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# DIRICHLET EIGENVALUES OF CONES IN THE SMALL APERTURE LIMIT

THOMAS OURMIÈRES-BONAFOS

## Abstract

We are interested in finite cones of fixed height 1 parametrized by their opening angle. We study the eigenpairs of the Dirichlet Laplacian in such domains when their apertures tend to 0. We provide multi-scale asymptotics for eigenpairs associated with the lowest eigenvalues of each fibers of the Dirichlet Laplacian. To do so, we investigate the family of their one-dimensional Born-Oppenheimer approximations. The eigenvalue asymptotics involves powers of the cube root of the aperture, while the eigenvectors include simultaneously two scales.

## 1 Introduction

### 1.1 Motivations and related questions

Finding an explicit expression of the first Dirichlet eigenvalues in two or three dimensional domains is not an easy task, in general. We know how to proceed when the domain is a tensor product reducing the problem to solving ordinary differential equations. Nevertheless, even for simple two dimensional domains like triangles this question is still complicated. This specific question is detailed in [14] where a finite term asymptotics is provided in the regime  $\theta$  goes to 0 (where  $\theta$  is the aperture of the triangle). More recently [10] gives a complete asymptotics for right-angled triangles.

In the latter paper, the aim of the authors was the study of a broken waveguide with corner in the small angle regime. The knowledge on triangles with small aperture leads to a comparison between the waveguide and a triangle. The question of waveguides with corners has already been investigated for the L-shape waveguide in [13]. For an arbitrary angle [3] provides an asymptotics of the lowest eigenvalues when the angle goes to  $\pi/2$ . The regime with small angle limit has been studied in [5] and more recently in [9, 10]. The question of waveguides with corner arises naturally because it is studied for smooth waveguide in [11, 6, 7] where we learn, among other things, that curvature induces bound states below the essential spectrum. The idea is that a corner can be seen as an infinite curvature.

The aim of the present paper is to obtain asymptotics for three dimensional cones in the small aperture limit. As in two dimensions, this question naturally appears when looking for the ground states in the small aperture regime of the conical layer studied in [12].

We can apply our result to obtain asymptotics for geometrical domains close enough of cones. Hence a spherical sector being the union of a cone and a spherical skullcap we have a finite terms asymptotics resulting of the asymptotics on cones. It echoes to [14] in a higher dimension : it is the three dimensional equivalent of the circular sector, the Bessel functions playing a similar role as trigonometric functions.

## 1.2 The Dirichlet Laplacian on conical families

Let us denote by  $(x_1, x_2, x_3)$  the Cartesian coordinates of the space  $\mathbb{R}^3$  and by  $\mathbf{0} = (0, 0, 0)$  the origin. The positive Laplace operator is given by  $-\partial_1^2 - \partial_2^2 - \partial_3^2$ . We are interested in domains delimited by a finite open cone. For  $\theta \in (0, \frac{\pi}{2}]$ , we introduce the cone  $\text{Co}(\theta)$  defined by:

$$\text{Co}(\theta) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -1 < x_3 < 0 \text{ and } \cot^2 \theta (x_1^2 + x_2^2) < (x_3 + 1)^2\}.$$

The angle  $\theta$  represents the half opening angle of the cone. The aim of this paper is the investigation of the lowest eigenvalues of each fibers of the positive Dirichlet Laplacian  $-\Delta_{\text{Co}(\theta)}^{\text{Dir}}$  in the small aperture limit.

**Remark 1**  $\text{Co}(\theta)$  being a convex domain, we know that  $\text{Dom} \left( -\Delta_{\text{Co}(\theta)}^{\text{Dir}} \right) = H^2(\text{Co}(\theta)) \cap H_0^1(\text{Co}(\theta))$ .

△

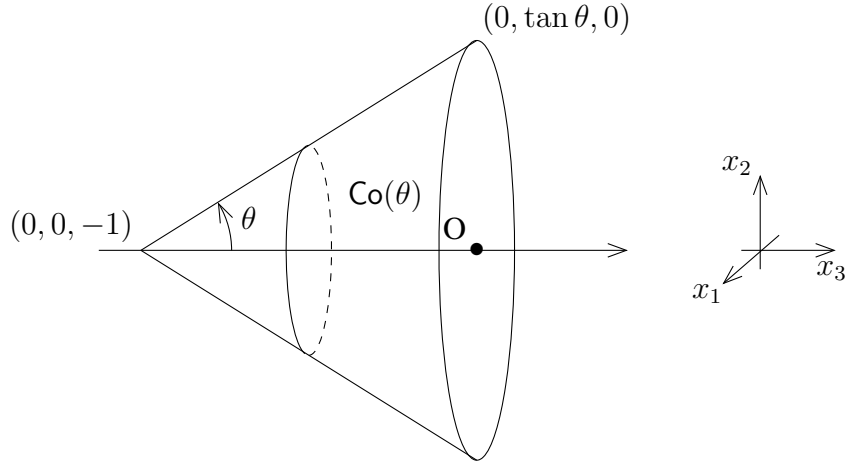


Figure 1: **The cone**  $\text{Co}(\theta)$

## 1.3 Structure of the paper

One can show that after the use of adapted coordinates and a Fourier transform the Dirichlet Laplacian on the cone  $\text{Co}(\theta)$  reduces to a countable family of two dimensional operators. This is discussed in Section 2.

In Section 3 we state the main theorem and we apply it to a spherical sector. We also go about the so called Born-Oppenheimer approximation [8]. Numerical experiments motivate and illustrate the study.

Afterwards, in Section 4, we perform a change of variables that transforms the meridian triangle into a rectangle. The operator is more complicated but we deal with a simpler geometrical domain. Thanks to this substitution we can construct quasimodes for each operator of the countable family using some lemmas which are adapted from the Fredholm alternative. We have Agmon estimates [1, 2] for those operators, localizing the eigenfunctions. The proof of Theorem 3 about the asymptotics of the first eigenvalues of the cone is over, when, using a Feshbach-Grushin projection, we justify that the Born-Oppenheimer approximation is actually an approximation of our problem and we apply the classical technical background for the separation of the eigenvalues.

We conclude by Section A illustrating by numerical experiments the shape of the eigenfunctions which illustrates some theoretical results obtained all along the paper.

## 2 Fiber decomposition

In this section, we describe the fiber decomposition of the Dirichlet Laplacian on the cone  $\text{Co}(\theta)$ . We use the terminology detailed in [15, Section XIII.16].

### 2.1 Partial wave decomposition

We are interested in the positive Laplace operator on the cone  $\text{Co}(\theta)$  which writes

$$-\Delta_{\text{Co}(\theta)}^{\text{Dir}} = -\partial_1^2 - \partial_2^2 - \partial_3^2.$$

We can describe the domain  $\text{Co}(\theta)$  using cylindrical coordinates. Let us perform the change of variables and introduce  $(r, \phi, z)$  such that

$$r = \sqrt{x_1^2 + x_2^2}, \quad \phi = \arctan \frac{x_2}{x_1}, \quad z = x_3. \quad (1)$$

The cartesian domain  $\text{Co}(\theta)$  is transformed into  $\text{Tri}(\theta) \times \mathbb{S}^1$  where the meridian domain  $\text{Tri}(\theta)$  is:

$$\text{Tri}(\theta) := \{(r, z) \in \mathbb{R}^2 : -1 < z < 0 \text{ and } 0 < r < \tan \theta (z + 1)\}.$$

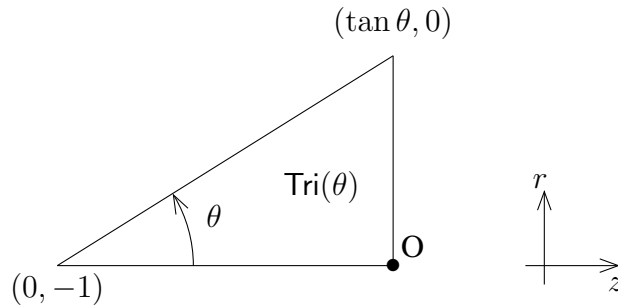


Figure 2: **Meridian domain**  $\text{Tri}(\theta)$

Performing the change of variables the Dirichlet Laplacian is written, on the geometrical domain  $\text{Tri}(\theta) \times \mathbb{S}^1$ , as the operator :

$$\mathcal{H}_{\text{Tri}(\theta) \times \mathbb{S}^1} := -\frac{1}{r} \partial_r (r \partial_r) - \frac{1}{r^2} \partial_\phi^2 - \partial_z^2,$$

its domain being deduced by the change of variables (1).

The domain  $\text{Tri}(\theta) \times \mathbb{S}^1$  being axisymmetric we perform a Fourier transform and we have the constant fiber direct sum:

$$L^2(\text{Tri}(\theta) \times \mathbb{S}^1, r dr d\phi dz) = L^2(\text{Tri}(\theta), r dr dz) \otimes L^2(\mathbb{S}^1) = \bigoplus_{m \in \mathbb{Z}} L^2(\text{Tri}(\theta), r dr dz),$$

where  $L^2(\mathbb{S}^1)$  refers to functions on the unit circle with the orthonormal basis  $\{e^{2i\pi m\phi} : m \in \mathbb{Z}\}$ . The operator  $\mathcal{H}_{\text{Tri}(\theta) \times \mathbb{S}^1}$  decomposes as

$$\mathcal{H}_{\text{Tri}(\theta) \times \mathbb{S}^1} = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_{\text{Tri}(\theta)}^{[m]}, \quad \text{with } \mathcal{H}_{\text{Tri}(\theta)}^{[m]} = -\frac{1}{r} \partial_r (r \partial_r) - \partial_z^2 + \frac{m^2}{r^2}, \quad (2)$$

where the  $\mathcal{H}_{\text{Tri}(\theta)}^{[m]}$  are the fibers of  $\mathcal{H}_{\text{Tri}(\theta)}$  and their domains  $\text{Dom}(\mathcal{H}_{\text{Tri}(\theta)}^{[m]})$  are implicitly defined by the decomposition. This yields :

$$\mathfrak{S}(\mathcal{H}_{\text{Tri}(\theta) \times \mathbb{S}^1}) = \bigcup_{m \in \mathbb{Z}} \mathfrak{S}(\mathcal{H}_{\text{Tri}(\theta)}^{[m]}). \quad (3)$$

And thus, if we denote by  $\mu_n^{[m]}(\theta)$  the  $n^{\text{th}}$  eigenvalue of  $\mathcal{H}_{\text{Tri}(\theta)}^{[m]}$  we have the following description of the spectum:

$$\mathfrak{S}(\mathcal{H}_{\text{Tri}(\theta) \times \mathbb{S}^1}) = \bigcup_{(n,m) \in \mathbb{N}^* \times \mathbb{Z}} \{\mu_n^{[m]}(\theta)\}.$$

**Remark 2** Let  $\psi \in \text{Dom}(\mathcal{H}_{\text{Tri}(\theta)}^{[m]})$ , then we have the Dirichlet boundary condition  $\psi(r, 0) = 0$  and  $\psi((z+1)\tan\theta, z) = 0$ .

If  $m \neq 0$ , we have for integrability reasons  $\psi(0, z) = 0$ . We refer to [4, Chapt. II] for more information.

△

## 2.2 Rescaling of the meridian domain $\text{Tri}(\theta)$

We rescale the integration domain in order to avoid its dependence on  $\theta$ . It transfers this dependence in the coefficients of the operator and it ease the study of the asymptotics  $\theta \rightarrow 0$ . For this reason, let us perform the following linear change of coordinates:

$$x = z, \quad y = \frac{1}{\tan\theta} r, \quad (4)$$

which maps  $\text{Tri}(\theta)$  onto  $\text{Tri}\left(\frac{\pi}{4}\right)$ . That is why we set for simplicity:

$$\text{Tri} := \text{Tri}\left(\frac{\pi}{4}\right). \quad (5)$$

Then, for each  $m \in \mathbb{Z}$ ,  $\mathcal{H}_{\text{Tri}(\theta)}^{[m]}$  is unitary equivalent to the operator with the new integration domain  $\text{Tri}$ :

$$\mathcal{D}^{[m]}(\theta) := -\partial_x^2 - \frac{1}{\tan^2\theta} \partial_y (y \partial_y) + \frac{m^2}{\tan^2\theta y^2}.$$

with implicit boundary conditions as in Remark 2. We let  $h = \tan\theta$ ; after a multiplication by  $\tan^2\theta$ , we get the new operator:

$$\mathcal{L}^{[m]}(h) := -h^2 \partial_x^2 - \frac{1}{y} \partial_y (y \partial_y) + \frac{m^2}{y^2}. \quad (6)$$

### 3 Numerical motivations and main results

#### 3.1 Asymptotic expansion of eigenvalues

According to the structure of the spectrum established in (3), we recall that we denote by  $\mu_n^{[m]}(\theta)$  the  $n^{\text{th}}$  eigenvalue of the  $m^{\text{th}}$  fiber of  $\mathcal{H}_{\text{Co}(\theta)}$ . In order to get more detailed information about the behavior of  $\mu_n^{[m]}(\theta)$  we can, at first, study it numerically. We carried out the computation with the operator  $\mathcal{L}^{[m]}(\tan \theta)$  defined in (6).

Figure 3 suggests that, for  $m = 0, 1, 2$ , the eigenvalues converge to a certain limit as the aperture  $\theta$  goes to 0. Moreover, this value is near  $j_{m,1}^2$ , where we denote by  $j_{m,1}$  the first zero of the  $m^{\text{th}}$  Bessel function of first kind, represented by black dots. This result have to be connected to the one established in [12] where, studying a conical layer, the value  $\frac{j_{0,1}^2}{\pi^2}$  play a similar role. In this paper, they only consider the operator from the fiber of order 0 because in this case the other fibers have only essential spectrum (the factor  $\frac{1}{\pi^2}$  being a normalization constant). One can see that for  $\theta$  large enough the eigenvalues cross and although  $\mu_n^{[m]}$  represents the  $n^{\text{th}}$  eigenvalue of the  $m^{\text{th}}$  fiber of  $\mathcal{H}_{\text{Co}(\theta)}$  it is clear that  $\mu_n^{[0]}(\theta)$  is not necessarily the  $n^{\text{th}}$  eigenvalue of  $\mathcal{H}_{\text{Co}(\theta)}$ . We will see in Appendix B that the other Bessel zeros  $j_{m,k}$  could explain some structure of the eigenvectors outside the semi-classical limit  $\theta \rightarrow 0$ .

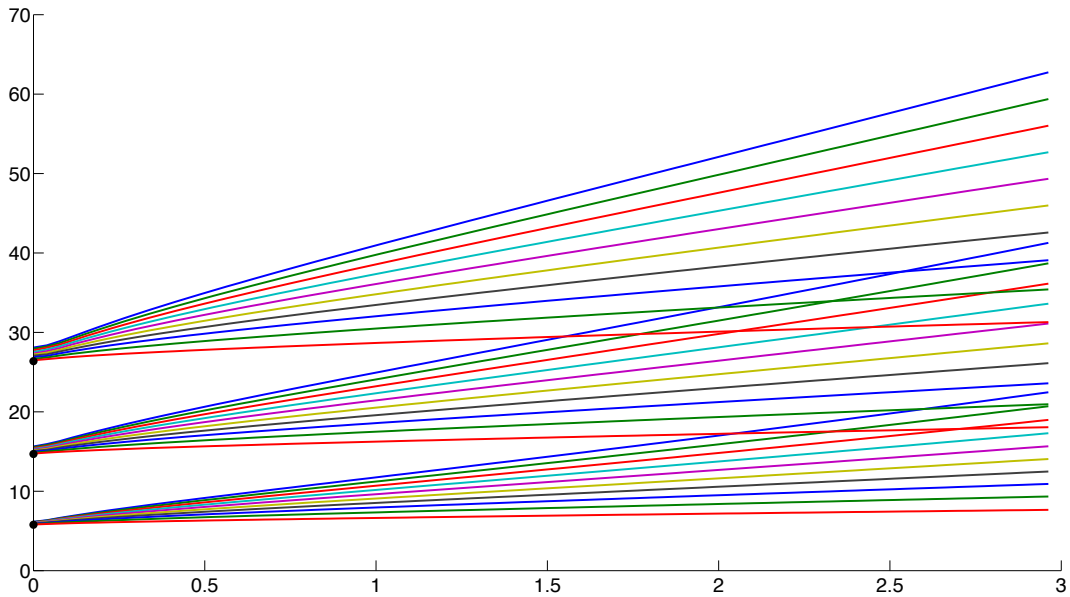


Figure 3: This figure represents the dependence of the first ten eigenvalues  $\mu_n^{[m]}$  ( $m = 0, 1, 2$ ) on the aperture  $\theta$  [°]. We computed each eigenvalue for 80 values of  $\theta$ .

The main result of this paper is not only the convergence illustrated in Figure 3 but an asymptotic expansion of these eigenvalues. Indeed, the lowest eigenvalues of each angular component of  $\mathcal{H}_{\text{Co}(\theta)}$  admit expansions at any order in powers of  $\theta^{1/3}$ . We first state the result for the scaled operators  $\mathcal{L}^{[m]}(h)$  introduced in (6):

**Theorem 3** *If we denote by  $z_A(n)$  the  $n^{\text{th}}$  zero of the reversed Airy function, the eigenvalues of  $\mathcal{L}^{[m]}(h)$ , denoted by  $\lambda_n^{[m]}(h)$ , admit the expansions:*

$$\lambda_n^{[m]}(h) \underset{h \rightarrow 0}{\sim} \sum_{k \geq 0} \beta_{k,n}^{[m]} h^{k/3} \quad \text{with } \beta_{0,n}^{[m]} = j_{m,1}^2, \beta_{1,n}^{[m]} = 0, \beta_{2,n}^{[m]} = (2 j_{m,1}^2)^{2/3} z_A(n),$$

*the terms of odd rank being zero for  $j \leq 8$ . The corresponding eigenvectors have expansions in powers of  $h^{1/3}$  with both scales  $x/h^{2/3}$  et  $x/h$ .*

In terms of the physical domain  $\text{Tri}(\theta)$ , we immediately deduce from the previous theorem that the eigenvalues of each angular component  $m$  of  $-\Delta_{\text{Co}(\theta)}^{\text{Dir}}$  admit the expansions:

$$\mu_n^{[m]}(\theta) \underset{\theta \rightarrow 0}{\sim} \frac{1}{\theta^2} \sum_{k \geq 0} \beta_{k,n}^{[m],\Delta} \theta^{k/3} \quad \text{with } \beta_{0,n}^{[m],\Delta} = j_{m,1}^2, \beta_{1,n}^{[m],\Delta} = 0, \beta_{2,n}^{[m],\Delta} = (2 j_{m,1}^2)^{2/3} z_A(n).$$

## 3.2 Application to the spherical cone

Theorem 3 on the cone  $\text{Co}(\theta)$  is closely related to the Dirichlet problem on a spherical cone. We denote by  $\text{Sph}(\theta)$  the spherical cone of radius 1 and aperture  $\theta$  with center in  $(0, 0, -1)$  depicted in Figure 4. We have

$$-\Delta_{\text{Sph}(\theta)}^{\text{Dir}} := -\partial_1^2 - \partial_2^2 - \partial_3^2.$$

We perform the change of variables

$$\rho = \sqrt{x_1^2 + x_2^2 + (x_3 + 1)^2}, \quad \alpha = \arccos\left(\frac{x_3 + 1}{\rho}\right), \quad \beta = \begin{cases} \arccos\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}\right) & \text{if } x_2 \geq 0, \\ 2\pi - \arccos\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}\right) & \text{if } x_2 < 0. \end{cases} \quad (7)$$

Hence the domain  $\text{Sph}(\theta)$  is transformed into

$$\widehat{\text{Sph}}(\theta) = \widehat{\text{Circ}}(\theta) \times \mathbb{S}^1,$$

where  $\widehat{\text{Circ}}(\theta)$  is the circular meridian domain in the coordinates  $(\rho, \alpha)$ .

**Remark 4** If instead of the change of variables (7) we change into cylindrical coordinates as in (1), the meridian circular sector  $\text{Circ}(\theta)$  associated is the one of Figure 5. One can pass from  $\widehat{\text{Circ}}(\theta)$  to  $\text{Circ}(\theta)$  by the change of variables

$$r = \rho \cos \alpha - 1, \quad z = \rho \sin \alpha,$$

which links those two domains without the intermediary cartesian domain.  $\triangle$

The Dirichlet Laplacian  $\mathcal{H}_{\widehat{\text{Sph}}(\theta)}$  writes in spherical coordinates

$$\mathcal{H}_{\widehat{\text{Sph}}(\theta)} := -\frac{1}{\rho^2} \partial_\rho (\rho^2 \partial_\rho) - \frac{1}{\rho^2 \sin \alpha} \partial_\alpha (\sin \alpha \partial_\alpha) - \frac{1}{\rho^2 \sin^2 \alpha} \partial_\beta^2,$$

on  $L^2(\widehat{\text{Sph}}(\theta), \rho^2 \sin \alpha d\rho d\alpha d\beta)$ . As in Section 2 we have the constant fiber direct sum:

$$L^2(\widehat{\text{Sph}}(\theta), \rho^2 \sin \alpha d\rho d\alpha d\beta) = \bigoplus_{m \in \mathbb{Z}} L^2(\widehat{\text{Circ}}(\theta), \rho^2 \sin \alpha d\rho d\alpha),$$

and  $\mathcal{H}_{\widehat{\text{Sph}}(\theta)}$  decomposes in fibers :

$$\mathcal{H}_{\widehat{\text{Sph}}(\theta)} = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_{\widehat{\text{Circ}}(\theta)}^{[m]},$$

where

$$\mathcal{H}_{\widehat{\text{Circ}}(\theta)}^{[m]} := -\frac{1}{\rho^2} \partial_\rho (\rho^2 \partial_\rho) - \frac{1}{\rho^2 \sin \alpha} \partial_\alpha (\sin \alpha \partial_\alpha) + \frac{m^2}{\rho^2 \sin^2 \alpha},$$

with implicit domains and boundary conditions. Let  $(\mu, \Psi)$  be an eigenpair of  $\mathcal{H}_{\widehat{\text{Circ}}(\theta)}^{[m]}$ , with  $\Psi(\rho, \alpha) = R(\rho)M(\alpha)$ . It should satisfy the following system of differential equations :

$$\begin{cases} \left[ \partial_\rho (\rho^2 \partial_\rho) + (c(\theta) - \mu \rho^2) \right] R(\rho) &= 0, \\ \left[ -\frac{1}{\sin \alpha} \partial_\alpha (\sin \alpha \partial_\alpha) + \frac{m^2}{\sin^2 \alpha} \right] M(\alpha) &= c(\theta)M(\alpha). \end{cases} \quad (8)$$

**Remark 5** We are not interested here in solving those equations. Nevertheless one can see that formally, when  $\theta \rightarrow 0$  the angle  $\alpha$  is small and the last equation of (8) looks like the Bessel equation. This could be a lead to find an asymptotic expansion at any order of  $\mu$  when  $\theta \rightarrow 0$ .  $\triangle$

However thanks to Theorem 3 we have easily a finite term asymptotic for the eigenvalues of  $-\Delta_{\text{Sph}(\theta)}^{\text{Dir}}$ . Let  $\text{Co}(\theta, \cos \theta)$  be the set  $\text{Co}(\theta)$  up to a dilatation of ratio  $\cos \theta$ . We have the set inclusion in  $\mathbb{R}^3$

$$\text{Co}(\theta, \cos \theta) \subset \text{Sph}(\theta) \subset \text{Co}(\theta).$$

Let  $\check{\mu}_n(\theta)$  be the  $n^{\text{th}}$  eigenvalue of the Dirichlet Laplacian on  $\text{Sph}(\theta)$  and  $\mu_n(\theta)$  the one on the cone  $\text{Co}(\theta)$ , the monotonicity of the Dirichlet Laplacian yields:

$$(1 + \tan^2 \theta) \mu_n(\theta) \geq \check{\mu}_n(\theta) \geq \mu_n(\theta). \quad (9)$$

If  $\check{\mu}_n^{[m]}(\theta)$  denotes the  $n^{\text{th}}$  eigenvalue of the  $m^{\text{th}}$  fiber of the Dirichlet Laplacian, (9) yields for small  $\theta$

$$(1 + \tan^2 \theta) \mu_n^{[0]}(\theta) \geq \check{\mu}_n^{[0]}(\theta) \geq \mu_n^{[0]}(\theta).$$

To deal with higher fibers we can apply the exact same argument on the meridian circular sector  $\text{Cir}(\theta)$  and the meridian triangle  $\text{Tri}(\theta)$  because :

$$\text{Tri}(\theta, \cos \theta) \subset \text{Cir}(\theta) \subset \text{Tri}(\theta).$$

As for  $m \geq 1$  there is a Dirichlet boundary condition everywhere (see Remark 2) we have :

$$(1 + \tan^2 \theta) \mu_n^{[m]}(\theta) \geq \check{\mu}_n^{[m]}(\theta) \geq \mu_n^{[m]}(\theta).$$

Those inequalities provide the first terms in the asymptotics of  $\check{\mu}_n^{[m]}(\theta)$ .



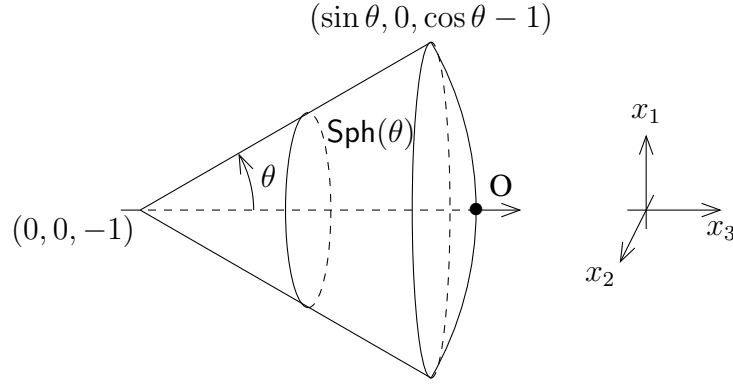


Figure 4: **The spherical cone**  $\text{Sph}(\theta)$

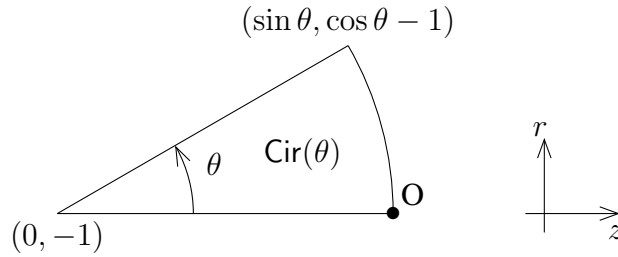


Figure 5: **Meridian domain**  $\text{Cir}(\theta)$

### 3.3 Schrödinger operators in one dimension

In the analysis of  $\mathcal{L}^{[m]}(h)$  we will see that its so called Born-Oppenheimer approximation will play an important role, that is why we define

$$l_{\text{BO}}^{[m]}(h) := -h^2 \partial_x^2 + v^{[m]}(x), \quad (10)$$

where the effective potential  $v^{[m]}$  is obtained by replacing  $-\frac{1}{y} \partial_y (y \partial_y) + \frac{m^2}{y^2}$  in the expression of  $\mathcal{L}^{[m]}(h)$  by its lowest eigenvalue on each slice of  $\text{Tri}$  at fixed  $x$ . One can see that

$$v^{[m]}(x) = \frac{j_{m,1}^2}{(x+1)^2} \quad \text{for } x \in (-1, 0).$$

By construction, for any  $m$ , the operator (10) can be seen as a lower bound of the operator  $\mathcal{L}^{[m]}(h)$ . As it is shown in subsection 4.4, the choice of this approximation gives us informations about the eigenvalues of  $\mathcal{H}_{\text{Tri}(\theta) \times \mathbb{S}^1}$ . We have the

**Proposition 6** *The eigenvalues of  $l_{\text{BO}}^{[m]}(h)$ , denoted by  $\lambda_{\text{BO},n}^{[m]}(h)$ , admit the expansion:*

$$\lambda_{\text{BO},n}^{[m]}(h) \underset{h \rightarrow 0}{\sim} \sum_{k \geq 0} \hat{\beta}_{k,n}^{[m]} h^{2j/3}, \quad \text{with } \hat{\beta}_{0,n}^{[m]} = j_{m,1}^2 \text{ and } \hat{\beta}_{1,n}^{[m]} = (2 j_{m,1}^2)^{2/3} z_A(n).$$

The shape of the effective potential  $v^{[m]}$  is the same as in [10, Section 3], we then deduce Proposition 6 from [10, Th 4.1.]. The key is the construction of quasimodes at the scale  $h^{2/3}$  which naturally arises

expanding the effective potential  $v^{[m]}$  and recognizing the Airy operator at first order. It yields an upper bound of the eigenvalues of  $l_{\text{BO}}^{[m]}(h)$ . To obtain a lower bound we then need the Agmon estimates of Propositions 7 and 8 (see [1, 2]) and to apply the min-max principle and get the separation of eigenvalues.

The Agmon estimates near  $x = 0$  being the

**Proposition 7** *Let  $\Gamma_0 > 0$ . There exist  $h_0 > 0$ ,  $C_0 > 0$  and  $\eta_0 > 0$  such that for any  $h \in (0, h_0)$  and all eigenpair  $(\lambda, \psi)$  of  $l_{\text{BO}}^{[m]}(h)$  satisfying  $|\lambda - j_{m,1}^2| \leq \Gamma_0 h^{2/3}$ , we have:*

$$\int_{-1}^0 e^{\eta_0 h^{-1}|x|^{3/2}} (|\psi|^2 + |h^{2/3} \partial_x \psi|^2) dx \leq C_0 \|\psi\|^2.$$

And the Agmon estimates near  $x = -1$  being the

**Proposition 8** *Let  $\Gamma_0 > 0$  and  $\rho_0 \in (0, j_{m,1})$ . There exist  $h_0 > 0$ ,  $C_0 > 0$  such that for any  $h \in (0, h_0)$  and all eigenpair  $(\lambda, \psi)$  of  $l_{\text{BO}}^{[m]}(h)$  satisfying  $|\lambda - j_{m,1}^2| \leq \Gamma_0 h^{2/3}$ , we have:*

$$\int_{-1}^0 (x+1)^{-\rho_0/h} (|\psi|^2 + |h \partial_x \psi|^2) dx \leq C_0 \|\psi\|^2.$$

We denote by  $\mu_{\text{BO},n}^{[m]}(\theta)$  the  $n^{\text{th}}$  eigenvalue of the operator  $l_{\text{BO}}^{[m]}(\tan \theta)$ . Hence, we can compare the eigenvalues  $\mu_n^{[m]}(\theta)$  and the one of the Born-Oppenheimer approximation denoted by  $\mu_{\text{BO},n}^{[m]}(\theta)$  in terms of geometrical datas. Figure 6 depicts that for small apertures  $\theta$ , the Born-Oppenheimer approximation seems to be an accurate approximation of the two dimensional operator (6). The director coefficient being numerically close of  $2/3$ , we deduce that their difference is of order  $\theta^{2/3}$ .

## 4 Meridian triangle Tri with Dirichlet boundary condition

The aim of this section is to prove Theorem 3. The proof will be divided into two main steps : a construction of quasimodes and the use of the true eigenfunctions of  $\mathcal{L}^{[m]}(h)$  as quasimodes for the Born-Oppenheimer approximation in order to obtain a lower bound for true eigenvalues. We first perform a change of variables to transform the triangle into a square:

$$u = x \in (-1, 0), \quad t = \frac{y}{x+1} \in (0, 1). \quad (11)$$

The meridian triangle Tri is transformed into a square Sq

$$\text{Sq} := (-1, 0) \times (0, 1). \quad (12)$$

The operator  $\mathcal{L}^{[m]}(h)$  becomes:

$$\begin{aligned} \mathcal{L}_{\text{Sq}}^{[m]}(h)(u, t; \partial_u, \partial_t) &:= \frac{1}{(u+1)^2} \left( -\frac{1}{t} \partial_t (t \partial_t) + \frac{m^2}{t^2} \right) - h^2 \partial_u^2 \\ &\quad - \frac{h^2 t^2}{(u+1)^2} \partial_t^2 + \frac{2h^2 t}{u+1} \partial_t \partial_u - \frac{2h^2 t}{(u+1)^2} \partial_t, \end{aligned} \quad (13)$$

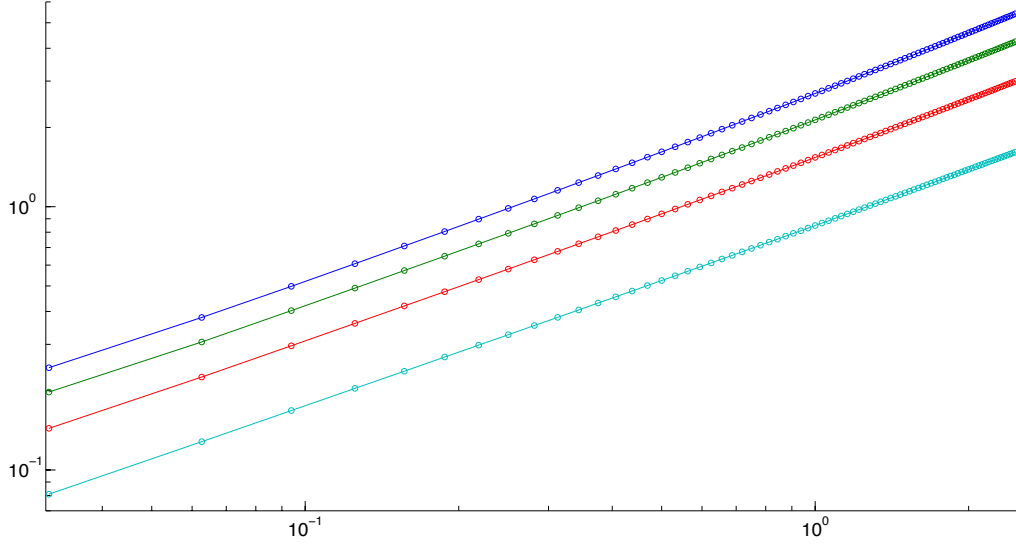


Figure 6: This figure represents the dependence of the difference between the four first eigenvalues  $\mu_n^{[0]}$  and  $\mu_{\text{BO},n}^{[0]}$  on the aperture  $\theta$  [°] in a log-log scale.

on  $L^2(\text{Sq}, t(u+1)^2 du dt)$  with Dirichlet boundary condition on the faces  $\{(0, t) : 0 < t < 1\}$  and  $\{(u, 1) : -1 < u < 0\}$ . The equation  $\mathcal{L}^{[m]}(h)\psi_h^{[m]} = \lambda_h^{[m]}\psi_h^{[m]}$  is transformed into the equation

$$\mathcal{L}_{\text{Sq}}^{[m]}(h)\hat{\psi}_h^{[m]} = \lambda_h^{[m]}\hat{\psi}_h^{[m]} \quad \text{with} \quad \hat{\psi}_h^{[m]}(u, t) = \psi_h^{[m]}(x, y).$$

In what follows we denote by  $\langle \cdot, \cdot \rangle_t$  the scalar product on  $L^2((0, 1), t dt)$ .

## 4.1 Quasimodes

This section is devoted to the proof of the following proposition.

**Proposition 9** *There are sequences  $(\beta_{j,n}^{[m]})_{j \geq 0}$  for any integer  $n \geq 1$  so that there holds: for all  $N_0 \in \mathbb{N}^*$  and  $J \in \mathbb{N}$ , there exist  $h_0 > 0$  and  $C > 0$  such that for  $h \in (0, h_0)$*

$$\text{dist} \left( \mathfrak{S}_{\text{dis}}(\mathcal{L}^{[m]}(h)), \sum_{j=0}^J \beta_{j,n}^{[m]} h^{j/3} \right) \leq C h^{(J+1)/3}, \quad n = 1, \dots, N_0. \quad (14)$$

Moreover, we have:  $\beta_{0,n}^{[m]} = j_{m,1}^2$ ,  $\beta_{1,n}^{[m]} = 0$ , and  $\beta_{2,n}^{[m]} = (2 j_{m,1}^2)^{2/3} z_{\text{A}}(n)$ .

*Proof:* The proof is divided into three parts. The first one explains the shape of the Ansatz chosen to construct quasimodes. The second part deals with three lemmas about operators which appears in the first part. The third part is the determination of the profiles of the Ansatz.

**Shape of the Ansatz** We want to construct quasimodes  $(\beta_h^{[m]}, \psi_h^{[m]})$  for the operator  $\mathcal{L}^{[m]}(h)(x, y; \partial_x, \partial_y)$ . It will be more convenient to work on the square Sq with the operator  $\mathcal{L}_{\text{Sq}}^{[m]}(h)(u, t; \partial_u, \partial_t)$ . We introduce the new scales

$$s = h^{-2/3}u \quad \text{and} \quad \sigma = h^{-1}u,$$

and we look for quasimodes  $(\beta_h^{[m]}, \hat{\psi}_h^{[m]})$  in the form series

$$\beta_h^{[m]} \sim \sum_{j \geq 0} \beta_j^{[m]} h^{j/3} \quad \text{and} \quad \hat{\psi}_h^{[m]}(u, t) \sim \sum_{j \geq 0} \left( \Psi_j^{[m]}(s, t) + \Phi_j^{[m]}(\sigma, t) \right) h^{j/3} \quad (15)$$

in order to solve  $\mathcal{L}_{\text{Sq}}^{[m]}(h)\hat{\psi}_h^{[m]} = \beta_h^{[m]}\hat{\psi}_h^{[m]}$  in the sense of formal series. An Ansatz containing only scale  $h^{-2/3}$  is not sufficient to construct quasimodes for  $\mathcal{L}_{\text{Sq}}^{[m]}(h)$  because one can see that the system is overdetermined. Expanding the operator in powers of  $h^{2/3}$  we obtain the formal series:

$$\mathcal{L}_{\text{Sq}}^{[m]}(h)(h^{2/3}s, t; h^{-2/3}\partial_s, \partial_t) \sim \sum_{j \geq 0} \mathcal{L}_{2j}^{[m]} h^{2j/3} \quad \text{with leading term} \quad \mathcal{L}_0^{[m]} = -\frac{1}{t}\partial_t(t\partial_t) + \frac{m^2}{t^2}, \quad (16)$$

and in power of  $h$ :

$$\mathcal{L}_{\text{Sq}}^{[m]}(h)(h\sigma, t; h^{-1}\partial_\sigma, \partial_t) \sim \sum_{j \geq 0} \mathcal{N}_{3j}^{[m]} h^j \quad \text{with leading term} \quad \mathcal{N}_0^{[m]} = \left( -\frac{1}{t}\partial_t(t\partial_t) + \frac{m^2}{t^2} \right) - \partial_\sigma^2. \quad (17)$$

In what follows, in order to ensure the Dirichlet condition on  $\text{Tri} \setminus (-\pi\sqrt{2}, 0) \times \{0\}$  we will require for our Ansatz the boundary condition, for any  $j \in \mathbb{N}$ :

$$\Psi_j^{[m]}(0, t) + \Phi_j^{[m]}(0, t) = 0, \quad 0 \leq t \leq 1, \quad (18)$$

$$\Psi_j^{[m]}(s, 1) = 0, \quad s < 0 \quad \text{and} \quad \Phi_j^{[m]}(\sigma, 1) = 0, \quad \sigma \leq 0. \quad (19)$$

More specifically, we are interested in the ground energy  $\lambda = j_{m,1}^2$  of the Dirichlet problem at 1 for  $\mathcal{L}_0^{[m]}$  on the interval  $(0, 1)$ . Thus, we have to solve the Dirichlet problem for the operators  $\mathcal{N}_0^{[m]} - j_{m,1}^2$  and  $\mathcal{L}_0^{[m]} - j_{m,1}^2$  on the half-strip

$$\text{Hst} = \mathbb{R}_- \times (0, 1), \quad (20)$$

and look for exponentially decreasing solutions. Our aim is to apply the spectral theorem to the truncated series  $\hat{\psi}_h^{[m]}(u, t)$  restricted in the square Sq thanks to a cut-off function.

**Lemmas** To start the construction of our Ansatz we will need the three next lemmas. Lemmas 11 and 12 are consequences of the Fredholm alternative.

**Lemma 10** We denote the  $n^{\text{th}}$  normalized eigenfunction of  $\mathcal{L}_0^{[m]}$  by  $b_n^{[m]}$ :

$$b_n^{[m]}(t) = C_n^{[m]} J_m(j_{m,n}t) \quad (C_n^{[m]} \in \mathbb{R} \text{ being a normalization constant}),$$

where  $J_m$  is the  $m^{\text{th}}$  Bessel function of first kind. Let  $F = F(\sigma, t)$  be a function in  $L^2(\text{Hst}, t d\sigma dt)$  with exponential decay with respect to  $\sigma$  and let  $G \in H^{3/2}((0, 1), t dt)$  be a function of  $t$  with  $G(1) = 0$ . Then there exists a unique  $\gamma \in \mathbb{R}$  such that the problem

$$\left( \mathcal{N}_0^{[m]} - j_{m,1}^2 \right) \Phi = F, \quad \Phi(\sigma, 1) = 0, \Phi(0, t) = G(t) + \gamma b_1^{[m]}(t)$$

admits a unique solution in  $H^2(\text{Hst}, t d\sigma dt)$  with exponential decay. There holds

$$\gamma = - \int_{-\infty}^0 \int_0^1 F(\sigma, t) \sigma b_1^{[m]}(t) t d\sigma dt - \int_0^1 G(t) b_1^{[m]}(t) t dt. \quad (21)$$

**Lemma 11** Let  $F = F(s, t)$  be a function in  $L^2(\text{Hst}, t ds dt)$  with exponential decay with respect to  $s$ . Then, there exists solution(s)  $\Psi$  such that:

$$\left( \mathcal{L}_0^{[m]} - j_{m,1}^2 \right) \Psi = F \quad \text{in} \quad \text{Hst}, \quad \Psi(s, 1) = 0$$

if and only if  $\langle F(s, \cdot), b_1^{[m]} \rangle_t = 0$  for all  $s < 0$ . In this case,  $\Psi(s, t) = \Psi^\perp(s, t) + g(s) b_1^{[m]}(t)$  where  $\Psi^\perp$  satisfies  $\langle \Psi(s, \cdot)^\perp, b_1^{[m]} \rangle_t \equiv 0$  and has also exponential decay.

**Lemma 12** Let  $n \geq 1$ . We recall that  $z_A(n)$  is the  $n^{\text{th}}$  zero of the Airy reverse function, and we denote by

$$a_n^{[m]}(s) = \tilde{C}_n^{[m]} \mathbf{A} \left( (2 j_{m,1}^2)^{1/3} s + z_A(n) \right) \quad \left( \tilde{C}_n^{[m]} \in \mathbb{R} \text{ being a normalization constant} \right)$$

a normalized eigenvector of the operator  $-\partial_s^2 - (2 j_{m,1}^2) s$  with Dirichlet condition on  $\mathbb{R}_-$  associated with the eigenvalue  $(j_{m,1}^2)^{2/3} z_A(n)$ . Let  $f = f(s)$  be a function in  $L^2(\mathbb{R}_-)$  with exponential decay and let  $c \in \mathbb{R}$ . Then there exists a unique  $\beta \in \mathbb{R}$  such that the problem:

$$\left( -\partial_s^2 - 2 j_{m,1}^2 s - (j_{m,1}^2)^{2/3} z_A(n) \right) g = f + \beta a_n^{[m]} \quad \text{in} \quad \mathbb{R}_-, \quad \text{with} \quad g(0) = c,$$

has a unique solution in  $H^2(\mathbb{R}_-)$  with exponential decay.

**Remark 13** The key in proving Lemmas 10 and 11 are the decomposition as a tensor products of  $L^2(\text{Hst}, t d\sigma dt)$  and  $L^2(\text{Hst}, t ds dt)$ . One can see that  $L^2(\text{Hst}, t d\sigma dt) = L^2(\mathbb{R}_-, d\sigma) \otimes L^2((0, 1), t dt)$  and  $L^2(\text{Hst}, t ds dt) = L^2(\mathbb{R}_-, ds) \otimes L^2((0, 1), t dt)$ . An orthonormal basis of  $L^2((0, 1), t dt)$  being  $(b_n^{[m]})_{n \in \mathbb{N}^*}$ . We then look for solution  $\Phi$  and  $\Psi$  decomposed in this orthonormal basis. Lemma 12 is an application of the Fredholm alternative after changing the function  $g$  to obtain an homogeneous Dirichlet condition at  $s = 0$ .  $\triangle$

**Determination of the profiles** Now we can start the construction of the Ansatz (15).

**Terms in  $h^0$**  The constant terms yields:

$$\mathcal{L}_0^{[m]} \Psi_0^{[m]} = \beta_0^{[m]} \Psi_0^{[m]}, \quad \mathcal{N}_0^{[m]} \Phi_0^{[m]} = \beta_0^{[m]} \Phi_0^{[m]}$$

with boundary conditions (18)-(19) for  $j = 0$ . We choose  $\beta_0^{[m]} = j_{m,1}^2$ , moreover  $\Psi_0^{[m]}$  is a tensor product so,  $\Psi_0^{[m]} = g_0^{[m]}(s) b_1^{[m]}(t)$ . The boundary condition (18) yields :  $\Psi_0^{[m]}(0, t) = -g_0^{[m]}(0) b_1^{[m]}(t)$ . Lemma 10 gives  $g_0^{[m]}(0) = 0$  and  $\Phi_0^{[m]} \equiv 0$ . The function  $g_0^{[m]}$  will be determined later.  $\square$

**Terms in  $h^{1/3}$**  Collecting the terms of order  $h^{1/3}$  we have:

$$\left(\mathcal{L}_0^{[m]} - \beta_0^{[m]}\right) \Psi_1^{[m]} = \beta_1^{[m]} \Psi_0^{[m]} - \mathcal{L}_1^{[m]} \Psi_0^{[m]} = \beta_1^{[m]} \Psi_0^{[m]}$$

and

$$\left(\mathcal{N}_0^{[m]} - \beta_0^{[m]}\right) \Phi_1^{[m]} = \beta_1^{[m]} \Phi_0^{[m]} - \mathcal{N}_1^{[m]} \Phi_0^{[m]} = 0,$$

with boundary conditions (18)-(19) for  $j = 1$ . Lemma 11 yields  $\beta_1^{[m]} = 0$  which leads to the following form for the function  $\Psi_1^{[m]}(s, t) = g_1^{[m]}(s)b_1^{[m]}(t)$ . The boundary condition (18) yields :  $\Psi_1^{[m]}(0, t) = -g_1^{[m]}(0)b_1^{[m]}(t)$ . Lemma 10 gives  $g_1^{[m]}(0) = 0$  and  $\Phi_1^{[m]} \equiv 0$ .

**Terms in  $h^{2/3}$**  Collecting terms of order  $h^{2/3}$  we have:

$$\left(\mathcal{L}_0^{[m]} - \beta_0^{[m]}\right) \Psi_2^{[m]} = \beta_2^{[m]} \Psi_0^{[m]} - \mathcal{L}_2^{[m]} \Psi_0^{[m]}$$

and

$$\left(\mathcal{N}_0^{[m]} - \beta_0^{[m]}\right) \Phi_2^{[m]} = 0$$

where  $\mathcal{L}_2^{[m]} = -\partial_s^2 + 2s \left( \frac{1}{t} \partial_t (t \partial_t) - \frac{m^2}{t^2} \right)$  and boundary conditions (18)-(19) for  $j = 2$ . Lemma 11 yields the following equation in the s-variable:

$$\left\langle (\beta_2^{[m]} - \mathcal{L}_2^{[m]}) \Psi_0^{[m]}(s, \cdot), b_1^{[m]} \right\rangle_t = 0, \quad s < 0.$$

Nevertheless  $\Psi_0(s, t) = g_0^{[m]}(s)b_1^{[m]}(t)$ , consequently this equation becomes:

$$(-\partial_s^2 - 2j_{m,1}^2 s) g_0^{[m]}(s) = \beta_2^{[m]} g_0^{[m]}(s), \quad s < 0.$$

This equation leads to  $\beta_2^{[m]} \equiv (2j_{m,1}^2)^{2/3} z_A(n)$  and  $g_0^{[m]} \equiv a_n^{[m]}$ .

We deduce that  $(\mathcal{L}_0^{[m]} - \beta_0^{[m]}) \Psi_2^{[m]} = 0$  and the  $\Psi_2^{[m]}$  has the form  $\Psi_2^{[m]} = g_2^{[m]}(s)b_1^{[m]}(t)$ . The boundary condition (18) yields :  $\Psi_2^{[m]}(0, t) = -g_2^{[m]}(0)b_1^{[m]}(t)$ . Lemma 10 gives  $g_2^{[m]}(0) = 0$  and  $\Phi_2^{[m]} \equiv 0$ .

**Terms in  $h^{3/3}$**  Collecting the terms of order  $h^{3/3}$  we have:

$$\left(\mathcal{L}_0^{[m]} - \beta_0^{[m]}\right) \Psi_3^{[m]} = \beta_3^{[m]} \Psi_0^{[m]} + \beta_2^{[m]} \Psi_1^{[m]} - \mathcal{L}_2^{[m]} \Psi_1^{[m]}$$

and

$$\left(\mathcal{N}_0^{[m]} - \beta_0^{[m]}\right) \Phi_3^{[m]} = 0$$

with boundary conditions (18)-(19) for  $j = 3$ . The scalar product with  $b_1^{[m]}$  (Lemma 11) then the scalar product with  $g_0^{[m]}$  (Lemma 12) give  $\beta_3^{[m]} = 0$  and  $g_1^{[m]}$  is parallel to  $g_0^{[m]}$ . We choose  $g_1^{[m]} \equiv 0$ . As a consequence  $\Psi_3^{[m]}$  has the form  $\Psi_3^{[m]}(s, t) = g_3^{[m]}(s)b_1^{[m]}(t)$ . Lemma 10 gives  $g_3^{[m]}(0) = 0$  and  $\Phi_3^{[m]} \equiv 0$ .

**Terms in  $h^{4/3}$**  Collecting the terms of order  $h^{4/3}$  we have:

$$\left(\mathcal{L}_0^{[m]} - \beta_0^{[m]}\right) \Psi_4^{[m]} = \beta_4^{[m]} \Psi_0^{[m]} + \beta_2^{[m]} \Psi_2^{[m]} - \mathcal{L}_4^{[m]} \Psi_0^{[m]} - \mathcal{L}_2^{[m]} \Psi_2^{[m]}$$

and

$$\left(\mathcal{N}_0^{[m]} - \beta_0^{[m]}\right) \Phi_4^{[m]} = 0$$

where

$$\mathcal{L}_4^{[m]} = 2\partial_t \partial_s - \frac{3s^2}{2} \left( \frac{1}{t} \partial_t (t \partial_t) - \frac{m^2}{t^2} \right)$$

with boundary conditions (18)-(19) for  $j = 4$ . The scalar product with  $b_1^{[m]}$  (Lemma 11) yields an equation for  $g_2^{[m]}$  and the scalar product with  $g_0^{[m]}$  (Lemma 12) determined  $\beta_4^{[m]}$ . Thanks to Lemma 11,  $\Psi_4^{[m]}$  has the form  $\Psi_4^{[m]} = \Psi_4^{[m]\perp} + g_4^{[m]}(s) b_1^{[m]}(t)$  with  $\Psi_4^{[m]\perp}$  which can be nonzero. Lemma 10 yields  $g_4^{[m]}(0) = 0$ , moreover  $\left\langle \Psi_4^{[m]\perp}(0, \cdot), b_1^{[m]} \right\rangle_t = 0$  and we have a solution  $\Phi_4^{[m]}$  with exponential decay.

**Continuation** We can construct the further terms by induction along the same lines. This leads to define the quasimodes for  $\mathcal{L}^{[m]}(h)$ :

$$\beta_h^{[m,J]} = \sum_{j=0}^J \beta_j^{[m]} h^{j/3}, \quad (22)$$

$$\psi_h(x, y)^{[m,J]} = \chi^{\text{lef}}(x) \sum_{j=0}^J \left( \Psi_j^{[m]} \left( \frac{x}{h^{2/3}}, \frac{y}{x+1} \right) + \Phi_j^{[m]} \left( \frac{x}{h}, \frac{y}{x+1} \right) \right) h^{j/3}, \quad (23)$$

where  $\chi^{\text{lef}}$  is a smooth cut-off function such that:

$$\chi^{\text{lef}}(x) = 1 \quad \text{for } x \in \left( -\frac{1}{2}, 0 \right) \quad \text{and} \quad \chi^{\text{lef}}(x) = 0 \quad \text{for } x \leq -\frac{3}{4}. \quad (24)$$

The spectral theorem yields the conclusion.

## 4.2 Agmon estimates

In our way to prove Theorem 3, we now state Agmon estimates like for  $l_{\text{BO}}^{[m]}(h)$ . Let us first notice that, due to Proposition 9, the  $N_0$  lowest eigenvalues  $\lambda$  of  $\mathcal{L}^{[m]}(h)$  satisfy:

$$|\lambda - j_{m,1}^2| \leq \Gamma_0 h^{2/3}, \quad (25)$$

for some positive constant  $\Gamma_0$  depending on  $N_0$ .

If we denote by  $Q_h^{[m]}$  the quadratic form associated to  $\mathcal{L}^{[m]}(h)$  we have, for all  $\psi \in \text{Dom}(Q_h^{[m]})$ , the following lower bound:

$$Q_h^{[m]}(\psi) \geq \int_{\text{Tri}} \left( h^2 |\partial_x \psi|^2 + \frac{j_{m,1}^2}{(x+1)^2} |\psi|^2 \right) y dx dy. \quad (26)$$

Thus, the analysis giving Propositions 7 and 8 applies exactly in the same way and we obtain:

**Proposition 14** Let  $\Gamma_0 > 0$ . There exist  $h_0 > 0$ ,  $C_0 > 0$  and  $\eta_0 > 0$  such that for  $h \in (0, h_0)$  and all eigenpair  $(\lambda, \psi)$  of  $\mathcal{L}^{[m]}(h)$  satisfying  $|\lambda - j_{m,1}^2| \leq \Gamma_0 h^{2/3}$ , we have:

$$\int_{\text{Tri}} e^{\eta_0 h^{-1}|x|^{3/2}} (|\psi|^2 + |h^{2/3} \partial_x \psi|^2) y dx dy \leq C_0 \|\psi\|^2.$$

**Proposition 15** Let  $\Gamma_0 > 0$ . There exist  $h_0 > 0$ ,  $C_0 > 0$  and  $\rho_0 \in (0, j_{m,1})$  such that for  $h \in (0, h_0)$  and all eigenpair  $(\lambda, \psi)$  of  $\mathcal{L}^{[m]}(h)$  satisfying  $|\lambda - j_{m,1}^2| \leq \Gamma_0 h^{2/3}$ , we have:

$$\int_{\text{Tri}} (x+1)^{-\rho_0/h} (|\psi|^2 + |h \partial_x \psi|^2) y dx dy \leq C_0 \|\psi\|^2.$$

**Remark 16** Propositions 14 and 15 are also verified for  $\psi$  being a finite linear combination of eigenfunctions associated to eigenvalues satisfying (25). One can show that considering both the positive quadratic forms on the left sides of the inequalities in Propositions 14 and 15 and using the Cauchy-Schwarz inequality for those quadratic forms.  $\triangle$

### 4.3 Approximation of the first eigenfunctions by tensor products

In this subsection we will work with the operator  $\mathcal{L}_{\text{Sq}}^{[m]}(h)$  rather than  $\mathcal{L}^{[m]}(h)$ . Let us consider the first  $N_0$  eigenvalues of  $\mathcal{L}_{\text{Sq}}^{[m]}(h)$  (shortly denoted  $\lambda_n(h)$ ). In each corresponding eigenspace we choose a normalized eigenfunction  $\hat{\psi}_n$  so that  $\langle \hat{\psi}_n, \hat{\psi}_p \rangle = 0$  if  $n \neq p$ . We introduce:

$$\widehat{\mathcal{S}}_{N_0}(h) = \text{span}(\hat{\psi}_1, \dots, \hat{\psi}_{N_0}).$$

Let us define  $Q_{\text{Sq}}^{0,[m]}$  the following quadratic form:

$$Q_{\text{Sq}}^{0,[m]}(\hat{\psi}) = \int_{\text{Sq}} \left( |\partial_t \hat{\psi}|^2 + \frac{m^2}{t^2} |\hat{\psi}|^2 - j_{m,1}^2 |\hat{\psi}|^2 \right) t(u+1)^2 du dt,$$

associated with the operator  $\mathcal{L}_{\text{Sq}}^{0,[m]} = \text{Id}_u \otimes \left( -\frac{1}{t} \partial_t (t \partial_t) + \frac{m^2}{t^2} - j_{m,1}^2 \right)$  on  $L^2(\text{Sq}, t(u+1)^2 du dt)$ .

We consider the Feshbach-Grushin projection on the eigenspace associated with the eigenvalue 0 of  $-\frac{1}{t} \partial_t (t \partial_t) + \frac{m^2}{t^2} - j_{m,1}^2$ :

$$\Pi_1^{[m]} \hat{\psi}(u, t) = \left\langle \hat{\psi}(u, \cdot), b_1^{[m]} \right\rangle_t b_1^{[m]}(t). \quad (27)$$

We can now state a first approximation result:

**Proposition 17** There exist  $h_0 > 0$  and  $C > 0$  such that for  $h \in (0, h_0)$  and all  $\hat{\psi} \in \widehat{\mathcal{S}}_{N_0}(h)$ :

$$0 \leq Q_{\text{Sq}}^{0,[m]}(\hat{\psi}) \leq C h^{2/3} \|\hat{\psi}\|^2$$

and

$$\left\| \left( \text{Id} - \Pi_1^{[m]} \right) \hat{\psi} \right\| + \left\| \frac{1}{t} \left( \text{Id} - \Pi_1^{[m]} \right) \hat{\psi} \right\| + \left\| \partial_t \left( \text{Id} - \Pi_1^{[m]} \right) \hat{\psi} \right\| \leq C h^{1/3} \|\hat{\psi}\|.$$

Moreover,  $\Pi_1^{[m]} : \widehat{\mathcal{S}}_{N_0}(h) \rightarrow \Pi_1^{[m]}(\widehat{\mathcal{S}}_{N_0}(h))$  is an isomorphism.



*Proof:* If  $\hat{\psi} = \hat{\psi}_n$  we have:

$$Q_{\text{Sq},h}^{[m]}(\hat{\psi}_n) = \lambda_n \|\hat{\psi}_n\|^2.$$

From this we infer:

$$Q_{\text{Sq},h}^{[m]}(\hat{\psi}_n) \leq (j_{m,1}^2 + Ch^{2/3}) \|\hat{\psi}_n\|^2.$$

The orthogonality of the  $\hat{\psi}_n$  (in  $L^2$  and for the quadratic form) allows to extend this inequality to  $\hat{\psi} \in \widehat{\mathcal{S}}_{N_0}(h)$ :

$$Q_{\text{Sq}}^{0,[m]}(\hat{\psi}) \leq Ch^{2/3} \|\hat{\psi}\|^2.$$

Moreover  $\Pi_1^{[m]}\hat{\psi}$  being in the kernel of  $\mathcal{L}_{\text{Sq}}^{0,[m]}$ , we have:

$$\begin{aligned} Q_{\text{Sq}}^{0,[m]}(\hat{\psi}) &= \left\langle \mathcal{L}_{\text{Sq}}^{0,[m]} \left( \Pi_1^{[m]}\hat{\psi} + (Id - \Pi_1^{[m]})\hat{\psi} \right), \hat{\psi} \right\rangle \\ &= \left\langle \mathcal{L}_{\text{Sq}}^{0,[m]} \left( Id - \Pi_1^{[m]} \right) \hat{\psi}, \hat{\psi} \right\rangle \\ &= Q_{\text{Sq}}^{0,[m]} \left( (Id - \Pi_1^{[m]})\hat{\psi} \right). \end{aligned}$$

If  $\mu_2$  denotes the second eigenvalue of the one dimensional operator  $-\frac{1}{t}\partial_t(t\partial_t) + \frac{m^2}{t^2} - j_{m,1}^2$  we have, for all  $u \in (-1, 0)$ , thanks to the min-max principle:

$$\int_0^1 \left| \partial_t \left( (Id - \Pi_1^{[m]})\hat{\psi} \right) \right|^2 + \frac{m^2}{t^2} \left| (Id - \Pi_1^{[m]})\hat{\psi} \right|^2 - j_{m,1}^2 \left| (Id - \Pi_1^{[m]})\hat{\psi} \right|^2 t dt \geq \mu_2 \int_0^1 \left| (Id - \Pi_1^{[m]})\hat{\psi} \right|^2 t dt.$$

Multiplying by  $(u+1)^2$  and taking the integral over  $u \in (-1, 0)$ , we obtain:

$$Q_{\text{Sq}}^{0,[m]} \left( (Id - \Pi_1^{[m]})\hat{\psi} \right) \geq \mu_2 \left\| (Id - \Pi_1^{[m]})\hat{\psi} \right\|^2.$$

We deduce that:

$$0 \leq Q_{\text{Rec}}^{0,[m]}(\hat{\psi}) \leq Ch^{2/3} \|\hat{\psi}\|^2 \quad ; \quad \left\| \frac{1}{t} \left( Id - \Pi_1^{[m]} \right) \hat{\psi} \right\| + \left\| \partial_t \left( Id - \Pi_1^{[m]} \right) \hat{\psi} \right\| \leq Ch^{1/3} \|\hat{\psi}\|.$$

We also have:

$$\left\| \left( Id - \Pi_1^{[m]} \right) \hat{\psi} \right\| \leq Ch^{1/3} \|\hat{\psi}\|,$$

which yields the Proposition 17. □

## 4.4 Reduction to the Born-Oppenheimer approximation

The aim of this subsection is to prove Theorem 3 using the projections of the true eigenfunctions  $(\Pi_1^{[m]}\hat{\psi}_n)$  as test functions for the quadratic form of the Airy operator. It justifies that  $l_{\text{BO}}^{[m]}(h)$  is an actual approximation of  $\mathcal{L}^{[m]}(h)$ . Let us consider  $\hat{\psi} \in \widehat{\mathcal{S}}_{N_0}(h)$ , we will need a few lemmas to estimate the quadratic form of the Airy operator tested on  $\Pi_1^{[m]}\hat{\psi}$ . The first lemma is an estimate in the triangle Tri, we let  $\hat{\psi}(u, t) = \psi(x, y)$  and we consider the space  $S_{N_0}(h)$

$$S_{N_0}(h) := \text{span}(\psi_1, \dots, \psi_{N_0}),$$

then we have the

**Lemma 18** For all  $\psi \in S_{N_0}(h)$  and for all  $k \in \mathbb{N}$ , there exist  $h_0 > 0$  and  $C > 0$  such that we have, for  $h \in (0, h_0)$ :

$$\int_{\text{Tri}} (x+1)^{-k} |\partial_y \psi|^2 y dx dy \leq C \|\psi\|^2.$$

*Proof:* First let  $\psi = \psi_j$  for some  $j \in \{1, \dots, N_0\}$ . It satisfies the equation:

$$\left( -h^2 \partial_x^2 - \frac{1}{y} \partial_y (y \partial_y) + \frac{m^2}{y^2} \right) \psi_j = \lambda_j(h) \psi_j.$$

Multiplying by  $(x+1)^{-k}$ , taking the scalar product with  $\psi_j$  and integrating by parts we find:

$$\int_{\text{Tri}} (x+1)^{-k} |\partial_y \psi_j|^2 y dx dy \leq C \int_{\text{Tri}} (x+1)^{-k} (|\psi_j|^2 + h^2 (x+1)^{-1} |\psi_j| |\partial_x \psi_j|) y dx dy.$$

Using the Agmon estimates of Propostion 15 with  $\rho_0 \geq k+1$  we deduce the lemma for  $\psi = \psi_j$ . For  $\psi \in S_{N_0}(h)$ , we proceed as explained in the Remark 16.  $\square$

We can now prove:

**Lemma 19** Let  $\hat{\psi}$  be in  $\widehat{S}_{N_0}(h)$ . It exists  $h_0 > 0$  and  $C > 0$  such that for all  $h \in (0, h_0)$

$$\left| h^2 \int_{\text{Sq}} \frac{1}{(u+1)^2} \partial_u \hat{\psi} \partial_t \hat{\psi} t (u+1)^2 dt du \right| \leq C h^{4/3} \|\hat{\psi}\|^2.$$

*Proof:* Thanks to the Cauchy-Schwartz inequality we have:

$$\left| h^2 \int_{\text{Sq}} \frac{1}{(u+1)^2} \partial_u \hat{\psi} \partial_t \hat{\psi} t (u+1)^2 dt du \right|^2 \leq h^4 \int_{\text{Sq}} |\partial_u \hat{\psi}|^2 (u+1)^2 t dt du \int_{\text{Sq}} \frac{1}{(u+1)^4} |\partial_t \hat{\psi}|^2 t (u+1)^2 dt du.$$

In the original coordinates on the meridian domain Tri we have:

$$\int_{\text{Sq}} \frac{1}{(u+1)^4} |\partial_t \hat{\psi}|^2 t (u+1)^2 dt du = \int_{\text{Tri}} (x+1)^{-4} |\partial_y \psi|^2 y dx dy.$$

Combining Lemma 18 with this equality we obtain

$$\int_{\text{Sq}} \frac{1}{(u+1)^4} |\partial_t \hat{\psi}|^2 t (u+1)^2 dt du \leq C_1 \|\hat{\psi}\|^2, \quad (28)$$

for some  $C_1 > 0$ . Using Proposition 14 expressed in the Square Sq coordinates, it exists  $C_2 > 0$  such that:

$$\int_{\text{Sq}} \left| \partial_u \hat{\psi} - \frac{1}{(u+1)^2} \partial_t \hat{\psi} \right|^2 t (u+1)^2 dt du \leq C_2 h^{-4/3} \|\hat{\psi}\|^2.$$

For some  $C_3 > 0$ , equation (28) yields

$$\int_{\text{Sq}} |\partial_u \hat{\psi}|^2 t (u+1)^2 dt du \leq C_3 h^{-4/3} \|\hat{\psi}\|^2,$$

which achieves the proof of the lemma.  $\square$

To have estimates in  $L^2(\text{Sq}, t dt du)$  instead of  $L^2(\text{Sq}, t(u+1)^2 dt du)$  we will need the

**Lemma 20** Let  $\hat{\psi}$  be in  $\widehat{S}_{N_0}(h)$ . It exists  $h_0 > 0$  and  $C > 0$  such that for all  $h \in (0, h_0)$

$$\left| h^2 \int_{\text{Sq}} |\partial_u \hat{\psi}|^2 utdtdu \right| \leq Ch^{4/3} \left\| \hat{\psi} \right\|^2; \quad \left| \int_{\text{Sq}} |u| |\hat{\psi}|^2 utdtdu \right| \leq Ch^{4/3} \left\| \hat{\psi} \right\|^2.$$

*Proof:* We express each integral in the meridian domain Tri and we use the Agmon estimate of Proposition 14 which gives us the lemma.  $\square$

We can now prove the

**Proposition 21** Let  $\hat{\psi} \in \widehat{S}_{N_0}(h)$ . It exists  $h_0 > 0$  and  $C > 0$  such that for all  $h \in (0, h_0)$

$$\int_{\text{Sq}} \left( h^2 |\partial_u \hat{\psi}|^2 + j_{m,1}^2 |u| |\hat{\psi}|^2 \right) tdtdu \leq (\lambda_{N_0}(h) - j_{m,1}^2) \left\| \hat{\psi} \right\|^2 + Ch^{4/3} \left\| \hat{\psi} \right\|^2.$$

*Proof:* Let  $\psi \in S_{N_0}(h)$ , the orthogonality of the  $(\psi_i)_{i \in \{1, \dots, N_0\}}$  yields

$$Q_h^{[m]}(\psi) \leq \lambda_{N_0}(h) \|\psi\|^2.$$

Equation (26) leads to

$$\int_{\text{Tri}} h^2 |\partial_x \psi|^2 + \frac{j_{m,1}^2}{(x+1)^2} |\psi|^2 y dx dy \leq \lambda_{N_0}(h) \|\psi\|^2.$$

Using the convexity of  $\left( x \mapsto \frac{1}{(x+1)^2} \right)$  we get

$$\int_{\text{Tri}} \left( h^2 |\partial_x \psi|^2 + j_{m,1}^2 |x| |\psi|^2 \right) y dx dy \leq (\lambda_{N_0}(h) - j_{m,1}^2) \|\psi\|^2.$$

Performing the change of variable (11) and thanks to Lemmas 19 and 20, we obtain in the square Sq:

$$\int_{\text{Sq}} \left( h^2 |\partial_u \hat{\psi}|^2 + j_{m,1}^2 |u| |\hat{\psi}|^2 \right) tdtdu \leq (\lambda_{N_0}(h) - j_{m,1}^2) \left\| \hat{\psi} \right\|^2 + Ch^{4/3} \left\| \hat{\psi} \right\|^2,$$

which ends the proof of the proposition.  $\square$

**Proof of Theorem 3** The inequality of Proposition 21 can be written as

$$Q_{A,h}^{[m]}(\hat{\psi}) \leq (\lambda_{N_0}(h) - j_{m,1}^2) \left\| \hat{\psi} \right\|^2 + Ch^{4/3} \left\| \hat{\psi} \right\|^2,$$

where  $Q_{A,h}^{[m]}$  is the positive quadratic form associated to an Airy operator defined by

$$Q_{A,h}^{[m]}(\hat{\psi}) := \int_{\text{Sq}} \left( h^2 |\partial_u \hat{\psi}|^2 + j_{m,1}^2 |u| |\hat{\psi}|^2 \right) tdtdu.$$

Proposition 17 and (25) give

$$Q_{A,h}^{[m]}(\hat{\psi}) \leq (\lambda_{N_0}(h) - j_{m,1}^2) \left\| \Pi_1^{[m]} \hat{\psi} \right\|_{L^2(\text{Sq}, tdtdu)}^2 + Ch^{4/3} \left\| \Pi_1^{[m]} \hat{\psi} \right\|_{L^2(\text{Sq}, tdtdu)}^2.$$

Moreover we obtain

$$Q_{A,h}^{[m]}(\hat{\psi}) = Q_{A,h}^{[m]} \left( \Pi_1^{[m]} \hat{\psi} \right) + Q_{A,h}^{[m]} \left( (Id - \Pi_1^{[m]}) \hat{\psi} \right) + 2b_{A,h}^{[m]} \left( \Pi_1^{[m]} \hat{\psi}, (Id - \Pi_1^{[m]}) \hat{\psi} \right),$$

where  $b_{A,h}^{[m]}$  is the bilinear form associated to  $Q_{A,h}^{[m]}$ . We remark that

$$b_{A,h}^{[m]} \left( \Pi_1^{[m]} \hat{\psi}, (Id - \Pi_1^{[m]}) \hat{\psi} \right) = \int_u \left\langle \Pi_1^{[m]} \left( (-h^2 \partial_u^2 + j_{m,1}^2 |u|) \hat{\psi} \right), (Id - \Pi_1^{[m]}) \hat{\psi} \right\rangle_t du = 0.$$

Finally we have

$$Q_{A,h}^{[m]} \left( \Pi_1^{[m]} \hat{\psi} \right) \leq (\lambda_{N_0}(h) - j_{m,1}^2) \left\| \Pi_1^{[m]} \hat{\psi} \right\|_{L^2(\text{Sq}, tdtdu)}^2 + Ch^{4/3} \left\| \Pi_1^{[m]} \hat{\psi} \right\|_{L^2(\text{Sq}, tdtdu)}^2.$$

Let us denote by  $\pi_1^{[m]} \hat{\psi} := \left\langle \Pi_1^{[m]} \hat{\psi}, b_1^{[m]} \right\rangle_t$ , it is a function in the only  $u$ -variable and we have in one dimension:

$$q_{A,h}^{[m]}(\pi_1^{[m]} \hat{\psi}) := \int_{-1}^0 h^2 \left| \partial_u \pi_1^{[m]} \hat{\psi} \right|^2 + j_{m,1}^2 |u| \left| \pi_1^{[m]} \hat{\psi} \right|^2 du \leq (\lambda_{N_0}(h) - j_{m,1}^2) \left\| \pi_1^{[m]} \hat{\psi} \right\|^2 + Ch^{4/3} \left\| \pi_1^{[m]} \hat{\psi} \right\|^2,$$

where the norms  $\left\| \pi_1^{[m]} \hat{\psi} \right\|$  are taken on  $L^2((-1, 0), du)$ .

Let us consider a smooth cut-off function such that:

$$\chi(u) = 1 \quad \text{for } x \in \left( -\frac{1}{2}, 0 \right) \quad \text{and} \quad \chi(u) = 0 \quad \text{for } u \leq -\frac{3}{4}.$$

Proposition 14 gives

$$q_{A,h}^{[m]} \left( \chi \pi_1^{[m]} \hat{\psi} \right) + \mathcal{O}(h^\infty) \left\| \pi_1^{[m]} \hat{\psi} \right\|^2 \leq q_{A,h}^{[m]}(\pi_1^{[m]} \hat{\psi}), \quad \text{and} \quad \left\| \pi_1^{[m]} \hat{\psi} \right\|^2 = (1 + \mathcal{O}(h^\infty)) \left\| \chi \pi_1^{[m]} \hat{\psi} \right\|^2.$$

So it holds

$$q_{A,h}^{[m]}(\chi \pi_1^{[m]} \hat{\psi}) \leq (\lambda_{N_0}(h) - j_{m,1}^2) \left\| \chi \pi_1^{[m]} \hat{\psi} \right\|^2 + Ch^{4/3} \left\| \chi \pi_1^{[m]} \hat{\psi} \right\|^2,$$

and we can consider all the integrals over  $\mathbb{R}_-$ . We then consider  $\widehat{s}_{N_0}(h) := \text{span} \left( \pi_1^{[m]} \hat{\psi}_1, \dots, \pi_1^{[m]} \hat{\psi}_{N_0} \right)$  and apply the min-max principal to the  $N_0$  dimensional space  $\chi \widehat{s}_{N_0}(h)$  which yields

$$j_{m,1}^2 + (2 j_{m,1}^2)^{2/3} z_A(N_0) h^{2/3} \leq \lambda_{N_0}(h) + Ch^{4/3},$$

and we have the separation of the eigenvalues.

## A Eigenvectors shape in the semi-classical limit

To illustrate some properties of the eigenvectors we compute some of them. These computations are performed in the scaled meridian domain  $\text{Tri}$  for the scaled operator  $\mathcal{L}^{[m]}(\tan \theta)$  defined in (6).

Figure 7 pictures the dominant term in the construction (23) : it is almost a tensor product of the Airy function of first kind and the Bessel function of first kind (respectively along the X-axis and Y-axis). In addition, the eigenvectors are localized near the right boundary and are ousted from the left corner, it ties in the Agmon estimates of Propositions 14 and 15.

Figure 8 enlighten the localization for increasing values of  $m$ . As explained in Remark 2 for nonzero  $m$ , there is a Dirichlet boundary condition along the X-axis which induces a repulsion of the eigenvector in the top right corner. We observe that this repulsion increases with the value of  $m$ . We can interpretate this phenomenon by the shape of the  $m^{\text{th}}$  Bessel function of first kind which leads the behavior along the Y-axis.

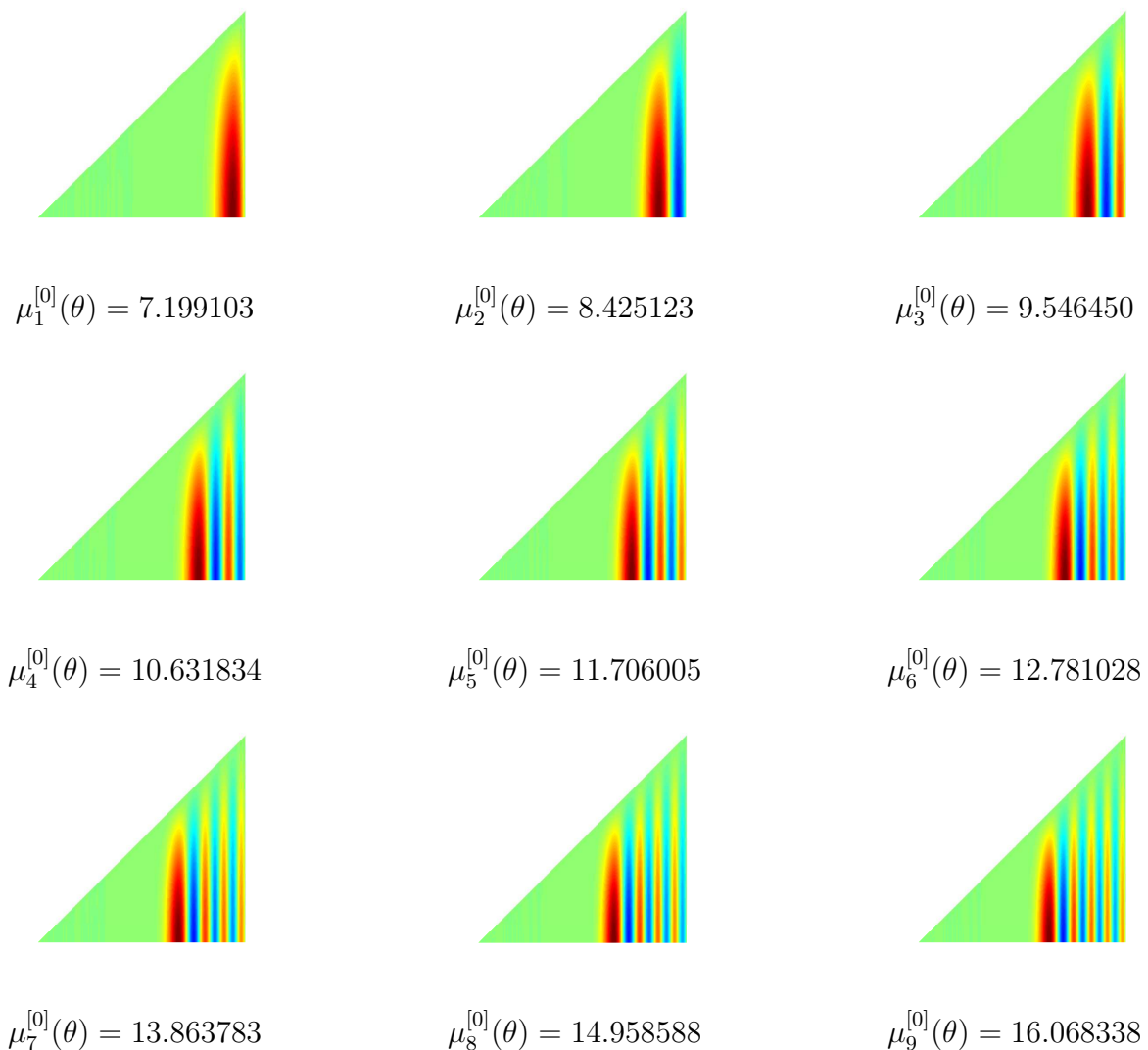


Figure 7: Computation for  $\theta = 0.0226 * \pi/2 \sim 2^\circ$ . Numerical values of the nine first eigenvalues for  $m = 0$ . Plots of the associated eigenvectors in the domain  $\text{Tri}$ .

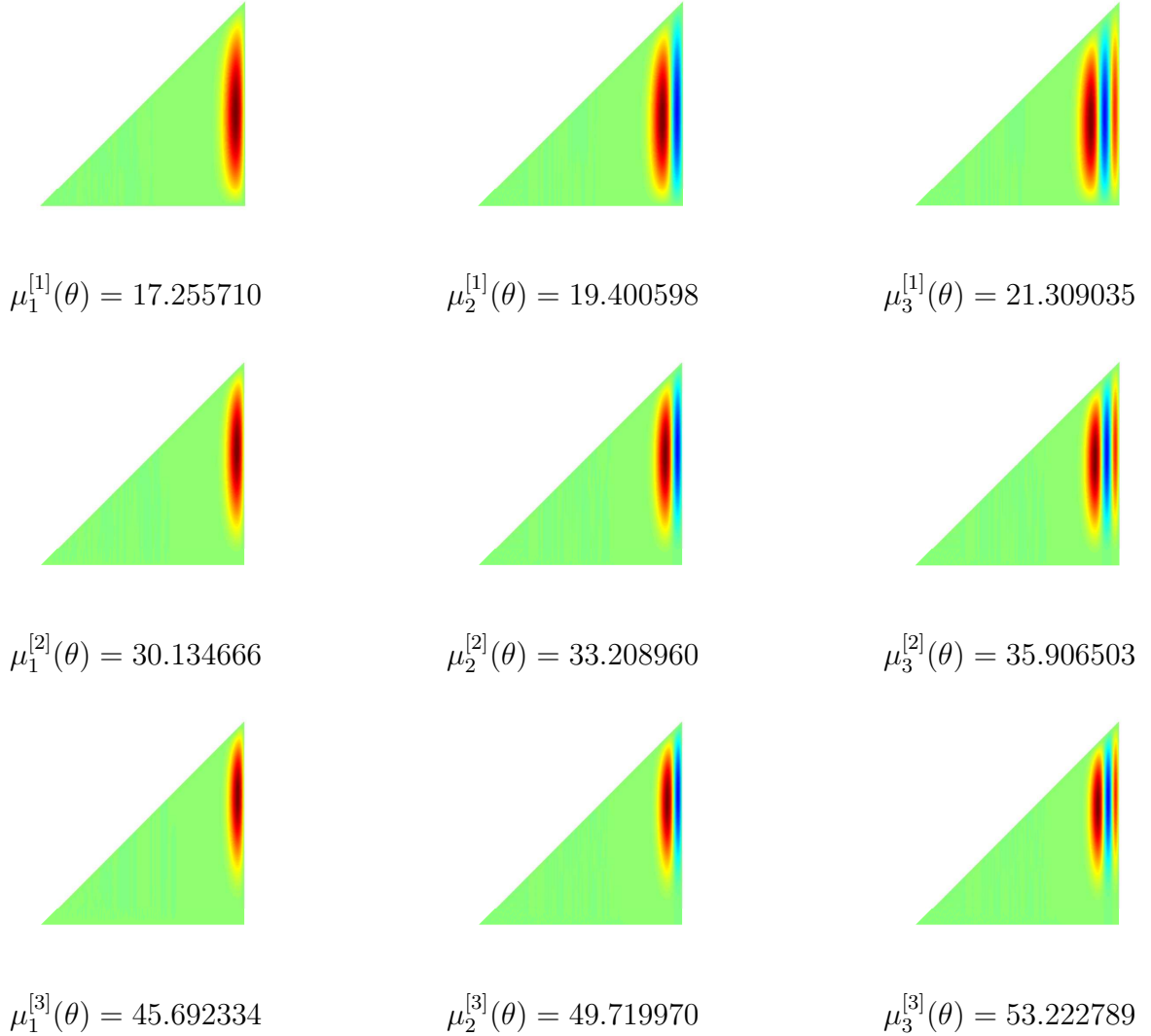


Figure 8: Computation for  $\theta = 0.0226 * \pi/2 \sim 2^\circ$ . Numerical values of the three first eigenvalues for  $m = 1$ ,  $m = 2$  and  $m = 3$ . Plots of the associated eigenvectors in the meridian domain Tri.

## B Behaviour for higher values of the opening angle $\theta$

Figure 9 brings out a phenomenon on the shape of the eigenvalues : the curves bend for some values of the opening angle  $\theta$ . At first glance we can think about crossing and avoidance of the eigenvalues. Figure 10 represents a zoom of the Figure 9 and we see that what may be a crossing is an avoidance.

To understand how those avoidances act on the eigenfunctions we picture on Figure 11, for  $m = 0$ , eigenfunctions after an avoidance. Even if we are not in the semi-classical regime  $\theta \rightarrow 0$  anymore, the sixth eigenfunction has still the shape expected. The seventh has two main nodal zones while there is the shadow of other nodal zones. The eighth eigenfunction have seven nodal zones and the shape of what is expected to be the seventh eigenfunction in the semi-classical limit. We observe the same kind of phenomenon for the ninth and tenth eigenfunctions. To have an idea of what happens, we can think about the first terms of Theorem 3. If  $j_{0,1}^2 + (2j_{0,1}^2)^{2/3} z_A(7)h^{2/3}$  is near  $j_{0,2}^2 + (2j_{0,2}^2)^{2/3} z_A(1)h^{2/3}$  we

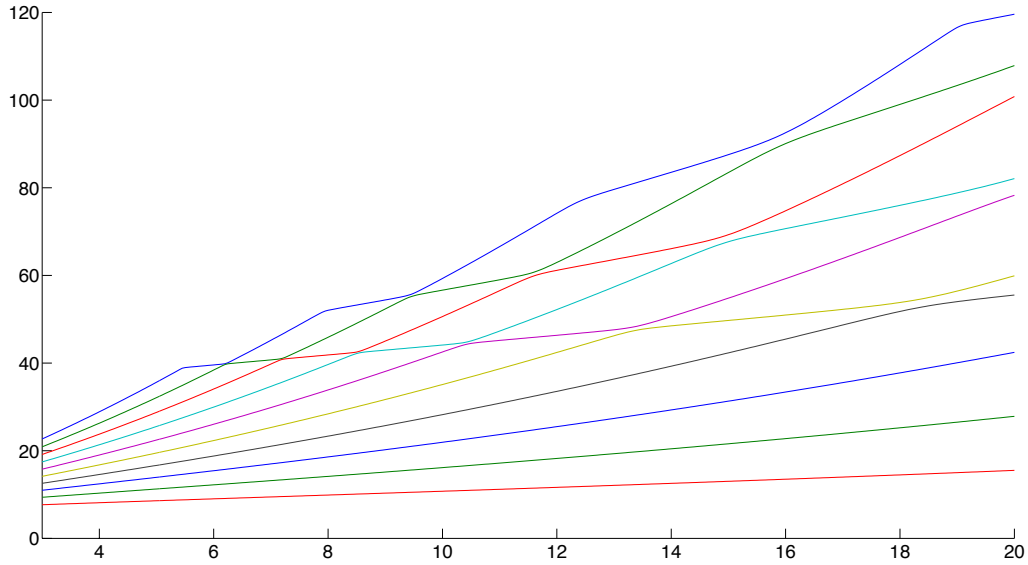


Figure 9: This figure represents the dependence of the first ten eigenvalues  $\lambda_n^{[0]}$  on the aperture  $\theta$  [°]. We computed each eigenvalue every  $0.05^\circ$ .

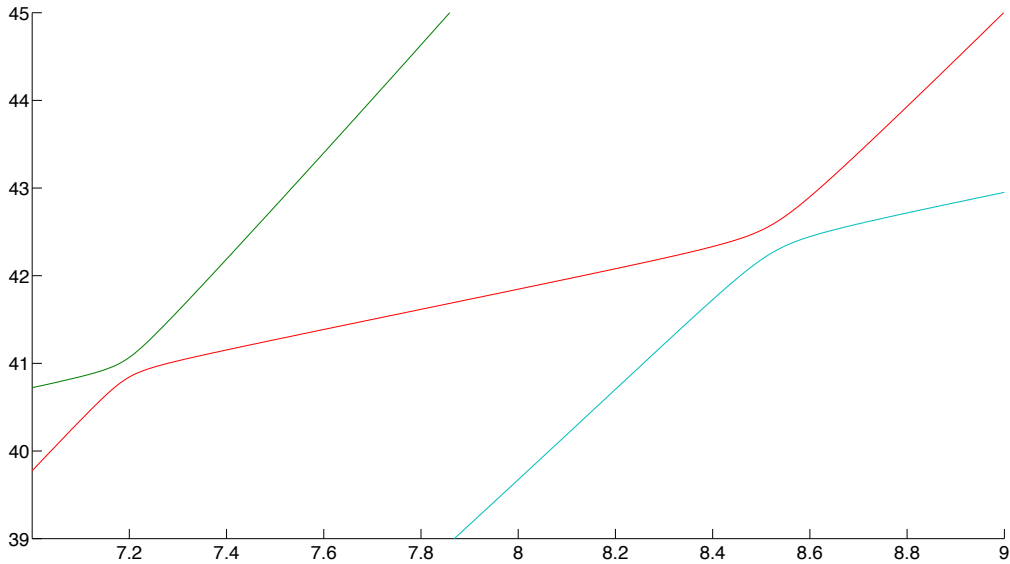


Figure 10: This figure is a zoom of Figure 9. We computed each eigenvalue every  $0.005^\circ$ .

can imagine that the other zeros of Bessel have to play a role. It could partially explain the structure of Figures 11 detailed before. We see that it also happens for the terms  $j_{0,1}^2 + (2j_{0,1}^2)^{2/3} z_A(8) h^{2/3}$  and  $j_{0,2}^2 + (2j_{0,2}^2)^{2/3} z_A(2) h^{2/3}$  for the ninth and tenth eigenfunctions.

Figure 12 represents the evolution of nodal zones along an avoidance in the meridian domain  $\text{Tri}(\theta)$ . We chose this domain to spotlight the smooth structure of the nodal lines. The three nodal zones on

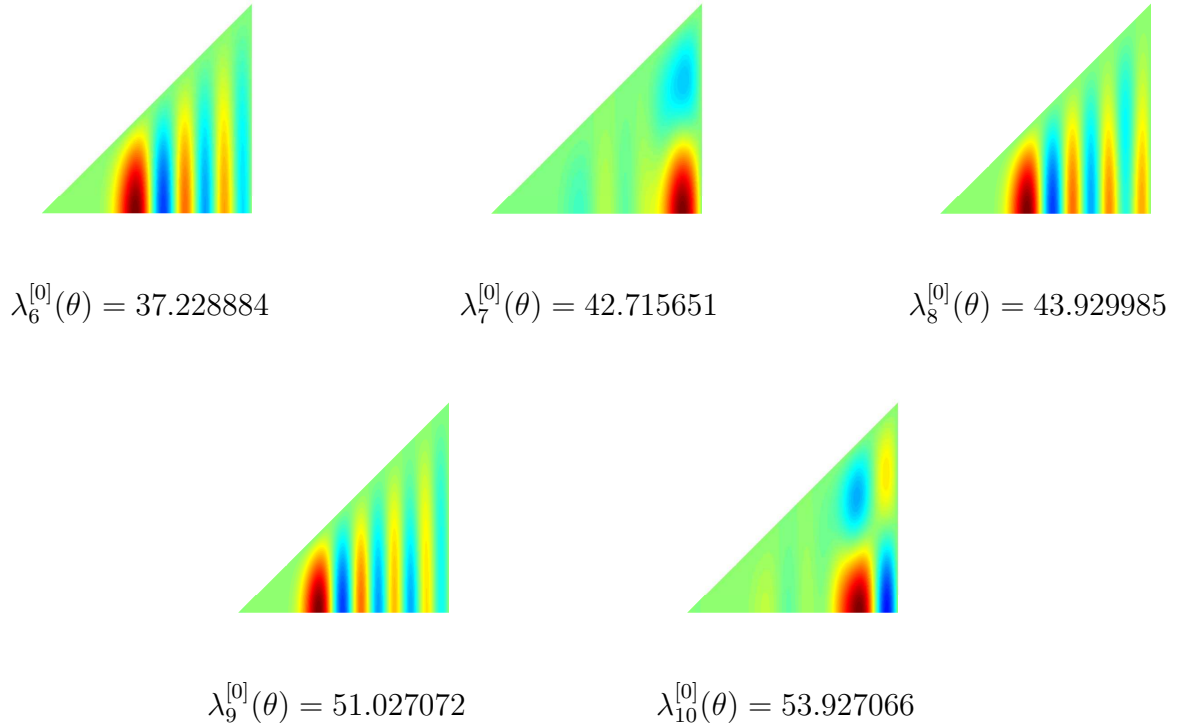


Figure 11: Computation for  $\theta = 8.8^\circ$ . Numerical values of corresponding eigenvalues. Plots of the associated eigenvectors in the meridian domain Tri.

the left side are unchanged. Nevertheless the two blue nodal zones of the right side are progressively turned into one unique. We then observe the same effect on the two red nodal zones of the right side. Finally we have on the left side the three initial nodal zones and two other, one above the other.

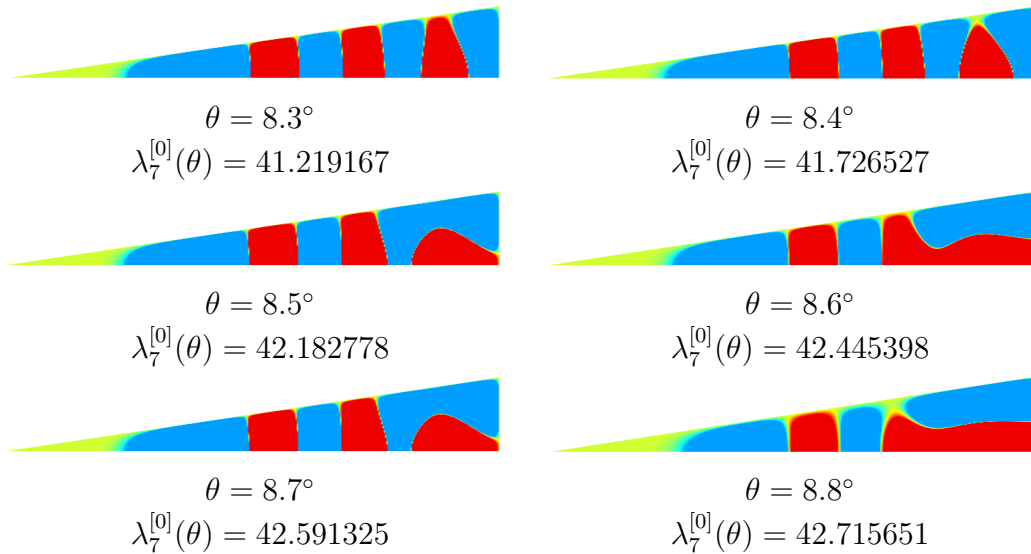


Figure 12: Computations for various  $\theta$ . Numerical values of corresponding eigenvalues. Plots of the associated nodal zones in the original meridian domain  $\text{Tri}(\theta)$ .



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