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Global-local optimizations on hierarchies of segmentations

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Abstract

Hierarchical segmentation is a multi-scale analysis of an image and provides a series of simplifying nested partitions. Such a hierarchy is rarely an end by itself and requires external criteria or heuristics to solve problems of image segmentation, texture extraction and semantic image labelling. In this theoretical paper we first propose a novel energy minimization framework to formulate optimization problems on hierarchies of segmentations. Second we provide the three important notions of *h-increasing*, *singular*, and *scale increasing* energies, necessary to solve the global combinatorial optimization problem of partition selection and which results in linear time dynamic programs. Common families of such energies are summarized, and also a method to generate new ones is described. Finally we demonstrate the application of this framework on problems of image segmentation and texture enhancement.

Keywords: Hierarchical segmentation, Optimization, Mathematical Morphology, Energy minimization, Dynamic programming

1. Introduction

To segment an image by a global constraint classically means to associate a numerical energy with every possible partition of the space where this image is defined. The best partition is then that which minimizes the energy. All that seems clear, up to a small problem. In practice, the energies range from 0 to 10^4 and often much less. Using the formula for the classical Bell's number, a digital square of 5×5 pixels has 4.6×10^{18} different partitions possible [10]. Each value of energy thus maps on, average to, millions of

billions(4.6×10^{15}) of partitions. What do we minimize here? Which implicit assumptions underlie the methods which give a unique minimal cut?

There are only two ways for obtaining (or hoping) uniqueness: by limiting the number of partitions, and by imposing constraints to the energy. For the first one, we can think to hierarchies, which provide strong restrictions. For the second one, we can try and replace the lattice of the integers by another one, more comprehensive, e.g. a lattice of partitions, and make hold the minimizations on it. But how to create a lattice of partitions from an energy? Which conditions must we introduce? And if uniqueness is finally ensured (the lattice structure is precisely made for that), how to reach the minimal one in the maze of all partitions? By means of which vital thread?

There have been multiple approaches for global constraints for optimization. There are two methods we contrast here: First, the graph cuts based optimization, second, partition selection from hierarchies of partitions. The former emphasize the use of seeds, e.g. the labels in graph-cuts, or the markers in watershed based methods. In addition, they view the space as a one scale structure. This perspective is illustrated by the search for a maximum flow in a directed graph, whose segmentation applications include the optimization of conditional random field (CRF) [21]. The latter approaches emphasize the scaling of the space by means of hierarchies, and attach less importance to labelling questions, in a first step at least.

This paper focusses on the second type of global constraints, which are approached from the viewpoint of mathematical morphology ¹. A hierarchy, or pyramid, of image segmentations is understood as a series of progressive simplified versions of an initial image, which result in increasing partitions of the space. How can these partitions cooperate and summarize the hierarchy into a unique cut, optimal in some sense. Three questions arise here, namely:

1. Given a hierarchy H of partitions and an energy ω on the partial partitions, how to combine the classes of this hierarchy for obtaining a new partition that minimizes ω , and which can be determined easily?
2. When one energy ω depends on an integer j , i.e. $\omega = \omega^j$, how to generate a sequence of optimal partitions that increase with j , which therefore should form a optimal hierarchy?

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3. Most of the segmentations involve several features (colour, shape, size, etc.) that we can handle with different energies. How to combine them?

These questions have been taken up by several authors, over many years, and by various methods. The most popular energies ω for hierarchical partitions derive from that of D. Mumford and J. Shah [24], in which a data fidelity term is summed up with a boundary regularization term. The optimization turns out to be a trade-off between these two constraints. In [19], for example, G. Koepfler, C. Lopez, and J.M. Morel build a pyramid of segmentations, from fine to coarse, by progressively giving more and more weight to the second term of Mumford and Shah functional. They stop the region growing when a certain number of regions is reached. The method initiated by P. Salembier and L. Garrido for generating thumbnails rests on the same type of energy [32]. They interpret the optimal cut as the most accurate image simplification for a given compression rate. The approach has been extended to additive energies by L. Guigues et Al. [17]. It is always assumed, in all these studies, that the energy of any partial partition equals the sum of the energies of its classes, which considerably simplifies the combinatorial complexity, and answers the above two questions 1 and 2.

However, one can wonder whether additivity is the very underlying cause of the nice properties, since P. Soille's constraint connectivity [38], where the addition is replaced by the supremum, satisfies similar properties. It is also the case for F. Zanoguerra's lasso, which labels a foreground inside a given contour [41]. Finally, one finds in literature a third type of energy, which holds on nodes only, and no longer on partial partitions. It appears in the method for labeling of P. Arbeláez et Al. [4], or in the studies of H.G. Akçay and S. Aksoy, in [1]. And again, it yields optimal cuts.

Is there a common denominator to all these approaches, more comprehensive than just additivity, and which explains why they always lead to unique optima? The following paper is a theoretical attempt to delimit this central concept, and to give answers to the above questions from 1) to 3). The theory is established in section 3 to 5, section 6 presents the algorithms, which are then applied to the main two families of climbing energies in section 7. Before the conclusion, the approach is extended to partial optimization in section 9, and some bridges between graph cuts and hierarchical optimizations are given in section 10.

2. Basic notions: Hierarchies and partitions

This section provides the necessary definitions and prerequisites needed in understanding the rest of the paper. The usual distinction between continuous and digital spaces is not appropriate for the general theory developed in sections 3 to 6. What is actually needed reduces to the two following hypotheses, which are assumed over the whole paper:

1. the space E to partition is topological,
2. the smallest partition π_0 has a finite number of classes.

The first assumption allows us to speak of frontiers between classes, or edges. The second one aims to avoid fractalities, and to permit various inductions, in Proposition 3.2 and in Algorithm 1, among others. Some additional hypotheses are introduced when the energies are particularized in section 7, e.g. "the classes are connected sets", or "the edges are simple arcs of \mathbb{R}^2 ". None of these assumptions are specific to image analysis. Space E may be the concern of semantic entities, grammar, NASDAQ quotations, or chamber music as well.

2.1. Partitions, partial partitions

Intuitively, a partition of E of the space under study (Euclidean, digital, graph, or else) is a division of this set into regions that do not overlap, and whose union restores E in its entirety. These regions are called classes. More formally, one obtains the classes of a partition by means of an extensive mapping $S : E \rightarrow \mathcal{P}(E)$ such that,

$$x, y \in E \Rightarrow S(x) = S(y) \text{ or } S(x) \cap S(y) = \emptyset.$$

Below, the symbols S, T stand for classes, and π for partitions. Partition π_1 is smaller than partition π_2 when each class of π_1 is included in a class of π_2 . This condition provides an ordering on the partitions, called refinement, which in turn induces a complete lattice.

Following Ch. Ronse [29], a partition $\pi(S)$ associated with a set $S \in \mathcal{P}(E)$ is called *partial partition (in short p.p.)* of E of support S . In particular, the partial partition of S into the single class S is denoted by $\{S\}$. The family of all partial partitions of set E is denoted by $\mathcal{D}(E)$, or simply by \mathcal{D} .

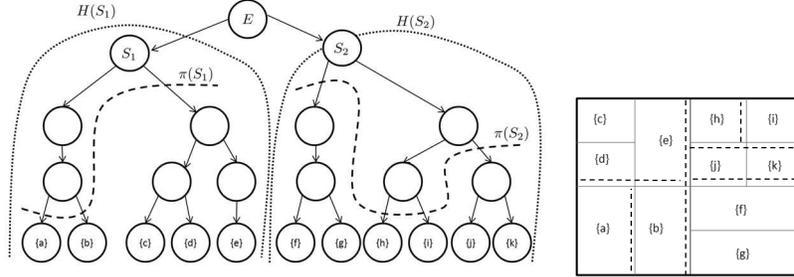


Figure 1: Hierarchy representation using a dendrogram and corresponding leaves partition shown in full lines, while the union of cuts $\pi(S_1) \cup \pi(S_2)$ in dotted lines.

2.2. Hierarchies of partitions

A hierarchy H is a chain of partitions π_i , i.e.

$$H = \{\pi_i, 0 \leq i \leq n \mid i \leq k \leq n \Rightarrow \pi_i \leq \pi_k\}, \quad (1)$$

where π_0 is the finest partition and π_n is the partition $\{E\}$ of E in a single class. The classes of π_0 are called the *leaves*, and E is the *root*. Since the number of leaves of π_0 is finite (as we have assumed above), the number n of different partitions of H is also finite. The intermediary classes are called *nodes*. If the q classes of the partition $\pi(S)$ are $\{T_u, 1 \leq u \leq q\}$, one writes

$$\pi(S) = T_1 \sqcup \dots \sqcup T_q,$$

where the symbol \sqcup indicates that the classes are concatenated. Given two p.p. $\pi(S_1)$ and $\pi(S_2)$ having disjoint supports, $\pi(S_1) \sqcup \pi(S_2)$ is the p.p. whose classes are either those of $\pi(S_1)$ or those of $\pi(S_2)$.

Let $S_i(x)$ be the class of partition π_i of H at point $x \in E$. Expression (1) means that at each point $x \in E$ the family $\{S_i(x), x \in E, 0 \leq i \leq n\}$ of those classes $S_i(x)$ that contain x forms a finite chain of nested sets from the leaf $S_0(x)$ to E .

$$\mathcal{S}(x) = \{S_i(x), 0 \leq i \leq n\}. \quad (2)$$

Conversely, according to a classical result [9], a family $\{S_i(x), x \in E, 0 \leq i \leq n\}$ of indexed sets generates the classes of a hierarchy iff, for $i \leq j$ and $x, y \in E$

$$S_i(x) \subseteq S_j(y) \text{ or } S_i(x) \supseteq S_j(y) \text{ or } S_i(x) \cap S_j(y) = \emptyset, \quad (3)$$

conditions which mean that the classes form an ultra-metric space [5], [22]. The partitions of a hierarchy may be represented by their classes, via a dendrogram, i.e. a tree where each node of bifurcation is a class S , or by their frontiers, via the saliency map of the edges, which indicates the level in the hierarchy when an edge disappears [26],[15]. The first representation is depicted in Figure 1, the second one in Figure 2. The classes of π_{i-1} at level $i - 1$ which are included in class S_i of level i are said to be *the sons* of S_i . Clearly, the descendants of each node S form in turn a hierarchy $H(S)$ of root S , which is included in the complete hierarchy $H = H(E)$. One denotes by $\mathcal{S}(E)$, or just \mathcal{S} , the set of all classes S of all partitions involved in H .

The hierarchy can be loosely seen as a set of partitions containing super-pixels of increasing sizes. Here we don't use the superpixel terminology, and prefer to distinguish between the class and partial partition.



Figure 2: Ducks: Initial RGB image, Luminance of RGB image, Saliency of partitions from Luminance(inverted to see with better contrast)

2.3. Generating hierarchies of segmentations

In the paper, the focus is not on the methods for obtaining hierarchies of segmentations, they are considered as inputs. The main techniques for hierarchical segmentation include the various Matheron semi-groups of connected filters (openings, alternating sequential filters) [31], the progressive floodings of watersheds [23], [15] and [3], the hierarchies obtained by increasing connections [2], [38], and the functional minimizations of Mumford and Shah type [24]. In addition, the learning strategies for segmentation, as developed by [5], among others, lead to very significant watersheds hierarchies. One can also quote the approach in [11] where additive functionals are enriched by the introduction of shape descriptors, which yield compact representations in [7].

2.4. Cuts in a hierarchy

Any partition π of E whose classes are taken in \mathcal{S} defines a *cut* π in a hierarchy H . The set of all cuts of E is denoted by $\Pi(E) = \Pi$. Every "horizontal" section $\pi_i(H)$ at level i is obviously a cut, but several levels can cooperate in a same cut, such as $\pi(S_1)$ and $\pi(S_2)$, drawn with thick dotted lines in Figure 1, where the partition $\pi(S_1) \sqcup \pi(S_2)$ generates a cut of $H(E)$. One can also define cuts inside any sub-hierarchy $H(S)$ of summit S , and similarly $\Pi(S)$ stands for the family of all cuts of $H(S)$.

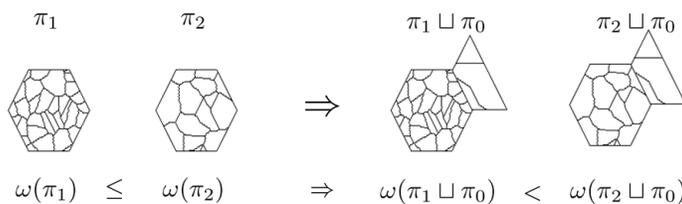


Figure 3: $\pi_1(S)$ and $\pi_2(s)$ are partial partitions on hexagonal support S , h -increasingness

3. Cuts and energies

3.1. Energies

An energy on the set \mathcal{D} of all p.p. of E is a numerical function $\omega : \mathcal{D} \rightarrow [0, \infty]$. In the following, \mathcal{D} will be provided with several energies ω , which may satisfy the following axioms:

i) ω is *h-increasing*, i.e.

$$\omega(\pi_1) \leq \omega(\pi_2) \Rightarrow \omega(\pi_1 \sqcup \pi_0) \leq \omega(\pi_2 \sqcup \pi_0), \quad (4)$$

where π_1 and π_2 are two p.p. of same support S , and π_0 a p.p. of support S_0 disjoint of S [35]. The geometrical meaning of Rel.(4) is depicted in Figure 3.

ii) ω is *singular* when the energy $\omega(\{S\})$ of class S is strictly smaller than, or larger than, any p.p. of S , i.e.

$$\omega(\{S\}) > \vee \{\omega(\pi(S))\} \quad \text{or} \quad \omega(\{S\}) < \wedge \{\omega(\pi(S))\}, \quad \pi(S) \text{ p.p. of } \{S\}. \quad (5)$$

3.2. Energetic orderings of the cuts

When an energy ω is allocated to the p.p. of E , cuts included, the cut(s) that minimizes ω provides an optimal segmentation of E relatively to ω . However, one can wonder about the meaning of such a best cut. The family of possible cuts being finite, and the energy ω being a positive number, we are sure to always find a cut that minimizes ω . And not only one but billions of billions, as we already saw.... The problem is ill-posed because the minimization holds of the lattice of the positive numbers. We must change the approach, well-pose the problem by equipping the set of all cuts of H with an ordering which involves ω , and make the minimization directly hold on the cuts. If we are able to associate a lattice with such an ordering, the uniqueness of the solution will then be ensured.

Theorem 3.1. *Let H be a hierarchy and ω be a h -increasing and singular energy. Energy ω induces an ordering on the set $\Pi(E)$ of all cuts of H . In this ordering, cut $\pi \in \Pi(E)$ is less energetic than cut $\pi' \in \Pi(E)$ w.r.t. ω , and one writes $\pi \leq_{\omega} \pi'$, when in each class S of the supremum by refinement given by $\pi \vee \pi'$ the p.p. of π inside S has an energy smaller or equal to that of π' inside S .*

Equivalently, for each leaf $x \in E$

a) either the class $S(x)$ of π is the support of a p.p. χ' of π' and $\omega(\{S\}) \leq \omega(\chi')$

b) or the class $S'(x)$ of π' is the support of a p.p. χ of π and $\omega(\chi) \leq \omega(\{S'\})$.

Proof. The equivalence of the two formulations is a consequence of Rel. 3, which shows that each class of $\pi \vee \pi'$ is either a class of π or of π' . The reflexivity, in statements a) and b) is obvious. For the transitivity, consider $\pi_1, \pi_2, \pi_3 \in \Pi$, of classes S_1, S_2 , and S_3 at x , with $S_1 \subseteq S_2 \subseteq S_3$ (Fig.4). The three energies can be ordered in six different manners, but it suffices to prove two cases only, the others configurations leading to similar proofs. Suppose firstly that $\pi_1 \leq_{\omega} \pi_3$ and $\pi_3 \leq_{\omega} \pi_2$. According to a) and b) we have

$$\omega(\{S_1\} \sqcup \chi \sqcup \chi') \leq \omega(\{S_3\}) \leq \omega(\{S_2\} \sqcup \chi''). \quad (6)$$

The singularity implies that these inequalities are strict. The set differences $S_2 \setminus S_1$ and $S_3 \setminus S_2$ are occupied by the partial partitions χ, χ' , and χ'' , as depicted in Figure 4. If $\omega(\{S_2\} \sqcup \chi') \leq \omega(\{S_1\} \sqcup \chi \sqcup \chi')$, then the double inequality $\omega(\{S_2\} \sqcup \chi') < \omega(\{S_3\}) < \omega(\{S_2\} \sqcup \chi'')$ contradicts the singularity

axiom. If $\omega(\{S_2\} \sqcup \chi') > \omega(\{S_1\} \sqcup \chi \sqcup \chi')$, then the h -increasingness implies that $\omega(\{S_2\}) > \omega(\{S_1\} \sqcup \chi)$, thus $\pi_1 \leq_\omega \pi_2$. Suppose secondly that $\pi_3 \leq_\omega \pi_1$ and $\pi_1 \leq_\omega \pi_2$, i.e. that

$$\omega(\{S_3\}) \leq \omega(\{S_1\} \sqcup \chi \sqcup \chi') \text{ and } \omega(\{S_1\} \sqcup \chi) \leq \omega(\{S_2\}),$$

hence, by h -increasingness $\omega(\{S_1\} \sqcup \chi \sqcup \chi') \leq \omega(\{S_2\} \sqcup \chi')$. As we also have $\omega(\chi') \leq \omega(\chi'')$, it comes $\omega(\{S_2\} \sqcup \chi') \leq \omega(\{S_2\} \sqcup \chi'')$, thus $\pi_3 \leq_\omega \pi_2$, which achieves the proof of the transitivity.

Conversely, consider three cuts $\pi_1 \leq_\omega \pi_2 \leq_\omega \pi_3$ which are identical everywhere except in the region S_3 of the figure, where they have the energies $\omega(\{S_1\} \sqcup \chi) \leq \omega(\{S_2\})$ and $\omega(\{S_2\} \sqcup \chi') = \omega(\{S_3\})$, with p.p. $\chi' = \chi''$. If ω is neither h -increasing and singular, we may have $\omega(\{S_1\} \sqcup \chi \sqcup \chi') > \omega(\{S_2\} \sqcup \chi') = \omega(\{S_3\})$, which contradicts the transitivity.

For the anti-symmetry, we must prove that $\pi \leq_\omega \pi'$ and $\pi' \leq_\omega \pi$ imply that $\pi = \pi'$. Suppose that the class $S'(x)$ of π' is the support of a p.p. χ made of more than one class of π . By applying the case b) of the theorem to the inequality $\pi \leq_\omega \pi'$, we have $\omega(\chi) \leq \omega(\{S'\})$. But we are also in case a) for $\pi' \leq_\omega \pi$, hence $\omega(\chi) \geq \omega(\{S'\})$, which implies the equality of the two members. But this contradicts the singularity of ω , so that S' is partitioned into a unique class of π , namely S . If we reverse the roles of π and π' , we obtain the same result, which is also independent of the choice of the leave x in E . This achieves the proof of anti-symmetry.

Conversely, consider an ordering \leq_ω whose energy would be non singular, and two cuts π and π' identical everywhere except in the class $S'(x)$ of π , where π is locally the p.p. χ . Supposed that $\omega(\chi) = \omega(S'(x))$. This implies $\pi \leq_\omega \pi'$ and also $\pi' \leq_\omega \pi$. However we do not have $\pi' = \pi$ since $\chi \neq S'(x)$. Thus singularity is needed, which achieves the proof. \square

A lattice structure directly derives from the ordering \leq_ω :

Proposition 3.2. *The set $\Pi(E)$ of all cuts of $H(E)$ forms a lattice for the energetic ordering \leq_ω . Given a family $\{\pi^j, 1 \leq j \leq p\}$ in $\Pi(E)$, the infimum $\bigwedge_\omega \pi^j$ (resp. supremum $\bigvee_\omega \pi^j$) is obtained by taking the p.p. of lowest energy (resp. highest energy) in each class of the refinement supremum $\bigvee \pi^j$.*

Proof. We give a proof by induction on the number of partitions:

a) Consider two partitions π^1 and π^2 . In each class S of the refinement supremum $\pi^1 \vee \pi^2$ one of the two partitions at least is $\{S\}$, and the other

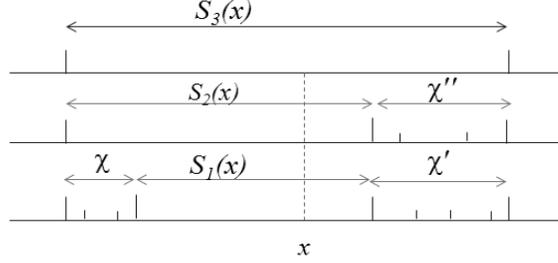


Figure 4: Energetic ordering

a finer p.p. of support S . By singularity of ω they have different energies, so that one can always choose the less (resp. most) energetic one. By doing the same for all classes of $\pi^1 \vee \pi^2$ we obtain the unique largest lower-bound $\pi^1 \wedge_\omega \pi^2$. (resp. smallest upper bound $\pi^1 \vee_\omega \pi^2$) of π^1 and π^2 .

b) Consider now three partitions π^1 , π^2 , and π^3 . Each class S of $\pi^1 \vee \pi^2 \vee \pi^3$ is a class of at least one of the three partitions, π^3 say. Then the restriction of $\pi^1 \vee \pi^2$ to S is a p.p. of S in which, according to a), π^1 and π^2 have a unique infimum (resp. supremum). By singularity of ω , the energy of this extremum is different from $\omega(\{S\})$, so that one can always choose a unique p.p. of smaller (resp. larger) energy.

Under iteration, the proof extends to any integer n , which achieves it. \square

Theorem 3.1 and proposition 3.2 enrich the structure for optimization by providing local interpretation of a global energy, since energetic extrema are now associated with *each class* of $\vee \pi^j$. Note also that any sub-hierarchy of root a node S of H forms in turn an energetic lattice for \leq_ω, \vee_ω , and \wedge_ω . The "if" statement of the theorem urges us to find out h -increasing and singular energies, and the "only if" one tells us that the described localization can only be reached h -increasing and singular energies. But of course, there may exist other nice orderings of the cuts, on the base of their energies as well. Though theorem 3.1 and proposition 3.2 could probably be formulated for more general families of partitions.

In the notation, we distinguish the refinement lattice from the ω -lattice by using for the former the three symbols \leq, \vee , and \wedge , without ω subscript.

4. Optimization and hierarchical increasingness

The optimization problem can now be stated more soundly. It consists in finding the minimum element of the energetic lattice $(\vee_\omega, \wedge_\omega)$. Three entities are involved, namely:

- A hierarchy/pyramid H of partitions of E which segment an input image,
- An energy ω , i.e. a non negative numerical function over the family $\mathcal{D}(E)$ of all partial partitions of E ,
- An "energetic" function f on E which may be the initial image, or another one, which parametrizes energy ω .

These three pieces of information are independent, and aim to determine the cuts that minimizes ω , i.e. such that $\omega(\pi^*) = \inf\{\omega(\pi) \mid \pi \in \Pi(E)\}$. they are called below the *optimal cuts*.

4.1. Optimal cut characterization

The h -increasingness (4), which was necessary to establish the energetic lattice structure, now turns out to be too general, as it does not specifically take hierarchies into account. We are thus led to replace it by the following weaker but more adapted version, which needs to introduce the set \mathcal{H} of all finite hierarchies of partitions of E .

Definition An energy ω on $\mathcal{D}(E)$ is *weakly h -increasing* when for any hierarchy $H \in \mathcal{H}$, any disjoint nodes S and S_0 of H , and any partition π_0 of S_0 , we have

$$\omega(\pi^*) = \inf\{\omega(\pi), \pi \in \Pi(S)\} \Rightarrow \omega(\pi^* \sqcup \pi_0) \leq \inf\{\omega(\pi \sqcup \pi_0), \pi \in \Pi(S)\} \quad (7)$$

where $\Pi(S)$ stands for the finite set of all p.p. of node S involved in hierarchy H .

Clearly, h -increasingness implies weak h -increasingness, i.e. Rel.(4) \Rightarrow Rel.(7). More precisely, Rel.(4) has been weakened just enough to obtain the theorem of optimal cut working in both senses. Indeed, we now have

Theorem 4.1. *Let $H \in \mathcal{H}$ be a finite hierarchy, and ω an energy on $\mathcal{D}(E)$, and S be a node of H of sons $T_1..T_p$. If $\pi_1^*, ..\pi_p^*$ are optimal cuts of $T_1..T_p$ respectively, then*

$$\pi_1^* \sqcup \pi_2^* .. \sqcup \pi_p^* \tag{8}$$

is an optimal cut of $\Pi(S) \setminus \{S\}$, for any $H \in \mathcal{H}$ and any $T_1..T_p$ in H , if and only if ω is weakly h -increasing.

Proof. Let us prove that Rel.(7) implies that $\pi_1^* \sqcup \pi_2^* .. \sqcup \pi_p^*$ is an optimal cut of S . We firstly limit ourselves to two classes, T_1 and T_2 , say, and consider the energy $\omega(\pi_1^* \sqcup \pi_2^*)$. The weak h -increasingness of ω implies that

$$\omega(\pi_1^* \sqcup \pi_2^*) \leq \inf\{\omega(\pi_1^* \sqcup \pi_2), \pi_2 \in \Pi(T_2)\}$$

and that

$$\omega(\pi_1^* \sqcup \pi_2) \leq \inf\{\omega(\pi_1 \sqcup \pi_2), \pi_1 \in \Pi(T_1)\}$$

hence

$$\omega(\pi_1^* \sqcup \pi_2^*) \leq \inf\{\omega(\pi_1 \sqcup \pi_2), \pi_1 \in \Pi(T_1), \pi_2 \in \Pi(T_2)\}$$

which shows that $\omega(\pi_1^* \sqcup \pi_2^*)$ is an optimal cut of $\pi(T_1 \cup T_2) \setminus \{T_1 \cup T_2\}$. Under finite iteration, the property extends to $S = \cup\{T_i, 1 \leq i \leq p\}$. Conversely, suppose that ω is not weakly h -increasing. It means that one can find a node S and a partial partition $\pi_0 \in S_0$, with both S and S_0 in H and such that

$$\omega(\pi^* \sqcup \pi_0) > \inf\{\omega(\pi \sqcup \pi_0), \pi \in \Pi(S)\}. \tag{9}$$

Consider the hierarchy H_0 which derives from H by replacing the partitions of S_0 by π_0 , at all levels \leq to that of S_0 . In H_0 , π_0 is a optimal cut of S_0 , although, because of Rel.(9) and because the finite number of elements of $\Pi(S)$, $\omega(\pi^* \sqcup \pi_0)$ is not a optimal cut of $S \cup S_0$. This counter example achieves the proof. \square

Presented in a stochastic framework where probabilities are assigned to energies, h -increasingness could be interpreted as a Markov chain property (of order one), since when the optimal energies of the sons of S are known, one does not need the knowledge of the descendants below the sons, to determine whether the energy of S is optimal.

Corollary 4.2. *When ω is h -increasing but not weakly, then the "only if" part of the theorem is no longer true.*

When the h -increasing energy ω is also singular, then Theorem 4.1 leads to the following key consequence

Proposition 4.3. *Let ω be h -increasing and singular energy. Then for any $H \in \mathcal{H}$ and any node S of H with p sons $T_1..T_p$ of optimal cuts $\pi_1^*, ..\pi_p^*$, there exists a unique optimal cut of the sub-hierarchy of root S . It is either the cut $\pi_1^* \sqcup \pi_2^*.. \sqcup \pi_p^*$, or the one class partition $\{S\}$ itself:*

$$\omega(\pi^*(S)) = \min\{\omega(\{S\}), \omega(\pi_1^* \sqcup \pi_2^*.. \sqcup \pi_p^*)\} \quad (10)$$

Proposition 4.3 is essential. It governs the choices of models for energies, and their implementations:

Firstly, the obtained optimal cut $\pi^*(E)$ is indeed globally less energetic than any other cut in H . In addition locally, each class $S \in \pi^*(E)$ is less energetic than any p.p. of S into classes of H , and also less energetic than any p.p. composed of classes of H and containing S . This is a strong property of regional minimum. Thus, this optimization is both local and global.

Secondly, the dynamic programming Rel.(10) allows us to find the optimal cut of H in one ascending pass. The nodes of H above the leaves have to be visited according to an order which respects the inclusions. One then compares the energy of each node with that of the p.p. of its sons, and the less energetic of the two is kept for continuing the ascending pass, and so on until the top node E is reached (the formal algorithm is described in section 6).

Thirdly, the condition (4) of h -increasingness for an energy being a notion independent of any hierarchy, one can use a different ω for each of the n levels of hierarchy H .

Finally, dealing with h -increasingness is sufficient. Fortunately so, because it is incomparably easier to check the h -increasingness of an energy than its possible weak h -increasingness.

4.2. uniqueness of the minimum cut

Are the assumptions of Corollary 4.3 compatible with each other? Can an energy be both h -increasing and singular? For answering the question, we must involve the minimum m of all the positive differences of energies involved in the p. p. of H , i.e.

$$m = \inf\{\omega(\pi) - \omega(\pi'), \omega(\pi) < \omega(\pi')\} \quad \pi, \pi' \text{ p.p. of } H.$$

As the number of p.p. of H is finite, m is strictly positive. Therefore, one can find a ε such as $0 < \varepsilon < m$, and state the following:

Proposition 4.4. *Let ω be a h -increasing energy over \mathcal{D} . Introduce the additional energy ω' for all $\{\pi(S) \in \Pi(S), S \in \mathcal{S}\}$*

$$\omega'[\pi(S)] = \varepsilon \text{ when } \pi(S) \neq \{S\} \quad \text{and} \quad \omega'[S] = 0 \text{ when not,}$$

with $0 < \varepsilon < m$. Then the sum $\omega + \omega'$ is h -increasing and provides a unique optimal cut with each sub-hierarchy $H(S)$, namely

$$\begin{aligned} \pi^*(S) & \text{ when } \omega[\pi^*(S)] \neq \omega[\{S\}] \\ \{S\} & \text{ when } \omega[\pi^*(S)] = \omega[\{S\}]. \end{aligned} \tag{11}$$

Proof. The energy ω' is h -increasing, since in implication (4) the two p.p. $\{S\} \sqcup \pi_0$ and $\pi(S) \sqcup \pi_0$ have always more than one class, so that $\omega'(\{S\} \sqcup \pi_0) = \omega'(\pi(S) \sqcup \pi_0) = \varepsilon$ and Rel.(4) is satisfied. Hence $\omega + \omega'$ is h -increasing. As ω' is smaller than m , the two energies ω and $\omega + \omega'$ yield the same minimum cut $\pi^*(S)$ when $\omega[\pi^*(S)] \neq \omega[\{S\}]$, but when not $(\omega + \omega')(\{S\})$ is less energetic than $(\omega + \omega')(\pi^*(S))$, which achieves the proof. \square

Mutatis mutandis, the same proof applies also if one takes systematically $\pi^*(S)$ rather than $\{S\}$ in case of equality, or again, given ω_0 , if one takes $\pi^*(S)$ when $\pi^*(S) \leq \omega_0$ and $\{S\}$ when not, etc..As ε is arbitrary small, $\omega + \omega'$ is computationally identical to ω , and ω' does not need to be explicitly introduced.

4.3. Identifying and generating h -increasing energies

An easy way to obtain a h -increasing energy consists in defining it, firstly, over all sets $S \in \mathcal{P}(E)$, considered as one class partial partitions $\{S\}$, and then in extending it to all partial partitions by some law of composition. Then, the h -increasingness is introduced by the law of composition, and not by $\omega[\mathcal{P}(E)]$.

The first two modes of composition which come to mind are, of course, addition and supremum. The additive mode was studied by L. Guigues under the name of *separable energies* [16], [17], a context in which he established the Rel.(15) below. All classes S of \mathcal{S} are supposed to be connected. Denote by $\{T_u, 1 \leq u \leq q\}$ the q sons which partition the node S , i.e. $\pi(S) = T_1 \sqcup \dots \sqcup T_q$. Provide the simply connected sets of $\mathcal{P}(E)$ with an arbitrary energy ω , and extend it from $\mathcal{P}(E)$ to the set $\mathcal{D}(E)$ of all partial partitions by using the sums

$$\omega(\pi(S)) = \omega(T_1 \sqcup \dots \sqcup T_q) = \sum_1^q \omega(T_u). \tag{12}$$

Just as the sum-generated ones, the \vee -generated energies on the partial partitions are defined from an energy ω on $\mathcal{P}(E)$ followed by a law of composition, which is now the supremum.

$$\omega(\pi) = \omega(T_1 \sqcup \dots \sqcup T_n) = \vee\{\omega(T_i)\}. \quad (13)$$

Proposition 4.5. *Let E be a set and $\omega : \mathcal{P}(E) \rightarrow \mathbb{R}^+$ an arbitrary energy defined on $\mathcal{P}(E)$, and let $\pi \in \mathcal{D}(E)$ be a partial partition of classes $\{S_i, 1 \leq i \leq n\}$. Then the two extensions of ω to the partial partitions $\mathcal{D}(E)$ by addition (Rel. (12)) and by supremum (Rel. (13)) define h -increasing energies.*

Proof. Let $S_{i,0}, S_{i,1}, S_{i,2}$ be the classes of π_0, π_1, π_2 respectively in implication (4), and let ω be an additive energy. We draw from the law of composition (12) that $\omega(\pi_1 \sqcup \pi_0) = \omega(\pi_1) + \omega(\pi_0)$, and $\omega(\pi_2 \sqcup \pi_0) = \omega(\pi_2) + \omega(\pi_0)$, which shows that the additive energy ω is h -increasing. Similarly, if ω is \vee -generated, then $\omega(\pi_1 \sqcup \pi_0) = \omega(\pi_1) \vee \omega(\pi_0)$, and $\omega(\pi_2 \sqcup \pi_0) = \omega(\pi_2) \vee \omega(\pi_0)$, which achieves the proof. \square

Corollary 4.6. *If $\{\alpha_j, j \in J\}$ stands for a family of non negative weights, then the weighed sum $\sum \alpha_j \omega_j$ and supremum $\vee \alpha_j \omega_j$ of h -increasing energies ω_j turn out to be h -increasing.*

A number of other laws are compatible with h -increasingness, such as the infimum or multiplication. One can also make ω depend on more than one class, on the proximity of the edges, on another hierarchy, etc.. Examples of composition by addition and supremum are demonstrated below, in Section 7.

5. Scale increasingness and climbing energies

5.1. Scale increasingness

Up to now, we did not pay attention to the *scaling* of the space. In fact, this sort of notion is less associated to a unique energy ω than to a family $\{\omega_\lambda, \lambda \in \Lambda\}$ of energies depending on a positive value λ . We will describe such a scaling in terms of progressively coarser optimal cuts (for the refinement ordering) with respect to λ . In literature, the idea of acting on the λ coefficient of Mumford and Shah functional in Eq.(18) goes back to G. Koepfler, C. Lopez, and J.M. Morel [19], who used λ for generating the input pyramid H .

The family $\{\omega_\lambda, \lambda \in \Lambda\}$ provides hierarchy H with a sequence of p optimal cuts π^{λ^*} , of labels $\lambda \in \Lambda$. *A priori*, the π^{λ^*} are not ordered, but if they were, then the optimal classes would more and more spread over the space, which would provide nice progressive simplifications to the optimal cuts. To achieve this goal, we need the supplementary axiom of *scale increasingness* (14), which yields the following axiomatic:

Definition We call *climbing energy* any family $\{\omega^j, 1 \leq j \leq p\}$ of energies over $\tilde{\Pi}$ which satisfies the three following axioms, valid for $\omega^j, 1 \leq j \leq p$ and for all $\pi \in \Pi(S), S \in \mathcal{S}$:

- i)* each ω^j is *h-increasing*,
- ii)* each ω^j is *singular*,
- iii)* the $\{\omega^j\}$ are *scale increasing*, i.e. for $j \leq k$, each support $S \in \mathcal{S}$ and each partition $\pi \in \Pi(S)$, we have that

$$j \leq k \text{ and } \omega^j(S) \leq \omega^j(\pi) \Rightarrow \omega^k(S) \leq \omega^k(\pi), \quad \pi \in \Pi(S), \quad S \in \mathcal{S}. \quad (14)$$

Axiom *i)* compares the same energy at two different levels, whereas axiom *iii)* compares two different energies at the same level. The relation (14) shows that as λ increases, the ω^λ preserves the ordering of the energies between the nodes of hierarchy H and their partial partitions. In particular, if ω_0 is *h-increasing* and *singular*, and if $\{\omega^\lambda, \lambda \in \Lambda\}$ is a climbing family, then the two families $\{\lambda\omega_0, \lambda \in \Lambda\}$ and $\{\omega^\lambda + \omega_0, \lambda \in \Lambda\}$ are climbing.

5.2. Ordering of the optimal cuts

The climbing energies satisfy the very nice property to order the optimal cuts with respect to the parameter λ :

Theorem 5.1. *Let $\{\omega^\lambda, \lambda \in \Lambda\}$ be a family of climbing energies, and let π^{λ^*} (resp. π^{μ^*}) denote the optimal cut of hierarchy H according to the energy ω^λ (resp. ω^μ). Then the family $\{\pi^{\lambda^*}, \lambda \in \Lambda\}$ of the optimal cuts generates a hierarchy H^* of partitions, i.e.*

$$\lambda \leq \mu \Rightarrow \pi^{\lambda^*} \leq \pi^{\mu^*}, \quad \lambda, \mu \in \Lambda. \quad (15)$$

Proof. Given λ , there exists a unique optimal cut π^{λ^*} of hierarchy H (axiom *ii)*). At the class S of π^{λ^*} that contains point x we have $\omega^\lambda(S) \leq \omega^\lambda(\pi)$ for every partial partition π of S . Then the scale increasingness (14) implies that $\omega^\mu(S) \leq \omega^\mu(\pi), \mu \geq \lambda$. By *h-increasingness* of ω^μ , and by uniqueness, class S is thus temporary optimal for ω^μ , which means that the class of the final optimal cut for ω^μ at point x covers S . \square

Family $\{\omega^\lambda, \lambda \in \Lambda\}$ is climbing in two senses: for each λ the energy climbs pyramid H up to its optimal cut (h -increasingness), and as λ varies, it generates a new pyramid to be climbed (scale-increasingness).

6. Algorithms for optimal cuts

This section is devoted to three algorithms. First, to find the optimal cut for a given h -increasing energy ω , while the second, is to determine the scales of appearance of the classes of the provisional optimal cut, and finally the third to find the family of optimal cuts for a *climbing* family energy.



Figure 5: Optimal Cuts Pyramids: Optimal cuts shown for different λ s. Original image 25098, $\lambda_{25098} = 0(\text{leaves}), 3000, 8000$ and Original image 169012, $\lambda_{169012} = 0(\text{leaves}), 400, 10000$

The hierarchy is organized as indexed set of classes present in each level (or partition). For each class S we introduce 2 operations. Firstly we can access children of S by the *ChildOf*(S) operation, and secondly, we can access the parent S by *ParentOf*(S) operation. Computationally, the h -increasingness condition (4) allows us to reach the optimal cut in one ascending pass (linear in the number of classes in H), by generalizing the algorithm by Guigues [17] to all h -increasing energies. The law of composition is referred to by *ComposeFunc* procedure and can be the sum, infimum or supremum with their scalar weighted versions as explained earlier in propositions 4.5 and 4.6. The *ComposeFunc* is used to calculate the composition of two energies as exemplified in Eq. (16), where the two energies $\omega_\phi(S)$ and $\omega_\theta(S)$ that represent fidelity term and a regularization term, respectively. These can be any pair of energies as further demonstrated in section 7.

Algorithm 1: Optimal Cut algorithm: $OptimalCut(H, \lambda, \omega^\lambda(S))$

Data: H represented by saliency s , Scale parameter λ , Energies $\omega_\phi(S)$, $\omega_\partial(S)$ for each class S .

Result: Optimal cut π^*

begin

$NumLevels \leftarrow |s|$ Assign the maximum number of levels (partitions) in the hierarchy H . This integer index is given by the number of unique values of saliency function s
 $NumClass \leftarrow 0$ Initialized to 0. Assign the maximum number of classes in a given partition (level) of the hierarchy H
 $\pi^* \leftarrow \emptyset$ Initialise the optimal cut to empty set
 $Q \leftarrow 0$ Quantization constant that is set to zero

for $level \in [2, NumLevels]$ **do**

$NumClass(level) \leftarrow |\pi(level)|$
 for $S \in [1, NumClass(level)]$ **do**

$ChildList \leftarrow ChildOf(S)$

$UpperParent \leftarrow ParentOf(S)$ Parent of Current class

$\Omega_{Class} \leftarrow \omega_\phi(S) + \lambda\omega_\partial(S)$

$\Omega_{Child} \leftarrow ComposeFunc(\omega_\phi^\lambda(S_{Child}), \omega_\partial^\lambda(S_{Child}), \lambda)$ Where $S_{Child} \in ChildList$.

if $(\Omega_{Class} \leq \Omega_{Child} + Q)$ **then**

$ChildOf(S) \leftarrow \emptyset$

else

$ChildOf(UpperParent) \leftarrow$

$ChildOf(UpperParent) \cup S_{child} \setminus S$ where

$S_{child} \in ChildList$

end

end

end

end

$\pi^* \leftarrow \bigcup S$ Where $S \in H$ represents classes that are not deleted.

end

Algorithm 2: Lambda List: $\text{LambdaList}(H, \omega_\phi(S), \omega_\partial(S))$

Data: H represented by saliency s , Energies $\omega_\phi(S)$, $\omega_\partial(S)$ for each class S .

Result: Lambda List Λ of values which correspond to λ s at which class S appears [17].

begin

```

    NumLevels  $\leftarrow$   $|s|$  Assign the maximum number of levels
    (partitions) in the hierarchy  $H$ . This integer index is given by the
    number of unique values of saliency function  $s$ 
    NumClass  $\leftarrow$  0 Initialized to 0. Assign the maximum number of
    classes in a given partition (level) of the hierarchy  $H$ 
     $\lambda \leftarrow$  0,  $\Delta\lambda \leftarrow \epsilon$  Where the step  $\epsilon$  is the smallest value for which
    there is a class that appears optimal in the place of its children
    while All classes have not appeared do
        for level  $\in [2, \text{NumLevels}]$  do
            NumClass(level)  $\leftarrow |\pi(\text{level})|$ 
            for  $S \in [1, \text{NumClass}(\text{level})]$  do
                ChildList  $\leftarrow \text{ChildOf}(S)$ 
                 $\Omega^\lambda(S) \leftarrow \omega_\phi(S) + \lambda\omega_\partial(S)$ 
                 $\Omega_{\text{child}}^\lambda(S) \leftarrow \text{ComposeFunc}(\omega_\phi(S_{\text{child}}), \omega_\partial(S_{\text{child}}), \lambda)$ 
                if  $\Omega^\lambda(S) \leq \Omega_{\text{child}}^\lambda(S)$  then
                    Update the Lambda List.
                     $\Lambda(S) \leftarrow \lambda$ 
                end
            end
        end
         $\lambda \leftarrow \lambda + \Delta\lambda$ 
    end
end

```

The two types of *ComposeFunc*: $\text{ComposeFunc}(\omega_\phi(S), \omega_\partial(S), \lambda)$ or $\text{ComposeFunc}(\omega(S), \lambda)$

- Addition:
$$\sum_{S_i \in \text{childof}(S)} \omega_\phi(S_i) + \lambda\omega_\partial(S_i)$$
- Supremum:
$$\bigvee_{S_i \in \text{childof}(S)} \omega(S_i)$$

Algorithm 3: Optimal Cut Pyramids

Data: Input hierarchy of partitions H , represented by saliency s ,
Energy $\omega^\lambda(S)$ parametrized by λ for each class S .

Result: Optimal cut pyramid H^*

```
begin
   $H^* \leftarrow \emptyset$ 
   $j \leftarrow 1, i \leftarrow 1$  indexing starts at 1
   $\Lambda \leftarrow \text{LambdaList}(H)$ 
   $\pi^i \leftarrow \text{OptimaCut}(H, \Lambda(i), \omega^{\Lambda(i)})$ 
  while  $|\pi^i| > 1$  do
     $i \leftarrow i + 1$ 
     $\pi^i \leftarrow \text{OptimaCut}(H, \Lambda(i), \omega^{\Lambda(i)})$ 
    if  $\pi^i \notin H^*$  then
       $H^*(j) \leftarrow \pi^i$ 
       $j \leftarrow j + 1$ 
    end
  end
end
```

- Infimum: $\bigwedge_{S_i \in \text{childof}(S)} \omega(S_i)$

The *LambdaList* algorithm implemented in this paper uses a quantization of the increment $\Delta\lambda$.

A set of the optimal cuts belonging to the optimal cut pyramid is shown for two images in figure 5. The leaves(finest partition) is obtained by assigning 0 in the optimal cut, since this produces the classes with the minimal variance. Partitions for higher values of λ are shown, which shows the series of partitions in the optimal cut pyramid. The optimal cut algorithm can be seen as a max-flow problem on directed planar graphs which are trees (see section 9). While for a directed planar graph the complexity is $n \log(n)$, the complexity of the dynamic program 1 is at worst linear. The complexity of the algorithm 3 is linear in the number of causal λ s of the input pyramid [17].

7. A few useful h -increasing energies and experiments

Following the two modes of composition given by proposition 4.5, we now review two families of climbing energies obtained by addition and by supremum.

7.1. Additive energies

Scale increasingness. Proposition 4.5 has shown that ω is h -increasing. Concerning the scale increasingness, a supplementary assumption of affinity is needed [16]. It is obtained by decomposing $\omega^\lambda(S)$ into a linear function of λ :

$$\omega^\lambda(S) = \omega_\varphi(S) + \lambda\omega_\partial(S) \quad S \in \mathcal{S}. \quad (16)$$

When the energy ω_∂ is sub-additive, i.e.

$$\omega_\partial\left(\bigcup_{1 \leq u \leq q} T_u\right) \leq \sum_{1 \leq u \leq q} \omega_\partial(T_u), \quad (17)$$

then the family is obviously scale increasing, since

$$\omega_\partial(S) = \omega_\partial\left(\bigcup_{1 \leq u \leq q} T_u\right) \leq \sum_{1 \leq u \leq q} \omega_\partial(T_u) = \omega_\partial(\pi(S)).$$

Conversely, L. Guigues showed that the condition (17) is necessary for scale increasingness [16] [17].

Mumford and Shah energy. It is the most popular additive energy, and historically the first [24]. One can find an exhaustive study of this functional in [25]. We write it for the Euclidean plane, and suppose that the edges are rectifiable simple arcs. Its first term, called *fidelity term*, (ω_φ in (16)) sums up the quadratic differences between f and its average $m(T_u)$ in the various T_u , and the second term, called *regularity term*, weights by λ^j the lengths ∂T^i of the frontiers of all T_u , i.e.

$$\omega^j(\pi(S)) = \sum_{1 \leq u \leq q} \int_{x \in T^u} \|f(x) - m(T_u)\|^2 + \lambda^j \sum_{1 \leq u \leq q} (\partial T_u) \quad (18)$$

where the weight λ^j is a numerical increasing function of the level number j . Both increasingness relations (4) and (14) are satisfied by the family of energies Rel.(18), which therefore are climbing. Here the term ω_∂ involves the arc length function, but it is not the only choice. One can also think about another $\omega_\partial(S)$, which reflects the convexity of A .

Additive energy by convexity. Consider in \mathbb{R}^2 a connected set S without holes and with a non zero curvature everywhere on ∂S . Let $d\alpha$ be the elementary rotation of its outward normal along the element du of the frontier ∂S . As the curvature $c(u)$ equals $d\alpha/du$, and as the total rotation of the normal around ∂S equals 2π , we have

$$2\pi = \int_{c \geq 0} c(u) du - \int_{c < 0} |c(u)| du.$$

When dealing with partitions, the distinction between outward and inward vanishes, but the parameter

$$\nu(X) = \frac{1}{2\pi} \int_{\partial S} |c(u)| du \quad (19)$$

still makes sense. It reaches the minimum value 1 when set S is convex, and increases with the degree of concavity. For a starfish with five pseudo-podes, it values around 5. Now $\nu(S)$ is sub-additive for the open parts of contours, therefore it can participate as a regularity term in an additive energy. In digital implementation, the angles between contour arcs must be treated separately (since sub-additivity applies on the open parts).

Additive energies by active contours. The active contours aim to match regular closed curves with the zones of maximum variation in an image [18] [12]. The energies we view are particular cases of active contours adapted to hierarchies, and derive from the approach proposed by Y. Xu at Al. [40]. The main idea is the following: each node $S \in H$ is dilated and eroded by a disc B , and the two crowns $S \oplus B \setminus S$, and $S \setminus S \ominus B$ are compared. This comparison stands for the fidelity term in Rel. (16), and a function of the curvature (e.g. Rel. (19)) stands for the regularity term. One goes from sets to partial partitions by additivity, according to the algorithm (12).

The simplest comparative energy is given by the difference of a given energetic function f on the two crowns:

$$\omega_\varphi(S) = \left| \int_{(S \oplus B \setminus S)} f(x) dx - \int_{(S \setminus S \ominus B)} f(x) dx \right|, \quad S \in \mathcal{P}(E). \quad (20)$$

It can be expressed in a dimensionless form by putting:

$$\tilde{\omega}_\varphi(S) = \left| \frac{\int_{(S \oplus B \setminus S)} f(x) dx - \int_{(S \setminus S \ominus B)} f(x) dx}{a(S)} \right|, \quad S \in \mathcal{P}(E),$$

where $a(S)$ denotes the area of S . When the absolute value bars are removed, the both energies ω_φ and $\tilde{\omega}_\varphi$ become sub-additive. Alternatively, the energy ω_φ proposed in [40] is the sum of the variances of f in the two crowns, divided by the variance of f in the union of these two crowns.

For the regularity term ω_∂ of the energy (16), one classically takes the above function ν of Rel. (19), which is scale increasing and generates the climbing family $\{\omega_\varphi + \lambda\omega_\partial\}$.

Optimal cut example for color image segmentation. The example of additive energy which terminates this section is a variant of the creation of thumbnails by Ph. Salembier and L. Garrido [32], but unlike in [32], the function which generates here the hierarchy of segmentations is not that used afterwards for the energy. We aim to obtain an optimal simplified version of a colour image f , constrained by compression rate. A hierarchy H has been obtained by previous segmentations of the scalar luminance $l = (r + g + b)/3$ based on flooded watersheds [15]. In each class S of H , the simplification consists in replacing the function f by its colour mean, i.e. the means of the three channels over S . Note that this colour mean does not intervene directly in the two energies (21) and (23), but rather in the display of the optimal cut.

In a first experiment depicted in figure 2, the data fidelity term (we refer in short as $\omega_\varphi(S)$) is given by the variance of the luminance in S . It yields the first term of the energy (21). The constraint function, expressed by the second term in (21) is the coding cost (we refer in short as $\omega_\partial(\pi)$). It is equal, for S , to the contour length $|\partial S|$, plus 24 bits for the average color of S .

$$\omega_{lum}(S) = \sum_{x \in S} \|l(x) - \bar{l}(S)\|^2 + \lambda(24 + |\partial S|), \quad (21)$$

In a second experiment depicted in figure 6, we separate each colour vector (r, g, b) into two components: the vector luminance $\vec{l} = (l, l, l)$ which gives the gray scale, plus the orthogonal chrominance vector $\vec{c} = (r - m, g - m, b - m) = (c_1, c_2, c_3)$ whose module is the saturation. The fidelity term of the energy is now the sum of the variances of the components of \vec{c} over S as shown in (22).

$$\omega_{chrom}(S) = \sum_{x \in S, 1 \leq i \leq 3} \|c_i(x) - \bar{c}_i(S)\|^2 + \lambda(24 + |\partial S|), \quad (22)$$

The principle idea here is to show that the parameters of the image involved in ω (namely the chrominance) can be completely de-correlated from

those used for generating the hierarchy (namely the luminance). We observe that the class variance is now calculated over the chrominance function of the image, which simplifies the image while keeping partitions which minimize the variance of the chrominance vector c .

$$\omega_{Texture}(S) = \omega_{chrom}(S) + \sum_{St \in siblings(S)} \frac{\mu}{\sigma^2(Area(St))}, \quad (23)$$



Figure 6: Optimal cuts by using variance of luminance(left), chrominance(right)



Figure 7: Optimal Cuts for texture using variance of chrominance for scale $\lambda = 100$: Left, input Image, middle and right, cuts for parameters at $\mu = 10^{12}$ and $\mu = 10^{14}$, in Eq. 23

This leads to a third experiment depicted in figure 7. The first term of (23) is the same as (22). The second term in (23) decreases when any child in the hierarchy whose siblings have low variance of component areas with respect to each other. This is done also with the constraint that the variance of the chrominance vector is reduced over the partitions of pyramid produced from the luminance vector l . Intuitively, texture features are formulated into this multi-scale framework where the optimal scale parameter combines the effect of chrominance and structure of texture into one global energy function, thus showing the flexibility of the framework.

According to Lagrange formalism, one classically reaches an optimum under constraint $\omega(S)$ by means of a system of partial derivatives. Now remarkably our approach, in both cases, replaces the computation of derivatives by a *climbing*. Indeed we can access the energy a cut π by summing up that of its classes, which leads to $\omega(\pi) = \omega_\varphi(\pi) + \lambda^j \omega_\partial(\pi)$. The cost $\omega_\partial(\pi)$ decreases as λ^j increases, therefore we can climb the pyramid of the optimal cuts and stop (thus optimal λ) when constraint is satisfied.

7.2. Sup-generated energies

The composition by supremum appears in several circumstances. For example, when dealing with the variation of a numerical function over a partial partition versus that of its classes, or in the problems of proximity to a ground truth, where the farthest distance from a point of some partial partition π to a ground truth set is the supremum of the farthest distances for the classes of π [20]. Proposition 4.5 has shown that ω is h -increasing.

Binary energies composed by supremum. The simplest \vee -energies are indeed the binary ones, which take values 1 and 0 only. Consider a binary \vee -energy ω such that for all $\pi, \pi_0, \pi_1, \pi_2 \in \mathcal{D}(E)$ we have

$$\begin{aligned} \omega(\pi) = 1 &\Rightarrow \omega(\pi \sqcup \pi_0) = 1, \\ \omega(\pi_1) = \omega(\pi_2) = 0 &\Rightarrow \omega(\pi_1 \sqcup \pi_2) = \omega(\pi_1 \sqcup \pi_0). \end{aligned}$$

This binary \vee -energy is obviously h -increasing. The Soille-Grazzini minimization provides an example of this type [38] [39]. A numerical function f is now associated with hierarchy H . Consider the range of variation $\delta(S) = \max\{f(x), x \in S\} - \min\{f(x), x \in S\}$ of f inside set S , and the h -increasing binary energy $\omega^k(\langle S \rangle) = 0$ when $\delta(S) \leq k$, and $\omega^k(\langle S \rangle) = 1$ when not. Compose ω according the law of the supremum, i.e. $\pi = \sqcup \langle S_i \rangle \Rightarrow \omega^k(\pi) = \bigvee_i \omega^k(\langle S_i \rangle)$. Then the class of the optimal cut at point $x \in E$ is the larger class of H whose range of variation is $\leq j$. When the energy ω^k of a father equals that of its sons, one keeps the father when $\omega^k = 0$, and the sons when not. As k varies a climbing family is generated.

Ordered energies. When they are not binary, some \vee -generated energies are presented via an ordering condition. As previously, an energy is still associated with each subset S of E . The axiom (4) of h -increasingness does not require we know the energy of all partial partitions. In particular, when

the comparison of the partial partitions π_1 and π_2 reduces to that of their classes, then a law of composition becomes useless.

Definition An energy ω on \mathcal{D} is said to be ordered when for all pairs $\pi_1, \pi_2 \in \mathcal{D}$ we have $\omega(\pi_1) \leq \omega(\pi_2)$ iff

- both π_1 and π_2 admit the same support $Supp$,
- for all points $x \in Supp$, of classes $S_1(x)$ and $S_2(x)$ in π_1 and π_2 respectively, the inequality $\omega[\{S_1(x)\}] \leq \omega[\{S_2(x)\}]$ holds.

An ordered energy ω is always h -increasing. When S is the support of the partition $\pi = \sqcup T_i$, then $\omega(S) \leq \omega(\pi)$ iff $\omega(S) \leq \vee \omega(T_i)$, and we find again the \vee -composition.

Here is an example of ordered energy due to H.G.Akçay and S. Aksoy [1] who study airborne multi-bands images and introduce (up to a small change) $\mu(S) = \text{Area}(S) \times (\text{mean of all standard deviations of all bands in } S)$. They work with energy *maximization*. Allocate a non negative measure $\mu(S)$ to each node of a hierarchy H , where μ takes its values in a partially ordered set M , such as a color space. The energy ω is ordered by the two conditions

$$\omega(S) \leq \omega(S') \Leftrightarrow S \supseteq S' \ \& \ \mu(S) \geq \mu(S') \quad S, S' \in \mathcal{P}(E), \mu \in M. \quad (24)$$

The node S^* of the optimal cut at point x is the highest more energetic than all its descendants. The optimal cut π^* is obtained in one pass, by Guigues' algorithm [17].

Figure 8 shows optimal cuts for three different laws of composition. In a) the additive mode chooses the father S , when $\omega(S) \leq \sum \omega(T_j)$. In b) the mode by supremum chooses the S , when $\omega(S) \leq \vee \omega(T_j)$. Finally, in c) one takes the largest node which is more energetic than all its descendants (maximization of ω).

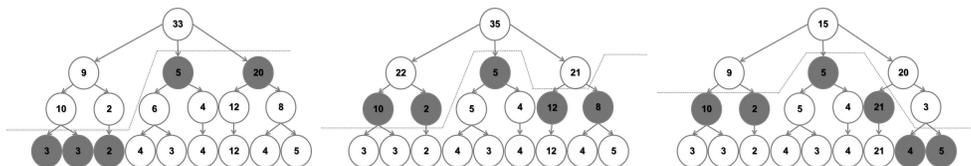


Figure 8: Optimal cuts for composition laws: addition, supremum and refinement

Composition of \vee -generated energies. Though the weighted supremum of \vee -generated energies is h -increasing (Prop. 4.5), the infimum is not. In practice, this half-result is nevertheless useful, since the \vee , paradoxically, expresses the intersection of criteria. For example, when the function f to optimize is colour, one can take for energies:

- $\omega_1(S) = 0$ when range of luminance in $S < k_1$, and $\omega_1(S) = 1$ when not,
- $\omega_2(S) = 0$ when range of saturation in $S < k_2$, and $\omega_2(S) = 1$ when not.

Then the h -increasing energy $\omega_1(S) \vee \omega_2(S) = 0$ when S is constant enough for both luminance and saturation.

8. Partial optimizations

Covering the whole space with some optimal partition is not always an aim. Some studies require doing it, but in others ones the regions of interest are limited, and clearly marked out by the context. Moreover, the leave partition often includes a good many classes due to noise. And thirdly, the hierarchies generated by connected filters may comprise a large number of singleton classes. For example, Figure 9 b) and c) depict the flat zones obtained by an alternating filter by reconstruction acting on the pepper image a) [31]. All black pixels indicate the singleton flat zones. When climbing the hierarchy, most of these point classes are covered by extended classes which are more significant.

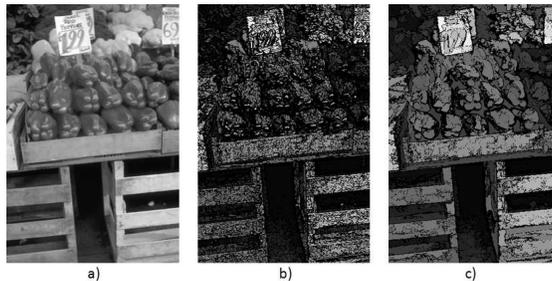


Figure 9: Alternated sequential filtering of sizes 1 and 5 of image 25098 from Berkeley Database

In other situations, some classes may be considered as non relevant because they are too small, or too large, or too far from the zone of interest, or of a non wanted hue, etc... In all cases, they are clearly identified, so that some label can indicate that they don't intervene when computing the optimal cut.

This subject extends the work on the theory of partial partitions [29]. Denote by $\mathcal{W}(E) \subseteq \mathcal{P}(E)$ the set of all these undesirable classes. The energies ω must satisfy the condition that, for all families $\{S_i\} \subseteq \mathcal{P}(E)$ and all families $\{W_j\} \in \mathcal{W}(E)$ such that $(\cup_i S_i) \cap (\cup_j W_j) = \emptyset$, we have

$$\omega((\sqcup_i S_i) \sqcup (\sqcup_j W_j)) = \omega(\sqcup_i S_i).$$

The energy of the partial partition of classes $\{S_i\}$ must not change when outside $\{W_j\}$ classes are added. It means that $\omega(W) = 0$ when the law of composition involved in ω is the sum or the supremum, and that $\omega(W) = \infty$ when it is the infimum. When ω is h -increasing, the computation of the optimal cut is unchanged, but now results in a partition which may contain W classes.

9. Flow or Optimal Cut on Hierarchy?

It is now instructive to go back to the alternative approaches by min-cut max-flow, or by conditional random fields(CRFs), that we quoted in the introduction. One can notice that:

1. The CRFs and min-cut max-flow formulation represent spatial interaction between pixels which is restricted to a unitary neighborhood, and the increasing complexity may not be always advantageous [21].
2. Hierarchical methods provide a lower combinatorial complexity while supplying intuitive segmentations. In addition, the construction of a hierarchy and of the ulterior energy ω may use independent pieces of information (e.g. in section 5, luminance based energy, versus chrominance and texture).
3. Hierarchical clustering induces larger and larger neighbourhoods. This helps differentiate textures at various scales.

Here we compare the two methods of optimization we contrasted in the introduction. Is it possible to interpret the structure of the data in a hierarchy H , and the notions involved in the search for its optimal cut, in terms of maximum flow problem, or dually of optimal cut, in an oriented graph G ?

The definition of a flow through G requires the data of a source and a sink. The particular shape of a pyramid leads us to take for source the family A of all leaves, and for sink the whole space E . In flows, capacities are often

allocated to the edges, and sometimes to the vertices. For the sake of comparison, in case of a hierarchy H , we will take the nodes. Now, in the graph case, one wants to maximize the flow, whereas above, both additive and super-generated energies were the matter of minimizations. We must choose, and from now on we decide to *maximize* the hierarchical energies, i.e. to invert the ordering relations (e.g. in comparisons father/sons of the h -increasing case).

In a pyramid, each leaf a is connected to the root E by a unique path $[a, \dots, E]$, strictly increasing, and different for each leaf. For example, in Figure 10 we demonstrate a toy example with sample energies shown on a dendogram. Each node is given a capacity, which appears within each node. As long as two paths in this graph(tree) have no common node, the flows they carry are independent, and upper bounded by the lowest capacity along the portion where they are disjoint. When two such lines meet at some node, e.g. the two paths $[a_1, \dots, S]$ and $[a_2, \dots, S]$ which meet in S in then one must adopt some law for composing them, which is exactly what the optimal cut algorithm performs in the dynamic program.

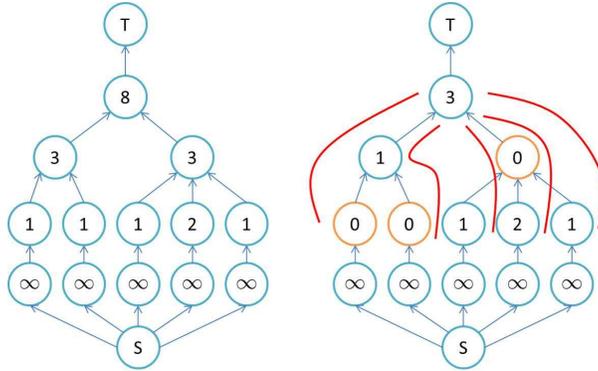


Figure 10: Flow on Hierarchy: Dendogram with each class assigned a positive value. One of the augmenting flows is showing the equivalence between the minimum cost cut of the flow (in red) and the optimal cut (in orange)

Consider for example the additive energies, which are the most similar to the flows over a directed graph. In this additive case, the capacities $\omega(T_i)$ of the sons $\{T_i\}$ of S are added, and compared to $\omega(S)$. The min of $\sum \omega(T_i)$ and $\omega(S)$ gives the provisional capacity of the flow in S , and one pursues

the climbing. At the end, the nodes of the optimal cut are those whose provisional capacity coincide with their own one, as depicted in Figure 10. Finally, we exactly obtain a min-cut in the graph-cut sense, but presented in another language.

10. Conclusion

We have established a method for finding optimal segmentations in hierarchies, which turns out to be both global and local. This point is the major theoretical contribution of the paper. The approach rests on the energetic lattice of the cuts of a hierarchy (proposition 3.2), which is obtained for h -increasing and singular energies. This lattice structure results in unique optimal cuts characterized by the proposition 4.3. The fast algorithms 1 to 3 also derive from the same basic concepts.

It appeared that these two ideas were a common basis for several partition optimization methods (those which precisely work well), developed by [24], [32], [17], [38], [1], [3], [7], [11], [19], [41], among others.

Then we introduced the climbing energies as families of energies which provide ordered optimal cuts, thus giving a scale space light to the approach.

Finally we demonstrated how to formulate multiple constraint functions over the image space and obtain different optimal segmentations. Two examples with colour image segmentation and texture enhancement were shown.

In future applications we aim to look at problems of evaluation of segmentation from hierarchies, labelling on hierarchies and also explicit the links to clustering.

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