



ISMM 2013  
Uppsala

LIMG  
ESIEE Université Paris-Est

- The hierarchical cuts theory
- Climbing energies and optimal cuts

B. Ravi Kiran et Jean Serra

# The goal

- We have a family of partitions that segment an image.
- How to combine them in order to obtain the best possible segmentation?
- Classically, one associates an energy  $w$  with each partition and one takes the partition with smallest energy (e.g. Mumford-Shah).

What does this mean really?

# Unicity problem

- For example, let us take a small  $5 \times 5$  picture and an energy whose dynamic range is 1000.

- As there are 4.61018 different partitions of the  $5 \times 5$  square, one finds on average :

4,600,000,000,000,000 partitions by energy !

i.e. 30 billions times the distance to the moon in kilometres :)

- The methods which work well introduce additional implicit assumptions

# How to get out ?

- We keep down the number of possible partitions by restricting them to the cuts of a hierarchy.
- We structure these cuts in a lattice which depends on the energy  $w$ , which ensures a unique minimum.
- We must find a way for reaching easily this minimum.
- When there are several energies, or an energy which depends on a positive parameter, we must find out how to combine them.

that will be the plan of the talk

# Plan

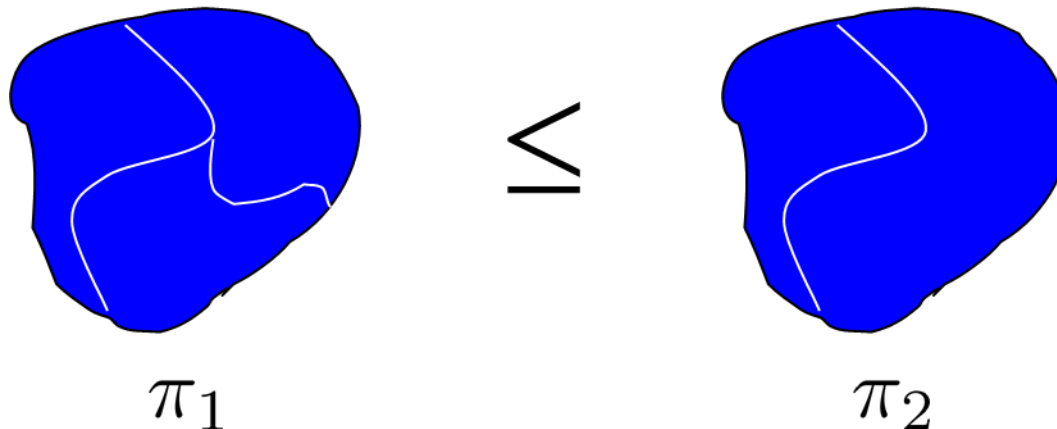
1. Hierarchies
2. Singular energies and lattices
3. Optimal cuts and hierarchical increasingness
4. Compositions of energy by sums and by suprema

# Hierarchy, or pyramid, of partitions

- A hierarchy of partitions is a chain of partitions

$$H = \{\pi_i, 0 \leq i \leq n\} \text{ with } i \leq j \Rightarrow \pi_i \leq \pi_j$$

- The partitions are ordered by refinement
- The assumption:  $\pi_0$  has a finite number of classes, called leaves.



# Hierarchy, or pyramid, of partitions

- Associate with hierarchy  $H$  the family  $\mathcal{S}$  of all classes  $S_i(x)$  for all partitions.

$$\mathcal{S} = \{S_i(x), x \in E, 0 \leq i \leq p\}$$

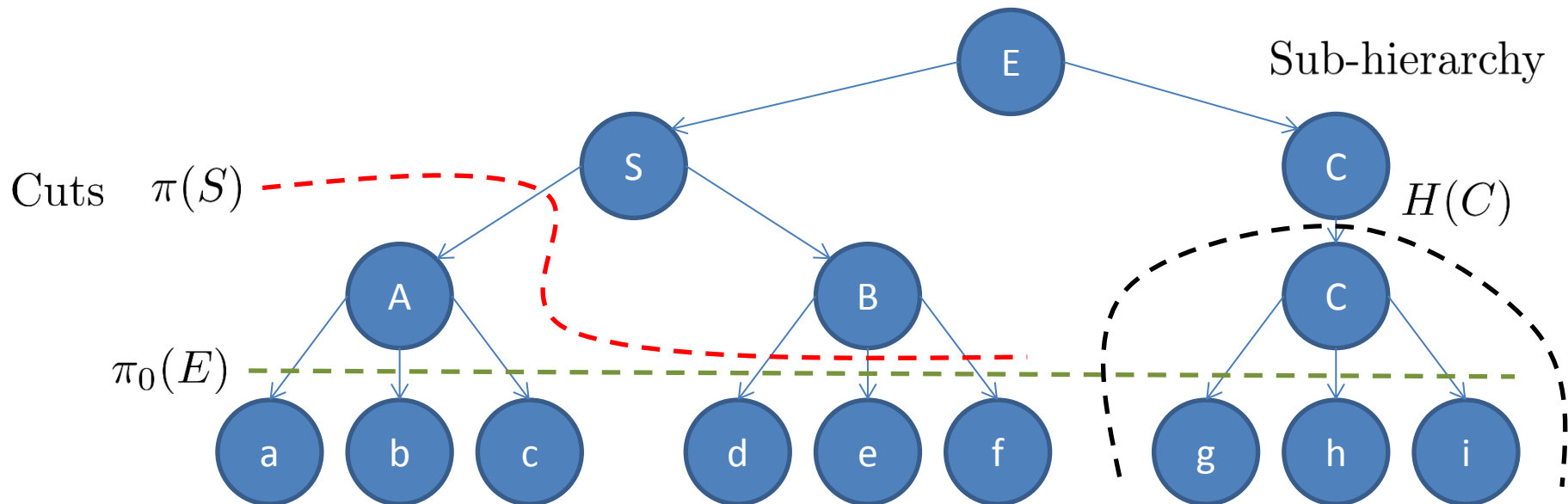
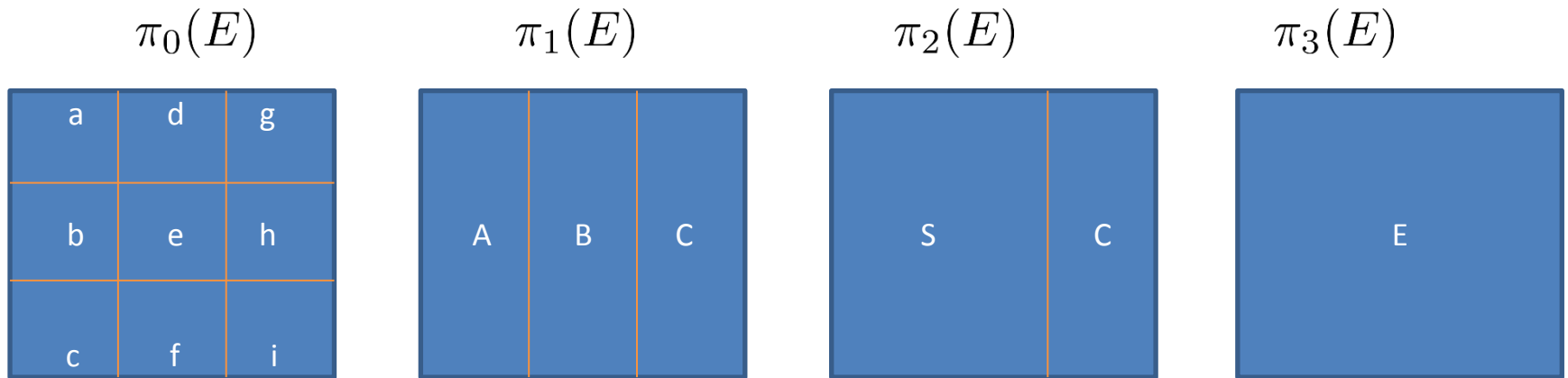
- Every family  $\mathcal{S}$  of indexed sets induces a hierarchy iff for  $i \leq j$

$$x, y \in E \Rightarrow S_i(x) \subseteq S_j(y) \quad \text{or} \quad S_i(x) \supseteq S_j(y) \quad \text{or} \quad S_i(x) \cap S_j(y) = \emptyset$$

A relation equivalent to an ultra-metric on the classes of  $\mathcal{S}$  .



# Representation of a hierarchy



# Energy and pyramid

The search for an optimal cut rests on three **independent** entities:

- a **pyramid**  $H$  of partitions of space  $E$
- a **function**  $f$  on  $E$   
(  $f$  may have been used, or not, to generate the pyramid),
- an **energy**  $\omega$  i.e. a non negative function  
 $\omega : \mathcal{D} \rightarrow \mathbb{R}^+$   
of the set  $\mathcal{D}$  of the partial partitions of  $E$  into  $\mathbb{R}^+$ .

# Plan

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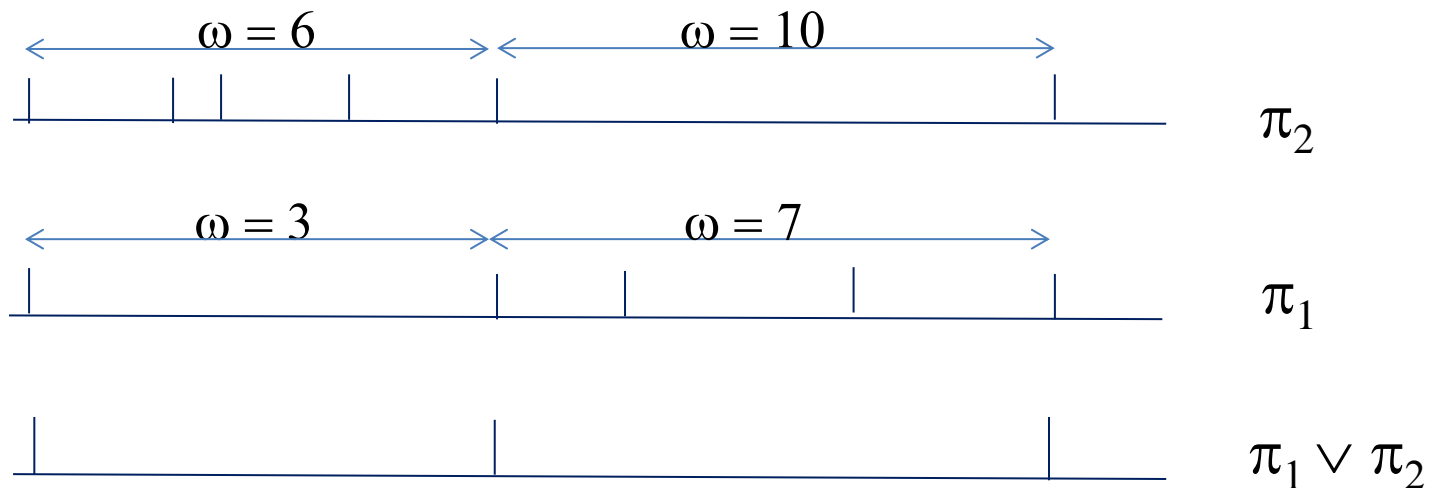
# Energetic ordering on cuts

Cut  $\pi_1$  is said to be less energetic than  $\pi_2$  when, in each class  $S$  of  $\pi = \pi_1 \vee \pi_2$  we have

$$\omega(\pi_1 \sqcap S) \leq \omega(\pi_2 \sqcap S)$$

( i.e. the restriction of  $\pi_1$  to  $S$  has less energy than that of  $\pi_2$  to  $S$ ).

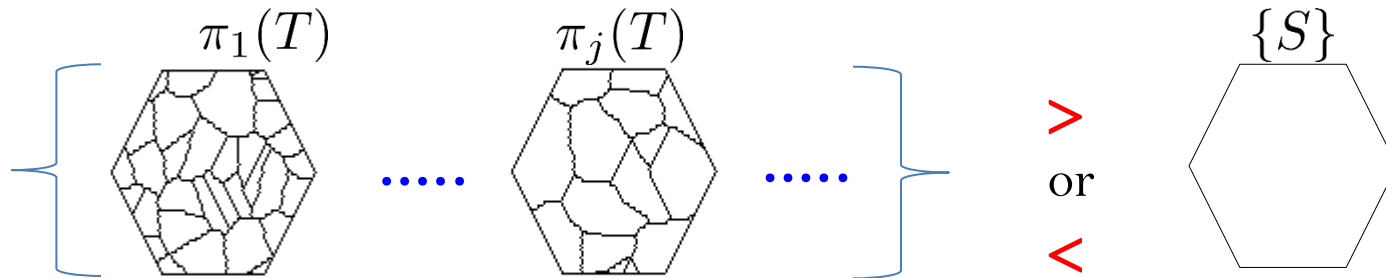
One writes  $\pi_1 \leq_{\omega} \pi_2$



# Energetic ordering and singular energy

Energy  $\omega$  is singular when

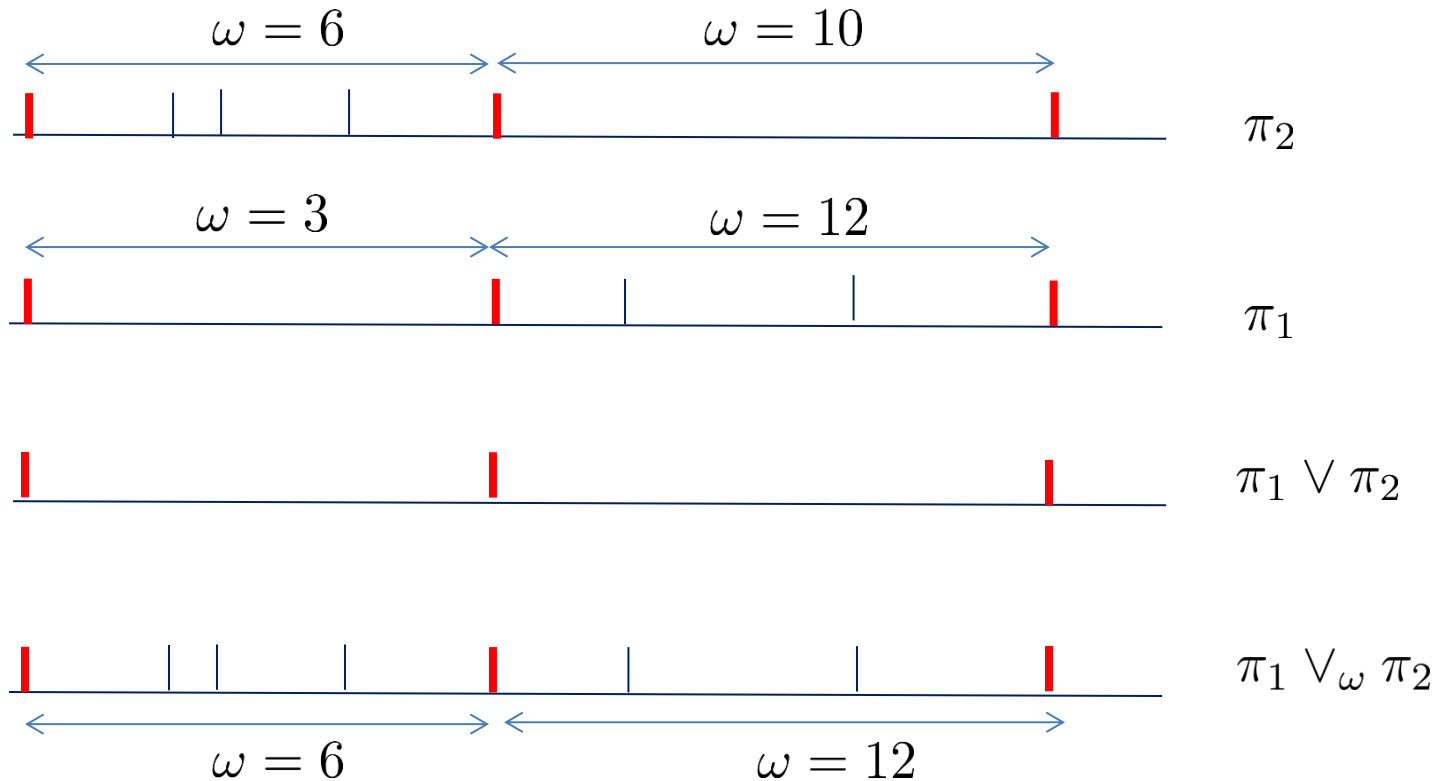
- either  $\omega(S) > \vee \omega(\pi(S))$
- or  $\omega(S) < \vee \omega(\pi(S))$



**Proposition:** The relation  $\pi_1 \leq_{\omega} \pi_2$  defines an ordering which is energetic, iff energy  $\omega$  is singular

# Energetic Lattice

The energetic ordering induces a lattice where, in each class of  $\pi_1 \vee \pi_2$  the most energetic partial partition is chosen.



# Energetic Lattice

- The energetic lattice  $(\leq_{\omega}, \vee_{\omega})$  answers the unicity question, since:  
when energy is singular then one cut only has a minimum energy.
- In this optimal cut, each class  $S$  is less energetic than all possible partial partition of support  $S$ .
- Such a minimum is thus stronger than the usual energetic minima since it is both **local** and **global**.
- It just remains to find out how to get it :)

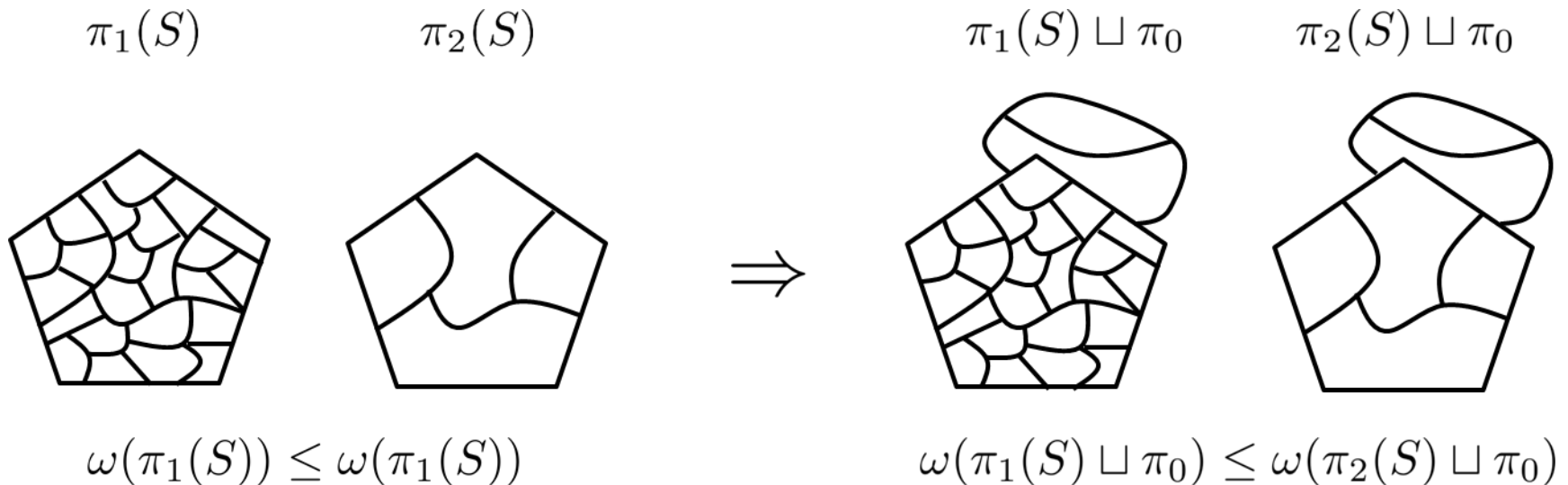
# Plan

1. Hierarchies
2. Singular energies and lattices
3. Optimal cuts and hierarchical increasingness
4. Compositions of energy by sums and by suprema



# Hierarchical increasingness

- How to reach the cut of minimal energy ?
- Introduce the **hierarchical increasingness (h-increasingness)** axiom between fathers and sons, as the implication:



# Climbing energies

Energy is said to be climbing when it is both

- **Singular** (unique optimal cut), and
- ***h*-increasing** (tractable access to the optimal cut).
- **Proposition:** When energy  $\omega$  is climbing then the optimal cut of the sub-hierarchy  $H(S)$  is  
either  $\pi(T_1) \sqcup \pi(T_2) \sqcup \pi(T_3)$  or  $S$  itself
- The optimal cut for the whole space  $E$  is then obtained by progressively climbing from the leaves level to the root.

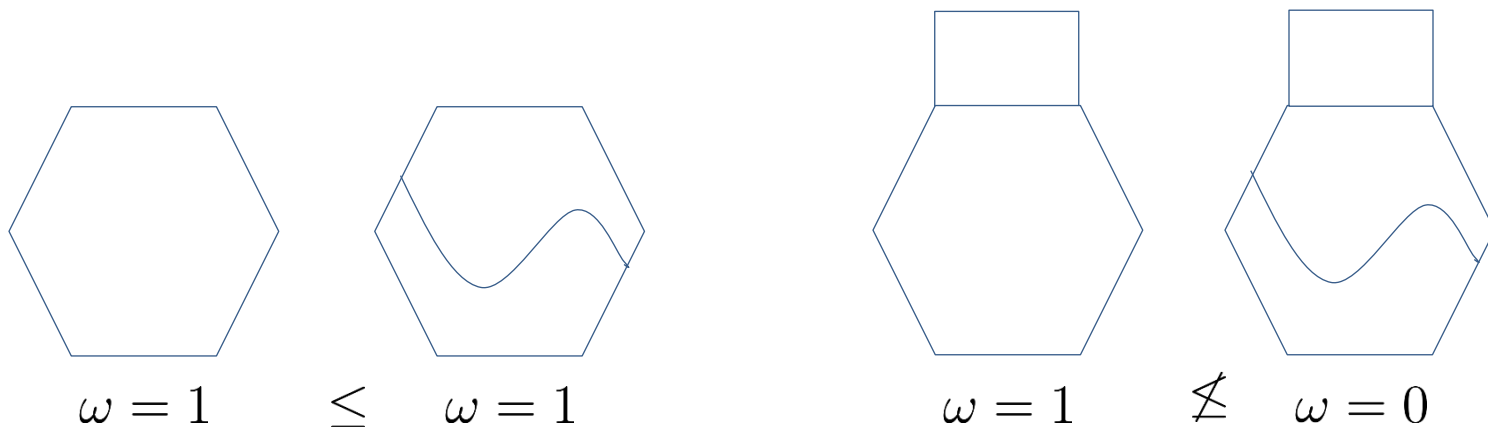
# Hierarchical increasingness

- The energies holding on partial partitions are far from being always  $h$ -increasing.
- Consider the partial partitions of support  $S$ .

$\omega(\pi(S)) = 1$  when  $\pi(S)$  has at most two components,

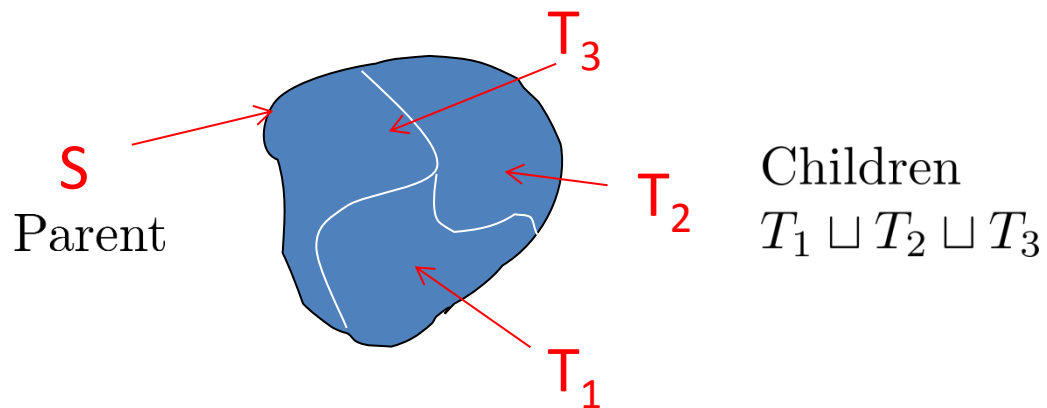
$\omega(\pi(S)) = 0$  when  $\pi(S)$  when not.

The energy  $\omega$  above is obviously not  $h$ -increasing:



# How to construct a climbing energy?

- To get an  $h$ -increasing energy, it suffices to start from an arbitrary energy on  $\mathcal{S}$



- and to extend it to the partial partitions of support  $S$  and of classes  $T_1, T_2, T_3$  by admissible composition rules, e.g.

$$\omega(\pi) = \omega(T_1) + \omega(T_2) + \omega(T_3) \text{ or } \omega(\pi) = \omega(T_1) \vee \omega(T_2) \vee \omega(T_3)$$

# How to construct a climbing energy?

- Examples of  $h$ -increasing energies:

Addition: Mumford-Shah: Salembier, Guigues

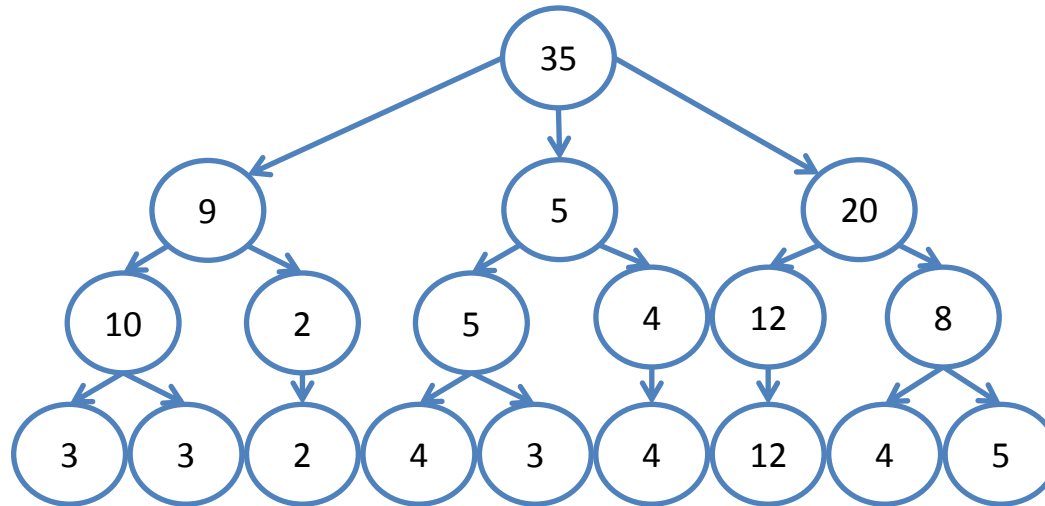
Supremum: Soille-Grazzini, Akcay-Aksoy, Wavelets.

- When  $\omega$  is  $h$ -increasing, and when

$$\omega(T_1 \sqcup T_2 \sqcup T_3) = \omega(S)$$

- then we generate a climbing energy by taking either the father or the sons by any external constraint independent of  $\omega$
- For example, by taking always the father, or choosing according to the number of sons (e.g. textures).

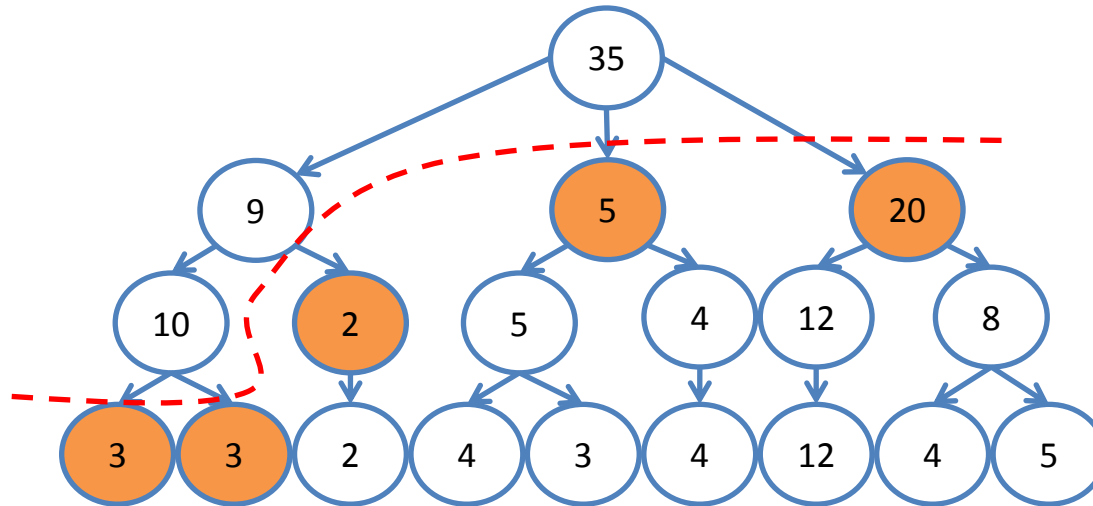
# Sum generated energies (Salembier-Guigues)



- The value  $\omega(S)$  at node  $S$  is compared to the sum  $\sum_k \omega(T_k)$  of the energies of the sons:
- if  $\omega(S) \leq \sum_k \omega(T_k)$  , one keeps the class  $S$ ,
- if not replace by its sons

**The optimal cut is then the union of the remaining classes.**

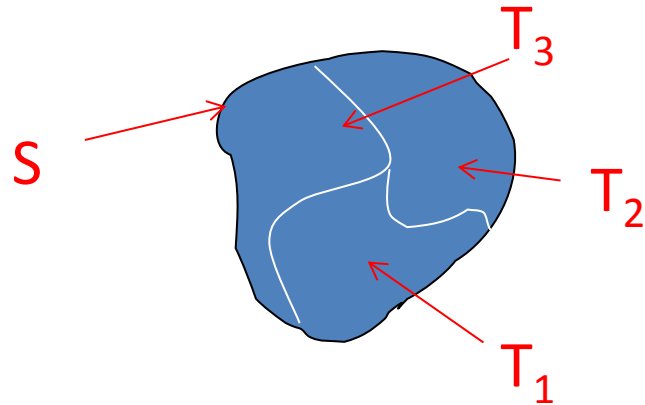
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# An example : Mumford-Shah



$$\omega(S, \lambda) = \sum_{1 \leq k \leq p} \omega_{\varphi}(T_k) + \lambda \sum_{1 \leq k \leq p} \omega_{\partial}(T_k)$$

with  $\omega_{\varphi}(T) = \int_{x \in T} \|f(x) - \mu(T)\|^2$  fidelity term.  
and  $\omega_{\partial}(T) = |\partial T|$  regularity term



# Optimal Cut: Luminance



$$\omega(S, \lambda) = \sum_{1 \leq k \leq p} \omega_{\varphi}(T_k) + \lambda \sum_{1 \leq k \leq p} \omega_{\partial}(T_k)$$

with luminance

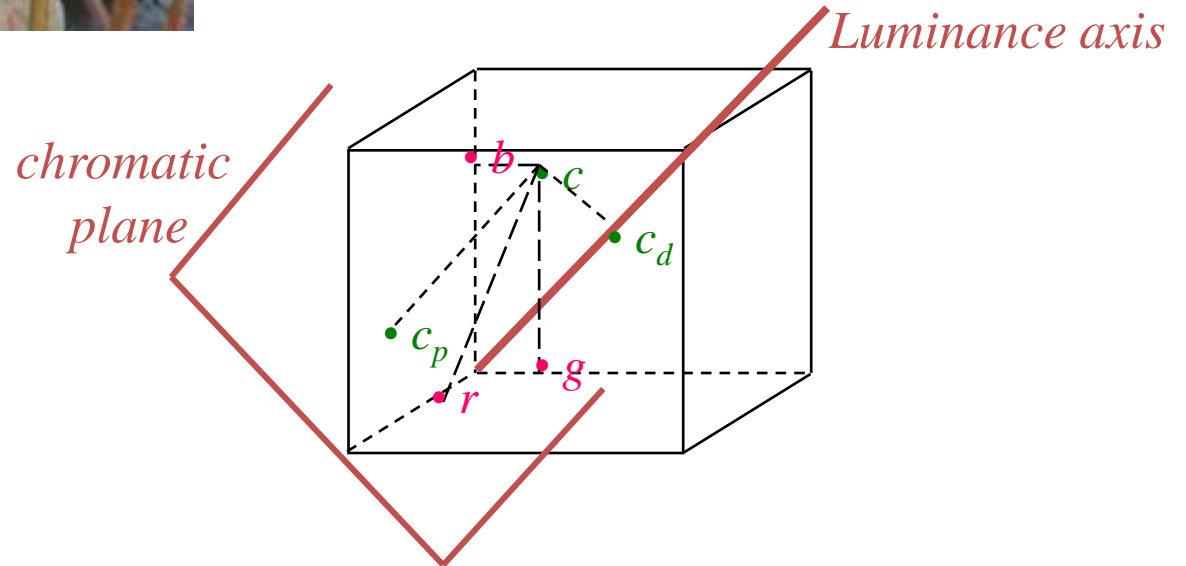
$$\omega_{\varphi}(T) = \int_{x \in T} \|l(x) - \mu(T)\|^2$$

fidelity term.

# Luminance-Chrominance



*Initial image*



# Optimal Cut: Luminance



$$\omega(S, \lambda) = \sum_{1 \leq k \leq p} \omega_{\varphi}(T_k) + \lambda \sum_{1 \leq k \leq p} \omega_{\partial}(T_k)$$

with luminance (top right)

$$\omega_{\varphi}(T) = \int_{x \in T} \|l(x) - \mu(T)\|^2$$

fidelity term.

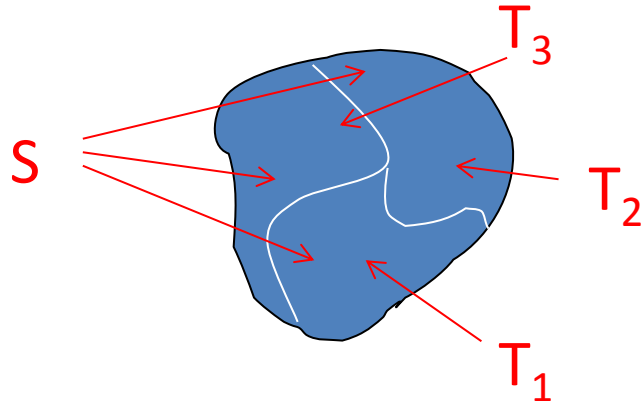
with chrominance (bottom right)

$$\omega_{\varphi}(T) = \sum_i \int_{x \in T} \|c_i(x) - \mu_i(T)\|^2$$

fidelity term.



# Another example: color and texture



$$\omega(S, \lambda) = \sum_{1 \leq k \leq p} \omega_{\varphi}(T_k) + \lambda \sum_{1 \leq k \leq p} \omega_{\partial}(T_k) + \mu \omega_{\rho}(\{T_k\})$$

with chrominance

$$\omega_{\varphi}(T) = \sum_i \int_{x \in T} \|c_i(x) - \mu_i(T)\|^2 \text{ fidelity term.}$$

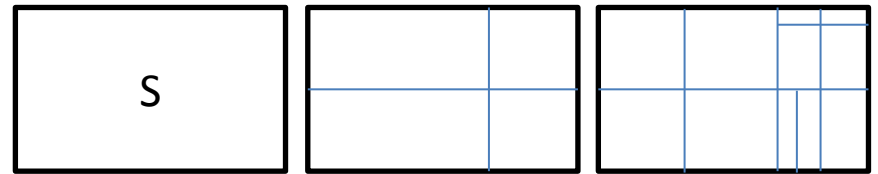
$$\omega_{\partial}(T) = |T| \text{ Regularization term - contour length}$$

$$\omega_{\rho}(\{T\}) = \sum (\{|T|\} - \mu(\{|T|\}))^2 \text{ Regularization term - texture regularity}$$

# Another example: color and texture



*Initial image*



*Partition with min variation  
in component sizes*

# Another example: color and texture



Initial image

$$\omega(S, \lambda) = \sum_{1 \leq k \leq p} \omega_{\varphi}(T_k) + \lambda \sum_{1 \leq k \leq p} \omega_{\partial}(T_k) + \mu \omega_{\rho}(\{T_k\})$$

Right: optimal cuts:  
top, very uniform textures ( high  $\mu$  )

# Another example: color and texture



Initial image



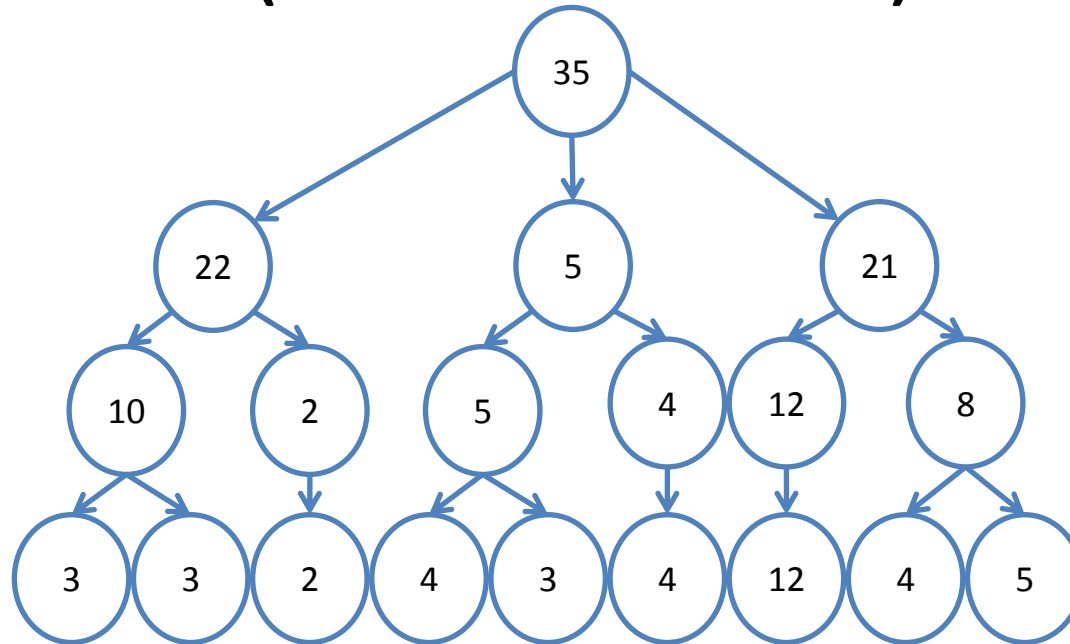
$$\omega(S, \lambda) = \sum_{1 \leq k \leq p} \omega_{\varphi}(T_k) + \lambda \sum_{1 \leq k \leq p} \omega_{\partial}(T_k) + \mu \omega_{\rho}(\{T_k\})$$

Right: optimal cuts:

- top, very uniform textures ( high  $\mu$  )
- bottom (weaker  $\mu$  )



# Suprema generated energies (Soille-Grazzini)

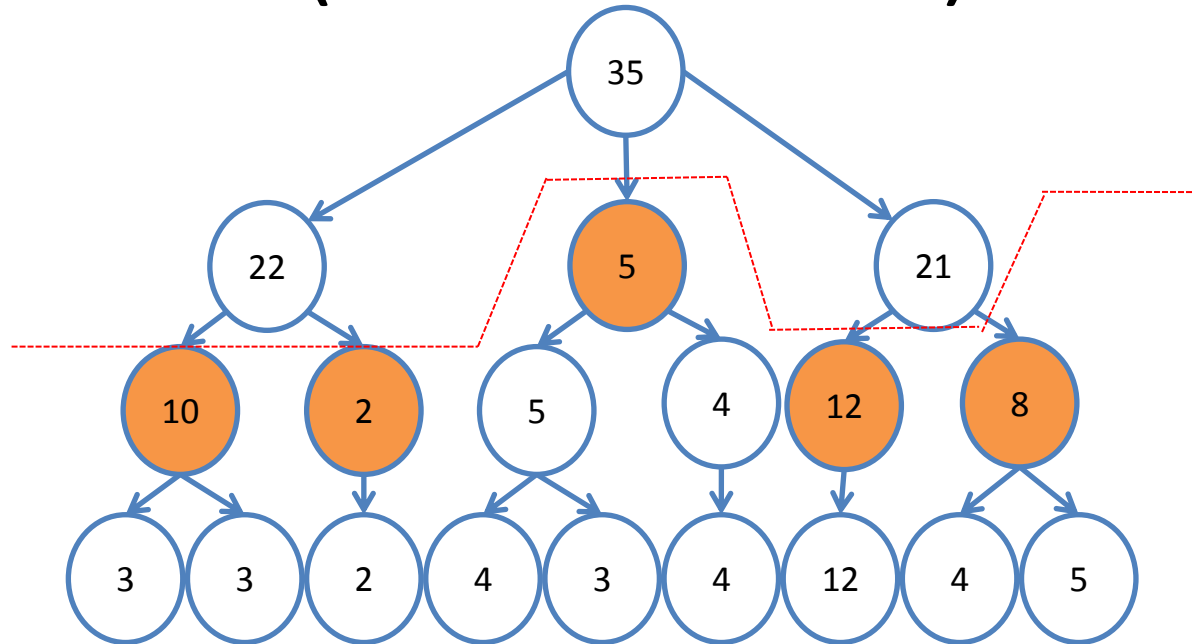


- The values of  $\omega(S)$  are supposed to increase as going up in the hierarchy. The value at node  $S$  is  $\max f(S) - \min f(S)$ .
- Node  $S$  is kept when  $\omega(S) \leq k$ . (here  $k = 20$ )

The optimal cut is the union of the largest remaining nodes.



# Suprema generated energies (Soille-Grazzini)



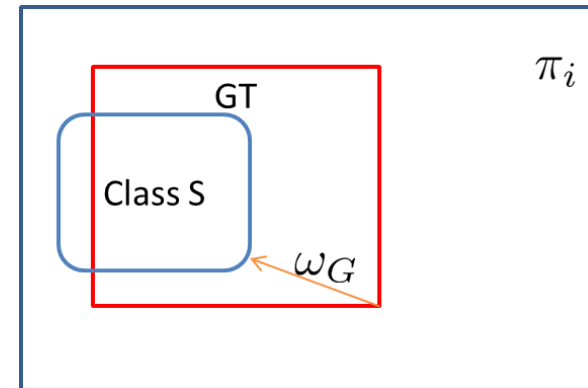
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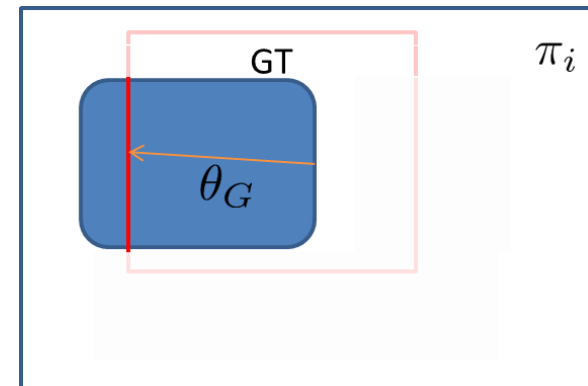
# Ground truth Evaluation

In the poster session we have an example that performs composition by suprema and infima applied to the problem of evaluation of hierarchies by ground truth.

- *Local measures*: Each class  $S$  in  $H$  is assigned 2 radii:  $\omega_G, \theta_G$
- Given a hierarchy  $H$  and ground truth partition  $G$  find the partition in  $H$  closest to  $G$ .
  - Closest from  $H \rightarrow G$
  - Closest from  $G \rightarrow H$



minimum radius of dilation of ground truth contour that covers the contour of S.



minimum radius of dilation of the contour of S to cover GT within S.

# Plan

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5. Climbing families of energies.

# Climbing families of energies

- The energy often depends on a positive parameter, i.e.  $\{\omega^\lambda, \lambda > 0\}$ . Is it then possible to order the optimal cuts according to  $\lambda$ ?
- The family  $\{\omega^\lambda, \lambda > 0\}$  is said to be climbing when:

each  $\omega^\lambda$  is climbing (i.e. singular and h-increasing) for any partial partition  $\pi$  of support  $S$  we have

$$\lambda \leq \mu \quad \text{and} \quad \omega^\lambda(S) \leq \omega^\lambda(\pi) \Rightarrow \omega^\mu(S) \leq \omega^\mu(\pi)$$

**Proposition:** When the family  $\{\omega^\lambda, \lambda > 0\}$  of energies is climbing, then the optimal cuts increase with  $\lambda$  (for the refinement ordering)

# Another example

*Initial image*



*Chrominance as fidelity  
term*

$$\lambda = 0$$

***N.B. no regularity term***



# Another example

*Initial image*



*Chrominance as fidelity  
term*

$$\lambda = 400$$



# Another example

*Initial image*



*Chrominance as fidelity  
term*

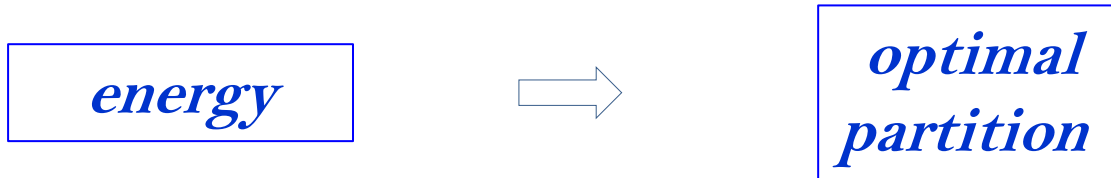
$$\lambda = 10,000$$

***Note: the textures have  
been filtered out.***



# Conclusions

- We replaced the numerical approach



by the lattice one



which adds a local meaning to the global energy  $\omega$ , (similar to the uniform convergence versus the simple one).



# Conclusions

- We replaced the variational approach by the axiomatics

*Singular and h-increasing energy = climbing energy*

which allows the fast computation

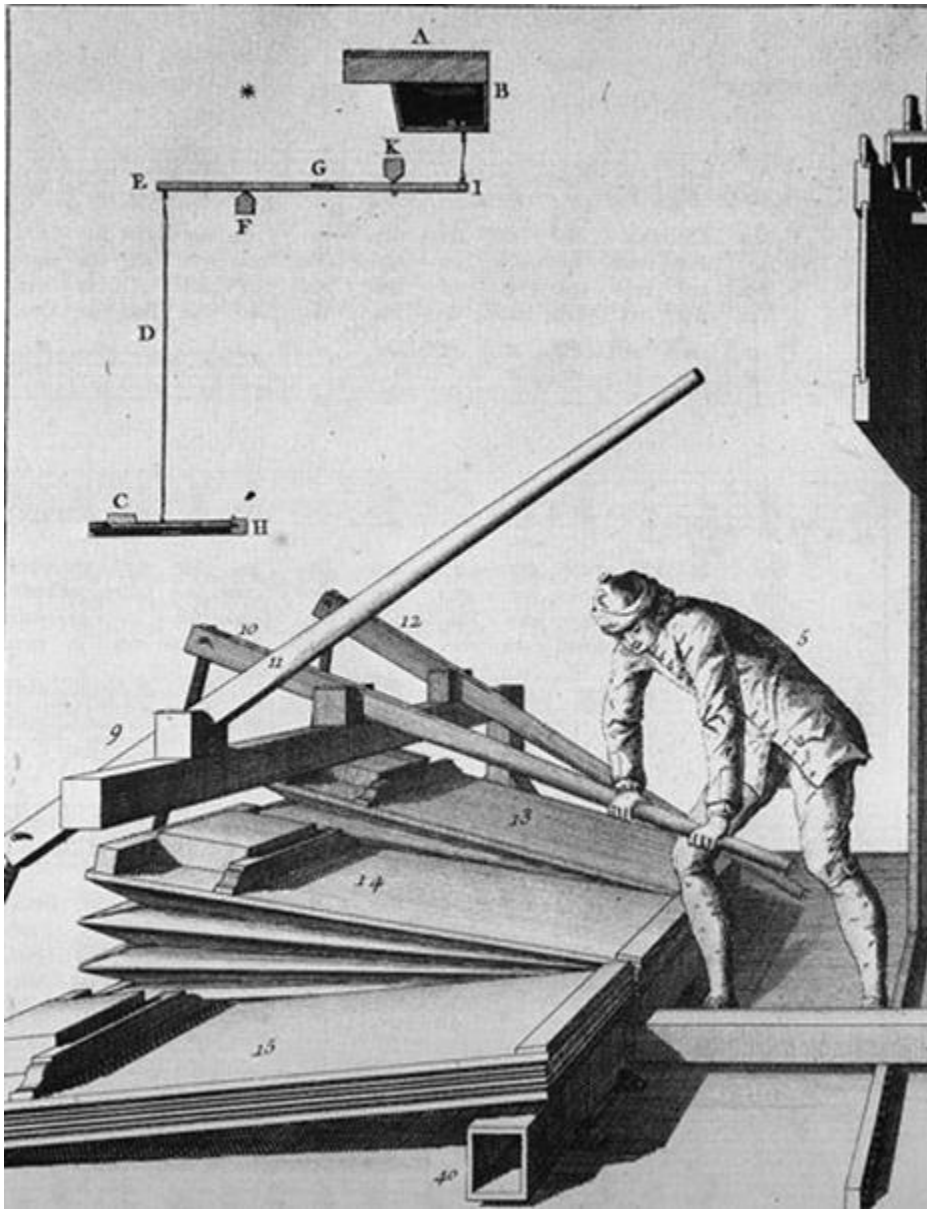
*climbing energy*  $\Rightarrow$  *optimal cut in one pass*

- We introduced the climbing families of energies  
Which results in

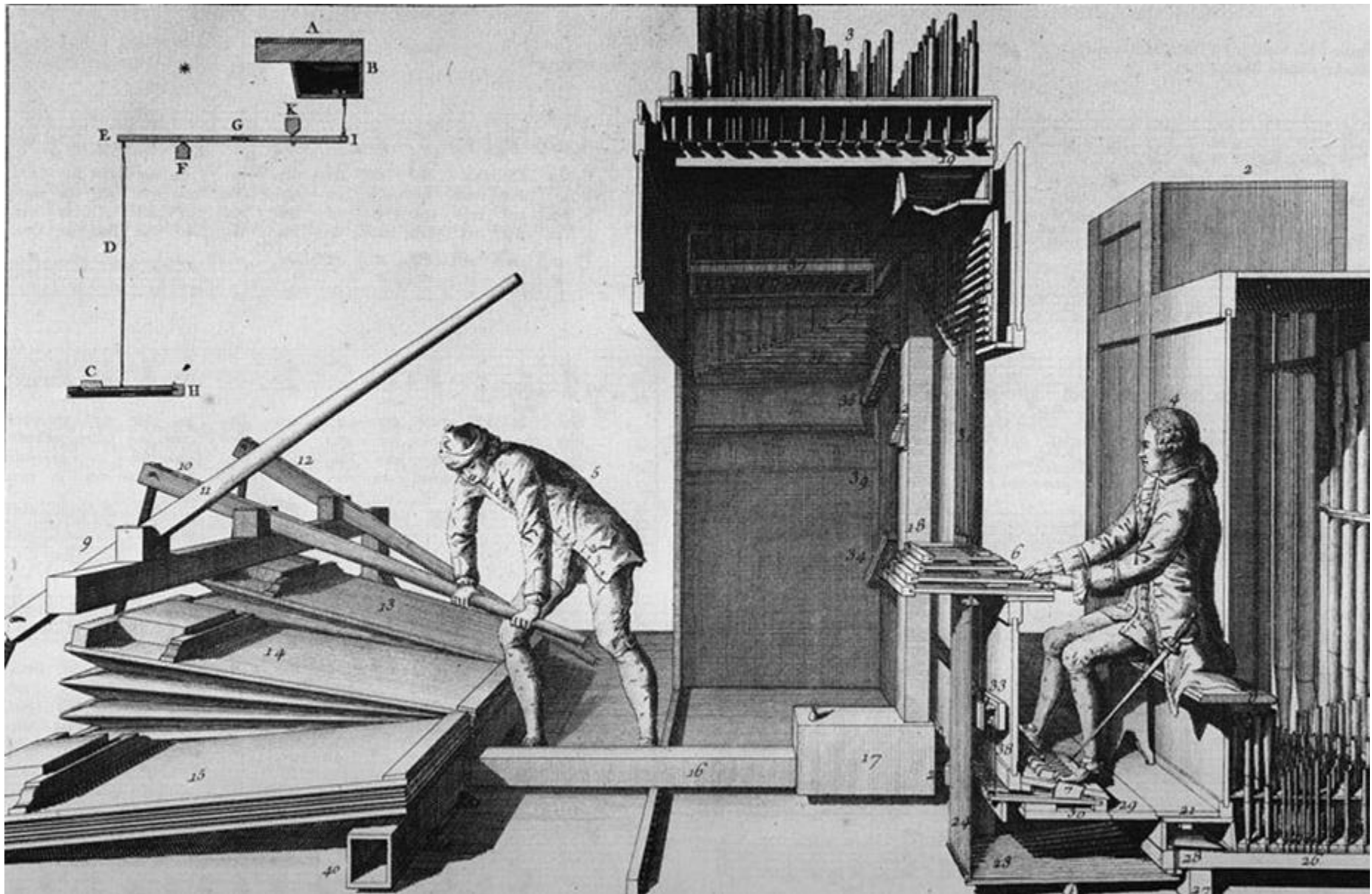
*Climbing  
families  
of energies*



*Hierarchies  
of  
optimal partitions*



.... A study by *me*



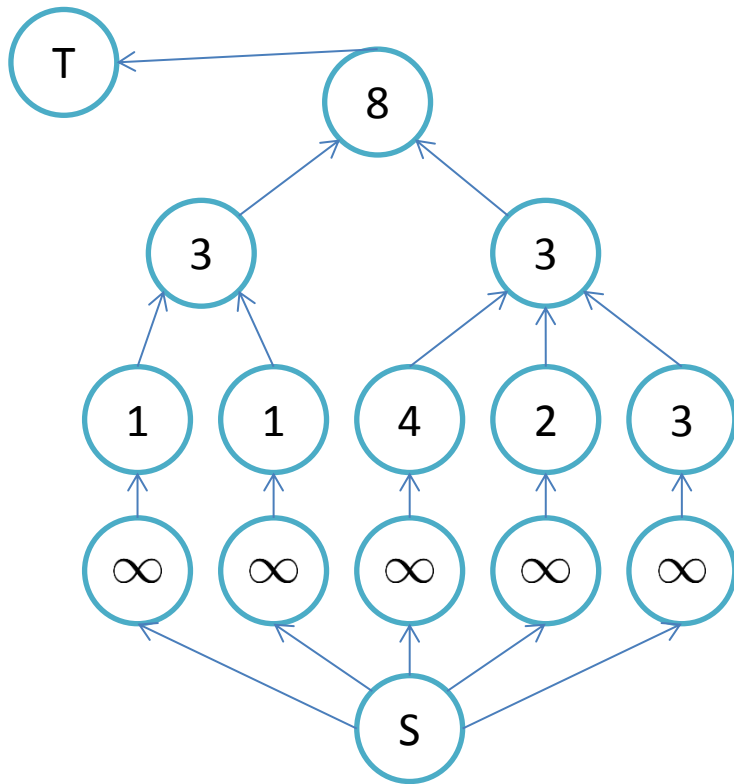
.... A study by *me*, and *my advisor*.



# Additive energies and graph-cuts

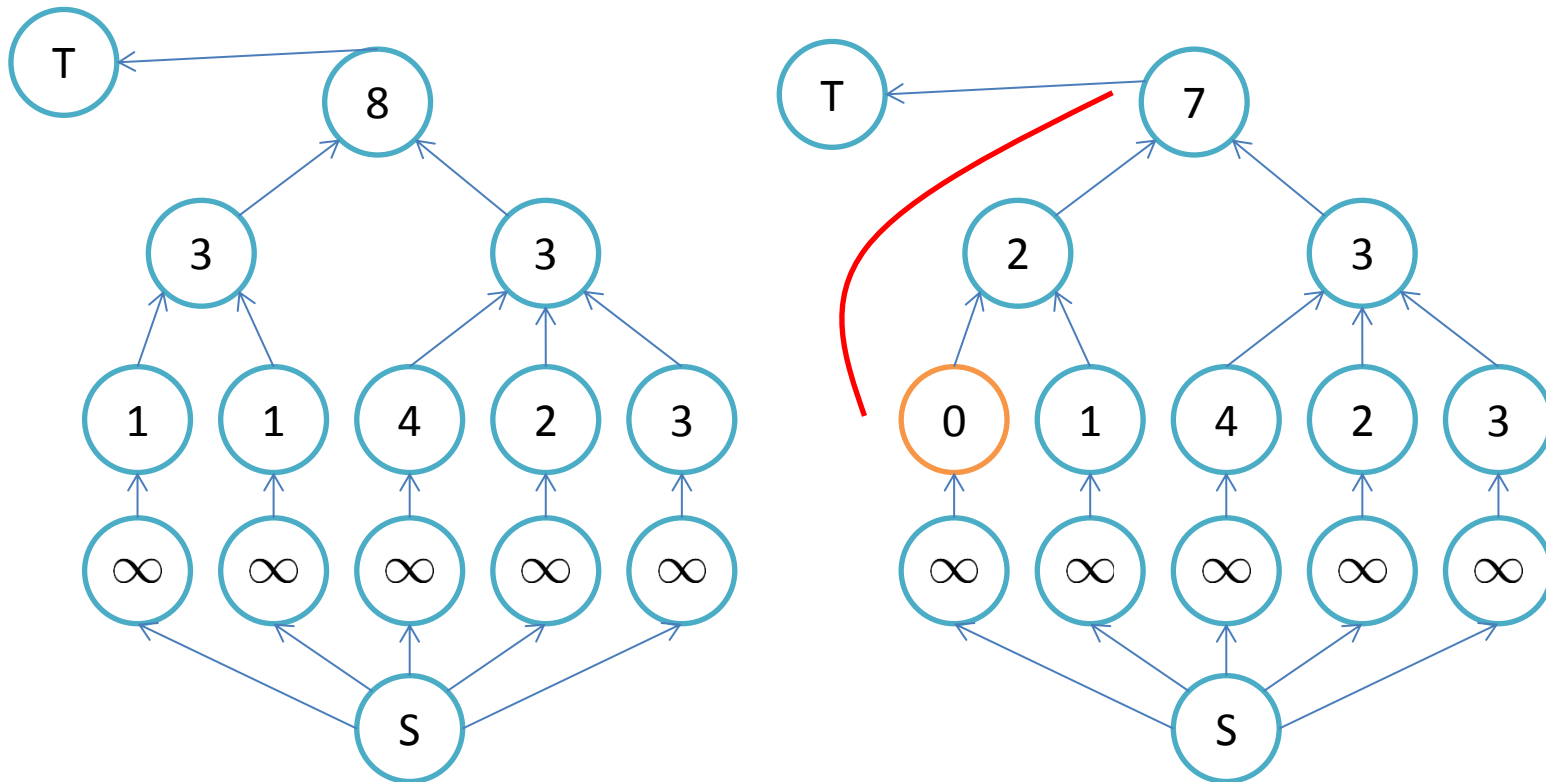
- The definition of a flow through  $G$  requires the data of
  - a source: the leaves, with infinite weight,
  - a sink: the root,
  - and a flow capacity at each node.
- The flows of two separated paths are
  - independent,
  - and upper-bounded by the lowest capacity along the path.
- When two lines meet at a (father) node, the capacities of the sons are added and compared to that of the father. One keeps the largest.

# min-cut versus optimal cut



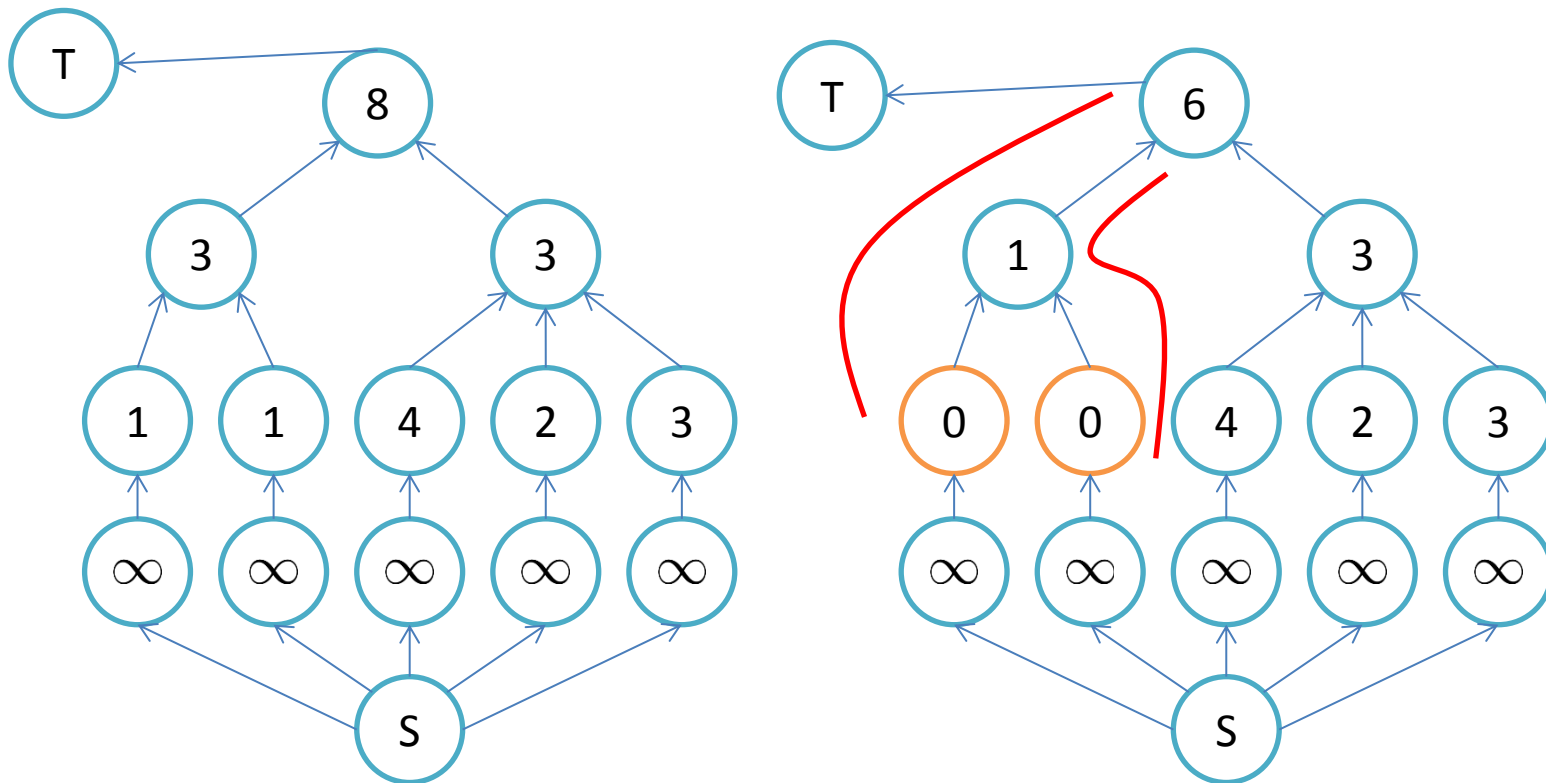
Initial hierarchy

# min-cut versus optimal cut



The minimum value on each path is subtracted from each node in the path, up till the point where we obtain a cut that separates S and T.

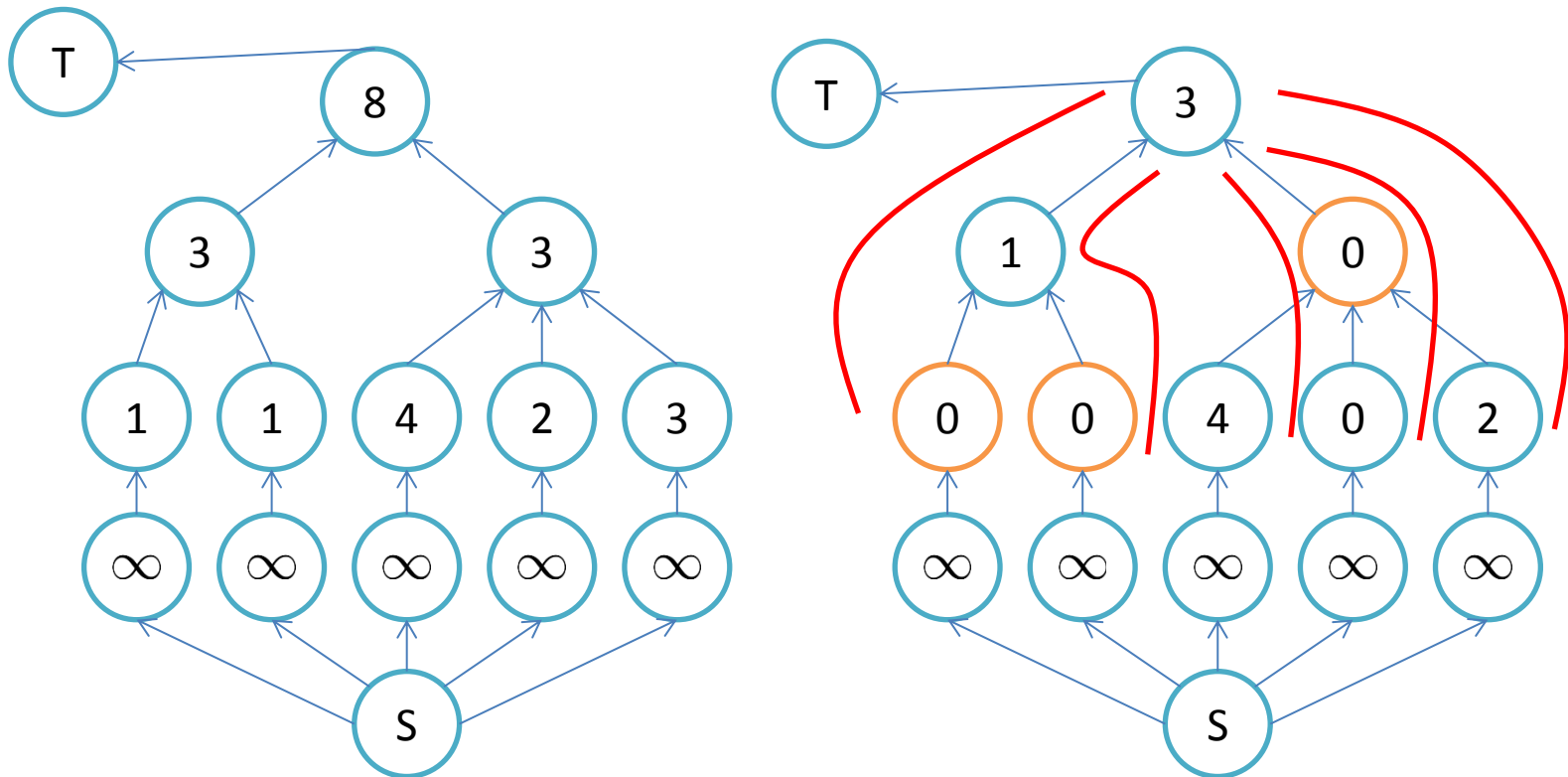
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# min-cut versus optimal cut



The minimum value on each path is subtracted from each node in the path, up till the point where we obtain a cut that separates S and T.

***This separation is exactly the optimal cut.***

# Composition of the $\vee$ -generated energies

- Let  $\{\omega_i, i \in I\}$  be a family of climbing energies and  $\{\lambda_i, i \in I\}$  A family of positive weights.
- Then the weighted supremum  $\omega = \vee \lambda_i \omega_i$  defines a climbing energy (but not the infimum).
- Paradoxically, the supremum can express an intersection of criteria. For example, if in  $S$ 
  - $\omega_1(S) = 0$  if the luminance range  $< k_1$ , and  $\omega_1(S) = 1$  if not,
  - $\omega_2(S) = 0$  if the saturation range  $< k_2$ , and  $\omega_2(S) = 1$  if not,
  - $\omega_3(S) = 0$  if the area of  $S$  is  $\geq k_3$ , and  $\omega_3(S) = 1$  if not,then the energy  $\vee \omega_i(S) = 0$  when  $S$  is not too small and rather constant in luminance and saturation.