(f), D(f^{-1})\}$. It is easily seen that \tilde{D} satisfies the triangle inequality from the following relation (which uses that D is left-invariant)

$$\begin{aligned} D(h) &= D((\mu_1 d_1)(\lambda_1, c_1), (1, 0)) \leq D((\mu_1 d_1)(\lambda_1, c_1), (\mu_1, d_1)) + D((\mu_1, d_1), (1, 0)) = \\ &= D((\lambda_1, c_1), (1, 0)) + D((\mu_1, d_1), (1, 0)) = D(f) + D(g). \end{aligned}$$

Consider the vector space of symmetric square matrices of size two with trace 0

$$Y = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, a, b \in \mathbb{R} \right\}.$$

The map $S \in Y \mapsto \exp(S) \bmod O$ (where O stands for all orthogonal matrices) is a diffeomorphism from Y to $SL_2\mathbb{R}/SO(2)$. Moreover, any matrix S defines a geodesic $t \mapsto e^{tS} \bmod O$, and

$$D(1, e^{tS}) = t\|S\| = t\sqrt{a^2 + b^2}.$$

Take any real u and v . So considering the inverse diffeomorphism, we get that there exist real a, b and an orthogonal matrix O such that

$$\begin{pmatrix} u & v \\ 0 & \frac{1}{u} \end{pmatrix} = e \begin{pmatrix} a & b \\ b & -a \end{pmatrix} O.$$

For the product of this matrix with the transposed one

$$\begin{pmatrix} u & v \\ 0 & \frac{1}{u} \end{pmatrix} \begin{pmatrix} u & 0 \\ v & \frac{1}{u} \end{pmatrix} = \begin{pmatrix} u^2 + v^2 & \frac{v}{u} \\ \frac{v}{u} & \frac{1}{u^2} \end{pmatrix} = e \begin{pmatrix} a & b \\ b & -a \end{pmatrix} O O^T e \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = e^2 \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

On the one hand, eigenvalues of this matrix are roots of the equation

$$x^2 - (u^2 + v^2 + \frac{1}{u^2})x + 1 = 0,$$

on the other hand as the matrix

$$\frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

is orthogonal with trace being equal to 0 we conclude that the eigenvalues of its exponential are $e^{\pm 2\sqrt{a^2 + b^2}}$. So we conclude that the distance $D = \sqrt{a^2 + b^2}$ is a function of $u^2 + v^2 + 1/u^2$.

Here we present an isomorphism for $\left\{ \begin{pmatrix} u & v \\ 0 & \frac{1}{u} \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} \lambda & c \\ 0 & 1 \end{pmatrix} \right\}$. It is given by the following formulas $u = \text{sign}(\lambda)\sqrt{|\lambda|}$ and $v = \text{sign}(\lambda)c/\sqrt{|\lambda|}$.

Finally, we see that

Conclusion 2. If f is an $(\lambda_1, c_1, \lambda_2, c_2)$ -quasi-isometric embedding, we set $D(f) = D((\lambda_1, c_1), (1, 0)) + D((\lambda_2, c_2), (1, 0))$. Then $D(f)$ is a function of $D_0 = \lambda_1 + c_1^2/\lambda_1 + 1/\lambda_1 + \lambda_2 + c_2^2/\lambda_2 + 1/\lambda_2$.

3.7. Statement of quantitative problem. We finally come up with a precise notion of quasi-isometric distortion growth.

Definition 4. Let (X, x_0) be a space with a base point, Y be another space. Then we call *quasi-isometric distortion growth* the function

$$D_G(X, x_0, Y)(R) = \inf \{ D | \exists f : B_{x_0}^X(R) \rightarrow Y \text{ is a quasi-isometric embedding, } D(\lambda, c) \leq D_0 \},$$

where $B_{x_0}^X(R)$ is a ball in X centred at x_0 of radius R .

We remind that $D_0(\lambda, c) = \lambda + c^2/\lambda + 1/\lambda$.

3.8. Example : maps to trees. In the following proposition we can take for example a hyperbolic plane as the space X .

Proposition 2. *Let X be a metric space. We suppose that for any points x, y and any positive real numbers R and $R' \leq R/2$ the set $B_x(R) \setminus B_y(R')$ is connected. Let Y be a tree, let $f : B_x(R) \rightarrow Y$ be a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding. Then $R \leq 8\lambda_2c_1 + 4c_2$.*

Proof. Let x_1, x_2 be two points of $B_x(R)$ with distance at least R between them. Denote $y_i = f(x_i)$ for $i = 1, 2$.

For any point y of a geodesic (y_1, y_2) there exists a point $z \in B_x(R)$ such that $d(f(z), y) \leq c_1$. It follows from the fact that the image of (x_1, x_2) is c_1 -connected by the definition of a quasi-isometric embedding (for any point $x \in B_x(R)$ $\text{Diam}(f(x)) \leq c_1$) and every c_1 -connected path between y_1 and y_2 includes the geodesic (y_1, y_2) in its c_1 -neighbourhood.

Now consider a chain of points $\{x_i\}$ connecting x_1, x_2 and such that $d(x_i, x_{i+1}) < c_1/\lambda_1$. Hence, in the image $d(f(x_i), f(x_{i+1})) < 2c_1$ and so there exists i such that $d(f(x_i), y) \leq 2c_1$. Notice that $Y \setminus B_y(2c_1)$ has several connected components and the distance between these components is at least $4c_1$.

Suppose that a point z is rather far from both x_1 and x_2 : $d(z, x_i) > 4\lambda_2c_1 + c_2, i = 1, 2$. In the set $B_x(R) \setminus B_z(4\lambda_2c_1 + c_2)$ we also find a c_1/λ_1 -chain. Hence, there exists a point $z' \notin B_z(4\lambda_2c_1 + c_2)$ of this path such that $d(f(z'), y) \leq 2c_1$. Hence, $d(f(z), f(z')) \leq 4c_1$ and by property of quasi-isometry $d(z, z') \leq 4\lambda_2c_1 + c_2$, so $z' \in B_z(4\lambda_2c_1 + c_2)$. What leads to the contradiction with our hypothesis. Hence, for any $y \in (y_1, y_2)$ there exists $z' \in B_{x_1}(4\lambda_2c_1 + c_2) \cup B_{x_2}(4\lambda_2c_1 + c_2)$ such that $d(f(z'), y) \leq 2c_1$.

Consider two points y', y'' with $d(y', y'') \leq c_2/\lambda_2$ such that $z' \in B_{x_1}(4\lambda_2c_1 + c_2)$ and $z'' \in B_{x_2}(4\lambda_2c_1 + c_2)$. So, $d(z', z'') \geq R - 8\lambda_2c_1 - 2c_2$ and $d(f(z'), f(z'')) \leq c_2/\lambda_2$. Hence $R - 8\lambda_2c_1 - 2c_2 \leq \lambda_2(c_2/\lambda_2) + c_2 = 2c_2$. So we get $R \leq 8\lambda_2c_1 + 4c_2$. \square

4. QUASI-ISOMETRIC CLASSIFICATION - SURVEY

One of the first appearances of quasi-isometries was the proof of the famous Mostow rigidity theorem. It is proved by showing that equivariant quasi-isometries are within bounded distance of isometries.

Theorem 7. (*G. Mostow [8]*) *Suppose that $n \geq 3$ and $\Gamma, \Gamma' \subset \text{Isom}(\mathbb{H}^n)$ are lattices and $\rho : \Gamma \rightarrow \Gamma'$ is an isomorphism. Then ρ is induced by an isometry, i.e. there exists an isometry $\alpha \in \text{Isom}(\mathbb{H}^n)$ such that $\alpha \circ \gamma = \rho(\gamma) \circ \alpha$ for all $\gamma \in \Gamma$.*

Mostow extended the previous theorem to all rank one symmetric spaces. In the course of the proof, he establishes the following fact.

Theorem 8. (*G. Mostow [8]*) *Let X and X' be two rank 1 symmetric Riemannian spaces of negative curvature. If X and X' are quasi-isometric then they are isometric.*

Mostow's theorem was followed by generalizations of P. Pansu [10] (case of rank one) and B. Kleiner and B. Leeb [12] (higher ranks) (see for example the lecture notes of C. Drutu and M. Kapovich [9] for a survey on quasi-isometric rigidity). These generalizations help to proceed in quasi-isometric classification of some important classes of metric spaces.

Theorem 9. (*B. Kleiner, B. Leeb [12]*) *For $1 \leq i \leq k, 1 \leq j \leq k'$ let each X_i, X'_j be either a nonflat irreducible symmetric space of noncompact type (in addition assume that X has rank 2) or an irreducible thick Euclidean Tits building with cocompact affine Weyl group (in addition assume that X has Moufang Tits boundary). Let $X = \mathbb{E}^n \times \prod_{i=1}^k X_i$ and $X' = \mathbb{E}^{n'} \times \prod_{j=1}^{k'} X'_j$ be metric products. Then there exists a bijection σ and homotheties $X_i \rightarrow X'_{\sigma(i)}$.*

The quasi-isometric classification of 3-manifolds is a hard and open problem, only partial results have been achieved yet. For example we do not know if the fundamental groups of all (closed) graph manifolds are quasi-isometric. At least, the following result reduces the problem to the case of non-positively curved manifolds.

Theorem 10. (*M. Kapovich, B. Leeb [13]*) *Let M be a Haken manifold of zero Euler characteristic (which is neither Nil nor Sol), equipped with a Riemannian metric. Then there exists a compact non-positively curved 3-manifold N with totally geodesic flat boundary and a bilipschitz homeomorphism between the universal covers of M and N which preserves the canonical decomposition. In particular, the fundamental groups $\pi_1(M)$ and $\pi_1(N)$ are quasi-isometric.*

Also a special case of Schwartz' theorem (with $n = 3$) gives some results for classification of 3-manifolds.

Theorem 11. (*R. Schwartz [14]*) *Let $G \neq \text{Isom}(\mathbb{H}^2)$ be a rank one Lie group. Suppose that Γ, Γ' are non-uniform lattices in G which are quasi-isometric to each other. Then there exists an isometry $g \in \text{Isom}(\mathbb{H}^n)$ such that the groups Γ' and $g\Gamma g^{-1}$ are commensurable.*

This theorem holds more generally for simple Lie groups of rank 1. For higher ranks we have Wortman's result.

Theorem 12. (*K. Wortman [15]*) *Let K be a global field and S a finite nonempty set of inequivalent valuations containing all of the Archimedean ones. Suppose G is a connected simple K -group of adjoint type that is placewise not rank one with respect to S . Let Λ be a finitely generated group, and assume there is a quasi-isometry $\phi : \Lambda \rightarrow G(\mathcal{O}_S)$. If G is K -isotropic and K is a number field, then there exists a finite index subgroup Λ_S of Λ and a homomorphism $\phi : \Lambda_S \rightarrow G(\mathcal{O}_S)$ with a finite kernel and finite co-image such that*

$$\sup_{\lambda \in \Lambda_S} d(\phi(\lambda), \psi(\lambda)) < \infty.$$

Wortman's theorem also covers non K -isotropic fields and function fields, but the result is not complete in this case.

Theorem 13. (*U. Hamenstädt [29]*) *Two negatively curved homogeneous spaces are quasi-isometric if and only if their isometry groups are cocompact subgroups of the same Lie group.*

A lot of results are obtained for solvable groups. For nilpotent groups we have the following theorems of P. Pansu and Y. Shalom.

Theorem 14. (*P. Pansu [10]*) *Let Γ and Γ' be two quasi-isometric finitely generated nilpotent groups. The associated graded Lie groups $gr(\Gamma \otimes \mathbb{R})$ and $gr(\Gamma' \otimes \mathbb{R})$ are isomorphic.*

Theorem 15. (*Y. Shalom [25]*) *Quasi-isometric finitely generated nilpotent groups have the same Betti numbers.*

The theorem of B. Farb and L. Mosher deals with solvable Baumslag-Solitar groups $BS(1, n)$ (n is an integer) which are given by the presentations

$$BS(1, n) = \langle a, b \mid aba^{-1} = b^n \rangle.$$

Theorem 16. (*B. Farb, L. Mosher [16]*) *Let $m, n \geq 2$ be two integers, then $BS(1, n)$ and $BS(1, m)$ are quasi-isometric if and only if they are commensurable. This holds if and only if there exist integers r, i, j such that $n = r^i$ and $m = r^j$.*

Further, A. Eskin, D. Fisher and K. Whyte proved the following theorems for solvable groups.

Theorem 17. (*A. Eskin, D. Fisher, K. Whyte*) *Let Γ be a finitely generated group quasi-isometric to Sol . Then Γ is virtually a lattice in Sol .*

They launched a program for analyzing quasi-isometries of Lie groups of the form $\mathbb{R}^m \rtimes_M \mathbb{R}_n$ whose completion is still in progress. Here is an instance of the expected results.

Theorem 18. *Suppose M, M' are diagonalisable matrices with no eigenvalues on the unit circle, and $G = \mathbb{R} \rtimes_M \mathbb{R}_n$, $G' = \mathbb{R} \rtimes_{M'} \mathbb{R}_n$. Then G and G' are quasi-isometric if and only if M' has the same absolute Jordan form as M^α for some $\alpha \in \mathbb{R}$.*

Parts and special cases of this theorem are proved in different articles of A. Eskin, D. Fisher, K. Whyte [17, 18, 19], T. Dymarz [20] and I. Peng [21, 22].

An alternate way of proving that two groups are not quasi-isometric is to show that certain algebraic features are quasi-isometry invariants. Results of that kind for solvable groups appear in Y. Shalom's paper [25].

Part 2. Morse Lemma

Hyperbolic metric spaces have recently appeared in discrete mathematics and computer science (see, e.g., [2]). The notion of δ -hyperbolicity turns out to be more appropriate than other previously used notions of approximation by trees (e.g., tree width). This motivates our search for optimal bounds for a cornerstone of hyperbolic group theory like the Morse lemma.

This part is devoted to the quantitative version of the Morse Lemma, its “anti”-variant and their applications.

In the published article [30], a quasi-isometric embedding was defined as

Definition 5. A map $f : E \rightarrow F$ between metric spaces is a *rough* (λ, c) -*quasi-isometric embedding* if for any two points x, y of E

$$\frac{1}{\lambda}|x - y|_E - c \leq |f(x) - f(y)|_F \leq \lambda|x - y|_E + c.$$

The difference is in the lower bound as the additive constant in it is c and not c/λ . We revised all proofs and examples using our new definition. All previously obtained results remain true. The main difference in our new proof of Morse Lemma appears in Lemma 10 on exponential contraction. It was previously stated only for Δ -connected curves, see Lemma 9 in [30]. Now we do not need to substitute a given quasi-geodesic with a continuous one any more.

5. BASICS OF HYPERBOLIC GEOMETRY

The contemporary research on hyperbolic groups and hyperbolic spaces was started in 1987 by M.Gromov in his paper [Gr].

5.1. Metric definition. In this text we will use following notations for distances between points and sets. Let E be a metric space with metric d . We write $|x - y|$ for the distance $d(x, y)$ between two points x and y of the space E . For a subset A of E and a point x , $d(x, A)$ denotes the distance from x to A .

Definition 6. Let X be a metric space and x, y, z be three points in X . The Gromov product $(x|y)_z$ of x and y at z is

$$(x|y)_z = \frac{1}{2} (|x - z| + |y - z| - |x - y|).$$

To explain the geometrical meaning of this definition, we introduce *tripods* which are presented as three points in a metric tree with the branches connecting these points (it is possible that the lengths of some edges is 0).

Proposition 3. *Let x, y, z be three points in some metric space X . Then there exists a tripod T and an isometry $f : x, y, z \rightarrow T$ such that $f(x), f(y)$ and $f(z)$ are the endpoints of the tripod T . Moreover, the lengths of the branches of T are exactly equals to corresponding Gromov’s product.*

The proof is evident, verify it directly by the definition of Gromov's product. Now we are ready to give the definition of δ -hyperbolic spaces.

Definition 7. A metric space X is called δ -hyperbolic if for any four points x, y, z, w the inequality

$$(x|z)_w \geq \min(x, y)_w, (y, z)_w - \delta$$

holds.

This definition can be rewritten in another form. There are three ways to divide these four points into pairs. Introduce the corresponding sums of distances

$$\begin{aligned} p &= |x - w| + |z - y| \\ m &= |x - y| + |z - w| \\ g &= |x - z| + |y - w|. \end{aligned}$$

Redenote the points to have that $p \leq m \leq g$. Then the definition can be rewritten in the following form

$$g \leq m + 2\delta.$$

That is the greatest sum cannot exceed the mean sum by more than by 2δ .

5.2. Case of geodesic metric spaces.

Definition 8. A geodesic (geodesic segment, geodesic ray) σ in a metric space E is an isometric embedding of a real line (real interval I , real half-line \mathbb{R}_+) in E .

We write xy for a geodesic segment between two points x and y (in general, there could exist several geodesic paths between two points; we assume any one of them by this notation). A geodesic triangle xyz is a union of three geodesic segments xy , yz , and xz .

If δ -hyperbolic space X is geodesic we can use one more equivalent definition of δ -hyperbolicity in terms of "thin triangles". For two given points x, y we will denote by xy a geodesic segment between them. In general such a geodesic segment is not necessarily unique so under this notation we assume one of these geodesic segments.

A geodesic metric space is a space such that there exists a geodesic segment xy between any two points x and y . Geodesic δ -hyperbolic spaces can be described in terms of thin triangles.

Definition 9. A geodesic triangle xyz is called δ -thin if the distance from any point p of xy to the union of xz and yz does not exceed δ :

$$d(p, xz \cup yz) \leq \delta.$$

Proposition 4. A geodesic metric space E is δ -hyperbolic if and only if every geodesic triangle is $\frac{1}{2}\delta$ -thin.

According to M. Bonk and O. Schramm [6], every δ -hyperbolic metric space embeds isometrically into a complete δ -hyperbolic geodesic metric space. So, many theorems can be reduced to the investigation of geodesic hyperbolic spaces using the definition of hyperbolicity in terms of δ -thin triangles. Usually the factor $\frac{1}{2}$ is dropped in the last definition.

- Example 4.*
- One of the most important examples of δ -hyperbolic spaces are metric trees, here $\delta = 0$.
 - Fundamental groups of compact Riemannian manifolds with negative (sectional) curvature are δ -hyperbolic.

Take some group G and a presentation $P = \langle X, R \rangle$ of G . Introduce a word metric on G . That is the length of any element g is the minimal length of a word (of generators) which is needed to write g in P . It is easy to check that it is indeed a metric. The next theorem shows that in some sense most finite presentations are hyperbolic.

Theorem 19 (Gromov). *Fix integers p and q . Consider all presentations P with p generators ($|X| = p$) and q relators ($|R| = q$). Denote by $N_{hyp}(n_1, \dots, n_q)$ the number of all hyperbolic presentations with the lengths of relators equal to n_1, \dots, n_q , by $N(n_1, \dots, n_q)$ the number of all presentations with the same property. Then*

$$\frac{N_{hyp}(n_1, \dots, n_q)}{N(n_1, \dots, n_q)} \rightarrow 1$$

as $n_i \rightarrow \infty$ ($i = 1, \dots, q$).

5.3. Divergence. Now we are going to introduce the notion of *divergence function* which allows us to estimate lengths of paths which leave a ball together with two diverging geodesics. Later this approach will help us show that the length of a curve lying far from a geodesic is very large.

Definition 10. Let F be a metric space. We say that $e : \mathbb{N} \rightarrow \mathbb{R}$ is a *divergence function* for the space F if any point $x \in F$ and any two geodesic segments $\gamma = (x, y)$ and $\gamma' = (x, z)$ it holds: for any $R, r \in \mathbb{N}$ such that $R+r$ does not exceed lengths of γ, γ' if $d(\gamma(R), \gamma'(R)) > e(0)$ and σ is a path from $\gamma(R+r)$ to $\gamma'(R+r)$ in the closure of the complement of a ball $B_{R+r}(x)$ (that is in $\overline{X \setminus B_{R+r}(x)}$) then the length of σ is at greater than $e(r)$.

While two points move along two geodesic rays, the distance between them grows linearly by the triangle inequality which is true in all metric spaces. Though we will see that if two such geodesics leave some bounded tube then the lengths of paths connecting two points on them and lying in the complement of the ball grow exponentially in any hyperbolic space (for example the length of a circle grows exponentially with the radius). If e is an exponential function then we say that geodesics diverge exponentially.

Theorem 20. *In a hyperbolic space geodesics diverge exponentially.*

An amazing fact is that the opposite statement is also true and even more: a non-linear divergence in a geodesic space implies that the divergence function is exponential and, finally, that the space is hyperbolic. Though here we are not going to prove this result.

5.4. Isoperimetry. An other important property, characterizing hyperbolic spaces, is that the isoperimetric inequality is linear for them,

$$Area(D) \leq A_{ip}l,$$

where l is the length of a closed curve filled by an optimal disk D and A_{ip} is some constant depending on a particular space. Let us explain how isoperimetric inequalities can be generalized to the case of groups.

Let G be a finitely generated group, $P = \langle X | R \rangle$ a finite presentation of G . Closed curves in the Cayley polyhedron correspond to words $w \in F(X)$ representing the unity of G and, hence, they can be expressed in $F(X)$ in the form

$$w = (u_1^{-1} r_1^{\alpha_1} u_1) \dots (u_n^{-1} r_n^{\alpha_n} u_n)$$

where $u_1 \in F(X)$, $r_i \in R$, $\alpha_i \in \{-1, 1\}$. Of course, in general there exists infinitely many of such decompositions.

Definition 11. The least value of n is called the *area* of w .

Definition 12. The Dehn (or isoperimetric) function $f : \mathbb{N} \rightarrow \mathbb{N}$ is

$$f(l) = \max \{Area(w) | w = 1, |w| = l\}.$$

Any finitely generated group has different presentations with different Dehn functions. The following lemma helps us to establish the relation between them.

Lemma 1. *Let G be a group and let P and Q be two finite presentations of G with Dehn functions f and g respectively. Then there exist constants $a_1, a_2, a_3, a_4 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$*

$$f(n) \leq a_1 g(a_2 n + a_3) + a_4.$$

Now we can conclude that if for some presentation, Dehn function is bounded by a linear (polynomial, exponential etc) function, then for any presentation of that group Dehn function is also bounded by a function of the same type. Moreover, the type of isoperimetric inequalities is invariant under quasi-isometries, see Definition 1.

5.5. Comparison with trees. A metric tree is one of the most important examples of hyperbolic spaces. Most properties of hyperbolic spaces can be illustrated in trees and theorems in this subject should be first verified for them. The following theorem establishes a close relation between general hyperbolic spaces and trees. It says that if we are looking from far away, then a hyperbolic space looks similar to a tree. We will write $|x|$ for the distance from x to the base point.

Theorem 21. *Let X be a δ -hyperbolic metric space with a base point w and k be a positive integer. If $|X| \leq 2^k + 2$ then there exist a finite metric tree with a base point t and a map $\Phi : X \rightarrow T$ such that*

- (1) Φ preserves distances to the base point,

$$|\Phi(x) - t| = |x|$$

for any point x of X .

- (2) $|y - x| - 2k\delta \leq |\Phi(y) - \Phi(x)| \leq |y - x|$ for any two points x, y of X .

6. THE GEOMETRY OF δ -HYPERBOLIC SPACES

In this section we will give some lemmas on geometry of triangles, perpendiculars and projections in δ -hyperbolic metric spaces.

Definition 13. In a metric space, a *perpendicular* from a point to a curve (in particular, a geodesic) is a shortest path from this point to the curve.

Of course, a perpendicular is not necessarily unique.

Lemma 2. *In a geodesic δ -hyperbolic space, let b be a point and σ be a geodesic such that $d(b, \sigma) = R$. Let ba be a perpendicular from b to σ , where $a \in \sigma$. Let c be a point of σ such that $|b - c| = R + 2\Delta$. Then $|a - c| \leq 2\Delta + 4\delta$.*

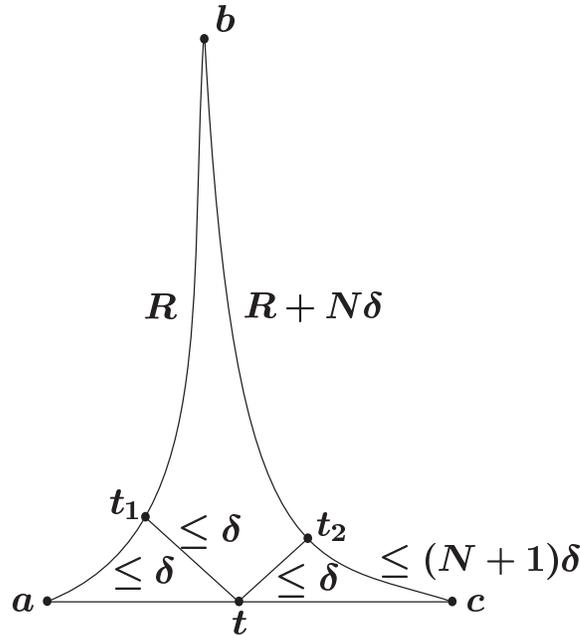


FIGURE 1. Illustration for Lemma 2.

Proof. The triangle abc (see Fig. 1) is δ -thin by the definition of a δ -hyperbolic space. Hence, there exists a point $t \in \sigma$ such that $d(t, ba) \leq \delta$ and $d(a, bc) \leq \delta$. Let t_1 and t_2 be the respective projections of t on ba and bc . By hypothesis, R is the minimum distance from b to the points of σ . Therefore, $R = |b - a| \leq |b - t_1| + |t_1 - t| \leq |b - t_1| + \delta$ and $R \leq |b - t_2| + |t_2 - t| \leq |b - t_2| + \delta$. Hence, $|a - t_1| \leq \delta$ and $|c - t_2| \leq 2\Delta + \delta$. By the triangle inequality, we obtain $|a - c| \leq |a - t_1| + |t_1 - t| + |t - t_2| + |t_2 - c| \leq 2\Delta + 4\delta$. \square

Remark 2. In particular, all the orthogonal projections of a point to a geodesic lie in a segment of length 4δ .

Lemma 3. *In a δ -hyperbolic space, let two points b and d be such that $|b - d| = \Delta$. Let σ be a geodesic and a and c be the respective orthogonal projections of b and d on σ . Let $|a - b| > 3\Delta + 6\delta$, and let $d(d, \sigma) > d(b, \sigma)$. Let two points $x_1 \in ab$ and $x_4 \in cd$ be such that $2\Delta + 5\delta < d(x_1, \sigma) = d(x_4, \sigma) < |a - b| - (\Delta + 2\delta)$. Then $|x_1 - x_4| \leq 4\delta$ and $|a - c| \leq 8\delta$.*

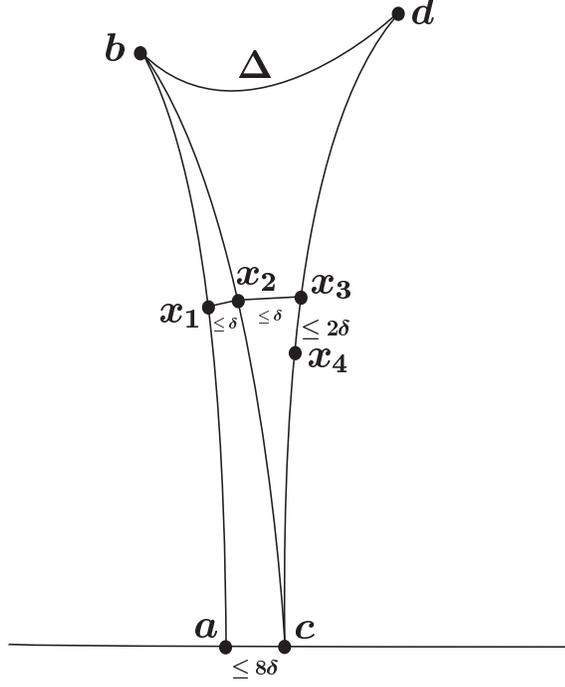


FIGURE 2. Illustration for Lemma 3.

Proof. (See Fig. 2.) By the triangle inequality and because cd is a perpendicular to σ , $|c - d| \leq |a - b| + |b - d|$, whence $|b - c| \leq |c - d| + |b - d| \leq |a - b| + 2|b - d|$. By Lemma 2, $|a - c| \leq 2\Delta + 4\delta$. The triangle abc is δ -thin, $|a - x_1| > |a - c| + \delta$. Therefore, by the triangle inequality, $d(x_1, ac) > \delta$, and hence $d(x_1, bc) \leq \delta$. Let x_2 denote the point of bc nearest x_1 . Because the triangle bcd is also δ -thin and $|b - x_2| \geq |b - x_1| - |x_1 - x_2| \geq \Delta + \delta$, there exists a point $x_3 \in cd$ such that $|x_3 - x_2| \leq \delta$. It follows from the triangle cx_1x_3 that $|x_3 - c| \geq |x_1 - c| - 2\delta \geq |x_1 - a| - 2\delta$. On the other hand, because x_5c is a perpendicular to σ , $|x_3 - c| \leq |x_3 - x_1| + |x_1 - a|$. Now, $|a - x_1| = |c - x_4|$, and hence $|x_4 - x_3| \leq 2\delta$. Finally, we obtain the statement in the lemma: $|x_1 - x_4| \leq 4\delta$.

By the triangle inequality and because $d(x_1, \sigma) = d(x_4, \sigma)$, we have $|x_1 - c| \leq |c - x_4| + |x_4 - x_1| \leq |a - x_1| + 4\delta$. Hence, using Lemma 2, we conclude that $|a - c| \leq 8\delta$. \square

Lemma 4. *Let σ be a geodesic segment, a be a point not on σ , and c be a projection of a on σ . Let $b \in \sigma$ be arbitrary, and let d denote the projection of b on ac . Then the $|c - d| \leq 2\delta$.*

Proof. By hypothesis, bd minimizes the distance from any its points to ac , and because the triangle bcd is δ -thin, there exists a point $e \in bd$ such that $d(e, ac) = |e - d| \leq \delta$ and $d(e, bc) \leq \delta$. Because ac is a perpendicular to σ , $|a - c| \leq |a - d| + |d - e| + d(e, bc) \leq |a - d| + 2\delta$. Hence $|c - d| \leq 2\delta$. \square

Lemma 5. *As in the preceding lemma, let σ be a geodesic segment, a be a point not on σ , c be a projection of a on σ , and b be some point on σ . Let d denote a point on ac such that $|d - c| = \delta$ and e denote a point on bc such that $|e - c| = 3\delta$. Then*

- $d(d, ab) \leq \delta$, $d(e, ab) \leq \delta$, $d(c, ab) \leq 2\delta$, and
- the length of ab differs from the sum of the lengths of the two other sides by at most 8δ ,

$$|a - c| + |b - c| - 2\delta \leq |a - b| \leq |a - c| + |b - c| + 8\delta.$$

Proof. The triangle abc is δ -thin. Therefore, obviously, $d(d, ab) \leq \delta$ (the distance from a point of ac to ab is a continuous function). We take a point $x \in bc$ such that $d(x, ca) \leq \delta$. Using Lemma 4, we obtain $|b - x| + d(x, ca) \geq |b - c| - 2\delta$, and hence $|c - x| \leq d(x, ca) + 2\delta \leq 3\delta$.

We now let d_1 and e_1 denote the respective projections of d and e on ab . Then by the triangle inequality, we have

- $|a - d| - \delta \leq |a - d_1| \leq |a - d| + \delta$,
- $|b - e| - \delta \leq |b - e_1| \leq |b - e| + \delta$, and
- $0 \leq |d_1 - e_1| \leq |d_1 - d| + |d - c| + |c - e| + |e - e_1| \leq 6\delta$.

Combining all these inequalities, we obtain the second point in the lemma. \square

Lemma 6. *Let σ be a geodesic and a and b be two points not on σ . Further, let a and b have a common projection c on σ . Let d be a point of σ and c_1 be the projection of d on ab . Then*

$$|d - c| \leq |d - c_1| + 6\delta.$$

Remark 3. Lemma 6 deals with a geodesic segment. The statement is not true for a complete geodesic passing through a and b , as can be seen from Fig. 3.

Proof. We take a point $e \in bc$ such that $|c - e| = \delta$ and consider the triangle bcd (see Fig. 4). Because bc is a perpendicular to dc , $d(e, bd) \leq \delta$. Let e_1 denote a projection of e on bd . Let e_2 and e_3 be the respective projections of e_1 on the geodesic segments dc_1 and bc_1 . Because the triangle dbc_1 is δ -thin, either $|e_1 - e_2| \leq \delta$ or $|e_1 - e_3| \leq \delta$.

I. If $|e_1 - e_2| \leq \delta$, then $|d - c| \leq |c - e| + |e - e_1| + |e_1 - e_2| + |e_2 - d| \leq |d - c_1| + 3\delta$.

II. If $|e_1 - e_2| > \delta$, then the length of the path cee_3 is at most 3δ . We apply the same arguments to ad (we assume that this is possible; otherwise, we could apply the first case to it). We obtain the points g , g_1 , and g_3 and the length of the path cgg_3 is also at most 3δ . If neither of these paths intersects cc_1 , then its length does not exceed 6δ (which follows from consideration of the triangle ce_3g_3). \square

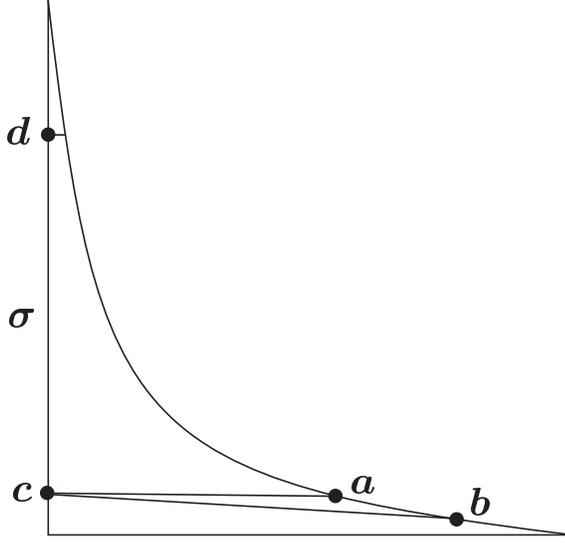


FIGURE 3. Illustration for Remark 3.

Lemma 7. *Let E be a δ -hyperbolic metric space and abc be a triangle in E . Then the diameter of the set S of points of the side ab such that distance to bc and ac does not exceed $2d$ is not greater than $C(d + \delta)$, where C is a constant.*

Proof. Let x be a point of ab such that $d(x, bc) \leq \delta$ and $d(x, ac) \leq \delta$ and y be a point of ab such that $d(y, bc) \leq d$ and $d(y, ac) < d$. Without loss of generality, we assume that $y \in (a, x)$. Because the triangle abc is δ -thin, one of these two distances does not exceed δ .

We first assume that $d(y, ac) \leq \delta$. Let x' and y' be points of ac such that $d(x, x') \leq \delta$ and $d(y, y') \leq \delta$. We let $t, t', s,$ and s' denote the respective projections of $x, x', y,$ and y' on bc . Because $x't'$ is a perpendicular to bc , $|x' - t'| \leq |x' - x| + |x - t| \leq 2\delta$, and hence $|t - t'| \leq 4\delta$. If y and y' are sufficiently far from bc , i.e., if $d \geq 9\delta$, then $|s - s'| \leq 6\delta$ by Lemma 3. Otherwise, we can give a rough estimate by the triangle inequality: $|s - s'| \leq |s - y| + |y - y'| + |y' - s'| \leq 19\delta$. Hence, in any case, $|s - s'| \leq 19\delta$. We consider two cases.

If s is in the segment $[b, t']$, then by applying the triangle inequality several times, we obtain

$$|b - y| \leq |b - s| + |s - y| \leq |b - t'| + |s - y| \leq |b - x| + |x - t| + |t - t'| + |s - y| \leq |b - x| + 5\delta + d.$$

And because $|b - y| = |b - x| + |x - y|$, we have $|x - y| \leq 5\delta + d$.

The same arguments we apply if $s \in [t', c]$. We merely note that we can replace y with y' and t with t' with respective errors less than δ and 19δ :

$$|c - y'| \leq |c - s'| + |s' - y'| \leq |c - s'| + |s' - y'| \leq |c - s| + 19\delta + |s - y| + \delta \leq |c - t'| + 20\delta + d.$$

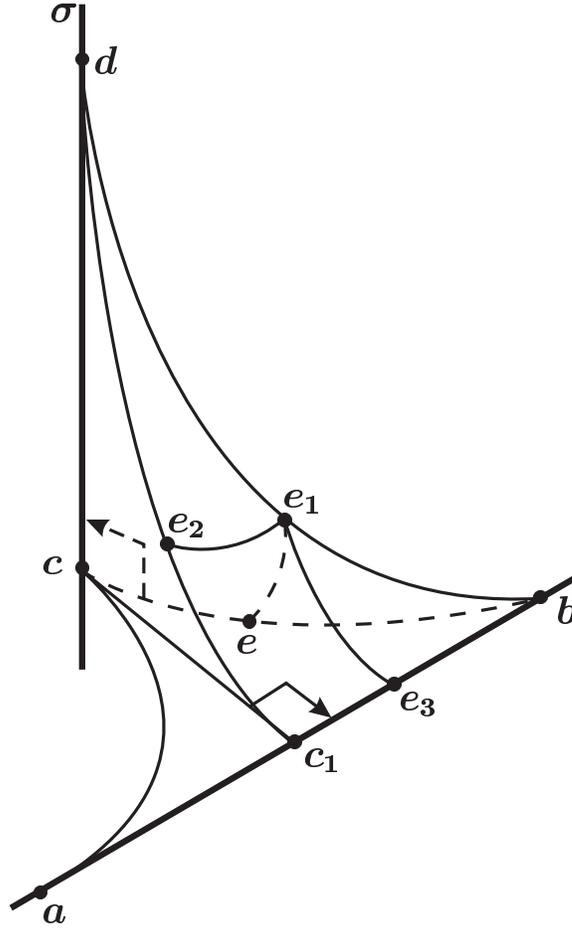


FIGURE 4. Illustration for Lemma 6.

Now, because $|c - t'| \leq |c - x'| + |x' - t'| \leq |c - x'| + 2\delta$, we have

$$|c - x'| + |x' - y'| = |c - y'| \leq |c - x'| + 22\delta + d.$$

Finally, $|x - y| \leq |y - y'| + |y' - x'| + |x - x'| \leq 24\delta + d$.

The case $d(y, bc) \leq \delta$ is treated identically with d and δ interchanged. □

7. QUASI-GEODESICS AND Δ -LENGTH

Definition 14. A $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic in F is a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding (in the sense of Definition 3) of a real interval $I = [0, l]$ into F .

Let $\gamma : I \rightarrow F$ be a curve. We assume that the interval $I = [x_0, x_n]$ of length $|I| = l$ gives the parametrization of the quasi-geodesic γ . We take a subdivision $T_n = (x_0, x_1, \dots, x_n)$ and let $y_i, i = 0, 1, \dots, n$, denote $\gamma(x_i)$. The *anti-mesh* of T_n is $d(T_n) = \min_{0 < i \leq n} |y_i - y_{i-1}|$.

Definition 15 (Δ -length). Let $\gamma : I \rightarrow F$ be a curve. The value

$$L_\Delta(\gamma) = \sup_{T_n: d(T_n) \geq \Delta} \sum_{i=1}^n |y_i - y_{i-1}|$$

is called the Δ -length of the quasi-geodesic γ .

We note that the values of Δ -length and classical length are the same for a geodesic.

Lemma 8. Let $\gamma : I \rightarrow F$ be a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic. For $\Delta \geq 2c_1$,

$$L_\Delta(\gamma) \leq 2\lambda_1 l.$$

Proof. By the definition of Δ -length, $\Delta \leq |y_i - y_{i-1}| \leq \lambda_1 |x_i - x_{i-1}| + c_1$. Hence, because $\Delta \geq 2c_1$, we obtain $|x_i - x_{i-1}| \geq (\Delta - c_1)/\lambda_1 \geq c_1/\lambda_1$.

Now, by definition of a quasi-geodesic, we have

$$\sup_{T_n} \sum_i |y_i - y_{i-1}| \leq \sup_{T_n} \sum_i (\lambda_1 |x_i - x_{i-1}| + c_1) \leq \sup_{T_n} \sum_i 2\lambda_1 |x_i - x_{i-1}| = 2\lambda_1 l,$$

where the last equality follows because the sum of $|x_i - x_{i-1}|$ for every subdivision of the interval I is exactly equal to the length of I . \square

Lemma 9. Let $\gamma : I \rightarrow F$ be a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic. Let $R \geq c_2/\lambda_2$ be the distance between the endpoints of γ , and let $\Delta \geq 2c_1$. Then $L_\Delta(\gamma) \leq 4\lambda_1 \lambda_2 R$.

Proof. By definition of a quasi-isometry, $(l - c_2)/\lambda_2 \leq R \leq \lambda_1 l + c_1$. Hence, $l \leq \lambda_2 R + c_2$. And by Lemma 8, $L_\Delta(\gamma) \leq 2(\lambda_2 R + c_2)\lambda_1$. In particular, $L_\Delta(\gamma) \leq 4\lambda_1 \lambda_2 R$ for $R \geq c_2/\lambda_2$. \square

8. EXPONENTIAL CONTRACTION

Lemma 10 (Exponential contraction). Let $\Delta > 0$. In a geodesic δ -hyperbolic space E , let γ be a $\Delta/2$ -connected curve at a distance not less than $R \geq 6\Delta + 116\delta$ from a geodesic σ . Let L_Δ be the Δ -length of γ . Then the length of the projection of γ on σ is not greater than

$$\max\left(\frac{4\delta}{\Delta} e^{-KR/38\delta} L_\Delta, 8\delta\right),$$

where $K = \ln 2/19$.

Proof. Let y_0, y_1, \dots, y_n be points on γ such that $\Delta \leq |y_i - y_{i-1}| \leq 2\Delta$ for $i = 1, 2, \dots, n$ and y_0 and y_n are the endpoints of γ . Let y_k be the point of this set that is nearest from σ . We take a perpendicular from y_k to σ and a point x_k on it with $|y_k - x_k| = 2\Delta + 3\delta$. Now, on the perpendiculars from all other points y_i , we take points x_i such that $d(x_i, \sigma) = d(x_k, \sigma)$ (see Fig. 5). By Lemma 3, $|x_i - x_{i-1}| \leq 4\delta$ for $i = 1, 2, \dots, n$. Therefore,

$$\sum_{i=1}^n |x_i - x_{i-1}| \leq n4\delta \leq n\Delta \frac{4\delta}{\Delta} \leq \frac{4\delta}{\Delta} L_\Delta.$$

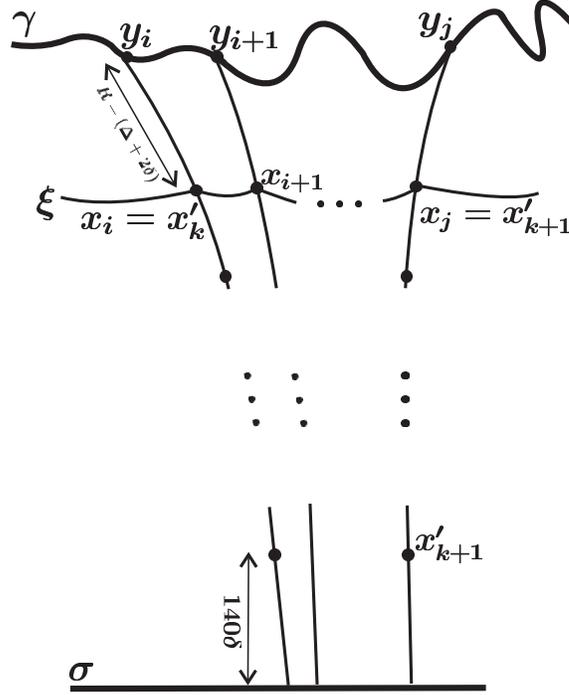


FIGURE 5. Exponential contraction of the length of a curve γ under projection on a geodesic σ .

We set $\bar{x}_0 = x_0$ and $\bar{x}_{n^1} = x_n$ and select points $\bar{x}_i \in \{x_1, x_2, \dots, x_{n-1}\}$ such that $8\delta \leq |\bar{x}_i - \bar{x}_{i-1}| \leq 16\delta$. For each $i = 0, 1, \dots, n^1$, we choose a perpendicular from \bar{x}_i to σ , move \bar{x}_i along it a distance $16\delta + 3\delta = 19\delta$ towards σ , and obtain x_i^1 . By Lemma 3, $|x_i^1 - x_{i-1}^1| \leq 4\delta$ and

$$\sum_{i=1}^{n^1} |x_i^1 - x_{i-1}^1| \leq n^1 4\delta \leq \frac{1}{2} \sum_{i=1}^{n^1} |\bar{x}_i - \bar{x}_{i-1}| \leq \frac{1}{2} \sum_{i=1}^n |x_i - x_{i-1}| \leq \frac{1}{2} \frac{4\delta}{\Delta} L_\Delta.$$

We can continue such a process while the distance from the set of points $\{x_i^m, i = 0, 1, \dots, n^m\}$ to σ is not less than 19δ and $|x_0^m - x_{n^m}^m| \geq 8\delta$. After k steps, we have

$$\sum_{i=1}^{n^k} |x_i^k - x_{i-1}^k| \leq \frac{1}{2^k} \frac{4\delta}{\Delta} L_\Delta = \frac{4\delta}{\Delta} e^{-((\ln 2)/19\delta)(19\delta k)} L_\Delta.$$

We set $r = 19\delta k$ and $K = (\ln 2)/19$. We need $8\delta \leq (4\delta/\Delta)e^{-Kr/\delta} L_\Delta$ and hence $r \leq (\delta/K) \ln(L_\Delta/2\Delta)$. Now, if the distance between the projections of the endpoints $|x_0^m - x_{n^m}^m|$ is not less than 8δ at some step m , then we use Lemma 3 to do the last projection on σ , and its length does not exceed 8δ . Otherwise, we must do the last descent to the distance 55δ using Lemma 3 (the estimate for the projection on a geodesic with $\Delta = 16\delta$

gives the necessary distance from the set of points to the geodesic to be greater than $3 \times 16\delta + 6\delta = 54\delta$) and intervals of a length not less than 8δ contract to intervals of a length not more than δ , and we hence have a contraction factor of unity at the last step.

We need just to notice now that our choice of R allows us to conclude that $R - (2\Delta + 3\delta) + 55\delta \geq R/2$, so the number of iterations $k \geq R/38\delta$. \square

9. QUANTITATIVE VERSION OF THE MORSE LEMMA

We are now ready to state and prove the quantitative version of the Morse lemma. In a δ -hyperbolic space E , any $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic γ belongs to the H -neighborhood of any geodesic σ connecting its endpoints, where the constant H depends only on the space E (in particular, on the constant δ) and the quasi-isometry constants $\lambda_1, \lambda_2, c_1$ and c_2 .

9.1. Attempts. To motivate our method, we describe a sequence of arguments yielding sharper and sharper estimates. Here, for simplicity, we will assume that $\lambda_1 = \lambda_2 = \lambda$ and additive constants are small relatively to λ . We start with the proof in [1], Chapter 5.1, Theorem 6 and Lemma 8, where the upper bound $H \leq \lambda^8 c^2 \delta$ was obtained (up to universal constants, factors of the order $\log_2(\lambda c \delta)$). The first weak step in this proof is replacing a (λ, c) -quasi-geodesic with a discrete (λ', c) -quasi-geodesic γ' parameterized by an interval $[1, 2, \dots, l]$ of integers, where $\lambda' \sim \lambda^2 c$. For a suitable $R \sim \lambda'^2$, we take an arc $x_u x_v$ of γ' and introduce a partition of that arc $x_u, x_{u+N}, x_{u+2N}, \dots, x_v$ for some well-chosen $N \sim \lambda'$. The approximation of a δ -hyperbolic space by a tree (see [1], Chapter 2.2, Theorem 12.ii) is used to obtain an estimate of the form $|y_{u+iN} - y'_{u+(i+1)N}| \leq c' \sim \ln \lambda'$. By the triangle inequality, $|x_u - x_v| \leq |x_u - y_u| + |y_u - y_{u+N}| + \dots + |y_v - x_v| \leq 2(R + \lambda') + (N^{-1}|u - v| + 1)c'$. On the other hand, $\lambda'^{-1}|u - v| \leq |x_u - x_v|$. Combining these two inequalities, we obtain an estimate for $|u - v|$ and hence for the distance from any point of the arc $x_u x_v$ to the point x_u . The second weak step in this argument is in the estimate of the length of projections, which can be improved significantly.

Another proof was given in [7]. It allows to obtain the estimate $\lambda^2 H_{\text{am}}$, where H_{am} is the constant of the anti-Morse lemma (see Section 11) and is given by the equation $H_{\text{am}} \simeq \ln \lambda + \ln H_{\text{am}}$.¹ It is very close to an optimal upper bound but still not sharp as the sharp estimate for $H_{\text{am}} \simeq \ln \lambda$. The proof uses the fact that in a hyperbolic space the divergence function is exponential.

To prove the anti-Morse lemma, the authors of [7] take a point p of the geodesic σ that is the distant from the quasi-geodesic γ and construct a path α between two points of γ such that α is in the complement of the ball of radius $d(p, \gamma)$ with the center p . Finally, they compare two estimates of the length: one estimate follows from the hypothesis that α is a quasi-geodesic, and the other is given by the exponential geodesic divergence. To prove the Morse lemma, they take a (connected) part γ_1 of γ that belongs to the complement of the H_{am} -neighborhood of the geodesic σ , and they show that the length of γ_1 does not exceed

¹Be careful while reading [7] because a slightly different definition of quasi-geodesics is used there with $\lambda_1 = \lambda^2$; cf. Lemma 9.

$2\lambda^2 H_{\text{am}}$ by the definition of a quasi-geodesic. In [7], they also use another definition of a quasi-geodesic, which is less general than our definition because, in particular, it assumes that a quasi-geodesic is a continuous curve. Consequently, some technical work is needed to generalize their results.

To improve these bounds, we use Lemma 10 (exponential contraction) instead of exponential geodesic convergence and Lemma 9, which do not require discretization as in [1] and provide a much more precise estimate for a length of a projection. We can then take $R = \ln \lambda$ and obtain $H \leq O(\lambda^2 \ln \lambda)$ by a similar triangle inequality.

Below, we prove the Morse and anti-Morse lemmas independently. We only mention that arguments in [7] can be used to deduce the optimal bound for the Morse lemma from the anti-Morse lemma. We can also obtain an optimal upper bound for H from Lemma 11.

We now sketch the proof of a stronger result (but still not optimal): $H \leq O(\lambda^2 \ln^* \lambda)$, where $\ln^* \lambda$ is the minimal number n of logarithms such that $\underbrace{\ln \dots \ln}_n \lambda \leq 1$.

The preceding argument is used as the initial step. It allows assuming that the endpoints x and x' of γ satisfy $|x - x'| \leq O(\ln \lambda)$. Then comes an iterative step. We prove that if xx' is an arc on γ and $|x - x'| = d_1$, then there exist two points y and y' at distance at most $C_2(c, \delta)\lambda^2$ from a geodesic σ_1 connecting x and x' such that $d_2 := |y - y'| \leq C_3(c, \delta) \ln d_1$. Indeed, we choose a point z of the arc xx' that is farthest from σ_1 and let σ' denote a perpendicular from z to σ_1 . If all points of the arc xx' (on either side of z) whose projection on σ' is at a distance $\leq \lambda^2$ from σ_1 are at a distance not less than $\ln d_1$ from σ' , then Lemma 10 implies that the length of the arc is much greater than $\lambda^2 \ln d_1$, contradicting the quasi-geodesic assumption. Hence, there are points y and y' that are near σ' . We can arrange that their projections on σ' are near each other, which yields $|y - y'| \leq \ln d_1$. We apply this relation several times starting with $d_1 = C_1(c, \delta) \ln \lambda$ until $d_i \leq 1$ for some $i = \ln^* \lambda$.

In summary, we use two key ideas to improve the upper bound of H : exponential contraction and consideration of a projection of γ on a different geodesic σ' .

9.2. Proof of the Morse lemma. We use the same ideas to prove the quantitative version of the Morse lemma, but we should do it more accurately.

Remark 4. In Section 10 we will give examples (properly parametrized and discretized rays in a tree) where $H = \lambda_1 \lambda_2 \min\{c_1, c_2\}/4$.

Proof of Theorem 2. First, we notice that a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic γ is a c_1 -connected curve. We will use Lemma 9 and Lemma 10 to get control on the Δ -length of γ with $\Delta = 2 \max\{c_1, \delta\}$.

We introduce the following construction for subdividing the quasi-geodesic γ . We let z denote the point of our quasi-geodesic that is farthest from σ . Let $\sigma_0 = \sigma$ be the geodesic connecting the endpoints of γ . Let σ'_0 be the geodesic minimizing the distance between z and σ_0 (because σ_0 is a geodesic segment, σ'_0 is not necessarily perpendicular to the complete geodesic carrying σ_0). Let s_0 denote the point of intersection of σ_0 and σ'_0 . Let s'_0 be the point of σ'_0 such that the length of the segment $[s_0, s'_0]$ is equal to 9δ . We consider

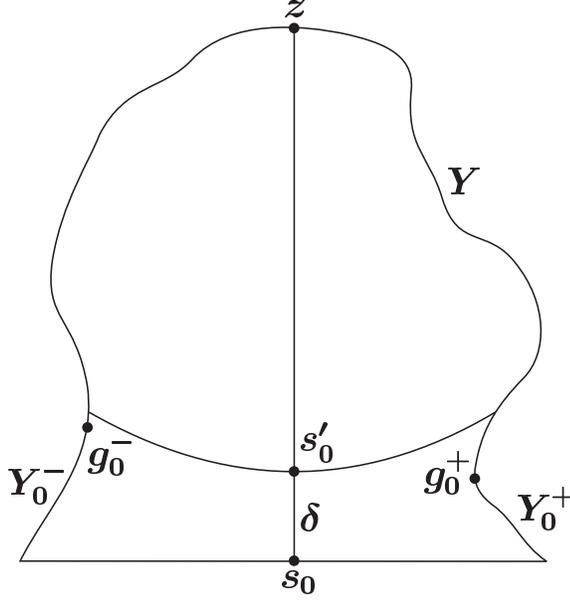


FIGURE 6. Illustration of proof of Theorem 2

the set of points of γ whose projections on σ'_0 belong to the segment $[s_0, s'_0]$. The point z separates this set into two subsets γ_0^+ and γ_0^- (see Fig. 6).

Let d_0^\pm denote the minimal distance of points of γ_0^\pm to σ'_0 . We also introduce the following notation:

- $d_0 = d_0^+ + d_0^- + \delta$;
- γ_1 is a c_1 -connected component of $\gamma \setminus (\gamma_0^+ \cup \gamma_0^-)$ containing z and is also a quasi-geodesic with the same constants and properties as γ ;
- σ_1 is a geodesic connecting the endpoints of the sub-quasi-geodesic γ_1 ;
- L_1 is the Δ -length of γ_1 .

Applying the same idea to the curve γ_1 , the same point z , and the geodesic σ_1 , we obtain the geodesic σ'_1 , the parts γ_1^\pm of the quasi-geodesic, and the distances d_1^\pm . We have $l(\sigma'_0) \leq l(\sigma'_1) + \delta + 6\delta$. To show this, we apply Lemma 6 assuming that $c = s'_0$, $d = z$, and a and b are the endpoints of γ_1 . Continuing the process, we obtain a subdivision of γ by γ_i^\pm and two families of geodesics σ_i and σ'_i . Finally, for some n , we obtain $d_n \leq c_2 + 6\Delta + 161\delta$ (the choice of such a bound will allow us to apply Lemma 10 on exponential contraction for all $i < n$).

The quantity L_i is the length of the subcurve γ_{i-1} , which is also a quasi-geodesic. Hence, $l(\sigma'_n) \leq L_n \leq 4d_n\lambda_1\lambda_2$ by construction. Therefore,

$$l(\sigma'_0) \leq \sum_{i=1}^n 9\delta + 4(c_2 + 6\Delta + 161\delta)\lambda_1\lambda_2.$$

Our goal is to estimate n .

By Lemma 10, we obtain

$$L_{\Delta}(\gamma_i^+ \cup \gamma_i^-) \geq 9\delta \frac{\Delta}{4\delta} \max(e^{Kd_{i+1}^+/38\delta}, e^{Kd_{i+1}^-/38\delta}) \geq \frac{9\Delta}{4} e^{K(d_{i+1}-\delta)/76\delta}.$$

On the other hand, $L_{\Delta}(\gamma_i^+ \cup \gamma_i^-) = L_i - L_{i+1} + 2\Delta$. Hence, setting $C_0 = (9\Delta/4)e^{-K/78}$, we have

$$(1) \quad C_0 e^{Kd_{i+1}/78\delta} \leq L_i - L_{i+1} + 2\Delta.$$

Let g_i^{\pm} be a point of γ_i^{\pm} that minimizes the distance to σ_i' . The part of the quasi-geodesic γ between g_i^+ and g_i^- is also a quasi-geodesic with the same constants and properties. By the triangle inequality, $|g_i^- - g_i^+| < d_i^+ + d_i^- + \delta$. Therefore, by construction (see the beginning of the proof) and because $d_i \geq c_2 + 6\Delta + 78\delta \geq c_2$ for $i < n$,

$$(2) \quad L_i \leq 4\lambda_1\lambda_2 d_i.$$

The function de^{-d} is decreasing. Therefore, because $d_i \geq \frac{1}{4\lambda_1\lambda_2}L_i$, we obtain

$$\frac{K}{2\delta} d_i e^{-Kd_i/2\delta} \leq \frac{K}{2\delta} \frac{1}{4\lambda_1\lambda_2} L_i e^{-(K/(8\delta\lambda_1\lambda_2))L_i}.$$

We are now ready to estimate n :

$$n = \sum_{i=1}^n 1 = \frac{1}{C_0} \sum_{i=1}^n e^{-Kd_i/2\delta} C_0 e^{Kd_i/2\delta} \leq \frac{1}{C_0} \frac{8\lambda_1\lambda_2\delta}{K} \sum_{i=1}^n e^{-(K/8\delta\lambda_1\lambda_2)L_i} \frac{K}{8\lambda_1\lambda_2\delta} (L_{i-1} - L_i + 2\Delta).$$

Setting $X_i = (K/8\lambda_1\lambda_2\delta)L_i$, we have

$$\sum_{i=1}^n 1 \leq \frac{\lambda_1\lambda_2\delta}{8C_0K} \sum_{i=1}^n e^{-X_i} (X_{i-1} - X_i) + 2\Delta/C_0 \sum_{i=1}^n e^{-X_i},$$

and because the function e^{-X} is decreasing for $X \geq 0$, we can use the estimate

$$\sum_{i=1}^n e^{-X_i} (X_{i-1} - X_i) \leq \int_0^{\infty} e^{-X} dX = -e^{-x}|_0^{\infty} = 1.$$

Summarizing all facts, we finally obtain the claimed result

$$H = 4\lambda_1\lambda_2 \left(\frac{\delta}{8C_0K} + c_2 + 12c_1 + 78\delta \right) + A_0,$$

we recall that $K = \ln 2/19$ and $C_0 = (9\Delta/4)e^{-K/38}$ and $A_0 = 8/9e^{K/38}$. Lastly we notice that $\delta/\Delta \leq 1$ and $A_0 \leq \lambda_1\lambda_2 A_0$. \square

10. OPTIMALITY OF THEOREM 2

Proposition 5. *Let T be a metric tree. Then for any constants $\lambda_1, \lambda_2 \geq 1$ and $c_1, c_2 \geq 0$ there exists a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic γ such that Morse constant $H \geq \lambda_1 \lambda_2 \min\{c_1, c_2\}/4 - \min\{c_1, c_2\}$.*

Proof. We will construct explicitly such a quasi-geodesic $\gamma : I \rightarrow T$, where I is a parametrization interval.

Consider a geodesic ray σ with base point s_1 in the tree T . Denote by $c = \min\{c_1, c_2\}$. Let I be an interval of length $l = \lambda_2 c/2$. Divide I by intervals I_1, I_2, \dots, I_n of length $l_1 = c/\lambda_1$. Let s_2, \dots, s_n be consequent points of σ such that $|s_i - s_{i-1}| = c/2$ for $i = 2, \dots, n$. We assume

- $\gamma(I_i) = s_i$ for any $i \leq n/2$,
- $\gamma(I_i) = s_{n-i}$ otherwise.

First, we check that γ is indeed a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry. If two points x_1, x_2 of I are in the same little interval I_i then

$$\frac{|x_1 - x_2| - c_2}{\lambda_2} \leq \frac{l_1 - c_2}{\lambda_2} \leq |\gamma(x_1) - \gamma(x_2)| = 0 \leq \lambda_1 |x_1 - x_2| + c_1.$$

If x_1 and x_2 are in different intervals I_i and I_k then the distance between their images is at least $c/2$ and for the left-hand inequality we have

$$\frac{|x_1 - x_2| - c_2}{\lambda_2} \leq \frac{l - c_2}{\lambda_2} \leq \frac{c}{2} \leq |\gamma(x_1) - \gamma(x_2)|.$$

Finally, we prove the right-hand inequality. We have $(|i - k| - 1)l_1 \leq |x_1 - x_2|$ and also $|\gamma(x_1) - \gamma(x_2)| \leq c|i - k|$. Hence,

$$\lambda_1 |x_1 - x_2| + c_1 \geq (|i - k| - 1)c + c_1 \geq |\gamma(x_1) - \gamma(x_2)|.$$

We see easily that $H \geq 1/2 \cdot c/2 \cdot l/l_1 - c = \lambda_1 \lambda_2 c/4 - c$. □

11. ANTI-MORSE LEMMA

We have already proved that any quasi-geodesic γ in a hyperbolic space is at distance not more than $\lambda_1 \lambda_2 (c_1 + c_2 + \delta)$ from a geodesic segment σ connecting its endpoints. This estimate cannot be improved. But the curious thing is that this geodesic belongs to a $\ln \lambda$ -neighborhood of the quasi-geodesic! We can therefore say that any quasi-geodesic is $\ln \lambda$ -quasiconvex. This upper bound can be improved in some particular spaces: for example, any quasi-geodesic is c_1 -quasiconvex in a tree.

The proof of Theorem 3 (see the introduction) that we give below is based on using

- Lemma 10 (exponential contraction) to prove that at the distance $\ln \lambda$ from the geodesic the length of σ is at most $\lambda^2 \ln \lambda$ and
- an analogue of Lemma 10 to prove that the length of a circle of radius R is at least e^R (up to some constants).

Lemma 11. *Let X be a hyperbolic metric space, γ be a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-geodesic, and σ be a geodesic connecting the endpoints of γ . Let (y_u, y_v) be an arc of γ such that no point of this arc is at distance less than $C_1 \ln \lambda_1 \lambda_2 + C_2$ from σ and y_u and y_v are the points of the arc nearest from σ . Then the length of the projection of the arc (y_u, y_v) on σ does not exceed $\max\{8\delta, C_3(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)\}$ (with some well-chosen constants $C_1 = O(\delta)$, $C_2 = O(\delta + c_1)$ and a universal constant C_3).*

Proof. By the definition of a quasi-geodesic, we have

$$\frac{|u - v| - c_2}{\lambda_2} \leq |y_u - y_v| \leq \lambda_1 |u - v| + c_1.$$

On the other hand,

$$|y_u - y_v| \leq |y_u - y'_u| + |y'_u - y'_v| + |y'_v - y_v|,$$

where y'_u and y'_v are the projections of y_u and y_v on σ . We adjust constants $C_1 = O(\delta)$ and $C_2 = O(\Delta + \delta)$ such that

$$R = C_1 \ln \lambda_1 \lambda_2 + C_2 = \frac{38\delta}{K} \ln 32(\lambda_1 \lambda_2)^2 + 6\Delta + 116\delta.$$

We assume $\Delta = 2 \max\{c_1, \delta\}$ (such a choice allows applying Lemma 9). We apply the lemma on exponential contraction (we assume that the length of the arc is rather large for using the estimate with an exponential factor and not to treat the obvious case where the length of the projection is 8δ). We let $l(y_u, y_v)$ denote the Δ -length of the arc (y_u, y_v) :

$$|y'_u - y'_v| \leq l(y_u, y_v) \frac{4\delta}{\Delta} e^{-KR/38\delta} = e^{-(a_1\delta + a_2\Delta)} \frac{4\delta}{\Delta} \cdot \frac{1}{32(\lambda_1 \lambda_2)^2} l(y_u, y_v),$$

where a_1 and a_2 are some positive universal constants, we denote $E_1 = \exp(-(a_1\delta + a_2\Delta)) < 1$. Combining all these inequalities and using Lemma 9, we obtain

$$\begin{aligned} \frac{|u - v| - c_2}{\lambda_2} \leq |y_u - y_v| &\leq 2R + E_1 \frac{4\delta}{\Delta} \cdot \frac{1}{32(\lambda_1 \lambda_2)^2} l(y_u, y_v) \\ &\leq 2R + \frac{1}{8(\lambda_1 \lambda_2)^2} 4\lambda_1 \lambda_2 |y_u - y_v| \\ &\leq 2R + \frac{1}{2\lambda_1 \lambda_2} (\lambda_1 |u - v| + c_1). \end{aligned}$$

We therefore conclude that $|y_u - y_v| \leq C_3 \lambda_1 \lambda_2 (\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)$, hence $l(y_u, y_v) \leq C_3 (\lambda_1 \lambda_2)^2 (\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)$, where C_3 is some universal constant and, finally, the length of the projection of the arc (y_u, y_v) of γ does not exceed

$$\max\{8\delta, C_3(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)\}$$

□

Proof of Theorem 3. The proof follows directly from Lemma 11. Because we have already proved that for every point $z' \in \sigma$, there exists a point $z \in \gamma$ such that the projection of z

on σ is at distance not more than several times $c_1 + \delta$ from z' . For simplicity, we therefore assume that for any point of σ , there exists a point of γ projecting on this point.

Assume $H_{am} = C_4(\delta \ln \lambda_1 \lambda_2 + \delta + c_1)$ where C_4 is just a universal constant which can be found from Lemma 11. If the distance between z and z' is less than H_{am} , then the statement is already proved. If not, then we take an arc (y_u, y_v) of γ containing the point z such that the endpoints y_u and y_v are at the distance H_{am} from σ and these points are the points of this arc that are nearest from σ . Hence, by Lemma 11, the length of the projection (which includes z) of the arc (y_u, y_v) does not exceed $\max\{8\delta, C_3(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)\}$. Therefore, the distance from z to y_u (and y_v) is not greater than $C_5(\delta \ln \lambda_1 \lambda_2 + \delta + c_1 + c_2)$ with some universal constant C_5 . \square

12. APPLICATION OF ANTI-MORSE LEMMA

Proposition 6. *Let X, Y be two geodesic hyperbolic spaces, let $f : X \rightarrow Y$ be a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry. Let σ be a geodesic in X . Then the distance from $\tilde{\gamma} = f(\sigma) \subset Y$ to any geodesic connecting its ends is at most $\lambda_1 H_{am}^X + c_1$, where H_{am}^X is a anti-Morse constant for the space X .*

We see that in case of a quasi-isometry instead of a quasi-isometric embedding we have a stronger result than Morse Lemma.

Proof. Let $\tilde{\sigma} \subset Y$ be a geodesic connecting the ends of $\tilde{\gamma}$. Define also a quasi-geodesic $\gamma = f^{-1}(\tilde{\sigma})$ in X . Because σ and γ share their ends, we can apply the Anti-Morse Lemma to them, so $\sigma \subset U_{H_{am}}(\gamma)$ lies in $H_{am}^X = (c_1 + c_2 + \delta) \log \lambda_1 \lambda_2$ -neighbourhood of γ . Now applying f to σ and γ we obtain that $\tilde{\gamma} \subset U_{\lambda_1 H_{am}^X + c_1}(\tilde{\sigma})$. \square

12.1. Proof of Proposition 1. Here, we prove Proposition 1 (see the introduction). We call any connected component of the ball $B = B(O, R)$ with deleted center O a *branch*. We call points that are sent to the branch containing the image of the center $f(O)$ green points and all other points of T red points.

Proof of Proposition 1. We show that there exist two red points r_1 and r_2 such that $d(O, r_1 r_2) \leq r = c_3 + 1$.

By Definition 1, a c -neighborhood of every point of the border should contain a point of the image. We must have at least $(d-1)d^{R-c_3-1}$ red points near the border (we exclude the green part). The number of points in each connected component of the complement of the ball of radius r is less than d^{R-r} . Therefore, there is a constant c_3 depending on the tree only such that if $r \geq c_3$, then one component contains an insufficient number of points to cover the boundary of B . Hence, there exist two red points r_1 and r_2 in different components of the complement of $B(O, r)$, which means that the geodesic $r_1 r_2$ passes at a distance less than r from the center O . It follows that the quasi-geodesic $f(r_1 r_2)$ passes at distance less than $\lambda_1 r + c_1$ from $f(O)$ and belongs to a $(\lambda_1 H_{am} + c_1)$ -neighborhood of the geodesic $f(r_1) f(r_2)$ by Proposition 6. Because every path from $f(O)$ to $f(r_1) f(r_2)$ passes through O , we conclude that $d(O, f(O)) < \lambda_1(H_{am} + c_1) + c_1 + \lambda_1 r$. We need only choose a good value for r . Simply calculating the number of points in above mentioned components

gives the estimate $1 + d + d^2 + \dots + d^{R-r} \leq (1/\ln d)d^{R-r+1}$. For $r = c_3 + 1$, we have $(1/\ln d)d^{R-r+1} \leq (d-1)d^{R-c_3-1}$, which completes the proof. \square

13. GEODESICALLY RICH SPACES

Definition 16. A metric space X is said to be geodesically rich if there exist constants r_0, r_1, r_2 such that for every pair of points p and q with $|p - q| \geq r_0$, there exists a geodesic γ (with ends at infinity) such that $d(p, \gamma) < r_1$ and $|d(q, \gamma) - |q - p|| < r_2$.

Remark 5. We introduced the notion of geodesically rich spaces in [30], see Definition 11. Still now we do not need the second condition to estimate a displacement of points under self-quasi-isometries fixing ideal boundary so we change the definition to a weaker form.

Example 5. A line and a ray are not geodesically rich.

Example 6. Nonelementary hyperbolic groups are geodesically rich. We prove this later.

Any δ -hyperbolic metric space H can be embedded isometrically in a geodesically-rich δ -hyperbolic metric space G (with the same constant of hyperbolicity). We take a 3-regular tree with a root (T, O) , assume that $G = H \times T$, and set the metric analogously to a real tree:

- the distance between points in the subspace (H, O) equals the distance between the corresponding points in H ;
- the distance between other points equals the sum of the three distances from the points to their projections on (H, O) and between their projections on (H, O) .

It is easy to show that the space G is δ -hyperbolic and geodesically rich. But such a procedure completely changes the ideal boundary of the space. We therefore ask another question:

Remark 6. It is not always possible to embed a δ -hyperbolic metric space H isometrically in a geodesically rich δ -hyperbolic metric space G with an isomorphic boundary. An example can be provided by a δ -hyperbolic space with an isolated point at the ideal boundary. As an illustration, consider a real line \mathbb{R} . Its ideal boundary contains only two points. Now consider a δ -hyperbolic space H with the same ideal boundary $\partial H = \{\xi_1, \xi_2\}$ and an isometric embedding $\gamma : \mathbb{R} \rightarrow H$ (hence, γ is a geodesic). We will show that every point $p \in \gamma$ lies at distance at most 2δ from any infinite geodesic σ what means that H is not geodesically rich. Because H is δ -hyperbolic, the triangle $p\xi_1\xi_2$ with sides coinciding with γ and σ is δ -thin. Hence, there exists a point $q \in \sigma$ such that $d(q, \gamma(\xi_1, p)) \leq \delta$ and $d(q, \gamma(\xi_2, p)) \leq \delta$. And we conclude that $d(p, \sigma) \leq 2\delta$ because γ is a geodesic.

Lemma 12. *Let G be a nonelementary hyperbolic group. Then there exist constants c_0, c_1 , and c_2 such that for every two points p and q in the group G with $|p - q| > r_0$, there exists a geodesic γ such that $d(p, \gamma) \leq r_1$ and $||p - q| - d(q, \gamma)| \leq r_2$.*

Proof. We first assume that p is the unity of the group. We argue by contradiction: we suppose that the statement is false, i.e., there exists a sequence of points q_n such that

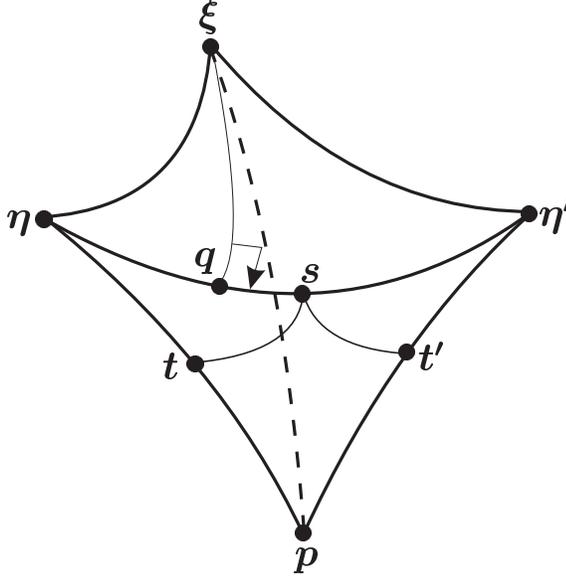


FIGURE 7. Illustration for Lemma 12.

$|q_n - p| \rightarrow \infty$ as $n \rightarrow \infty$, and all pairs p and q_n do not satisfy the conditions in the lemma. We suppose that ξ is a limit point of this sequence. We supply the boundary of the group with a visual metric.

We prove by contradiction that there exists $\varepsilon > 0$ such that for any point ξ there exist points η and η' on the ideal boundary $G(\infty)$ such that the pairwise visual distances between ξ , η , and η' are greater than ε (see Fig. 7). In other words, we will prove that the union of the three balls of radius ε with centers ξ , η and η' does not cover the whole ideal boundary. On the contrary, we suppose that there exist three sequences of points ξ_n , η_n and η'_n such that the union of $B(\xi_n, 1/n)$, $B(\eta_n, 1/n)$ and $B(\eta'_n, 1/n)$ includes $G(\infty)$. By compactness, we can assume that $\xi_n \rightarrow \xi$, $\eta_n \rightarrow \eta$ and $\eta'_n \rightarrow \eta'$, and we find that $G(\infty)$ belongs to the union of $B(\xi, 2/n)$, $B(\eta, 2/n)$ and $B(\eta', 2/n)$. Hence, the ideal boundary contains only the three points ξ , η and η' , which contradicts the assumption that G is nonelementary. We show that the geodesic γ with the endpoints η and η' satisfies the conditions in the lemma, which leads to the contradiction.

In what follows, we write ξ , η , and η' but assume that we consider three sequences of points converging to the corresponding points of the ideal boundary. The triangle $p\eta\eta'$ is δ -thin. We take a point s of $\eta\eta'$ such that $d(s, p\eta) \leq \delta$ and $d(s, p\eta') \leq \delta$. We let t and t' denote projections of s respectively on $p\eta$ and $p\eta'$. By the triangle inequality, we have

$$|\eta - t| + |\eta' - t'| - 2\delta \leq |\eta - \eta'| \leq |\eta - t| + |\eta' - t'| + 2\delta.$$

By hypothesis,

$$\text{visdist}_p(\eta, \eta') = e^{-(\eta|\eta')_p} > \varepsilon.$$

Hence,

$$|p - \eta| + |p - \eta'| - |\eta - \eta'| < 2\varepsilon_0,$$

where $\varepsilon_0 = -\ln \varepsilon$

Combining the two inequalities, we obtain $|p - t| + |p - t'| \leq 2(\varepsilon_0 + \delta)$ and $d(p, \eta\eta') \leq 2\varepsilon_0 + 3\delta$. The same arguments applied to the triangles $p\eta\xi$ and $p\eta'\xi$ show that the distance from the point p to the geodesics $\eta\xi$ and $\eta'\xi$ also does not exceed $2\varepsilon_0 + 3\delta$. We let p_1, p_2 , and p_3 denote the respective projections of p on $\eta\eta'$, $\eta\xi$, and $\eta'\xi$ and q denote the projection of ξ on $\eta\eta'$. By the triangle inequality, $|p_1 - p_2| \leq |p_1 - p| + |p - p_2| \leq 2(2\varepsilon_0 + 3\delta)$. Applying Lemma 5 to the triangles $q\xi\eta$ and $q\xi\eta'$, we find that the point q is not farther than 2δ from both $\eta\xi$ and $\eta'\xi$. Therefore, both p_1 and q are at bounded distances from $\eta\xi$ and $\eta'\xi$, and we can apply Lemma 7, whence it follows that p_1 and q are near each other at a distance of the order $\varepsilon_0 + \delta$. \square

This Lemma proves that a nonelementary hyperbolic group satisfies the definition of geodesically rich space.

14. QUASI-ISOMETRIES FIXING THE IDEAL BOUNDARY

We now give some estimates of the displacement of points in geodesically rich spaces under quasi-isometries that fix the ideal boundary. We do not yet know whether these results are optimal.

Theorem (see Theorem 4 in the introduction). *Let X be a geodesically rich hyperbolic metric space. Let $f : X \rightarrow X$ be a $(\lambda_1, \lambda_2, c_1, c_2)$ -self-quasi-isometry fixing the boundary ∂X . Then any point $O \in X$ can be displaced at most at distance $d(f(O), O) \leq \lambda_1(H_{am} + r_1) + 2c_1 + r_0 + r_2$.*

Proof. Consider a point O and its image $f(O)$. If $d(O, f(O)) < r_0$, there is nothing to prove. Otherwise, let γ be a geodesic such that $d(O, \gamma) \leq r_1$ and $|d(f(O), \gamma) - d(O, f(O))| \leq r_2$ and in particular, $d(f(O), \gamma) \geq |d(O, f(O))| - r_2$. Such a geodesic exists by definition of geodesic ally rich.

Because $f(\gamma)$ is a quasi-geodesic with the same endpoints as γ , the quasi-geodesic lies near γ : $f(\gamma) \subset U_{\lambda_1 H_{am} + c_1}(\gamma)$ by Proposition 6. Also since $d(O, \gamma) \leq r_1$, in the image $d(f(O), f(\gamma)) \leq \lambda_1 r_1 + c_1$. Combining all the arguments, we obtain

$$d(O, f(O)) \leq d(f(O), \gamma) + r_2 \leq \lambda_1 H_{am} + c_1 + r_2 + \lambda_1 r_1 + c_1.$$

\square

The property of being geodesically rich plays crucial role here. For example a translation of a real line \mathbb{R} (which is an isometry) fixes its ideal boundary but still moves its point to any pregiven distance.

Part 3. Poincaré inequalities and quasi-isometries

14.1. **The critical exponent for L^p -cohomology.** L^p -cohomology groups provides invariants for quasi-isometries. The continuous first L^p -cohomology group of a hyperbolic metric space X is

$$L^p H_{cont}^1(X) := \{[f] \in L^p H^1(X) \mid f \text{ extends continuously to } X \cup \partial X\},$$

where $X \cup \partial X$ is Gromov's compactification of X . Following the works of Pierre Pansu, and Marc Bourdon and Bruce Kleiner [26], we define the following quasi-isometric numerical invariant of X

$$p_{\neq 0}(X) = \inf \{p \geq 1 \mid L^p H_{cont}^1(X) \neq 0\}.$$

If $p_{\neq 0}$ achieves different values for two spaces X and Y , then X and Y are not quasi-isometric. We expect that the difference $|p_{\neq 0}(X) - p_{\neq 0}(Y)|$ also bounds for below the quasi-isometrical distortion growth. We are able to prove this only for a family of examples.

Let Z_μ and $Z_{\mu'}$ be two variants of the space $\mathbb{T}^n \times (-\infty, \infty)$ with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$ and $dt^2 + \sum e^{2\mu'_i t} dx_i^2$ respectively. The main result of this part is a sharp lower bound for the quasi-isometrical distortion growth between Z_μ and $Z_{\mu'}$, of the form

$$const (p_{\neq 0}(Z_{\mu'}) - p_{\neq 0}(Z_\mu)) R.$$

14.2. **Scheme of proof.** Constants in Poincaré inequalities are the quantitative incarnation of L^p -cohomology.

Let X be a Riemannian manifold. Let p be a number in $[1, \infty]$. Then the \mathbb{L}^p -norm $|\cdot|_p$ of functions and vectorfields make sense. We will say that $C = C(X, p)$ is a Poincaré constant for X and \mathbb{L}^p if for any function f in \mathbb{L}^p there exists such a constant c (which is in fact a mean value of f over X) such that the Poincaré inequality holds

$$|f - c|_p \leq C |\nabla f|_p.$$

Variants of this definition appear in the litterature. In an appendix 21, we shall check that these definitions are equivalent, up to universal constants.

For the family of spaces Z_μ , it is known that $p_{\neq 0}(Z_\mu) = \frac{\sum \mu_i}{\max \mu_i}$. We show that

- if $p > p_{\neq 0}(Z_\mu)$, then the Poincaré constant for a ball of radius R satisfies

$$C_p(B^{Z_\mu}(R)) \geq const. (Vol B(R))^{1/p};$$

- if $p \leq p_{\neq 0}(Z_\mu)$, then

$$C_p(B^{Z_\mu}(R)) = o\left((Vol B(R))^{1/p}\right).$$

Next, we show that under transport by a (λ, c) -quasi-isometry, C_p is multiplied or divided by at most $e^{(\lambda+c)/a}$ for some positive constant a . Transport under quasi-isometric embeddings is more delicate, this is why our arguments work only for a family of examples. For these examples, we are able to get a lower bound. Roughly speaking, it states

Assume that $p_{\neq 0}(Z_{\mu'}) < p < p_{\neq 0}(Z_{\mu})$. If there exists a (λ, c) -quasi-isometric embedding $B^{Z_{\mu}}(R) \rightarrow Z_{\mu'}$, which induces an isomorphism on fundamental groups, then

$$C_p(B^{Z_{\mu}}(R)) \geq \text{const.} e^{-(\lambda+c)/a} C_p(B^{Z_{\mu'}}(R)).$$

This yields

$$\begin{aligned} \lambda + c &\geq a(\log(C_p(B^{Z_{\mu'}}(R))) - \log(C_p(B^{Z_{\mu}}(R)))) \\ &\sim (p_{\neq 0}(Z_{\mu'}) - p_{\neq 0}(Z_{\mu}))R. \end{aligned}$$

which is the announced lower bound on quasi-isometric distortion growth.

15. REGULARISATION AND QUASI-ISOMETRIES

In this section we will study how Poincaré inequalities are transformed under quasi-isometries. For this purpose we will introduce the notion of cross-kernels, which will help us to regularize transported functions.

15.1. Kernels. First we recall what are classical kernels.

Definition 17. Let X be a geodesic space, dx a measure on X . A kernel ψ is a non-negative function on $X \times X$ such that

- ψ is bounded, $\psi \leq S^\psi$;
- for every $x \in X$ $\int_X \psi(x, x') dx' = 1$;
- the support of ψ is concentrated near the diagonal: there exist constants $\varepsilon^\psi > 0$, $\tau^\psi > 0$ and $R^\psi < \infty$ such that $\psi(x, y) > \tau^\psi$ if $d(x, y) \leq \varepsilon^\psi$; $\psi(x, y) = 0$ if $d(x, y) > R^\psi$.

R^ψ is called the *width*, ε^ψ the *radius of positivity*, S^ψ the *supremum* and τ^ψ the *margin* of ψ .

The convolution of two kernels is

$$\psi_1 * \psi_2 = \int_X \psi_1(x, z) \psi_2(z, y) dz,$$

the result is also a kernel. The convolution of a kernel and a function is

$$g * \psi(x) = \int_X g(z) \psi(x, z) dz.$$

Lemma 13. For any $\varepsilon > 0$ there exists a kernel ψ on $X \times X$ and a constant τ such that for any two points x_1, x_2 such that $d(x_1, x_2) < \varepsilon$ we have $\psi(x_1, x_2) > \tau$. Moreover, τ depends exponentially on ε

$$\tau = c_\tau e^{-\varepsilon},$$

where c_τ depends only on the local geometry of the space X .

Proof. We start from kernel

$$\psi'(x, x') = \text{vol}(B(x, 1))^{-1} 1_{\{d(x, x') \leq 1\}}$$

with radius of positivity $\varepsilon' = 1$ and margin $\tau' = v(1)^{-1}$, where, for $r > 0$, $v(r)$ denotes the infimum of volumes of balls of radius r in X . We know from the proof of Lemma 1.2 in [11] that the m -th convolution ψ'^{*m} has radius of positivity $\varepsilon'_m \geq m(\varepsilon'/2) = m/2$ and margin $\tau'_m \geq \tau'^m v(\frac{1}{2})^{m-1}$. \square

Definition 18. A *cocycle* on Y is a map $a : Y \times Y \rightarrow \mathbb{R}$ such that for every y_1, y_2, y_3 in Y ,

$$a(y_1, y_2) = a(y_1, y_3) + a(y_2, y_3).$$

The convolution of a cocycle with a kernel is defined by

$$a * \phi(x, x') = \int_{Y \times Y} a(y, y') \phi(x, y) \phi(x', y') dy dy'.$$

Definition 19. Let ψ be a kernel and a a cocycle on X . The semi-norm $N_{p, \psi}$ is defined by

$$N_{p, \psi}(a) = \left(\int_{X \times X} |a(x_1, x_2)|^p \psi(x_1, x_2) dx_1 dx_2 \right)^{1/p}.$$

The following facts are known, see [11].

Lemma 14. 1) *Semi-norms $N_{p, \psi}$ are pairwise equivalent. More precisely, let ψ_1 and ψ_2 be two kernels. Then*

$$N_{\psi_2} \leq \hat{C} N_{\psi_1},$$

where

$$\hat{C} = \frac{\sup \psi_1 \sup \psi_2}{c_\tau} \frac{R^{\psi_2}}{\varepsilon^{\psi_1}} (2e)^{R^{\psi_2}/\varepsilon^{\psi_1}}.$$

Let the space X be a Riemannian manifold and have the following properties: (1) its injectivity radius is bounded below, (2) its Ricci curvature is bounded from below. Then the volumes of balls are bounded below (Croke inequality [3]) and above (Bishop inequality).

2) For any function g define a cocycle $u(x, y) = g(x) - g(y)$. Then for any p and any kernel ψ' with bounded derivatives there exists a kernel ψ_1 such that the \mathbb{L}^p -norm of $\nabla(g * \psi')$ (we regularise g) is bounded from above by a ψ_1 -seminorm of the corresponding cocycle u

$$\|\nabla(g * \psi')\|_p \leq N_{p, \psi_1}(u)$$

with the kernel ψ_1 defined as follows

$$\psi_1 = \frac{\sup \nabla \psi' \sup \psi'}{\text{vol}(B(z', R^{\psi'}))} 1_{\{d(z, z') \leq R^{\psi'}\}}.$$

3) Conversely, there also exists a kernel ψ_2 such that

$$N_{p, \psi_2}(u) \leq C \|\nabla g\|_p,$$

where C depends only on dimension. Here the kernel ψ_2 can be taken as

$$\psi_2(x, y) = \max\{1, \Theta(x, y)^{-1}\} 1_{\{d(x, y) \leq R\}},$$

where $\Theta(x, y)$ is the density of the volume element in polar coordinates and $R > 0$ can be chosen arbitrarily.

In the third hypothesis we propose to use $R = 1$, then ψ_2 is bounded by 1 and the width of its support is also 1. For reader's convenience, we include the proof of the second statement of this Lemma, following [11].

Proof. Denote by α the cocycle $u * \psi'$. Then for any y ,

$$\nabla(\alpha * \psi')(x) = \frac{\partial \alpha(x, y)}{\partial x} = \int (g(z') - g(z)) d_x \psi'(z, x) \psi'(z', y) dz dz'.$$

Choose $y = x$. Then we obtain

$$|\nabla(g * \psi'(x))| \leq \sup \nabla \psi' \sup \psi \int_{B(x, R^\psi) \times B(x, R^\psi)} |g(z') - g(z)| dz dz'.$$

Now applying Hölder inequality we get the needed statement with the kernel

$$\psi_1 = \frac{\sup \nabla \psi' \sup \psi'}{\text{vol}(B(z', R^{\psi'}))} \mathbf{1}_{\{d(z, z') \leq R^{\psi'}\}}.$$

□

15.2. Cross-kernels. Let X, Y be two metric spaces, let $f : X \rightarrow Y$ and $f' : Y \rightarrow X$ be (K, c) -quasi-isometries between them such that for any $x \in X$, $d(x, f' \circ f(x)) \leq c$ and vice versa (that is, they are inverse in the quasi-isometrical sense). Let g be a measurable function on Y . We want to find a way to transport g by our quasi-isometry (using the regularisation) to obtain a similar measurable function on X . We will take

$$h(x) = \int_Y g(z) \psi(f(x), z) dz$$

as a function on X corresponding to g .

We are going to construct a numerical function on $X \times Y$ which will play the role of a kernel. Indeed, a *cross-kernel* can be considered as the composition (relatively to the first variable) of a quasi-isometry from X to Y and a kernel on $Y \times Y$. Conversely, a cross-kernel generates a quasi-isometry.

Definition 20. A *cross-kernel* is a bounded non-negative function $\phi : X \times Y \rightarrow \mathbb{R}$ such that

- for all $x \in X$, $\int_Y \phi(x, y) dy = 1$;
- for all $R > 0$ there exists $Q_1^\phi > 0$ such that if $d(y, y') \geq Q_1^\phi$ and $d(x, x') \leq R$, then that $\phi(x, y)\phi(x, y') = 0$;
- for all $R > 0$ there exists $Q_2^\phi > 0$ such that if $d(y, y') \leq R$ and $d(x, x') \geq Q_2^\phi$, then $\phi(x, y)\phi(x', y') = 0$;
- there exists a constant S^ϕ such that for any $y \in Y$, $\int_X \phi(x, y) dx \leq S^\phi$;
- there exist τ^ϕ, D^ϕ such that for any $y \in Y$ the set $\{x \in X | \phi(x, y) > \tau^\phi\}$ contains a ball of radius D^ϕ .

Remark 7. For our purposes, the third axiom could be replaced with a weaker one: there exists $R > 0$ such that for any $y \in Y$ there exist $x_0 \in X$ such that for any $x \in X$ with $d(x, x_0) > R$, $\phi(x, y) = 0$. But we prefer our definition as it is more symmetric and easier to apply.

Before we construct a cross-kernel with a quasi-isometry and a kernel, we will show that a cross-kernel ϕ defines a quasi-isometry. Simply let $f : X \rightarrow Y$ be defined as follows $x \mapsto \{y | \phi(x, y) > \tau^\phi\}$. We notice that if we remove the last hypothesis in the definition, we get a quasi-isometric embedding instead of a quasi-isometry.

Lemma 15. *If ψ is a kernel on $Y \times Y$ and f is a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry or a quasi-isometric embedding from X to Y . In case of a quasi-isometry we also assume that the radius of positivity of ψ is at least $\zeta\lambda_2 + c_2$ with $\zeta > 0$, then $\phi(x, y) = \psi(f(x), y)$ is a cross-kernel on $X \times Y$ and $Q_1^\phi(R) \leq 2R^\psi + \lambda_1 R + c_1$, $Q_2^\phi(R) \leq \lambda_2(2R^\psi + R + c_2)$ and $S^\phi \leq (2\lambda_1 R^\psi + c_1) \sup_{Y \times Y} \psi$. In case of a quasi-isometry $D^\phi \geq \zeta$ and $\tau^\phi = \tau^\psi \geq c_\tau e^{-\varepsilon^\psi}$.*

Proof. 1) Evidently, for any $x \in X$ $\int_Y \phi(x, y) dy = 1$ by the definition of kernels.

2) Check the second axiom. Take two points x_1, x_2 such that $d(x_1, x_2) \leq R_1$ and two points y_1, y_2 such that $d(y_1, y_2) \geq 2R^\psi + \lambda_1 R_1 + c_1$. If $d(f(x_1), y_1) \geq R^\psi$, there is nothing to prove as $\psi(f(x_1), y_1) = 0$. Otherwise $d(f(x_2), y_2) \geq d(y_1, y_2) - d(f(x_2), y_1) \geq d(y_1, y_2) - (d(f(x_2), f(x_1)) + d(f(x_1), y_1)) \geq d(y_1, y_2) - (\lambda_1 d(x_1, x_2) + c_1 + R^\psi) \geq R^\psi$. Hence, $\psi(f(x_2), y_2) = 0$.

3) Check the third axiom. Take two points y_1, y_2 such that $d(y_1, y_2) \leq Q_1$ and two points x_1, x_2 such that $d(x_1, x_2) \geq \lambda_2(2R^\psi + Q_1 + c_2)$. If $d(f(x_1), y_1) \geq R^\psi$, there is nothing to prove as $\psi(f(x_1), y_1) = 0$. Otherwise $d(f(x_2), y_2) \geq d(f(x_1), f(x_2)) - d(f(x_1), y_2) \geq d(f(x_1), f(x_2)) - (d(f(x_1), y_1) + d(y_1, y_2)) \geq d(x_1, x_2)/\lambda_2 - c_2 - (R^\psi + Q_1) \geq R$. Hence, $\psi(f(x_2), y_2) = 0$.

4) Check the fourth axiom. For any $y \in Y$, if $d(f(x), y) > R^\psi$ then $\psi(f(x), y) = 0$. Hence, the diameter of the set of points $X_y \in X$ such that for any $x \in X_y$ $d(f(x), y) \leq R^\psi$, is less than $\lambda_1 2R^\psi + c_1$. Hence, $\int_X \phi(x, y) dx \leq (2\lambda_1 R^\psi + c_1) \sup_{Y \times Y} \psi$.

5) If $d(f(x), y) < \zeta\lambda_2 + c_2$ then $\phi(x, y) > \tau^\psi$. Hence, the diameter of the set of points of X with this property is at least ζ . \square

Remark 8. If, in the previous Lemma f is a quasi-isometric embedding, then we do not need the condition on radius of positivity of ψ .

15.3. Transporting cocycles.

Definition 21. Let a be a cocycle on Y and ϕ a cross-kernel on $X \times Y$. The convolution of a with ϕ is the cocycle defined on X by

$$a * \phi(x, x') = \int_{Y \times Y} a(y, y') \phi(x, y) \phi(x', y') dy dy'.$$

Lemma 16. *Let $\phi : X \times Y \rightarrow \mathbb{R}$ be a cross-kernel, let a be a cocycle on Y and let ψ be a kernel on X . Then*

$$N_\psi(a * \phi) \leq CN_{\tilde{\psi}}(a),$$

where $\tilde{\psi}$ is a kernel on Y and

$$C \leq \left(\frac{\sup \psi}{\tau} \right)^{1/p} (S^\phi)^{2/p},$$

where $\tau = c_\tau^Y e^{-Q_1^\phi(R^\psi)}$ (for the definition of constant c_τ^Y see lemma 13, it depends on the local geometry of the space Y only).

In particular, if ϕ is associated with a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry or a quasi-isometric embedding,

$$C \leq \frac{1}{c_\tau^Y} (\sup \psi)^{3/p} e^{((2+\lambda_1)R^\psi + c_1)/p} (2\lambda_1 R^\psi + c_1)^{2/p}.$$

Proof.

$$\begin{aligned} (N_\psi(a * \phi))^p &= \int_{X \times X} |a * \phi(x, x')|^p \psi(x, x') dx dx' = \\ &= \int_{X \times X} \left| \int_{Y \times Y} a(y, y') \phi(x, y) \phi(x', y') dy dy' \right|^p \psi(x, x') dx dx' \end{aligned}$$

By Hölder inequality

$$\leq \int_{X \times X} \int_{Y \times Y} |a(y, y')|^p |\phi(x, y) \phi(x', y')| dy dy' \psi(x, x') dx dx'$$

Assume $\psi'(y, y') = \int_{X \times X} \phi(x, y) \phi(x', y') \psi(x, x') dx dx'$

$$= \int_{Y \times Y} |a(y, y')|^p \psi'(y, y') dy dy'.$$

Now we need to show that ψ' is dominated by some kernel ψ'' .

First we will prove that $\psi'(y, y') = 0$ if $d(y, y') > R^{\psi'}$ for some $R^{\psi'}$.

If $d(x, x') > R^\psi$ then by the definition of kernels $\psi(x, x') = 0$, hence

$$\begin{aligned} \psi'(y, y') &= \int_{X \times X} \phi(x, y) \phi(x', y') \psi(x, x') dx dx' = \\ &= \int_{(x_1, x_2) \in X \times X, d(x_1, x_2) \leq R^\psi} \phi(x, y) \phi(x', y') \psi(x, x') dx dx'. \end{aligned}$$

If $d(x, x') < R^\psi$ then by definition of cross-kernels there exists a number $Q_1^\phi(R^\psi)$ such that if $d(y, y') > Q_1^\phi$ we have $\phi(x, y) \phi(x', y') = 0$. We estimate $\psi'(y, y')$ from above in an evident way

$$\psi'(y, y') \leq \sup \psi \int_{X \times X} \phi(x, y) \phi(x', y') dx dx' \leq \sup \psi (S^\phi)^2.$$

By Lemma 13 we conclude that there exists a kernel $\tilde{\psi}$ such that $\tilde{\psi}(y, y') \geq \tau = c_\tau^Y e^{-Q_1^\phi}$ whenever the distance between y, y' does not exceed Q_1^ϕ . Hence,

$$\psi'(y, y') \leq \frac{\sup \psi}{\tau} (S^\phi)^2 \tilde{\psi}(y, y').$$

□

Theorem 22. *Let X, Y be two quasi-isometric spaces, let ϕ be a cross-kernel on $X \times Y$, let C_X denote the p -Poincaré constant for X . Then the Poincaré constant C_Y for Y is bounded from above by*

$$C_Y \leq \text{const}(X) \text{const}(Y) \frac{\tilde{C}(Y)^{Q_1^\phi(0)} (\sup \phi)^2 + C_X \sup \phi V(Q_2^\phi(0))}{D^{\phi\tau\phi}},$$

where multiplicative constants depends only on the local geometry of X and Y .

Proof. The idea of our proof is the following. By Minkowski inequality we will show that

$$\|g\|_p \leq N_\psi(dg) + \|h\|_p.$$

We know that the semi-norm N_ψ is bounded from above by the \mathbb{L}_p norm. So, applying Poincaré inequality to h and then Lemma 16 for cocycles to ∇h we will get the upper-bound for Poincaré constant for g . In other words,

$$\|g\|_p \leq N_\psi(dg) + \|h\|_p \leq \|\nabla g\|_p + C_{\text{poincare}} \|\nabla h\|_p \leq \|\nabla g\|_p + C_{\text{cocycle}} C_{\text{poincare}} \|\nabla g\|_p.$$

First step. We have

$$\int_Y |g(z)|^p dz \leq \frac{1}{D^{\phi\tau\phi}} \int_{X \times Y} |g(z)|^p \phi(x, z) dx dz.$$

Here we need just to notice that for any $z \int_X \phi(x, z) dx \geq D^{\phi\tau\phi}$.

Second step. Now by Minkowski inequality applied to $(\int |g(z)|^p \phi(x, z) dx dz)^{1/p}$ we get

$$\begin{aligned} \left(\int |g(z)|^p \phi(x, z) dz dx \right)^{1/p} &\leq \left(\int |g(z) - h(x)|^p \phi(x, z) dz dx \right)^{1/p} + \left(\int |h(x)|^p \phi(x, z) dz dx \right)^{1/p} = \\ &= \left(\int |g(z) - h(x)|^p \phi(x, z) dz dx \right)^{1/p} + \left(\int |h(x)|^p dx \right)^{1/p} \end{aligned}$$

Third step. For any points $z \in Y$ and $x \in X$ we have

$$g(z) - h(x) = g(z) - \int_Y g(y) \phi(x, y) dy =$$

by definition of cross-kernel $\int_Y \phi(x, y) dy = 1$ so we go on

$$= g(z) \int_Y \phi(x, y) dy - \int_Y g(y) \phi(x, y) dy = \int_Y (g(z) - g(y)) \phi(x, y) dy.$$

Now we apply Hölder inequality

$$|g(z) - h(x)|^p \leq \int_Y |g(z) - g(y)|^p \phi(x, y) dy.$$

So,

$$\left(\int |g(z) - h(x)|^p \phi(x, z) dz dx \right)^{1/p} \leq \left(\int |g(z) - g(y)|^p \phi(x, y) \phi(x, z) dz dx dy \right)^{1/p}.$$

Fourth step. Evidently, $\int_X \phi(x, y) \phi(x, z) dx$ is uniformly bounded, and it vanishes outside of a strip of width $Q_1^\phi(0)$ (take $R = 0$ for the second property of cross-kernel). Hence, there exists a kernel ψ on $Y \times Y$ and constants $C_1 = C_1(\phi, Y)$ and $C_2 = C_2(Y)$ such that

$$\begin{aligned} \left(\int |g(z) - g(y)|^p \phi(x, y) \phi(x, z) dz dx dy \right)^{1/p} &\leq C_1 \left(\int |g(z) - g(y)|^p \psi(z, y) dz dy \right)^{1/p} = \\ &= C_1 N_{p, \psi}(g(z) - g(y)) \leq C_1 C_2 \|\nabla g\|_p. \end{aligned}$$

Let $\tilde{\psi}_Y$ be a model kernel on Y with the margin $\tilde{\tau}_Y$ and radius of positivity $\tilde{\varepsilon}_Y$. Fix some $r < \tilde{\varepsilon}_Y$. Assume m to be the least integer such that $m(\tilde{\varepsilon}_Y - r) \geq Q_1^\phi(0)$. Hence, if we take $\psi = \tilde{\psi}_Y^m$, we set

$$C_1 = \frac{(\sup \phi)^2}{\tilde{\tau}_Y^{m/p} v(r)^{(m-1)/p}},$$

where $v(r)$ is the infimum of volumes of balls of radius r .

Fifth step. We apply Poincaré inequality to h

$$\|h\|_p \leq C_X \|\nabla h\|_p.$$

We have already discussed that any semi-norm defined by a kernel and an \mathbb{L}^p -norm are equivalent. Hence, we can apply Lemma 16 for cocycles to $\|\nabla h\|$ and $\|\nabla g\|$ just adding some multiplicative constant $C_3 = C_3^X C_3^Y$

$$\|h\|_p \leq C_X C_3 \sup \phi V(Q_2^\phi(0)) \|\nabla g\|_p,$$

where $V(Q_2^\phi(0))$ is supremum of volumes of all balls of radius $Q_2^\phi(0)$ in X .

Final step. Combining all these results we conclude that

$$\begin{aligned} \|g\|_p &\leq \frac{1}{\tau C} (\|h\|_p + C_1 C_2 \|\nabla g\|_p) \leq \frac{1}{D^{\phi \tau \phi}} (C_X C_3 \|\nabla g\|_p + C_1 C_2 \|\nabla g\|_p) = \\ &= \frac{C_1 C_2 + C_3 C_X \sup \phi V(Q_2^\phi(0))}{D^{\phi \tau \phi}} \|\nabla g\|_p. \end{aligned}$$

□

Now summarizing all the results of this section we know that quasi-isometries preserve Poincaré inequalities. Moreover, if C_X is a Poincaré constant for the domain, then the Poincaré constant C_Y for the range does not exceed $C(1 + C_X)$, where C is exponential in function of the quasi-isometric distortion, up to some multiplicative constants which depend on the local geometry of X and Y .

16. POINCARÉ INEQUALITY FOR EXPONENTIAL METRIC

We will give an upper bound for the Poincaré constant in a ball of radius in a space with the metric $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$.

Theorem 23. *Let $\tilde{X} = \mathbb{R}_+ \times \mathbb{R}^n$ with the metric $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$. Let $X = \tilde{X}/\Gamma$ where Γ is a lattice of translations in the factor \mathbb{R}^n . Then the Poincaré constant for a ball $B(R)$ in X is*

$$C_p(\mu) \leq C(p, \mu) + C_p(\mathbb{T}^n) e^{\mu_n R},$$

where $C(p, \mu)$ is a constant depending only on p and $\mu = \sum \mu_i$, $C_p(\mathbb{T}^n)$ is a Poincaré constant for a torus \mathbb{T}^n .

First, we fix the direction $\theta = (x_1, \dots, x_n)$.

16.1. Poincaré inequality for fixed direction. Let f be a function such that its partial derivative $\partial f / \partial t$ is in $\mathbb{L}^p(e^{\mu t} dt, [0, +\infty))$. By Hölder inequality we get

$$\int_0^{+\infty} \left| \frac{\partial f}{\partial t} \right| dt \leq \left(\int_0^{+\infty} \left| \frac{\partial f}{\partial t} \right|^p e^{\mu t} dt \right)^{1/p} \left(\int_0^{+\infty} e^{-(\mu t/p)(p/(p-1))} dt \right)^{1-1/p} < +\infty.$$

Hence, for every fixed direction θ there exists a limit $\lim_{t \rightarrow \infty} f(t, \theta)$.

Here we will prove the Poincaré inequality for the fixed direction θ in a ball of radius R (we allow R to be infinity, in this case we deal with a complete space). Assume $c_\theta = f(R, \theta)$ or $c_\theta = \lim_{t \rightarrow \infty} f(t, \theta)$ if $R = \infty$. We write

$$\int_a^R |f(t) - c_\theta|^p e^{\mu t} dt \leq C(p, \mu) \int_a^R |f'(t)|^p e^{\mu t} dt,$$

where $C(p, \mu)$ is a constants depending only on p and μ .

First, if $R = \infty$, prove that $|f(t) - c_\theta|^p e^{\mu t} \rightarrow 0$ as $t \rightarrow \infty$. Apply the Newton-Leibniz theorem and then Hölder inequality to $|f(t) - c_\theta|$. We have

$$(3) \quad \begin{aligned} |f(t) - c_\theta| &= \left| \int_t^\infty \frac{\partial f}{\partial s} ds \right| \leq \int_t^\infty \left| \frac{\partial f}{\partial s} \right| ds \leq \\ &\leq \left(\int_t^\infty \left| \frac{\partial f}{\partial s} \right|^p e^{\mu s} ds \right)^{1/p} \left(\int_t^\infty e^{-\mu s/(p-1)} ds \right)^{1-1/p}. \end{aligned}$$

Calculate the last integral

$$\int_t^\infty e^{-\mu s/(p-1)} ds = -\frac{p-1}{\mu} e^{-\frac{\mu s}{p-1}} \Big|_t^\infty = \frac{p-1}{\mu} e^{-\frac{\mu t}{p-1}}.$$

Denote the constant $D_0 = \left(\frac{p-1}{\mu} \right)^{1-1/p}$

$$|f(t) - c_\theta|^p \leq D_0 e^{-\mu t} \int_t^{+\infty} \left| \frac{\partial f}{\partial s} \right|^p e^{\mu s} ds.$$

Hence

$$|f(t) - c_\theta|^p e^{\mu t} \leq D_0 \int_t^{+\infty} \left| \frac{\partial f}{\partial s} \right|^p e^{\mu s} ds \rightarrow 0$$

as $t \rightarrow +\infty$.

Now we integrate by parts

$$\int_a^R |f(t) - c_\theta|^p e^{\mu t} dt = \left[|f(t) - c_\theta|^p \frac{e^{\mu t}}{\mu} \right]_a^R - \int_a^R f'(t) |f(t) - c_\theta|^{p-1} \frac{e^{\mu t}}{\mu} dt.$$

As $c_\theta = f(R)$

$$\int_a^R |f(t) - c_\theta|^p e^{\mu t} dt = -|f(a) - c_\theta|^p \frac{e^{\mu a}}{\mu} - \int_a^R f'(t) |f(t) - c_\theta|^{p-1} \frac{e^{\mu t}}{\mu} dt.$$

By Hölder inequality

$$\int_a^R f'(t) |f(t) - c_\theta|^{p-1} \frac{e^{\mu t}}{\mu} dt \leq \left(\int_a^R |f'(t)|^p e^{\mu t} dt \right)^{1/p} \left(\int_a^R |f(t) - c_\theta|^p e^{\mu t} dt \right)^{(p-1)/p}$$

Introduce following notations

$$X = \int_a^R |f(t) - c_\theta|^p e^{\mu t} dt$$

$$Y = \int_a^R |f'(t)|^p e^{\mu t} dt$$

So,

$$X \leq |f(0) - c_\theta|^p + Y^{1/p} X^{(p-1)/p} \leq Y + Y^{1/p} X^{(p-1)/p}.$$

Dividing by Y we obtain

$$\frac{X}{Y} \leq \frac{1}{\mu} + \left(\frac{X}{Y} \right)^{(p-1)/p}.$$

This inequality is true only if $X \leq C(p, \mu)Y$ which proves Poincaré inequality for fixed direction.

16.2. Poincaré inequality for exponential metric. Introduce the following notations $\tilde{f}_r(t, \theta) = f(r, \theta)$ (the function is considered as a function of two variables), $f_r(\theta) = f(r, \theta)$ (the function is considered as a function of one variable).

We have already proved that for any $\theta \in \mathbb{T}^n$

$$\int_0^R |f(t, \theta) - f(R, \theta)|^p e^{ht} dt \leq c \int_0^R \left| \frac{\partial f}{\partial t} \right|^p e^{ht} dt.$$

We change metric

$$\int_{R-1}^R \|\nabla_e f_r\|_{\mathbb{L}^p(\mathbb{T}^n)}^p dr \leq e^{p\mu R} \int_{B(R) \setminus B(R-1)} |\nabla f|^p dr.$$

Write Poincaré inequality on torus for the function $f_r(\theta)$. There exists a number c_r such that

$$\int_{\mathbb{T}^n} |f_r(\theta) - c_r|^p d\theta \leq (C_p(\mathbb{T}^n))^p \int_{\mathbb{T}^n} |\nabla_e f_r(\theta)|^p d\theta,$$

where $C_p(\mathbb{T}^n)$ is a Poincaré constant for \mathbb{T}^n .

Assume $c = \int_{R-1}^R c_r dr$ and we obtain

$$\begin{aligned} \|f - c\|_{\mathbb{L}^p(B(R))} &= \left\| \int_{R-1}^R (f - c_r) dr \right\|_{\mathbb{L}^p(B(R))} \\ &\leq \left\| \int_{R-1}^R (f - f_r) dr \right\|_{\mathbb{L}^p(B(R))} + \left\| \int_{R-1}^R (f_r - c_r) dr \right\|_{\mathbb{L}^p(B(R))} \\ &\leq \int_{R-1}^R (\|f - f_r\|_{\mathbb{L}^p(B(R))} + \|f_r - c_r\|_{\mathbb{L}^p(B(R))}) dr \\ &\leq C(p, \mu) \|\nabla f\|_{\mathbb{L}^p(B(R))} + C_p(\mathbb{T}^n) e^{\mu_n R} \|\nabla f\|_{\mathbb{L}^p(B(R) \setminus B(R-1))} \end{aligned}$$

17. LOWER BOUND ON POINCARÉ CONSTANT

Let Z_μ denote $\mathbb{T}^n \times \mathbb{R}$ equipped with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$, where we suppose $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$. In this section we will give a lower bound for the quasi-isometric distortion growth between two spaces $Z = Z_\mu$ and $Z' = Z_{\mu'}$, using our results on transported Poincaré inequalities. Let $O, O' = (0, \dots, 0)$ be base points of Z and Z' respectively. First we notice that the width of $\mathbb{T}^n \times (-\infty, 0]$ is finite so it is at finite distance from a ray $(-\infty, 0]$, so from now on, we shall focus our attention on the part of $B_Z(O, R)$ where $t \geq 0$.

Theorem 24. *Let Z, Z' be two locally homogeneous hyperbolic metric spaces with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$ and $dt^2 + \sum e^{2\mu'_i t} dx_i^2$ respectively, $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ and $0 < \mu'_1 \leq \mu'_2 \leq \dots \leq \mu'_n$. Assume also that $\sum \mu_i / \mu_n > \sum \mu'_i / \mu'_n$. Suppose that there exist constants a and b such that for any i $b \leq \mu_i, \mu'_i \leq a$. Then there exist constants $G_0(a, b), G_1(a, b)$ and $G_2(a, b)$ such that the following holds.*

- Let $\Theta : B_Z(R) \rightarrow Z'$ be a continuous $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding, inducing an isomorphism on fundamental groups. Suppose that Θ sends base point to base point, $\Theta(O) = O'$ and that $R > \text{const}(\lambda_1 + c_1 + 1)(\lambda_2 + c_2 + 1) / \mu'_n$ (with universal multiplicative constant). If $p > \sum \mu'_i / \mu'_n$, up to replacing Z with a connected 2-sheeted covering, Poincaré constant $C_p(\mu)$ for a ball of radius R in the space Z is bounded from below by

$$C_p(\mu) \geq (G_0(a, b))^{1/p} (\lambda_1 + c_1)^{-3/p-2/p^2} e^{-(9/p+3/p^2)(\lambda_1+c_1)} e^{(\sum \mu_i/p)R} \left(p - \sum \mu'_i / \mu'_n \right)^{1/p}.$$

- The distortion growth (see Definition 4) for quasi-isometrical embedding of $B_Z(R)$ into Z' is bounded from below by

$$D_G(R) \geq G_1 \left(\frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_i}{\mu'_n} \right) R - G_2,$$

provided that $R \geq R_0$ for some ineffective constant R_0 .

Remark 9. The assumption that Θ be continuous is not that restrictive: every quasi-isometric embedding is within bounded distance of a continuous quasi-isometric embedding, with a slight loss on additive constants.

If $\dim(Z) \geq 3$, the assumption that Θ be isomorphic on fundamental groups is not that restrictive either. In Lemma 17, we shall show that this is automatic, but unfortunately the argument introduces an ineffective constant, which therefore arises in the distortion growth estimate.

We will prove this theorem in several steps. First we introduce non-trivial double-covering spaces of \tilde{Z} and \tilde{Z}' of Z and Z' . We prove that Θ lifts to a $(\lambda_1, \lambda_2, 2c_1, c_2)$ -quasi-isometry. Then we take a test-function $e^{\pi i x_n}$ on \tilde{Z}' which depends only on one coordinate x_n . It varies very slowly outside of some ball, so the absolute value of the transported and regularised function v on \tilde{Z} stays near to 1. Lemmas 14 and 16 help us to control how the lower bound of Poincaré constant changes under transport. This helps us get a lower bound for Poincaré constant of \tilde{Z} in function of $\{\mu_i\}, \{\mu'_i\}$ and the constants of quasi-isometric embedding. We also have an upper bound for the Poincaré constant of \tilde{Z} by Theorem 23. The combination of these results provides a lower bound for the distortion growth for Z and Z' .

17.1. Quasi-isometric embeddings and fundamental groups.

Lemma 17. *Let Z, Z' be two spaces of the described form with equal dimensions $n+1 \geq 3$. Then for any $\lambda_1 \geq 1, \lambda_2 \geq 1, c_1 \geq 0, c_2 \geq 0$ there exists $R_0 = R_0(\lambda_1, \lambda_2, c_1, c_2)$ such that if $R > R_0$ and a continuous map $f : B_{Z_\mu}(O, R_0) \rightarrow Z_{\mu'}$ is a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding, then f induces an isomorphism on the fundamental groups $\pi_1(Z_\mu) \rightarrow \pi_1(Z_{\mu'})$.*

Proof. We provide a proof by contradiction. Assume that for arbitrarily large values of R , there exists a map $f_R : B_Z(R) \rightarrow Z'$ which is a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding which is not isomorphic on fundamental groups. Pick a $2c_1/\lambda_1$ -dense and c_1/λ_1 -discrete subset Λ of Z . Notice that if f_R is a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometry, then f_R is bi-Lipschitz on $B_Z(R) \cap \Lambda$. Conversely, if a map defined on $B(R) \cap \Lambda$ is bi-Lipschitz, then it can be continuously extended on $B(R)$ as a quasi-isometric embedding. Indeed, away from a ball, Z' is contractible up to scale c_1 .

If $d(O', f_R(O)) \rightarrow \infty$ then for R large enough f_R is homotopic to 0, hence f_R lifts to $\tilde{f}_R : B_Z(R) \rightarrow \tilde{Z}' = X_{\mu'}$ which is homogeneous. Now up to composing \tilde{f}_R with an isometry we can suppose that it preserves the center $\tilde{f}_R(O) = O'$. By Ascoli's theorem, we can find a sequence $\tilde{f}_{R_j}|_\Lambda$ which uniformly converges to $\tilde{f}|_\Lambda : Z \cap \Lambda \rightarrow \tilde{Z}'$ which is also bi-Lipschitz. We continuously extend $\tilde{f}|_\Lambda$ to $\tilde{f} : Z \rightarrow \tilde{Z}'$, \tilde{f} is a quasi-isometric embedding. Its extension to ideal boundaries is continuous and injective. By the theorem of invariance of domain, $\partial \tilde{f} : T^n \simeq \partial X_\mu = S^n$ is open, and thus a homeomorphism. This provides a contradiction if $n \geq 2$.

If $d(O', f_R(O))$ stays bounded, we can directly use Ascoli's theorem, and get a limiting continuous quasi-isometric embedding f . Again, f extends to the ideal boundary,

$\partial f : \partial Z \rightarrow \partial Z'$, the map ∂f is continuous and injective. Because ∂Z and $\partial Z'$ have the same dimension, ∂f is an open map by the theorem of invariance of domain and ∂f is a homeomorphism. Hence, ∂f induces an isomorphism on fundamental groups. If R_j is sufficiently large, then f_{R_j} is at bounded distance from f and hence f_{R_j} also induces an isomorphism $\pi_1(B_Z(R)) \rightarrow \pi_1(Z')$. This contradiction completes the proof. \square

Remark 10. The proof does not provide an effective value of R_0 .

17.2. Lifting to a double covering space. Introduce a double covering of Z' . Let $\tilde{Z}' = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1} \times \mathbb{R}/2\mathbb{Z} \times [0; +\infty)$. Consider the map $\tilde{Z}' \rightarrow Z'$ defined by

$$(x_1, x_2, \dots, x_n, t) \mapsto (x_1, x_2, \dots, x_n \pmod{1}, t).$$

So we identify $(x_1, x_2, \dots, x_n, t)$ and $(x_1, x_2, \dots, x_n + 1, t)$ in \tilde{Z}' . Consider a complex function $u(x_1, x_2, \dots, x_n, t) = e^{\pi i x_n}$ on \tilde{Z}' .

Composition of u with deck transformation $\iota' : \tilde{Z}' \rightarrow \tilde{Z}'$

$$\iota' : (x_1, x_2, \dots, x_n, t) \mapsto (x_1, x_2, \dots, x_n + 1, t)$$

gives $u \circ \iota' = -u$.

We have $\Theta : Z \rightarrow Z'$ which is a continuous map inducing an isomorphism in fundamental groups, and we have \tilde{Z}' which is a covering space of Z' . We need to show that there exists a non-trivial covering space $\tilde{Z} \rightarrow Z$ such that the following diagram commutes.

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{\Theta}} & \tilde{Z}' \\ \pi_Z \downarrow & & \downarrow \pi_{Z'} \\ Z & \xrightarrow{\Theta} & Z' \end{array}$$

Define

$$\tilde{Z} = \{(z, \tilde{z}') \mid z \in Z, \tilde{z}' \in \pi_{Z'}^{-1}(\Theta(z))\},$$

that is $\tilde{Z} \subset Z \times \tilde{Z}'$. Let $[\gamma']$ be a loop in Z' which does not lift to a loop in \tilde{Z}' . By hypothesis, there exists a loop γ in Z such that $\Theta(\gamma)$ is homotopic to γ' . Then γ does not lift to a loop in \tilde{Z} . There exists an isometry ι of order 2 on \tilde{Z} such that $\tilde{\Theta} \circ \iota = \iota' \circ \tilde{\Theta}$.

17.3. Θ lifts to a quasi-isometric embedding. Here we will prove that in the constructed double coverings Θ lifts to a $(\lambda_1, \lambda_2, 2c_1, c_2)$ -quasi-isometric embedding. We need two preliminary lemmas concerning distances in two-fold coverings.

Lemma 18. *Let $Z = Z_\mu$ be a locally homogeneous space. Let z be a point in Z in the region where $t \geq 0$. Let $c = t(z)$. Every loop based at z' of length less than $\frac{c}{4}$ is null-homotopic.*

Proof. Let γ be a non null-homotopic geodesic loop at z . Assume that its length is $< \frac{c}{4}$. Let $\sigma :]-\infty, c]$ be a geodesic ray joining the unique ideal boundary point of the part of Z where $t \leq 0$ to z . Let γ_t be the geodesic loop based at $\sigma(t)$, homotopic to $\sigma_{[t,c]}^{-1} \gamma \sigma_{[t,c]}$. The length of γ_t is a nonnegative, convex function of t , which tends to 0 as t tends to infinity, so it is nondecreasing. Therefore $\text{length}(\gamma_0) \leq \frac{c}{4}$. The null-homotopic loop $\gamma_0^{-1} \sigma^{-1} \gamma \sigma$ lifts to a geodesic quadrangle $ABCD$ in the universal covering X of Z , with $t(A) = t(B) = 0$, $t(C) = t(D) = c$. Let δ denote the hyperbolicity constant of X . By definition, there exist

points P and Q in X such that P (resp. Q) sits at distance at most δ from all sides of the triangle ABC (resp. ACD). Since $t(P) \leq \frac{c}{4}$, $t(Q) \geq c - \frac{c}{4}$, there exist points P' and $Q' \in AD$ such that $d(P', BC) \leq 2\delta$, $d(Q', BC) \leq 2\delta$ and $d(P', Q') \geq \frac{c}{2} - 4\delta$.

Let $-a$ be the upper bound of the sectional curvature of X . Then.... \square

Lemma 19. *Let γ be a geodesic in Z with end-points z_3, z_4 , the length of γ be b . Let z_2 be a point at distance a from the base point $z_1 = O$. Assume also that $d(z_1, z_4) = d(z_2, z_3) = t$. Then there exist two points u_1 on z_1z_4 and u_2 on z_2, z_3 such that*

$$d(u_1, u_2) \leq .$$

Proof. Up to a normalisation, we can suppose that the curvature of Z $K \leq -1$. Then we can compare it with a $CAT(-1)$ space X . The metric on X in polar coordinates is

$$ds^2 = dr^2 + \sinh^2 r d\theta^2.$$

Consider a triangle $z_1z_3z_4$, let a triangle $x_1x_3x_4 \subset X$ be a triangle of comparison for it. The geodesic x_3x_4 lies in the area $\{t - b \leq r \leq t + b\}$. Then we have for the angle $\angle x_3x_1x_4 = \theta$

$$\theta \leq \int_{\gamma} d\theta \leq \int_{\gamma} \frac{ds}{\sinh r} \leq \frac{1}{\sinh(t-b)} \int_{\gamma} ds = \frac{b}{\sinh(t-b)}.$$

Let u_1 be a mid-point of z_1z_4 , that is $d(z_1, u_1) = t/2$, v_1 is a corresponding point on x_1x_4 . Let u' be a point of z_1z_3 such that $d(z_1, u') = t/2$, its corresponding point on x_1x_3 is v' . Then

$$d(u_1, u') \leq d(v_1, v') \leq \int \sinh \frac{t}{2} d\theta = \theta \sinh \frac{t}{2} \leq b \frac{\sinh(t/2)}{\sinh(t-b)},$$

where we integrated along an arc of a circle between v_1 and v' centred in x_1 .

In the same manner we consider the triangle $z_1z_2z_3$. Take a point u_2 on z_2z_3 such that $d(z_3u_2) = d(z_3u') = d(z_1z_3) - t/2$. Then

$$d(u'u_2) \leq a \frac{\sinh(t' - t/2)}{\sinh(t-a)}.$$

\square

Lemma 20. *Let z_1, z_2 be two points in Z such that $d(O', \Theta(z_1)) > c_1$ or $d(O', \Theta(z_2)) > c_1$ and $d(z_1, z_2) \leq c_1/\lambda_1$. Then $d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) = d(\Theta(z_1), \Theta(z_2))$.*

Proof. Let $\tilde{z}_1 \in \tilde{Z}$ be such that $d(\tilde{O}, \tilde{z}_1) > c_1$. Set

$$\begin{aligned} W &= \{\tilde{z}_2 \in \tilde{Z} \mid d(\tilde{z}_1, \tilde{z}_2) \leq c_1\}, \\ U &= \{\tilde{z}_2 \in W \mid d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) = d(\Theta(z_1), \Theta(z_2))\} \subset W, \\ V &= \{\tilde{z}_2 \in W \mid d(\tilde{\Theta}(\tilde{z}_1), \iota' \circ \tilde{\Theta}(\tilde{z}_2)) = d(\Theta(z_1), \Theta(z_2))\} \subset W. \end{aligned}$$

By construction, $W = U \cup V$. Let us show that the intersection of U and V is empty

$$U \cap V = \{\tilde{z}_2 \in W \mid d(\tilde{\Theta}(\tilde{z}_1), \iota' \circ \tilde{\Theta}(\tilde{z}_2)) = d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2))\}.$$

If $(\tilde{z}_1, \tilde{z}_2) \in U \cap V$, then the geodesic segments connecting $\tilde{\Theta}(\tilde{z}_1)$ with $\tilde{\Theta}(\tilde{z}_2)$ and $\tilde{\Theta}(\tilde{z}_1)$ with $\iota' \circ \tilde{\Theta}(\tilde{z}_2)$ induce a loop γ in Z' of length $2d(\Theta(z_1), \Theta(z_2)) \leq 2(\lambda_1(c_1/\lambda_1) + c_1) = 4c_1$ which

is not homotopic to 0. According to Lemma 18, this is incompatible with the assumption that $d(O', \Theta(z_1)) > c_1$. Hence, $U \cap V$ is empty. Since U is non-empty (it contains at least \tilde{z}_1) and closed in W , V is closed in W and W is connected, we conclude that $U = W$, which finishes the proof. \square

Lemma 21. *A $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding $\Theta : Z \rightarrow Z'$ lifts to a $(\lambda_1, \lambda_2, 2c_1, c_2)$ -quasi-isometric embedding $\tilde{\Theta} : \tilde{Z} \rightarrow \tilde{Z}'$.*

Proof. The lefthand inequality in the definition of quasi-isometric embedding is evident as in a covering space distances cannot diminish. Let $\tilde{\gamma} \subset \tilde{Z}$ be a geodesic between \tilde{z}_1 and \tilde{z}_2 . Let t_1 be the first point such that $d(\tilde{\Theta}\tilde{\gamma}(t), \tilde{O}') \leq c_1$ and t_2 be the last point with such a property. Then

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) \leq d(\tilde{\Theta}\tilde{\gamma}(t_1), \tilde{\Theta}\tilde{\gamma}(t_2)) + d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_1)) + d(\tilde{\Theta}(\tilde{z}_2), \tilde{\Theta}\tilde{\gamma}(t_2)).$$

By definition of t_1 and t_2 $d(\tilde{\Theta}\tilde{\gamma}(t_1), \tilde{\Theta}\tilde{\gamma}(t_2)) \leq 2c_1$. Now divide parts of γ between $\tilde{\Theta}(\tilde{z}_1)$ and $\tilde{\Theta}\tilde{\gamma}(t_1)$ and between $\tilde{\Theta}(\tilde{z}_1)$ and $\tilde{\Theta}\tilde{\gamma}(t_2)$ by segments of length c_1/λ_1 . We apply the previous lemma to them, so

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_1)) + d(\tilde{\Theta}(\tilde{z}_2), \tilde{\Theta}\tilde{\gamma}(t_2)) \leq N \left(\lambda_1 \frac{c_1}{\lambda_1} + c_1 \right),$$

where $N \leq d(\tilde{z}_1, \tilde{z}_2)/(c_1/\lambda_1)$ is a number of segments in the subdivision. So,

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) \leq 2c_1 + 2\lambda_1 d(\tilde{z}_1, \tilde{z}_2).$$

\square

17.4. Proof of the first statement of Theorem 24. Let ψ' be a kernel on \tilde{Z} invariant by isometry, that is for any isometry ι

$$\psi'(\iota(\tilde{z}_1), \iota(\tilde{z}_2)) = \psi'(\tilde{z}_1, \tilde{z}_2).$$

Let also ϕ be a cross-kernel constructed with the quasi-isometry $\tilde{\Theta}$ and a kernel ζ on \tilde{Z}' which is also invariant by isometries. Define a complex function v on \tilde{Z} as follows

$$v = (u * \phi) * \psi'.$$

Then $v \circ \iota = -v$. Indeed,

$$v \circ \iota = (u * \phi) * \psi' \circ \iota = (u * \phi \circ \iota) * \psi'.$$

On the other hand,

$$\begin{aligned} u * \phi \circ \iota &= \int u(\tilde{z}') \phi(\iota\tilde{z}, \tilde{z}') d\tilde{z}' = \int u(\tilde{z}') \zeta(\iota'\tilde{\Theta}(\tilde{z}), (\iota')^2\tilde{z}') d\tilde{z}' = \\ &= \int u(\tilde{z}') \zeta(\tilde{\Theta}(\tilde{z}), \iota'\tilde{z}') d\tilde{z}' = \int u(\iota'\tilde{z}') \zeta(\tilde{\Theta}(\tilde{z}), \tilde{z}') d\tilde{z}' = -u * \phi \end{aligned}$$

hence, v is skewsymmetric with respect to ι . We get immediately that $\int v = 0$. Now we apply successively Lemma 14 and Lemma 16.

Step 1. By Lemma 14 there exists a kernel ψ_1 on \tilde{Z} which is controlled by a and b and such that

$$\left(\int |\nabla(u * \phi * \psi')|^p \right)^{1/p} \leq N_{\psi_1}(u * \phi),$$

where for ψ_1 we have the width of support is $R^{\psi_1} = R^{\psi'}$ and

$$\sup \psi_1 \leq \frac{\sup \nabla \psi' \sup \psi'}{\inf_z \text{vol} B(\tilde{z}, R^{\psi'})}.$$

Step 2. By Lemma 16 there exists a kernel ζ_1 on \tilde{Z}' such that

$$N_{\psi_1}(u * \phi) \leq \tilde{C} N_{\zeta_1}(u),$$

where the width of support of ζ_1 is $2R^\zeta + \lambda_1 R^{\psi'} + c_1$, the supremum of ζ_1 is

$$\sup \zeta_1 = \frac{\sup \psi_1}{c_\tau^Y} e^{2R^\zeta + \lambda_1 R^{\psi'} + c_1} (2\lambda_1 R^\zeta + c_1)^2$$

and

$$\tilde{C} = \frac{1}{c_\tau^Y} (\sup \psi_1)^{3/p} e^{((2+\lambda_1)R^{\psi'} + c_1)/p} \left((2 + \lambda_1)R^{\psi'} + c_1 \right)^{2/p}.$$

Step 3. Applying Lemma 14 we get that there exists a kernel ζ_2 on \tilde{Z}' such that

$$N_{\zeta_2}(u) \leq C(n) \|\nabla u\|_p,$$

we remind that the constant $C(n)$ depends only on the dimension of \tilde{Z}' if the Ricci curvature is bounded from below, that is $\sup \mu_i$ is bounded.

Step 4. Here we merely need to pass from N_{ζ_1} to N_{ζ_2} . We apply Lemma 14 once more

$$N_{\zeta_1} \leq \hat{C} N_{\zeta_2},$$

where

$$\hat{C} = \frac{\sup \zeta_1 \sup \zeta_2}{c_\tau^Y} \frac{R^{\zeta_2}}{\varepsilon^{\zeta_2}} (2e)^{(2R^\zeta + \lambda_1 R^{\psi'} + c_1)/\varepsilon^{\zeta_2}}.$$

Choose ψ' and ζ such that $R^{\psi'} = 1$ and $R^\zeta = 1$. Then $\sup \psi'$ and $\sup \zeta$ are controlled by a and b . We note also that $\varepsilon^{\zeta_2} = 1$. So combining all inequalities we get

$$\int_{B(R)} |\nabla v|^p \leq C_1(a, b) (\lambda_1 + c_1)^{3+2/p} e^{(9+3/p)(\lambda_1 + c_1)} \int_{\mathbb{T}^n \times [0, +\infty]} |\nabla u|^p,$$

where $C_1(a, b)$ is a constant depending only on a , b and dimension n . Denote $Q = \lambda_1 + c_1$. The distortion growth $D_G \geq 1/2Q$ so we will establish a lower bound for Q now. Assume

$$C(Q) = (\lambda_1 + c_1)^{3+2/p} e^{(9+3/p)(\lambda_1 + c_1)}.$$

Let us compute $|v(\tilde{z})|$ for \tilde{z} which is rather far from the center. In fact we require that $d(\tilde{O}, \tilde{z}) \geq R_0 = \lambda_2/\mu'_n \log 8\pi + 2\lambda_2(\lambda_1 + c_1) + c_2 + 1$.

$$\begin{aligned}
|(u * \phi) * \psi'(\tilde{z})| &= \left| \int_X \int_Y u(\tilde{z}') \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\
&\geq \left| \int_X \int_Y (u(\tilde{z}') - u(\tilde{\Theta}(\tilde{z})) + u(\tilde{\Theta}(\tilde{z}))) \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\
&\geq \left| \int_X \int_Y (u(\tilde{\Theta}(\tilde{z}))) \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\
&\quad - \left| \int_X \int_Y (u(\tilde{z}') - u(\tilde{\Theta}(\tilde{z}))) \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\
&\geq 1 - \int_X \int_Y |u(\tilde{z}') - u(\tilde{\Theta}(\tilde{z}))| \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') \psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1.
\end{aligned}$$

$\psi'(\tilde{z}, \tilde{z}_1)$ is non-zero if $d(\tilde{z}, \tilde{z}_1) \leq R^{\psi'} = 1$ and $\zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}')$ is non-zero if $d(\tilde{z}', \tilde{\Theta}(\tilde{z}_1)) \leq R^\zeta = 1$. So the diameter of the set \hat{S} of points \tilde{z}' such that the integrand is non-zero, is at most $2\lambda_1 + c_1 + 2 \leq 4(\lambda_1 + c_1)$. Assume $\hat{z}' = \tilde{\Theta}(\tilde{z}) \in \hat{S}$. Then by the mean value theorem, for any point $\tilde{z}' \in \hat{S}$,

$$\begin{aligned}
|u(\tilde{z}') - u(\hat{z}')| &\leq |\tilde{z}' - \hat{z}'| \sup_{\tilde{z}' \in \hat{S}} |\nabla u(\tilde{z}')| \leq 4(\lambda_1 + c_1) \sup_{\tilde{z}' \in \hat{S}} \left| \frac{\partial u}{\partial \tilde{x}_n} \right| e^{-\mu'_n t} \leq 4\pi(\lambda_1 + c_1) e^{-\mu'_n t} \\
&\leq 4\pi(\lambda_1 + c_1) \sup_{\hat{z}' \in \hat{S}} e^{-\mu'_n d(\hat{O}', \hat{z}')} \leq 4\pi(\lambda_1 + c_1) e^{-\mu'_n ((R_0 - 1 - c_2)/\lambda_2 - 2(\lambda_1 + c_1))} \leq \frac{1}{2}.
\end{aligned}$$

Hence we have proved that

$$\frac{1}{2} \leq |(u * \phi) * \psi'(z)| \leq 1.$$

And we conclude from this relation that for $R > R_0$

$$\int_{B(R)} |v|^p \geq \frac{1}{2^p} \text{vol}(B(R)) - \text{vol}(B(R_0)) \geq e^{(\sum \mu_i)R} / 2^{p+1}.$$

Let us compute the integral $\int |\nabla u|^p$.

$$\int |\nabla u|^p = \int \left| \frac{\partial u}{\partial x_n} \right|^p e^{-\mu'_n p t} e^{(\sum \mu'_i) t} dt dx_n = \pi \int_0^{+\infty} e^{(\sum \mu'_i - p \mu'_n) t} dt = \frac{\mu'_n \pi}{-\sum \mu'_i / \mu'_n + p}.$$

Hence the Poincaré constant $C_p(\mu)$ for Z satisfies

$$\begin{aligned}
(C_p(\mu))^p &\geq \frac{\|v\|^p}{\|\nabla v\|^p} \geq \frac{\|v\|^p}{C_1(a, b) C(Q) \|\nabla u\|^p} \\
&\geq (\mu'_n \pi 2^{p+1} C_1(a, b) C(Q))^{-1} e^{(\sum \mu_i)R} (p - \sum \mu'_i / \mu'_n).
\end{aligned}$$

This proves the first claim in Theorem 24.

17.5. Proof of the second statement of Theorem 24. Let $\Theta : B_Z(R) \rightarrow Z'$ be a $(\lambda_1, \lambda_2, c_1, c_2)$ -quasi-isometric embedding. Approximate Θ with a continuous quasi-isometric embedding. According to Lemma 17, for R large enough (the bound is not effective, however), Θ is isomorphic on fundamental groups. Lemma 18 implies that Θ moves the origin a bounded distance away. Indeed, a non null-homotopic loop of length 1 based at O is mapped to a non null-homotopic loop of length $\leq Q = \lambda_1 + c_1$ based at $\Theta(O)$. This implies that $t(\Theta(O)) \leq 4Q$ and $d(O', \Theta(O)) \leq 4Q + 1$.

The space \tilde{Z} is of the form $\tilde{T} \times \mathbb{R}$ where $\tilde{T} \rightarrow T$ is a connected 2-sheeted covering space of torus, that is \tilde{T} is also a torus. Hence we can apply Theorem 23. We have $C_p(\mu) \leq C_2(a, b)e^{\mu_n R}$. So we arrive to

$$(\mu'_n \pi 2^{p+1} C_1(a, b) C(Q))^{-1/p} e^{(\sum \mu_i/p)R} \left(p - \sum \mu'_i/\mu'_n \right)^{1/p} \leq C_2(a, b) e^{\mu_n R}.$$

Hence with $C_3(a, b) = (\mu'_n \pi 2^{p+1} C_1(a, b))^{1/p} C_2(a, b)$,

$$C_3(a, b) C(Q) \geq e^{(\sum \mu_i/p - \mu_n)R} \left(p - \frac{\sum \mu'_i}{\mu'_n} \right)^{1/p}.$$

We have calculated that $C(Q) = Q^{3+2/p} e^{(9+3/p)Q}$. Combining these results, assuming $p = \sum \mu'_i/\mu'_n + 1/R$ and using the fact that $Q \geq \log Q$, we get for R large enough

$$Q \geq G_1(a, b) \left(\frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_n}{\mu'_n} \right) R - G_2(a, b)$$

with $G_1(a, b)$ and $G_2(a, b)$ being constants depending only on a and b .

This finishes the proof of Theorem 24.

Part 4. Examples of different distortion growths

18. APPROXIMATION OF DISTANCES AND AN EXAMPLE OF QI

Let X, Y be two geodesic hyperbolic metric spaces with base points $x_0 \in X, y_0 \in Y$. Let $\theta : \partial X \rightarrow \partial Y$ be a homeomorphism between ideal boundaries.

Hypothesis 1. Assume that there exists a constant D such that for any $x \in X$ there exists a geodesic ray γ from the base point $\gamma(0) = x_0$ and passing near x : $d(x, \gamma) < D$.

We are going to construct approximatively (up to D) a map Θ extending the boundary homeomorphism θ . Take some point x and a geodesic ray γ from x_0 passing near x : $d(\gamma, x) < D$. Then $\gamma(\infty)$ is a point on ideal boundary ∂X . The corresponding point $\theta(\gamma(\infty)) \in \partial Y$ defines a geodesic ray γ' such that $\gamma'(0) = y_0$ and $\gamma'(\infty) = \theta(\gamma(\infty))$. Set $\Theta(x) = \gamma'(d(x_0, x))$. So, by construction, Θ preserves the distance to the base point. Still, it depends on the choices of γ and γ' .

Definition 22. Define the following quantity

$$K(R) = \sup \left\{ \left| \log \frac{d_{y_0}(\theta(\xi_1), \theta(\xi_2))}{d_{x_0}(\xi_1, \xi_2)} \right| \mid d_{x_0}(\xi_1, \xi_2) \geq e^{-R} \vee d_{x_0}(\theta(\xi_1), \theta(\xi_2)) \geq e^{-R} \right\}.$$

We are going to prove that Θ is a $\left(1 + 2\frac{K(R)}{D+\delta}, D + \delta + 2K(R)\right)$ -quasi-isometry. We begin with a Lemma which gives an approximation (up to an additive constant) of the distance between two points in a hyperbolic metric space. In its proof, all equalities hold with a bounded additive error depending linearly on δ .

Lemma 22. *Let P_1, P_2 be two points in a hyperbolic metric space Z . Let P_0 be a base point (possibly at infinity). Let distances (horo-distances if P_0 is at infinity) from P_1 and P_2 to P_0 be $d(P_1, P_0) = t_1$ and $d(P_2, P_0) = t_2$. Assume that there exists points P_1^∞ and P_2^∞ such that P_1 (resp. P_2) belongs to the geodesic ray defined by P_0 and P_1^∞ (resp. P_2^∞). Denote by*

$$t_\infty = -\log \text{visdist}_{P_0}(P_1^\infty, P_2^\infty)$$

the logarithm of visual distance seen from P_0 . Then up to adding a multiple of δ ,

$$d(P_1, P_2) = t_1 + t_2 - 2 \min\{t_1, t_2, t_\infty\}.$$

Proof. Let P'_0 be a projection of P_0 on the geodesic $P_1^\infty P_2^\infty$. By Lemma 5, P'_0 lies at distance at most 2δ from both $P_0 P_1^\infty$ and $P_0 P_2^\infty$. Hence, up to an additive constant bounded by 4δ the distance between P_0 and P'_0 is equal to Gromov's product of P_1^∞ and P_2^∞ . It follows that $t_\infty = d(P_0, P'_0) = -\log \text{visdist}(P_1, P_2)$.

The triangle $P_0 P_1^\infty P_2^\infty$ is δ -thin. Notice that if P_1 (or P_2) lies near the side $P_1^\infty P_2^\infty$ then $t_1 \geq t_\infty$. Otherwise, $t_1 \leq t_\infty$ (both inequalities are understood up to an additive error δ). This follows from the definition of the point P'_0 as a projection and Lemma 5.

Hence, if $t_1, t_2 \geq t_\infty$, $d(P_1, P_2) = d(P_1, P_0) + d(P_2, P_0) - 2d(P_0, P'_0) = t_1 + t_2 - 2t_\infty$.

If $t_1 \leq t_\infty \leq t_2$, $d(P_1, P_2) = d(P_1, P'_0) + d(P'_0, P_2) = t_2 - t_1$.

Finally, if $t_1, t_2 \leq t_\infty$, we get $d(P_1, P_2) = |t_1 - t_2| = t_1 + t_2 - 2 \min\{t_1, t_2\}$ as P_1 lies near $P_0 P_2^\infty$. □

Lemma 23. *Let Z and Z' be two hyperbolic metric spaces. Let Θ be the radial extension of a boundary homeomorphism θ , as described at the beginning of this section. Then for any two points $P_1, P_2 \in B(P_0, R) \subset Z$ such that $d(P_1, P_2) > c$, we have*

$$\frac{d_{Z'}(\Theta(P_1), \Theta(P_2))}{d_Z(P_1, P_2)} \leq 1 + 2\frac{K(R)}{c}.$$

If $d(P_1, P_2) < c$,

$$d_{Z'}(\Theta(P_1), \Theta(P_2)) < 2K(R) + c.$$

Proof. We will use the same notations as in Lemma 22. Visual distance d_Z^∞ between P_1^∞ and P_2^∞ and the (horo-)distance t_∞ from P_0 to $P_1^\infty P_2^\infty$ are connected by the relation $e^{-t_\infty} = d_\infty(P_1^\infty, P_2^\infty)$. In the same way we define t'_∞ as the (horo-)distance for corresponding images.

By Lemma 22 we know that $d(P_1, P_2) = t_1 + t_2 - 2 \min\{t_1, t_2, t_\infty\}$.

Assume first $d(P_1, P_2) > c$. We will write $d_Z = d(P_1, P_2)$ for the distance between P_1 and P_2 and $d_{Z'} = d(\Theta(P_1), \Theta(P_2))$ for the distance between their images.

We have to consider four cases depending on the relative sizes of t_1, t_2, t_0 and t'_∞ as they determine values of minima defining d_Z and $d_{Z'}$. Without loss of generality, we may assume that $t_1 \leq t_2$.

1st case If both $t_1 < t_\infty$ and $t_1 < t'_\infty$, then

$$\frac{d_{Z'}}{d_Z} = \frac{t_2 - t_1}{t_2 - t_1} = 1,$$

and this case is trivial.

2nd case If $t_\infty < t_1$ and $t'_\infty < t_1$. We have to give an upper bound for

$$\frac{d_{Z'}}{d_Z} = \frac{t_1 + t_2 - 2t'_\infty}{t_1 + t_2 - 2t_\infty}.$$

Consider

$$t'_\infty - t_\infty = \log \frac{d_\infty(\theta(P_1^\infty), \theta(P_2^\infty))}{d_\infty(P_1^\infty, P_2^\infty)}.$$

Because $d_Z > c$, we have $t_1 + t_2 - 2t_\infty > c$ hence $e^{(t_1+t_2)/2}e^{-t_\infty} > e^{c/2}$. And as $t_1, t_2 \leq R$ we obtain for visual distance $d_Z^\infty \geq e^{c/2}e^{-R} \geq e^{-R}$. We conclude that

$$|t'_\infty - t_\infty| \leq K(R).$$

Finally,

$$\frac{d_{Z'}}{d_Z} = \frac{d_{Z'} - d_Z + d_Z}{d_Z} = 1 + \frac{t'_\infty - t_\infty}{t_1 + t_2 - t_\infty} \leq 1 + \frac{1}{c}|t'_\infty - t_\infty|.$$

3d case Now let $t_\infty < t_1 < t'_\infty$. Then

$$d_{Z'} - d_Z = t_2 - t_1 - (t_1 + t_2 - 2t_\infty) = 2(t_\infty - t_1) \leq 0,$$

which leads to

$$\frac{d_{Z'}}{d_Z} \leq 1.$$

4th case Finally if $t'_\infty < t_1 < t_0^\infty$ then

$$d_{Z'} - d_Z = (t_1 + t_2 - 2t'_\infty) - (t_2 - t_1) = 2(t_1 - t'_\infty) \leq 2(t_0^\infty - t'_\infty).$$

We know that $t_1 \leq R$ and at the same time we have $t'_\infty < t_1$, hence $t'_\infty < R$ and visual distance between $P_1^{\infty'}$ and $P_2^{\infty'}$ is at least e^{-R} . Now as in the 2nd case we obtain that $t_0^\infty - t'_\infty \leq K(R)$ and hence

$$\frac{d_{Z'}}{d_Z} \leq 1 + 2\frac{K(R)}{c}.$$

Now assume that $d_Z(P_1, P_2) \leq c$ (we still suppose $t_1 \leq t_2$), hence the distance $t_\infty > t_2$ and we are either in first or fourth situation. In the first case, $t_1 < t_\infty$ and $t_1 < t'_\infty$ so $d_{Z'} = d_Z \leq c$. In the fourth case, we have still $d_{Z'} - d_Z \leq 2K(R)$ and hence $d'_{Z'} \leq c + 2K(R)$. \square

Applying the Lemma both to Θ and Θ^{-1} , we get the following Theorem.

Theorem 25. *Let X, Y be two geodesic hyperbolic metric spaces with base points $x_0 \in X$, $y_0 \in Y$. Assume that there exists a constant D such that for any $x \in X$ there exists a geodesic ray γ from the base point $\gamma(0) = x_0$ and passing near x : $d(x, \gamma) < D$ (Hypothesis 1). Let the restriction of $\Theta : \partial X \rightarrow \partial Y$ be a homeomorphism between ideal boundaries. Then Θ is a (λ, C_q) -quasi-isometry, where $\lambda = 1 + 2\frac{K(R)}{c}$ and $C_q = 2K(R) + c$. The constant c can be chosen as $c = D + \delta$ where δ is the hyperbolicity constant.*

19. EXAMPLES

19.1. **Bi-Hölder maps.** Let θ be a bi-Hölder map:

$$\begin{aligned} d(\theta(\xi_1), \theta(\xi_2)) &\leq cd(\xi_1, \xi_2)^\alpha, \alpha < 1, \\ d(\theta(\xi_1), \theta(\xi_2)) &\geq \frac{1}{c}d(\xi_1, \xi_2)^\beta, \beta > 1. \end{aligned}$$

Assume first that for two points ξ_1, ξ_2 of the ideal boundary, the visual distance $d(\xi_1, \xi_2) > e^{-R}$. Then we have

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \leq \log cd(\xi_1, \xi_2)^{\alpha-1} = -(1-\alpha) \log d(\xi_1, \xi_2) \lesssim (1-\alpha)R.$$

Now, if the visual distance between images of ξ_1 and ξ_2 satisfies $d(\theta(\xi_1), \theta(\xi_2)) > e^{-R}$, we get

$$d(\xi_1, \xi_2) \geq \frac{1}{c^{1/\alpha}} e^{-R/\alpha}$$

and hence

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \gtrsim \frac{1-\alpha}{\alpha} R.$$

We obtain the lower bound for $\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)}$ just in the same way as the upper-bound. If $d(\xi_1, \xi_2) > e^{-R}$

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \geq \log \frac{1}{c} d(\xi_1, \xi_2)^{\beta-1} = -(1-\beta) \log d(\xi_1, \xi_2) \lesssim (1-\beta)R.$$

If $d(\theta(\xi_1), \theta(\xi_2)) > e^{-R}$

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \geq \log \frac{1}{c} d(\theta(\xi_1), \theta(\xi_2))^{(\beta-1)/\beta} = -\frac{1-\beta}{\beta} \log d(\theta(\xi_1), \theta(\xi_2)) \gtrsim \frac{1-\beta}{\beta} R.$$

This gives

$$K(R) \lesssim \max\{1-\alpha, 1-\beta\}R.$$

In particular, consider two variants of the space $T^n \times [0, +\infty)$ Z and Z' with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$ and $dt^2 + \sum e^{2\mu'_i t} dx_i^2$ respectively. The visual distance between points P_1 and P_2 is given by

$$d_\infty(P_1, P_2) \sim \max |x_i^1 - x_i^2|^{1/\mu_i}.$$

Pick the identity map $\theta : \partial Z \rightarrow \partial Z'$. Then

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \sim \frac{\max_i |x_i^1 - x_i^2|^{1/\mu'_i}}{\max_i |x_i^1 - x_i^2|^{1/\mu_i}} \leq \max_i |x_i^1 - x_i^2|^{1/\mu'_i - 1/\mu_i}.$$

Suppose that $d(P_1, P_2) > e^{-R}$. Then

$$\begin{aligned} \left| \log \frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \right| &\leq \left| \log \max_i |x_i^1 - x_i^2|^{1/\mu'_i - 1/\mu_i} \right| = \\ &= \max_i \left(\mu_i \left| \frac{1}{\mu'_i} - \frac{1}{\mu_i} \right| \left| \log |x_i^1 - x_i^2|^{1/\mu_i} \right| \right) \leq \max_i \left| \frac{\mu_i}{\mu'_i} - 1 \right| R. \end{aligned}$$

So, we conclude that $K(R) = |\max_i (\mu_i/\mu'_i) - 1| R$.

Remark 11. More generally, such bi-Hölder maps exist between boundaries of arbitrary simply connected Riemannian manifolds with bounded negative sectional curvature. The Hölder exponent is controlled by sectional curvature bounds.

19.2. Unipotent locally homogeneous space. Now assume the space Z is a quotient $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}$ of the space $\mathbb{R}^2 \times \mathbb{R}$ with the metric $dt^2 + e^{2t}(dx^2 + dy^2)$. Consider the space $Z' = \mathbb{R}^2/\mathbb{Z}^2 \times_\alpha \mathbb{R}$, quotient of the space $\mathbb{R}^2 \times_\alpha \mathbb{R}$, where α is the 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The locally homogeneous metric is of the form $dt^2 + g_t$ where $g_t = (e^{t\alpha})^* g_0$

$$e^{t\alpha} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t x + te^t y \\ e^t y \end{pmatrix}$$

and so $g_t = d(e^t x + te^t y)^2 + d(e^t y)^2 = e^{2t}(dx^2 + 2tdxdy + (t^2 + 1)dy^2)$.

Let $\theta : \partial Z \rightarrow \partial Z'$ be the identity. Consider two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in Z . We will write $x = x_1 - x_2$ and $y = y_1 - y_2$. For the visual distance between P_1, P_2 we have

$$d_\infty(P_1, P_2) = \max\{|x|, |y|\}.$$

For their images $\theta(P_1)$ and $\theta(P_2)$ (see section 5 of [27] and [28])

$$d_\infty(\theta(P_1), \theta(P_2)) = \max\{|y|, |x - y \log |y|\}.$$

First we will give an upper-bound for $\log(d_\infty(\theta(P_1), \theta(P_2))/d_\infty(P_1, P_2))$. We have four different cases.

1st case. If $|x| < |y|$ and $|x - y \log |y|| < |y|$,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = 1.$$

2nd case. If $|x - y \log |y|| < |y| < |x|$,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} < 1.$$

3d case. If $|x| < |y| < |x - y \log |y||$.

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y \log |y||}{|y|} \leq \frac{|x|}{|y|} + |\log |y||.$$

If $d_\infty(P_1, P_2) > e^{-R}$ we have $e^{-R} < |y| \leq 1$ (the upper bound follows from the fact that y is a coordinate of a point of a torus) and hence $|\log |y|| \leq R$ and we finish as follows,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \leq \frac{|x|}{|y|} + |\log |y|| \leq 1 + R.$$

If $d_\infty(\theta(P_1), \theta(P_2)) > e^{-R}$ we will consider two situations.

- If $|x| > |y \log |y||$ then $|x - y \log |y|| < 2|x|$ and as $|x| < |y|$

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \leq 2.$$

- If $|x| < |y \log |y||$ then $e^{-R} < |x - y \log |y|| < 2|y \log |y||$ and hence $|\log |y|| < R$, so

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \leq 1 + R.$$

4th case. Let now $|y| < |x|$ and $|y| < |x - y \log |y||$

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y \log |y||}{|x|} \leq 1 + \frac{|y \log |y||}{|x|}.$$

We will check two possibilities.

- If $|y| \leq |x|^2$ then

$$\frac{|y \log |y||}{|x|} = \frac{|y|^{1/2}}{|x|} \left| |y|^{1/2} \log |y| \right| \leq 1.$$

- Now suppose that $|y| \geq |x|^2$. If $d_\infty(P_1, P_2) > e^{-R}$, we see easily that $|y| \geq e^{-2R}$ and hence

$$\frac{|y \log |y||}{|x|} \leq \frac{|x \log |y||}{|x|} \leq |\log |y|| \leq 2R.$$

If $d_\infty(\theta(P_1), \theta(P_2)) > e^{-R}$ we use the fact that $|a + b| \geq 2 \max\{|a|, |b|\}$. Hence, either $|x| > e^{-R}/2$ or $|y \log |y|| > e^{-R}/2$ and so $|y| \gtrsim e^{-R}$ and we finish the estimation as earlier.

So in the fourth case we have also

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \leq 2R.$$

Here, we have proved that $\log(d_\infty(\theta(P_1), \theta(P_2))/d_\infty(P_1, P_2)) \leq \log R$. Now we proceed to give also a lower bound for this expression.

1st case. If $|x| < |y|$ and $|x - y \log |y|| < |y|$

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = 1.$$

2nd case. If $|x - y \log |y|| < |y| < |x|$

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|y|}{|x|}.$$

Without loss of generality, assume $x > 0$. By the construction of Z , $|y| < 1$ hence $\log |y| < 0$. If $0 < x \leq y \log |y|$, we have $y < 0$. Now transform $x \leq y \log |y|$ as $1 \leq -\log |y|(-y)/x$, hence

$$-\frac{y}{x} \geq -\frac{1}{\log |y|}.$$

Now either $d_\infty(\theta(P_1), \theta(P_2)) = |y| > e^{-R}$ or $e^{-R} \leq d_\infty(P_1, P_2) = |x| \leq y \log |y|$ which also means that $|y| \gtrsim e^{-R}$. So,

$$\frac{|y|}{|x|} \geq \frac{1}{R}.$$

If on the contrary $y \log |y| \leq x$ we have

$$(4) \quad x - y \log |y| < |y| < x.$$

First we notice that $y \log |y| > x - |y| > 0$. As $|y| < 1$ for any point of our space, $\log |y| < 0$ and we conclude that $y < 0$. Now from (4) we obtain that $x < -y(1 - \log |y|)$. As $1 - \log |y| > 0$ we obtain

$$-\frac{y}{x} > \frac{1}{1 - \log |y|}.$$

If $d_\infty(\theta(P_1), \theta(P_2)) = |y| > e^{-R}$, we trivially get that

$$\frac{|y|}{|x|} > \frac{1}{R}.$$

If $e^{-R} \leq d_\infty(P_1, P_2) = |x|$ we write $e^{-R} < x < -y(1 - \log |y|)$ and hence $y \gtrsim e^{-R}$, so we obtain the same result. So, in both cases we come to the same result

$$\left| \log \frac{|y|}{|x|} \right| < R.$$

3d case. Assume $|x| < |y| < |x - y \log |y||$, this case is trivial as

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y \log |y||}{|y|} \geq 1.$$

4th case. Let now $|y| < |x|$ and $|y| < |x - y \log |y||$. We also suppose that $x > 0$ to save notation.

$$(5) \quad \frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y \log |y||}{|x|} = \left| 1 - \frac{y \log |y|}{x} \right|.$$

If (5) is greater than $1/2$ then we have nothing to prove. So suppose that (5) is less than $1/2$

$$-\frac{x}{2} \leq x - y \log |y| \leq \frac{x}{2}$$

and so

$$\frac{x}{2} \leq y \log |y| \leq \frac{3x}{2}.$$

The last inequality shows that if either $d_\infty(\theta(P_1), \theta(P_2)) \geq e^{-R}$ or $d_\infty(P_1, P_2) \geq e^{-R}$, $|y| \gtrsim e^{-R}$ and so we have

$$\frac{|y \log |y||}{x} \geq \frac{|y \log |y||}{y} = |\log |y|| \geq \frac{1}{R},$$

which completes our discussion of this example. We have proved that

$$K(R) \lesssim \log R.$$

Part 5. Appendix

20. POINCARÉ INEQUALITY FOR \mathbb{H}^n

Let \mathbb{H}^n be n -dimensional hyperbolic space. The metric is written as $dr^2 + \sinh^2(r)d\theta^2$ in polar coordinates, this is very close to the exponentially growing metrics studied in section 16. From the results of section 16, little effort is needed to get the Poincaré inequality for balls in \mathbb{H}^n ,

$$\left(\int_{B_{\mathbb{H}^n}(R)} |f(x) - c|^p d\mu \right)^{1/p} \leq C_p^{hyp}(R) \left(\int_{B_{\mathbb{H}^n}(R)} |\nabla f|^p d\mu \right)^{1/p}.$$

Theorem 26. *Let \mathbb{H}^n be n -dimensional hyperbolic space. Then for a ball $B(R)$ of \mathbb{H}^n the Poincaré constant does not exceed*

$$C_p^{hyp}(R) \leq C(p, n)(1 + e^R),$$

where $C(p, n)$ depends only on p and dimension n .

Proof. We will provide the proof by comparing the hyperbolic metric with an exponential metric $dr^2 + e^{2r}d\theta^2$. To pass from the exponential to \sinh , we will divide the ball $B(R)$ in two parts: a little ball near the center and its complement. Finally we will compare the initial inequality with the Euclidean Poincaré inequality on this small ball and with our "exponential" inequality (Theorem 23) on the complement.

Let the volume element be $d\mu = \sinh^{n-1} r dr d\theta$. We will also write $dvol_{hyp}$ for $d\mu$, $dvol_{eucl}$ for euclidean volume element and $dvol_{exp}$ for exponential volume element $d\mu_{exp} = e^{(n-1)r} dr d\theta$. The idea of the proof is following. First we notice that outside of a ball $B(1)$ exponential and hyperbolic metrics are equivalent. On the other hand inside of a ball $B(2)$ hyperbolic metric is equivalent with euclidean metric. This motivates us to use the partition of unity to prove the initial Poincaré inequality for hyperbolic metric.

Let $b = \int_{B(2)} f dvol_{eucl}$, χ be the continuous function

- $\chi(x) = 1$ if $x \in B(1)$
- $\chi(x) = 0$ if $x \in H \setminus B(2)$
- $\chi(x) = 2 - r$ if $x \in B(2) \setminus B(1)$

We notice that

- $\sinh^{n-1} r \leq e^{(n-1)r}$ for $r \geq 0$,
- $e^{(n-1)r} \leq c_e \sinh^{n-1} r$ for $r \geq 1$ where the constant c_e is equal to $e^{n-1}/\sinh^{n-1} 1$,
- in $B(2)$ ($r \leq 2$) the hyperbolic and euclidean metrics are equivalent

$$1 \leq \frac{\sinh^{n-1} r dr d\theta}{r^{n-1} dr d\theta} \leq \left(\frac{\sinh 2}{2} \right)^{n-1}.$$

Now present $f - b$ as follow $f - b = \chi(f - b) + (1 - \chi)(f - b)$. First we consider the function $b + (1 - \chi)(f - b)$. We notice that $\nabla(b + (1 - \chi)(f - b))$ equals to 0 on $B(1)$, hence

$$\int_{B(R)} |\nabla(1 - \chi)(f - b)|^p \sinh^{(n-1)} r dr d\theta = \int_{B(R) \setminus B(1)} |\nabla(1 - \chi)(f - b)|^p \sinh^{(n-1)} r dr d\theta.$$

And we can write

$$\begin{aligned} \int_{B(R)} |b + (1 - \chi)(f - b) - c_1|^p \sinh^{(n-1)} r dr d\theta &\leq \int_{B(R)} |b + (1 - \chi)(f - b) - c_1|^p e^{(n-1)r} dr d\theta \\ &\leq (C_{p,n}^{exp})^p \int_{B(R)} |\nabla(1 - \chi)(f - b)|^p e^{(n-1)r} dr d\theta \\ &\leq c_e (C_{p,n}^{exp}(R))^p \int_{B(R)} |\nabla(1 - \chi)(f - b)|^p \sinh^{(n-1)} r dr d\theta, \end{aligned}$$

where the second inequality is a Poincaré inequality for exponential metric.

Now we will apply to the righthand part of the inequality the following formulas

$$\nabla(1 - \chi)(f - b) = (1 - \chi)\nabla f + (f - b)\nabla(1 - \chi).$$

and

$$\|f_1 + f_2\|_{\mathbb{L}^p}^p \leq 2(\|f_1\|_{\mathbb{L}^p}^p + \|f_2\|_{\mathbb{L}^p}^p).$$

We get that

$$\begin{aligned} &\int_{B(R)} |b + (1 - \chi)(f - b) - c_1|^p \sinh^{(n-1)} r dr d\theta \leq \\ &\leq c_e p (C_{p,n}^{exp})^p \left(\int_{B(R)} |\nabla f|^p \sinh^{(n-1)} r dr d\theta + \int_{B(2)} |f - b|^p \sinh^{(n-1)} r dr d\theta \right) \end{aligned}$$

Now we write euclidean Poincare inequality in $B(2)$ with euclidean constant $C_{p,n}^{eucl}$ (it depends only on dimension)

$$\begin{aligned}
\int_{B(2)} |f - b|^p dvol_{hyp} &\leq \left(\frac{\sinh 2}{2}\right)^{n-1} \int_{B(2)} |f - b|^p dvol_{eucl} \\
&\leq \left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \int_{B(2)} |\nabla f|_{eucl}^p dvol_{eucl} \\
&\leq \left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \int_{B(2)} |\nabla f|_{hyp}^p dvol_{hyp}.
\end{aligned}$$

Consider the function $\chi(f - b)$. It equals to 0 on the complement of $B(2)$ so we can easily treat this case involving euclidean Poincaré inequality as two metrics are equivalent there.

$$\begin{aligned}
&\int_{B(2)} |\chi(f - b) - c_2|^p dvol_{hyp} \leq \left(\frac{\sinh 2}{2}\right)^{n-1} \int_{B(2)} |\chi(f - b) - c_2|^p dvol_{eucl} \leq \\
&\leq \left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \int_{B(2)} |\nabla f|^p dvol_{eucl} \leq \left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \int_{B(2)} |\nabla f|^p dvol_{hyp}.
\end{aligned}$$

Now we need to combine all these results. First, we have

$$\int_{B(R)} |f - c_1 - c_2|^p d\mu \leq p \int_{B(R)} \left(|b + (1 - \chi)(f - b) - c_1|^p + |\chi(f - b) - c_2|^p \right) d\mu,$$

remind that $d\mu = dvol_{hyp}$. Further, we note that for big enough R

$$\left(\frac{\sinh 2}{2}\right)^{n-1} (C_{p,n}^{eucl})^p \leq c_e p (C_{p,n}^{exp}(R))^p$$

hence

$$\begin{aligned}
\int_{B(R)} |f - c_1 - c_2|^p d\mu &\leq c_e p^2 (C_{p,n}^{exp}(R))^p 2c_e (C_{p,n}^{eucl})^p \left(\int_{B(R)} |\nabla f|^p d\mu + \int_{B(2)} |\nabla f|^p d\mu \right) \\
&\leq 4c_e^2 p^2 (C_{p,n}^{exp}(R) C_{p,n}^{eucl})^p \int_{B(R)} |\nabla f|^p d\mu.
\end{aligned}$$

□

21. EQUIVALENCE OF THREE FORMS OF THE POINCARÉ INEQUALITY

In the literature, we can meet three different definitions of Poincaré inequalities. We will show that they are equivalent.

Definition 23. • There exists a constant C_p^1 such that for any function f with $\nabla f \in \mathbb{L}^p$ and its mean value $\tilde{c}_f = \oint f$

$$\|f - \tilde{c}_f\|_{\mathbb{L}^p} \leq C_p^1 \|\nabla f\|_{\mathbb{L}^p};$$

- there exists a constant C_p^2 such that for any function f with $\nabla f \in \mathbb{L}^p$ there exists a constant c_f

$$\|f - \tilde{c}_f\|_{\mathbb{L}^p} \leq C_p^2 \|\nabla f\|_{\mathbb{L}^p};$$

- there exists a constant C_p^3 such that for any function f with $\nabla f \in \mathbb{L}^p$

$$\left(\oint \oint_{X \times X} |f(x) - f(y)|^p dx dy \right)^{1/p} \leq C_p^3 \left(\oint_X |\nabla f(x)|^p dx \right)^{1/p}.$$

Proposition 7. *All three definitions are equivalent in the sense that C_p^1, C_p^2 and C_p^3 differs only by universal multiplicative constants.*

Proof. 1 \Rightarrow 2 Evident, just assume $c_f = \tilde{c}_f$.

2 \Rightarrow 3 Assume $g = f - c_f$. Hence $\nabla g \in \mathbb{L}^p$ and we have $\|g\|_p \leq C_p^2 \|\nabla g\|_p$. So,

$$\oint |f(x) - f(y)|^p dx dy \leq 2 \left(\oint |f(x)|^p dx + \oint |f(y)|^p dy \right) \leq 4 \oint C_p^2 |\nabla f|^p.$$

We just proved the third definition with $C_p^3 \leq 4C_p^2$.

3 \Rightarrow 1 Now consider $\|f - \tilde{c}_f\|_{\mathbb{L}^p}$

$$\begin{aligned} \|f - \tilde{c}_f\|_{\mathbb{L}^p} &= \left(\int_X \left| f(x) - \frac{\int_X f(y) dy}{\int_X dy} \right|^p dx \right)^{1/p} = \left(\oint_X \left| \int_X (f(x) - f(y)) dy \right|^p dx \right)^{1/p} \\ &\leq \left(Vol(x) \oint |f(x) - f(y)|^p dx dy \right)^{1/p} \leq \left(Vol(X) C_p^3 \oint |\nabla f|^p dx \right)^{1/p} = C_p^3 \|\nabla f\|_{\mathbb{L}^p}. \end{aligned}$$

Hence, $C_p^1 \leq C_p^3$.

□

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