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REMARKS ON FUNCTIONAL CALCULUS FOR PERTURBED FIRST ORDER DIRAC OPERATORS

PASCAL AUSCHER AND SEBASTIAN STAHLHUT

ABSTRACT. We make some remarks on earlier works on R -bisectoriality in L^p of perturbed first order differential operators by Hytönen, McIntosh and Portal. They have shown that this is equivalent to bounded holomorphic functional calculus in L^p for p in any open interval when suitable hypotheses are made. Hytönen and McIntosh then showed that R -bisectoriality in L^p at one value of p can be extrapolated in a neighborhood of p . We give a different proof of this extrapolation and observe that the first proof has impact on the splitting of the space by the kernel and range.

1. INTRODUCTION

Recall that an unbounded operator \mathcal{A} on a Banach space \mathcal{X} is called *bisectorial* of angle $\omega \in [0, \pi/2)$ if it is closed, its spectrum is contained in the closure of $S_\omega := \{z \in \mathbb{C}; |\arg(\pm z)| < \omega\}$, and one has the resolvent estimate

$$\|(I + \lambda\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq C_{\omega'} \quad \forall \lambda \notin S_{\omega'}, \quad \forall \omega' > \omega.$$

Assuming reflexivity of \mathcal{X} , this implies that the domain is dense and also the fact that the null space and the closure of the range split. More precisely, we say that the operator \mathcal{A} *kernel/range decomposes* if $\mathcal{X} = \mathbf{N}(\mathcal{A}) \oplus \overline{\mathbf{R}(\mathcal{A})}$ (\oplus means that the sum is topological). Here $\mathbf{N}(\mathcal{A})$ denotes the kernel or null space and $\mathbf{R}(\mathcal{A})$ its range, while the domain is denoted by $\mathbf{D}(\mathcal{A})$. Bisectoriality in a reflexive space is stable under taking adjoints.

For any bisectorial operator, one can define a calculus of bounded operators by the Cauchy integral formula,

$$\psi(\mathcal{A}) := \frac{1}{2\pi i} \int_{\partial S_{\omega'}} \psi(\lambda) (I - \frac{1}{\lambda}\mathcal{A})^{-1} \frac{d\lambda}{\lambda},$$

$$\psi \in \Psi(S_{\omega''}) := \{\phi \in H^\infty(S_{\omega''}) : \phi \in O(\inf(|z|, |z^{-1}|)^\alpha), \alpha > 0\},$$

with $\omega'' > \omega' > \omega$. If this calculus may be boundedly extended to all $\psi \in H^\infty(S_{\omega''})$, the space of bounded holomorphic functions in $S_{\omega''}$ for all $\omega'' > \omega$, then \mathcal{A} is said to have an H^∞ -calculus of angle ω .

Assume $\mathcal{X} = L^q$ of some σ -finite measure space. A closed operator \mathcal{A} is called *R -bisectorial* of angle ω if its spectrum is contained in S_ω and for all $\omega' > \omega$, there

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exists a constant $C > 0$ such that

$$\left\| \left(\sum_{j=1}^k |(I + \lambda_j \mathcal{A})^{-1} u_j|^2 \right)^{1/2} \right\|_q \leq C \left\| \left(\sum_{j=1}^k |u_j|^2 \right)^{1/2} \right\|_q$$

for all $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k \notin S_{\omega'}$ and $u_1, \dots, u_k \in L^q$. This is the so called R -boundedness criterion applied to the resolvent family. Note that the definition implies that \mathcal{A} is bisectorial. This notion can be defined on any Banach space but we do need this here.

In [10] and [11], the equivalence between bounded H^∞ -calculus and R -bisectoriality is studied for some perturbed first order Hodge-Dirac and Dirac type bisectorial operators in L^p spaces (earlier work on such operators appear in [1]). It is known that the former implies the latter in subspaces of L^p [14, Theorem 5.3]. But the converse is not known. In this specific case, the converse was obtained but for p in a given open interval, not just one value of p . Subsequently, in [9], the R -bisectoriality on L^p for these first order operators was shown to be stable under perturbation of p , allowing to apply the above mentioned results and complete the study. The proof of this result in [9] uses an extrapolation “à la” Calderón and Zygmund, by real methods. Here, we wish to observe that there is an extrapolation “à la” Šneĭberg using complex function theory. Nevertheless, the argument in [9] is useful to obtain further characterization of R -bisectoriality in L^p in terms of kernel/range decomposition. Indeed, we shall see that for the first order operators in L^p considered in [9], this property remains true by perturbation of p in the same interval as for perturbation of R -bisectoriality.

Our plan is to first review properties of perturbed Dirac type operators at some abstract level of generality. Then we consider the first order differential operators of [11, 9]. We next show the Šneĭberg extrapolation for (R -)bisectoriality of such operators and conclude for the equivalence of R -bisectoriality and H^∞ -calculus. We then show that of H^∞ -calculus, R -bisectoriality, bisectoriality hold simultaneously to kernel/range decomposition on a certain open interval. We interpret this with the motivating example coming from a second order differential operator in divergence form, showing that this interval agrees with an interval studied in [2].

2. ABSTRACT RESULTS

In this section, we assume without mention the followings: \mathcal{X} is a reflexive complex Banach space. The duality between \mathcal{X} and its dual \mathcal{X}^* is denoted $\langle u^*, u \rangle$ and is anti-linear in u^* and linear in u . Next, D is a closed, densely defined operator on \mathcal{X} and B is a bounded operator on \mathcal{X} . We state a first proposition on properties of BD , DB and their duals under various hypotheses.

Proposition 2.1. (1) BD with $D(BD) = D(D)$ is densely defined. Its adjoint, $(BD)^*$, is closed, and $D((BD)^*) = \{u \in \mathcal{X}; B^*u \in D(D^*)\} = D(D^*B^*)$ with $(BD)^* = D^*B^*$.

(2) Assume that $\|Bu\| \gtrsim \|u\|$ for all $u \in \overline{R(D)}$. Then,

(i) $B|_{\overline{R(D)}}: \overline{R(D)} \rightarrow \overline{R(BD)}$ is an isomorphism,

(ii) BD and D^*B^* are both densely defined and closed.

(iii) $DB|_{\overline{R(D)}}$ and $BD|_{\overline{R(BD)}}$ are similar under conjugation by $B|_{\overline{R(D)}}$.

- (3) Assume that $\|Bu\| \gtrsim \|u\|$ for all $u \in \overline{R(D)}$ and $\mathcal{X} = N(D) \oplus \overline{R(D)}$. Then $N(D) = N(BD)$.
- (4) Assume that $\|Bu\| \gtrsim \|u\|$ for all $u \in \overline{R(D)}$ and $\mathcal{X} = N(D) \oplus \overline{R(BD)}$. Then,
 (i) $\mathcal{X} = N(DB) \oplus \overline{R(D)}$.
 (ii) $R(DB) = R(D)$.
- (5) Assume that $\|Bu\| \gtrsim \|u\|$ for all $u \in \overline{R(D)}$ and $\mathcal{X} = N(D) \oplus \overline{R(BD)}$. Then,
 (i) $\mathcal{X}' = N(D^*B^*) \oplus \overline{R(D^*)}$.
 (ii) $(\overline{R(BD)})^* = \overline{R(D^*)}$ in the duality $\langle \cdot, \cdot \rangle$, with comparable norms.
 (iii) $\|B^*u^*\| \gtrsim \|u^*\|$ for all $u^* \in \overline{R(D^*)}$, hence $B^*|_{\overline{R(D^*)}}: \overline{R(D^*)} \rightarrow \overline{R(B^*D^*)}$ is an isomorphism.
 (iv) $(DB)^* = B^*D^*$.
 (v) $D^*B^*|_{\overline{R(D^*)}}$ and $B^*D^*|_{\overline{R(B^*D^*)}}$ are similar under conjugation by $B^*|_{\overline{R(D^*)}}$.
 (vi) $\overline{R(B^*D^*)} = (\overline{R(D)})^*$ in the duality $\langle \cdot, \cdot \rangle$, with comparable norms.
 (vii) $D^*B^*|_{\overline{R(D^*)}}$ is the adjoint of $BD|_{\overline{R(BD)}}$ in the duality $\langle \cdot, \cdot \rangle$.
 (viii) $B^*D^*|_{\overline{R(B^*D^*)}}$ is the adjoint to $DB|_{\overline{R(D)}}$ in the duality $\langle \cdot, \cdot \rangle$.

Proof. We skip the elementary proofs of (1) and (2) except for (2iii). See the proof of [8, Lemma 4.1] where this is explicitly stated on a Hilbert space. The reflexivity of \mathcal{X} is used to deduce that $D^*B^* = (BD)^*$ is densely defined. We next show (2iii). Note that $\overline{R(D)}$ is an invariant subspace for DB . Let $\beta = B|_{\overline{R(D)}}$. If $u \in \mathcal{D}(BD|_{\overline{R(BD)}}) = \overline{R(BD)} \cap \mathcal{D}(BD) = \overline{R(BD)} \cap \mathcal{D}(D)$, then $\beta^{-1}u \in \overline{R(D)} \cap \mathcal{D}(D\beta) = \overline{R(D)} \cap \mathcal{D}(DB) = \mathcal{D}(DB|_{\overline{R(D)}})$ and

$$BDu = \beta Du = \beta(DB)(\beta^{-1}u).$$

We now prove (3). Clearly $N(D) \subset N(BD)$. Conversely, let $u \in N(BD)$. From $\mathcal{X} = N(D) \oplus \overline{R(D)}$ write $u = v + w$ with $v \in N(D)$ and $w \in \overline{R(D)}$. It follows that $Du = Dw$ and $0 = BDu = BDw$. As $B|_{\overline{R(D)}}: \overline{R(D)} \rightarrow \overline{R(BD)}$ is an isomorphism, we have $w = 0$. Hence, $u = v \in N(D)$.

We next prove (4). We know that DB is closed. Its null space is $N(DB) = \{u \in \mathcal{X}; Bu \in N(D)\}$.

Let us first show (i), namely that $\mathcal{X} = N(DB) \oplus \overline{R(D)}$. As $\mathcal{X} = N(D) \oplus \overline{R(BD)}$ by assumption, the projection P_1 on $\overline{R(BD)}$ along $N(D)$ is bounded on \mathcal{X} . Let $u \in \mathcal{X}$. As $P_1Bu \in \overline{R(BD)}$, there exists $v \in \overline{R(D)}$ such that $P_1Bu = Bv$ and $\|v\| \lesssim \|Bv\| = \|P_1Bu\| \lesssim \|u\|$. Since $Bu = (I - P_1)Bu + P_1Bu$ and $(I - P_1)Bu \in N(D)$, we have $B(u - v) \in N(D)$, that is $u - v \in N(DB)$. It follows that $u = u - v + v \in N(DB) + \overline{R(D)}$ with $\|v\| + \|u - v\| \lesssim \|u\|$.

Next, we see that $R(DB) = R(D)$. Indeed, the inclusion $R(DB) \subseteq R(D)$ is trivial. For the other direction, if $v \in R(D)$, then one can find $u \in \mathcal{D}(D)$ such that $v = Du$. Using $\mathcal{X} = N(D) \oplus \overline{R(BD)}$, one can select $u \in \overline{R(BD)} = \overline{BR(D)}$ and write $u = Bw$ with $w \in \overline{R(D)}$. Hence $v = DBw \in R(DB)$.

We turn to the proof of (5). Item (i) is proved as Lemma 6.2 in [11]. To see (ii), we observe that if $u^* \in \overline{R(D^*)}$, then $\overline{R(BD)} \ni u \mapsto \langle u^*, u \rangle$ is a continuous linear functional. Conversely, if $\ell \in (\overline{R(BD)})^*$, then by the Hahn-Banach theorem, there is $u^* \in \mathcal{X}^*$ such that $\ell(u) = \langle u^*, u \rangle$ for all $u \in \overline{R(BD)}$. Write $u^* = v^* + w^*$ with $v^* \in N(D^*B^*)$ and $w^* \in \overline{R(D^*)}$ by (i). Since $\langle v^*, u \rangle = 0$ for all $u \in \overline{R(BD)}$, we have $\ell(u) = \langle w^*, u \rangle$ for all $u \in \overline{R(BD)}$ with $w^* \in \overline{R(D^*)}$.

To see (iii), consider again $\beta = B|_{\overline{\mathcal{R}(D)}}$. Let $u^* \in \overline{\mathcal{R}(D^*)}$, $u \in \overline{\mathcal{R}(D)}$. Then

$$\langle B^*u^*, u \rangle = \langle u^*, Bu \rangle = \langle u^*, \beta u \rangle.$$

Using (ii), we have proved $B^*|_{\overline{\mathcal{R}(D^*)}} = \beta^*$ and the conclusion follows.

To see item (iv), we remark that combining (iii) and item (2) applied to B^*D^* , we have $(B^*D^*)^* = DB$, hence $(DB)^* = B^*D^*$ by reflexivity.

Item (v) follows from item (iii) as for item (2iii).

Item (vi) follows from $\overline{\mathcal{R}(D)} = \beta^{-1}\overline{\mathcal{R}(BD)}$ with β as above: $(\overline{\mathcal{R}(D)})^* = \beta^*(\overline{\mathcal{R}(BD)})^*$ and we conclude using item (ii) and $B^*|_{\overline{\mathcal{R}(D^*)}} = \beta^*$.

Item (vii) follows from the dualities $(BD)^* = D^*B^*$ and $(\overline{\mathcal{R}(BD)})^* = \overline{\mathcal{R}(D^*)}$.

To prove item (viii), we recall that $DB|_{\overline{\mathcal{R}(D)}} = \beta^{-1}BD|_{\overline{\mathcal{R}(BD)}}\beta$. Thus using what precedes,

$$(DB|_{\overline{\mathcal{R}(D)}})^* = \beta^*(BD|_{\overline{\mathcal{R}(BD)}})^*(\beta^*)^{-1} = B^*(D^*B^*|_{\overline{\mathcal{R}(D^*)}})(\beta^*)^{-1} = B^*D^*|_{\overline{\mathcal{R}(B^*D^*)}}.$$

□

Remark 2.2. Note that the property $\|Bu\| \gtrsim \|u\|$ for all $u \in \overline{\mathcal{R}(D)}$ alone does not seem to imply $\|B^*u^*\| \gtrsim \|u^*\|$ for all $u^* \in \overline{\mathcal{R}(D^*)}$. Hence the situation for BD and B^*D^* is not completely symmetric without further hypotheses.

Here is an easy way to check the assumptions above from kernel/range decomposes assumptions.

Corollary 2.3. *Assume that $\|Bu\| \gtrsim \|u\|$ for all $u \in \overline{\mathcal{R}(D)}$. If D and BD kernel/range decompose, then $\mathcal{X} = \mathcal{N}(D) \oplus \overline{\mathcal{R}(BD)}$. In particular this holds if D and BD are bisectorial.*

Proof. By Proposition 2.1, (3), $\mathcal{N}(D) = \mathcal{N}(BD)$. We conclude from $\mathcal{X} = \mathcal{N}(BD) \oplus \overline{\mathcal{R}(BD)}$. □

Corollary 2.4. *Assume that $\|Bu\| \gtrsim \|u\|$ for all $u \in \overline{\mathcal{R}(D)}$ and that D kernel/range decomposes. If BD kernel/range decomposes so does DB . If BD is bisectorial, so is DB , with same angle as BD . The same holds if R -bisectorial replaces bisectorial everywhere.*

Proof. The statement about kernel/range decomposition is a consequence of Corollary 2.3 and Proposition 2.1, item (4). Assume next that BD is bisectorial and let us show that DB is bisectorial. By Proposition 2.1, item (2), $DB|_{\overline{\mathcal{R}(D)}}$ and $BD|_{\overline{\mathcal{R}(BD)}}$ are similar, thus $DB|_{\overline{\mathcal{R}(D)}}$ is bisectorial. Trivially $DB|_{\mathcal{N}(DB)} = 0$ is also bisectorial. As $\mathcal{X} = \mathcal{N}(DB) \oplus \overline{\mathcal{R}(D)}$ by Corollary 2.3 and Proposition 2.1, item (4), we conclude that DB is bisectorial in \mathcal{X} .

The proof for R -bisectoriality is similar. □

Remark 2.5. The converse DB (R -)bisectorial implies BD (R -)bisectorial seems unclear under the above assumptions on B and D , even if \mathcal{X} is reflexive which we assumed. So it appears that the theory is not completely symmetric for BD and for DB under such assumptions.

Corollary 2.6. *Assume that D kernel/range decomposes. The followings are equivalent:*

- (1) $\|Bu\| \gtrsim \|u\|$ for all $u \in \overline{\mathcal{R}(D)}$ and BD bisectorial in \mathcal{X} .

(2) $\|B^*u^*\| \gtrsim \|u^*\|$ for all $u^* \in \overline{\mathcal{R}(D^*)}$ and B^*D^* bisectorial in \mathcal{X}^* .

Moreover the angles are the same. If either one holds, then DB and D^*B^* are also bisectorial, with same angle. The same holds with R -bisectorial replacing bisectorial everywhere if \mathcal{X} is an L^p space with σ -finite measure and $1 < p < \infty$.

Proof. It is enough to assume (1) by symmetry (recall that we assume \mathcal{X} reflexive). That $\|B^*u^*\| \gtrsim \|u^*\|$ for all $u^* \in \overline{\mathcal{R}(D^*)}$ follows from Corollary 2.3 and Proposition 2.1, item (5). Next, as $B^*D^* = (DB)^*$ by Proposition 2.1, item (5), and as DB is bisectorial by Corollary 2.4, B^*D^* is also bisectorial by general theory. This proves the equivalence. Checking details, one sees that the angles are the same. Bisectoriality of DB and D^*B^* are already used in the proofs. The proof is the same for R -bisectoriality, which is stable under taking adjoints on reflexive L^p space with σ -finite measure (see [15, Corollary 2.1]). \square

3. FIRST ORDER CONSTANT COEFFICIENTS DIFFERENTIAL SYSTEMS

Assume now that D is a first order differential operator on \mathbb{R}^n acting on functions valued in \mathbb{C}^N whose symbol satisfies the conditions (D0), (D1) and (D2) in [9]. We do not assume that D is self-adjoint. Let $1 < q < \infty$ and $\mathcal{D}_q(D) = \{u \in L^q; Du \in L^q\}$ with $L^q := L^q(\mathbb{R}^n; \mathbb{C}^N)$ and $D_q = D$ on $\mathcal{D}_q(D)$. We keep using the notation D instead of D_q for simplicity. The followings properties have been shown in [11].

- (1) D is a R -bisectorial operator with H^∞ -calculus in L^q .
- (2) $L^q = N_q(D) \oplus \overline{\mathcal{R}_q(D)}$.
- (3) $N_q(D)$ and $\overline{\mathcal{R}_q(D)}$, $1 < q < \infty$, are complex interpolation families.
- (4) D has the coercivity condition

$$\|\nabla u\|_q \lesssim \|Du\|_q \quad \text{for all } u \in \mathcal{D}_q(D) \cap \overline{\mathcal{R}_q(D)} \subset W^{1,q}.$$

Here, we use the notation ∇u for $\nabla \otimes u$.

- (5) The same properties hold for D^* .

Let us add one more property.

Proposition 3.1. *Let $t > 0$. The spaces $\mathcal{D}_q(D)$, $1 < q < \infty$, equipped with the norm $\|f\|_{q,t} := \|f\|_q + t\|Df\|_q$, form a complex interpolation family. The same holds for D^* .*

Proof. Since D is bisectorial in L^q , we have $\|(1 + itD)^{-1}u\|_q \leq C\|u\|_q$ with C independent of t . [To be precise, we should write D_q for q and use that the resolvents are compatible for different values of q , that is the resolvents for different q agree on the intersection of the L^q 's.] Thus $(I + itD)^{-1}: (L^q, \|\cdot\|_q) \rightarrow (\mathcal{D}_q(D), \|\cdot\|_{q,t})$ is an isomorphism with uniform bounds with respect to t :

$$\|u\|_q \leq \|(I + itD)^{-1}u\|_q + t\|D(I + itD)^{-1}u\|_q \leq (C + 1)\|u\|_q.$$

The conclusion follows by the functoriality of complex interpolation. \square

4. PERTURBED FIRST ORDER DIFFERENTIAL SYSTEMS

Let $B \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N))$. Identified with the operator of multiplication by $B(x)$, $B \in \mathcal{L}(L^q)$ for all q . Its adjoint B^* has the same property. With D as before, introduce the set

$$\mathcal{I}(BD) = \{q \in (1, \infty); \|Bu\|_q \gtrsim \|u\|_q \text{ for all } u \in \mathcal{R}_q(D)\}.$$

By density, we may replace $\mathcal{R}_q(D)$ by its closure. For $q \in \mathcal{I}(BD)$, $B|_{\overline{\mathcal{R}_q(D)}} : \overline{\mathcal{R}_q(D)} \rightarrow \overline{\mathcal{R}_q(BD)}$ is an isomorphism. Let

$$b_q = \inf \left(\frac{\|Bu\|_q}{\|u\|_q}; u \in \overline{\mathcal{R}_q(D)}, u \neq 0 \right) > 0.$$

Lemma 4.1. *The set $\mathcal{I}(BD)$ is open.*

Proof. We have for all $1 < q < \infty$, $\|Bu\|_q \leq \|B\|_\infty \|u\|_q$. Thus, the bounded map $B: \overline{\mathcal{R}_q(D)} \rightarrow L^q$ is bounded below by b_q for each $q \in \mathcal{I}(BD)$. Using that $\overline{\mathcal{R}_q(D)}$ and L^q are complex interpolation families, the result follows from a result of Šneĭberg [16] (see also Kalton-Mitrea [12]). \square

Remark 4.2. If B is invertible in $L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N))$, then B is invertible in $\mathcal{L}(L^q)$ and its inverse is the operator of multiplication by B^{-1} . In this case, $\mathcal{I}(BD) = (1, \infty)$.

For next use, let us recall the statement of Šneĭberg (concerning lower bound) and Kalton-Mitrea (concerning invertibility even in the quasi-Banach case).

Proposition 4.3. *Let (X_s) and (Y_s) be two complex interpolation families of Banach spaces for $0 < s < 1$. Let T be an operator with $C = \sup_{0 < s < 1} \|T\|_{\mathcal{L}(X_s, Y_s)} < \infty$. Assume that for $s_0 \in (0, 1)$ and $\delta > 0$, $\|Tu\|_{Y_{s_0}} \geq \delta \|u\|_{X_{s_0}}$ for all $u \in X_{s_0}$. Then, there is an interval J around s_0 whose length is bounded below by a number depending on C, δ, s_0 on which $\|Tu\|_{Y_s} \geq \frac{\delta}{2} \|u\|_{X_s}$ for all $u \in X_s$ and all $s \in J$. If, moreover, T is invertible in $\mathcal{L}(X_{s_0}, Y_{s_0})$ with lower bound δ then there is an interval J whose length is bounded below by a number depending on C, δ, s_0 on which T is invertible with $\|Tu\|_{Y_s} \geq \frac{\delta}{2} \|u\|_{X_s}$ for all $u \in X_s$ and all $s \in J$.*

Our point is that the lower bound on the size of J is universal for complex families. Define two more sets related to the operator BD :

$$\begin{aligned} \mathcal{B}(BD) &= \{q \in \mathcal{I}(BD); BD \text{ bisectorial in } L^q\} \\ \mathcal{R}(BD) &= \{q \in \mathcal{I}(BD); BD \text{ } R\text{-bisectorial in } L^q\} \end{aligned}$$

Note that these are subsets of $\mathcal{I}(BD)$. We can define the analogous sets for B^*D^* .

Proposition 4.4. *Let $1 < p < \infty$. Then $p \in \mathcal{R}(BD)$ (resp. $\mathcal{B}(BD)$) if and only if $p' \in \mathcal{R}(B^*D^*)$ (resp. $\mathcal{B}(B^*D^*)$).*

Proof. This is Corollary 2.6. \square

Our next results are the new observation of this paper, simplifying the approach of [9].

Proposition 4.5. *These sets are open.*

Proof. Let us consider the openness of $\mathcal{B}(BD)$ first. We know that for all $p \in \mathcal{I}(BD)$, BD is densely defined and closed on L^p from Proposition 2.1, item (2). Fix $q \in \mathcal{B}(BD)$. Let ω be the angle of bisectoriality in L^q and $\omega < \mu < \pi/2$. Let $C_\mu = \sup_{\lambda \notin S_\mu} \|(I + \lambda BD)^{-1}\|_{\mathcal{L}(L^q)}$. Fix $\lambda \notin S_\mu$. Then $\|\lambda BD(I + \lambda BD)^{-1}\|_{\mathcal{L}(L^q)} \leq C_\mu + 1$. Thus for all $u \in D_q(D)$,

$$\begin{aligned} \|(I + \lambda BD)u\|_q &\geq (2C_\mu)^{-1} \|u\|_q + (2C_\mu + 2)^{-1} |\lambda| \|BDu\|_q \\ &\geq (2C_\mu)^{-1} \|u\|_q + (2C_\mu + 2)^{-1} b_q |\lambda| \|Du\|_q \\ &\geq \delta \|u\|_{q, |\lambda|} \end{aligned}$$

with $\delta = \inf((2C_\mu)^{-1}, (2C_\mu + 2)^{-1}b_q) > 0$. Also

$$\|(I + \lambda BD)u\|_q \leq C \|u\|_{q,|\lambda|}$$

with $C = \sup(1, \|B\|_\infty)$. Applying Proposition 4.3 thanks to Proposition 3.1, we obtain an open interval J about q contained in $\mathcal{I}(BD)$ such that for all $\lambda \notin S_\mu$ and $p \in J$, $(I + \lambda BD)^{-1}$ is bounded on L^p with bound $2/\delta$.

The proof for perturbation of R -bisectoriality is basically the same, with C_μ being the R -bound of $(I + \lambda BD)^{-1}$, that is the best constant in the inequality

$$\left\| \left(\sum_{j=1}^k |(I + \lambda_j BD)^{-1} u_j|^2 \right)^{1/2} \right\|_q \leq C \left\| \left(\sum_{j=1}^k |u_j|^2 \right)^{1/2} \right\|_q$$

for all $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k \notin S_\mu$ and $u_1, \dots, u_k \in L^q$. One works in the sums $L^q \oplus \dots \oplus L^q$ equipped with the norm of the right hand side and $\mathbb{D}_q(D) \oplus \dots \oplus \mathbb{D}_q(D)$ equipped with

$$\left\| \left(\sum_{j=1}^k |u_j|^2 \right)^{1/2} \right\|_q + \left\| \left(\sum_{j=1}^k |\lambda_j|^2 |Du_j|^2 \right)^{1/2} \right\|_q.$$

To obtain the R -lower bound (replacing δ), one linearizes using the Kahane-Kintchine inequality with the Rademacher functions

$$\left(\sum_{j=1}^k |u_j|^2 \right)^{1/2} \sim \left(\int_0^1 \left| \sum_{j=1}^k r_j(t) u_j \right|^q dt \right)^{1/q},$$

valid for any $q \in (1, \infty)$ (see, for example, [15] and follow the argument above). Details are left to the reader. \square

Remark 4.6. These sets may not be intervals. They are (possibly empty) intervals when restricted to each connected component of $\mathcal{I}(BD)$ because (R -)bisectoriality interpolates in L^p scales. See [13, Corollary 3.9] for a proof concerning R -bisectoriality. In particular, if $\mathcal{I}(BD) = (1, \infty)$ these sets are (possibly empty) open intervals.

Theorem 4.7. *For $p \in \mathcal{I}(BD)$, the following assertions are equivalent:*

- (i) BD is R -bisectorial in L^p .
- (ii) BD is bisectorial and has an H^∞ -calculus in L^p .

Moreover, the angles in (i) and (ii) are the same. Furthermore, if one of the items holds, then they hold as well for DB , and also for B^*D^* and D^*B^* in $L^{p'}$.

Proof. The implication (ii) \Rightarrow (i) is a general fact proved in [14]. Assume conversely that (i) holds. Then, there is an interval (p_1, p_2) around p for which (i) holds with the same angle by Proposition 4.5. Note also that (2) and (3) of Proposition 2.1 apply with $\mathcal{X} = L^q$ for each $q \in (p_1, p_2)$. Hence, B^* has a lower bound on $\overline{\mathcal{R}_{q'}(D^*)}$. We may apply Corollary 8.17 of [11], which states that D^*B^* satisfies (ii) on $L^{q'}$. By duality, we conclude that BD satisfies (ii) in L^q .

The last part of the statement now follows from Corollary 2.6. \square

Remark 4.8. As $p \in \mathcal{R}(BD)$ if and only if $p \in \mathcal{R}(B^*D^*)$, Proposition 4.5 and Theorem 4.7 can be compared to Theorem 2.5 of [9] for the stability of R -bisectoriality and the equivalence with H^∞ -calculus. The argument here is much easier and fairly general once we have Proposition 3.1. However, the argument in [9] is useful since

it contains a quantitative estimate on how far one can move from q . We come back to this below. Recall that the motivation of [9, Theorem 2.5], thus reproved here, is to complete the theory developed in [11].

5. RELATION TO KERNEL/RANGE DECOMPOSITION

For a closed unbounded operator \mathcal{A} on a Banach space \mathcal{X} , recall that \mathcal{A} kernel/range decomposes if $\mathcal{X} = \mathbf{N}(\mathcal{A}) \oplus \overline{\mathbf{R}(\mathcal{A})}$ and that it is implied by bisectoriality. The converse is not true (the shift on $\ell^2(\mathbb{Z})$ is invertible, so the kernel/range decomposition is trivial, but it is not bisectorial as its spectrum is the unit circle). For the class of BD operators in the previous section, we shall show that a converse holds.

For a set $A \subseteq (1, \infty)$, let $A' = \{q'; q \in A\}$.

Consider D and B as in Section 4. Recall that $p \in \mathcal{R}(BD)$ if and only if $p' \in \mathcal{R}(B^*D^*)$. That is $\mathcal{R}(B^*D^*)' = \mathcal{R}(BD)$. Recall also that $\mathcal{R}(BD) \subseteq \mathcal{I}(BD)$, hence $\mathcal{R}(BD) \subseteq \mathcal{I}(B^*D^*)'$ as well.

Assume $p_0 \in \mathcal{R}(BD)$ and let I_0 be the connected component of $\mathcal{I}(BD) \cap \mathcal{I}(B^*D^*)'$ that contains p_0 . It is an open interval.

Let

$$\begin{aligned} \mathcal{B}_0(BD) &= \{q \in I_0; BD \text{ bisectorial in } L^q\} \\ \mathcal{R}_0(BD) &= \{q \in I_0; BD \text{ } R\text{-bisectorial in } L^q\} \\ \mathcal{H}_0(BD) &= \{q \in I_0; BD \text{ bisectorial in } L^q \text{ with } H^\infty\text{-calculus}\} \\ \mathcal{S}_0(BD) &= cc_{p_0}\{q \in I_0; BD \text{ kernel/range decomposes in } L^q\} \end{aligned}$$

The notation cc_{p_0} means the connected component that contains p_0 .

Theorem 5.1. *The four sets above are equal open intervals.*

It should be noted that the theorem assumes non emptiness of $\mathcal{R}_0(BD)$.

Proof. It is clear that $\mathcal{H}_0(BD) \subseteq \mathcal{R}_0(BD) \subseteq \mathcal{B}_0(BD)$. By Proposition 4.5 and the discussion in Remark 4.6, $\mathcal{R}_0(BD)$ and $\mathcal{B}_0(BD)$ are open subintervals of I_0 . By Theorem 4.7, we also know that $\mathcal{H}_0(BD) = \mathcal{R}_0(BD)$.

As bisectoriality implies kernel/range decomposition, $\mathcal{B}_0(BD)$ is contained in the set $\{q \in I_0; BD \text{ kernel/range decomposes in } L^q\}$. As $\mathcal{B}_0(BD)$ contains p_0 , we have $\mathcal{B}_0(BD) \subseteq \mathcal{S}_0(BD)$. Thus it remains to show that $\mathcal{S}_0(BD) \subseteq \mathcal{R}_0(BD)$, which is done in the next results. \square

For $1 < p < \infty$, let p^*, p_* be the upper and lower Sobolev exponents: $p^* = \frac{np}{n-p}$ if $p < n$ and $p^* = \infty$ if $p \geq n$, while $p_* = \frac{np}{n+p}$.

Lemma 5.2. *Let $p \in \mathcal{R}_0(BD)$. Then $BD|_{\overline{\mathcal{R}_q(BD)}}$ is R -bisectorial (in $\overline{\mathcal{R}_q(BD)}$) for $q \in I_0 \cap (p_*, p^*)$.*

Proof. The (non-trivial) argument to extrapolate R -bisectoriality at p to R -bisectoriality at any $q \in I_0 \cap (p_*, p)$ is exactly what is proved in Sections 3 and 4 of [9], taken away the arguments related to kernel/range decomposition which are not assumed here. We next provide the argument for $q \in I_0 \cap (p, p^*)$. By duality, $p' \in \mathcal{R}(B^*D^*)$. By symmetry of the assumptions, $B^*D^*|_{\overline{\mathcal{R}_{q'}(B^*D^*)}}$ is R -bisectorial. By duality of R -bisectoriality in subspaces of reflexive Lebesgue spaces and Proposition 2.1, item (5), $DB|_{\overline{\mathcal{R}_q(D)}}$ is R -bisectorial. By Proposition 2.1, item (2), this implies that $BD|_{\overline{\mathcal{R}_q(BD)}}$ is R -bisectorial. \square

Corollary 5.3. $\mathcal{S}_0(BD) \subseteq \mathcal{R}_0(BD)$.

Proof. The set $\{q \in I_0; BD \text{ kernel/range decomposes in } L^q\}$ is open (this was observed in [9], again as a consequence of Šneĭberg's result). Thus, as a connected component, $\mathcal{S}_0(BD)$ is an open interval. Write $\mathcal{R}_0(BD) = (r_-, r_+)$ and $\mathcal{S}_0(BD) = (s_-, s_+)$ and recall that $(r_-, r_+) \subseteq (s_-, s_+)$. Assume $s_- < r_-$. One can find p, q with $q \in I_0 \cap (p_*, p)$ and $s_- < q \leq r_- < p < r_+$. By the previous lemma, we have that $BD|_{\overline{\mathcal{R}_q(BD)}}$ is R -bisectorial in $\overline{\mathcal{R}_q(BD)}$. Also $BD|_{\mathcal{N}_q(BD)} = 0$ is R -bisectorial. As $q \in \mathcal{S}_0(BD) = (s_-, s_+)$, we have $L^q = \overline{\mathcal{R}_q(BD)} \oplus \mathcal{N}_q(BD)$. Hence, BD is R -bisectorial in L^q . This is a contradiction as $q \notin \mathcal{R}_0(BD)$. Thus $r_- \leq s_-$. The argument to obtain $s_+ \leq r_+$ is similar. \square

Remark 5.4. It was observed and heavily used in [9] that for a given p , L^p boundedness of the resolvent of BD self-improves to off-diagonal estimates. Thus, the set of those $p \in I_0$ for which one has such estimates in addition to bisectoriality in L^p is equal to $\mathcal{B}_0(BD)$ as well.

6. SELF-ADJOINT D AND ACCRETIVE B

The operators D and B are still as in Section 4. In addition, assume that D is self-adjoint on L^2 and that B is strictly accretive in $\overline{\mathcal{R}_2(D)}$, that is for some $\kappa > 0$,

$$\operatorname{Re}\langle u, Bu \rangle \geq \kappa \|u\|_2^2, \quad \forall u \in \overline{\mathcal{R}_2(D)}.$$

Then, B and B^* have lower bound κ on $\overline{\mathcal{R}_2(D)}$ and $\overline{\mathcal{R}_2(D^*)} = \overline{\mathcal{R}_2(D)}$. In this case, BD and $DB = (B^*D)^*$ (replacing B by B^*) are bisectorial operators in L^2 . Moreover, using that B is multiplication and D a coercive first order differential operator with constant coefficients, [8, Theorem 3.1] (see [4] for a direct proof) shows that BD and DB have H^∞ -calculus in L^2 . Thus, Theorem 5.1 applies and one has the

Theorem 6.1. *There exists an open interval $I(BD) = (q_-(BD), q_+(BD)) \subseteq (1, \infty)$, containing 2, with the following dichotomy: H^∞ -calculus, R -bisectoriality, bisectoriality and kernel/range decomposition hold for BD in L^p if $p \in I(BD)$ and all fail if $p = q_\pm(BD)$. The same property hold for DB with $I(DB) = I(BD)$. The same property hold for B^*D and DB^* in the dual interval $I(DB^*) = I(B^*D) = (I(BD))'$.*

In applications, one tries to find an interval of p for bisectoriality, which is the easiest property to check.

The example that motivated the study of perturbed Dirac operators in the following setup, introduced in [7] and exploited in [8] to reprove the Kato square root theorem obtained in [5] for second order operators and in [6] for systems. Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^m \otimes \mathbb{C}^n))$ satisfy

$$(1) \quad \int_{\mathbb{R}^n} \nabla \bar{u}(x) \cdot A(x) \nabla u(x) dx \gtrsim \|\nabla u\|_2^2,$$

for all $u \in W^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$. Then BD , with $B = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$ and $D = \begin{pmatrix} 0 & -\operatorname{div} \\ \nabla & 0 \end{pmatrix}$, has a bounded H^∞ -calculus in $L^p(\mathbb{R}^n; \mathbb{C}^m \oplus [\mathbb{C}^m \otimes \mathbb{C}^n])$ for all $p \in (q_-(BD), q_+(BD))$, with angle at most equal to the accretivity angle of A .

Let us finish with the interpretation of the kernel/range decomposition in this particular example. As $BD = \begin{pmatrix} 0 & -\operatorname{div} \\ A\nabla & 0 \end{pmatrix}$, we see that

$$\mathbf{N}_p(BD) = \{u = (0, g) \in L^p(\mathbb{R}^n; \mathbb{C}^m \oplus [\mathbb{C}^m \otimes \mathbb{C}^n]); \operatorname{div} g = 0\}$$

and

$$\overline{\mathbf{R}_p(BD)} = \{u = (f, g) \in L^p(\mathbb{R}^n; \mathbb{C}^m \oplus [\mathbb{C}^m \otimes \mathbb{C}^n]); g = A\nabla h, h \in \dot{W}^{1,p}(\mathbb{R}^n; \mathbb{C}^m)\},$$

where $\dot{W}^{1,p}(\mathbb{R}^n; \mathbb{C}^m)$ is the homogeneous Sobolev space. Thus,

$$(2) \quad L^p(\mathbb{R}^n; \mathbb{C}^m \oplus [\mathbb{C}^m \otimes \mathbb{C}^n]) = \mathbf{N}_p(BD) \oplus \overline{\mathbf{R}_p(BD)}$$

is equivalent to the *Hodge splitting adapted to A for vector fields*

$$(3) \quad L^p(\mathbb{R}^n; \mathbb{C}^m \otimes \mathbb{C}^n) = \mathbf{N}_p(\operatorname{div}) \oplus A\nabla \dot{W}^{1,p}(\mathbb{R}^n; \mathbb{C}^m).$$

Writing details for DB instead we arrive the equivalence between

$$(4) \quad L^p(\mathbb{R}^n; \mathbb{C}^m \oplus [\mathbb{C}^m \otimes \mathbb{C}^n]) = \mathbf{N}_p(DB) \oplus \overline{\mathbf{R}_p(DB)}$$

and a second *Hodge splitting adapted to A for vector fields*

$$(5) \quad L^p(\mathbb{R}^n; \mathbb{C}^m \otimes \mathbb{C}^n) = \mathbf{N}_p(\operatorname{div} A) \oplus \nabla \dot{W}^{1,p}(\mathbb{R}^n; \mathbb{C}^m).$$

As $q_{\pm}(BD) = q_{\pm}(DB)$, we obtain that (3) and (5) hold for $p \in (q_-(BD), q_+(BD))$ and fail at the endpoints.

Let $L = -\operatorname{div} A\nabla$. It was shown in [2, Corollary 4.24] that (5) holds for $p \in (q_+(L^*)', q_+(L))$, where the number $q_+(L)$ is defined as the supremum of those $p > 2$ for which $t^{1/2}\nabla e^{-tL}$ is uniformly bounded on L^p for $t > 0$ (Strictly speaking, this is done when $m = 1$, and Section 7.2 in [2] gives an account of the extension to systems). As a consequence, we have shown that $q_+(BD) = q_+(DB) = q_+(L)$ and $q_-(BD) = q_-(DB) = q_+(L^*)'$.

In the previous example, the matrix B is block-diagonal. If B is a full matrix, then DB and BD happen to be in relation with a second order system in \mathbb{R}_+^{n+1} as first shown in [3]. Their study brought new information to the boundary value problems associated to such systems when $p = 2$. Details when $p \neq 2$ will appear in the forthcoming PhD thesis of the second author.

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REFERENCES

- [1] Sergey S. Ajiev. Extrapolation of the functional calculus of generalized Dirac operators and related embedding and Littlewood-Paley-type theorems. I. *J. Aust. Math. Soc.*, 83(3):297–326, 2007. [2](#)
- [2] Pascal Auscher. On necessary and sufficient conditions for L^p -estimates of Riesz transforms associated to elliptic operators on \mathbb{R}^n and related estimates. *Mem. Amer. Math. Soc.*, 186(871):xviii+75, 2007. [2](#), [10](#)
- [3] Pascal Auscher, Andreas Axelsson, and Alan McIntosh. Solvability of elliptic systems with square integrable boundary data. *Ark. Mat.* 48 (2010), 253–287. [10](#)

- [4] Pascal Auscher, Andreas Axelsson, and Alan McIntosh. On a quadratic estimate related to the Kato conjecture and boundary value problems. *Contemp. Math.*, 505:105–129, 2010. [9](#)
- [5] Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh, and Philippe Tchamitchian. The solution of the Kato square root problem for second order elliptic operators on \mathbb{R}^n . *Ann. of Math. (2)* 156, 2 (2002), 633–654. [9](#)
- [6] Pascal Auscher, Steve Hofmann, Alan McIntosh, and Philippe Tchamitchian. The Kato square root problem for higher order elliptic operators and systems on \mathbb{R}^n . *J. Evol. Equ.*, 1(4):361–385, 2001. Dedicated to the memory of Tosio Kato. [9](#)
- [7] Pascal Auscher, Alan McIntosh and Andrea Nahmod. The square root problem of Kato in one dimension, and first order elliptic systems. *Indiana Univ. Math. J.* 46, 3 (1997), 659–695. [9](#)
- [8] Andreas Axelsson, Stephen Keith, and Alan McIntosh. Quadratic estimates and functional calculi of perturbed Dirac operators. *Invent. Math.*, 163(3):455–497, 2006. [3](#), [9](#)
- [9] Tuomas Hytönen and Alan McIntosh. Stability in p of the H^∞ -calculus of first-order systems in L^p . *The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis*, 167–181, Proc. Centre Math. Appl. Austral. Nat. Univ., 44, Austral. Nat. Univ., Canberra, 2010. [2](#), [5](#), [6](#), [7](#), [8](#), [9](#)
- [10] Tuomas Hytönen, Alan McIntosh, and Pierre Portal. Kato’s square root problem in Banach spaces. *J. Funct. Anal.*, 254(3):675–726, 2008. [2](#)
- [11] Tuomas Hytönen, Alan McIntosh, and Pierre Portal. Holomorphic functional calculus of Hodge–Dirac operators in L^p . *J. Evol. Equ.*, 11 (2011), 71–105. [2](#), [3](#), [5](#), [7](#), [8](#)
- [12] Nigel Kalton and Marius Mitrea. Stability results on interpolation scales of quasi-Banach spaces and applications. *Trans. Amer. Math. Soc.*, 350(10):3903–3922, 1998. [6](#)
- [13] Nigel Kalton, Peer Kunstmann and Lutz Weis. Perturbation and interpolation theorems for the H^∞ -calculus with applications to differential operators. *Math. Ann.* 336 (2006), no. 4, 747–801. [7](#)
- [14] Nigel Kalton and Lutz Weis. The H^∞ -calculus and sums of closed operators. *Math. Ann.* 321 (2001), no. 2, 319–345. [2](#), [7](#)
- [15] Peer Kunstmann and Lutz Weis. Maximal L_p -regularity for parabolic equations, Fourier multiplier theorems and H^∞ -functional calculus. *Functional analytic methods for evolution equations*, 65–311, Lecture Notes in Math., 1855, Springer, Berlin, 2004. [5](#), [7](#)
- [16] I. Ja. Šneĭberg. Spectral properties of linear operators in interpolation families of Banach spaces. *Mat. Issled.*, 9(2(32)):214–229, 254–255, 1974. [6](#)

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