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Mathematical modeling of volumetric material growth

Jean-Franois Ganghoffer · Pavel I. Plotnikov · Jan Sokołowski

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Abstract Growth (resp. atrophy) describes the physical processes by which a material of solid body increases (resp. decreases) its size by addition (resp. removal) of mass. In the present contribution, we propose a sound mathematical analysis of growth, relying on the decomposition of the geometric deformation tensor into the product of a growth tensor describing the local addition of material and an elastic tensor which is characterizing the reorganization of the body. The Blatz-Co hyperelastic constitutive model is adopted for an isotropic body, satisfying convexity conditions (resp. concavity conditions) with respect to the transformation gradient (resp. temperature). The evolution law for the transplant is obtained from the natural assumption that the evolution of the material is independent of the reference frame. It involves a modified Eshelby tensor based on the specific free energy density. The heat flux is dependent upon the transplant. The model consists of the constitutive equation, the energy balance, and the evolution law for the transplant. It is completed by suitable boundary conditions for the displacement, temperature and transplant tensor. The existence of locally unique solutions is obtained, for sufficiently smooth data close to the stable equilibrium. The ques-

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tion of the global existence is examined in the simplified situation of quasistatic isothermal equations of linear elasticity under the assumption of isotropic growth.

Keywords Volumetric growth · Mathematical analysis · local existence of solutions

1 Introduction

Growth (resp. atrophy) describes the physical processes by which a material of solid body increases (resp. decreases) its size by addition (resp. removal) of mass. A clear distinction is generally made between growth per se, remodeling (change of properties), and morphogenesis (shape changes), a classification suggested by [17] Taber (1995). The advantages and drawbacks of the existing growth models are exposed in the recent contribution [10](Menzel and Kuhl, 2012). They can be classified as follows: The kinematic models with an evolution towards an homeostatic state rely on the kinematic decomposition of the transformation gradient into a generally incompatible mapping and an elastic mapping; they were historically introduced by [15] Rodriguez et al. (1994). The growth transformation evolves in time as a function of the difference between a stress measure and a corresponding measure associated to the surmised homeostatic state ([17] Taber, 1998 ; [16] Rodriguez et al., 2007 ; [1] Alford et al., 2008 ; [21] Vignes et Papadopoulos, 2010). This first class of models is criticized due to the absence of a rational mechanical framework. Approaches analogous to elastoplasticity have been developed in a rational framework basing on the writing of the second principle of thermodynamics for open systems, in order to identify the evolution laws of growth ([7] Kuhl et al., 2007 ; [9] Menzel, 2007 ; [11] Olsson et Klarbring, 2008). It is important to note the prominent role of Eshelby stress in relation to the material driving forces for growth ([5], [6] Ganghoffer, 2010, 2011; [7] Kuhl et al., 2007), relying on Eshelby pioneering approach ([4] Eshelby, 1957). Central here is the idea to separate the shape variation due to the physical motion from the microstructural evolutions due to growth and remodeling phenomena occurring in the evolutive reference configuration.

The mathematical analysis of nonlinear models is described in great detail in the monograph [12] for the fluid dynamics equations. We thereby advocate a novel contribution, since this is the first attempt to lay down a sound mathematical framework for growth models in solid continuum mechanics.

2 Problem formulation

The present work deals with the mathematical modeling of volumetric growth in thermoelastic bodies. The mechanical models are based on the general idea that growth can be taken into account by considering that deformations of a solid body can be due to both changes of mass end elastic deformation. The most important statement of the theory ([14]) is that the geometric deformation tensor is decomposed into the product of a growth tensor describing the local addition of material and an elastic tensor characterizing the reorganization of the body. The rigorous foundation of the volumetric growth theory was given in [3]; our considerations are based on equations formulated in this last work. The state of

the material is characterized by a displacement vector field $\mathbf{u}$, the density ϱ, the temperature θ, and a matrix-valued transplant $\mathbf{K}$, from the reference crystal to a tangent neighbourhood of any material point $\mathbf{X}$. It is supposed that the state variables are functions of the reference configuration variable $X \in \Omega \subset \mathbb{R}^3$ and the temporal variables $t \in [0,T]$. Furthermore, we assume that the reference set Ω is a bounded domain in $\mathbb{R}^d$ with a boundary of class C^∞. In this framework, the motion of material is a one-parametric family of diffeomorphisms $\Omega \times (0,T) \ni (X,t) \mapsto \mathbf{u}(x,t)$. At every moment, $x = \mathbf{u}(X,t)$ is the physical position in the real Euclidian space of the material particle labeled by the reference coordinate X. Hence, the evolution of the material is completely determined by the vector field

$$\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^d,$$

the Kelvin temperature,

$$\theta : \Omega \times \mathbb{R} \to \mathbb{R}_+$$

and the transplant tensor field

$$\mathbf{K} : \Omega \times (0,T) \to \mathbb{R}^{d^2}.$$

The important kinematic characteristics are the transformation gradient $\mathbf{F}(X,t)$ and the velocity field $\mathbf{v}(X,t)$, defined by

$$\mathbf{F} = \nabla \mathbf{u} = \Big(\frac{\partial u_i}{\partial X_j}\Big)_{i,j}, \quad \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}, \tag{2.1}$$

with $\mathbf{u}$ the displacement field. In order to derive the governing equations, we have to formulate the constitutive relations which gives the expressions for stresses, internal energy, and heat flux in terms of $\mathbf{u}$, θ, $\mathbf{K}$. Furthermore, we assume that the material is hyperelastic.

The behavior of a hyperelastic material is completely described by the specific free energy density function $\Psi(\mathbf{F},\theta)$. Usually, $\Psi : \mathbb{R}^{d^2} \times \mathbb{R} \to \mathbb{R}$ is smooth, convex in $\mathbf{F}$ and concave in θ. A typical example is given by the following expression

$$\Psi(\mathbf{F},\theta) = -c_T\,\theta\,\ln\theta + \theta\,W(\mathbf{F}), \tag{2.2}$$

where the convex or polyconvex scalar valued function W represents the stored elastic energy density per unit referential volume. For example, for biological tissues, the Blatz-Co hyperelastic model is commonly adopted with the stored energy density given by

$$W(\mathbf{F}) = \frac{\nu}{2}\Big[\,\big(\mathrm{tr}\,(\mathbf{F}^T\mathbf{F}) - 3\,\big)) - \frac{2}{q}\Big(\big(\,\det\,\mathbf{F})^q - 1\Big)\Big].$$

The Blatz-Co stored energy is polyconvex for $q < 1$, and it is coercitive for $q < 2/3$.

The specific free energy density Ψ is a thermodynamical potential, and all thermodynamical quantities can be represented in terms of Ψ; we accordingly have the following formulae for the internal energy E and entropy S

$$E = \Psi(\mathbf{F},\theta) - \theta\frac{\partial\Psi(\mathbf{F},\theta)}{\partial\theta}, \quad S = -\frac{\partial\Psi(\mathbf{F},\theta)}{\partial\theta}. \tag{2.3}$$

The most important postulate of the hyperelasticity theory is that the nominal stress tensor $\mathbf{T}$ conjugate to $\mathbf{F}$ is defined by the formula

$$\mathbf{T} = \frac{\partial \Psi(\mathbf{F}, \vartheta)}{\partial \mathbf{F}} \tag{2.4}$$

Here, the notation $\partial \Psi / \partial \mathbf{F}$ stands for the matrix

$$\frac{\partial \Psi(\mathbf{F}, \vartheta)}{\partial \mathbf{F}} = \Big(\frac{\partial \Psi(\mathbf{F}, \vartheta)}{\partial F_{i,j}} \Big)_{i,j}.$$

The mass flux for an hyperelastic material is zero, and the heat flux Q can be taken in the standard form of the Fourier law

$$Q = \kappa \nabla \Big(\frac{1}{\theta} \Big)$$

with κ the scalar heat conductivity for an isotropic heat conduction model.

The main assumptions of volumetric growth theory is that the strain tensor is a product of the elastic strain tensor ant the transplant tensor, which leads to the following representation for the density of the specific free energy Ψ_g of the growing material

$$\Psi_g(\mathbf{F}, \theta, \mathbf{K}) = \frac{1}{J_K} \Psi(\mathbf{F}\mathbf{K}, \theta), \tag{2.5}$$

with $J_K = \det\,(\mathbf{K})$ the Jacobian of the transplant tensor. Previous expression of the density means that the constitutive law is now written from the fixed reference crystal, which entails consequences as to the constitutive law. The formulae for the internal energy and stress tensor are now

$$E_g = \Psi_g - \theta \frac{\partial \Psi_g}{\partial \theta}, \quad \mathbf{T}_g = \frac{1}{J_K} \frac{\partial \Psi(\mathbf{F}\mathbf{K}, \theta)}{\partial \mathbf{F}}. \tag{2.6}$$

If we now introduce the tensor $\Phi =: \mathbf{F}\mathbf{K}$, then the stress tensor can be rewritten in the form

$$\mathbf{T}_g = \frac{1}{J_K} \frac{\partial \Psi(\Phi, \theta)}{\partial \Phi} \mathbf{K}^\top. \tag{2.7}$$

The heat flux for the growing material becomes

$$Q = \frac{\kappa}{J_K} \nabla \Big(\mathbf{K}^\top \mathbf{K} \frac{1}{\theta} \Big). \tag{2.8}$$

Following [3] the governing equations represents the mass conservation law, the linear momentum conservation law, and the energy balance equation. Notice that in accordance with [3], the mass balance equation for the first order model has the form

$$\frac{\partial}{\partial t} \big(\varrho J_K \big) = 0 \quad \text{in} \quad \Omega \times (0, T). \tag{2.9}$$

In other words, the product $\varrho(x,t) J_K(x,t)$ equals some function which is independent of the temporal variable and completely defined by the initial data. This equality means that the density of the reference crystal is constant; it results from the consideration of the mass balance in the reference configuration, equality

$$\frac{\partial}{\partial t} \big(\varrho_0 \big) + \varrho_0 tr \mathbf{L}_\mathbf{K} = 0 \quad \text{in} \quad \Omega \times (0, T). \tag{2.10}$$

together with the equalities $\varrho_0 = \varrho_c {J_K}^- 1$ and $\frac{\partial}{\partial t} J_K = J_K tr \mathbf{L_K}$. Without loss of generality and for simplicity of the analysis, we can assume that $\varrho J_K = 1$, and by doing so eliminate the material density ϱ. In this case, the governing equations of the first order growth theory read

Model A

$$\frac{1}{J_K}\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div}\Big(\frac{1}{J_K}\frac{\partial \Psi(\Phi,\theta)}{\partial \Phi}\mathbf{K}^\top\Big) = \mathbf{f} \ \text{ in } \ \Omega\times(0,T), \tag{2.11a}$$

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\frac{1}{J_K}\Big(\Psi(\Phi,\theta) - \theta\frac{\partial \Psi(\Phi,\theta)}{\partial \theta}\Big)\Big) + \operatorname{div}\Big(\frac{\kappa}{J_K}\,\mathbf{K}^\top\mathbf{K}\,\nabla\Big(\frac{1}{\theta}\Big)\Big) = \\ \operatorname{tr}\Big(\frac{1}{J_K}\,\frac{\partial \Psi(\Phi,\theta)}{\partial \Phi}\mathbf{K}^\top\frac{\partial \nabla\mathbf{u}}{\partial t}\Big) \ \text{ in } \ \Omega\times(0,T),\end{aligned} \tag{2.11b}$$

where

$$\Phi = \nabla\mathbf{u}\,\mathbf{K}, \quad J_K = \det\mathbf{K}, \tag{2.11c}$$

$\mathbf{f}$ is a given vector field of exterior bulk forces. Notice that the angular momentum conservation law is equivalent to the equality

$$\frac{\partial \Psi(\Phi,\theta)}{\partial \Phi}\Phi^\top = \Phi\Big(\frac{\partial \Psi(\Phi,\theta)}{\partial \Phi}\Big)^\top, \tag{2.11d}$$

which can be regarded as the structural restriction to the density of specific free energy. The balance equations should be supplemented with an evolution equation for the transplant $\mathbf{K}$. For a nonconservative model, this equation can be taken in the form

$$\frac{\partial \mathbf{K}}{\partial t} = \mathbf{K}\,\mathbf{g}(\nabla\mathbf{u},\theta,\mathbf{K}) \tag{2.11e}$$

The main problem is a specification of the function $\mathbf{g}$: some structural conditions can be obtained from the natural assumption that the material evolution is independent of the reference frame, see [3]. It was shown in [3] that the frame-independence principle leads to the representation

$$\mathbf{g} = \mathbf{K}\Big(\mathbf{g}_0(I_i,\mathbf{K})\mathbf{I} + \mathbf{g}_1(I_i,\mathbf{K})\mathbf{B} + \mathbf{g}_2(I_i,\mathbf{K})\mathbf{B}^2\Big), \tag{2.11f}$$

where $\mathbf{B}$ is the modified Eshelby tensor

$$\mathbf{B} = \mathbf{\Psi}(\Phi,\theta)I - \frac{\partial \Psi(\Phi,\theta)}{\partial \Phi}\Phi, \tag{2.11g}$$

I_i are the principal invariants of the tensor $\mathbf{B}$, and g_i is an arbitrary scalar function.

The previously defined Eshelby stress is highlighted as the driving force for growth; observe that it incorporates both a mechanical and a thermal contribution, as reflected by the form of the strain energy density. The set of equations (2.11) forms the complete thermodynamically consistent system of differential equations for modeling of the volumetric grows. In most applications, the most important is the quasistatic problem since the inertia forces are not essential for the growth of biological tissues (they can be discarded). Next, we can linearize the heat flux in the energy balance equation near some equilibrium value of the temperature θ_{eq}, that can be taken as $\theta_{eq} = 1$ in a dimensionless formulation. Thus, we come to the following more precisely

Model B

$$\operatorname{div}\Big(\frac{1}{J_K}\frac{\partial\Psi(\Phi,\theta)}{\partial\Phi}\mathbf{K}^\top\Big)+\mathbf{f}=0 \quad \text{in} \quad \Omega\times(0,T), \tag{2.12a}$$

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\frac{1}{J_K}\big(\Psi(\Phi,\theta)-\theta\frac{\partial\Psi(\Phi,\theta)}{\partial\theta}\big)\Big)+\operatorname{div}\Big(\frac{\kappa}{J_K}\,\mathbf{K}^\top\mathbf{K}\,\nabla\theta\Big)=\\ \operatorname{tr}\Big(\frac{1}{J_K}\frac{\partial\Psi(\Phi,\theta)}{\partial\Phi}\mathbf{K}^\top\Big)\frac{\partial\nabla\mathbf{u}}{\partial t}\Big) \quad \text{in} \quad \Omega\times(0,T),\end{aligned} \tag{2.12b}$$

$$\frac{\partial\mathbf{K}}{\partial t}=\mathbf{g}(\nabla\mathbf{u},\theta,\mathbf{K}). \tag{2.12c}$$

The governing equations should be supplemented by boundary and initial conditions. The first is the mixed boundary condition for strain and stresses, which can be formulated as follows. Assume that the boundary of $\partial\Omega$ is decomposed into two disjoint sets $\partial\Omega=\Sigma_0\cap\Sigma_1$. It is supposed that the displacement field is prescribed at Σ_0 and the normal stresses are prescribed on Σ_n, i.e.

$$\mathbf{u}=\mathbf{U} \quad \text{in} \quad \Sigma_0\times(0,T), \qquad \frac{1}{J_K}\Big(\frac{\partial\Psi(\Phi,\theta)}{\partial\Phi}\mathbf{K}^\top\Big)\mathbf{n}=\mathbf{h} \quad \text{on} \quad \Sigma_1\times(0,T), \tag{2.12d}$$

where $\mathbf{U}(X,t)$ is a given displacement field, $\mathbf{h}$ is a given normal stress, and $\mathbf{n}$ is the outward normal vector to $\partial\Omega$.

The most general condition for the temperature is the *radiation condition*

$$\frac{1}{J_k}\mathbf{K}^T\mathbf{K}\nabla\theta\cdot\mathbf{n}+\lambda\theta=q, \tag{2.12e}$$

where q is the given exterior heat flux. At initial time, the temperature and transplant distributions are prescribed, thus

$$\theta(X,0)=\theta_0(X), \quad \mathbf{K}(X,0)=\mathbf{K}_0(X) \quad \text{in} \quad \Omega. \tag{2.12f}$$

Here, θ_0 and $\mathbf{K}_0$ are given functions. The further simplification can be obtained if we neglect the heat effect and consider the isothermal traction problem, assuming that $\Psi=\Psi(\Phi)$. In this case, equations (2.12) become

Model C

$$\operatorname{div}\Big(\frac{1}{J_K}\frac{\partial\Psi(\Phi)}{\partial\Phi}\mathbf{K}^\top\Big)=\mathbf{f} \quad \text{in} \quad \Omega\times(0,T), \tag{2.13a}$$

$$\frac{\partial\mathbf{K}}{\partial t}=\mathbf{g}(\nabla\mathbf{u},\mathbf{K}). \tag{2.13b}$$

$$\frac{1}{J_K}\Big(\frac{\partial\Psi(\mathbf{\Phi})}{\partial\Phi}\mathbf{K}^\top\Big)\mathbf{n}=\mathbf{h} \quad \text{on} \quad \partial\Omega, \tag{2.13c}$$

$$\mathbf{K}(X,0)=\mathbf{K}_0(x) \quad \text{in} \quad \Omega. \tag{2.13d}$$

3 Local existence and uniqueness of solutions. Preliminaries

We refer to the basic question if Model C is well posed: recall that the problem is well posed if it has, at least locally, a unique solution for all smooth data. In this paper, we give a positive answer to this question. Namely, we establish the local existence and uniqueness theorem for problem (2.13) for sufficiently smooth data close to the stable equilibrium. In order to formulate the result, we recall the basic hypotheses of the finite elasticity; we refer to monographs [2] and [20] for precise definitions and proofs.

Finite elasticity. Constitutive assumptions. Linearized equations. We will assume that the reference configuration $\Omega \subset \mathbf{R}^3$ is a bounded domain with the boundary $\partial\Omega$ of the class C^∞. Without loss of generality, we may assume that the mass center is located at the origin

$$\int_\Omega X_i \, d\mathbf{X} = 0, \quad i = 1, 2, 3. \tag{3.1}$$

Recall that for any deformation $\mathbf{u} : \Omega \to \mathbb{R}^3$, $\nabla\mathbf{u}$ denotes the Jacobian matrix with the entries $(\nabla\mathbf{u})_{ij} = \partial_{x_j} u_i$, $i, j = 1, 2, 3$. Introduce the elastic stress tensor

$$\mathbf{T} = \frac{\partial\Psi(\Phi)}{\partial\Phi} \quad \text{with the entries} \;\; T_{ij} = \frac{\partial\Psi(\Phi)}{\partial\Phi_{ij}}, \quad \Phi = \nabla\mathbf{u}. \tag{3.2}$$

Next, we introduce the quadratic form of the second differential of the elastic stored energy

$$\mathcal{L}(\Phi)\xi \cdot \xi \equiv L_{ij,pq}(\Phi)\, \xi_{ij}\, \xi_{pq} \quad \text{for all matrices} \;\; \xi, \tag{3.3}$$

where

$$L_{ij,pq}(\Phi) \;=\; \frac{\partial^2\Psi(\Phi)}{\partial\Phi_{ij}\;\partial\Phi_{pq}} \tag{3.4}$$

Condition 1 *We will make the following assumptions.*

H.1 *The function $\Psi(\Phi)$ belongs to the class $C^\infty(\mathcal{A})$, where $\mathcal{A}$ is the domain*

$$\mathcal{A} = \{\Phi : 1/2 \le |\Phi| \le 1, \;\det \Phi > 0\}.$$

H.2 *For every $\Phi \in \mathcal{A}$, the tensor $\mathbf{T}$ satisfies the* symmetry condition

$$\mathbf{T}(\Phi)\Phi^\top = \Phi\mathbf{T}(\Phi)^\top. \tag{3.5}$$

H.3 *For any orthogonal matrix $\mathbf{R} \in \mathcal{A}$ and $\Phi \in \mathcal{A}$,*

$$\mathbf{T}(\mathbf{R}\Phi) = \mathbf{R}\mathbf{T}(\Phi). \tag{3.6}$$

H.4 *The reference configuration Ω is unstressed, i.e., $\mathbf{T}(\mathbf{I}) = 0$. It follows from this and (**H.3**) that $\mathbf{T}(\mathbf{R}) = 0$ for all orthogonal matrices $\mathbf{R} \in \mathcal{A}$.*

H.5 *There is $c > 0$ such that for all symmetric matrices $\mathbf{S}$,*

$$\mathcal{L}(\mathbf{I})S \cdot S \;\ge c\,|\mathbf{S}|^2. \tag{3.7}$$

Notice that Condition 1 (**H.2**) and (**H.3**) express the *momentum balance* and the *"observer independence" principle*, see [20]. It follows from Condition 1 (**H.2**)-(**H.4**) that the elastic stress tensor has the following symmetry properties

$$\frac{\partial\mathbf{T}(\mathbf{R})}{\partial R_{qp}}\mathbf{R} \;\; \text{is symmetric if} \quad \mathbf{R} \;\; \text{is an orthogonal matrix}, \tag{3.8}$$

$$\frac{\partial\mathbf{T}}{\partial\Phi_{qp}}(\mathbf{I}) = \frac{\partial\mathbf{T}^\top}{\partial\Phi_{qp}}(\mathbf{I}), \quad \frac{\partial\mathbf{T}}{\partial\Phi_{qp}}(\mathbf{I}) = \frac{\partial\mathbf{T}}{\partial\Phi_{pq}}(\mathbf{I}), \tag{3.9}$$

which directly follow from Condition 1 (**H.2**) and (**H.3**), see Chapter III in [20]. Notice also that relations (3.8), (3.9) and inequality (3.7) imply the nonnegativity of the form $\mathcal{L}(\mathbf{I})$ on the space of all 3×3 matrices:

$$\mathcal{L}(\mathbf{I})\,\xi \cdot \xi \ge c\,|\xi + \xi^\top|^2. \tag{3.10}$$

The symmetry conditions (3.5) and (3.8) imply the more general inequality

$$\mathcal{L}(\mathbf{R})\,\xi \cdot \xi \ge c\,|\mathbf{R}^\top\xi + (\mathbf{R}^\top\xi)^\top|^2. \tag{3.11}$$

3.1 The Korn inequality. Linearized problem.

The local analysis of nonlinear elasticity problems relies on the well-posedness of basic boundary problems of linear elastostatic. In turn, the proof of solvability and uniqueness of solutions to linear elasticity boundary value problems is based on the Korn-type inequalities. In this paper, we will use the following version of the second Korn inequality. Let Ω be a bounded Lipschitz domain Ω, there is a constant $c > 0$, depending only on Ω, such that the inequality

$$\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} \leq c \int_\Omega |\nabla \mathbf{v} + (\nabla \mathbf{v})^\top|^2 \, dX \tag{3.12}$$

holds for all vector fields $\mathbf{v} : \Omega \to \mathbb{R}^3$ satisfying

$$\int_\Omega \mathbf{v} \, dX = 0, \quad \int_\Omega (\partial_{X_i} v_j - \partial_{X_j} v_i) \, dX = 0. \tag{3.13}$$

The proof is given in Chapter III of [20].

The Korn inequality leads to the following result on solvability of the linear boundary problem

$$\begin{aligned} \partial_{X_j} \big(L_{ij,pq}(\mathbf{I}) \, \partial_{X_q} v_p \big) &= f_i, \quad \text{in } \Omega, \\ \big(L_{ij,pq}(\mathbf{I}) \, \partial_{X_q} v_p \big) \, n_j &= h_i, \quad \text{on } \partial\Omega. \end{aligned} \tag{3.14}$$

These equations can be equivalently rewritten in the following vector form

$$\begin{aligned} \operatorname{div} \big(\mathcal{L}(\mathbf{I}) \, \nabla \mathbf{v} \big) &= \mathbf{f}, \quad \text{in } \Omega, \\ \big(\mathcal{L}(\mathbf{I}) \, \nabla \mathbf{v} \big) \, \mathbf{n} &= \mathbf{h}, \quad \text{on } \partial\Omega. \end{aligned} \tag{3.15}$$

Assume that the given forces $\mathbf{f}$ and $\mathbf{h}$ satisfy the following solvability condition:

Condition 2

$$\begin{aligned} \int_\Omega \mathbf{f} \, dX + \int_{\partial\Omega} \mathbf{h} \, ds = 0, \\ \int_\Omega (f_i X_j - f_j X_i) \, dX + \int_{\partial\Omega} (h_i X_j - h_j X_i) \, ds = 0, \end{aligned} \tag{3.16}$$

for all $i, j = 1, 2, 3$. In this case, we say that the couple $(\mathbf{f}, \mathbf{h})$ is equilibrated.

As it was shown in Chapter III of [20], Condition 2 guarantees the solvability of problem (3.14); more precisely, the following existence result is obtained.

Proposition 1 *Let Ω be a bounded domain with C^∞ boundary $\partial\Omega$ and let an integer $m \geq 0$, and $p > 3$. Then for every $(\mathbf{f}, \mathbf{h}) \in W^{m,p}(\Omega) \times W^{m+1-1/p,p}(\partial\Omega)$ satisfying the orthogonality conditions* (3.16) *problem* (3.14) *has a unique solution $\mathbf{v} \in W^{m+2,p}(\Omega)$ satisfying condition* (3.13).

3.2 Growth stress tensor.

For growing material, the stress tensor depends on the transplant tensor $\mathbf{K}$, and it is defined by

$$\mathbf{T}_g(\nabla\mathbf{u},\mathbf{K}) = \frac{1}{J_K}\frac{\partial\Psi(\nabla\mathbf{u}\,\mathbf{K})}{\partial\nabla\mathbf{u}} = \frac{1}{J_K}\frac{\partial\Psi(\Phi)}{\partial\Phi}\,\mathbf{K}^\top \equiv \frac{1}{J_K}\mathbf{T}(\Phi)\,\mathbf{K}^\top, \tag{3.17}$$

where $\Phi = \nabla u\mathbf{K}$. Notice that in the general case the reference configuration is not unstressed due to residual stresses, since $\mathbf{T}_g(\mathbf{I},\mathbf{K}) \neq 0$ for general matrices $\mathbf{K}$. The only exception is the case of orthogonal transplant tensors $\mathbf{K}(\mathbf{X}) = \mathbf{R}(\mathbf{X})$ with $\mathbf{R}(\mathbf{X})^{-1} = \mathbf{R}(\mathbf{X})^\top$.

The quadratic form of the second differential for the mapping $\nabla\mathbf{u} \to \Psi(\nabla\mathbf{u}\mathbf{K})$ reads

$$\mathcal{L}_g(\nabla\mathbf{u},\mathbf{K})\xi\cdot\xi = L_{g,ij,pq}(\nabla\mathbf{u},\mathbf{K})\,\xi_{ij}\,\xi_{pq} \quad \text{for all matrices } \xi, \tag{3.18}$$

where

$$L_{g,ij,pq}(\Phi) = \frac{1}{J_K}\frac{\partial^2\Psi(\Phi)}{\partial\Phi_{il}\,\partial\Phi_{pm}}K_{qm}K_{jl}, \quad \Phi = \nabla\mathbf{u}\mathbf{K}. \tag{3.19}$$

The expression for $\mathcal{L}_g$ can be rewritten in the equivalent form

$$\mathcal{L}_g(\nabla\mathbf{u},\mathbf{K})\xi\cdot\xi = \frac{1}{J_K}\mathcal{L}(\Phi)\,(\xi\mathbf{K})\cdot(\xi\mathbf{K}), \tag{3.20}$$

It follows this and (3.11) that for an orthogonal transplant $\mathbf{K}(\mathbf{X}) = \mathbf{R}(\mathbf{X})$ with $\mathbf{R}(\mathbf{X})^{-1} = \mathbf{R}(\mathbf{X})^\top$, that corresponds to the unstressed reference configuration, we have

$$\mathcal{L}_g(\mathbf{I},\mathbf{K})\xi\cdot\xi \geq c|\mathbf{K}^\top\xi\mathbf{K} + (\mathbf{K}^\top\xi\mathbf{K})^\top|^2, \tag{3.21}$$

where $c > 0$ is independent of $\mathbf{K}$. In particular, we have for any $\mathbf{v} \in W^{1,2}(\Omega)$ and orthogonal $\mathbf{K}$,

$$\int_\Omega \mathcal{L}_g(\mathbf{I},\mathbf{K})\nabla\mathbf{v}\cdot\nabla\mathbf{v}\,dx \geq c\int_\Omega |\mathbf{K}^\top\nabla\mathbf{v}\mathbf{K} + (\mathbf{K}^\top\nabla\mathbf{v}\mathbf{K})^\top|^2\,dX. \tag{3.22}$$

Unfortunately, the Korn inequality (3.12)-(3.13) can not be directly extended to the case of variable coefficients and the question on well-posedness of linearized traction problem for growing materials remains open. However, the Korn inequality with variable coefficients holds true for vector fields v satisfying the Dirichlet boundary condition, see [13]. In particular, for any continuous orthogonal $\mathbf{K}(x)$, there is a positive constant c, depending only on Ω and $\mathbf{K}$, such that the inequality

$$\|\mathbf{v}\|^2_{H^{1,2}(\Omega)} \leq c\int_\Omega |\mathbf{K}^\top\nabla\mathbf{v}\mathbf{K} + (\mathbf{K}^\top\nabla\mathbf{v}\mathbf{K})^\top|^2\,dX \tag{3.23}$$

holds for all functions $v \in H^{1,2}_0(\Omega)$. This leads to the following version of Proposition 1 for the Dirichlet boundary value problem

Proposition 2 *Let Ω be a bounded domain with C^∞ boundary $\partial\Omega$ and let an integer $m \geq 0$, and $p > 3$. Furthermore, assume that $\mathbf{R} \in W^{m+1,p}(\Omega)$, and $\mathbf{R}(\mathbf{X})$ is orthogonal for every $\mathbf{X} \in \Omega$. Then, for every $(\mathbf{f},\mathbf{h}) \in W^{m,p}(\Omega) \times W^{m+2-1/p,p}(\partial\Omega)$ the problem*

$$\begin{aligned} div\,\big(\mathcal{L}_g(\mathbf{I},\mathbf{R})\,\nabla\mathbf{v}\big) &= \mathbf{f}, \quad in\ \ \Omega,\\ \mathbf{v} &= \mathbf{h}, \quad on\ \ \partial\Omega \end{aligned} \tag{3.24}$$

has a unique solution $\mathbf{v} \in W^{m+2,p}(\Omega)$.

3.3 A transplant rate

The evolution of transplant is described by equation (2.13b). Furthermore, we assume that the function $\mathbf{g}$ in this equation is smooth.

Condition 3 *The function* $\mathbf{g}$ *belongs to the class* $C^{\infty}(\mathcal{A}\times\mathcal{A})$

For the traction problem, we assume in addition that the growth is isotropic.

Condition 4 *The function* $\mathbf{g}$ *admits the representation*

$$\mathbf{g} = g_I(\nabla\mathbf{u},\mathbf{K})\,\mathbf{I}, \tag{3.25}$$

where the scalar function $g_I \in C^{\infty}(\mathcal{A}\times\mathcal{A})$.

4 Traction and Dirichlet problems. Local existence and uniqueness of solutions.

We are now in a position to formulate and prove the local solvability of the traction problem (2.13) with isotropic growth. Let us consider the boundary value problem for a deformation $\mathbf{u}$ and a transplant $\mathbf{K}$

$$\operatorname{div}\mathbf{T}_g(\nabla\mathbf{u},\mathbf{K}) = \lambda\mathbf{f} \quad \text{in} \quad \Omega\times(0,T), \tag{4.1a}$$

$$\mathbf{T}_g(\nabla\mathbf{u},\mathbf{K})\mathbf{n} = \lambda\mathbf{h} \quad \text{on} \quad \partial\Omega\times(0,T), \tag{4.1b}$$

$$\mathbf{K} = K\mathbf{I}, \quad \frac{\partial K}{\partial t} = g_I(\nabla\mathbf{u},K) \quad \text{in} \quad \Omega\times(0,T), \tag{4.1c}$$

$$K(\mathbf{X},0) = K_0(\mathbf{X}) \quad \text{in} \quad \Omega. \tag{4.1d}$$

Here, the stress tensor $\mathbf{T}_g$ is defined by the relations (3.17), with λ is a small parameter. For every integer $m \geq 0$ and $p > 1$, let denote by $U^{m,p}$ the closed subspace of $W^{m,p}(\Omega)$ which consists of all vector fields $\mathbf{v} \in W^{m,p}(\Omega)$ satisfying the orthogonality condition (3.13). We also denote by $V^{m,p}$ the closed subspace of $W^{m,p}(\Omega)\times W^{m+1-1/p}(\partial\Omega)$, which consists of all equilibrated couples $(\mathbf{f},\mathbf{h}) \in W^{m,p}(\Omega)$ satisfying the orthogonality conditions (3.16). The following theorem is the main result of this section.

Theorem 1 *Let conditions 1-4 be satisfied. Furthermore, we assume that the vector functions* $(\mathbf{f},\mathbf{h})$ *belong to the class* $C(0,T;W^{m,p}(\Omega))\times C(0,T;W^{m+1-1/p,p}(\Omega))$, $m \geq 0$, $p > 3$, *and satisfy the nondegeneracy condition*

$$\inf_{[0,T]} \left|C_{ii}(t) + C_{jj}(t)\right| \geq \nu > 0 \quad \textit{for} \quad i \neq j, \tag{4.2}$$

where $\mathbf{C}(t)$ *is the matrix with the entries*

$$C_{ij}(t) = \int_{\Omega} (X_i f_j(\mathbf{X},t) + X_j f_i(\mathbf{X},t))\,dX + \int_{\partial\Omega} X_i h_j(\mathbf{X},t) + X_j h_i(\mathbf{X},t)\,dX.$$

Then, there is $\varepsilon > 0$ *and* $T^* \in (0,T)$ *such that for every* K_0 *and* λ *satisfying the conditions*

$$\|K_0 - 1\|_{W^{m+1,p}(\Omega)} \leq \epsilon, \quad \lambda \in [-\varepsilon,\varepsilon],$$

problem (4.1) *has a solution* $\mathbf{u} \in C(0,T^*;W^{m+2,p}(\Omega))$, $K \in C^1(0,T^*;W^{m+1,p}(\Omega))$ *such that*

$$\int_\Omega \mathbf{u}\,dx = 0.$$

Moreover, this solution is locally unique.

Remark 1 The nondegeneracy condition for the stationary traction problem was proposed in [19] in the form

$$|\mu_i + \mu_j| > 0 \quad \text{for} \quad i \neq j, \tag{4.3}$$

where μ_j are the eigenvalues of the symmetric matrix C. This means that the load $(\mathbf{f},\mathbf{h})$ admits no axis of equilibrium. Notice that the matrix $\mathbf{C}$ can be transformed into the diagonal form, $\mathbf{C} = \mathbf{O}\text{diag}\,(\mu_1,\mu_2,\mu_3)\mathbf{O}^\top$. For diagonal matrices, conditions (4.2) and (4.3) coincide; hence, conditions (4.2) can be replaced by the condition (4.3) in the stationary case.

In the nonstationary case, the orthogonal matrix O depends on t and these conditions are not equivalent.

The proof of Theorem 2 is based on the following proposition on solvability of the stationary traction problem for the deformation $\nabla\mathbf{u}$, while the transplant K is fixed.

$$\text{div}\ \mathbf{T}_g(\nabla\mathbf{u},\mathbf{K}) = \lambda\mathbf{f} \quad \text{in} \quad \Omega, \tag{4.4a}$$

$$\mathbf{T}_g(\nabla\mathbf{u},\mathbf{K})\mathbf{n} = \lambda\mathbf{h} \quad \text{on} \quad \partial\Omega. \tag{4.4b}$$

Proposition 3 *Let the couple* $(\mathbf{f},\mathbf{h}) \in W^{m,p}(\Omega) \times W^{m+1-1/p}(\partial\Omega)$, $m \geq 0$, $p > 3$, *be equilibrated and satisfy nondegeneracy condition*

$$\left|C_{ii} + C_{jj}\right| \geq \nu > 0 \quad \textit{for} \quad i \neq j, \tag{4.5}$$

where $\mathbf{C}$ *is the matrix with the entries*

$$C_{ij} = \int_\Omega (X_i f_j(\mathbf{X}) + X_j f_i(\mathbf{X}))\,dX + \int_{\partial\Omega} X_i h_j(\mathbf{X},t) + X_j h_i(\mathbf{X})\,dX.$$

Then there is $\varepsilon > 0$ *such that for all* $\lambda \in [-\varepsilon,\varepsilon]$ *and*

$$K \in B_K(\varepsilon) = \{K : \|1 - K\|_{W^{m+1,p}(\Omega)} \leq \varepsilon\}$$

problem (4.4) *has a solution* $\mathbf{u} = Id + \mathbf{w}$ *such that*

$$\mathbf{w} \in B_w(\delta) = \{\mathbf{w} \in W^{m+2,p}(\Omega) : \|\mathbf{w}\|_{W^{m+2,p}(\Omega)} \leq \delta, \quad \int_\Omega \mathbf{w}\,dX = 0\},$$

where $\delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. *The solution is unique in the ball* $B_w(\delta)$. *Moreover, there is a* C^1 *mapping* $\mathfrak{U} : B_K(\varepsilon) \times [-\varepsilon,\varepsilon] \to B_u(\delta)$ *such that* $\mathbf{w} = \mathfrak{U}(K,\lambda)$.

This proposition is a modification of the existence theorem for the traction problem in nonlinear elasticity given in [8]. The proof of the proposition imitates the proof of Theorem 3.1 in [8]; hence, we give only a sketch of the proof.

First, we establish the notation and present some auxiliary results.

For every $m \geq 0$ and $p > 3$ denote by $U^{m,p}$ the closed subspace of $W^{m,p}(\Omega)$ which consists of all vector fields $v \in W^{m,p}(\Omega)$ satisfying the orthogonality condition (3.13).

We also denote by $V^{m,p}$ the closed subspace of $W^{m,p}(\Omega) \times W^{m+1-1/p}(\partial\Omega)$ which consists of all equilibrated couples $(\mathbf{f}, \mathbf{h}) \in W^{m,p}(\Omega) \times W^{m+1-1/p}(\partial\Omega)$ satisfying the orthogonality conditions (3.16).

Denote by $\mathcal{S}$ the Banach space of all skew-symmetric matrices $S = -S^\top$ supplemented with the norm

$$\|S\|_{\mathcal{S}} = \Big(\sum_{i,j} S_{ij}^2\Big)^{1/2}.$$

We look for a solution to problem (4.1a)-(4.1b) in the form

$$\mathbf{u}(\mathbf{X}) = \mathbf{X} + \mathbf{v}(\mathbf{X}) + S\mathbf{X}, \quad \mathbf{v} \in U^{m+2,p}, \quad S \in \mathcal{S}. \tag{4.6}$$

Following [8], let choose an auxiliary vector field $\varphi \in C^\infty(\Omega)$ satisfying the conditions

$$\int_\Omega \varphi \, dX = 0, \quad \int_\Omega X_i \varphi_j \, dx = 0, \quad (i \neq j),$$
$$\int_\Omega (X_i\varphi_i + X_j\varphi_j) \, dX \neq 0, (i \neq j).$$

Next, introduce the matrix $\mathbf{E}(\nabla\mathbf{u}, K)$ with the entries

$$E_{ij} = \Big(\int_\Omega (X_i\varphi_i + X_j\varphi_j) \, dX\Big)^{-1} \int_\Omega (T_{g,ij}\big(\nabla\mathbf{u}, K) - T_{g,ji}(\nabla\mathbf{u}, K)\big) \, dX. \tag{4.7}$$

for $i \neq j$, and $E_{ii} = 0$. In order to apply the Lyapunov-Schmidt reduction method, we consider the modified problem

$$\text{div } \mathbf{T}_g(\nabla\mathbf{u}, \mathbf{K}) - \mathbf{E}(\nabla\mathbf{u}, K) = \lambda\mathbf{f} \quad \text{in} \quad \Omega, \tag{4.8}$$

$$\mathbf{T}_g(\nabla\mathbf{u}, \mathbf{K})\mathbf{n} = \lambda\mathbf{h} \quad \text{on} \quad \partial\Omega.$$

Recalling decomposition (4.6), we can rewrite this boundary value problem in the form of the operator equation

$$\mathfrak{F}(\mathbf{v}, S, K, \lambda) \equiv \Big(\text{div } \mathbf{T}_g(\nabla\mathbf{u}, K) - \mathbf{E}(\nabla\mathbf{u}, K) - \lambda\mathbf{f}, \ \mathbf{T}_g(\nabla\mathbf{u}, \mathbf{K})\mathbf{n} - \lambda\mathbf{h}\Big) = 0. \tag{4.9}$$

Here $\mathbf{u}$ is defined by the decomposition (4.6). For every $r > 0$, let define the balls

$$B_v(r) = \big\{ \mathbf{v} \in U^{m+2,p} : \|\mathbf{v}\|_{W^{1,2}(\Omega)} \leq r \big\},$$

$$B_K(r) = \big\{ K \in W^{m+1,p}(\Omega) : \|1 - K\|_{W^{1,2}(\Omega)} \leq r \big\},$$

$$B_S(r) = \big\{ S \in \mathcal{S} : \|S\|_{\mathcal{S}} \leq r \big\}, \quad B_\lambda(r) = [-r, r].$$

Lemma 1 *There is $r > 0$ such that the operator*

$$\mathcal{F} : B_v(r) \times B_S(r) \times B_K(r) \times B_\lambda(r) \to V^{m,p}$$

is continuously differentiable. Moreover, its derivatives with respect to $\mathbf{v}$ *and* S *admit the representation*

$$D_{\mathbf{v}}\mathfrak{F}(0,0,1,\lambda)\delta\mathbf{v} = \Big(\operatorname{div}\ \big(\mathcal{L}(\mathbf{I})\,\nabla\delta\mathbf{v}\,\big),\ \big(\mathcal{L}(\mathbf{I})\,\nabla\delta\mathbf{v}\,\big)\,\mathbf{n}\Big), \tag{4.10}$$

$$D_S\mathfrak{F}(0,0,1,\lambda) = 0. \tag{4.11}$$

Proof The differentiability of the operator $\mathfrak{F}$ directly follows from the condition 1 (**H.1**) combined with the fact that $W^{m+1,p}(\Omega)$ is compactly embedded into $C^1(\Omega)$, hence $W^{m+1,p}(\Omega)$ is a Banach algebra, see [2] and [20] for the details and complete proofs.

In view of (4.9), for every $\mathbf{v}$, S, K and λ, the function $\mathfrak{F}(\mathbf{v}, S, K, \lambda)$ is simply a couple of vector fields. It follows from the definition of the matrix E that this couple is equilibrated. Hence, $\mathfrak{F}$ takes it values in $V^{m,p}$. The formula for $D_{\mathbf{v}}\mathfrak{F}$ is a straightforward consequence of the expression for the quadratic forms $\mathcal{L}$ and $\mathcal{L}_g$. Next, it follows from the representation (4.6) and (4.9) that the derivative $D_{\mathbf{v}}\mathfrak{F}$ and $D_S\mathfrak{F}$ coincide. From this and the identity $\mathcal{L}(\mathbf{I})\delta S \equiv 0$, we obtain (4.11). □

The next Lemma reduces the boundary value problem (4.4) to a system of nonlinear equations in three dimensional space $\mathcal{S}$.

Lemma 2 *Under the assumptions of Proposition 3, there exists $\rho > 0$ and a continuously differentiable mapping $\mathfrak{V} : B_S(\rho) \times B_K(\rho) \times B_\lambda(\rho) \to B_v(r)$ such that*

$$\mathfrak{F}\left(\mathfrak{V}(S,K,\lambda),\ S,\ K,\ \lambda\right) = 0.$$

Moreover, $\mathfrak{V}(0,1,0) = 0$ *and* $D_S\mathfrak{V}(0,1,0) = 0$.

Proof It follows from (4.10) and Proposition 1 that there exists the bounded operator $(D_{\mathbf{v}}\mathfrak{F}(0,0,1,\lambda))^{-1} : V^{m,p} \to U^{m+2,p}$. Hence the existence of the C^1 mapping $\mathfrak{V}$ is a straightforward consequence of the Implicit function theorem. It remains to note that in view of Lemma 1 and (4.11),

$$D_S\mathfrak{V}(0,1,0) = -(D_{\mathbf{v}}\mathfrak{F}(0,0,1,\lambda))^{-1} D_S\mathfrak{F}(0,0,1,0) = 0.$$

□

We are now in a position to complete the proof of Proposition 3.

Proof Notice that the function

$$\mathbf{u} = Id + \mathfrak{V}(S,K,\lambda) + S\mathbf{X}$$

satisfies the equations and boundary conditions (4.4) if and only if

$$\mathbf{E}\left(\mathfrak{V}(S,K,\lambda) + S\mathbf{X},\ K\right)\varphi = 0. \tag{4.12}$$

We can consider this relation as a finite system of scalar equations for the skew symmetric matrix S. It is a remarkable fact of the theory, see [8], Th. 6.1, that this equation can be written in the form

$$\mathfrak{T}(S,K,\lambda) \equiv \mathfrak{B}(S) + \mathfrak{P}(S,K,\lambda) = 0. \tag{4.13}$$

Here the operators $\mathfrak{B}$ and $\mathfrak{P}$ are defined by the equalities

$$\mathfrak{B}(S)_{ij} = (C_{ii} + C_{jj})S_{ij}$$

$$\mathfrak{P} = \int_\Omega ((\mathbf{v}_j^* f_i - \mathbf{v}_i^* f_j)\, dX + \int_{\partial\Omega} ((\mathbf{v}_j^* h_i - \mathbf{v}_i^* h_j)\, ds,$$

$$\mathbf{v}^* = \mathfrak{V}(S, K, \lambda).$$

In view of Lemma 2, the operator $\mathfrak{T} : B_S(\rho) \times B_K(\rho) \times B_\lambda(\rho) \to \mathcal{S}$ is continuously differentiable and $D_S\mathfrak{T}(0,1,0)\delta S = \mathfrak{B}(\delta S)$. On the other hand, the nondegenaracy condition (4.5) implies the existence of the bounded operator $\mathfrak{B}^{-1} : \mathcal{S} \to \mathcal{S}$. Applying the Implicit function theorem, we conclude that there exist $\varepsilon > 0$ and a continuously differentiable operator $\mathfrak{S} : B_K(\varepsilon) \times B_\lambda(\varepsilon)$ such that

$$\mathfrak{T}\big(\mathfrak{S}(K,\lambda),\, K,\, \lambda\big) = 0.$$

It remains to note that

$$\mathbf{u} = Id + \mathfrak{U}(K,\lambda) \quad \text{with} \quad \mathfrak{U}(K,\lambda) = \mathfrak{V}\big(\mathfrak{S}(K,\lambda), K, \lambda\big) + \mathfrak{S}(K,\lambda)\,\mathbf{X}$$

is a desired solution to problem (4.4). □

The following propositions constitutes the existence and local uniqueness of the solution to the traction problem (4.4) for the load $(\mathbf{f}, \mathbf{h})$ depending on t.

Proposition 4 *Let the couple* $(\mathbf{f}, \mathbf{h}) \in C(0,T; W^{m,p}(\Omega)) \times C(0,T; W^{m+1-1/p}(\partial\Omega))$, $m \geq 0$, $p > 3$, *be equilibrated and satisfies the nondegeneracy condition* (4.5) *for every* $t \in [0,T]$. *Then there is* $\varepsilon > 0$ *such that for all* $\lambda \in (-\varepsilon, \varepsilon)$ *and*

$$K \in C(0,T; W^{m+1,p}(\Omega)), \quad K(t) \in B_K(\varepsilon) = \{K : \|1 - K\|_{W^{m+1,p}(\Omega)} \leq \varepsilon\},$$

problem (4.4) *has a solution* $\mathbf{u} = Id + \mathbf{w}$ *such that for every* $t \in [0,T]$,

$$\mathbf{w}(t) \in B_w(\delta) = \{\mathbf{w} \in W^{m+2,p}(\Omega) : \|\mathbf{w}\|_{W^{m+2,p}(\Omega)} \leq \delta, \quad \int_\Omega \mathbf{w}\, dX = 0\},$$

where $\delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. *The solution is unique in the ball* $B_w(\delta)$. *Moreover, there is a mapping* $\mathfrak{U} : B_K(\varepsilon) \times [-\varepsilon, \varepsilon] \times [0,T] \to B_u(\delta)$ *such that* $\mathbf{w}(t) = \mathfrak{U}(K(t), \lambda, t)$. *The mapping* $\mathfrak{U}$ *is continuous in* t *and continuously differentiable with respect to* K *and* λ.

Proof It suffices to note that for every $t \in [0,T]$, the load $(\mathbf{f}(\mathbf{X},t), \mathbf{h}(\mathbf{X},t))$ satisfies all conditions of Proposition 3. □

It follows from Proposition 4 that equations (4.1) can be written in the form of the Cauchy problem for differential operator equation in the Banach space $W^{m+1,p}(\Omega)$

$$\frac{\partial K}{\partial t} = g_I(\mathbf{I} + \nabla\mathfrak{U}(K,\lambda,t), K) \quad \text{on} \quad [0,T], \qquad K(0) = K_0 \tag{4.14}$$

It follows from Proposition 4 and Condition 4 that the operator $K \to g_I(\mathbf{I} + \nabla\mathfrak{U}(K,\lambda,t), K)$ satisfies the Lipschitz condition in the ball $B_K(\varepsilon)$. Moreover, the Lipschitz constant is independent of t.

Hence, the existence of solutions to problem (4.1) on the small interval $[0, T^*]$ is a consequence of the Picard Theorem. This completes the proof of Theorem 2. □

4.1 Dirichlet problem

In the case of the Dirichlet problem

$$\operatorname{div} \mathbf{T}_g(\nabla \mathbf{u}, \mathbf{K}) = \lambda \mathbf{f} \quad \text{in} \quad \Omega \times (0,T), \tag{4.15a}$$

$$\mathbf{u} = \lambda \mathbf{h} \quad \text{on} \quad \partial\Omega \times (0,T), \tag{4.15b}$$

$$\frac{\partial \mathbf{K}}{\partial t} = \mathbf{g}(\nabla \mathbf{u}, \mathbf{K}) \quad \text{in} \quad \Omega \times (0,T), \tag{4.15c}$$

$$\mathbf{K}(\mathbf{X}, 0) = \mathbf{K}_0(\mathbf{X}) \quad \text{in} \quad \Omega, \tag{4.15d}$$

We can get more general result than that for the traction problem. In particular, for Problem (4.15), we do not need the isotropy assumptions for the transplant and the solvability condition for the load $\mathbf{f}$. The following theorem constitutes the existence and local uniqueness of solutions to problem (4.15).

Theorem 2 *Let Conditions 1, 3 be satisfied and vector functions* $(\mathbf{f}, \mathbf{h})$ *belong to the class* $C(0,T; W^{m,p}(\Omega)) \times C(0,T; W^{m+2-1/p,p}(\Omega))$, $m \geq 0$, $p > 3$. *Let* $\mathbf{R} \in W^{m+1,p}(\Omega)$ *be a matrix-valued function such that* $\mathbf{R}(\mathbf{X})$ *is orthogonal for all* $\mathbf{X} \in \Omega$.

Then there is $\varepsilon > 0$ *and* $T^* \in (0,T)$ *such that for every* $\mathbf{K}_0$ *and* λ *satisfying the conditions*

$$\|\mathbf{K}_0 - \mathbf{R}\|_{W^{m+1,p}(\Omega)} \leq \epsilon, \quad \lambda \in [-\varepsilon, \varepsilon],$$

problem (4.15) *has a solution* $\mathbf{u} \in C(0,T^*; W^{m+2,p}(\Omega))$. *Moreover this solution is locally unique.*

Proof The proof of this Theorem is the simplification of the proof of Theorem 2. The only remark is that we have to replace the operator equation (4.9) by the operator equation

$$\mathfrak{D}(\mathbf{u}, \mathbf{K}, \lambda) \equiv \Big(\operatorname{div} \mathbf{T}_g(\nabla \mathbf{u}, \mathbf{K}) - \lambda \mathbf{f},\ \mathbf{u} - \lambda \mathbf{h} \Big) = 0.$$

It is easy to see that

$$D_{\mathbf{u}} \mathfrak{D}(Id, \mathbf{R}, \lambda) \delta \mathbf{u} = \Big(\operatorname{div} \ \big(\mathcal{L}_g(\mathbf{I}, \mathbf{R}) \nabla \delta \mathbf{u} \big),\ \delta \mathbf{u} \Big) \tag{4.16}$$

The existence of the bounded inverse operator

$$D_{\mathbf{u}} \mathfrak{D}(Id, \mathbf{R}, \lambda)^{-1} : W^{m,p}(\Omega) \times W^{m+2-1/p,p}(\partial\Omega) \to W^{m+2,p}(\Omega)$$

follows from Proposition 2. Applying the Implicit function theorem, we conclude that operator equation (4.16) has a solution in the form $\mathbf{u} = \mathfrak{C}(\mathbf{K}, \lambda, t)$. Here the operator $\mathfrak{C}$ is continuously differentiable in the neighborhood of $(\mathbf{R}, 0)$. Thus we come to the operator equation for $\mathbf{K}$,

$$\frac{\partial \mathbf{K}}{\partial t} = \mathbf{g}(\mathfrak{C}(\mathbf{K}, \lambda, t), \mathbf{K}).$$

The application of the Picard Theorem completes the proof. □

5 Conclusion

In this contribution, we have set up models for the growth of continuum solid bodies, viewed as a reorganization of the reference configuration leading to an overall change of mass. The constitutive model for the growing body is expressed in a reference crystal which is mapped to the reference configuration by a transplant operation, in line with ideas advanced in [3]. The growth model is expressed in terms of an evolution law for the transplant, which involves a modified Eshelby stress as a driving force for growth. The Blatz-Co hyperelastic constitutive law is adopted for the growing body. The model is formulated as a boundary value problem consisting of the balance of momentum and energy, the constitutive equation, and the growth law. We have assumed that the heat flux depends on the transplant. As a main thrust of this contribution, the mathematical aspects of a growth model has been investigated. The existence of local solutions has been obtained, for sufficiently smooth data close to the stable equilibrium.

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