



Pricing and hedging contingent claims with liquidity costs and market impact

Frédéric Abergel, Grégoire Loeper

► To cite this version:

Frédéric Abergel, Grégoire Loeper. Pricing and hedging contingent claims with liquidity costs and market impact. 2013. hal-00802402v1

HAL Id: hal-00802402

<https://hal.science/hal-00802402v1>

Preprint submitted on 22 Mar 2013 (v1), last revised 9 Sep 2013 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Pricing and hedging contingent claims with liquidity costs and market impact

Frederic Abergel and Grégoire Loeper
Chair of Quantitative Finance
Laboratory of Mathematics Applied to Systems,
École Centrale Paris, 92290 Châtenay Malabry, France

March 19, 2013

Abstract

We study the influence of taking liquidity costs and market impact into account when hedging a contingent claim, first in the discrete time setting, then in continuous time. In the latter case and in a complete market, we derive a fully non-linear pricing partial differential equation, and characterizes its parabolic nature according to the value of a numerical parameter naturally interpreted as a *relaxation coefficient* for market impact. We then investigate the more challenging case of stochastic volatility models, and prove the parabolicity of the pricing equation in a particular case.

Introduction

There is a long history of studying the effect of transaction costs and liquidity costs in the context of derivative pricing and hedging. Transaction costs due to the presence of a Bid-Ask spread are well understood in discrete time, see [5]. In continuous time, they lead to quasi-variational inequalities, see e.g. [12], and to imperfect claim replication due to the infinite cost of hedging continuously over time. In this work, the emphasis is put rather on **liquidity costs**, that is, the extra price one has to pay over the theoretical price of a tradable asset, due to the finiteness of available liquidity at the best possible price. A reference work for the modelling and mathematical study of liquidity in the context of a dynamic hedging strategy is [3], see also [10], and our results can be seen as partially building on the same approach.

It is however unfortunate that a major drawback occurs when adding liquidity costs: as can easily be seen in [3] [9] [10], the pricing and hedging equation are not unconditionally parabolic anymore and, therefore, only a local existence and uniqueness of smooth solutions may be available. Note that this drawback can easily be inferred from the very early heuristics in [6]: the formula suggested by Leland makes perfectly good sense for small perturbation of the initial volatility, but is meaningless when the modified volatility becomes negative. A partial conclusion is that incorporating liquidity cost leads to ill-posed pricing equation for large option positions, a situation which cannot be considered satisfactory and hints at the fact that some ingredient may be missing: this missing ingredient is precisely the market impact of the delta-hedger, as will become clear from our results.

Motivated by the need for quantitative approaches to algorithmic trading, the study of **market impact** in order-driven markets has become a very active research subject in the past decade. In a very elementary way, there always is an instantaneous market impact - the **virtual impact** in [11] - whenever a transaction takes place, in the sense that the best available price immediately after a transaction may be modified if the size of the transaction is larger than the quantity available at the best limit in the order book. As many empirical works show, see e.g. [2] [11], a relaxation phenomenon then takes place: after a trade, the instantaneous impact decreases to a smaller value, the permanent impact. This phenomenon is named **resilience** in [11], it can be interpreted as a rapid, negatively correlated response of the market to large price changes due to

liquidity effects. In the context of derivative hedging, very little work has been done to take market impact into account. It is however clear that there are realistic situations - e.g., a large option on an illiquid stock - where the market impact of an option hedging strategy is significant. One may refer to [4] [7] [10] for early attempts, although in these references, market impact and liquidity costs are not taken jointly into account, a situation that leads again to ill-posed problem. As we shall demonstrate, the level of **permanent** impact plays a key role in the well-posedness of the pricing and hedging equation, and the fact that it was overlooked in the previous work on liquidity costs is the reason why the pricing equations in those models were not parabolic.

This paper aims at filling this gap by laying the foundation for a reasonable model of liquidity costs and market impact for derivative hedging. We start in a discrete time setting, where notions are best introduced and properly defined, and then move on to the continuous time case, restricting ourselves to continuous, Itô semi-martingales. Liquidity costs are modelled by a simple, stationary order book, characterized by its shape around the best price, and permanent market impact is measured by a numerical parameter γ , $0 \leq \gamma \leq 1$: $\gamma = 0$ means no permanent impact, whereas $\gamma = 1$ means no relaxation. This simplified representation of market impact rests on the hypothesis that the characteristic time of the derivative hedger may be different than the relaxation time of the order book, a realistic hypothesis since delta-hedge generally occurs at a lower frequency than does liquidity providing.

What we consider as our main result is Theorem 3.3, which states that, in the complete market case, the range of parameter for which the pricing equation is unconditionally parabolic is $\frac{2}{3} \leq \gamma \leq 1$. This result, which we find quite nice in that it is explicit in terms of the parameter γ , obviously explains the ill-posedness of the pricing equations in the references [3] [9] or [4] [7], since they all correspond to the case $\gamma = 0$ within our formulation. In particular, Theorem 3.3 implies that when re-hedging occurs at the same frequency as that at which liquidity is provided to the order book - that is, when $\gamma = 1$ - the pricing equation is well-posed. The paper is organized as follows: after recalling some classical notations and concepts, Section 1 presents the order book model that will be used to describe liquidity costs. Then, in Section 2, we write down the model for the observed price dynamics and study the associated risk-minimizing strategy taking into account liquidity costs and market impact. Section 3 is devoted to the continuous time version of the results in Section 2, in particular, to the case of a complete market. Finally, we address in Section 4 the difficult and interesting case of stochastic volatility models, for which partial results are presented.

1 Basic notations and definitions

To ease notations, we will assume that the risk-free interest rate is always 0.

Discrete time setting

The tradable asset price is modelled by a stochastic process S_k , ($k = 0, \dots, T$) on a probability space (Ω, \mathcal{F}, P) . \mathcal{F}_k denotes the σ -field of events observable up to and including time k . S_k is assumed to be adapted and square-integrable.

A contingent claim is a square-integrable random variable $H \in L^2(P)$ of the following form $H = \delta^H S_T + \beta^H$ with δ^H and β^H , \mathcal{F}_T -measurable random variables.

A trading strategy Φ is given by two stochastic processes δ_k , ($k = 0, \dots, T$) and β_k , ($k = 0, \dots, T$). δ_k (resp. β_k) is the amount of stock (resp. cash) held during period k , ($= [t_k, t_{k+1})$) and is fixed at the beginning of that period, *i.e.* we assume that δ_k (resp. β_k) is \mathcal{F}_k -measurable ($k = 0, \dots, T$). Moreover, δ and β are in $L^2(P)$.

The theoretical value of the portfolio at time k is given by

$$V_k = \delta_k S_k + \beta_k, (k = 1, \dots, T).$$

A strategy is *H-admissible* iff each V_k is square-integrable and $V_T = H$.

Continuous time setting

In the continuous case, (Ω, \mathcal{F}, P) is a probability space with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right-continuity and completeness. $T \in \mathbb{R}^{*+}$ denotes a fixed and finite time horizon. Moreover, \mathcal{F}_0 is trivial and $\mathcal{F}_T = \mathcal{F}$.

The risky asset $S = (S_t)_{0 \leq t \leq T}$ is a strictly positive, continuous \mathcal{F}_t -semimartingale, and a trading strategy Φ is a pair of càdlàg and adapted processes $\delta = (\delta_t)_{0 \leq t \leq T}$, $\beta = (\beta_t)_{0 \leq t \leq T}$, while a contingent claim is described by a random variable $H \in L^2(P)$, with $H = \delta^H S_T + \beta^H$, δ^H and β^H being \mathcal{F}_T -measurable random variables. H -admissible strategies are defined as follows:

Definition 1.0.1 *A trading strategy will be called H -admissible iff*

$$\left\{ \begin{array}{l} \delta_T = \delta^H \quad P - a.s. \\ \beta_T = \beta^H \quad P - a.s. \\ \delta \text{ has finite and integrable quadratic variation} \\ \beta \text{ has finite and integrable quadratic variation} \\ \delta \text{ and } \beta \text{ have finite and integrable quadratic covariation.} \end{array} \right.$$

Order book, transaction cost and impact

A constant, symmetric order-book profile is considered around the price \hat{S}_t of the asset S at a given time t **before** the option position is delta-hedged - think of \hat{S}_t as a theoretical price in the absence of the option hedger. $\mu(x)$ is the relative **density** (assumed to be nonnegative) of the order book, namely, the derivative of the function $M(x) \equiv \int_0^x \mu(t) dt \equiv$ number of shares one can buy (resp. sell) between the prices \hat{S}_t and $\hat{S}_t(1+x)$ for positive (resp. negative) x .

The instantaneous **market impact** of a transaction of size ϵ is then

$$I(\epsilon) = \hat{S}_t M^{-1}(\epsilon), \quad (1.1)$$

it is precisely the difference between the price before and after the transaction is completed.

The actual **cost** of the same transaction is

$$C(\epsilon) = \hat{S}_t \int_0^{M^{-1}(\epsilon)} (1+x) \mu(x) dx \equiv \hat{S}_t \int_0^\epsilon (1 + M^{-1}(y)) dy. \quad (1.2)$$

We denote by κ the function M^{-1} .

2 Cost process with market impact in discrete time

In this section, we focus on the discrete time case.

2.1 The observed price dynamics

The model for the dynamics of the observed price - that is, the price S_k that the market can see at every time t_k after the re-hedging is complete - is now presented.

A natural modelling assumption is that the price moves according to the following sequence of events:

- First, it changes under the action of the "market" according to some (positive) stochastic dynamics for the theoretical price increment $\Delta \hat{S}_k$

$$\hat{S}_k \equiv S_{k-1} + \Delta \hat{S}_k \equiv S_{k-1}(1 + \Delta M_k + \Delta A_k), \quad (2.1)$$

where ΔM_k (resp. ΔA_k) is the increment of an \mathcal{F} -martingale (resp. an \mathcal{F} -predictable process).

- Then, the hedger applies some extra pressure by re-hedging her position, being thereby subject to liquidity costs and market impact as introduced in Section 1. As a consequence, the dynamics of the observed price is

$$S_k = S_{k-1}(1 + \Delta M_k + \Delta A_k)(1 + \gamma\kappa(\delta_k - \delta_{k-1})), \quad (2.2)$$

where γ , $0 \leq \gamma \leq 1$, measures the permanent impact.

2.2 Incremental cost of the hedging strategy

Following the approach developed in [9], the incremental cost ΔC_k of re-hedging at time t_k is now studied: the strategy consists in buying $\delta_k - \delta_{k-1}$ shares of the asset, and rebalancing the cash account from β_{k-1} to β_k . With the notations just introduced in Section 2.1, there holds

$$\Delta C_k = S_k \int_0^{\delta_k - \delta_{k-1}} (1 + \kappa(u)) du + (\beta_k - \beta_{k-1}). \quad (2.3)$$

Using the value process

$$V_k = \beta_k + \delta_k S_k \equiv \beta_k + \delta_k \hat{S}_k (1 + \gamma\kappa(\delta_k - \delta_{k-1})), \quad (2.4)$$

one can then rewrite the incremental cost between t_{k-1} and t_k as

$$\Delta C_k = (V_k - V_{k-1}) - (\delta_k S_k - \delta_{k-1} S_{k-1}) + \hat{S}_k \int_0^{\delta_k - \delta_{k-1}} (1 + \kappa(u)) du. \quad (2.5)$$

Straightforward computations lead to

$$\Delta C_k = (V_k - V_{k-1}) - \delta_{k-1}(S_k - S_{k-1}) + S_k \left(\frac{\int_0^{\delta_k - \delta_{k-1}} (1 + \kappa(u)) du}{1 + \gamma\kappa(\delta_k - \delta_{k-1})} - (\delta_k - \delta_{k-1}) \right), \quad (2.6)$$

or equivalently

$$\Delta C_k = (V_k - V_{k-1}) - \delta_{k-1}(S_k - S_{k-1}) + S_k \left(\frac{\int_0^{\delta_k - \delta_{k-1}} \kappa(u) du - \gamma(\delta_k - \delta_{k-1})\kappa(\delta_k - \delta_{k-1})}{1 + \gamma\kappa(\delta_k - \delta_{k-1})} \right). \quad (2.7)$$

To ease the notations, let us define, for $x \in \mathbf{R}$,

$$\mathbf{g}(x) \equiv \frac{\int_0^x \kappa(u) du - \gamma x \kappa(x)}{1 + \gamma \kappa(x)}. \quad (2.8)$$

\mathbf{g} is a smooth function satisfying

$$\mathbf{g}(0) = \mathbf{g}'(0) = 0, \quad \mathbf{g}''(0) = (1 - 2\gamma)\kappa'(0),$$

thanks to the natural assumption on the instantaneous price impact function $\kappa(0) = 0$.

We summarize our results in the

Proposition 2.1 *The incremental cost of implementing a hedging strategy at time t_k has the following expression*

$$\Delta C_k = (V_k - V_{k-1}) - \delta_{k-1}(S_k - S_{k-1}) + S_k \mathbf{g}(\delta_k - \delta_{k-1}), \quad (2.9)$$

where the function \mathbf{g} is given in (2.8).

2.3 Optimality conditions for quadratic local-risk minimization

Proposition (2.1) is the starting point for the characterization of a local-risk minimizing strategy. Upon using a quadratic criterion, and under some assumptions ensuring the convexity of the quadratic risk, see e.g. [9], one easily derives the two (pseudo-)optimality conditions for the value process V_{k-1} and the hedge ratio δ_{k-1} :

$$E(\Delta C_k | \mathcal{F}_{k-1}) = 0 \quad (2.10)$$

and

$$E((\Delta C_k)(S_k - S_{k-1} + S_k \mathbf{g}'(\delta_k - \delta_{k-1})) | \mathcal{F}_{k-1}) = 0. \quad (2.11)$$

Equation (2.11) can be better understood - especially when passing to the continuous time limit - by introducing a modified price process accounting for the cumulated effect of liquidity costs and market impact, as in [9] [3]. To this end, we introduce the

Definition 2.3.1 *The supply price \bar{S} is the process defined by*

$$\bar{S}_0 = S_0 \quad (2.12)$$

and, for $k \geq 1$,

$$\bar{S}_k - \bar{S}_{k-1} = S_k(1 + \mathbf{g}'(\delta_k - \delta_{k-1})) - S_{k-1}, \quad (2.13)$$

where \mathbf{g} is defined in Equation (2.8).

Using \bar{S} instead of S , Equation (2.11) can be rewritten as

$$E((\Delta C_k)(\bar{S}_k - \bar{S}_{k-1}) | \mathcal{F}_{k-1}) = 0, \quad (2.14)$$

again, a familiar expression in the context of local-risk minimization.

One can easily notice that Equations (2.10) and (2.11) reduce exactly to Equations (2.1) in [9] when market impact is neglected ($\gamma = 0$) and the risk function is quadratic.

3 The continuous-time setting

This section is devoted to the characterization of the limiting equation for the value and the hedge parameter when the time step goes to zero. Since the proofs are identical to those given in [1] [9], we shall only provide formal derivations, limiting ourselves to the case of (continuous) Itô semi-martingales for the driving stochastic equations

3.1 The observed price dynamics

A first result concerns the dynamics of the observed price. Assuming that the underlying processes are continuous and taking limits in *ucp* topology, one shows that the continuous-time equivalent of (2.2) is

$$dS_t = S_t(dX_t + dA_t + \gamma \kappa'(0) d\delta_t) \quad (3.1)$$

where X is a continuous martingale and A is a continuous, predictable process of bounded variation. Equation (3.6) is fundamental in that it contains the information on the strategy-dependent volatility of the observed price that will lead to fully non-linear parabolic pricing equation. In fact, the following result holds true:

Lemma 3.1 Consider a hedging strategy δ which is a function of time and the observed price S at time t : $\delta_t \equiv \delta(S_t, t)$. Then, the observed price dynamics (3.6) can be rewritten as

$$(1 - \gamma\kappa'(0)S_t \frac{\partial \delta}{\partial S}) \frac{dS_t}{S_t} = dX_t + dA'_t, \quad (3.2)$$

where A' is another predictable, continuous process of bounded variation.

Proof: use Itô's lemma in Equation (3.6).

3.2 Cost of a strategy and optimality conditions

At this stage, we are not concerned with the actual optimality - with respect to local-risk minimization - of pseudo-optimal solutions, but rather, with pseudo-optimality in continuous time. Hence, we shall use Equations (2.10)(2.14) as a starting point when passing to the continuous time limit.

Thanks to $\mathbf{g}'(0) = 0$, there holds the

Proposition 3.2 The cost process of an admissible hedging strategy (δ, V) is given by

$$C_t \equiv \int_0^t (dV_u - \delta dS_u + \frac{1}{2} S_u \mathbf{g}''(0) d\langle \delta, \delta \rangle_u). \quad (3.3)$$

Moreover, an admissible strategy is (pseudo-) **optimal** iff it satisfies the two conditions

- C is a martingale
- C is orthogonal to the supply price process \bar{S} , with

$$d\bar{S}_t = dS_t + S_t(\mathbf{g}''(0)d\delta_t + \frac{1}{2} \mathbf{g}^{(3)}(0) d\langle \delta, \delta \rangle_t). \quad (3.4)$$

In particular, if C is pseudo-optimal, there holds that

$$d\langle C, \bar{S} \rangle_t \equiv d\langle V, S \rangle_t - \delta d\langle S, S \rangle_t + \mathbf{g}''(0)S_t d\langle V, \delta \rangle_t - \delta S_t \mathbf{g}''(0) d\langle \delta, S \rangle_t = 0. \quad (3.5)$$

3.3 The case of a complete market

3.3.1 The case of a single asset

It is of course interesting and useful to fully characterize the results of Section 3.2 in the case of a complete market. Hence, we assume in this section that the driving factor X is a one-dimensional Wiener process W and that \mathcal{F} is its natural filtration, so that the increment of the observed price is simply

$$dS_t = S_t(\sigma dW_t + \gamma\kappa'(0)d\delta_t + dA_t) \quad (3.6)$$

where the "unperturbed" volatility σ is supposed to be constant. We also make the **markovian** assumption that the strategy is a function of the state variable S and of time.

Under this set of assumptions, the orthogonality condition becomes trivial: the cost process C has to be identically 0, Equation (3.5) yields

$$\frac{\partial V}{\partial S} = \delta, \quad (3.7)$$

while the martingale condition for the cost process C_t reads

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial S^2} + S_t \mathbf{g}''(0) \left(\frac{\partial^2 V}{\partial S^2} \right)^2 \right) \frac{d\langle S, S \rangle_t}{dt} = 0. \quad (3.8)$$

Applying Lemma 3.1 yields

$$(1 - \gamma\kappa'(0)S_t \frac{\partial \delta}{\partial S}) \frac{dS_t}{S_t} = \sigma dW_t + dA'_t, \quad (3.9)$$

that is:

$$\frac{d \langle S, S \rangle_t}{dt} = \frac{\sigma^2 S_t^2}{(1 - \gamma\kappa'(0)S_t \frac{\partial \delta}{\partial S})^2}. \quad (3.10)$$

Hence, taking (3.7) into account, there holds

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial S^2} + \mathbf{g}''(0)S_t \left(\frac{\partial^2 V}{\partial S^2} \right)^2 \right) \frac{\sigma^2 S_t^2}{(1 - \gamma\kappa'(0)S_t \frac{\partial \delta}{\partial S})^2} = 0 \quad (3.11)$$

or, using (3.7) and the identity $\mathbf{g}''(0) = (1 - 2\gamma)\kappa'(0)$:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial S^2} (1 + (1 - 2\gamma)\kappa'(0)S_t \frac{\partial^2 V}{\partial S^2}) \right) \frac{\sigma^2 S_t^2}{(1 - \gamma\kappa'(0)S_t \frac{\partial^2 V}{\partial S^2})^2} = 0. \quad (3.12)$$

Equation (3.12) can be seen as the pricing equation in our model: any contingent claim can be perfectly replicated at zero cost, as long as one can exhibit a solution to (3.12). Consequently, of the utmost importance is the parabolicity of the pricing equation (3.12).

For instance, the case $\gamma = 1$ corresponding to a full market impact (no relaxation) yields the following equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \frac{\sigma^2 S_t^2}{(1 - \gamma\kappa'(0)S_t \frac{\partial^2 V}{\partial S^2})} = 0, \quad (3.13)$$

which can be shown to be parabolic, see [8]. In fact, there holds the sharp result

Theorem 3.3 *The non-linear backward partial differential operator*

$$V \rightarrow \frac{\partial V}{\partial t} + \frac{1}{2} \left((1 + (1 - 2\gamma)\kappa'(0)S_t \frac{\partial^2 V}{\partial S^2}) \right) \frac{\sigma^2 S_t^2}{(1 - \gamma\kappa'(0)S_t \frac{\partial^2 V}{\partial S^2})^2} \frac{\partial^2 V}{\partial S^2} \quad (3.14)$$

is parabolic whenever $\frac{2}{3} \leq \gamma \leq 1$.

Proof: the parabolic nature of the operator is determined by the monotonicity of the function

$$p \rightarrow F(p) = \frac{p(1 + (1 - 2\gamma)p)}{(1 - \gamma p)^2}. \quad (3.15)$$

A direct computation shows that $F'(p)$ has the sign of $1 + (2 - 3\gamma)p$, so that F is globally (in p) monotonic increasing on its domain of definition $]-\infty, \frac{1}{\gamma}[$ whenever $\frac{2}{3} \leq \gamma \leq 1$. Therefore, the pricing equation is globally well-posed in this range of parameters.

3.3.2 The multi-asset case

For the sake of completeness, the case of several assets is worked out.

Assuming that the i^{th} asset, $1 \leq i \leq D$, has its own transaction cost structure and impact parameter characterized by (κ_i, γ_i) , one can easily extend the previous results and write down the modified dynamics for each observed price S_t^i

$$dS_t^i = S_t^i (\sigma_i dW_t^i + \gamma_i \kappa'_i(0) d\delta_t^i + dA_t^i) \quad (3.16)$$

with notations similar to those introduced in the one-dimensional case.

In fact, one can derive an even more general formulation, taking into account the potential market impact of one stock on another and obtain

$$dS_t^i = S_t^i (\sigma_i dW_t^i + \sum_{j=1}^D M_{ij} d\delta_t^j + dA_t^i), \quad (3.17)$$

where M_{ij} is the $D \times D$ matrix representing the respective market impact of asset j on asset i - Equation (3.17) corresponding to the diagonal case $M = \text{Diag}(\gamma_i \kappa'_i(0))_{1 \leq i \leq D}$. Denote by $\mathbf{S} \equiv (S^1, \dots, S^D)$ the vector-valued process of the D asset prices. Then, the completeness of the market allows one to use the perfect hedge

$$\delta_t^i = \frac{\partial V}{\partial S^i} \quad (3.18)$$

and the dynamics of \mathbf{S} modified by the hedging strategy can be written as

$$d\mathbf{S}_t = (\mathbf{I} - \mathcal{M})^{-1} \text{Diag}(\sigma_i S^i) d\mathbf{W}_t + d\mathbf{A}_t, \quad (3.19)$$

with

$$\mathcal{M}_{ik} = S^i \sum_{j=1}^D M_{ij} \frac{\partial^2 V}{\partial S^j \partial S^k}, \quad (3.20)$$

where $\mathbf{W} \equiv (W^1, \dots, W^D)$. The parabolic nature of the pricing equation can therefore be studied by means similar to the single asset case, but with more cumbersome computations.

4 Extension to stochastic volatility models

In this section, the case of stochastic volatility is considered. The results in Section 3.3.2 can be used in this context whenever the market is assumed to be completed, in the usual yet artificial way, *via* an option-based hedging strategy. In this case, the orthogonality conditions reduce to the usual δ -hedge, and the pricing equation can be studied as in Proposition ???. However, as we shall soon see, a realistic hedging strategy based on the tradable asset only leads to more difficult problems.

Let then the observed price process be a solution to the following set of SDE's

$$dS_t = S_t(\sigma_t dW_t^1 + \gamma \kappa'(0) d\sigma_t + \mu_t dt) \quad (4.1)$$

$$d\sigma_t = \nu_t dt + \Sigma_t dW_t^2 \quad (4.2)$$

where (W^1, W^2) is a two-dimensional Wiener process under \mathcal{P} with correlation ρ :

$$d \langle W^1, W^2 \rangle_t = \rho dt,$$

and the processes μ_t , ν_t and Σ_t are actually functions of the state variables S, σ .

We assume that the system (4.1, 4.2) admits a unique strong continuous solution with $S_t, \sigma_t > 0$, and consider a markovian framework, thereby looking for the value process V and the optimal strategy δ as smooth functions of the state variables

$$\begin{aligned} \delta_t &= \delta(S_t, \sigma_t, t) \\ V_t &= V(S_t, \sigma_t, t). \end{aligned}$$

Then, the dynamics of the observed price becomes

$$dS_t = \frac{S_t}{1 - \gamma \kappa'(0) S_t \frac{\partial \delta}{\partial S}} (\sigma_t dW_t^1 + \gamma \kappa'(0) \frac{\partial \delta}{\partial \sigma} d\sigma_t + dQ_t), \quad (4.3)$$

the orthogonality condition reads

$$\left(\frac{\partial V}{\partial S} - \delta \right) d \langle S, \bar{S} \rangle_t + \frac{\partial V}{\partial \sigma} d \langle \sigma, \bar{S} \rangle_t = 0 \quad (4.4)$$

and the pricing equation for the value function V is

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial S^2} - \gamma \kappa'(0) S_t \left(\frac{\partial \delta}{\partial S} \right)^2 \right) \frac{d \langle S, S \rangle_t}{dt} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial \sigma^2} - \gamma \kappa'(0) S_t \left(\frac{\partial \delta}{\partial \sigma} \right)^2 \right) \frac{d \langle \sigma, \sigma \rangle_t}{dt} +$$

$$+ \left(\frac{\partial^2 V}{\partial \sigma \partial S} - \gamma \kappa'(0) S_t \frac{\partial \delta}{\partial \sigma} \frac{\partial \delta}{\partial S} \right) \frac{d \langle S, \sigma \rangle_t}{dt} + \mathcal{L}_1 V = 0, \quad (4.5)$$

where \mathcal{L}_1 is a first-order partial differential operator.

Equations (4.4) and (4.5) are obviously quite complicated. In the next paragraph, we focus on a particular case that allows one to fully assess their well-posedness.

4.1 The case $\gamma = 1, \rho = 0$

When $\gamma = 1$, the martingale component of the supply price does not depend on the strategy anymore. As a matter of fact, the supply price dynamics is given by

$$d\bar{S}_t = dS_t + S_t(\mathbf{g}''(0)d\delta_t + \frac{1}{2}\mathbf{g}^{(3)}(0)d \langle \delta, \delta \rangle_t)$$

and therefore, using

$$\mathbf{g}''(0) = (1 - 2\gamma)\kappa'(0) = -\kappa'(0), \quad (4.6)$$

and the dynamics of S in (4.1), there holds that

$$d\bar{S}_t = S_t(\sigma_t dW_t^1 + dR_t), \quad (4.7)$$

where R is a predictable process of bounded variation.

If in addition, the Wiener processes for the asset and the volatility are supposed to be decorrelated: $\rho = 0$, the tedious computations leading to the optimal hedge and value function simplify, and one can study in full generality the well-posedness of the pricing and hedging equations (4.4)(4.5).

First and foremost, the orthogonality condition (4.4) simply reads in this case

$$\delta = \frac{\partial V}{\partial S}, \quad (4.8)$$

exactly as in the complete market case.

As for the pricing equation (4.5), one first works out using (4.8) the various brackets in (4.5) and finds that

$$\frac{d \langle S, S \rangle_t}{dt} = (1 - \kappa'(0) S_t \frac{\partial^2 V}{\partial S^2})^{-2} (\sigma_t^2 S_t^2 + \kappa'(0)^2 S_t^2 (\frac{\partial^2 V}{\partial S \partial \sigma})^2 \Sigma_t^2), \quad (4.9)$$

$$\frac{d \langle \sigma, \sigma \rangle_t}{dt} = \Sigma^2 \quad (4.10)$$

and

$$\frac{d \langle S, \sigma \rangle_t}{dt} = (1 - \kappa'(0) S_t \frac{\partial^2 V}{\partial S^2})^{-1} \kappa'(0) S_t \Sigma_t^2 \frac{\partial^2 V}{\partial S \partial \sigma}. \quad (4.11)$$

Plugging these expressions in (4.5) yields the pricing equation for V

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (1 - \kappa'(0) S_t (\frac{\partial^2 V}{\partial S^2}))^{-1} (\sigma_t^2 S_t^2 + \kappa'(0)^2 S_t^2 (\frac{\partial^2 V}{\partial S \partial \sigma})^2 \Sigma_t^2) + \frac{1}{2} (\frac{\partial^2 V}{\partial \sigma^2} - \kappa'(0) S_t (\frac{\partial^2 V}{\partial S \partial \sigma})^2) \Sigma^2 + \\ \kappa'(0) S_t \Sigma_t^2 (\frac{\partial^2 V}{\partial S \partial \sigma})^2 + \mathcal{L}_1 V = 0, \end{aligned} \quad (4.12)$$

or, after a few final rearrangements:

$$\frac{\partial V}{\partial t} + \frac{\sigma_t^2 S_t^2}{2(1 - \kappa'(0) S_t (\frac{\partial^2 V}{\partial S^2}))} \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma^2 + \frac{1}{2} \frac{\kappa'(0) S_t \Sigma^2}{(1 - \kappa'(0) S_t (\frac{\partial^2 V}{\partial S^2}))} (\frac{\partial^2 V}{\partial \sigma \partial S})^2 + \mathcal{L}_1 V = 0. \quad (4.13)$$

The main result of this section is the

Proposition 4.1 *Equation (4.13) is of parabolic type.*

Proof: one has to study the monotocity of the operator

$$\mathcal{L} : V \rightarrow \mathcal{L}(V) \equiv \frac{\sigma_t^2 S_t^2}{2(1 - \kappa'(0)S_t(\frac{\partial^2 V}{\partial S^2}))} \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma^2 + \frac{1}{2} \frac{\kappa'(0)S_t \Sigma^2}{(1 - \kappa'(0)S_t(\frac{\partial^2 V}{\partial S^2}))} \left(\frac{\partial^2 V}{\partial \sigma \partial S} \right)^2. \quad (4.14)$$

Introducing the classical notations

$$p \equiv \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \quad (4.15)$$

with $p_{11} = \frac{\partial^2 V}{\partial S^2}$, $p_{12} = p_{21} = \frac{\partial^2 V}{\partial S \partial \sigma}$ and $p_{22} = \frac{\partial^2 V}{\partial \sigma^2}$ and defining

$$\mathbf{L}(S, \mathbf{p}) \equiv \frac{\sigma_t^2 S_t^2 p_{11}}{(1 - \kappa'(0)S_t p_{11})} + \Sigma^2 p_{22} + \frac{\kappa'(0)S_t \Sigma^2}{(1 - \kappa'(0)S_t p_{11})} p_{12}^2, \quad (4.16)$$

one is led to study the positivity of the 2×2 matrix

$$\begin{pmatrix} \frac{\partial \mathbf{L}}{\partial p_{11}} & \frac{1}{2} \frac{\partial \mathbf{L}}{\partial p_{12}} \\ \frac{1}{2} \frac{\partial \mathbf{L}}{\partial p_{12}} & \frac{\partial \mathbf{L}}{\partial p_{22}} \end{pmatrix}. \quad (4.17)$$

Setting $F(p_{11}) = \frac{\sigma^2 S^2 p_{11}}{1 - \kappa'(0)S p_{11}}$ and $D(p_{11}) = 1 - \kappa'(0)S p_{11}$, one needs to show that the matrix $\mathbf{H}(\mathbf{p})$

$$\begin{pmatrix} F'(p_{11}) + (\kappa'(0)S \Sigma)^2 \frac{p_{12}^2}{D^2} & \kappa'(0)S \Sigma^2 \frac{p_{12}}{D} \\ \kappa'(0)S \Sigma^2 \frac{p_{12}}{D} & \Sigma^2 \end{pmatrix} \quad (4.18)$$

is positive. This result is trivially shown to be true by computing the trace and determinant of $\mathbf{H}(\mathbf{p})$:

$$Tr(\mathbf{H}(\mathbf{p})) = F'(p_{11}) + \Sigma^2 + (\kappa'(0)S \Sigma)^2 \frac{p_{12}^2}{D^2} \quad (4.19)$$

and

$$Det(\mathbf{H}(\mathbf{p})) = \Sigma^2 F'(p_{11}) \quad (4.20)$$

and using the fact that F is a monotonically increasing function.

This ends the proof of Proposition 4.13.

4.2 The correlated case

We now drop the assumption that $\rho = 0$, while still assuming that $\gamma = 1$. The supply price dynamics remains independent from the strategy, but δ cannot be eliminated from the pricing equation (4.5).

Using Lemma 2.2, one can write (4.3) as

$$\frac{dS_t}{S_t} = \frac{\sigma_t + \rho \kappa'(0) \frac{\partial \delta}{\partial \sigma} \Sigma}{1 - \kappa'(0)S_t \frac{\partial \delta}{\partial S}} dW_t^1 + \frac{\sqrt{1 - \rho^2} \kappa'(0) \frac{\partial \delta}{\partial \sigma} \Sigma}{1 - \kappa'(0)S_t \frac{\partial \delta}{\partial S}} dZ_t + dQ_t, \quad (4.21)$$

where W, Z are uncorrelated Wiener processes and Q is predictable, of finite variation. On the other hand, the supply price dynamics is again given by (4.7), so that the orthogonality condition becomes

$$\left(\frac{\partial V}{\partial S} - \delta \right) \frac{S(\sigma + \rho \kappa'(0) \Sigma \frac{\partial \delta}{\partial \sigma})}{1 - \kappa'(0)S \frac{\partial \delta}{\partial S}} + \rho \Sigma \frac{\partial V}{\partial \sigma} = 0 \quad (4.22)$$

or

$$\begin{vmatrix} \frac{\partial V}{\partial S} - \delta & -\rho \Sigma + \rho \kappa'(0) \Sigma S \frac{\partial \delta}{\partial \sigma} \\ \frac{\partial V}{\partial \sigma} & \sigma S + \rho \kappa'(0) \Sigma S \frac{\partial \delta}{\partial S} \end{vmatrix} = 0. \quad (4.23)$$

The expressions of the various angle brackets become

$$\frac{d \langle S, S \rangle_t}{dt} = (1 - \kappa'(0)S_t \frac{\partial^2 V}{\partial S^2})^{-2} (\sigma_t^2 S_t^2 + \kappa'(0)^2 S_t^2 (\frac{\partial \delta}{\partial \sigma})^2 \Sigma_t^2 + 2\rho \kappa'(0)^2 S_t^2 \Sigma_t \frac{\partial \delta}{\partial \sigma}), \quad (4.24)$$

$$\frac{d \langle \sigma, \sigma \rangle_t}{dt} = \Sigma^2 \quad (4.25)$$

and

$$\frac{d \langle S, \sigma \rangle_t}{dt} = (1 - \kappa'(0)S_t \frac{\partial^2 V}{\partial S^2})^{-1} \rho \Sigma_t (\kappa'(0)S_t \Sigma_t^2 \frac{\partial \delta}{\partial \sigma} + \sigma_t S_t). \quad (4.26)$$

References

- [1] F. Abergel and N. Millot. Non quadratic local risk-minimization for hedging contingent claims in incomplete markets. *SIAM Journal on Financial Mathematics*, 2(1):342–356, 2011.
- [2] R. Almgren, C. Thum, E. Hauptmann, and H. Li. Direct estimation of equity market impact. *working paper*.
- [3] U. Cetin, R. Jarrow, and P. Protter. Liquidity risk and arbitrage pricing theory. *Finance and Stochastics*, 8:311–341, 2004.
- [4] P. Henry-Labordère. *working paper*.
- [5] D. Lamberton, H. Pham, and M. Schweizer. Local risk-minimization under transaction costs. *Mathematics of Operations Research*, 23:585–612, 1997.
- [6] H. E. Leland. Option pricing and replication with transactions costs. *Journal of Finance*, 40(5):1283–1301, 1985.
- [7] H. Liu and J. M. Yong. Option pricing with an illiquid underlying asset market. *Journal of Economic Dynamics and Control*, 29:2125–2156, 2005.
- [8] G. Loeper. *in preparation*.
- [9] N. Millot and F. Abergel. Non quadratic local risk-minimization for hedging contingent claims in the presence of transaction costs. *Available at SSRN 1881175*, 2011.
- [10] A. Roch. Liquidity risk, volatility and financial bubbles. *PhD Thesis*.
- [11] P. Weber and B. Rosenow. Order book approach to price impact. *Quantitative Finance*, 5(4):357–364, 2005.
- [12] V. Zakamouline. European option pricing and hedging with both fixed and proportional transaction costs. *working paper*.