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Institut de Mathématiques et de Modélisation de Montpellier, Université Montpellier II,

34095 Montpellier, France

Abstract

From the asymptotic expansion of the ground state of the Gross–Pitaevskii equation in the Thomas–Fermi limit given by Gallo and Pelinovsky [GP], we infer an asymptotic expansion of the kinetic, potential and total energy of the ground state. In particular, we give a rigorous proof of the expansion of the kinetic energy calculated by Dalfovo, Pitaevskii and Stringari [DPS] in the case where the space dimension is 3. Moreover, we calculate one more term in this expansion, and we generalize the result to space dimensions 1 and 2.

1 Introduction

After recent experiments with Bose–Einstein condensates, new interest has been stimulated in the Gross–Pitaevskii equation with a harmonic potential, taken here in its adimensional form

$$i\varepsilon u_t + \varepsilon^2 \Delta u + (1 - |x|^2)u - |u|^2 u = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}_+, \tag{1.1}$$

where the space dimension d is one, two or three, $u(t,x) \in \mathbb{C}$ describes the wave function of a repulsive Bose gas, and ε is a small parameter that corresponds to the Thomas–Fermi approximation of a nearly compact atomic cloud [F, T].
A ground state of the Bose-Einstein condensate is a positive, time-independent solution $u(t,x) = \eta_\varepsilon(x)$ of the Gross–Pitaevskii equation (1.1). Namely, $\eta_\varepsilon : \mathbb{R}^d \mapsto \mathbb{R}$ satisfies the stationary Gross–Pitaevskii equation

$$\varepsilon^2 \Delta \eta_\varepsilon(x) + (1 - |x|^2)\eta_\varepsilon(x) - \eta_\varepsilon^3(x) = 0, \quad x \in \mathbb{R}^d, \tag{1.2}$$

$\eta_\varepsilon(x) > 0$ for all $x \in \mathbb{R}^d$, and η_ε has a finite energy $E_\varepsilon(\eta_\varepsilon)$, where E_ε is given by

$$E_\varepsilon(u) = \int_{\mathbb{R}^d} \left(\varepsilon^2 |\nabla u|^2 + (|x|^2 - 1)u^2 + \frac{1}{2}u^4 \right) dx.$$

In dimensions one, two and three, provided ε is sufficiently small, existence and uniqueness of a radial ground state η_ε is known [IM, GP]. It is also well known [IM, AAB] that $\eta_\varepsilon(x)$ converges to $\eta_0(x)$ as $\varepsilon \to 0$ for all $x \in \mathbb{R}^2$, where η_0 is the so called Thomas–Fermi approximation

$$\eta_0(x) = \begin{cases} (1-|x|^2)^{1/2} & \text{for } |x| < 1, \\ 0 & \text{for } |x| > 1. \end{cases} \tag{1.3}$$

Several quantities such as the kinetic energy of the ground state

$$E_{k,\varepsilon}(\eta_\varepsilon) = \varepsilon^2 \int_{\mathbb{R}^d} |\nabla \eta_\varepsilon|^2 dx,$$

can not be accurately approximated just by replacing the ground state η_ε by η_0, because of a logarithmic divergence at the boundary region $|x| = 1$. In [DPS], in the three-dimensional case, Dalfovo, Pitaevskii and Stringari give the behavior of the order parameter in the boundary region, which for instance provides the first term in the asymptotic expansion of $E_{k,\varepsilon}(\eta_\varepsilon)$ close to $\varepsilon = 0$. The comparison of this term with the energy obtained by solving numerically the Gross–Pitaevskii equation enables them to confirm the correctness of their prediction about the behaviour of the ground state at the boundary. The precise knowledge of quantities such as the kinetic energy of the ground state $E_{k,\varepsilon}(\eta_\varepsilon)$, its potential energy

$$E_p(\eta_\varepsilon) = \int_{\mathbb{R}^d} (|x|^2 - 1)\eta_\varepsilon^2 dx,$$

or its total energy $E_\varepsilon(\eta_\varepsilon)$, is an interesting information about the ground state, in particular because of the characterization of η_ε as a minimizer of E_ε. The purpose of this work is to show how we can obtain rigorously asymptotic expansions (theoretically at arbitrarily high order of accuracy) of such quantities. In particular, for $d = 3$, we give a rigorous proof of the expansion of the kinetic energy of the ground state given in [DPS], we calculate one more term in this expansion, and we generalize the result to one and two-dimensional cases. The extra term we calculate in the three-dimensional case also gives an idea about the range of values of the physical parameters for which the approximation of the kinetic energy given in [DPS] is valid. Namely, we show that this extra term has to be taken into account as soon as the ratio $\varepsilon^{1/2} = a_{HO}/R$ between the harmonic oscillator length and the radius of the potential (see Section 4) is not negligeable in front of $5 \cdot 10^{-2}$. The calculation of other terms in the expansion, on top of giving a more accurate approximation of the kinetic energy for small values of ε, would also give better approximations for larger values of ε.

The calculation of the expansions of the different energies rely on the expansion of η_ε into powers of ε in the limit $\varepsilon \to 0$, which was established in [GP]. So, let us first summarize the main ideas which provide this expansion of η_ε in [GP]. Since η_ε is radially symmetric, we can define a function ν_ε on $J_\varepsilon := (-\infty, \varepsilon^{-2/3}]$ by

$$\eta_\varepsilon(x) = \varepsilon^{1/3} \nu_\varepsilon \left(\frac{1-|x|^2}{\varepsilon^{2/3}} \right), \quad x \in \mathbb{R}^d. \tag{1.4}$$

Next, we rewrite equation (1.2) in terms of the new real variable $y = (1 - |x|^2)/\varepsilon^{2/3}$. It is equivalent for η_ε to solve (1.2) and for ν_ε to solve the differential equation

$$4(1 - \varepsilon^{2/3}y)\nu_\varepsilon''(y) - 2\varepsilon^{2/3}d\nu_\varepsilon'(y) + y\nu_\varepsilon(y) - \nu_\varepsilon^3(y) = 0, \quad y \in J_\varepsilon. \tag{1.5}$$

Let $N \geqslant 0$ be an integer. We look for ν_ε using the form

$$\nu_\varepsilon(y) = \sum_{n=0}^{N} \varepsilon^{2n/3}\nu_n(y) + \varepsilon^{2(N+1)/3}R_{N,\varepsilon}(y), \quad y \in J_\varepsilon. \tag{1.6}$$

Expansion (1.6) provides a solution of equation (1.5) if $\{\nu_n\}_{0\leqslant n\leqslant N}$ and $R_{N,\varepsilon}$ satisfy equations (1.7), (1.8) and (1.9) below.

- ν_0 solves the Painlevé-II equation

$$4\nu_0''(y) + y\nu_0(y) - \nu_0^3(y) = 0, \quad y \in \mathbb{R}, \tag{1.7}$$

- for $1 \leqslant n \leqslant N$, ν_n solves

$$-4\nu_n''(y) + W_0(y)\nu_n(y) = F_n(y), \quad y \in \mathbb{R}, \tag{1.8}$$

where

$$W_0(y) = 3\nu_0^2(y) - y$$

and

$$F_n(y) = -\sum_{\substack{n_1,\,n_2,\,n_3\,<\,n\\ n_1+n_2+n_3=n}} \nu_{n_1}(y)\nu_{n_2}(y)\nu_{n_3}(y) - 2d\nu_{n-1}'(y) - 4y\nu_{n-1}''(y),$$

- $R_{N,\varepsilon}$ solves

$$-4(1 - \varepsilon^{2/3}y)R_{N,\varepsilon}'' + 2\varepsilon^{2/3}dR_{N,\varepsilon}' + W_0R_{N,\varepsilon} = F_{N,\varepsilon}(y, R_{N,\varepsilon}), \quad y \in J_\varepsilon, \tag{1.9}$$

where

$$\begin{aligned} F_{N,\varepsilon}(y,R) \;=\; & -(4y\nu_N'' + 2d\nu_N') - \sum_{n=0}^{2N-1} \varepsilon^{2n/3} \sum_{\substack{n_1+n_2+n_3=n+N+1\\ 0\leqslant n_1,n_2,n_3\leqslant N}} \nu_{n_1}\nu_{n_2}\nu_{n_3} \\ & - \left(3\sum_{n=1}^{2N} \varepsilon^{2n/3} \sum_{\substack{n_1+n_2=n\\ 0\leqslant n_1,n_2\leqslant N}} \nu_{n_1}\nu_{n_2}\right) R - \left(3\sum_{n=N+1}^{2N+1} \varepsilon^{2n/3}\nu_{n-(N+1)}\right) R^2 - \varepsilon^{4(N+1)/3}R^3. \end{aligned}$$

Notice that for $0 \leqslant n \leqslant N$, $\nu_n(y)$ is defined for all $y \in \mathbb{R}$ and does not depend on ε, whereas $R_{N,\varepsilon}(y)$ is only defined for $y \in J_\varepsilon$.

In order to describe accurately the convergence of η_ε to η_0 by the first term (corresponding to $n = 0$) in expansion (1.6), ν_0 shall be chosen in such a way that

$$\varepsilon^{1/3}\nu_0\left(\frac{1-x^2}{\varepsilon^{2/3}}\right) \longrightarrow \begin{cases} \sqrt{1-|x|^2} & \text{if} |x| < 1 \\ 0 & \text{if} |x| > 1, \end{cases}$$

which means that ν_0 has to satisfy the asymptotic behaviour

$$\nu_0(y) \sim y^{1/2} \quad \text{as} \quad y \to +\infty \quad \text{and} \quad \nu_0(y) \to 0 \quad \text{as} \quad y \to -\infty. \tag{1.10}$$

The Painlevé-II equation is known to have a unique solution ν_0 with the asymptotic behaviour (1.10). This is the so-called Hastings-McLeod solution. Moreover, the behaviour of $\nu_0(y)$ as $y \to \pm\infty$ has been studied in details, for instance in [HM], [M], [FIKN]. Some of its properties are summarized in the next proposition.

Proposition 1.1 *[HM, M, FIKN] The Painlevé-II equation (1.7) admits a unique solution $\nu_0 \in \mathcal{C}^\infty(\mathbb{R})$ which satisfies (1.10). This solution ν_0 is strictly increasing on $\mathbb{R}$. The behaviour of ν_0 as $y \to -\infty$ is described by*

$$\nu_0(y) = \frac{1}{\sqrt{\pi}(-y)^{1/4}} \exp\left(-\frac{1}{3}(-y)^{3/2}\right)\left(1 + \mathcal{O}(|y|^{-3/4})\right) \underset{y\to-\infty}{\approx} 0, \tag{1.11}$$

whereas as $y \to +\infty$,

$$\nu_0(y) \underset{y\to+\infty}{\approx} y^{1/2} \sum_{n=0}^{\infty} \frac{b_n}{(2y)^{3n/2}}, \tag{1.12}$$

where $b_0 = 1$, $b_1 = 0$, and for $n \geqslant 0$,

$$b_{n+2} = 4(9n^2-1)b_n - \frac{3}{2}\sum_{m=1}^{n+1} b_m b_{n+2-m} - \frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{n+1-l} b_l b_m b_{n+2-l-m}.$$

Once ν_0 has been chosen to be the Hastings-McLeod solution of the Painlevé-II equation (1.7), we show by induction on n that there is a unique way to construct the sequence $(\nu_n)_{n\geqslant 1} \subset H^\infty(\mathbb{R})$ such that (1.8) is satisfied for every $n \geqslant 1$. Moreover, like for ν_0, the asymptotic behaviour of the ν_n's can be precisely described, as it is shown in the next proposition.

Proposition 1.2 *[GP] For every $n \geqslant 1$,*

$$\nu_n(y) \underset{y\to+\infty}{\approx} y^{\beta-2n} \sum_{m=0}^{\infty} g_{n,m} y^{-3m/2} \text{ for some } \{g_{n,m}\}_{m\in\mathbb{N}},$$

and

$$\nu_n(y) \underset{y\to-\infty}{\approx} 0,$$

where

$$\beta = \begin{cases} -5/2 & \text{if } d = 1, \\ 1/2 & \text{if } d = 2, 3. \end{cases}$$

Remark 1.3 *The coefficients $g_{n,m}$ can be calculated explicitely by plugging this expansion of ν_n into (1.8). For instance, $\nu_1(y) \underset{y\to+\infty}{\sim} \frac{5(7-d)}{4} y^{-9/2}$ if $d = 1$, whereas $\nu_1(y) \underset{y\to+\infty}{\sim} = \frac{1-d}{2} y^{-3/2}$ if $d = 2, 3$.*

Finally, we close the argument by constructing a remainder term $R_{N,\varepsilon}$ that solves equation (1.9). We also prove suitable estimates on $R_{N,\varepsilon}$ which ensure that the last term in the right hand side of (1.6) is indeed small compared to the other ones, and that the solution of the stationary Gross-Pitaevskii equation (1.2) that we have constructed by (1.4), (1.6) and our choices of the ν_n's and $R_{N,\varepsilon}$ is indeed the unique ground state of (1.2) (in particular, it is positive). More precisely, we have the following result.

Proposition 1.4 *[GP] For every $N \geqslant 0$, there exists $\varepsilon_N > 0$ and $C_N > 0$ such that for every $0 < \varepsilon < \varepsilon_N$, there is a solution $R_{N,\varepsilon} \in \mathcal{C}^\infty \cap L^\infty(J_\varepsilon)$ of equation (1.9) with*

$$\|R_{N,\varepsilon}\|_{L^\infty(J_\varepsilon)} \leqslant C_N, \qquad \int_{-\infty}^{\varepsilon^{-2/3}} R_{N,\varepsilon}(y)^2 W_0(y)(1-\varepsilon^{2/3}y)^{d/2-1} dy \leqslant C_N$$

$$\text{and} \quad S_{N,\varepsilon} : x \mapsto R_{N,\varepsilon}\left(\frac{1-|x|^2}{\varepsilon^{2/3}}\right) \in H^2(\mathbb{R}^d),$$

such that the unique radially symmetric ground state of equation (1.2) in $L^2(\mathbb{R}^d)$ writes

$$\eta_\varepsilon(x) = \varepsilon^{1/3} \sum_{n=0}^{N} \varepsilon^{2n/3} \nu_n\left(\frac{1-|x|^2}{\varepsilon^{2/3}}\right) + \varepsilon^{2N/3+1} R_{N,\varepsilon}\left(\frac{1-|x|^2}{\varepsilon^{2/3}}\right), \quad x \in \mathbb{R}^d. \tag{1.13}$$

Remark 1.5 *The estimate on the L^∞ norm of $R_{N,\varepsilon}$ written in [GP] was $\|R_{N,\varepsilon}\|_{L^\infty(J_\varepsilon)} \leqslant C_N \varepsilon^{-(d-1)/3}$. The estimate we have in Proposition 1.4 above is a direct consequence of this inequality with $R_{N,\varepsilon}$ replaced by $R_{N+1,\varepsilon}$, taking into account that $d \leqslant 3$, $R_{N,\varepsilon} = \nu_{N+1} + \varepsilon^{2/3} R_{N+1,\varepsilon}$ and $\nu_{N+1} \in L^\infty(\mathbb{R})$. The other estimate on $R_{N,\varepsilon}$ is a byproduct of the proof of Proposition 1.4 given in [GP]. Indeed, $R_{N,\varepsilon}$ is obtained there thanks to a fix point argument in a ball $B_{H^1_\varepsilon}(R^0_{N,\varepsilon}, C\varepsilon^{2/3})$, where $\|R^0_{N,\varepsilon}\|_{H^1_\varepsilon} \lesssim 1$ and*

$$\|u\|^2_{H^1_\varepsilon} = \int_{-\infty}^{\varepsilon^{-2/3}} \left(4(1-\varepsilon^{2/3}y)^{d/2}|u'|^2 + (1-\varepsilon^{2/3}y)^{d/2-1} W_0(y) u(y)^2\right) dy.$$

The asymptotic expansion of η_ε given by (1.4)-(1.6) as well as the precise description of the behaviour of $\nu_n(y)$ as $y \to \pm\infty$ (for $n \geqslant 0$) given in Propositions 1.1 and 1.2, enable us to calculate expansions for kinetic, potential and full energy of the ground state. Concerning the full energy, we get the following expansion.

Theorem 1.6 *For $d = 1$,*

$$\begin{aligned} E_\varepsilon(\eta_\varepsilon) &= -\frac{8}{15} - \frac{2}{3}\varepsilon^2 \ln \varepsilon + \left[\int_0^1 \frac{(1-t)^{-\frac{1}{2}} - 1}{t} dt - \frac{1}{2}\int_{-\infty}^{+\infty} \left(\nu_0(y)^4 - y_+^2 + \frac{2}{y}\mathbf{1}_{\{y\geqslant 1\}}\right) dy\right] \varepsilon^2 \\ &+ \left[-\frac{1}{2} - \frac{1}{4}\int_{-\infty}^{+\infty} y\left(\nu_0(y)^4 - y_+^2 + \frac{2}{y}\mathbf{1}_{\{y\geqslant 1\}}\right) dy - 2\int_{-\infty}^{+\infty} \nu_0(y)^3 \nu_1(y) dy\right] \varepsilon^{8/3} + O(\varepsilon^3). \end{aligned}$$

For $d = 2$,

$$\begin{aligned} E_\varepsilon(\eta_\varepsilon) &= -\frac{\pi}{6} - \frac{2\pi}{3}\varepsilon^2 \ln \varepsilon + \left[-\frac{\pi}{2} \int_{-\infty}^{+\infty} \left(\nu_0(y)^4 - y_+^2 + \frac{2}{y} \mathbf{1}_{\{y \geqslant 1\}} \right) dy + \pi \right] \varepsilon^2 \\ &\quad -2\pi \left[\int_{-\infty}^{+\infty} \left(\nu_0(y)^3 \nu_1(y) + \frac{1}{2} \mathbf{1}_{\{y \geqslant 0\}} \right) dy \right] \varepsilon^{8/3} + O(\varepsilon^3). \end{aligned}$$

For $d = 3$,

$$\begin{aligned} E_\varepsilon(\eta_\varepsilon) &= -\frac{16\pi}{105} - \frac{4\pi}{3}\varepsilon^2 \ln \varepsilon \\ &\quad + \left[2\pi \int_0^1 \frac{(1-t)^{\frac{1}{2}} - 1}{t} dt - \pi \int_{-\infty}^{+\infty} \left(\nu_0(y)^4 - y_+^2 + \frac{2}{y} \mathbf{1}_{\{y \geqslant 1\}} \right) dy + \frac{8\pi}{3} \right] \varepsilon^2 \\ &\quad + \left[\pi + \frac{\pi}{2} \int_{-\infty}^{+\infty} y \left(\nu_0(y)^4 - y_+^2 + \frac{2}{y} \mathbf{1}_{\{y \geqslant 1\}} \right) dy - 4\pi \int_{-\infty}^{+\infty} \left(\nu_0(y)^3 \nu_1(y) + \mathbf{1}_{\{y \geqslant 0\}} \right) dy \right] \varepsilon^{8/3} + O(\varepsilon^3). \end{aligned}$$

Remark 1.7 *Note that the notation ν_1 does not represent the same function in the three expansions given in Theorem 1.6. Indeed, the equation (1.8) satisfied by ν_1 depends on the dimension d through F_1.*

The rest of the paper is organized as follows. In section 2 we calculate asymptotic expansions of $E_\varepsilon(\eta_\varepsilon)$ and prove Theorem 1.6. In section 3 we calculate the asymptotic expansion of the potential energy $E_p(\eta_\varepsilon)$. In section 4, we deduce the expansion of the kinetic energy from the results of the two previous section, and we rediscover the expansion found by Dalfovo, Pitaevskii and Stringari in [DPS] on a formal level. In the appendix, we prove a key lemma which is used on many occasions in the calculation.

2 Expansion of $E_\varepsilon(\eta_\varepsilon)$

We are interested here in the behaviour of $E_\varepsilon(\eta_\varepsilon)$ as $\varepsilon \to 0$. First, if we multiply (1.2) by η_ε and sum over $\mathbb{R}^d$, we get

$$E_\varepsilon(\eta_\varepsilon) = -\frac{1}{2} \int_{\mathbb{R}^d} \eta_\varepsilon(x)^4 dx. \tag{2.1}$$

From the convergence of η_ε to η_0 in $L^p(\mathbb{R}^d)$ (for any $p \in [1, +\infty]$) as $\varepsilon \to 0$ [GP], we already know

$$E_\varepsilon(\eta_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} -\frac{1}{2} \int_{\mathbb{R}^d} \eta_0(x)^4 dx = \begin{cases} -8/15 & \text{if } d = 1 \\ -\pi/6 & \text{if } d = 2 \\ -16\pi/105 & \text{if } d = 3. \end{cases}$$

Next, we calculate some correction terms in the asymptotic expansion of $E_\varepsilon(\eta_\varepsilon)$ as $\varepsilon \to 0$. From (2.1), (1.4) and (1.6) we infer

$$
\begin{aligned}
&E_\varepsilon(\eta_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^d} \eta_0(x)^4 dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^d} \left(\eta_\varepsilon(x)^4 - \eta_0(x)^4\right) dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^d} \varepsilon^{4/3} \left(\nu_\varepsilon\left(\frac{1-|x|^2}{\varepsilon^{2/3}}\right)^4 - \sqrt{\left(\frac{1-|x|^2}{\varepsilon^{2/3}}\right)_+}^4 \right) dx \\
&= -\frac{\varepsilon^{4/3}}{2}\left|\mathbb{S}^{d-1}\right| \int_0^{+\infty} \left(\nu_\varepsilon\left(\frac{1-r^2}{\varepsilon^{2/3}}\right)^4 - \sqrt{\left(\frac{1-r^2}{\varepsilon^{2/3}}\right)_+}^4 \right) r^{d-1} dr \\
&= -\frac{\varepsilon^2}{4}\left|\mathbb{S}^{d-1}\right| \int_{-\infty}^{\varepsilon^{-2/3}} \left(\nu_\varepsilon(y)^4 - \sqrt{y_+}^4\right)(1-\varepsilon^{2/3}y)^{d/2-1} dy \\
&= -\frac{\varepsilon^2}{4}\left|\mathbb{S}^{d-1}\right| \int_{-\infty}^{\varepsilon^{-2/3}} \left((\nu_0(y) + \varepsilon^{2/3} R_{0,\varepsilon}(y))^4 - \sqrt{y_+}^4\right)(1-\varepsilon^{2/3}y)^{d/2-1} dy \\
&= -\frac{\varepsilon^2}{4}\left|\mathbb{S}^{d-1}\right| \sum_{j=1}^{6} I_j,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\varepsilon^{-2/3}} -\frac{2}{y}\mathbf{1}_{\{y\geqslant 1\}}(1-\varepsilon^{2/3}y)^{d/2-1} dy, \\
I_2 &= \int_{-\infty}^{\varepsilon^{-2/3}} \left(\nu_0(y)^4 - \sqrt{y_+}^4 + \frac{2}{y}\mathbf{1}_{\{y\geqslant 1\}}\right)(1-\varepsilon^{2/3}y)^{d/2-1} dy, \\
I_3 &= 4\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}} \nu_0(y)^3 R_{0,\varepsilon}(y)(1-\varepsilon^{2/3}y)^{d/2-1} dy, \\
I_4 &= 6\varepsilon^{4/3}\int_{-\infty}^{\varepsilon^{-2/3}} \nu_0(y)^2 R_{0,\varepsilon}(y)^2(1-\varepsilon^{2/3}y)^{d/2-1} dy, \\
I_5 &= 4\varepsilon^{2}\int_{-\infty}^{\varepsilon^{-2/3}} \nu_0(y) R_{0,\varepsilon}(y)^3(1-\varepsilon^{2/3}y)^{d/2-1} dy, \\
I_6 &= \varepsilon^{8/3}\int_{-\infty}^{\varepsilon^{-2/3}} R_{0,\varepsilon}(y)^4(1-\varepsilon^{2/3}y)^{d/2-1} dy.
\end{aligned}
$$

Next, we give an asymptotic expansion as $\varepsilon \to 0$ of each of the I_j's, with a $O(\varepsilon)$ remainder. For this purpose, the following lemma, which is proved in the appendix, will be convenient.

Lemma 2.1 *Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a bounded function, such that $g(y) \underset{y\to-\infty}{=} O(\exp(y))$, and $g(y) \underset{y\to+\infty}{=} O(y^{-\alpha})$, where $\alpha \in \mathbb{R}$. Then*

$$\int_{-\infty}^{\varepsilon^{-2/3}} g(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy$$
$$\underset{\varepsilon\to 0}{=} \begin{cases} O(\varepsilon^{-1/3}) & \text{if } \alpha \geqslant 1/2 \\ \int_{-\infty}^{+\infty} g(y)dy + O(\varepsilon^{1/3}) & \text{if } \alpha \geqslant 3/2 \\ \int_{-\infty}^{+\infty} g(y)dy - \left(\frac{d}{2}-1\right)\varepsilon^{2/3}\int_{-\infty}^{+\infty} yg(y)dy + O(\varepsilon) & \text{if } \alpha \geqslant 5/2. \end{cases} \tag{2.2}$$

Expansion of I_1. From the change of variable $t = \varepsilon^{2/3}y$, we get

$$\begin{aligned} I_1 &= -2\int_1^{\varepsilon^{-2/3}} (1-\varepsilon^{2/3}y)^{d/2-1}\frac{dy}{y} \\ &= -2\int_{\varepsilon^{2/3}}^1 (1-t)^{d/2-1}\frac{dt}{t} \\ &= -2\int_{\varepsilon^{2/3}}^1 \frac{dt}{t} - 2\int_0^1 \frac{(1-t)^{d/2-1}-1}{t}dt + 2\int_0^{\varepsilon^{2/3}} \frac{(1-t)^{d/2-1}-1}{t}dt \\ &= \frac{4}{3}\ln\varepsilon - 2\int_0^1 \frac{(1-t)^{d/2-1}-1}{t}dt - 2\left(\frac{d}{2}-1\right)\varepsilon^{2/3} + O(\varepsilon^{4/3}). \end{aligned}$$

Expansion of I_2. We apply Lemma 2.1 to the function

$$g_0(y) := \nu_0(y)^4 - \sqrt{y_+}^4 + \frac{2}{y}\mathbf{1}_{\{y\geqslant 1\}}.$$

Note that Proposition 1.1 yields $g_0(y) \underset{y\to-\infty}{=} O(\exp(y))$, and $g_0(y) \underset{y\to+\infty}{=} O(y^{-5/2})$. Thus,

$$I_2 = \int_{-\infty}^{\varepsilon^{-2/3}} g_0(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy = \int_{-\infty}^{+\infty} g_0(y)dy - \left(\frac{d}{2}-1\right)\varepsilon^{2/3}\int_{-\infty}^{+\infty} yg_0(y)dy + O(\varepsilon).$$

Expansion of I_3. From (1.6), we get

$$\begin{aligned} I_3 &= 4\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}} \nu_0(y)^3 R_{0,\varepsilon}(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy \\ &= 4\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}} \nu_0(y)^3\left(\sum_{j=0}^{k}\varepsilon^{2j/3}\nu_{j+1}(y) + \varepsilon^{2(k+1)/3}R_{k+1,\varepsilon}(y)\right)(1-\varepsilon^{2/3}y)^{d/2-1}dy. \end{aligned}$$

From Propositions 1.1 and 1.2, we have

$$\nu_0(y) \underset{y\to+\infty}{=} y^{1/2} - \frac{1}{2}y^{-5/2} + O(y^{-4}),$$

$$\nu_1(y) \underset{y\to+\infty}{=} \begin{cases} \frac{1-d}{2}y^{-3/2} + O(y^{-3}) & \text{if } d = 2, 3 \\ O(y^{-9/2}) & \text{if } d = 1, \end{cases}$$

and $\nu_0(y), \nu_1(y) \underset{y\to-\infty}{\approx} 0$. In particular, for $d = 2, 3$,

$$\nu_0(y)^3\nu_1(y) \underset{y\to+\infty}{=} \left(y^{1/2} + O(y^{-5/2})\right)^3 \left(\frac{1-d}{2}y^{-3/2} + O(y^{-3})\right) = \frac{1-d}{2} + O(y^{-3/2}),$$

a result which also holds for $d = 1$. Thus, Lemma 2.1 implies

$$\begin{aligned} & 4\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}} \nu_0(y)^3\nu_1(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy \\ &= 4\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}} g_1(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy - 4\varepsilon^{2/3}\int_0^{\varepsilon^{-2/3}} \frac{d-1}{2}(1-\varepsilon^{2/3}y)^{d/2-1}dy \\ &= 4\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}} g_1(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy - 4\frac{d-1}{d} \\ &= 4\frac{1-d}{d} + 4\varepsilon^{2/3}\int_{-\infty}^{+\infty} g_1(y)dy + O(\varepsilon), \end{aligned} \tag{2.3}$$

where

$$g_1(y) = \nu_0(y)^3\nu_1(y) + \frac{d-1}{2}\mathbf{1}_{\{y\geqslant 0\}} \underset{y\to+\infty}{=} O(y^{-3/2}).$$

Next Proposition 1.2 provides, for $j \geqslant 1$, $\nu_{j+1}(y) \underset{y\to+\infty}{=} O(y^{-7/2})$, and therefore $\nu_0(y)^3\nu_{j+1}(y) = O(y^{-2})$. Thus, Lemma 2.1 implies

$$4\varepsilon^{2(1+j)/3}\int_{-\infty}^{\varepsilon^{-2/3}} \nu_0(y)^3\nu_{j+1}(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy = O(\varepsilon^{2(1+j)/3}) = O(\varepsilon^{4/3}). \tag{2.4}$$

Finally, if $\varepsilon \leqslant 1$, thanks to Proposition 1.4,

$$\begin{aligned} & \left|4\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}} \nu_0(y)^3\varepsilon^{2(k+1)/3}R_{k+1,\varepsilon}(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy\right| \\ &\leqslant 4\varepsilon^{2(k+2)/3}\|R_{k+1,\varepsilon}\|_{L^\infty(J_\varepsilon)}\int_{-\infty}^{\varepsilon^{-2/3}} \nu_0(y)^3(1-\varepsilon^{2/3}y)^{d/2-1}dy \\ &\lesssim \varepsilon^{2(k+2)/3}\left(\int_{-\infty}^0 \nu_0(y)^3(1+|y|)^{1/2}dy + (\varepsilon^{-2/3})^{3/2}\int_0^{\varepsilon^{-2/3}} (1-\varepsilon^{2/3}y)^{d/2-1}dy\right) \\ &\lesssim \varepsilon^{2(k+2)/3}(1+\varepsilon^{-5/3}) = O(\varepsilon), \end{aligned} \tag{2.5}$$

provided $k \geqslant 2$. With such a choice of k, the combination of estimates (2.3), (2.4) and (2.5) yields

$$I_3 = 4\frac{1-d}{d} + 4\varepsilon^{2/3}\int_{-\infty}^{+\infty} g_1(y)dy + O(\varepsilon).$$

Estimates on I_4,I_5,I_6. As mentionned in Proposition 1.1, ν_0 is an increasing function on $\mathbb{R}$, $\nu_0(y) \underset{y\to-\infty}{\longrightarrow} 0$ and $\nu_0(y) \underset{y\to+\infty}{\sim} y^{1/2}$. On the other side, it is proved in [GP] that $W_0(y) = 3\nu_0(y)^2 - y > 0$ for $y \in \mathbb{R}$. Thus, there exists $C > 0$ such that for every $y \in \mathbb{R}$,

$$\max(\nu_0(y)^2, \nu_0(y), 1) \leqslant CW_0(y).$$

Thus, it follows from the estimates on $R_{0,\varepsilon}$ stated in Proposition 1.4 that

$$I_4 = O(\varepsilon^{4/3}), \quad I_5 = O(\varepsilon^2), \quad I_6 = O(\varepsilon^{8/3}).$$

Combining the asymptotic expansions of all of the I_j's, we obtain finally

$$\begin{aligned} &E_\varepsilon(\eta_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^d} \eta_0(x)^4 dx \\ &= -\frac{\varepsilon^2}{4}\left|\mathbb{S}^{d-1}\right|\left[\frac{4}{3}\ln\varepsilon + \left(4\frac{1-d}{d} - 2\int_0^1 \frac{(1-t)^{d/2-1}-1}{t}dt + \int_{-\infty}^{+\infty} g_0(y)dy\right)\right. \\ &\quad \left. + \left((1-\frac{d}{2})(2 + \int_{-\infty}^{+\infty} y g_0(y)dy) + 4\int_{-\infty}^{+\infty} g_1(y)dy\right)\varepsilon^{2/3} + O(\varepsilon)\right]. \end{aligned}$$

3 Expansion of $E_p(\eta_\varepsilon)$

A calculation similar to the one we made to compute $E_\varepsilon(\eta_\varepsilon) + \frac{1}{2}\|\eta_0\|^4_{L^4(\mathbb{R}^d)}$ gives

$$\begin{aligned} &E_p(\eta_\varepsilon) - \int_{\mathbb{R}^d} \left(|x|^2 - 1\right)\eta_0(x)^2 dx \\ &= -\frac{\varepsilon^2}{2}\left|\mathbb{S}^{d-1}\right|\int_{-\infty}^{\varepsilon^{-2/3}} y\left((\nu_0(y) + \varepsilon^{2/3}R_{0,\varepsilon}(y))^2 - \sqrt{y_+}^2\right)(1-\varepsilon^{2/3}y)^{d/2-1}dy \\ &= -\frac{\varepsilon^2}{2}\left|\mathbb{S}^{d-1}\right|\sum_{j=1}^4 J_j, \end{aligned}$$

where

$$\begin{aligned} J_1 &= \int_{-\infty}^{\varepsilon^{-2/3}} -\frac{1}{y}\mathbf{1}_{\{y\geqslant 1\}}(1-\varepsilon^{2/3}y)^{d/2-1}dy, \\ J_2 &= \int_{-\infty}^{\varepsilon^{-2/3}} \left(y(\nu_0(y)^2 - y_+) + \frac{1}{y}\mathbf{1}_{\{y\geqslant 1\}}\right)(1-\varepsilon^{2/3}y)^{d/2-1}dy, \\ J_3 &= 2\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}} y\nu_0(y)R_{0,\varepsilon}(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy, \\ J_4 &= \varepsilon^{4/3}\int_{-\infty}^{\varepsilon^{-2/3}} yR_{0,\varepsilon}(y)^2(1-\varepsilon^{2/3}y)^{d/2-1}dy. \end{aligned}$$

The method used to calculate asymptotic expansions of the J_j's for $j = 1, 2, 3, 4$ is similar to the one we used for the I_j's. More precisely,

Expansion of J_1.

$$J_1 = \frac{1}{2} I_1 = \frac{2}{3} \ln \varepsilon - \int_0^1 \frac{(1-t)^{d/2-1} - 1}{t} dt - \left(\frac{d}{2} - 1 \right) \varepsilon^{2/3} + O(\varepsilon^{4/3}).$$

Expansion of J_2. From Lemma 2.1,

$$J_2 \;=\; = \int_{-\infty}^{+\infty} g_2(y) dy - \left(\frac{d}{2} - 1 \right) \varepsilon^{2/3} \int_{-\infty}^{+\infty} y g_2(y) dy + O(\varepsilon),$$

where

$$g_2(y) = y(\nu_0(y)^2 - y_+) + \frac{1}{y} \mathbf{1}_{\{y \geqslant 1\}} \underset{y \to +\infty}{=} O(y^{-5/2}).$$

Expansion of J_3. We infer from Propositions (1.1) and (1.2) that

$$y \nu_0(y) \nu_1(y) \underset{y \to +\infty}{=} \frac{1-d}{2} + O(y^{-3/2}).$$

We put

$$g_3(y) := y \nu_0(y) \nu_1(y) + \frac{d-1}{2} \mathbf{1}_{\{y \geqslant 0\}} \underset{y \to +\infty}{=} O(y^{-3/2}).$$

Like in the calculation of I_3, we notice that for every $j \geqslant 1$, $y \nu_0(y) \nu_{j+1}(y) \underset{y \to +\infty}{=} O(y^{-2})$, and we deduce

$$J_3 = 2 \frac{1-d}{d} + 2 \varepsilon^{2/3} \int_{-\infty}^{+\infty} g_3(y) dy + O(\varepsilon).$$

Expansion of J_4. Like in the estimate on I_4, we have

$$J_4 = O(\varepsilon^{4/3}).$$

As a conclusion, we get

$$\begin{aligned} & E_p(\eta_\varepsilon) + \int_{\mathbb{R}^d} \eta_0(x)^4 dx \\ &= -\frac{\varepsilon^2}{2} \left| \mathbb{S}^{d-1} \right| \left[\frac{2}{3} \ln \varepsilon + \left(2 \frac{1-d}{d} - \int_0^1 \frac{(1-t)^{d/2-1} - 1}{t} dt + \int_{-\infty}^{+\infty} g_2(y) dy \right) \right. \\ &\quad \left. + \left((1 - \frac{d}{2})(1 + \int_{-\infty}^{+\infty} y g_2(y) dy) + 2 \int_{-\infty}^{+\infty} g_3(y) dy \right) \varepsilon^{2/3} + O(\varepsilon) \right]. \end{aligned}$$

4 Expansion of $E_{k,\varepsilon}(\eta_\varepsilon)$

Let us multiply (1.2) by η_ε and sum over $\mathbb{R}^d$. We get

$$E_{k,\varepsilon}(\eta_\varepsilon) \;=\; 2E_\varepsilon(\eta_\varepsilon) - E_p(\eta_\varepsilon)$$

Thus, the asymptotic expansions of $E_\varepsilon(\eta_\varepsilon)$ and $E_p(\eta_\varepsilon)$ obtained in the previous sections ensure

$$\begin{aligned}
&E_{k,\varepsilon}(\eta_\varepsilon)\\
&= -\frac{\varepsilon^2}{2}\left|\mathbb{S}^{d-1}\right|\left[\frac{2}{3}\ln\varepsilon + \left(2\frac{1-d}{d} - \int_0^1 \frac{(1-t)^{d/2-1}-1}{t}dt + \int_{-\infty}^{+\infty}(g_0(y)-g_2(y))dy\right)\right.\\
&\quad\left. + \left((1-\frac{d}{2})(1+\int_{-\infty}^{+\infty} y(g_0(y)-g_2(y))dy) + \int_{-\infty}^{+\infty}(4g_1(y)-2g_3(y))dy\right)\varepsilon^{2/3} + O(\varepsilon)\right]\\
&= -\frac{\varepsilon^2}{2}\left|\mathbb{S}^{d-1}\right|\left[\frac{2}{3}\ln\varepsilon + \left(2\frac{1-d}{d} - \int_0^1 \frac{(1-t)^{d/2-1}-1}{t}dt\right.\right.\\
&\quad\left. + \int_{-\infty}^{+\infty}(\nu_0(y)^2(\nu_0(y)^2-y) + \frac{1}{y}\mathbf{1}_{\{y\geqslant 1\}})dy\right)\\
&\quad + \left((1-\frac{d}{2})(\int_{-\infty}^{+\infty}\left(y\nu_0(y)^2(\nu_0(y)^2-y) + \mathbf{1}_{\{y\geqslant 0\}}\right)dy)\right.\\
&\quad\left.\left. + \int_{-\infty}^{+\infty}\left(2(2\nu_0(y)^2-y)\nu_0(y)\nu_1(y) + (d-1)\mathbf{1}_{\{y\geqslant 0\}}\right)dy\right)\varepsilon^{2/3} + O(\varepsilon)\right].
\end{aligned}$$

In [DPS], the author study the kinetic energy

$$E_{kin} = \frac{\hbar^2}{2m}\int_{\mathbb{R}^3}|\nabla\psi(\mathbf{r})|^2\,d\mathbf{r}$$

of the ground state ψ of the Gross–Pitaevskii equation with an isotropic harmonic trap

$$\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + \left(\mu - \frac{1}{2}m\omega_{HO}^2|\mathbf{r}|^2\right)\psi(\mathbf{r}) - \frac{4\pi\hbar^2 a}{m}|\psi(\mathbf{r})|^2\psi(\mathbf{r}) = 0, \tag{4.1}$$

where $\mu > 0$ is a chemical potential, $a > 0$ is the scattering length. Like in [DPS], we denote by R the radius of the condensate, defined by

$$\mu = \frac{1}{2}m\omega_{HO}^2 R^2,$$

and we introduce the harmonic oscillator length

$$a_{HO} = \left(\frac{\hbar}{m\omega_{HO}}\right)^{1/2},$$

as well as the maximal value α of the wave function ψ_{TF} of the Thomas-Fermi approximation:

$$\alpha = \frac{R}{(8\pi a_{HO}^4 a)^{1/2}}, \quad \psi_{TF}(\mathbf{r}) = \begin{cases} \alpha \left(1 - \frac{\mathbf{r}^2}{R^2}\right)^{1/2} & \text{if } |\mathbf{r}| \leqslant 1 \\ 0 & \text{if } |\mathbf{r}| > 1. \end{cases}$$

Then, the total number of particles is

$$N = \int_{|\mathbf{r}|\leqslant 1} \psi_{TF}(\mathbf{r})^2 d\mathbf{r} = \frac{R^5}{15 a a_{HO}^4}.$$

The change of variables

$$\psi(\mathbf{r}) = \alpha u(\mathbf{r}/R),$$

maps the ground state ψ of (4.1) to the solution $u = \eta_\varepsilon$ of (1.2), where

$$\varepsilon = \frac{a_{HO}^2}{R^2}.$$

Let us now use our expansion of $E_{k,\varepsilon}(\eta_\varepsilon)$ to calculate the expansion of E_{kin}.

$$\begin{aligned} E_{kin} &= \frac{\hbar^2}{2m} \alpha^2 R \int_{\mathbb{R}^3} |\nabla \eta_\varepsilon(x)|^2 \, dx \\ &= \frac{\hbar^2}{2m} \frac{R^3}{8\pi a_{HO}^4 a} \varepsilon^{-2} E_{k,\varepsilon}(\eta_\varepsilon) \\ &= \frac{5N\hbar^2}{2mR^2} \left[\ln \frac{R}{a_{HO}} + A + B \left(\frac{a_{HO}}{R}\right)^{4/3} + O\left(\left(\frac{a_{HO}}{R}\right)^2\right)\right], \end{aligned}$$

where

$$A = 1 + \frac{3}{4} \int_0^1 \frac{(1-t)^{1/2} - 1}{t} dt - \frac{3}{4} \int_{-\infty}^{+\infty} (\nu_0(y)^4 - y\nu_0(y)^2 + \frac{\mathbf{1}_{\{y\geqslant 1\}}}{y}) dy,$$

$$B = \frac{3}{8} \int_{-\infty}^{+\infty} \left(y\nu_0(y)^2(\nu_0(y)^2 - y) + \mathbf{1}_{\{y\geqslant 0\}}\right) dy - \frac{3}{2} \int_{-\infty}^{+\infty} \left((2\nu_0(y)^2 - y)\nu_0(y)\nu_1(y) + \mathbf{1}_{\{y\geqslant 0\}}\right) dy.$$

Let us now show that the constant A we get in the expansion of E_{kin} is indeed the same as the one obtained by Dalfovo, Pitaevskii and Stringari in [DPS]. First, taking into account the properties of the solution ν_0 of the Painlevé II equation mentioned in Proposition 1.1, we infer

$$\frac{1}{4} \int_{-\infty}^{+\infty} \left(\nu_0(y)^4 - y\nu_0(y)^2 + \frac{\mathbf{1}_{\{y\geqslant 1\}}}{y}\right) dy = \frac{1}{2} - \int_{-\infty}^{+\infty} \left(\nu_0'(y)^2 - \frac{\mathbf{1}_{\{y\geqslant 1\}}}{4y}\right) dy.$$

In order to compare our result with the one in [DPS], we introduce the function $\varphi(\xi) = 2^{-1/3}\nu_0(-2^{2/3}\xi)$, which is the solution to

$$\varphi'' - \xi\varphi - \varphi^3 = 0, \quad \varphi(\xi) \underset{\xi\to-\infty}{\sim} \sqrt{-\xi}, \quad \varphi(\xi) \underset{\xi\to+\infty}{\longrightarrow} 0.$$

Thus,

$$\begin{aligned} A &= 1+\frac{3}{4}(-2+2\ln 2)-\frac{3}{2}+3\int_{-\infty}^{+\infty}\left(\nu_0'(y)^2-\frac{\mathbf{1}_{\{y\geqslant 1\}}}{4y}\right)dy \\ &= -2+\frac{3}{2}\ln 2+3\int_{-\infty}^{+\infty}\left(\varphi'(\xi)^2+\frac{\mathbf{1}_{\{\xi\leqslant -2^{-2/3}\}}}{4\xi}\right)d\xi \\ &= -2+\frac{3}{2}\ln 2+3\lim_{M\to+\infty}\left(\int_{-M}^{+\infty}\varphi'(\xi)^2d\xi+\frac{1}{4}\ln 2^{-2/3}-\frac{1}{4}\ln M\right) \\ &= -2+\frac{7}{4}\ln 2+3C, \end{aligned}$$

where like in [DPS], we have denoted

$$C=\lim_{M\to+\infty}\left(\int_{-M}^{+\infty}\varphi'(\xi)^2d\xi-\frac{1}{4}\ln M\right)-\frac{1}{4}\ln 2.$$

By numerical calculations, we obtain $A\simeq -0.24$ and $B\simeq 15.3$. Thus, our additionnal term $B(a_{\rm HO}/R)^{4/3}$ in the expansion of E_{kin} has to be taken into account as soon as the small parameter $a_{\rm HO}/R$ is not negligible compared to $(|A|/B)^{3/4}\simeq 0.05$.

5 Appendix: Proof of Lemma 2.1

If $\alpha\geqslant 1/2$, we split the integral in three pieces. On the one side, since $d\leqslant 3$ and since the maps g and $y\mapsto |y|^{1/2}g(y)$ are in $L^1(\mathbb{R}_-)$, we have

$$\left|\int_{-\infty}^{0}g(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy\right| \leqslant \int_{-\infty}^{0}|g(y)|\max(2,2\varepsilon^{2/3}|y|)^{1/2}dy=O(1).$$

Then, there exists $C>0$ such that $g(y)\leqslant C(1+|y|)^{-1/2}$ for every $y\geqslant 0$, and since $d\geqslant 1$,

$$\left|\int_{0}^{\varepsilon^{-2/3}/2}g(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy\right|\leqslant\int_{0}^{\varepsilon^{-2/3}/2}C(1+|y|)^{-1/2}2^{1/2}dy=O(\varepsilon^{-1/3}).$$

Finally,

$$\begin{aligned}\left|\int_{\varepsilon^{-2/3}/2}^{\varepsilon^{-2/3}}g(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy\right| &= \left|\int_{1/2}^{1}g(t/\varepsilon^{2/3})(1-t)^{d/2-1}\frac{dt}{\varepsilon^{2/3}}\right| \\ &\leqslant C\varepsilon^{1/3}\int_{1/2}^{1}t^{-1/2}(1-t)^{d/2-1}\frac{dt}{\varepsilon^{2/3}}=O(\varepsilon^{-1/3}).\end{aligned}$$

If $\alpha \geqslant 5/2$, using the Taylor formula,

$$\begin{aligned}
&\int_{-\infty}^{\varepsilon^{-2/3}} g(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy \\
&= \int_{-\infty}^{+\infty} g(y)dy + \int_{-\infty}^{\varepsilon^{-2/3}/2} g(y)\left((1-\varepsilon^{2/3}y)^{d/2-1}-1\right)dy \\
&\quad + \int_{\varepsilon^{-2/3}/2}^{\varepsilon^{-2/3}} g(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy - \int_{\varepsilon^{-2/3}/2}^{+\infty} g(y)dy \\
&= \int_{-\infty}^{+\infty} g(y)dy + \int_{-\infty}^{\varepsilon^{-2/3}/2} g(y)\left(-(\frac{d}{2}-1)\varepsilon^{2/3}y + (\frac{d}{2}-1)(\frac{d}{2}-2)\frac{1}{2}(1-\xi_{\varepsilon,y})^{d/2-3}\varepsilon^{4/3}y^2\right)dy \\
&\quad + \int_{1/2}^{1} g(t/\varepsilon^{2/3})(1-t)^{d/2-1}\frac{dt}{\varepsilon^{2/3}} + O((\varepsilon^{-2/3})^{-5/2+1}),
\end{aligned}$$

where $\xi_{\varepsilon,y} \in [0, \varepsilon^{2/3}y)$ or $\xi_{\varepsilon,y} \in (\varepsilon^{2/3}y, 0)$, depending on the sign of y. In particular, if $y \leqslant \varepsilon^{-2/3}/2$, $\xi_{\varepsilon,y} \leqslant 1/2$ and since $d/2-3<0$, $(1-\xi_{\varepsilon,y})^{d/2-3} \leqslant 2^{3-d/2}$. Additionnally,

$$\int_{-\infty}^{\varepsilon^{-2/3}/2} g(y)y^2dy \leqslant \int_{-\infty}^{0} g(y)y^2dy + \int_{0}^{\varepsilon^{-2/3}/2} C(1+y)^{-5/2}y^2dy = O(\varepsilon^{-1/3})$$

Moreover, there exists a positive constant C such that for $t \in [1/2, 1]$, $g(t/\varepsilon^{2/3}) \leqslant (\varepsilon^{2/3}/t)^{5/2}$. Thus,

$$\begin{aligned}
\int_{-\infty}^{\varepsilon^{-2/3}} g(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy &= \int_{-\infty}^{+\infty} g(y)dy - (\frac{d}{2}-1)\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}/2} yg(y)dy + O(\varepsilon) \\
&= \int_{-\infty}^{+\infty} g(y)dy - (\frac{d}{2}-1)\varepsilon^{2/3}\int_{-\infty}^{+\infty} yg(y)dy + O(\varepsilon).
\end{aligned}$$

If $\alpha \geqslant 3/2$, we start the calculation like in the case $\alpha \geqslant 5/2$. Then, again thanks to the Taylor formula,

$$\begin{aligned}
\left|\int_{-\infty}^{\varepsilon^{-2/3}/2} g(y)\left((1-\varepsilon^{2/3}y)^{d/2-1}-1\right)dy\right| &= \left|-(\frac{d}{2}-1)\int_{-\infty}^{\varepsilon^{-2/3}/2} g(y)(1-\xi_{\varepsilon,y})^{d/2-2}\varepsilon^{2/3}ydy\right| \\
&\leqslant \left|\frac{d}{2}-1\right| 2^{2-d/2}\varepsilon^{2/3}\int_{-\infty}^{\varepsilon^{-2/3}/2} |g(y)||y|dy = O(\varepsilon^{1/3}),
\end{aligned}$$

for some $\xi_{\varepsilon,y} \in [0, \varepsilon^{2/3}y)$ or $\xi_{\varepsilon,y} \in (\varepsilon^{2/3}y, 0)$. Then,

$$\int_{\varepsilon^{-2/3}/2}^{\varepsilon^{-2/3}} g(y)(1-\varepsilon^{2/3}y)^{d/2-1}dy = \int_{1/2}^{1} g(t/\varepsilon^{2/3})(1-t)^{d/2-1}\frac{dt}{\varepsilon^{2/3}} = O(\varepsilon^{1/3}).$$

Finally,

$$\int_{\varepsilon^{-2/3}/2}^{+\infty} g(y)dy = O(\varepsilon^{1/3}),$$

which gives the result in the case $\alpha \geqslant 3/2$.

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References

[AAB] A. Aftalion, S. Alama, and L. Bronsard, *Giant Vortex and the Breakdown of Strong Pinning in a Rotating Bose-Einstein Condensate*, Arch. Rat. Mech. Anal., **178**, 247–286 (2005)

[DPS] F. Dalfovo, L. Pitaevskii, S. Stringari, *Order Parameter at the Boundary of a Trapped Bose Gas*, Phys. Rev. A 54, 4213-4217 (1996)

[F] E. Fermi, *Statistical method of investigating electrons in atoms*, Z. Phys. **48**, 73–79 (1928)

[FIKN] A.S. Fokas, A.R. Its, A.A. Kapaev and V.Y. Novokshenov, *Painlevé Transcendents, The Riemann-Hilbert Approach*, Mathematical Surveys and Monographs, 128. AMS, Providence, RI, 2006.

[GP] C. Gallo and D. Pelinovsky, *On the Thomas-Fermi ground state in a harmonic potential*, preprint.

[HM] S.P. Hastings and J.B. McLeod, *A boundary Value Problem Associated with the Second Painlevé Transcendent and the Korteweg-de Vries Equation*, Arch. Rat. Mec. Anal., **73**, 31–51 (1980)

[IM] R. Ignat and V. Millot, *The critical velocity for vortex existence in a two-dimensional rotating Bose–Einstein condensate*, J. Funct. Anal. **233**, 260–306 (2006)

[M] D. Margetis, *Asymptotic formula for the condensate wave function of a trapped Bose gas*, Phys. Rev. A **61**, 055601 (2000)

[T] L.H. Thomas, *The calculation of atomic fields*, Proc. Cambridge Philos. Soc. **23**, 542 (1927)