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To cite this version:

Michel Bellieud, Christian Licht, Somsak Orankitjaroen. Nonlinear Capacitary Problems for a General Distribution of Fibers. Applied Mathematics Research eXpress, 2013, pp.voir DOI. 10.1093/amrx/abt002 . hal-00801985

HAL Id: hal-00801985

https://hal.science/hal-00801985v1

Submitted on 18 Mar 2013

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Nonlinear capacitary problems for a general distribution of fibers

Michel Bellieud *, Christian Licht*, and Somsak Orankitjaroen †

December 5, 2012

Abstract

We determine the effective electric properties of a composite with high contrast. The energy density is given locally in terms of a convex function of the gradient of the potential. The permittivity may take very large values in a fairly general distribution of parallel fibers of tiny cross sections. For a critical size of the cross sections, we show that a concentration of electric energy may arise in a small region of space surrounding the fibers. This extra contribution is caused by the discrepancy between the behaviors of the potential in the matrix and in the fibers and is characterized by the density of the cross sections of the fibers with respect to the cross section of the body in terms of some suitable notion of capacity. Our results extend those established in [7] in the periodic case for the p-Laplacian to a general nonlinear framework and a non-periodic distribution of fibers.

AMS *subject classifications*: 32U20, 49J45, 74Q05.

Keywords: asymptotic analysis, Γ-convergence, capacity theory, homogenization, fibered media

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1 Introduction and setting out of the problem

Composites comprising traces of materials with extreme physical properties have been investigated by several authors over the past decades in various contexts, such as diffusion equations [7, 11, 16, 26, 28], fluid mechanics [12], electromagnetic theory [9], linearized elasticity [6, 8]. The common feature of this body of work is the emergence of a concentration of energy in a small region of space surrounding the strong components. This extra contribution is characterized by a local density of the geometric perturbations in terms of an appropriate capacity depending on the type of equations.

In this paper, we determine the effective electric properties of an electrified composite whereby a set of extremely thin fibers with very large permittivities is embedded in a matrix with permittivity of order 1. This study may as well concern various steady-state situations in Physics like heat diffusion for instance. It is interesting to refer to Electricity where capacity has a specific meaning. A similar problem has been studied by one of the authors with G. Bouchitté [7] in the periodic quasilinear case for fibers of circular cross section. In what follows, we investigate the non periodic case and consider a more general non linear framework and also fibers with arbitrarily shaped cross sections. This is worthwhile because fibers stem from draw plates and therefore are likely to display anisotropic behaviors governed by general convex functions. Let us notice that in the linear case, M. Briane and J. Casado-Díaz [15] obtained nonlocal effects with fibers the cross section of which is merely a bounded connected open subset of $\mathbb{R}^2$. Dropping the assumption of periodicity is a challenging task which may lead to quite different effective problems when composites with high contrast are considered. In our specific study, the effective problem turns out to show the same general features as in the periodic case, provided the fibers are not too closely spaced (see (1.6)).

We turn now to a more detailed introduction of the paper. Let $\mathcal{O} = \widehat{\mathcal{O}} \times (0, L)$, where $\widehat{\mathcal{O}}$ is a bounded smooth open subset of $\mathbb{R}^2$. We consider the boundary value problem in Electrostatics

$$(\mathcal{P}_\varepsilon) \begin{cases} \min\limits_{u \in u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})} F_\varepsilon(u) - \displaystyle\int_{\mathcal{O}} q_b u \, dx - \int_{\Gamma_1} q_s u \, d\mathcal{H}^2, \\ W^{1,p}_{\Gamma_0}(\mathcal{O}) = \left\{ \varphi \in W^{1,p}(\mathcal{O}) : \varphi = 0 \text{ on } \Gamma_0 \right\}, \quad \Gamma_0 \subset \partial\mathcal{O}, \quad \mathcal{H}^2(\Gamma_0) > 0, \quad \Gamma_1 = \partial\mathcal{O} \setminus \Gamma_0, \\ (q_b, q_s) \in L^{p'}(\mathcal{O}) \times L^{p'}(\Gamma_1), \quad u_0 \in C^1(\overline{\mathcal{O}}), \quad \left(\dfrac{1}{p} + \dfrac{1}{p'} = 1 \right), \\ F_\varepsilon(u) = \displaystyle\int_{\mathcal{O} \setminus T_{r_\varepsilon}} f(\nabla u) \, dx + \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u) \, dx. \end{cases} \tag{1.1}$$

The solution u_ε to $(\mathcal{P}_\varepsilon)$ describes the electric potential of an electrified fibered composite insulator, where the distributions of body and surface charges are denoted by q_b and q_s. The non periodic set T_{r_ε} occupied by the fibers is defined in terms of a bounded domain $S \subset \mathbb{R}^2$ with a Lipschitz boundary, of two small positive parameter $\varepsilon, r_\varepsilon$ such that $0 < r_\varepsilon \ll \varepsilon \ll 1$, and of a finite family $(\omega^j_\varepsilon)_{j \in J_\varepsilon}$ ($J_\varepsilon \subset \mathbb{N}$) of points in $\widehat{\mathcal{O}}$. We set

$$\Omega_\varepsilon = \left\{ \omega^j_\varepsilon \in \mathbb{R}^2, \ j \in J_\varepsilon \right\} \subset \widehat{\mathcal{O}}, \tag{1.2}$$

$$T_{r_\varepsilon} := \bigcup_{j \in J_\varepsilon} T^j_{r_\varepsilon}, \quad T^j_{r_\varepsilon} := S^j_{r_\varepsilon} \times (0, L), \quad S^j_{r_\varepsilon} := (\omega^j_\varepsilon + r_\varepsilon S). \tag{1.3}$$

The parameter r_ε describes the size of the sections of the fibers, which are homothetical to S, whereas the parameter ε accounts for the local density of the distribution of the fibers in $\mathcal{O}$ through the function n_ε defined by

$$\begin{aligned} & n_\varepsilon(x) := \sum_{z \in I_\varepsilon} (\sharp J^z_\varepsilon) \ 1_{Y^z_\varepsilon}(x_1, x_2), \quad J^z_\varepsilon := \left\{ j \in J_\varepsilon, \ \omega^j_\varepsilon \in Y^z_\varepsilon \right\}, \\ & Y^z_\varepsilon := \varepsilon z + \varepsilon Y, \quad Y := [-1/2, 1/2)^2, \quad I_\varepsilon := \left\{ z \in \mathbb{Z}^2, \quad Y^z_\varepsilon \subset \widehat{\mathcal{O}} \right\}, \end{aligned} \tag{1.4}$$

where $\sharp A$ denotes the cardinal of a set A. Given $x \in \mathcal{O}$, the scalar $n_\varepsilon(x)$ is the number of points of Ω_ε included in the cell Y^z_ε such that $(x_1, x_2) \in Y^z_\varepsilon$ and $z \in I_\varepsilon$, if this cell exists at all. Therefore, $n_\varepsilon(x)$ is

an approximation of the number of fibers included in the parallelepiped $Y_\varepsilon^z \times (0,L)$ containing x. The assumption

$$0 \le n_\varepsilon(x) \le N \quad \text{in } \mathcal{O}, \quad N \in \mathbb{N}, \quad n_\varepsilon \overset{\star}{\rightharpoonup} n \ \text{ weak star in } \ L^\infty(\mathcal{O}), \tag{1.5}$$

ensures that the fibers do not concentrate in some lower dimensional subset of $\mathcal{O}$ (see Remark 3.1 (ii)). We also suppose that

$$\min_{j,j' \in J_\varepsilon, j \neq j'} |\omega_\varepsilon^i - \omega_\varepsilon^{j'}| > R_\varepsilon, \qquad \mathrm{dist}(\Omega_\varepsilon, \partial\widehat{\mathcal{O}}) > 5\sqrt{2}\varepsilon, \tag{1.6}$$

for some sequence of positive reals (R_ε) satisfying

$$r_\varepsilon \ll R_\varepsilon \ll \varepsilon, \quad 1 \ll \gamma_\varepsilon^{(p)}(R_\varepsilon), \quad \gamma_\varepsilon^{(p)}(t) := \frac{t^{2-p}}{\varepsilon^2} \quad \text{if } p \neq 2, \quad \gamma_\varepsilon^{(2)}(t) := \frac{1}{\varepsilon^2 |\log t|}. \tag{1.7}$$

The hypothesis (1.6) guarantees that each fiber is separated by a sufficient distance from the other fibers and from the lateral boundary of $\mathcal{O}$ (the constant "$5\sqrt{2}$" in (1.6) is chosen in order to get (6.37)). The periodic case corresponds to $\Omega_\varepsilon = \{\varepsilon z,\ z \in I_\varepsilon\}$ and n_ε given by $n_\varepsilon(x) = 1$ if $x \in \bigcup_{z \in I_\varepsilon} Y_\varepsilon^z \times (0,L)$, $n_\varepsilon(x) = 0$ otherwise. With no loss of generality, we assume that

$$0 \in \widehat{\mathcal{O}}, \qquad \overline{D} \subset S, \tag{1.8}$$

where D denotes the open unit ball of $\mathbb{R}^2$. The density of electric energy is given in terms of a sequence of positive reals (λ_ε) and of two strictly convex functions f, g satisfying a growth condition of order $p \in (1, +\infty)$ of the type

$$a|\xi|^p \le f(\xi), g(\xi) \le b|\xi|^p \quad \forall \xi \in \mathbb{R}^3, \quad (a, b > 0). \tag{1.9}$$

We suppose that

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon \frac{r_\varepsilon^2 |S|}{\varepsilon^2} = \bar{k} \in (0, +\infty], \tag{1.10}$$

thus the density of electric energy is assumed to take large values in the fibers. For simplicity, we suppose that (see Remark 3.1 (iii))

$$u_0 = 0, \qquad \text{if } \bar{k} = +\infty. \tag{1.11}$$

2 Notations

For any weakly differentiable function $\varphi : \mathbb{R}^N \to \mathbb{R}$ ($N \in \{2,3\}$), we set

$$\widehat{\nabla}\varphi := (\partial_1\varphi, \partial_2\varphi, 0). \tag{2.1}$$

We denote by $f^{\infty,p}$ the "p-recession" function of f, defined by

$$f^{\infty,p}(\xi) := \limsup_{t \to +\infty} \frac{f(t\xi)}{t^p} \quad \forall \xi \in \mathbb{R}^3. \tag{2.2}$$

Our results are obtained under the hypothesis:

$$\exists \alpha' > 0,\ \exists \beta' \in (0,p),\ |f(\xi) - f^{\infty,p}(\xi)| \le \alpha'(1 + |\xi|^{\beta'}) \quad \forall \xi \in \mathbb{R}^3. \tag{2.3}$$

For any $\alpha \in \mathbb{R}$, we set

$$\mathrm{sgn}(\alpha) := 1 \ \text{ if } \ \alpha \ge 0, \quad \mathrm{sgn}(\alpha) := -1 \ \text{ if } \ \alpha < 0. \tag{2.4}$$

For all couples (U,V) of open subsets of $\mathbb{R}^2$ such that $\overline{U}\subset V$ and for all $\alpha\in\mathbb{R}$, we set

$$\begin{aligned}&\mathrm{cap}^f(U,V;\alpha)=\inf\mathcal{P}^f(U,V;\alpha),\\&\mathcal{P}^f(U,V;\alpha):\inf\left\{\int_V f\left(\widehat{\nabla}\varphi\right)\,dx_1dx_2:\varphi\in W_0^{1,p}(V);\ \varphi=\alpha\text{ in }U\right\}.\end{aligned}\tag{2.5}$$

The letter C denotes different constants whose precise values may vary. We employ the usual convention $\infty.0=0$. We denote the Lebesgue measure on $\mathbb{R}^N$ by $\mathcal{L}^N$, the Hausdorff k-dimensional measure on $\mathbb{R}^N$ by $\mathcal{H}^k$, the space of Radon measures on $\overline{\mathcal{O}}$ by $\mathcal{M}(\overline{\mathcal{O}})$, the space of Borel functions on $\mathcal{O}$ by $L^0(\mathcal{O})$, respectively.

3 Main result

We assume that the sequence $(\gamma_\varepsilon^{(p)}(r_\varepsilon))$ defined by (1.7) is convergent and set

$$\gamma^{(p)}:=\lim_{\varepsilon\to 0}\gamma_\varepsilon^{(p)}(r_\varepsilon)\in[0,+\infty].\tag{3.1}$$

The effective behavior depends on the order of magnitude of the parameter $\gamma^{(p)}$. A critical case occurs when $0<\gamma^{(p)}<+\infty$. Then, a gap between the mean potential of the constituent parts of the composite may appear, giving rise to a concentration of electric energy stored in a thin region of space enveloping the fibers. The effective electric energy then takes the form of a sum of three terms like

$$\Phi(u,v)=\int_{\mathcal{O}}f(\nabla u)\,dx+\Phi_{cap}(v-u)+\Phi_{fibers}(v).\tag{3.2}$$

The function u stands for the weak limit in $W^{1,p}(\mathcal{O})$ of the sequence (u_ε) of the solutions to (1.1), and v represents a local approximation of the effective potential in the fibers. More precisely the function nv, where n is defined by (1.5), is the weak-$*$ limit in $\mathcal{M}(\overline{\mathcal{O}})$ of the sequence of measures $(u_\varepsilon\mu_\varepsilon)$, being μ_ε the measure defined by

$$\mu_\varepsilon:=\frac{\varepsilon^2}{r_\varepsilon^2|S|}\mathbb{1}_{T_{r_\varepsilon}}(x)\mathcal{L}^3_{\lfloor\mathcal{O}}.\tag{3.3}$$

The functional Φ_{fibers} accounts for the effective electric energy stored in the fibers and is given by

$$\Phi_{fibers}(v)=\int_{\mathcal{O}}g^{hom}(\partial_3 v)n\,dx,\tag{3.4}$$

where n and $g^{hom}:\mathbb{R}\to\mathbb{R}$ are respectively defined by (1.5) and

$$g^{hom}(a):=\min\left\{g(q):q\in\mathbb{R}^3,\ q_3=a\right\}.\tag{3.5}$$

The second term of Φ describes the last mentioned concentration of energy in terms of the gap between the effective potential in the fibers and in the matrix. We obtain

$$\Phi_{cap}(v-u)=\int_{\mathcal{O}}c^f(S;v-u)n\,dx,$$

where

$$\begin{aligned}&c^f(S;\alpha)=c^f(S;\mathrm{sgn}(\alpha))|\alpha|^p,\qquad c^f(S;\pm1)=\lim_{\varepsilon\to0}c_\varepsilon^f(S;\pm1),\\&c_\varepsilon^f(S;\pm1):=\frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon S,\widehat{\mathcal{O}};\pm1).\end{aligned}\tag{3.6}$$

The sequences $(c_\varepsilon^f(S;\pm1))$ are assumed to be convergent if $p=2$. A study of cap^f (see Section 5) yields

$$c^f(S;\pm1)=\begin{cases}\gamma^{(p)}\mathrm{cap}^{f^{\infty,p}}(S,\mathbb{R}^2;\pm1) & \text{if }\ p<2,\\ \gamma^{(2)}c^{f^{\infty,2}}(\pm1) & \text{if }\ p=2,\\ +\infty & \text{if }\ p>2,\end{cases}\tag{3.7}$$

for some positive reals $c^{f^{\infty,2}}(\pm 1)$ independent of S (see Remark 3.1 (iv)).
We prove that the limiting problem in a variational sense associated with (1.1) is given by

$$(\mathcal{P}^{hom}): \quad \min\left\{F^{hom}(u) - \int_{\mathcal{O}} q_b u \, dx - \int_{\Gamma_1} q_s u \, d\mathcal{H}^2 : u \in u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})\right\}, \tag{3.8}$$

where

$$\begin{aligned}
&F^{hom}(u) = \inf\left\{\Phi(u,v) : v \in L^p(\mathcal{O})\right\},\\
&\Phi(u,v) = \begin{cases} \displaystyle\int_{\mathcal{O}} f(\nabla u)\, dx + \int_{\mathcal{O}} c^f(S; v-u) n dx + \bar{k}\int_{\mathcal{O}} g^{hom}(\partial_3 v) n dx & \\ & \text{if } (u,v) \in \left(u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})\right) \times V_p, \\ +\infty & \text{otherwise,} \end{cases}\\
&V_p := \left\{v \in L^0(\mathcal{O}),\ vn \in L^p(\mathcal{O}) : \partial_3 vn \in L^p(\mathcal{O}), vn = u_0 n \text{ on } \Gamma_0 \cap \left(\widehat{\mathcal{O}} \times \{0, L\}\right)\right\}.
\end{aligned} \tag{3.9}$$

Theorem 3.1. *Assume (1.2-1.10), (2.3), (3.1), then the unique solution u_ε to (1.1) converges weakly in $W^{1,p}(\mathcal{O})$ as ε tends to 0 toward the unique solution u to (3.8). Moreover, there holds*

$$\lim_{\varepsilon\to 0}\left\{F_\varepsilon(u_\varepsilon) - \int_{\mathcal{O}} q_b u_\varepsilon \, dx - \int_{\Gamma_1} q_s u_\varepsilon \, d\mathcal{H}^2\right\} = F^{hom}(u) - \int_{\mathcal{O}} q_b u \, dx - \int_{\Gamma_1} q_s u \, d\mathcal{H}^2. \tag{3.10}$$

Assume in addition that $\gamma^{(p)} > 0$ and let μ_ε be the measure defined by (3.3). Then the sequence of measures $(u_\varepsilon \mu_\varepsilon)$ weak $*$ converges in $\mathcal{M}(\overline{\mathcal{O}})$ to $nv\mathcal{L}^3_{\lfloor\mathcal{O}}$, where n is defined by (1.5) and v is the unique element of V_p, given by (3.9), such that $F^{hom}(u) = \Phi(u,v)$.

Remark 3.1. *(i) If $\gamma^{(p)} = 0$, the variables u, v are independent and the effective energy simply reads*

$$F^{hom}(u) = \int_{\mathcal{O}} f(\nabla u)\, dx + C, \quad C := \inf_{v\in V_p} \bar{k}\int_{\mathcal{O}} g^{hom}(\partial_3 v) dx \qquad (\gamma^{(p)} = 0).$$

If $\gamma^{(p)} = +\infty$ (in particular if $p > 2$), the functional $\Phi(u,v)$ takes infinite values unless $u = v$, hence

$$F^{hom}(u) = \int_{\mathcal{O}} f(\nabla u)\, dx + \bar{k}\int_{\mathcal{O}} g^{hom}(\partial_3 u)\, n dx \qquad (\gamma^{(p)} = +\infty),$$

and the effective energy is that of the matrix augmented by a permittivity term in the direction of the fibers.
If $0 < \gamma^{(p)} < +\infty$, the effective electric energy is not a local functional. This means that it can not be written as the integration over $\mathcal{O}$ of a density of electric energy of the form $h(x, u(x), \nabla u(x), ...)$. By introducing the additional state variable v, we can write the effective energy under the form of a local functional of the couple (u,v). This internal or hidden state variable is the limit of a suitable scaling of the electric potential in the sole fibers and accounts for the micro-structure. The total effective electric energy is that of a body totally filled up by the matrix material augmented by a term which is the infimal convolution of the last mentioned permittivity term supplied by the distribution of fibers and a bonding term depending on the gap of electric potentials in the matrix and in the fibers. This bonding term describes a concentration of electric energy in the matrix in the immediate vicinity of the fibers, which may occur only when $p \le 2$. It induces a total effective energy lower than $\Phi(u,u)$. The structure of Φ stems from the contribution of each term entering the decomposition:

$$F_\varepsilon(u) = \int_{\mathcal{O}\setminus(D_{R_\varepsilon}\times(0,L))} f(\nabla u)\, dx + \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u)\, dx + \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u)\, dx, \tag{3.11}$$

where, given (R_ε) satisfying (1.7), the set $D_{R_\varepsilon} \times (0,L)$ is the R_ε-neighborhood of the fibers defined by (6.3). The set $(D_{R_\varepsilon} \times (0,L)) \setminus T_{r_\varepsilon}$ is a small portion of the matrix surrounding the fibers where electric

energy may concentrate due to the gap between the mean electric potentials in the fibers and in the matrix. This will provide a limit capacitary term associated with $f^{\infty,p}(\widehat{\nabla} u)$ on $R_\varepsilon D \setminus r_\varepsilon S$. The contribution of $\mathcal{O} \setminus (D_{R_\varepsilon} \times (0,L))$ is obvious and the contribution of the fibers is classical (see [1, 29]).

(ii) The extension of our results to the case when the sequence (n_ε) is not bounded in $L^\infty(\mathcal{O})$ but only in $L^1(\mathcal{O})$ and weak-$*$ converges in $\mathcal{M}(\overline{\mathcal{O}})$ to some measure μ is a challenging mathematical problem. The effective energy stored in the fibers is then likely to be simply deduced from (3.4) by substituting $d\mu$ for ndx. As regards the concentration of electric energy around the fibers, we expect it to take the form $\Phi_{cap}(v-u) = \int c^f(S; v-u) d\mu_0$ for some suitable measure μ_0 absolutely continuous with respect to μ and satisfying $\mu_0(E \times (0,L)) = 0$ for all sets $E \subset \widehat{\mathcal{O}}$ such that $\mathrm{cap}^f(E, \widehat{\mathcal{O}}; 1) = 0$. Similar classes of measures arise in the study of Dirichlet problems on varying domains [19], [20], [21]. Computing this measure μ_0, if possible in terms of μ, seems to be a big task.

(iii) The simplifying assumption (1.11) ensures that the effective electric energy stored in the fibers vanishes if $\bar{k} = +\infty$. An alternative is to assume that u_0 takes the same values on the intersection of the opposite bases of $\mathcal{O}$ with Γ_0.

(iv) If $p = 2$, the constants $c^{f^{\infty,2}}(\pm 1)$ are simply defined by

$$c^{f^{\infty,2}}(\pm 1) = \frac{1}{\gamma^{(2)}} \lim_{\varepsilon \to 0} \frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S; \widehat{\mathcal{O}})}{\varepsilon^2} \quad \text{if } 0 < \gamma^{(2)} < +\infty, \quad c^{f^{\infty,2}}(\pm 1) = 1 \quad \text{otherwise.} \tag{3.12}$$

These constants can not be explicitely determined in terms of $\mathrm{cap}^{f^{\infty,2}}$. However, they can be calculated if $f(.) = \frac{1}{2}|.|^2$ (see (5.20, 5.36)):

$$c^{\frac{1}{2}|.|^2}(\pm 1) = \pi.$$

(iv) The phenomenon observed in the critical case does not appear in dimension 2 whenever the sequence of conductivities (λ_ε) is supposed to be uniformly bounded from below. Indeed, M. Briane and J. Casado-Díaz showed [13, 14] that in that case the nature of the problem is preserved through the homogenization process.

4 Conjecture for the case of a random distribution of fibers

In this section, we indicate a possible generalization of the periodic model to the case of parallel fibers randomly distributed in accordance with a stationary point process. In the model under consideration, the cross sections are not uniformly (i.e., periodically) distributed but their distribution is periodic in law i.e., the probability of presence of the sections is invariant under a suitable group $(\tau_z)_{z \in \mathbb{Z}^2}$ defined below. In the stochastic homogenization framework, the distribution of the sections is then said to be statistically homogeneous. We are going to give some precisions on this model.
Let us first define the discrete dynamical system $(\Omega, \boldsymbol{P}, (\tau_z)_{z \in Z^2})$ that models the distribution of the sections of the fibers. Given $d > 0$, we set

$$\Omega := \left\{ (\omega_i)_{i \in \mathbb{N}} : \omega_i \in \mathbb{R}^2,\ |\omega_k - \omega_l| \geq d \text{ for } k \neq l \right\}, \tag{4.1}$$

and denote by Σ the trace of the Borel σ-algebra of $(\mathbb{R}^2)^\infty$ on Ω. We equip Ω with the group $(\tau_z)_{z \in \mathbb{Z}^2}$ defined by

$$\tau_z \omega = \omega - z,$$

where $\omega - z$ must be understood as $(\omega_i - z)_{i \in \mathbb{N}}$, and we denote by $\mathcal{F}$ the σ-algebra made up of all the events of Σ which are invariant under the group $(\tau_z)_{z \in \mathbb{Z}^2}$. We assume the existence of a probability measure $\boldsymbol{P}$ on (Ω, Σ) for which $(\tau_z)_{z \in \mathbb{Z}^2}$ is a measure preserving transformation, i.e.,

$$\boldsymbol{P} \# \tau_z = \boldsymbol{P} \text{ for all } z \in \mathbb{Z}^2,$$

where $\boldsymbol{P} \# \tau_z$ denotes the pushforward of the probability measure $\boldsymbol{P}$ by the map τ_z. For any measurable function $X : \Omega \to \mathbb{R}$, we denote by $\boldsymbol{E}^{\mathcal{F}} X$ its conditional expectation given $\mathcal{F}$, i.e., the unique $\mathcal{F}$-measurable function satisfying

$$\int_E \boldsymbol{E}^{\mathcal{F}} X \, d\boldsymbol{P} = \int_E X \, d\boldsymbol{P} \text{ for every } E \in \mathcal{F}.$$

Note that $\boldsymbol{E}^{\mathcal{F}}X$ is τ_z-invariant (hence periodic) and that under the additional ergodic hypothesis which asserts that $\mathcal{F}$ is trivial, that is made up of events with probability measure 0 or 1, $\boldsymbol{E}^{\mathcal{F}}X$ is constant and nothing but the expectation $\boldsymbol{E}(X) := \int_\Omega X\ d\boldsymbol{P}$. Note also that the following asymptotic independance hypothesis

$$\lim_{|z|\to+\infty} \boldsymbol{P}(E \cap \tau_z E') = \boldsymbol{P}(E)\boldsymbol{P}(E'), \tag{4.2}$$

is a stronger but more intuitive condition yielding ergodicity.

The random set of fibers is defined by

$$T_{r_\varepsilon}(\omega) := \bigcup_{j\in J_\varepsilon(\omega)} T^j_{r_\varepsilon}, \quad T^j_{r_\varepsilon} := (\varepsilon\omega_j + r_\varepsilon S)\times(0,L), \quad J_\varepsilon(\omega) := \left\{j\in\mathbb{N},\ \ \omega_j\in\widehat{\mathcal{O}}\right\}. \tag{4.3}$$

We will denote by $(\mathcal{P}_\varepsilon(\omega))$ the problem associated with the random functional $F_\varepsilon(\omega,.)$.
Consider the random function

$$n_0 : \Omega \to \mathbb{N}, \quad \omega \mapsto n_0(\omega) := \#\left\{i\in\mathbb{N}:\ \omega_i\in\widehat{Y}\right\}, \quad \widehat{Y} := [0,1[^2. \tag{4.4}$$

In all likelyhood, the conditional expectation $\boldsymbol{E}^{\mathcal{F}}n_0(\omega)$ is the only additional corrector of the limit energy obtained in the periodic case. More precisely let us denote by $\Phi(\omega,.)$ the random functional:

$$\Phi(\omega,u,v) = \begin{cases} \displaystyle\int_{\mathcal{O}} f(\nabla u)\ dx + \boldsymbol{E}^{\mathcal{F}}n_0(\omega)\bar{k}\int_{\mathcal{O}} g^{hom}(\partial_3 v)\ dx + \boldsymbol{E}^{\mathcal{F}}n_0(\omega)\int_{\mathcal{O}} c^f(S; v-u)dx; \\ \qquad \text{if } (u,v)\in\left(u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})\right)\times V_p, \\ +\infty \text{ otherwise,} \end{cases} \tag{4.5}$$

and set $F^{hom}(\omega,u) = \inf\{\Phi(\omega,u,v) : v\in L^p(\mathcal{O}\}$. Then one may reasonably conjecture that

Conjecture 4.1. *Under the assumptions stated above, when ε tends to 0, the unique random solution $u_\varepsilon(\omega)$ to the problem $(\mathcal{P}_\varepsilon(\omega))$, deduced from (1.1) by substituting (4.3) for (1.3), almost surely weakly converges in $W^{1,p}(\mathcal{O})$ toward the unique solution $u(\omega)$ to*

$$(\overline{\mathcal{P}}(\omega)) \quad \min\left\{F^{hom}(\omega,u) - \int_{\mathcal{O}} q_b u\ dx - \int_{\Gamma_1} q_s u\ d\mathcal{H}^2 : u\in u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})\right\}.$$

Moreover,

$$\lim_{\varepsilon\to 0}\left\{F_\varepsilon(\omega,u_\varepsilon) - \int_{\mathcal{O}} q_b u_\varepsilon\ dx - \int_{\Gamma_1} q_s u_\varepsilon\ d\mathcal{H}^2\right\} = F^{hom}(\omega,u) - \int_{\mathcal{O}} q_b u\ dx - \int_{\Gamma_1} q_s u\ d\mathcal{H}^2,$$

and, if $\gamma^{(p)} > 0$, $v_\varepsilon(\omega) := \dfrac{\varepsilon^2}{r^2|D|}\mathbb{1}_{T_{r_\varepsilon}(\omega)}u_\varepsilon(\omega)$ almost surely weak $*$ converges in $\mathcal{M}(\overline{\mathcal{O}})$ to some $v(\omega)$ belonging to V_p such that $F^{hom}(\omega,u) = \Phi(\omega,u(\omega),v(\omega))$. Furthermore, under the ergodic hypothesis (for instance under condition (4.2)), there holds $\boldsymbol{E}^{\mathcal{F}}n_0(\omega) = \boldsymbol{E}n_0$ so that the functionals Φ, F^{hom} and the functions u and v are deterministic.

We hope to treat the mathematical analysis in a forthcoming paper.

5 Study of the capacity cap^f

Given a strictly convex function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying a growth condition of order $p\in(1,+\infty)$ of the type (1.9), our main objective in this section is to analyze the behavior with respect to certain small subsets of $\mathbb{R}^2$ of the mapping cap^f defined by (2.5). A similar study has already been performed in the setting of linear elasticity in [6, Section 3]. In [22], G. Dal Maso and I.V. Skrypnik have studied the capacity for monotone operators which are closely related to the ones considered in our paper. Also, their study has been extended to pseudo-monotone operators by J. Casado Díaz in [17]. Further results

concerning capacities and many references on this subject may be found for instance in [2, 25, 27, 30, 32]. In what follows, the letter U denotes a non-empty bounded connected Lipschitz open subset of $\mathbb{R}^2$ and V an open subset of $\mathbb{R}^2$ such that $\overline{U} \subset V$.

The proof of the following Lemma is similar to that of [6, Lemma 1]:

Lemma 5.1. *The problem (2.5) has a minimizing sequence in $\mathcal{D}(V)$.*

If $p < 2$, we denote by $K^p(V)$ the set of functions $\psi \in L^{p^*}(V)$ ($p^* := \frac{2p}{2-p}$) for which all the partial derivatives $\partial_1\psi$, $\partial_2\psi$ (in the sense of distributions) belong to $L^p(V)$. It is easy to check that, equiped with the norm

$$|\psi|_{K^p(V)} := \left(\int_V |\psi|^{p^*} dx\right)^{\frac{1}{p^*}} + \left(\int_V |\nabla\psi|^p dx\right)^{\frac{1}{p}}, \tag{5.1}$$

the space $K^p(V)$ is a reflexive Banach space. Therefore the closure of $\mathcal{D}(V)$ in $K^p(V)$, which will be denoted by $K_0^p(V)$, is also a reflexive Banach space. Gagliardo-Nirenberg-Sobolev inequality (see for instance [10, Theorem 9.9]), namely

$$\int_V |\psi|^{p^*} dx \le C \int_V |\nabla\psi|^p dx \quad \forall \psi \in K_0^p(V) \quad (p < 2), \tag{5.2}$$

holds true whatever the choice of the open set V, with a constant C depending only on p (we can take for instance $C = \frac{p}{2-p}$ but this constant is not optimal, see [10, footenote, p. 278]), unlike Poincaré inequality in $W_0^{1,p}(V)$, which may fail to hold when V is unbounded. The space $K_0^p(V)$ coincides with $W_0^{1,p}(V)$ if V is bounded and may be strictly larger otherwise. There holds $K_0^p(\mathbb{R}^2) = K^p(\mathbb{R}^2)$.

The next lemma marks a noteworthy difference between the case $p < 2$ and the case $2 \le p$: if $p \ge 2$, the infimum problem $\mathcal{P}^f(U, V; \alpha)$ (see (2.5)) is not achieved in general if V is unbounded (see Remark 5.1), whereas $\mathcal{P}^f(U, V; \alpha)$ is always achieved, if $p < 2$, provided we substitute $K_0^p(V)$ for $W_0^{1,p}(V)$ in (2.5).

Lemma 5.2. *(i) Assume that $p < 2$, and let $\alpha \in \mathbb{R}$. Then the problem*

$$\begin{aligned} &\mathcal{P}^f_{K_0^p}(U, V; \alpha) : \inf_{\psi \in \mathcal{K}_\alpha(U,V)} \int_V f(\partial_1\psi, \partial_2\psi, 0) dx, \\ &\mathcal{K}_\alpha(U, V) := \{\psi \in K_0^p(V),\ \psi = \alpha \quad in\ U\}, \end{aligned} \tag{5.3}$$

has a unique solution and

$$\mathrm{cap}^f(U, V; \alpha) = \min \mathcal{P}^f_{K_0^p}(U, V; \alpha). \tag{5.4}$$

Moreover, the solution ψ to (5.4) satisfies, for a.e. $x \in V$,

$$0 \le \psi(x) \le \alpha \quad if \quad \alpha \ge 0; \qquad \alpha \le \psi(x) \le 0 \quad if \quad \alpha \le 0. \tag{5.5}$$

(ii) Assume that $2 \le p$ and that V is bounded in one direction, and let $\alpha \in \mathbb{R}$. Then the problem (2.5) has a unique solution.

Proof. (i) By (2.5) and Lemma 5.1, we have

$$\mathrm{cap}^f(U, V; \alpha) = \inf \left\{ \int_V f(\partial_1\psi, \partial_2\psi, 0) dx, \quad \psi \in \mathcal{K}_\alpha(U, V) \cap \mathcal{D}(V) \right\}. \tag{5.6}$$

By repeating the argument of the proof of [6, Lemma 2], we find that

$$\overline{\mathcal{K}_\alpha(U, V) \cap \mathcal{D}(V)}^{K^p(V)} = \mathcal{K}_\alpha(U, V), \tag{5.7}$$

where $\overline{\mathcal{K}_\alpha(U, V) \cap \mathcal{D}(V)}^{K^p(V)}$ denotes the closure of $\mathcal{K}_\alpha(U, V) \cap \mathcal{D}(V)$ in $K^p(V)$. Since f is convex and satisfies the growth condition (1.9), the functional $\psi \to \int_V f(\partial_1\psi, \partial_2\psi, 0) dx$ is continuous on $K_0^p(V)$. We deduce that

$$\mathrm{cap}^f(U, V; \alpha) = \inf \left\{ \int_V f(\partial_1\psi, \partial_2\psi, 0) dx, \quad \psi \in \mathcal{K}_\alpha(U, V) \right\} = \inf \mathcal{P}^f_{K_0^p}(U, V; \alpha). \tag{5.8}$$

By (1.9) and Gagliardo-Nirenberg-Sobolev inequality there holds, for all $\psi \in \mathcal{D}(V)$ (extending ψ to $\mathbb{R}^2$ by setting $\psi = 0$ in $\mathbb{R}^2 \setminus V$)

$$\begin{aligned}\left(\int_V |\psi|^{p^*} dx\right)^{\frac{1}{p^*}} &= \left(\int_{\mathbb{R}^2} |\psi|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C\left(\int_{\mathbb{R}^2} |\nabla\psi|^p dx\right)^{\frac{1}{p}} = C\left(\int_V |\nabla\psi|^p dx\right)^{\frac{1}{p}} \\ &\leq C\left(\int_V f(\partial_1\psi, \partial_2\psi, 0)dx\right)^{\frac{1}{p}},\end{aligned} \tag{5.9}$$

yielding (see (5.1))

$$|\psi|^p_{K^p(V)} \leq C\int_V f(\partial_1\psi, \partial_2\psi, 0)dx \quad \forall \psi \in \mathcal{D}(V). \tag{5.10}$$

By Lemma 5.1, there exists a minimizing sequence (ψ_n) in $\mathcal{D}(V)$ to Problem (2.5). By (5.10), the sequence (ψ_n) is bounded in the reflexive Banach space $K^p_0(V)$, hence weakly converges in $K^p_0(V)$, up to a subsequence, to some ψ. As each function ψ_n belongs to the convex strongly closed (hence weakly closed) subset $\mathcal{K}_\alpha(U,V)$ of $K^p(V)$, we infer that ψ also belongs to $\mathcal{K}_\alpha(U,V)$. The functional $\varphi \to \int_V f(\partial_1\varphi, \partial_2\varphi, 0)dx$ is convex and strongly continuous on $K^p_0(V)$, hence weakly lower semi-continuous. We deduce that

$$\text{cap}^f(U,V;\alpha) = \liminf_{n\to+\infty}\int_V f(\partial_1\psi_n, \partial_2\psi_n, 0)dx \geq \int_V f(\partial_1\psi, \partial_2\psi, 0)dx \geq \inf \mathcal{P}^f_{K^p_0}(U,V;\alpha).$$

Taking (5.8) into account we infer that ψ is a solution to (5.4). The uniqueness of this solution follows from the strict convexity of f. The "markovian" property (5.5) results from the last mentioned uniqueness, and from the fact that for any $\psi \in \mathcal{K}_\alpha(U,V)$, the function defined by $\overline{\psi} := (\psi \vee 0) \wedge \alpha$ if $\alpha \geq 0$ and $\overline{\psi} := (\psi \wedge 0) \vee \alpha$ if $\alpha < 0$ belongs to $\mathcal{K}_\alpha(U,V)$ and satisfies $\int_V f(\partial_1\overline{\psi}, \partial_2\overline{\psi}, 0) \leq \int_V f(\partial_1\psi, \partial_2\psi, 0)dx$.
(ii) If $2 \leq p$ and V is bounded in one direction, by Poincaré inequality in $W^{1,p}_0(V)$ we have

$$|\psi|^p_{W^{1,p}_0(V)} \leq C\int_V |\nabla(\psi)|^p dx \leq C\int_V f(\partial_1\psi, \partial_2\psi, 0)dx \quad \forall \psi \in W^{1,p}_0(V). \tag{5.11}$$

Then we repeat the argument of the case $p < 2$, substituting (5.11) for (5.10), $W^{1,p}_0(V)$ for $K_0(V)$, and $\left\{\psi \in W^{1,p}_0(V),\ \psi = \alpha \text{ in } U\right\}$ for $\mathcal{K}_\alpha(U,V)$.

The next Lemma, whose proof is straightforward, states that the map $(f,U,V,\alpha) \to \text{cap}^f(U,V;\alpha)$ is convex with respect to α, decreasing with respect to V and increasing with respect to U and f.

Lemma 5.3. *(i) The map $\alpha \in \mathbb{R} \to \text{cap}^f(S,V;\alpha)$ is convex.*
(ii) Let V_1 and V_2 be two open subsets of $\mathbb{R}^N$ such that $\overline{U} \subset V_1 \subset V_2$. Then

$$\text{cap}^f(U,V_1;\alpha) \geq \text{cap}^f(U,V_2;\alpha). \tag{5.12}$$

(iii) Let U_1 and U_2 be two bounded connected open subsets of $\mathbb{R}^2$ such that $\overline{U}_1 \subset \overline{U}_2 \subset V$. Then

$$\text{cap}^f(U_1,V;\alpha) \leq \text{cap}^f(U_2,V;\alpha). \tag{5.13}$$

(iv) There holds

$$\text{cap}^{\lambda f}(U,V;\alpha) = \lambda\text{cap}^f(U,V;\alpha) \quad \forall\ \lambda > 0. \tag{5.14}$$

In addition, if $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ are two strictly convex functions satisfying a growth condition of order $p \in (1,+\infty)$ of the type (1.9) and if $f_1 \leq f_2$ in $\mathbb{R}^3$, then

$$\text{cap}^{f_1}(U,V;\alpha) \leq \text{cap}^{f_2}(U,V;\alpha). \tag{5.15}$$

In the following lemma, we investigate the continuity properties of $\text{cap}^f(U,V;\alpha)$ with respect to U and V.

Lemma 5.4. *Let (V_n) be an increasing sequence of open subsets of $\mathbb{R}^2$ such that $\overline{U} \subset V_1$ and $\bigcup_{n=1}^{+\infty} V_n = V$.*
(i) We have

$$\lim_{n\to+\infty} \mathrm{cap}^f(U, V_n; \alpha) = \mathrm{cap}^f(U, V; \alpha). \tag{5.16}$$

Furthermore there holds, if $0 \in V$,

$$\lim_{\lambda\to 0} \mathrm{cap}^f\left(U, \frac{1}{\lambda}V; \alpha\right) = \mathrm{cap}^f\left(U, \mathbb{R}^2; \alpha\right). \tag{5.17}$$

(ii) Assume that $p < 2$, and let ψ_n be the unique solution to $\mathcal{P}^f_{K^p_0}(U, V_n; \alpha)$ (see (5.3)) extended to V by setting $\psi_n = 0$ in $V \setminus V_n$. Then (ψ_n) converges weakly in $K^p(V)$ to the unique solution to $\mathcal{P}^f_{K^p_0}(U, V; \alpha)$.
(iii) Assume that $2 \le p$ and that V is bounded in one direction, and let ψ_n be the solution to $\mathcal{P}^f(U, V_n; \alpha)$ (see (2.5) and Lemma 5.2 (ii)), extended to V in the same way. Then (ψ_n) converges weakly in $W^{1,p}_0(V)$ to the unique solution to $\mathcal{P}^f(U, V; \alpha)$.
(iv) If $p < 2$ and if (U_n) is an increasing sequence of bounded open subsets of $\mathbb{R}^2$ such that $\bigcup_{n=1}^{+\infty} U_n = U$, then

$$\lim_{n\to+\infty} \mathrm{cap}^f(U_n, V; \alpha) = \mathrm{cap}^f(U, V; \alpha) \quad \forall \alpha \in \mathbb{R}. \tag{5.18}$$

Proof. (i) Let us fix $t > 0$. By Lemma 5.1, there exists $\psi \in \mathcal{D}(V)$ such that $\psi = \alpha$ in U and $\int_V f(\partial_1\psi, \partial_2\psi, 0)dx \le \mathrm{cap}^f(U, V; \alpha) + t$. Let us fix $n_0 \in \mathbb{N}$ such that $\mathrm{spt}\psi \subset V_{n_0}$. By (5.12) there holds, for all $n \ge n_0$: $\mathrm{cap}^f(U, V_n; \alpha) \le \mathrm{cap}^f(U, V_{n_0}; \alpha) \le \int_V f(\partial_1\psi, \partial_2\psi, 0)dx$. Taking again (5.12) into account, we deduce

$$\begin{aligned}\mathrm{cap}^f(U, V; \alpha) &\le \liminf_{n\to+\infty} \mathrm{cap}^f(U, V_n; \alpha)\\ &\le \limsup_{n\to+\infty} \mathrm{cap}^f(U, V_n; \alpha) \le \mathrm{cap}^f(U, V; \alpha) + t.\end{aligned}$$

By the arbitrary choice of t, Assertion (5.16) is proved.
If $0 \in V$, we can assume without loss of generality that $D \subset V$. Since the sequence (nD) is increasing, we deduce from (5.16) that $\lim_{n\to+\infty} \mathrm{cap}^f(U, nD; \alpha) = \mathrm{cap}^f\left(U, \bigcup_{n=1}^{+\infty} nD; \alpha\right) = \mathrm{cap}^f(U, \mathbb{R}^2; \alpha)$. Taking (5.12) into account, we then easily infer that $\lim_{\lambda\to 0} \mathrm{cap}^f\left(U, \frac{1}{\lambda}D; \alpha\right) = \mathrm{cap}^f(U, \mathbb{R}^2; \alpha)$. By (5.12) there holds $\mathrm{cap}^f(U, \mathbb{R}^2; \alpha) \le \mathrm{cap}^f(U, \frac{1}{\lambda}V; \alpha) \le \mathrm{cap}^f(U, \frac{1}{\lambda}D; \alpha)$. By passing to the limit as $\lambda \to 0$ in the third term of the last double inequality, we obtain (5.17). □
(ii) If $p < 2$ and ψ_n is the solution to $\mathcal{P}^f_{K^p_0}(U, V_n; \alpha)$, then by (5.10) we have

$$|\psi_n|^p_{K^p(V)} = |\psi_n|^p_{K^p(V_n)} \le C \int_{V_n} f(\partial_1\psi_n, \partial_2\psi_n, 0)dx = C\mathrm{cap}^f(U, V_n; \alpha).$$

It then follows from (5.16) that

$$\limsup_{n\to+\infty} |\psi_n|^p_{K^p(V)} \le C(\mathrm{cap}^f(U, V; \alpha)) < +\infty.$$

Therefore the sequence (ψ_n) is bounded in $K^p(V)$ and converges weakly in $K^p(V)$, up to a subsequence, to some function ψ. As each function ψ_n (extended by 0 to V) belongs to the weakly closed subset $\mathcal{K}_\alpha(U, V)$ of $K^p(V)$ (see (5.3)), we deduce that ψ also belongs to $\mathcal{K}_\alpha(U, V)$, hence (see (5.4))

$$\int_V f(\partial_1\psi, \partial_2\psi, 0)dx \ge \mathrm{cap}^f(U, V; \alpha).$$

On the other hand, by (5.16) and by the weak lower semi-continuity in $K_0^p(V)$ of the functional $\varphi \to \int_V f(\partial_1\varphi, \partial_2\varphi, 0)dx$, we have

$$\begin{aligned}\mathrm{cap}^f(U,V;\alpha) &= \lim_{n\to+\infty} \mathrm{cap}^f(U,V_n;\alpha) = \lim_{n\to+\infty} \int_{V_n} f(\partial_1\psi_n, \partial_2\psi_n, 0)dx \\ &= \lim_{n\to+\infty} \int_V f(\partial_1\psi_n, \partial_2\psi_n, 0)dx \geq \int_V f(\partial_1\psi, \partial_2\psi, 0)dx.\end{aligned}$$

Therefore, ψ is the solution to $\mathcal{P}^f_{K_0^p}(U,V;\alpha)$. □

(iii) Same argument as in the proof of Lemma 5.2 (ii).

(iv) Let ψ_n be the unique solution to $\mathcal{P}^f_{K_0^p}(U_n,V;\alpha)$ (see Lemma 5.2). By (5.10) and (5.13), there holds

$$|\psi_n|_{K^p(V)} \leq C\mathrm{cap}^f(U_n,V;\alpha) \leq C\mathrm{cap}^f(U,V;\alpha),$$

hence (ψ_n) is bounded in $K^p(V)$ and converges weakly, up to a subsequence, to some function ψ. Since each ψ_n belongs to $K_0^p(V)$, we have $\psi \in K_0^p(V)$. Moreover, it is easy to check that $\psi = \alpha$ in U, therefore $\psi \in \mathcal{K}_\alpha(U,V)$ (see (5.3)). We deduce from (5.13) and from the weak lower semi-continuity in $K_0^p(V)$ of the map $\varphi \to \int_V f(\partial_1\varphi, \partial_2\varphi, 0)dx$ that

$$\begin{aligned}\mathrm{cap}^f(U,V;\alpha) &\geq \limsup_{n\to+\infty} \mathrm{cap}^f(U_n,V;\alpha) \geq \liminf_{n\to+\infty} \mathrm{cap}^f(U_n,V;\alpha) = \liminf_{n\to+\infty} \int_V f(\partial_1\psi_n, \partial_2\psi_n, 0)dx \\ &\geq \int_V f(\partial_1\psi, \partial_2\psi, 0)dx \geq \mathrm{cap}^f(U,V;\alpha).\end{aligned}$$

Assertion (5.18) is proved. □

In the next Lemma, we investigate the case when f is positively homogeneous of degree p. Next properties (5.19) are easily deduced from Lemma 5.4 and from the change of variable formula. Formula (5.20) is deduced from the explicit computation performed in [7, p. 432] of the radial solution to the problem associated with $\mathrm{cap}^{\frac{1}{p}|\cdot|^p}\big(r_\varepsilon D, R_\varepsilon D;\alpha\big)$.

Lemma 5.5. *Assume that f is positively homogeneous of degree p and let $\lambda > 0$, $\alpha \in \mathbb{R}$. Then*

$$\begin{aligned}&\mathrm{cap}^f(\lambda U,V;\alpha) = \lambda^{2-p}\mathrm{cap}^f\left(U, \frac{1}{\lambda}V;\alpha\right) \quad \text{if} \quad \lambda U \subset V, \\ &\mathrm{cap}^f(U,V;\alpha) = |\alpha|^p \mathrm{cap}^f(U,V;\mathrm{sgn}(\alpha)).\end{aligned} \tag{5.19}$$

If $f(.) = \frac{1}{p}|\cdot|^p$, then

$$\mathrm{cap}^{\frac{1}{p}|\cdot|^p}\big(r_\varepsilon D, R_\varepsilon D;\alpha\big) = \begin{cases} \dfrac{2\pi}{p}\left|\dfrac{\frac{p-2}{p-1}}{R_\varepsilon^{\frac{p-2}{p-1}} - r_\varepsilon^{\frac{p-2}{p-1}}}\right|^{p-1} |\alpha|^p & \text{if } p \neq 2, \\ \\ \frac{\pi}{\log(R_\varepsilon/r_\varepsilon)}\alpha^2 & \text{if } p = 2. \end{cases} \tag{5.20}$$

Proof. Let us fix $t > 0$. By Lemma 5.1, there exists $\psi \in \mathcal{D}(V)$ such that $\psi = \alpha$ in λU and $\mathrm{cap}^f(\lambda U,V;\alpha) + t \geq \int_V f(\partial_1\psi, \partial_2\psi, 0)dx$. The function $\varphi(y) := \psi(\lambda y)$ belongs to $\mathcal{D}\left(\frac{1}{\lambda}V\right)$ and satisfies $\varphi = \alpha$ in U and $(\partial_1\varphi, \partial_2\varphi, 0)(y) = \lambda(\partial_1\psi, \partial_2\psi, 0)(\lambda y)$. By applying the change of variables formula, taking the positive homogeneousness of degree p of f into account, we obtain

$$\begin{aligned}\mathrm{cap}^f(\lambda U,V;\alpha) + t &\geq \int_V f(\partial_1\psi, \partial_2\psi, 0)(x)dx = \lambda^2 \int_{\frac{1}{\lambda}V} f(\partial_1\psi, \partial_2\psi, 0)(\lambda y)dy \\ &= \lambda^{2-p} \int_{\frac{1}{\lambda}V} f(\partial_1\varphi, \partial_2\varphi, 0)(y)dy \geq \lambda^{2-p}\mathrm{cap}^f\left(U, \frac{1}{\lambda}V;\alpha\right).\end{aligned}$$

By the arbitrary choice of t, we deduce that $\mathrm{cap}^f(\lambda U, V; \alpha) \geq \lambda^{2-p}\mathrm{cap}^f\left(U, \frac{1}{\lambda}V\right)$. The inverse inequality can be proved in a similar way. The first line of (5.19) is established. The second line of (5.19) is obtained in a analogous manner, by setting $\varphi(y) := \frac{1}{|\alpha|}\psi(y)$. □

The next Lemma illustrates the contrasting behavior of the capacity cap^f in the case $p < 2$ and in the case $2 \leq p$.

Lemma 5.6. *We have*

$$\begin{aligned} &\mathrm{cap}^f(U, \mathbb{R}^2; \alpha) > 0 \qquad \forall \alpha \in \mathbb{R} \setminus \{0\} \qquad && \text{if} \quad 1 < p < 2, \\ &\mathrm{cap}^f(U, \mathbb{R}^2; \alpha) = 0 \qquad \forall \alpha \in \mathbb{R} && \text{if} \quad 2 \leq p < +\infty. \end{aligned} \tag{5.21}$$

Proof. Assume that $p < 2$ and let ψ be the solution to $\mathcal{P}^f_{K^0_p}(U, \mathbb{R}^2)$ (see Lemma 5.2). Then $\psi = \alpha$ in U hence, since $\alpha \neq 0$ there holds $\psi \neq 0$ (recall that $U \neq \emptyset$). Therefore by (5.10) we have

$$\mathrm{cap}^f(U, \mathbb{R}^2; \alpha) = \int_{\mathbb{R}^2} f(\partial_1\psi, \partial_2\psi, 0)dx \geq C|\psi|_{K^p(\mathbb{R}^2)} > 0.$$

Suppose now that $p \geq 2$ and fix $r > 0$ such that $U \subset rD$. By (1.9), (5.13), (5.14), (5.15), (5.17), and (5.20), we have

$$\begin{aligned} \mathrm{cap}^f(U, \mathbb{R}^2; \alpha) &\leq \mathrm{cap}^{b|\cdot|^p}(U, \mathbb{R}^2; \alpha) = pb\mathrm{cap}^{\frac{1}{p}|\cdot|^p}(U, \mathbb{R}^2; \alpha) \leq pb\mathrm{cap}^{\frac{1}{p}|\cdot|^p}(rD, \mathbb{R}^2; \alpha) \\ &= \lim_{R\to+\infty} pb\mathrm{cap}^{\frac{1}{p}|\cdot|^p}(rD, RD; \alpha) \\ &= \begin{cases} \lim_{R\to+\infty} pb\frac{\pi}{\log\left(\frac{R}{r}\right)}\alpha^2 = 0 & \text{if } \ p = 2, \\ \lim_{R\to+\infty} pb\frac{2\pi}{p}\left|\frac{\frac{p-2}{p-1}}{R^{\frac{p-2}{p-1}} - r^{\frac{p-2}{p-1}}}\right|^{p-1}|\alpha|^p = 0 & \text{if } \ 2 < p. \end{cases} \end{aligned}$$

□

In the next two lemmas, we investigate the asymptotic behavior of $\mathrm{cap}^f(r_\varepsilon U; R_\varepsilon D)$, being (r_ε), (R_ε) any bounded sequences of positive reals such that $r_\varepsilon \ll R_\varepsilon$. Then, we establish (3.7). We start with the case $p \neq 2$:

Lemma 5.7. *Assume that $p \neq 2$, let U be a bounded connected Lipschitz open subset of $\mathbb{R}^2$ such that $0 \in U$, and let (r_ε) and (R_ε) be any sequences of positive reals such that $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$. Then,*
(i) for all $\alpha \in \mathbb{R} \setminus \{0\}$, there holds

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{cap}^f(r_\varepsilon U, R_\varepsilon D; \alpha) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon U, R_\varepsilon D; \alpha) \\ &= \begin{cases} \gamma^{(p)} & \text{if } \ \gamma^{(p)} \in \{0, +\infty\}, \\ \gamma^{(p)}\mathrm{cap}^{f^{\infty,p}}(U, \mathbb{R}^2; \alpha) & \text{if } \ 0 < \gamma^{(p)} < +\infty, \end{cases} \end{aligned} \tag{5.22}$$

where $\gamma^{(p)}$ is defined by (3.1).
(ii) For all $\alpha \in \mathbb{R} \setminus \{0\}$, there holds

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon U, \widehat{\mathcal{O}}; \alpha) = \begin{cases} \gamma^{(p)} & \text{if } \ \gamma^{(p)} \in \{0, +\infty\}, \\ \gamma^{(p)}\mathrm{cap}^{f^{\infty,p}}(U, \mathbb{R}^2; \alpha) & \text{if } \ 0 < \gamma^{(p)} < +\infty. \end{cases} \tag{5.23}$$

Proof. (i) It is easy to check that $f^{\infty,p}$ also verifies the growth condition (1.9). Hence by (5.14) and (5.15) there holds

$$\frac{pa}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|\cdot|^p}(r_\varepsilon U, R_\varepsilon D; \alpha) \leq \frac{1}{\varepsilon^2}\mathrm{cap}^h(r_\varepsilon U, R_\varepsilon D; \alpha) \leq \frac{pb}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|\cdot|^p}(r_\varepsilon U, R_\varepsilon D; \alpha) \quad (h \in \{f, f^{\infty,p}\}). \tag{5.24}$$

Let us fix $r, R > 0$ such that $rD \subset U \subset RD$. By (5.13) we have

$$\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon rD, R_\varepsilon D; \alpha) \leq \mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon U, R_\varepsilon D; \alpha) \leq \mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon RD, R_\varepsilon D; \alpha). \tag{5.25}$$

Thanks to (5.20), we can easily verify that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon rD, R_\varepsilon D; \alpha) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon RD, R_\varepsilon D; \alpha) = \gamma^{(p)} \qquad \text{if } \gamma^{(p)} \in \{0, +\infty\}. \tag{5.26}$$

The estimate (5.22) is proved in the case $\gamma^{(p)} \in \{0, +\infty\}$ (in particular if $p > 2$).
Assume that $0 < \gamma^{(p)} < +\infty$ (hence $p < 2$). Since $f^{\infty,p}$ is positively homogeneous of degree p, we can apply Lemma 5.5: we infer from (3.1), (5.17) and (5.19) that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon U, R_\varepsilon D; \alpha) = \lim_{\varepsilon\to 0} \frac{r_\varepsilon^{2-p}}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}\left(U, \frac{R_\varepsilon}{r_\varepsilon}D; \alpha\right) = \gamma^{(p)}\mathrm{cap}^{f^{\infty,p}}(U, \mathbb{R}^2; \alpha).$$

The proof of Lemma 5.7 (i) is achieved provided we show that $\lim_{\varepsilon\to 0} \Delta_\varepsilon = 0$, where

$$\Delta_\varepsilon := \frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon U, R_\varepsilon D; \alpha) - \frac{1}{\varepsilon^2}\mathrm{cap}^{f}(r_\varepsilon U, R_\varepsilon D; \alpha). \tag{5.27}$$

First we prove that $\limsup_{\varepsilon\to 0} \Delta_\varepsilon \leq 0$. To that aim, we consider the solution φ to $\mathcal{P}^f_{K_0^p}(r_\varepsilon U, R_\varepsilon D; \alpha)$ (see (5.3)). There holds $\varphi \in \mathcal{K}_\alpha(r_\varepsilon U, R_\varepsilon D; \alpha)$ and

$$\int_{R_\varepsilon D} f(\partial_1\varphi, \partial_2\varphi, 0)dx = \mathrm{cap}^f(r_\varepsilon U, R_\varepsilon D; \alpha); \quad \int_{R_\varepsilon D} f^{\infty,p}(\partial_1\varphi, \partial_2\varphi, 0)dx \geq \mathrm{cap}^{f^{\infty,p}}(r_\varepsilon U, R_\varepsilon D; \alpha). \tag{5.28}$$

We deduce from (2.3), (5.27) and (5.28) that

$$\begin{aligned}\Delta_\varepsilon &\leq \frac{1}{\varepsilon^2}\int_{R_\varepsilon D}(f^{\infty,p}(\partial_1\varphi, \partial_2\varphi, 0) - f(\partial_1\varphi, \partial_2\varphi, 0))dx \leq \alpha'\frac{1}{\varepsilon^2}\int_{R_\varepsilon D}(1 + |\partial_1\varphi, \partial_2\varphi, 0|^{\beta'})dx \\ &\leq C\frac{R_\varepsilon^2}{\varepsilon^2} + C\frac{1}{\varepsilon^2}\left(\int_{R_\varepsilon D}|\partial_1\varphi, \partial_2\varphi, 0|^p dx\right)^{\frac{\beta'}{p}}\left(R_\varepsilon^2\right)^{1-\frac{\beta'}{p}}.\end{aligned} \tag{5.29}$$

On the other hand, by (1.9), (5.14), (5.15) and (5.20), there holds

$$\begin{aligned}\int_{R_\varepsilon D}|\partial_1\varphi, \partial_2\varphi, 0|^p dx &\leq C\int_{R_\varepsilon D} f(\partial_1\varphi, \partial_2\varphi, 0)dx = C\mathrm{cap}^f(r_\varepsilon U, R_\varepsilon D; \alpha) \\ &\leq C\mathrm{cap}^{|.|^p}(r_\varepsilon U, R_\varepsilon D; \alpha) \leq Cr_\varepsilon^{2-p}.\end{aligned} \tag{5.30}$$

Joining (3.1), (5.29) and (5.30), we infer

$$\Delta_\varepsilon \leq C\frac{R_\varepsilon^2}{\varepsilon^2} + C\frac{1}{\varepsilon^2}\left(r_\varepsilon^{2-p}\right)^{\frac{\beta'}{p}}\left(R_\varepsilon^2\right)^{1-\frac{\beta'}{p}} \leq C\frac{R_\varepsilon^2}{\varepsilon^2} + C\gamma_\varepsilon^{(p)}(r_\varepsilon)\left(R_\varepsilon^2 r_\varepsilon^{2-p}\right)^{1-\frac{\beta'}{p}}.$$

It follows that $\limsup_{\varepsilon\to 0} \Delta_\varepsilon \leq 0$ (because $0 < \gamma^{(p)} < +\infty$, $p < 2$, $\beta' < p$). By repeating the same argument, considering the solution to $\mathcal{P}^{f^{\infty,p}}_{K_0^p}(r_\varepsilon U, R_\varepsilon D; \alpha)$ instead of that of $\mathcal{P}^f_{K_0^p}(r_\varepsilon U, R_\varepsilon D; \alpha)$, we find that $\liminf_{\varepsilon\to 0} \Delta_\varepsilon \geq 0$. The proof of (5.22) is achieved. □

(ii) By (1.9), (5.14) and (5.15) there holds

$$\frac{pa}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon U, \widehat{\mathcal{O}}; \alpha) \leq \frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon U, \widehat{\mathcal{O}}; \alpha) \leq \frac{pb}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon U, \widehat{\mathcal{O}}; \alpha). \tag{5.31}$$

Let us fix two positive reals d_1, d_2 such that $d_1 D \subset \widehat{\mathcal{O}} \subset d_2 D$. Then by (5.12) we have

$$\frac{1}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon U, d_2 D; \alpha) \leq \frac{1}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon U, \widehat{\mathcal{O}}; \alpha) \leq \frac{1}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon U, d_1 D; \alpha). \tag{5.32}$$

By (5.20) there holds

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon U, d_2 D;\alpha)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{cap}^{\frac{1}{p}|.|^p}(r_\varepsilon U, d_1 D;\alpha)=\gamma^{(p)}\quad \text{if}\ \ \gamma^{(p)}\in\{0,+\infty\}. \tag{5.33}$$

Joining (5.31)-(5.33), we get

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon U,\widehat{\mathcal{O}};\alpha)=\gamma^{(p)}\ \text{if}\ \ \gamma^{(p)}\in\{0,+\infty\}.$$

If $0<\gamma^{(p)}<+\infty$, we infer from (3.1), (5.17) and (5.19) that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}(r_\varepsilon U,\widehat{\mathcal{O}};\alpha)=\lim_{\varepsilon\to 0}\frac{r_\varepsilon^{2-p}}{\varepsilon^2}\mathrm{cap}^{f^{\infty,p}}\left(U,\frac{1}{r_\varepsilon}\widehat{\mathcal{O}};\alpha\right)=\gamma^{(p)}\mathrm{cap}^{f^{\infty,p}}(U,\mathbb{R}^2;\alpha).$$

Assertion (5.23) is proved. □

The case $p=2$ is appreciably more involved.

Lemma 5.8. *Assume that $p=2$, let S be a bounded connected Lipschitz open subset of $\mathbb{R}^2$ such that $0\in S$, and let (r_ε) and (R'_ε) be any sequences of positive reals satisfying (1.7), (3.1).*
(i) There holds

$$\begin{aligned}
&C_1\gamma_\varepsilon^{(2)}(r_\varepsilon)\le\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,R'_\varepsilon D;\pm 1)}{\varepsilon^2}\le C_2\gamma_\varepsilon^{(2)}(r_\varepsilon)\ \left(=C_2\frac{1}{\varepsilon^2|\log r_\varepsilon|}\right),\\
&C_1\gamma_\varepsilon^{(2)}(r_\varepsilon)\le\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\widehat{\mathcal{O}};\pm 1)}{\varepsilon^2}\le C_2\gamma_\varepsilon^{(2)}(r_\varepsilon),
\end{aligned}\tag{5.34}$$

for some positive constants C_1,C_2.

(ii) Assume that $0<\gamma^{(2)}<+\infty$ and that the sequences $\left(\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\widehat{\mathcal{O}};\pm 1)}{\varepsilon^2}\right)$ are convergent. Let $c^{f^{\infty,2}}(1)$ and $c^{f^{\infty,2}}(-1)$ be defined by

$$c^{f^{\infty,2}}(\pm 1):=\frac{1}{\gamma^{(2)}}\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\widehat{\mathcal{O}};\pm 1)}{\varepsilon^2}. \tag{5.35}$$

Then $c^{f^{\infty,2}}(1)$ and $c^{f^{\infty,2}}(-1)$ are positive reals, are independent of S, and satisfy

$$\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,R'_\varepsilon D;\pm 1)}{\varepsilon^2}=\gamma^{(2)}c^{f^{\infty,2}}(\pm 1). \tag{5.36}$$

Proof. See Section 8.2.

Remark 5.1. *(i) If $p\ge 2$ and $\alpha\in\mathbb{R}\setminus\{0\}$, then the infimum for problem $\mathcal{P}^f\left(U,\mathbb{R}^2;\alpha\right)$ (see (2.5)) is not achieved. Otherwise, should $\psi\in W_0^{1,p}(\mathbb{R}^2)$ be a minimum, then by (1.9) and the second line of (5.21), $|\nabla\psi|^p_{L^p(\mathbb{R}^2;\mathbb{R}^2)}\le C\int_{\mathbb{R}^2}f(\psi)dx=C\mathrm{cap}^f(U,\mathbb{R}^2;\alpha)=0$, hence $\psi=0$, in contradiction with the fact that $\psi=\alpha$ in U. This lack of solution is similar to Stokes' paradox in fluid Mechanics [31].*
(ii) If $V=\mathbb{R}^2$, weighted Sobolev spaces provide an interesting alternative approach to the questions of existence of a solution to $\mathcal{P}^f\left(U,\mathbb{R}^2;\alpha\right)$ (see [3] for more details on this subject). Indeed, it can be shown that

$$(\mathrm{cap}^f(U,\mathbb{R}^2;\alpha)=)\inf\mathcal{P}^f\left(U,\mathbb{R}^2;\alpha\right)=\min\left\{\int_{\mathbb{R}^2}f(\partial_1\psi,\partial_2\psi,0)dx,\quad \psi\in W^{1,p}_{\mu_p}(\mathbb{R}^2),\quad \psi=\alpha\ in\ U\right\},$$

where $W^{1,p}_{\mu_p}(\mathbb{R}^2)$ is the weighted Sobolev space defined by

$$W^{1,p}_{\mu_p}(\mathbb{R}^2):=\left\{\psi\in L^p_{\mu_p}(\mathbb{R}^2),\quad (\partial_1\psi,\partial_2\psi,0)\in L^p(\mathbb{R}^2;\mathbb{R}^3)\right\},$$

being μ_p the measure on $\mathbb{R}^2$ given by

$$\mu_p = \frac{1}{w_p(x)}\mathcal{L}^2, \quad w_2(x) = (1+|x|^2)\big(1+\log(1+|x|^2)\big)^2, \quad w_p(x) = (1+|x|)^p \text{ otherwise.}$$

The property $\mathrm{cap}^f(U,\mathbb{R}^2;\alpha) = 0$ if $p \geq 2$, stated in Lemma 5.6, can be recovered from the fact that if $\alpha \neq 0$, then the constant function $\psi = \alpha$ belongs to $W^{1,p}_{\mu_p}(\mathbb{R}^2)$ if and only if $p \geq 2$.

6 Technical preliminaries and a priori estimates

The proof of Theorem 3.1 rests on an extensive investigation into the asymptotic behavior of the sequence of the solutions to $(\mathcal{P}_\varepsilon)$ and, more generally, of sequences (u_ε) satisfying

$$\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty. \tag{6.1}$$

A commonly used method consists in introducing auxiliary sequences designed to characterize the comportment of the diverse constituents of the composite. The delicate step lies in the analysis of the behavior of the fibers. An interesting approach consists in investigating the sequence $(u_\varepsilon\mu_\varepsilon)$, where μ_ε denotes the measure with support included in the fibers defined by (3.3). To that aim, given a sequence (R_ε) satisfying (1.7), we introduce the operators $\langle\cdot\rangle_{R_\varepsilon}, \langle\cdot\rangle_{r_\varepsilon}, \langle\langle\cdot\rangle\rangle_\varepsilon$ defined on $L^p((0,L);W^{1,p}(\mathcal{O}))$ by setting

$$\begin{aligned}
&\langle\varphi\rangle_{R_\varepsilon}(x) := \sum_{j\in J_\varepsilon} \langle\varphi\rangle^j_{R_\varepsilon}(x_3)\mathbb{1}_{D^j_{R_\varepsilon}}(x_1,x_2), \quad \langle\varphi\rangle^j_{R_\varepsilon}(x_3) := \fint_{\partial D^j_{R_\varepsilon}} \varphi(s_1,s_2,x_3)\, d\mathcal{H}^1(s_1,s_2),\\
&\langle\varphi\rangle_{r_\varepsilon}(x) := \sum_{j\in J_\varepsilon} \langle\varphi\rangle^j_{r_\varepsilon}(x_3)\mathbb{1}_{D^j_{R_\varepsilon}}(x_1,x_2), \quad \langle\varphi\rangle^j_{r_\varepsilon}(x_3) := \fint_{\partial D^j_{r_\varepsilon}} \varphi(s_1,s_2,x_3)\, d\mathcal{H}^1(s_1,s_2),\\
&\langle\langle\varphi\rangle\rangle_\varepsilon(x) := \sum_{z\in I_\varepsilon} \left(\fint_{Y^z_\varepsilon} \varphi(s_1,s_2,x_3)ds_1ds_2\right)\mathbb{1}_{Y^z_\varepsilon}(x_1,x_2),
\end{aligned} \tag{6.2}$$

where

$$D^j_{R_\varepsilon} = \omega^j_\varepsilon + R_\varepsilon D, \qquad D_{R_\varepsilon} = \bigcup_{j\in J_\varepsilon} D^j_{R_\varepsilon}, \tag{6.3}$$

and analogously for $D^j_{r_\varepsilon}$. The series of estimates stated below will take a crucial part in the proof of Theorem 3.1 (the proof of Lemma 6.1 is situated at the end of Section 6).

Lemma 6.1. *There exists a constant C such that for all $\varphi \in L^p((0,L);W^{1,p}(\widehat{\mathcal{O}}))$,*

$$\begin{aligned}
&\int |\langle\varphi\rangle_{R_\varepsilon} - \langle\varphi\rangle_{r_\varepsilon}|^p\, d\mu_\varepsilon \leq \begin{cases} \frac{C}{\gamma^{(p)}_\varepsilon(r_\varepsilon)}\int_{\mathcal{O}} |\widehat{\nabla}\varphi|^p dx, & \text{if } p \leq 2,\\ \frac{C}{\gamma^{(p)}_\varepsilon(R_\varepsilon)}\int_{\mathcal{O}} |\widehat{\nabla}\varphi|^p dx, & \text{if } p > 2,\end{cases}\\
&\int |\langle\langle\varphi\rangle\rangle_\varepsilon - \langle\varphi\rangle_{r_\varepsilon}|^p\, d\mu_\varepsilon \leq \frac{C}{\gamma^{(p)}_\varepsilon(r_\varepsilon)}\int_{\mathcal{O}} |\widehat{\nabla}\varphi|^p dx,\\
&\int |\langle\langle\varphi\rangle\rangle_\varepsilon - \langle\varphi\rangle_{R_\varepsilon}|^p\, d\mu_\varepsilon \leq \frac{C}{\gamma^{(p)}_\varepsilon(R_\varepsilon)}\int_{\mathcal{O}} |\widehat{\nabla}\varphi|^p dx,\\
&\int_{Y^z_\varepsilon\times(0,L)} |\varphi - \langle\langle\varphi\rangle\rangle_\varepsilon|^p\, dx \leq C\varepsilon^p \int_{Y^z_\varepsilon\times(0,L)} |\widehat{\nabla}\varphi|^p dx \quad \forall z \in I_\varepsilon,\\
&\int |\varphi - \langle\varphi\rangle_{r_\varepsilon}|^p d\mu_\varepsilon \leq C r^p_\varepsilon \int |\widehat{\nabla}\varphi|^p d\mu_\varepsilon,
\end{aligned} \tag{6.4}$$

where $\widehat{\nabla}$ and $\gamma^{(p)}_\varepsilon(.)$ are defined, respectively, by (2.1) and (3.1).

The next Lemma states a lower bound inequality for convex functionals on measures.

Lemma 6.2. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^N$ and let μ_ε and μ be bounded Radon measures in $\overline{\mathcal{O}}$ such that μ_ε weak $*$ converges in $\mathcal{M}(\overline{\mathcal{O}})$ toward μ and f_ε a sequence of μ_ε-measurable functions such that $\sup_\varepsilon \int_{\mathcal{O}} |f_\varepsilon|^p \, d\mu_\varepsilon < +\infty$. Then*

i) the sequence of measures $(f_\varepsilon \mu_\varepsilon)$ is weak $*$ relatively compact in $\mathcal{M}(\overline{\mathcal{O}})$ and every cluster point ν is of the form $\nu = f\mu$ for some $f \in L^p_\mu$.

ii) If $f_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} f\mu$, then $\liminf_{\varepsilon \to 0} \int j(f_\varepsilon) \, d\mu_\varepsilon \geq \int j(f) \, d\mu$ for all convex and lower semi-continuous functions j on $\mathbb{R}$ satisfying a growth condition of order p. In addition

$$\begin{aligned} \liminf_{\varepsilon \to 0} \int |f_\varepsilon^+|^p \, d\mu_\varepsilon &\geq \int |f^+|^p \, d\mu, \\ \liminf_{\varepsilon \to 0} \int |f_\varepsilon^-|^p \, d\mu_\varepsilon &\geq \int |f^-|^p \, d\mu. \end{aligned} \tag{6.5}$$

Proof. The proof of this lemma is given in [7] with $j = \frac{1}{p}|\cdot|^p$ but the duality argument can be extended to any convex lower semi-continuous function satisfying a growth condition of order p. Assertion (6.5) results from the fact that if $f_\varepsilon^+ \mu_\varepsilon \overset{\star}{\rightharpoonup} g\mu$ and $f_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} f\mu$, then $g \geq f^+ \; \mu-a.e.$, which can be easily checked by using positive continuous test functions (notice that in general, $g \neq f^+$). □

The main results of Section 6 are stated in the next Proposition, where the asymptotic behavior of several sequences associated with some sequence (u_ε) satisfying (6.1) is specified.

Proposition 6.1. *Assume (1.9), (1.10), (3.1), (7.1). Let (u_ε) be a sequence in $W^{1,p}(\mathcal{O})$ satisfying (6.1) and let (μ_ε), $(\langle u_\varepsilon \rangle_{R_\varepsilon})$ and $(\langle\langle u_\varepsilon \rangle\rangle_\varepsilon)$ be defined by (3.3), (6.2). Then the next estimates hold true*

$$\begin{aligned} &\int_{\mathcal{O}} |u_\varepsilon|^p + |\nabla u_\varepsilon|^p dx \leq C, \\ &\int |\nabla u_\varepsilon|^p + |u_\varepsilon|^p + |\langle u_\varepsilon \rangle_{r_\varepsilon}|^p + |\langle u_\varepsilon \rangle_{R_\varepsilon}|^p d\mu_\varepsilon \leq C, \end{aligned} \tag{6.6}$$

and there exists $u \in u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})$ and $v \in V_p$ such that, up to a subsequence, the next convergences take place

$$\begin{array}{llll} u_\varepsilon \rightharpoonup u & & \text{weakly in} & W^{1,p}(\mathcal{O}) \\ u_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} nv\mathcal{L}^3_{\lfloor \mathcal{O}}, & \partial_3 u_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} n\partial_3 v \mathcal{L}^3_{\lfloor \mathcal{O}} & \text{weak} * \text{ in} & \mathcal{M}(\overline{\mathcal{O}}), \\ \langle u_\varepsilon \rangle_{R_\varepsilon} \mu_\varepsilon \overset{\star}{\rightharpoonup} nu\mathcal{L}^3_{\lfloor \mathcal{O}}, & \langle u_\varepsilon \rangle_{r_\varepsilon} \mu_\varepsilon \overset{\star}{\rightharpoonup} nv\mathcal{L}^3_{\lfloor \mathcal{O}} & \text{weak} * \text{ in} & \mathcal{M}(\overline{\mathcal{O}}). \end{array} \tag{6.7}$$

In addition, $v = u$ if $\gamma^{(p)} = +\infty$ (in particular when $p > 2$).

Proof. The first line of (6.6) follows from (6.1), from the Dirichlet condition on Γ_0 and from Poincaré inequality. We deduce that, up to a subsequence,

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly in } W^{1,p}(\mathcal{O}), \tag{6.8}$$

for some $u \in W^{1,p}(\mathcal{O})$. We infer from the weak continuity of the trace mapping in $W^{1,p}(\mathcal{O})$ that $u \in u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})$. It follows from the fourth line of (6.4) that the sequence $(\langle\langle u_\varepsilon \rangle\rangle_\varepsilon)$ defined by (6.2) strongly converges to u in $L^p(\mathcal{O})$. We deduce then from (1.5) that

$$\langle\langle u_\varepsilon \rangle\rangle_\varepsilon n_\varepsilon \rightharpoonup un \qquad \text{weakly in } L^p(\mathcal{O}). \tag{6.9}$$

For each $z \in I_\varepsilon$, there holds $\sharp\{z' \in I_\varepsilon, \overline{Y}_\varepsilon^z \cap \overline{Y}_\varepsilon^{z'} \neq \emptyset\} \leq 9$, therefore each set Y_ε^z has a non empty intersection with at most $9|n_\varepsilon|_{L^\infty(\mathcal{O})}$ different sets $S_{r_\varepsilon}^j$ (see (1.4)). We then infer from (1.5), (3.3), (6.2), and (6.9) that

$$\begin{aligned}
\int |\langle\langle u_\varepsilon\rangle\rangle_\varepsilon|^p d\mu_\varepsilon &= \sum_{j\in J_\varepsilon} \int_{\mathcal{O}} \left(\sum_{z\in I_\varepsilon} \left| \fint_{Y_\varepsilon^z} u_\varepsilon(s_1,s_2,x_3) ds_1 ds_2 \right|^p \mathbb{1}_{Y_\varepsilon^z}(x_1,x_2) \right) \frac{\varepsilon^2}{r_\varepsilon^2|S|} \mathbb{1}_{S_{r_\varepsilon}^j}(x_1,x_2) dx \\
&= \sum_{z\in I_\varepsilon} \sum_{\{j\in J_\varepsilon, S_{r_\varepsilon}^j \cap Y_\varepsilon^z \neq \emptyset\}} \int_0^L \left| \fint_{Y_\varepsilon^z} u_\varepsilon(s_1,s_2,x_3) ds_1 ds_2 \right|^p \frac{\varepsilon^2}{r_\varepsilon^2|S|} \mathcal{L}^2(S_{r_\varepsilon}^j \cap Y_\varepsilon^z) dx_3 \\
&\leq 9|n_\varepsilon|_{L^\infty(\mathcal{O})} \sum_{z\in I_\varepsilon} \int_0^L \varepsilon^2 \left| \fint_{Y_\varepsilon^z} u_\varepsilon(s_1,s_2,x_3) ds_1 ds_2 \right|^p dx_3 \\
&= 9|n_\varepsilon|_{L^\infty(\mathcal{O})} \int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_\varepsilon|^p dx \leq C.
\end{aligned} \tag{6.10}$$

On the other hand, by (1.5) and (3.3) we have

$$\mu_\varepsilon \overset{\star}{\rightharpoonup} n\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{ in} \quad \mathcal{M}(\overline{\mathcal{O}}). \tag{6.11}$$

By applying Lemma 6.2, taking (6.10) and (6.11) into account, we deduce that there exists $f \in L^0(\mathcal{O})$ such that $fn \in L^p(\mathcal{O})$ and that, up to a subsequence,

$$\langle\langle u_\varepsilon\rangle\rangle_\varepsilon \mu_\varepsilon \rightharpoonup fn\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{ in} \quad \mathcal{M}(\overline{\mathcal{O}}). \tag{6.12}$$

Testing the convergences (6.9) and (6.12) with a given $\varphi \in \mathcal{D}(\mathcal{O})$, taking the estimate $|\langle\langle\varphi\rangle\rangle_\varepsilon - \varphi| \leq C\varepsilon$ in $\mathcal{O}$ into account (this estimate is satisfied provided ε is sufficiently small, see (1.4), (6.2)), we get

$$\lim_{\varepsilon\to 0} \int_{\mathcal{O}} \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon n_\varepsilon dx = \int_{\mathcal{O}} \varphi u n dx; \qquad \lim_{\varepsilon\to 0} \int \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon d\mu_\varepsilon = \int_{\mathcal{O}} \varphi f n dx.$$

We prove below that

$$\lim_{\varepsilon\to 0} \left| \int_{\mathcal{O}} \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon n_\varepsilon dx - \int \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon d\mu_\varepsilon \right| = 0. \tag{6.13}$$

We deduce that $\int_{\mathcal{O}} \varphi u n dx = \int_{\mathcal{O}} \varphi f n dx$ and then, by the arbitrary choice of φ, that $fn = un$ a.e. in $\mathcal{O}$. Therefore,

$$\langle\langle u_\varepsilon\rangle\rangle_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} nu\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{ in} \quad \mathcal{M}(\overline{\mathcal{O}}). \tag{6.14}$$

By (1.9), (1.10), (3.3), and (6.1), we have

$$\int |\nabla u_\varepsilon|^p \, d\mu_\varepsilon \leq C. \tag{6.15}$$

From (3.1), (6.4), (6.6), and (6.10), we derive

$$\int |u_\varepsilon|^p + |\langle u_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle u_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \leq C,$$

which, joined with (6.15), yields (6.6). We deduce from (1.7), the third line of (6.4), and (6.14), that

$$\langle u_\varepsilon\rangle_{R_\varepsilon} \mu_\varepsilon \overset{\star}{\rightharpoonup} nu\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{ in} \quad \mathcal{M}(\overline{\mathcal{O}}). \tag{6.16}$$

By (6.6) and Lemma 6.2,

$$\langle u_\varepsilon\rangle_{r_\varepsilon} \mu_\varepsilon \overset{\star}{\rightharpoonup} nv\mathcal{L}^3_{\lfloor\mathcal{O}}, \quad u_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} nv_1\mathcal{L}^3_{\lfloor\mathcal{O}}, \quad \partial_3 u_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} nw\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak} * \text{ in} \quad \mathcal{M}(\overline{\mathcal{O}}), \tag{6.17}$$

for some $(v, v_1, w) \in (L^p(\mathcal{O}))^3$. It follows from the estimate stated in the fifth line of (6.4) that

$$nv = nv_1 \text{ a.e. in } \mathcal{O}. \tag{6.18}$$

To show that

$$nw = n\partial_3 v \text{ a.e. in } \mathcal{O} \text{ and } nv = nu_0 \text{ on } \Gamma_0 \cap \widehat{\mathcal{O}} \times \{0, L\}), \tag{6.19}$$

it suffices (as in [7]) to pass to the limit in $\int \varphi \partial_3 u_\varepsilon \, d\mu_\varepsilon$ by integrating by parts with first $\varphi \in \mathcal{D}(\mathcal{O})$, next φ of the form $\varphi(x) = \theta(x_1, x_2)\psi(x_3)$ with $\theta \in \mathcal{D}(\mathcal{O}_0)$, $\mathcal{O}_0 = \{(x_1, x_2) \in \omega : (x_1, x_2, 0) \in \Gamma_0\}$, $\psi(0) = 1$, $\psi(L) = 0$ and finally $\theta \in \mathcal{D}(\mathcal{O}_L)$, $\mathcal{O}_L = \{(x_1, x_2) \in \omega : (x_1, x_2, L) \in \Gamma_0\}$, $\psi(0) = 1$, $\psi(L) = 0$.
Collecting (6.8), (6.16), (6.17), (6.18), (6.19), the convergences (6.7) are proved. It remains to notice that the first line of (6.4) yields $v = u$ when $p > 2$ or $\gamma^{(p)} = +\infty$: introducing an additional state variable to account for the asymptotic behavior of the electric potential in the fibers is not necessary!

Proof of (6.13). By (1.3), (1.4), (3.3), and (6.2) there holds

$$\int_{\mathcal{O}} \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon n_\varepsilon dx = \int \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon d\mu_\varepsilon, \qquad \text{if } T_{r_\varepsilon} \cap \textstyle\bigcup_{z \in I_\varepsilon} \partial Y_\varepsilon^z \times (0, L) = \emptyset, \tag{6.20}$$

hence in this case there is nothing to prove. However, the equality (6.13) may fail to hold in the general case, because the border of some cells Y_ε^z can possibly intersect some of the sections of the fibers. To circumvent this difficulty, we introduce the operator $\langle\langle . \rangle\rangle_{1,\varepsilon}$ defined by (see (1.4))

$$\begin{aligned} \langle\langle\varphi\rangle\rangle_{1,\varepsilon} &:= \sum_{z \in I_\varepsilon} \left(\fint_{Y_\varepsilon^z} \varphi(s_1, s_2, x_3) ds_1 ds_2 \right) \mathbb{1}_{G_\varepsilon^z}(x_1, x_2), \\ G_\varepsilon^z &:= \left(Y_\varepsilon^z \cup \bigcup_{j \in J_\varepsilon^z} S_{r_\varepsilon}^j \right) \setminus \bigcup_{j \in J_\varepsilon \setminus J_\varepsilon^z} S_{r_\varepsilon}^j. \end{aligned} \tag{6.21}$$

We deduce from (3.3), (6.2) and (6.21) that

$$\begin{aligned} \int \langle\langle\varphi\rangle\rangle_{1,\varepsilon} \langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} d\mu_\varepsilon &= \sum_{j \in J_\varepsilon} \int_0^L \int_{S_{r_\varepsilon}^j} \langle\langle\varphi\rangle\rangle_{1,\varepsilon} \langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} \frac{\varepsilon^2}{r_\varepsilon^2 |S|} dx \\ &= \sum_{z \in I_\varepsilon} \sum_{j \in J_\varepsilon^z} \int_0^L \int_{S_{r_\varepsilon}^j} \langle\langle\varphi\rangle\rangle_{1,\varepsilon} \langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} \frac{\varepsilon^2}{r_\varepsilon^2 |S|} dx \\ &= \sum_{z \in I_\varepsilon} \sum_{j \in J_\varepsilon^z} \int_0^L \int_{S_{r_\varepsilon}^j} \left(\fint_{Y_\varepsilon^z} \varphi(s_1, s_2, x_3) ds_1 ds_2 \right) \left(\fint_{Y_\varepsilon^z} u_\varepsilon(s_1, s_2, x_3) ds_1 ds_2 \right) \frac{\varepsilon^2}{r_\varepsilon^2 |S|} dx \\ &= \sum_{z \in I_\varepsilon} \sum_{j \in J_\varepsilon^z} \int_0^L \int_{Y_\varepsilon^z} \left(\fint_{Y_\varepsilon^z} \varphi(s_1, s_2, x_3) ds_1 ds_2 \right) \left(\fint_{Y_\varepsilon^z} u_\varepsilon(s_1, s_2, x_3) ds_1 ds_2 \right) dx \\ &= \sum_{z \in I_\varepsilon} \int_0^L \int_{Y_\varepsilon^z} \left(\fint_{Y_\varepsilon^z} \varphi(s_1, s_2, x_3) ds_1 ds_2 \right) \left(\fint_{Y_\varepsilon^z} u_\varepsilon(s_1, s_2, x_3) ds_1 ds_2 \right) \sharp J_\varepsilon^z dx \\ &= \int_{\mathcal{O}} \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon n_\varepsilon dx. \end{aligned} \tag{6.22}$$

By (6.22), the proof of (6.13) is achieved provided we establish that

$$\lim_{\varepsilon \to 0} \left| \int_{\mathcal{O}} \left(\langle\langle\varphi\rangle\rangle_{1,\varepsilon} \langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle\varphi\rangle\rangle_\varepsilon \langle\langle u_\varepsilon\rangle\rangle_\varepsilon \right) d\mu_\varepsilon \right| = 0. \tag{6.23}$$

To that aim, we notice that since $\varphi \in \mathcal{D}(\mathcal{O})$, by (6.2) and (6.21) the following estimate holds true:

$$|\langle\langle\varphi\rangle\rangle_{1,\varepsilon} - \langle\langle\varphi\rangle\rangle_\varepsilon| \leq C\varepsilon. \tag{6.24}$$

We deduce that

$$\begin{aligned}
&\left|\int_{\mathcal{O}} \left(\langle\langle\varphi\rangle\rangle_{1,\varepsilon}\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle\varphi\rangle\rangle_{\varepsilon}\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}\right) d\mu_\varepsilon\right| \\
&\qquad \le \int_{\mathcal{O}} |\langle\langle\varphi\rangle\rangle_{1,\varepsilon}||\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon + \int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}||\langle\langle\varphi\rangle\rangle_{1,\varepsilon} - \langle\langle\varphi\rangle\rangle_{\varepsilon}|d\mu_\varepsilon \qquad (6.25)\\
&\qquad \le C\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon + C\varepsilon\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon.
\end{aligned}$$

We prove below that

$$\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon \le C\varepsilon, \tag{6.26}$$

$$\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon \le C. \tag{6.27}$$

Since Assertion (6.23) results from (6.25), (6.26), and (6.27), Assertion (6.13) is proved. □

Proof of (6.26). By (1.4), (3.3), there holds

$$\int_{\mathcal{O}} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|d\mu_\varepsilon = \sum_{z\in I_\varepsilon}\sum_{j\in J_\varepsilon^z} \frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_0^L\int_{S_{r_\varepsilon}^j} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|dx. \tag{6.28}$$

By (6.21), the function $\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon}$ takes constant values on each set $S_{r_\varepsilon}^j\times\{x_3\}$, whereas the function $\langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}$, defined by (6.2), may take up to four different values on $S_{r_\varepsilon}^j\times\{x_3\}$ if $S_{r_\varepsilon}^j\cap\partial Y_\varepsilon^z\neq\emptyset$ and $j\in J_\varepsilon^z$.
For each $z\in I_\varepsilon$, we denote by Z_ε^z the union of the cells $Y_\varepsilon^{z'}$ whose adherence has a non empty intersection with $\overline{Y_\varepsilon^z}$. The set Z_ε^z is the subset of $\varepsilon(z+[-1,2[^2)$ defined by

$$Z_\varepsilon^z := \bigcup_{k\in A_z} Y_\varepsilon^{z+k}, \quad A_z := (\mathbb{Z}^2\cap[-1,1]^2)\cap\left\{k\in\mathbb{Z}^2, z+k\in I_\varepsilon\right\}. \tag{6.29}$$

Let us fix $z\in I_\varepsilon$. For each $j\in J_\varepsilon^z$ and for a.e. $x_3\in(0,L)$, we have

$$\begin{aligned}
&\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S_{r_\varepsilon}^j} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_{\varepsilon}|(x_1,x_2,x_3)dx_1dx_2 \\
&= \sum_{k\in A_z}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S_{r_\varepsilon}^j\cap Y_\varepsilon^{z+k}}\left|\left(\fint_{Y_\varepsilon^z} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right) - \left(\fint_{Y_\varepsilon^{z+k}} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right)\right|dx_1dx_2 \\
&\le \sum_{k\in A_z}\varepsilon^2\left|\left(\fint_{Y_\varepsilon^z} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right) - \left(\fint_{Y_\varepsilon^{z+k}} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right)\right| \\
&\le C\sum_{k\in A_z}\int_{Z_\varepsilon^z}\left|\left(\fint_{Y_\varepsilon^z} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right) - \left(\fint_{Y_\varepsilon^{z+k}} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right)\right|dx_1dx_2.
\end{aligned}$$

Noticing that $\#A_z \leq 9$, we infer

$$
\begin{aligned}
&\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S^j_{r_\varepsilon}} |\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_\varepsilon|(x_1,x_2,x_3)dx_1dx_2 \\
&\leq C\sum_{k\in A_z}\int_{Z^z_\varepsilon}\left|u_\varepsilon(x_1,x_2,x_3) - \left(\fint_{Y^z_\varepsilon} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right)\right| \\
&\qquad\qquad + \left|u_\varepsilon(x_1,x_2,x_3) - \left(\fint_{Y^{z+k}_\varepsilon} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right)\right| dx_1dx_2 \qquad (6.30)\\
&\leq C\sum_{k\in A_z}\int_{Z^z_\varepsilon}\left|u_\varepsilon(x_1,x_2,x_3) - \left(\fint_{Y^{z+k}_\varepsilon} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right)\right| dx_1dx_2 \\
&\leq \sum_{k\in A_z} C_k\varepsilon\int_{Z^z_\varepsilon}\left|\widehat{\nabla}u_\varepsilon(x_1,x_2,x_3)\right| dx_1dx_2 \leq C\varepsilon\int_{Z^z_\varepsilon}\left|\widehat{\nabla}u_\varepsilon(x_1,x_2,x_3)\right| dx_1dx_2.
\end{aligned}
$$

The next to last inequality in (6.30) is deduced from a change of variables in Poincaré-Wirtinger inequality $\int_{]-1,2[^2}\left|\varphi - \fint_{k+]0,1[^2}\varphi ds_1ds_2\right| dx_1dx_2 \leq C_k\int_{]-1,2[^2}|\nabla\varphi|dx_1dx_2$ in $W^{1,1}(]-1,2[^2)$. Noticing that by (1.4) and (1.5) there holds $\sharp J^z_\varepsilon \leq N$, we deduce from (6.28), and (6.30) that

$$
\int_{\mathcal{O}}|\langle\langle u_\varepsilon\rangle\rangle_{1,\varepsilon} - \langle\langle u_\varepsilon\rangle\rangle_\varepsilon|d\mu_\varepsilon \leq C\varepsilon\sum_{z\in I_\varepsilon}\sharp J^z_\varepsilon\int_0^L\int_{Z^z_\varepsilon}\left|\widehat{\nabla}u_\varepsilon\right| dx \leq C\varepsilon\sum_{z\in I_\varepsilon}\int_0^L\int_{Z^z_\varepsilon}\left|\widehat{\nabla}u_\varepsilon\right| dx. \qquad (6.31)
$$

By (6.29), each set Y^z_ε is included in at most 9 distinct sets $Z^{z'}_\varepsilon$, therefore by (6.6) we have

$$
\begin{aligned}
\varepsilon\sum_{z\in I_\varepsilon}\int_0^L\int_{Z^z_\varepsilon}\left|\widehat{\nabla}u_\varepsilon\right| dx &\leq 9\varepsilon\sum_{z\in I_\varepsilon}\int_0^L\int_{Y^z_\varepsilon}\left|\widehat{\nabla}u_\varepsilon\right| dx \\
&\leq 9\varepsilon\int_{\mathcal{O}}\left|\widehat{\nabla}u_\varepsilon\right| dx \leq C\varepsilon\left(\int_{\mathcal{O}}\left|\widehat{\nabla}u_\varepsilon\right|^p dx\right)^{\frac{1}{p}} \leq C\varepsilon.
\end{aligned} \qquad (6.32)
$$

The estimate (6.26) follows from (6.31) and (6.32). □

Proof of (6.27). By (3.3), (6.2), (6.6), and (6.29), there holds

$$
\begin{aligned}
\int_{\mathcal{O}}|\langle\langle u_\varepsilon\rangle\rangle_\varepsilon|d\mu_\varepsilon &= \sum_{z\in I_\varepsilon}\sum_{j\in J^z_\varepsilon}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_0^L\int_{S^j_{r_\varepsilon}}|\langle\langle u_\varepsilon\rangle\rangle_\varepsilon|dx \\
&= \sum_{z\in I_\varepsilon}\sum_{j\in J^z_\varepsilon}\sum_{k\in A_z}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_0^L\int_{S^j_{r_\varepsilon}\cap Y^{z+k}_\varepsilon}\left|\fint_{Y^{z+k}_\varepsilon} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right| dx \\
&\leq C\sum_{z\in I_\varepsilon}\sum_{k\in A_z}\int_0^L\left|\int_{Y^{z+k}_\varepsilon} u_\varepsilon(s_1,s_2,x_3)ds_1ds_2\right| dx_3 \leq C\int_{\mathcal{O}}|u_\varepsilon|dx \leq C,
\end{aligned}
$$

because $\sharp A_z$ and $\sharp J^z_\varepsilon$ are uniformely bounded. Assertion (6.27) is proved. □

Proof of Lemma 6.1. By (3.3) and (6.2), we have

$$
\begin{aligned}
\int|\langle\varphi\rangle_{R_\varepsilon} - \langle\varphi\rangle_{r_\varepsilon}|^p d\mu_\varepsilon &= \sum_{j\in J_\varepsilon}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_0^L\int_{S^j_{r_\varepsilon}}|\langle\varphi\rangle_{R_\varepsilon} - \langle\varphi\rangle_{r_\varepsilon}|^p dx \\
&= \sum_{j\in J_\varepsilon}\varepsilon^2\int_0^L\left|\langle\varphi\rangle^j_{R_\varepsilon}(x_3) - \langle\varphi\rangle^j_{r_\varepsilon}(x_3)\right|^p dx_3 \qquad (6.33)\\
&= \sum_{j\in J_\varepsilon}\frac{\varepsilon^2}{R_\varepsilon^2|D|}\int_0^L\int_{D^j_{R_\varepsilon}}|\langle\varphi\rangle_{R_\varepsilon} - \langle\varphi\rangle_{r_\varepsilon}|^p dx,
\end{aligned}
$$

because $\langle\varphi\rangle^j_{r_\varepsilon}(x_3)$ and $\langle\varphi\rangle^j_{R_\varepsilon}(x_3)$ take *constant* values on $D^j_{R_\varepsilon}\times\{x_3\}$. The next inequality is proven in [7, Lemma A4]:

$$\begin{aligned}&\forall(R,\alpha)\in\mathbb{R}_+\times(0,1],\quad \forall\psi\in W^{1,p}(D_R),\quad \int_{D_R}\left|\psi-\fint_{\partial D_{\alpha R}}\psi\,d\mathcal{H}^1\right|^p dx\le C\,\frac{R^p}{h(\alpha)}\int_{D_R}|\nabla\psi|^p dx,\\ &h(\alpha)=\alpha^{2-p}\ \text{ if }\ 1<p<2,\quad h(\alpha)=\frac{1}{1+|\log\alpha|}\ \text{ if }\ p=2,\quad h(\alpha)=1\ \text{ if }\ p>2.\end{aligned}\tag{6.34}$$

By (6.34) there holds, for a.e. $x_3\in(0,L)$ and all $j\in J_\varepsilon$,

$$\begin{aligned}\int_{D^j_{R_\varepsilon}}|\langle\varphi\rangle_{R_\varepsilon}-\langle\varphi\rangle_{r_\varepsilon}|^p(x_1,x_2,x_3)dx_1dx_2&\le C\int_{D^j_{R_\varepsilon}}|\varphi-\langle\varphi\rangle_{R_\varepsilon}|^p(x_1,x_2,x_3)dx_1dx_2\\&\quad+C\int_{D^j_{R_\varepsilon}}|\varphi-\langle\varphi\rangle_{r_\varepsilon}|^p(x_1,x_2,x_3)dx_1dx_2\\&\le\begin{cases}C\frac{R^p_\varepsilon}{h\left(\frac{r_\varepsilon}{R_\varepsilon}\right)}\int_{D^j_{R_\varepsilon}}\left|\widehat{\nabla}\varphi\right|^p(x_1,x_2,x_3)dx_1dx_2, & \text{if } p\le 2,\\ CR^p_\varepsilon\int_{D^j_{R_\varepsilon}}\left|\widehat{\nabla}\varphi\right|^p(x_1,x_2,x_3)dx_1dx_2, & \text{if } p>2.\end{cases}\end{aligned}\tag{6.35}$$

The first line of (6.4) follows from (3.1), (6.33), and (6.35). Similarly, denoting by $\langle\langle\varphi\rangle\rangle^z_\varepsilon(x_3)$ the constant value taken by $\langle\langle\varphi\rangle\rangle_\varepsilon$ in $Y^z_\varepsilon\times\{x_3\}$, we get (see (6.2))

$$\begin{aligned}\int|\langle\langle\varphi\rangle\rangle_\varepsilon-\langle\varphi\rangle_{r_\varepsilon}|^p\,d\mu_\varepsilon&=\sum_{z\in I_\varepsilon}\sum_{j\in J^z_\varepsilon}\int_0^L\left|\langle\langle\varphi\rangle\rangle^z_\varepsilon(x_3)-\langle\varphi\rangle^j_{r_\varepsilon}(x_3)\right|^p\varepsilon^2dx_3\\&=\sum_{z\in I_\varepsilon}\sum_{j\in J^z_\varepsilon}\int_0^L\left|\fint_{Y^z_\varepsilon}(\varphi(x_1,x_2,x_3)-\langle\varphi\rangle^j_{r_\varepsilon}(x_3))dx_1dx_2\right|^p\varepsilon^2dx_3\\&\le C\sum_{z\in I_\varepsilon}\sum_{j\in J^z_\varepsilon}\int_0^L\int_{Y^z_\varepsilon}\left|\varphi(x_1,x_2,x_3)-\langle\varphi\rangle^j_{r_\varepsilon}(x_3)\right|^p dx.\end{aligned}\tag{6.36}$$

Noticing that by (1.4) and (1.6), we have

$$Y^z_\varepsilon\subset D^j_{\sqrt{2}\varepsilon}\subset Q^z_\varepsilon\subset\widehat{\mathcal{O}}\quad\forall z\in I_\varepsilon,\ \forall j\in J^z_\varepsilon,\qquad Q^z_\varepsilon:=\varepsilon(z+5Y),\tag{6.37}$$

we infer from (6.2), (6.34) that for a.e. $x_3\in(0,L)$, all $z\in I_\varepsilon$ and all $j\in J^z_\varepsilon$, there holds

$$\begin{aligned}\int_{Y^z_\varepsilon}\left|\varphi-\langle\varphi\rangle^j_{r_\varepsilon}\right|^p(s_1,s_2,x_3)ds_1ds_2&\le\int_{D^j_{\sqrt{2}\varepsilon}}\left|\varphi-\langle\varphi\rangle^j_{r_\varepsilon}\right|^p(s_1,s_2,x_3)ds_1ds_2\\&\le C\frac{\varepsilon^p}{h\left(\frac{r_\varepsilon}{\varepsilon}\right)}\int_{D^j_{\sqrt{2}\varepsilon}}\left|\widehat{\nabla}\varphi\right|^p(s_1,s_2,x_3)ds_1ds_2\\&\le C\frac{\varepsilon^p}{h\left(\frac{r_\varepsilon}{\varepsilon}\right)}\int_{Q^z_\varepsilon}\left|\widehat{\nabla}\varphi\right|^p(s_1,s_2,x_3)ds_1ds_2.\end{aligned}\tag{6.38}$$

By (1.5) we have

$$\sharp J^z_\varepsilon\le N\quad\forall z\in I_\varepsilon.\tag{6.39}$$

By (6.37) and (6.39), there holds

$$\sum_{\{z\in I_\varepsilon,\ J^z_\varepsilon\neq\emptyset\}}\int_0^L\int_{Q^z_\varepsilon}\left|\widehat{\nabla}\varphi\right|^p dx\le 25\int_{\mathcal{O}}\left|\widehat{\nabla}\varphi\right|^p dx.$$

We deduce from (6.36) and (6.38) that

$$\int |\langle\langle\varphi\rangle\rangle_\varepsilon - \langle\varphi\rangle_{r_\varepsilon}|^p \, d\mu_\varepsilon \leq C \frac{\varepsilon^p}{h\left(\frac{r_\varepsilon}{\varepsilon}\right)} \sum_{z\in I_\varepsilon} \int_0^L \int_{Q_\varepsilon^z} \left|\widehat{\nabla}\varphi\right|^p dx \leq \frac{C}{\gamma_\varepsilon^{(p)}(r_\varepsilon)} \int_{\mathcal{O}} |\widehat{\nabla}\varphi|^p dx,$$

hence the second line of (6.4) is proved. The third one is obtained in the same way and the fourth one is straightforward. The fifth one is easily derived by choosing $(R, \alpha) = (r_\varepsilon, 1)$ in (6.34). □

7 Proof of the main result

The demonstration of Theorem 3.1 is based on the Γ-convergence method (for precise details about this method, we refer the reader to [4, 5, 18]). The "lowerbound" and the "upperbound" stated respectively in Proposition 7.1 and Proposition 7.2, indicate in particular that the sequence of functionals (F_ε) Γ-converges with respect to the strong topology of $L^p(\mathcal{O})$ to the functional F^{hom} defined by (3.9). The proof of Theorem 3.1 is deduced in the following manner from the two last mentioned propositions and from the a priori estimates established in Proposition 6.1:

7.1 Proof of Theorem 3.1.

We will only prove Theorem (3.1) in the most interesting case

$$\gamma^{(p)} > 0. \tag{7.1}$$

Let (u_ε) be the sequence of the solutions to (1.1). By (1.11), and since u_0 is continuous on $\overline{\mathcal{O}}$ (see (1.1)), there holds $F_\varepsilon(u_\varepsilon) - \int_{\mathcal{O}} q_b u_\varepsilon dx - \int_{\Gamma_1} q_s u_\varepsilon d\mathcal{H}^2 \leq F_\varepsilon(u_0) - \int_{\mathcal{O}} q_b u_0 dx - \int_{\Gamma_1} q_s u_0 d\mathcal{H}^2 \leq C$. As $\left|\int_{\mathcal{O}} q_b u_\varepsilon dx + \int_{\Gamma_1} q_s u_\varepsilon d\mathcal{H}^2\right|^p \leq C F_\varepsilon(u_\varepsilon)$, we deduce that (u_ε) satisfies (6.1). Therefore, we can apply Proposition 6.1 and, after possibly extracting a subsequence, assume that (u_ε) converges weakly in $W^{1,p}(\mathcal{O})$ to some u, and that the sequence $(u_\varepsilon \mu_\varepsilon)$ weak $*$ converges in $\mathcal{M}(\overline{\mathcal{O}})$ to $v\mathcal{L}^3_{\lfloor\mathcal{O}}$ for some $v \in V_p$. We just have to prove that (u, v) is the solution to (3.8). To that aim, we first apply Proposition 7.1, to get

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) - \int_{\mathcal{O}} q_b u_\varepsilon dx - \int_{\Gamma_1} q_s u_\varepsilon d\mathcal{H}^2 \geq \Phi(u, v) - \int_{\mathcal{O}} q_b u dx - \int_{\Gamma_1} q_s u d\mathcal{H}^2. \tag{7.2}$$

By Proposition 7.2, there exists a sequence (φ_ε) such that,

$$\varphi_\varepsilon \rightharpoonup u \quad \text{strongly in } L^p(\mathcal{O}), \quad \varphi_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} vn\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak } * \text{ in } \mathcal{M}(\overline{\mathcal{O}}), \quad \limsup_{\varepsilon\to 0} F_\varepsilon(\varphi_\varepsilon) \leq \Phi(u, v). \tag{7.3}$$

Since u_ε is the solution to (1.1), there holds

$$F_\varepsilon(u_\varepsilon) - \int_{\mathcal{O}} q_b u_\varepsilon dx - \int_{\Gamma_1} q_s u_\varepsilon d\mathcal{H}^2 \leq F_\varepsilon(\varphi_\varepsilon) - \int_{\mathcal{O}} q_b \varphi_\varepsilon dx - \int_{\Gamma_1} q_s \varphi_\varepsilon d\mathcal{H}^2. \tag{7.4}$$

We infer from (7.2), (7.3), (7.4) and from the weak continuity on $W^{1,p}(\mathcal{O})$ of the linear form $\varphi \to \int_{\mathcal{O}} q_b \varphi dx - \int_{\Gamma_1} q_s \varphi d\mathcal{H}^2$ that

$$\Phi(u, v) - \int_{\mathcal{O}} q_b u dx - \int_{\Gamma_1} q_s u d\mathcal{H}^2 \leq \min(\mathcal{P}^{hom}),$$

hence (u, v) is the unique solution to $(\mathcal{P}^{hom})$ (the uniqueness results from the strict convexity of f and g). □

7.2 Lower bound

Proposition 7.1. *Under the assumptions of Theorem 3.1, for all* $(u,v) \in (u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})) \times V_p$ *and for all sequence* (u_ε) *in* $u_0 + W^{1,p}_{\Gamma_0}(\mathcal{O})$ *which weakly converges in* $W^{1,p}(\mathcal{O})$ *toward* u *and such that* $(u_\varepsilon \mu_\varepsilon)$ *weak* $*$ *converges in* $\mathcal{M}(\overline{\mathcal{O}})$ *to* $vn\mathcal{L}^3_{\lfloor \mathcal{O}}$*, we have*

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u,v). \tag{7.5}$$

Proof. We can suppose that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty$, otherwise there is nothing to prove. Accordingly, after possibly extracting a subsequence, we can assume that (6.1) is verified and that the estimates (6.6) and the convergences (6.7) established in Proposition 6.1 take place. We choose a suitable sequence (R_ε) of positive reals satisfying (1.7) (the choice of (R_ε) will be made more precise in Lemma 8.2), and establish (see below) that

$$\liminf_{\varepsilon \to 0} \int_{\mathcal{O}\setminus(D_{R_\varepsilon}\times(0,L))} f(\nabla u_\varepsilon)\ dx \geq \int_{\mathcal{O}} f(\nabla u)\ dx, \tag{7.6}$$

$$\liminf_{\varepsilon \to 0} \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon)\ dx \geq \bar{k} \int_{\mathcal{O}} g^{hom}(\partial_3 v)\ ndx, \tag{7.7}$$

$$\liminf_{\varepsilon \to 0} \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)\ dx \geq \int_{\mathcal{O}} c^f(S; v-u)\ ndx, \tag{7.8}$$

where g^{hom} and c^f are defined by (3.5) and (3.6). Collecting (7.6), (7.7), (7.8) we obtain (7.5) which, joined with (6.7), achieves the proof of Proposition 7.1. □

Proof of (7.6). By (1.7) and (6.3) we have $|D_{R_\varepsilon} \times (0,L)| \to 0$, hence the sequence $\left(\mathbb{1}_{\mathcal{O}\setminus(D_{R_\varepsilon}\times(0,L))}\nabla u_\varepsilon\right)$ weakly converges in $L^p(\mathcal{O};\mathbb{R}^3)$ toward ∇u. Assertion (7.6) then follows from the weak lower semi-continuity in $L^p(\mathcal{O};\mathbb{R}^3)$ of the functional $q \mapsto \int_{\mathcal{O}} f(q)\ dx$. □

Proof of (7.7). If $\bar{k} < +\infty$, by (1.10), (3.5), (6.7) and Lemma 6.2, we have:

$$\begin{aligned}\liminf_{\varepsilon \to 0} \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon)\ dx &\geq \liminf_{\varepsilon \to 0} \lambda_\varepsilon \frac{r_\varepsilon^2 |S|}{\varepsilon^2} \int g^{hom}(\partial_3 u_\varepsilon)\ d\mu_\varepsilon \\ &\geq \bar{k} \int_{\mathcal{O}} g^{hom}(\partial_3 v)\ ndx.\end{aligned}$$

Otherwise, if $\bar{k} = +\infty$, it is enough to notice that $\lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon)\ dx$ is bounded from below by 0. □

Proof of (7.8). If $\gamma^{(p)} = +\infty$ (in particular if $p > 2$), there is nothing to prove because then, by Proposition 6.1, $v = u$. From now on, we assume that $0 < \gamma^{(p)} < +\infty$ (hence $p \leq 2$). First, we show (see Lemma 8.1) that there exists an approximation $(\widehat{u}_\varepsilon)$ of u_ε piecewise constant in x_3 satisfying

$$\begin{aligned}&\lambda_\varepsilon \int_{T_{r_\varepsilon}} |\widehat{\nabla}\widehat{u}_\varepsilon|^p\ dx \leq \lambda_\varepsilon \int_{T_{r_\varepsilon}} |\nabla u_\varepsilon|^p\ dx, \quad \int_{\mathcal{O}} |\widehat{\nabla}\widehat{u}_\varepsilon|^p\ dx \leq \int_{\mathcal{O}} |\nabla u_\varepsilon|^p\ dx, \\ &\int |\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle \widehat{u}_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \leq C, \\ &\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon}\mu_\varepsilon \overset{\star}{\rightharpoonup} nv\mathcal{L}^3_{\lfloor \mathcal{O}}, \qquad \langle \widehat{u}_\varepsilon\rangle_{R_\varepsilon}\mu_\varepsilon \overset{\star}{\rightharpoonup} nu\mathcal{L}^3_{\lfloor \mathcal{O}} \quad \text{weak } * \text{ in} \quad \mathcal{M}(\overline{\mathcal{O}}), \\ &\liminf_{\varepsilon \to 0} \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)\ dx \geq \liminf_{\varepsilon \to 0} \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\ dx.\end{aligned} \tag{7.9}$$

Next, we fix a positive real δ satisfying

$$1 < \delta < 2, \tag{7.10}$$

and define the set $S_{r_\varepsilon}^{-r_\varepsilon^\delta}$ by setting $(U,\alpha) = (S_{r_\varepsilon}, r_\varepsilon^\delta)$ in

$$\begin{aligned}U^{-\alpha} &:= \{(x_1,x_2) \in U, \quad \operatorname{dist}((x_1,x_2),\partial U) > \alpha\}, \\ U^{+\alpha} &:= \left\{(x_1,x_2) \in \mathbb{R}^2, \quad \operatorname{dist}((x_1,x_2),\partial U) < \alpha\right\} \cup U.\end{aligned} \tag{7.11}$$

Notice that by (1.8) we have, for ε small enough (see (6.3)),

$$D^j_{r_\varepsilon} \subset S^{j,-r^\delta_\varepsilon}_{r_\varepsilon} \quad \forall\, j \in J_\varepsilon. \tag{7.12}$$

We prove (see Lemma 8.3) that for a suitable choice of the sequence (R_ε) satisfying (1.7) (the choice of this sequence is determined by Lemma 8.2), there exists an approximation $\widehat{u}_\varepsilon$ of $\widehat{\widehat{u}}_\varepsilon$ verifying

$$\begin{aligned} &\widehat{u}_\varepsilon = \langle \widehat{u}_\varepsilon \rangle_{r_\varepsilon} = \langle \widehat{\widehat{u}}_\varepsilon \rangle_{r_\varepsilon} \quad \text{on} \quad S^{-r^\delta_\varepsilon}_{r_\varepsilon} \times (0,L), \qquad \widehat{u}_\varepsilon = \langle \widehat{u}_\varepsilon \rangle_{R_\varepsilon} = \langle \widehat{\widehat{u}}_\varepsilon \rangle_{R_\varepsilon} \quad \text{on} \quad \partial D_{R_\varepsilon} \times (0,L), \\ &\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\, dx \geq \int_{(D_{R_\varepsilon}\times(0,L))\setminus(S^{-r^\delta_\varepsilon}_{r_\varepsilon}\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\, dx + o(1). \end{aligned} \tag{7.13}$$

The properties of $\widehat{u}_\varepsilon$ allow us to make good use of the capacitary problem (2.5). More precisely, by (6.2) and (7.13), for each $(j, x_3) \in J_\varepsilon \times (0,L)$, the function $\widehat{u}_\varepsilon$ takes the constant value $\langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{R_\varepsilon}(x_3)$ on $\partial D^j_{R_\varepsilon} \times \{x_3\}$ and the constant value $\langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{r_\varepsilon}(x_3)$ on $\partial S^{j,-r^\delta_\varepsilon}_{r_\varepsilon} \times \{x_3\}$ (see (7.11)). Therefore there holds, for all $j \in J_\varepsilon$ and for a.e. $x_3 \in (0,L)$ (see (2.5))

$$\int_{D^j_{R_\varepsilon}\setminus S^{j,-r^\delta_\varepsilon}_{r_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)(x)\, dx_1 dx_2 \geq \mathrm{cap}^{f^{\infty,p}}\big(S^{j,-r^\delta_\varepsilon}_{r_\varepsilon}, D^j_{R_\varepsilon}; \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{r_\varepsilon}(x_3) - \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{R_\varepsilon}(x_3)\big).$$

We deduce that

$$\begin{aligned} \int_{(D_{R_\varepsilon}\times(0,L))\setminus S^{-r^\delta_\varepsilon}_{r_\varepsilon}\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\, dx &\geq \int_0^L \Big(\sum_{j\in J_\varepsilon} \int_{D^j_{R_\varepsilon}\setminus S^{j,-r^\delta_\varepsilon}_{r_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\, dx_1 dx_2 \Big)\, dx_3 \\ &\geq \int_0^L \Big(\sum_{j\in J_\varepsilon} \mathrm{cap}^{f^{\infty,p}}\big(S^{j,-r^\delta_\varepsilon}_{r_\varepsilon}, D^j_{R_\varepsilon}; \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{r_\varepsilon}(x_3) - \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{R_\varepsilon}(x_3)\big)\Big)\, dx_3. \end{aligned} \tag{7.14}$$

Because $f^{\infty,p}$ is positively homogeneous of degree p, we can apply (5.19) and, for each $(j, x_3) \in J_\varepsilon \times (0,L)$, obtain (see (2.5), (7.11))

$$\begin{aligned} \mathrm{cap}^{f^{\infty,p}}\big(S^{j,-r^\delta_\varepsilon}_{r_\varepsilon}, D^j_{R_\varepsilon}; \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{r_\varepsilon} - \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{R_\varepsilon}\big) &= r_\varepsilon^{2-p} \mathrm{cap}^{f^{\infty,p}}\big(S^{-r^{\delta-1}_\varepsilon}, (R_\varepsilon/r_\varepsilon)D; \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{r_\varepsilon} - \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{R_\varepsilon}\big) \\ &= \frac{r_\varepsilon^{2-p}}{\varepsilon^2} \frac{\varepsilon^2}{r_\varepsilon^2|S|} \int_{S^j_{r_\varepsilon}} \mathrm{cap}^{f^{\infty,p}}\big(S^{-r^{\delta-1}_\varepsilon}, (R_\varepsilon/r_\varepsilon)D; \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{r_\varepsilon} - \langle \widehat{\widehat{u}}_\varepsilon \rangle^j_{R_\varepsilon}\big) dx_1 dx_2. \end{aligned} \tag{7.15}$$

Let us fix a bounded Lipschitz domain S' such that

$$\overline{S}' \subset S. \tag{7.16}$$

For small ε's, there holds $\overline{S}' \subset S^{-r^{\delta-1}_\varepsilon}$, therefore by (3.3), (5.13), (7.14), and (7.15) we have

$$\int_{(D_{R_\varepsilon}\times(0,L))\setminus S^{-r^\delta_\varepsilon}_{r_\varepsilon}\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\, dx \geq \frac{r_\varepsilon^{2-p}}{\varepsilon^2} \int \mathrm{cap}^{f^{\infty,p}}\big(S', (R_\varepsilon/r_\varepsilon)D; \langle \widehat{\widehat{u}}_\varepsilon \rangle_{r_\varepsilon} - \langle \widehat{\widehat{u}}_\varepsilon \rangle_{R_\varepsilon}\big) d\mu_\varepsilon. \tag{7.17}$$

We then distinguish two cases.

Case $p < 2$**.** Collecting (3.1), (5.12), (7.9), (7.13), and (7.17), we deduce that

$$\liminf_{\varepsilon\to 0} \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)\, dx \geq \gamma^{(p)} \liminf_{\varepsilon\to 0} \int_{\mathcal{O}} \mathrm{cap}^{f^{\infty,p}}\big(S', \mathbb{R}^2; \langle \widehat{\widehat{u}}_\varepsilon \rangle_{r_\varepsilon} - \langle \widehat{\widehat{u}}_\varepsilon \rangle_{R_\varepsilon}\big)\, d\mu_\varepsilon.$$

By applying Lemma 6.2 (ii) to the convex function $j(.) = \mathrm{cap}^{f^{\infty,p}}\big(S, \mathbb{R}^2; \cdot\big)$ which, for $p < 2$, has a growth of order p (see Lemma 5.3 and (5.19), (5.21)), taking (6.7) and (6.11) into account, we infer

$$\liminf_{\varepsilon\to 0} \int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)\, dx \geq \gamma^{(p)} \int_{\mathcal{O}} \mathrm{cap}^{f^{\infty,p}}\big(S', \mathbb{R}^2; v - u\big)\, n dx, \tag{7.18}$$

for all Lipschitz domain S' satisfying (7.16). Fixing an increasing sequence $(S_n)_{n\in\mathbb{N}}$ of Lipschitz domains such that $\overline{S}_n \subset S$ and $\bigcup_{n\in\mathbb{N}} S_n = S$, substituting S_n for S' in (7.18) and passing to the limit as $n \to +\infty$, thanks to (5.13), (5.18) and to the Monotone Convergence Theorem, we get (7.8).
Case $p = 2$**.** We fix two positive reals r, R such that

$$r\overline{D} \subset S \subset RD, \tag{7.19}$$

and specify the choice of S' by setting

$$S' := rD. \tag{7.20}$$

By (8.19) we have $\frac{R}{r}R_\varepsilon \le R'_\varepsilon$, therefore we infer from (5.12), (5.13), (5.19), and (7.19) that

$$\begin{aligned}\mathrm{cap}^{f^{\infty,2}}\left(r_\varepsilon rD, R_\varepsilon D; \pm 1\right) &= \mathrm{cap}^{f^{\infty,2}}\left(r_\varepsilon RD, R_\varepsilon \frac{R}{r}D; \pm 1\right) \ge \mathrm{cap}^{f^{\infty,2}}\left(r_\varepsilon S, R_\varepsilon \frac{R}{r}D; \pm 1\right)\\ &\ge \mathrm{cap}^{f^{\infty,2}}\left(r_\varepsilon S, R'_\varepsilon D; \pm 1\right).\end{aligned} \tag{7.21}$$

We deduce from (3.6), (3.7), (5.36), and (7.21) that

$$\liminf_{\varepsilon\to 0} \frac{\mathrm{cap}^{f^{\infty,2}}\left(r_\varepsilon rD, R_\varepsilon D; \pm 1\right)}{\varepsilon^2} \ge \gamma^{(2)} c^{f^{\infty,2}}(\pm 1). \tag{7.22}$$

By (5.19) and (7.17) , there holds

$$\begin{aligned}\int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\, dx \ge &\frac{\mathrm{cap}^{f^{\infty,2}}\left(r_\varepsilon rD, R_\varepsilon D; 1\right)}{\varepsilon^2}\int |(\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon} - \langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon})^+|^2 d\mu_\varepsilon\\ &+ \frac{\mathrm{cap}^{f^{\infty,2}}\left(r_\varepsilon rD, R_\varepsilon D; -1\right)}{\varepsilon^2}\int |(\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon} - \langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon})^-|^2 d\mu_\varepsilon.\end{aligned} \tag{7.23}$$

Thanks to (6.7), (6.11), and (7.22), by passing to the limit inferior in (7.23), taking (3.6), (3.7) and the lower semi-continuity property (6.5) into account, we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0} &\int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\, dx\\ &\ge \gamma^{(2)}\int_{\mathcal{O}} \left(c^{f^{\infty,2}}(1)|(v-u)^+|^2 + c^{f^{\infty,2}}(-1)|(v-u)^-|^2\right) n dx\\ &= \int_{\mathcal{O}} c^f(S; v-u) n dx.\end{aligned} \tag{7.24}$$

Joining (7.9), (7.13), (7.24), Assertion (7.8) is proved. □

7.3 Upper bound

Proposition 7.2. *Under the assumptions of Theorem 3.1, for all $(u, v) \in W^{1,p}_{\Gamma_0}(\mathcal{O}) \times V_p$, there exists a sequence (u_ε) such that*

$$\begin{aligned}&u_\varepsilon \rightharpoonup u \ \text{ strongly in } L^p(\mathcal{O}), \quad u_\varepsilon\mu_\varepsilon \overset{\star}{\rightharpoonup} vn\mathcal{L}^3_{\lfloor\mathcal{O}} \ \text{ weak} * \text{ in } \ \mathcal{M}(\overline{\mathcal{O}}),\\ &\limsup_{\varepsilon\to 0}\ F_\varepsilon(u_\varepsilon) \le \Phi(u, v).\end{aligned} \tag{7.25}$$

Proof. By density and diagonalization arguments, (see [7, pp. 424–429] for more details), we are reduced to prove that for all $(u, v) \in (C^1(\overline{\mathcal{O}}))^2$ such that

$$\Phi(u, v) < +\infty, \tag{7.26}$$

there exists a sequence (u_ε) in $W^{1,p}(\mathcal{O})$ (thanks to the truncature argument employed in [7, p. 428], we can forget the boundary constraint on Γ_0) such that

$$\begin{aligned} &u_\varepsilon \rightharpoonup u \ \text{ strongly in } L^p(\mathcal{O}), \quad u_\varepsilon \mu_\varepsilon \overset{\star}{\rightharpoonup} vn\mathcal{L}^3_{\lfloor \mathcal{O}} \ \text{ weak} * \text{ in } \ \mathcal{M}(\overline{\mathcal{O}}), \\ &\limsup_{\varepsilon\to 0} \int_{\mathcal{O}\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)dx + \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon)dx \le \Phi(u,v). \end{aligned} \tag{7.27}$$

Accordingly, let us fix $(u,v) \in (C^1(\overline{\mathcal{O}}))^2$ satisfying (7.26). By (1.9), (3.5) and the strict convexity of g, there exists a unique field $\varphi \in C(\overline{\mathcal{O}};\mathbb{R}^2)$ such that

$$g(\varphi_1(x), \varphi_2(x), \partial_3 v(x)) = g^{hom}(\partial_3 v(x)) \quad \forall x \in \overline{\mathcal{O}}. \tag{7.28}$$

We fix *any* sequence (R_ε) satisfying (1.7). We denote by $\theta_\varepsilon : \widehat{\mathcal{O}} \to \mathbb{R}$ the unique solution to the problem

$$\min\left\{ \int_{\widehat{\mathcal{O}}} f^{\infty,p}(\widehat{\nabla}\theta(x_1,x_2))\ dx_1dx_2, \quad \theta \in W^{1,p}(\widehat{\mathcal{O}}),\ \theta = 1 \text{ in } S_{r_\varepsilon}, \quad \theta = 0 \ \text{ in } \widehat{\mathcal{O}} \setminus D_{R_\varepsilon} \right\}.$$

Since $f^{\infty,p}$ is positively homogeneous of degree p, by (2.5) and (5.19) there holds, for all $j \in J_\varepsilon$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} \int_{D^j_{R_\varepsilon}} f^{\infty,p}(\alpha\widehat{\nabla}\theta_\varepsilon)\ dx_1dx_2 &= \mathrm{cap}^{f^{\infty,p}}\left(S^j_{r_\varepsilon}, D^j_{R_\varepsilon};\alpha\right) = \mathrm{cap}^{f^{\infty,p}}\left(r_\varepsilon S, R_\varepsilon D;\alpha\right) \\ &= r_\varepsilon^{2-p}\mathrm{cap}^{f^{\infty,p}}\left(S, R_\varepsilon/r_\varepsilon D; \mathrm{sgn}(\alpha)\right)|\alpha|^p. \end{aligned} \tag{7.29}$$

We set

$$u_\varepsilon(x) = \theta_\varepsilon(x_1,x_2)\chi_\varepsilon(x) + (1-\theta_\varepsilon(x_1,x_2))u(x), \tag{7.30}$$

where

$$\begin{aligned} \chi_\varepsilon(x) = \sum_{j\in J_\varepsilon} \Bigg(&\fint_{S^j_{r_\varepsilon}} v(s_1,s_2,x_3)\ ds_1ds_2 \\ &+ \left(\fint_{S^j_{r_\varepsilon}} \varphi(s_1,s_2,x_3)\ ds_1ds_2\right)(x_1-(\omega^j_\varepsilon)_1 + x_2 - (\omega^j_\varepsilon)_2)\Bigg)\mathbb{1}_{D^j_{R_\varepsilon}}(x_1,x_2). \end{aligned} \tag{7.31}$$

It is easy to check that the convergences stated in (7.27) hold true. We have

$$\begin{aligned} &\int_{\mathcal{O}\setminus T_{r_\varepsilon}} f(\nabla u_\varepsilon)dx + \lambda_\varepsilon \int_{T_{r_\varepsilon}} g(\nabla u_\varepsilon)dx := I_{\varepsilon 1} + I_{\varepsilon 2} + I_{\varepsilon 3}; \\ &I_{\varepsilon 1} = \int_{\mathcal{O}\setminus(D_{R_\varepsilon}\times(0,L))} f(\nabla u)dx, \\ &I_{\varepsilon 2} = \int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}\times(0,L)} f\left((\chi_\varepsilon - u)\widehat{\nabla}\theta_\varepsilon + (1-\theta_\varepsilon)\nabla u + \theta_\varepsilon\nabla\chi_\varepsilon\right)dx, \\ &I_{\varepsilon 3} = \lambda_\varepsilon \int_{T_{r_\varepsilon}} g\big(\nabla\chi_\varepsilon\big)\ dx. \end{aligned} \tag{7.32}$$

The proof of (7.27) is achieved provided we show that

$$\limsup_{\varepsilon\to 0} I_{\varepsilon 1} \le \int_{\mathcal{O}} f(\nabla u)dx, \tag{7.33}$$

$$\limsup_{\varepsilon\to 0} I_{\varepsilon 2} \le \int_{\mathcal{O}} c^f(S; v-u)ndx, \tag{7.34}$$

$$\lim_{\varepsilon\to 0} I_{\varepsilon 3} \le \overline{k}\int_{\mathcal{O}} g^{hom}(\partial_3 v)ndx. \tag{7.35}$$

The proof of (7.33) is straightforward.

Proofs of (7.34). Assuming first that $\gamma^{(p)} < +\infty$ (hence $p \le 2$ and (θ_ε) is bounded in $W^{1,p}(\widehat{\mathcal{O}})$) and applying (8.22) to $(h, A) = (f, D_{R_\varepsilon} \times (0, L) \setminus S_{r_\varepsilon} \times (0, L))$, noticing that by (1.5) there holds $|D_{R_\varepsilon}| \le C\frac{r_\varepsilon^2}{\varepsilon^2} = o(1)$ and $|\theta_\varepsilon| \le 1$ (see (5.5)), we get

$$\left| I_{\varepsilon 2} - \int_{(D_{R_\varepsilon} \times (0,L)) \setminus S_{r_\varepsilon} \times (0,L)} f\left((\chi_\varepsilon - u)\widehat{\nabla}\theta_\varepsilon\right) dx \right| = o(1), \tag{7.36}$$

and then deduce from (8.11) that

$$\left| I_{\varepsilon 2} - \int_{(D_{R_\varepsilon} \times (0,L)) \setminus S_{r_\varepsilon} \times (0,L)} f^{\infty,p}\big((\chi_\varepsilon - u)\widehat{\nabla}\theta_\varepsilon\big) dx \right| = o(1). \tag{7.37}$$

It follows from (7.37), (8.22) and the estimate (see (7.31))

$$|(\chi_\varepsilon - u)(x) - (v - u)(\omega_\varepsilon^j, x_3)| \le CR_\varepsilon \quad \text{in } D^j_{R_\varepsilon} \times (0, L), \quad \forall j \in J_\varepsilon,$$

that

$$\left| I_{\varepsilon 2} - \sum_{j \in J_\varepsilon} \int_{(D^j_{R_\varepsilon} \times (0,L)) \setminus S_{r_\varepsilon} \times (0,L)} f^{\infty,p}((v - u)(\omega_\varepsilon^j, x_3)\widehat{\nabla}\theta_\varepsilon)\, dx \right| = o(1). \tag{7.38}$$

By (3.3), (5.19), and (7.29) there holds

$$\begin{aligned} \sum_{j \in J_\varepsilon} \int_{(D^j_{R_\varepsilon} \times (0,L)) \setminus S^j_{r_\varepsilon} \times (0,L)} f^{\infty,p}((v - u)(\omega_\varepsilon^j, x_3)\widehat{\nabla}\theta_\varepsilon) dx &= \sum_{j \in J_\varepsilon} \int_0^L \operatorname{cap}^{f^{\infty,p}}\big(S_{r_\varepsilon}, R_\varepsilon D; (v - u)(\omega_\varepsilon^j, x_3)\big) dx_3 \\ &= \frac{r_\varepsilon^{2-p}}{\varepsilon^2} \int \operatorname{cap}^{f^{\infty,p}}\big(S, R_\varepsilon/r_\varepsilon D; \zeta_\varepsilon(x)\big) d\mu_\varepsilon, \end{aligned} \tag{7.39}$$

where

$$\zeta_\varepsilon(x) := \sum_{j \in J_\varepsilon} (v - u)(\omega_\varepsilon^j, x_3) \mathbb{1}_{D^j_{R_\varepsilon}}(x_1, x_2). \tag{7.40}$$

We distinguish then two cases.

Case $p < 2$, $\gamma^{(p)} < +\infty$. Let us fix some bounded open subset V of $\mathbb{R}^2$ such that $\overline{S} \subset V$. For small ε's there holds $V \subset R_\varepsilon/r_\varepsilon D$, hence by (5.12) there holds $\operatorname{cap}^{f^{\infty,p}}\big(S, R_\varepsilon/r_\varepsilon D; \zeta_\varepsilon(x)\big) \le \operatorname{cap}^{f^{\infty,p}}\big(S, V; \zeta_\varepsilon(x)\big)$, therefore by (3.1), (7.38) and (7.39) we have, since $0 < \gamma^{(p)} < +\infty$,

$$\limsup_{\varepsilon \to 0} I_{\varepsilon 2} \le \gamma^{(p)} \limsup_{\varepsilon \to 0} \int \operatorname{cap}^{f^{\infty,p}}\big(S, V; \zeta_\varepsilon(x)\big) d\mu_\varepsilon. \tag{7.41}$$

By Lemma 5.3 (i), the mapping $\operatorname{cap}^{f^{\infty,p}}\big(S, V; .\big)$ is locally Lipschitz continuous and by (7.40) the estimate $|\zeta_\varepsilon - (v - u)| \le C\varepsilon$ holds true in $D_{R_\varepsilon} \times (0, L)$, because $v - u$ is continous. We deduce that $\operatorname{cap}^{f^{\infty,p}}\big(S, V; \zeta_\varepsilon(x)\big) - \operatorname{cap}^{f^{\infty,p}}\big(S, V; v - u\big) \le C\varepsilon$ in D_{R_ε} and then infer from (6.11) and (7.41) that

$$\limsup_{\varepsilon \to 0} I_{\varepsilon 2} \le \gamma^{(p)} \int_{\mathcal{O}} \operatorname{cap}^{f^{\infty,p}}\big(S, V; v - u\big) n dx. \tag{7.42}$$

Let us substitute V_n for V in (7.42), where (V_n) denotes an increasing sequence of bounded open subsets of $\mathbb{R}^2$ such that $\overline{S} \subset V_1$ and $\bigcup_{n \in \mathbb{N}} V_n = \mathbb{R}^2$. Noticing that by (5.12) and (5.16) there holds $\operatorname{cap}^{f^{\infty,p}}\big(S, V_n; v - u\big) \le \operatorname{cap}^{f^{\infty,p}}\big(S, V_1; v - u\big)$ and $\lim_{n \to +\infty} \operatorname{cap}^{f^{\infty,p}}\big(S, V_n; v - u\big) = \operatorname{cap}^{f^{\infty,p}}\big(S, \mathbb{R}^2; v - u\big)$, by applying the Dominated Convergence Theorem, we get

$$\begin{aligned} \limsup_{\varepsilon \to 0} I_{\varepsilon 2} &\le \gamma^{(p)} \lim_{n \to +\infty} \int_{\mathcal{O}} \operatorname{cap}^{f^{\infty,p}}\big(S, V_n; v - u\big) n dx = \gamma^{(p)} \int_{\mathcal{O}} \operatorname{cap}^{f^{\infty,p}}\big(S, \mathbb{R}^2; v - u\big) n dx \\ &= \int_{\mathcal{O}} c^f(S; v - u)\, n dx. \qquad (p < 2) \end{aligned} \tag{7.43}$$

The proof of (7.34) is achieved in the case $p<2$, $0<\gamma^{(p)}<+\infty$.
Case $p=2$, $\gamma^{(2)}<+\infty$. By (7.39) and by the second line of (5.19), we have

$$\begin{aligned}\sum_{j\in J_\varepsilon}\int_{(D^j_{R_\varepsilon}\times(0,L))\setminus S^j_{r_\varepsilon}\times(0,L)} & f^{\infty,p}((v-u)(\omega^j_\varepsilon,x_3)\widehat{\nabla}\theta_\varepsilon)\ dx \\ &= \frac{\mathrm{cap}^{f^{\infty,2}}\big(r_\varepsilon S,R_\varepsilon D;1\big)}{\varepsilon^2}\sum_{j\in J_\varepsilon}\int|\zeta_\varepsilon|^2\mathbb{1}_{\zeta_\varepsilon>0}d\mu_\varepsilon \\ &\quad+\frac{\mathrm{cap}^{f^{\infty,2}}\big(r_\varepsilon S,R_\varepsilon D;-1\big)}{\varepsilon^2}\sum_{j\in J_\varepsilon}\int|\zeta_\varepsilon|^2\mathbb{1}_{\zeta_\varepsilon<0}d\mu_\varepsilon.\end{aligned}\tag{7.44}$$

By (5.36) there holds

$$\lim_{\varepsilon\to0}\frac{\mathrm{cap}^{f^{\infty,2}}\big(r_\varepsilon S,R_\varepsilon D;\pm1\big)}{\varepsilon^2}=\gamma^{(2)}c^{f^{\infty,2}}(\pm1).\tag{7.45}$$

By (7.40), the next estimates are satisfied

$$\begin{aligned}&||\zeta_\varepsilon|^2\mathbb{1}_{\zeta_\varepsilon>0}-|v-u|^2\mathbb{1}_{v-u>0}|\le C\varepsilon\quad\text{ in } D_{R_\varepsilon}\times(0,L),\\ &||\zeta_\varepsilon|^2\mathbb{1}_{\zeta_\varepsilon<0}-|v-u|^2\mathbb{1}_{v-u<0}|\le C\varepsilon\quad\text{ in } D_{R_\varepsilon}\times(0,L).\end{aligned}\tag{7.46}$$

We deduce from (3.6), (3.7), (6.11), (7.38), (7.44), (7.45), and (7.46) that

$$\begin{aligned}\lim_{\varepsilon\to0}I_{\varepsilon2}&=\gamma^{(2)}c^{f^{\infty,2}}(1)\int_{\mathcal{O}}|v-u|^2\mathbb{1}_{v-u>0}ndx+\gamma^{(2)}c^{f^{\infty,2}}(-1)\int_{\mathcal{O}}|v-u|^2\mathbb{1}_{v-u<0}ndx\\ &=\int_{\mathcal{O}}c^f(S,v-u)\ ndx.\end{aligned}\tag{$p=2$}$$

The proof of (7.34) is achieved in the case $p=2$, $0<\gamma^{(2)}<+\infty$.
Case $\gamma^{(p)}=+\infty$. We choose a sequence (R_ε) satisfying, besides (1.7), the estimate

$$R^p_\varepsilon\gamma^{(p)}_\varepsilon(r_\varepsilon)\ll1.\tag{7.47}$$

By (7.26) there holds $u=v$ and by (7.31) we have $|\chi_\varepsilon-u|<CR_\varepsilon$ in D_{R_ε}. Taking (1.9) into account, we infer that

$$\left|\int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}\times(0,L)}f\left((\chi_\varepsilon-u)\widehat{\nabla}\theta_\varepsilon\right)dx\right|\le CR^p_\varepsilon\int_{\mathcal{O}}|\widehat{\nabla}\theta_\varepsilon|^pdx\le CR^p_\varepsilon\gamma^{(p)}_\varepsilon(r_\varepsilon).\tag{7.48}$$

It follows then from (7.36), (7.48), and (7.47) that $\lim_{\varepsilon\to0}I_{\varepsilon2}=0$. □
Proof of (7.35). If $\bar k<+\infty$, noticing that by (7.31) there holds $|\nabla\chi_\varepsilon-(\varphi_1,\varphi_2,\partial_3v)|\le cr_\varepsilon$ in T_{r_ε}, we deduce from (1.10), (6.11), and (7.28) that

$$\begin{aligned}\limsup_{\varepsilon\to0}\ \lambda_\varepsilon\int_{T_{r_\varepsilon}}g(\nabla\chi_\varepsilon)\ dx&=\limsup_{\varepsilon\to0}\lambda_\varepsilon\frac{r^2_\varepsilon|S|}{\varepsilon^2}\int g(\varphi,\partial_3v)\ d\mu_\varepsilon=\bar k\int_{\mathcal{O}}g(\varphi,\partial_3v)ndx\\ &=\bar k\int_{\mathcal{O}}g^{hom}(\partial_3v)\ ndx.\end{aligned}$$

Otherwise, if $\bar k=+\infty$, then by (7.26) we have $\partial_3v=0$, therefore $\varphi=0$ (because by (1.9) there holds $g(0)=0$) and $\chi_\varepsilon=0$. Accordingly, $I_{\varepsilon3}=0$ and (7.35) is proved. □
□

8 Appendix.

8.1 Some technical lemmas related to the lower bound

Lemma 8.1. *Assume that $1 < p < 3$, and let (u_ε) be a sequence satisfying (6.1), (6.7). Then there exists a sequence $(\widehat{u}_\varepsilon)$ verifying (7.9).*

Proof. We fix two sequences (a_ε) and (b_ε) of positive reals such that

$$1 \gg a_\varepsilon \gg b_\varepsilon, \qquad a_\varepsilon b_\varepsilon^2 \gg \frac{R_\varepsilon^2}{\varepsilon^2}. \tag{8.1}$$

By means of De Giorgi's slicing argument (see Remark 8.1), we can choose for each ε a finite sequence $(l_{k,\varepsilon})_{k\in\{1,\dots,m_\varepsilon\}}$ such that $0 = l_{0,\varepsilon} < l_{1,\varepsilon} < \dots < l_{m_\varepsilon,\varepsilon} < l_{m_\varepsilon+1,\varepsilon} = L$ and

$$\begin{aligned}
&\left(k - \frac{1}{4}\right) a_\varepsilon \le l_{k,\varepsilon} \le \left(k + \frac{1}{4}\right) a_\varepsilon, \quad m_\varepsilon \sim \frac{L}{a_\varepsilon},\\
&\int_{H_\varepsilon} |\nabla u_\varepsilon|^p dx \le C \frac{b_\varepsilon}{a_\varepsilon} \int_{\mathcal{O}} |\nabla u_\varepsilon|^p dx \quad (= o(1)),\\
&\int_{H_\varepsilon} |\langle u_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle u_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \le C \frac{b_\varepsilon}{a_\varepsilon} \int |\langle u_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle u_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \quad (= o(1)),\\
&H_\varepsilon := D_{R_\varepsilon} \times \bigcup_{k=1}^{m_\varepsilon} \left(l_{k,\varepsilon} - \frac{1}{2} b_\varepsilon; l_{k,\varepsilon} + \frac{1}{2} b_\varepsilon\right) \cap \mathcal{O}.
\end{aligned} \tag{8.2}$$

Then, given a sequence $(\varphi_\varepsilon) \subset \mathcal{D}(0, L)$ such that

$$\begin{aligned}
&\varphi_\varepsilon = 1 \quad \text{in} \quad (0, L) \setminus \bigcup_{k=1}^{m_\varepsilon} \left(l_{k,\varepsilon} - \frac{1}{2} b_\varepsilon; l_{k,\varepsilon} + \frac{1}{2} b_\varepsilon\right), \quad \varphi_\varepsilon = 0 \quad \text{on} \quad \bigcup_{k=0}^{m_\varepsilon+1} \{l_{k,\varepsilon}\},\\
&0 \le \varphi_\varepsilon \le 1, \quad |\varphi_\varepsilon'| < \frac{C}{b_\varepsilon},
\end{aligned} \tag{8.3}$$

we set

$$\widehat{u}_\varepsilon(x_1, x_2, x_3) := \sum_{k=1}^{m_\varepsilon+1} \left(\fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \varphi_\varepsilon(s_3) u_\varepsilon(x_1, x_2, s_3) ds_3 \right) \mathbb{1}_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})}(x_3), \tag{8.4}$$

and claim that the sequence $(\widehat{u}_\varepsilon)$ defined by (8.4) satisfies (7.9).

By Jensen's inequality and (2.1) we have, since $0 \le \varphi_\varepsilon \le 1$,

$$\begin{aligned}
\int_{T_{r_\varepsilon}} |\widehat{\nabla} \widehat{u}_\varepsilon|^p dx &= \sum_{k=1}^{m_\varepsilon+1} \int_{S_{r_\varepsilon}} \int_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \left| \fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \varphi_\varepsilon(s_3) \widehat{\nabla} u_\varepsilon(x_1, x_2, s_3) ds_3 \right|^p dx\\
&= \sum_{k=1}^{m_\varepsilon+1} \int_{S_{r_\varepsilon}} \int_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \fint_{(l_{k-1,\varepsilon}; l_{k,\varepsilon})} \left| \widehat{\nabla} u_\varepsilon(x_1, x_2, s_3) \right|^p ds_3 dx \le \int_{T_{r_\varepsilon}} |\nabla u_\varepsilon|^p dx,
\end{aligned}$$

which proves the first inequality of the first line of (7.9). The second one is obtained in the same way.

Since the sequence (u_ε) satisfies (6.1), we can apply Proposition (6.1). We infer from (6.6) and (8.4) that

$$\int |\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon}|^p + |\langle \widehat{u}_\varepsilon\rangle_{R_\varepsilon}|^p d\mu_\varepsilon \le C. \tag{8.5}$$

The estimate stated in the second line of (7.9) is proved.
By applying Lemma 6.2, we deduce from (6.11) and (8.5) that

$$\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon} \mu_\varepsilon \overset{\star}{\rightharpoonup} wn\mathcal{L}^3_{\lfloor \mathcal{O}} \quad \text{weak-}* \text{ in } \mathcal{M}(\overline{\mathcal{O}}), \tag{8.6}$$

up to a subsequence, for some measurable function w such that $wn \in L^p(\mathcal{O})$. We have to prove that $wn = vn$ a.e. in $\mathcal{O}$. By (6.7), the sequence $(\langle u_\varepsilon\rangle_{r_\varepsilon}\mu_\varepsilon)$ weak-$*$ converges in $\mathcal{M}(\overline{\mathcal{O}})$ to $vn\mathcal{L}^3_{\lfloor\mathcal{O}}$. Since the support of $(1-\varphi_\varepsilon)\langle u_\varepsilon\rangle_{r_\varepsilon}$ is included in H_ε and since by (8.3) there holds $0 \le \varphi_\varepsilon \le 1$, we infer from (6.6), (8.1) and the third line of (8.2) that the sequence $((1-\varphi_\varepsilon)\langle u_\varepsilon\rangle_{r_\varepsilon}\mu_\varepsilon)$ weak-$*$ converges in $\mathcal{M}(\overline{\mathcal{O}})$ to 0. It then follows that

$$\varphi_\varepsilon\langle u_\varepsilon\rangle_{r_\varepsilon}\mu_\varepsilon \overset{\star}{\rightharpoonup} vn\mathcal{L}^3_{\lfloor\mathcal{O}} \quad \text{weak-}* \text{ in } \mathcal{M}(\overline{\mathcal{O}}). \tag{8.7}$$

Let us fix $\psi \in C(\overline{\mathcal{O}})$ and set

$$\overline{\psi}_\varepsilon(x) := \sum_{k=1}^{m_\varepsilon+1}\left(\fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\psi(x_1,x_2,s_3)ds_3\right)\mathbb{1}_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}(x_3). \tag{8.8}$$

It is easy to check that $|\psi - \overline{\psi}_\varepsilon|_{L^\infty(\mathcal{O})} \le Ca_\varepsilon \ll 1$ (see (8.1), (8.2)), consequently by (8.6) and (8.7) we have

$$\lim_{\varepsilon\to 0}\int \overline{\psi}_\varepsilon\varphi_\varepsilon\langle u_\varepsilon\rangle_{r_\varepsilon}d\mu_\varepsilon = \int_{\mathcal{O}}\psi vndx, \qquad \lim_{\varepsilon\to 0}\int \overline{\psi}_\varepsilon\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon}d\mu_\varepsilon = \int_{\mathcal{O}}\psi wndx. \tag{8.9}$$

On the other hand, by Fubini Theorem, (6.2) and (8.4) there holds

$$\langle \widehat{u}_\varepsilon\rangle^j_{r_\varepsilon}(x_3) = \left(\fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\varphi_\varepsilon(s_3)\langle u_\varepsilon\rangle^j_{r_\varepsilon}(s_3)ds_3\right) \quad \forall x_3 \in (l_{k-1,\varepsilon};l_{k,\varepsilon}), \quad \forall j \in J_\varepsilon,$$

therefore, by (3.3), (6.2), (8.4), and (8.8), we have

$$\begin{aligned}
&\int \overline{\psi}_\varepsilon\varphi_\varepsilon\langle u_\varepsilon\rangle_{r_\varepsilon}d\mu_\varepsilon = \sum_{j\in J_\varepsilon}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S^j_{r_\varepsilon}\times(0,L)}\overline{\psi}_\varepsilon(x)\varphi_\varepsilon(x_3)\langle u_\varepsilon\rangle^j_{r_\varepsilon}(x_3)dx\\
&= \sum_{j\in J_\varepsilon}\sum_{k=1}^{m_\varepsilon+1}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S^j_{r_\varepsilon}}dx_1dx_2\int_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\left(\fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\psi(x_1,x_2,s_3)ds_3\right)\varphi_\varepsilon(x_3)\langle u_\varepsilon\rangle^j_{r_\varepsilon}(x_3)dx_3\\
&= \sum_{j\in J_\varepsilon}\sum_{k=1}^{m_\varepsilon+1}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S^j_{r_\varepsilon}}dx_1dx_2\int_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}dx_3\left(\fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\psi(x_1,x_2,s_3)ds_3\right)\left(\fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\varphi_\varepsilon(s_3)\langle u_\varepsilon\rangle^j_{r_\varepsilon}(s_3)ds_3\right)\\
&= \sum_{j\in J_\varepsilon}\sum_{k=1}^{m_\varepsilon+1}\frac{\varepsilon^2}{r_\varepsilon^2|S|}\int_{S^j_{r_\varepsilon}\times(l_{k-1,\varepsilon};l_{k,\varepsilon})}\overline{\psi}_\varepsilon(x)\langle \widehat{u}_\varepsilon\rangle^j_{r_\varepsilon}(x_3)dx = \int \overline{\psi}_\varepsilon\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon}d\mu_\varepsilon.
\end{aligned} \tag{8.10}$$

Joining (8.9) and (8.10), we deduce that $\int_{\mathcal{O}}\psi vndx = \int_{\mathcal{O}}\psi wndx$ and then, by the arbitrary choice of ψ, infer that $nv = nw$ a.e. in $\mathcal{O}$ and that the convergence (8.6) takes place for the entire sequence $(\langle \widehat{u}_\varepsilon\rangle_{r_\varepsilon}\mu_\varepsilon)$. The first of the two convergences stated in the third line of (7.9) is proved. The second one is obtained in the same manner.

Let us prove the lower bound stated in the fourth line of (7.9). By (2.3) and Hölder's inequality, for any measurable subset $A \subset \mathbb{R}^3$, there holds

$$\int_A |f^{\infty,p}(\varphi) - f(\varphi)|\ dx \le \alpha'\int_A(1+|\varphi|^{\beta'})dx \le \alpha'\left(|A| + |A|^{1-\frac{\beta'}{p}}|\varphi|^{\frac{\beta'}{p}}_{L^p(A)}\right) \quad \forall\varphi \in L^p(A),$$

for some positive reals α', $\beta' \in (0,p)$, therefore

$$\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}|f^{\infty,p}(\nabla u_\varepsilon) - f(\nabla u_\varepsilon)|\ dx \le C\left(\frac{R_\varepsilon^2}{\varepsilon^2} + \left(\frac{R_\varepsilon^2}{\varepsilon^2}\right)^{1-\frac{\beta'}{p}}|\nabla u_\varepsilon|^{\frac{\beta'}{p}}_{L^p((D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon})}\right), \tag{8.11}$$

yielding, by (6.6),

$$\lim_{\varepsilon\to 0}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} |f^{\infty,p}(\nabla u_\varepsilon)-f(\nabla u_\varepsilon)|\ dx=0. \tag{8.12}$$

By (8.2) and (8.3) we have $u_\varepsilon(x)=\varphi_\varepsilon(x_3)u_\varepsilon(x)$ in $\mathcal{O}\setminus H_\varepsilon$ and $|\varphi_\varepsilon'|\le \frac{C}{b_\varepsilon}$, hence by (1.9) there holds

$$\begin{aligned}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} |f^{\infty,p}(\nabla u_\varepsilon)-f^{\infty,p}(\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x)))|dx &\le \int_{H_\varepsilon} |f^{\infty,p}(\nabla u_\varepsilon)|+|f^{\infty,p}(\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x)))|dx\\ &\le C\int_{H_\varepsilon}|\nabla u_\varepsilon|^p+\left|\frac{u_\varepsilon(x)}{b_\varepsilon}\right|^p dx.\end{aligned} \tag{8.13}$$

Since $1<p<3$, the space $W^{1,p}(\mathcal{O})$ is continuously imbedded in $L^{p^*}(\mathcal{O})$ (see [10, Corollary 9.14]), therefore

$$\left(\int_{H_\varepsilon}|u_\varepsilon(x)|^{p^*}\,dx\right)^{\frac{1}{p^*}}\le\left(\int_{\mathcal{O}}|u_\varepsilon(x)|^{p^*}\,dx\right)^{\frac{1}{p^*}}\le C|u_\varepsilon|_{W^{1,p}(\mathcal{O})},\qquad \frac{1}{p^*}=\frac{1}{p}-\frac{1}{3}. \tag{8.14}$$

By (8.2) we have $\mathcal{L}^3(H_\varepsilon)\le C\frac{R_\varepsilon^2}{\varepsilon^2}\frac{b_\varepsilon}{a_\varepsilon}$, hence by Hölder inequality, (6.6), and (8.14), there holds

$$\begin{aligned}\int_{H_\varepsilon}\left|\frac{u_\varepsilon(x)}{b_\varepsilon}\right|^p dx &\le \frac{1}{b_\varepsilon^p}\left(\int_{H_\varepsilon}|u_\varepsilon(x)|^{p^*}\,dx\right)^{\frac{p}{p^*}}\mathcal{L}^3(H_\varepsilon)^{\left(1-\frac{p}{p^*}\right)}\le \frac{C}{b_\varepsilon^p}|u_\varepsilon|^{\frac{p}{p^*}}_{W^{1,p}(\mathcal{O})}\left(\frac{R_\varepsilon^2}{\varepsilon^2}\frac{b_\varepsilon}{a_\varepsilon}\right)^{\frac{p}{3}}\\ &\le C\left(\frac{R_\varepsilon^2}{\varepsilon^2}\frac{1}{a_\varepsilon b_\varepsilon^2}\right)^{\frac{p}{3}}.\end{aligned} \tag{8.15}$$

We deduce from the second line of (8.2), (8.13) and (8.15) that

$$\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} |f^{\infty,p}(\nabla u_\varepsilon)-f^{\infty,p}(\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x)))|dx\le C\frac{b_\varepsilon}{a_\varepsilon}\int_{\mathcal{O}}|\nabla u_\varepsilon|^p dx+C\left(\frac{R_\varepsilon^2}{\varepsilon^2}\frac{1}{a_\varepsilon b_\varepsilon^2}\right)^{\frac{p}{3}},$$

and then from (6.6) and (8.1) that

$$\lim_{\varepsilon\to 0}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} |f^{\infty,p}(\nabla u_\varepsilon)-f^{\infty,p}(\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x)))|dx=0. \tag{8.16}$$

Since $\varphi_\varepsilon=0$ on $\bigcup_{k=0}^{m_\varepsilon+1}\{l_{k,\varepsilon}\}$, we have

$$\fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\partial_3(\varphi_\varepsilon(s_3)u_\varepsilon(x_1,x_2,s_3))\,ds_3=0\quad\forall k\in\{1,...,m_\varepsilon+1\},$$

therefore by (2.1) and (8.4) there holds, for all $k\in\{1,...,m_\varepsilon+1\}$ and all $x_3\in(l_{k-1,\varepsilon};l_{k,\varepsilon})$,

$$\begin{aligned}\fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\nabla(\varphi_\varepsilon(s_3)u_\varepsilon(x_1,x_2,s_3))\,ds_3 &= \fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\widehat{\nabla}(\varphi_\varepsilon(s_3)u_\varepsilon(x_1,x_2,s_3))\,ds_3\\ &=\widehat{\nabla}\left(\fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\varphi_\varepsilon(s_3)u_\varepsilon(x_1,x_2,s_3))ds_3\right)\\ &=\widehat{\nabla}\widehat{u}_\varepsilon(x_1,x_2,x_3).\end{aligned} \tag{8.17}$$

We deduce from Jensen's inequality and (8.17) that

$$
\begin{aligned}
&\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f^{\infty,p}(\nabla(\varphi_\varepsilon(s_3)u_\varepsilon(x_1,x_2,s_3)))\,dx_1dx_2ds_3\\
&\quad=\int_{D_{R_\varepsilon}\setminus S_{r_\varepsilon}} dx_1dx_2\sum_{k=1}^{m_\varepsilon+1}\int_{(l_{k-1,\varepsilon};l_{k,\varepsilon})} f^{\infty,p}(\nabla(\varphi_\varepsilon(s_3)u_\varepsilon(x_1,x_2,s_3)))\,ds_3\\
&\quad\geq\int_{D_{R_\varepsilon}\setminus S_{r_\varepsilon}} dx_1dx_2\sum_{k=1}^{m_\varepsilon+1}(l_{k,\varepsilon}-l_{k-1,\varepsilon})f^{\infty,p}\left(\fint_{(l_{k-1,\varepsilon};l_{k,\varepsilon})}\nabla(\varphi_\varepsilon(x_3)u_\varepsilon(x_1,x_2,s_3))\,ds_3\right)\\
&\quad=\int_{D_{R_\varepsilon}\setminus S_{r_\varepsilon}} dx_1dx_2\sum_{k=1}^{m_\varepsilon+1}\int_{(l_{k,\varepsilon},l_{k-1,\varepsilon})} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon(x_1,x_2,x_3))dx_3\\
&\quad=\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\,dx.
\end{aligned}
\tag{8.18}
$$

The estimate stated in the last line of (7.9) results from (8.12), (8.16) and (8.18).

□

The proof of the next Lemma relies on De Giorgi's slicing argument (see Remark 8.1).

Lemma 8.2. *Let (u_ε) be a bounded sequence in $W^{1,p}(\mathcal{O})$, and let (R'_ε) be an arbitrary sequence satisfying (1.7). Let r,R be two positive reals such that $r\overline{D}\subset S\subset RD$. Then, there exists a sequence (R_ε) satisfying (1.7) and*

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_{(D_{R_\varepsilon}\setminus D_{R_\varepsilon/2})\times(0,L)}|\nabla u_\varepsilon|^p\;dx=0,\\
&\frac{R}{r}R_\varepsilon\leq R'_\varepsilon\quad \text{if}\;\; p=2\;\;\text{and}\;\; 0<\gamma^{(2)}<+\infty.
\end{aligned}
\tag{8.19}
$$

Proof. We fix a sequence of positive real numbers (Q_ε) satisfying

$$
r_\varepsilon\ll Q_\varepsilon\ll\varepsilon,\qquad 1\ll\gamma_\varepsilon^{(p)}(Q_\varepsilon)\quad(\text{respectively, } Q_\varepsilon\ll R'_\varepsilon\;\text{ if } p=2 \text{ and } 0<\gamma^{(2)}<+\infty).
$$

where $\gamma_\varepsilon(.)$ is defined by (1.7) (if $p<2$ and $r_\varepsilon\ll\varepsilon^{\frac{2}{2-p}}$, we can set for instance $Q_\varepsilon=\varepsilon^h$ with $1<h<\frac{2}{2-p}$ and choose any sequence (Q_ε) such that $r_\varepsilon\ll Q_\varepsilon\ll\varepsilon$ otherwise). Next, we choose a sequence of positive integers (q_ε) such that $\lim_{\varepsilon\to 0}q_\varepsilon=+\infty$, $r_\varepsilon\ll 2^{q_\varepsilon}Q_\varepsilon\ll\varepsilon$ (respectively, $r_\varepsilon\ll 2^{q_\varepsilon}Q_\varepsilon\ll R'_\varepsilon$ if $p=2$ and $0<\gamma^{(2)}<+\infty$). For each $\varepsilon>0$, the family of sets $(D_{2^mQ_\varepsilon}\setminus D_{2^{m-1}Q_\varepsilon})_{m\in\mathbb{N},1\leq m\leq q_\varepsilon}$, where $D_{2^mQ_\varepsilon}$ is defined by setting $R_\varepsilon=2^mQ_\varepsilon$ in (6.3), is disjoint, therefore

$$
\sum_{m=1}^{q_\varepsilon}\int_{(D_{2^mQ_\varepsilon}\setminus D_{2^{m-1}Q_\varepsilon})\times(0,L)}|\nabla u_\varepsilon|^p\;dx\leq\int_{\mathcal{O}}|\nabla u_\varepsilon|^p\;dx\leq C,
$$

because (u_ε) is bounded in $W^{1,p}(\mathcal{O})$. Hence, for each $\varepsilon>0$, there exists an integer m_ε such that $1\leq m_\varepsilon\leq q_\varepsilon$ and

$$
\int_{(D_{2^{m_\varepsilon}Q_\varepsilon}\setminus D_{2^{m_\varepsilon-1}Q_\varepsilon})\times(0,L)}|\nabla u_\varepsilon|^p\;dx\leq\frac{C}{q_\varepsilon}.
$$

The sequence (R_ε) defined by $R_\varepsilon=2^{m_\varepsilon}Q_\varepsilon$ satisfies (1.7) and (8.19). □

Lemma 8.3. *Assume that $0<\gamma^{(p)}<+\infty$, let (u_ε) be a sequence satisfying (6.1), (R'_ε) a sequence of positive reals verifying (1.7) and (R_ε) a sequence of positive reals satisfying (1.7) and (8.19) in accordance with Lemma 8.2. Let $\widehat{u}_\varepsilon$ be defined by (8.4). Then, there exists an approximation $\widehat{u}_\varepsilon$ of $\widehat{u}_\varepsilon$ verifying (7.13).*

Proof. Fixing a positive real δ such that $1<\delta<2$ and two functions $\zeta_\varepsilon,\xi_\varepsilon\in\mathcal{C}^\infty(\overline{\mathcal{O}})$ such that

$$\begin{aligned}&\zeta_\varepsilon=0 \text{ in } D_{R_\varepsilon/2}\times(0,L),\quad \zeta_\varepsilon=1 \text{ on } \partial D_{R_\varepsilon}\times(0,L),\quad |\widehat{\nabla}\zeta_\varepsilon|\le\frac{C}{R_\varepsilon},\\ &\xi_\varepsilon=0 \text{ in } (D_{R_\varepsilon}\setminus S_{r_\varepsilon})\times(0,L),\quad \xi_\varepsilon=1 \text{ in } S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L),\quad |\widehat{\nabla}\xi_\varepsilon|\le\frac{C}{r_\varepsilon^\delta},\end{aligned}\tag{8.20}$$

where the set $S_{r_\varepsilon}^{-r_\varepsilon^\delta}=\bigcup_{j\in J_\varepsilon}S_{r_\varepsilon}^{j,-r_\varepsilon^\delta}$ is defined by (7.11), we set

$$\widehat{u}_\varepsilon:=\widehat{u}_\varepsilon+\zeta_\varepsilon(\langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}-\widehat{u}_\varepsilon)+\xi_\varepsilon(\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}-\widehat{u}_\varepsilon).\tag{8.21}$$

The function $\widehat{u}_\varepsilon$ coincides with $\langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}$ on $\partial D_{R_\varepsilon}\times(0,L)$ and with $\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}$ on $S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L)$, hence by (6.2) $\widehat{u}_\varepsilon$ takes constant values on each sets $\partial D_{R_\varepsilon}^j\times\{x_3\}$ and $S_{r_\varepsilon}^{j,-r_\varepsilon^\delta}\times\{x_3\}$. Since $D_{r_\varepsilon}\subset S_{r_\varepsilon}^{-r_\varepsilon^\delta}$, it easily follows from (6.2) that $\widehat{u}_\varepsilon=\langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}=\langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}$ on $\partial D_{R_\varepsilon}\times(0,L)$ and $\widehat{u}_\varepsilon=\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}=\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}$ on $S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L)$, as stated in the first line of (7.13).
Let us recall (see [23, Proposition 2.32]) that any convex function h on $\mathbb{R}^3$ satisfying (1.9) also verifies

$$|h(a)-h(b)|\le C|a-b|(1+|a|^{p-1}+|b|^{p-1})\quad\forall\ a,\ b\in\mathbb{R}^3,$$

hence by Hölder inequality, for any bounded measurable set $A\subset\mathbb{R}^3$ and all $\varphi,\varphi'\in L^p(A)$, there holds

$$\begin{aligned}\int_A|h(\varphi)-h(\varphi')|\ dx&\le C\left(\int_A|\varphi-\varphi'|^pdx\right)^{\frac1p}\left(\int_A\left|1+|\varphi|^{p-1}+|\varphi|^{p-1}\right|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\\&\le C|\varphi-\varphi'|_{L^p(A)}\left(|A|^{\frac{p-1}{p}}+|\varphi|_{L^p(A)}^{p-1}+|\varphi'|_{L^p(A)}^{p-1}\right).\end{aligned}\tag{8.22}$$

Applying (8.22) to $h=f^{\infty,p}$ and $A=(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}$, setting

$$E_\varepsilon:=L^p((D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon};\mathbb{R}^3),\tag{8.23}$$

we infer

$$\begin{aligned}&\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}\left|f^{\infty,p}\big(\widehat{\nabla}\widehat{u}_\varepsilon\big)-f^{\infty,p}\big(\widehat{\nabla}\widehat{u}_\varepsilon\big)\right|dx\\&\qquad\le C|\widehat{\nabla}(\widehat{u}_\varepsilon-\widehat{u}_\varepsilon)|_{E_\varepsilon}\left(\left|\frac{R_\varepsilon^2}{\varepsilon^2}\right|^{\frac{p-1}{p}}+|\widehat{\nabla}\widehat{u}_\varepsilon|_{E_\varepsilon}^{p-1}+|\widehat{\nabla}\widehat{u}_\varepsilon|_{E_\varepsilon}^{p-1}\right)\\&\qquad\le C|\widehat{\nabla}(\widehat{u}_\varepsilon-\widehat{u}_\varepsilon)|_{E_\varepsilon}\left(\left|\frac{R_\varepsilon^2}{\varepsilon^2}\right|^{\frac{p-1}{p}}+|\widehat{\nabla}\widehat{u}_\varepsilon|_{E_\varepsilon}^{p-1}+|\widehat{\nabla}(\widehat{u}_\varepsilon-\widehat{u}_\varepsilon)|_{E_\varepsilon}^{p-1}\right).\end{aligned}\tag{8.24}$$

On the other hand, by (8.20) and (8.21) we have

$$\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}|\widehat{\nabla}(\widehat{u}_\varepsilon-\widehat{u}_\varepsilon)|^p\ dx\le C\int_{(D_{R_\varepsilon}\setminus D_{R_\varepsilon/2})\times(0,L)}\frac{|\widehat{u}_\varepsilon-\langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}|^p}{R_\varepsilon^p}\ dx+\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}|\widehat{\nabla}\widehat{u}_\varepsilon|^pdx.\tag{8.25}$$

The next estimate is obtained in a similar way as the fifth estimate of (6.4):

$$\int_{(D_{R_\varepsilon}\setminus D_{R_\varepsilon/2})\times(0,L)}|\widehat{u}_\varepsilon-\langle\widehat{u}_\varepsilon\rangle_{R_\varepsilon}|^p\ dx\le CR_\varepsilon^p\int_{(D_{R_\varepsilon}\setminus D_{R_\varepsilon/2})\times(0,L)}|\widehat{\nabla}\widehat{u}_\varepsilon|^p\ dx.\tag{8.26}$$

Joining (8.4), (8.25) and (8.26), we obtain

$$\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}|\widehat{\nabla}(\widehat{u}_\varepsilon-\widehat{u}_\varepsilon)|^p\ dx\le C\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}|\widehat{\nabla}\widehat{u}_\varepsilon|^pdx\le C\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}|\nabla u_\varepsilon|^p\ dx,$$

then deduce from (8.19) that

$$\lim_{\varepsilon\to 0}|\widehat{\nabla}(\widehat{u}_\varepsilon-\widehat{u}_\varepsilon)|^p_{E_\varepsilon}=0;\qquad \lim_{\varepsilon\to 0}|\widehat{\nabla}\widehat{u}_\varepsilon|^p_{E_\varepsilon}=0, \tag{8.27}$$

and infer from (8.24) and (8.27) that

$$\lim_{\varepsilon\to 0}\int_{(D_{R_\varepsilon}\times(0,L))\setminus T_{r_\varepsilon}}\Big|f^{\infty,p}\big(\widehat{\nabla}\widehat{u}_\varepsilon\big)-f^{\infty,p}\big(\widehat{\nabla}\widehat{u}_\varepsilon\big)\Big|\,dx=0. \tag{8.28}$$

On the other hand, by (1.9), (8.20) and (8.21) there holds

$$f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\le C\big(|\widehat{\nabla}\widehat{u}_\varepsilon|^p+|\widehat{u}_\varepsilon-\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}|^p/r_\varepsilon^{\delta p}\big)\qquad \text{in } (S_{r_\varepsilon}\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta})\times(0,L).$$

Accordingly, by (3.3), the last line of (6.4), (6.6), and (7.9), we have

$$\begin{aligned}\int_{(S_{r_\varepsilon}\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta})\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\,dx &\le C\frac{r_\varepsilon^2}{\varepsilon^2}\int\Big(|\widehat{\nabla}\widehat{u}_\varepsilon|^p+|\widehat{u}_\varepsilon-\langle\widehat{u}_\varepsilon\rangle_{r_\varepsilon}|^p/r_\varepsilon^{\delta p}\Big)\,d\mu_\varepsilon\\ &\le C\left(1+r_\varepsilon^{p(1-\delta)}\right)\frac{r_\varepsilon^2}{\varepsilon^2}\int|\widehat{\nabla}\widehat{u}_\varepsilon|^p\,d\mu_\varepsilon\\ &\le C\left(1+r_\varepsilon^{p(1-\delta)}\right)\frac{r_\varepsilon^2}{\varepsilon^2}\int|\nabla u_\varepsilon|^p\,d\mu_\varepsilon\le\frac{Cr_\varepsilon^{2-p}}{\varepsilon^2}r_\varepsilon^{p(2-\delta)}.\end{aligned}$$

Since we have assumed that $\gamma^{(p)}<+\infty$ and $1<\delta<2$ (see (1.7), (3.1), (7.10)), we infer

$$\lim_{\varepsilon\to 0}\int_{(S_{r_\varepsilon}\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta})\times(0,L)} f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\,dx=0. \tag{8.29}$$

The estimate stated in the second line of (7.13) results from (8.28) and (8.29). □

Remark 8.1. *De Giorgi's slicing argument [24] is based on the following observation: if for each $\varepsilon>0$, $(A^i_\varepsilon)_{i\in\{1,\dots,l_\varepsilon\}}$ denotes a family of disjoint μ-measurable subsets of a set A equipped with a measure μ, and if (f_ε) is a sequence in $L^1_\mu(A)$ such that $|f_\varepsilon|_{L^1_\mu(A)}\le C$, then for each $\varepsilon>0$, there exists $i_\varepsilon\in\{1,\dots,l_\varepsilon\}$ such that $\int_{A^{i_\varepsilon}_\varepsilon}|f_\varepsilon|d\mu\le\frac{C}{l_\varepsilon}$. This argument is especially useful when non uniformly integrable sequences bounded in L^1_μ are considered. We employ this argument in the proof of Lemma 8.1 to establish the existence of the set H_ε satisfying (8.2) and in the proof of Lemma 8.2.*

8.2 Proof of Lemma 5.8

The assertion (i) of Lemma 5.8 is simply obtained by repeating the proof of Lemma 5.7 (i): the first line of (5.34) follows from (3.1), (5.20), (5.24), (5.25) and the second line is obtained in a similar manner.

To prove the assertion (ii), we fix two sequences of positive reals (r_ε) and (R'_ε) satisfying (1.7), (3.1). By (5.34), the sequence $\left(\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,R'_\varepsilon D;\pm1)}{\varepsilon^2}\right)$ is bounded from above and below by positive reals, hence after possibly extracting a subsequence we can suppose that

$$\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,R'_\varepsilon D;\pm1)}{\varepsilon^2}=\gamma^{(2)}c(\pm1), \tag{8.30}$$

for some positive reals $c(\pm1)$. The proof of Lemma 5.8 is achieved provided we show that

$$c(\pm1)=c^{f^{\infty,2}}(\pm1), \tag{8.31}$$

and that the reals $c^{f^{\infty,2}}(\pm1)$ are independent of S. To that aim, we establish the two lemmas.

Lemma 8.4. *Assume (8.30) and let Φ_c denote the functional defined by substituting $c(\pm1)$ for $c^{f^{\infty,2}}$ in (3.7). Then the results deduced from propositions 7.1 and 7.2 by substituting Φ_c for Φ hold true.*

Lemma 8.5. *Assume (8.30). Then*

$$\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S, R_\varepsilon D;\pm 1)}{\varepsilon^2}=\gamma^{(2)}c(\pm 1)\quad \textit{for all sequence } (R_\varepsilon) \textit{ satisfying (1.7).} \tag{8.32}$$

The proofs of lemmas 8.4 and 8.5 are situated at the end of Section 8.2. Since $\gamma^{(2)}<+\infty$, we have $r_\varepsilon << \varepsilon^2 << \varepsilon$, hence the sequence $R_\varepsilon := \varepsilon^2$ satisfies (1.7). Noticing that by (5.19) there holds $\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\varepsilon^2 D;\pm 1)=\mathrm{cap}^{f^{\infty,2}}\left(\varepsilon^3 r_\varepsilon S,\varepsilon^5 D;\pm 1\right)$, and setting $\tilde r_\varepsilon := \varepsilon^3 r_\varepsilon$, we deduce from Lemma 8.5 that

$$\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\varepsilon^2 D;\pm 1)}{\varepsilon^2}=\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(\tilde r_\varepsilon S,\varepsilon^5 D;\pm 1)}{\varepsilon^2}=\gamma^{(2)}c(\pm 1). \tag{8.33}$$

The assumption $\gamma^{(2)}<+\infty$ also implies that $r_\varepsilon << \varepsilon^3$, hence by (1.7) there holds

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon^{(2)}(\tilde r_\varepsilon)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2|\log(\varepsilon^3 r_\varepsilon)|}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2|\log r_\varepsilon|}=\gamma^{(2)}. \tag{8.34}$$

Moreover, we have

$$\tilde r_\varepsilon \ll \varepsilon^5 \ll \varepsilon \quad \text{and} \quad \gamma_\varepsilon^{(2)}(\varepsilon^5)\gg 1. \tag{8.35}$$

By (8.34) and (8.35), the sequences $(\tilde r_\varepsilon)$ and (ε^5) (in place of (r_ε) and (R'_ε)) satisfy the assumptions of Lemma 5.8. By (8.33), the assertion deduced from (8.30) by substituting $(\tilde r_\varepsilon,\varepsilon^5)$ for $(r_\varepsilon, R'_\varepsilon)$ holds true. Therefore we can apply Lemma 8.5 with $(\tilde r_\varepsilon,\varepsilon^5)$ in place of $(r_\varepsilon, R'_\varepsilon)$. We obtain

$$\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(\varepsilon^3 r_\varepsilon S, R_\varepsilon D;\pm 1)}{\varepsilon^2}=\gamma^{(2)}c(\pm 1)\quad \text{for all } (R_\varepsilon) \text{ s.t. } \tilde r_\varepsilon \ll R_\varepsilon \ll \varepsilon \text{ and } \gamma_\varepsilon^{(2)}(R_\varepsilon)\gg 1. \tag{8.36}$$

Choosing $R_\varepsilon=\varepsilon^2$ in (8.36), taking (5.19) into account, we infer

$$\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\frac{1}{\varepsilon}D;\pm 1)}{\varepsilon^2}=\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(\varepsilon^3 r_\varepsilon S,\varepsilon^2 D;\pm 1)}{\varepsilon^2}=\gamma^{(2)}c(\pm 1). \tag{8.37}$$

By (1.8) we have $R'_\varepsilon D\subset\widehat{\mathcal{O}}\subset\frac{1}{\varepsilon}D$ provided ε is small enough, hence by (5.12) there holds:

$$\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S, R'_\varepsilon D;\pm 1)}{\varepsilon^2}\geq\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\widehat{\mathcal{O}};\pm 1)}{\varepsilon^2}\geq\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\frac{1}{\varepsilon}D;\pm 1)}{\varepsilon^2}.$$

By passing to the limit as $\varepsilon\to 0$ in the last inequalities, thanks to (5.35), (8.30) and (8.37), we get $\gamma^{(2)}c(\pm 1)\geq\gamma^{(2)}c^{f^{\infty,2}}(\pm 1)\geq\gamma^{(2)}c(\pm 1)$, and infer (8.31). It remains to show that $c^{f^{\infty,2}}(1)$ and $c^{f^{\infty,2}}(-1)$ are independent of the choice of S. To that aim, we fix two positive reals r, R such that $rS\subset D\subset RS$. By (5.13) and (5.19), there holds

$$\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\frac{R'_\varepsilon}{r}D;\pm 1)}{\varepsilon^2}\leq\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon D, R'_\varepsilon D;\pm 1)}{\varepsilon^2}\leq\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S,\frac{R'_\varepsilon}{R}D;\pm 1)}{\varepsilon^2}.$$

By (8.32), the first and third terms of the above double inequality converge to $\gamma^{(2)}c^{f^{\infty,2}}(\pm 1)$, therefore

$$c^{f^{\infty,2}}(\pm 1)=\frac{1}{\gamma^{(2)}}\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon D, R'_\varepsilon D;\pm 1)}{\varepsilon^2}.$$

The proof of Lemma 5.8 is achieved. □

Proof of Lemma 8.4. We revisit the proofs of propositions 7.1 and 7.2, substituting the assumption (8.30) for (5.36). Starting with Proposition 7.1, we fix some positive reals r and R verifying $r\overline{D}\subset S\subset RD$

and choose, in accordance with Lemma 8.2, a sequence (R_ε) satisfying (1.7) and (8.19). The proof then remains unchanged until formula (7.22) which, by virtue of (8.30) becomes

$$\liminf_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon rD, R_\varepsilon D;\pm 1)}{\varepsilon^2}\geq \gamma^{(2)}c(\pm 1). \tag{8.38}$$

By passing to the limit inferior as $\varepsilon\to 0$ in Formula (7.23), which also holds true, thanks to (8.38) we find

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_{(D_{R_\varepsilon}\times(0,L))\setminus S_{r_\varepsilon}^{-r_\varepsilon^\delta}\times(0,L)} & f^{\infty,p}(\widehat{\nabla}\widehat{u}_\varepsilon)\ dx \\ & \geq \gamma^{(2)}\int_{\mathcal{O}}\ \left(c(1)|(v-u)^+|^2+c(-1)|(v-u)^-|^2\right)ndx,\end{aligned} \tag{8.39}$$

yielding

$$\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)\geq \Phi_c(u,v). \tag{8.40}$$

As regards Proposition 7.2), we set $R_\varepsilon = R'_\varepsilon$. Then, substituting (8.30) for (7.45) and repeating the argument of the proof of Proposition 7.2, we find that

$$\limsup_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)\leq \Phi_c(u,v). \tag{8.41}$$

Lemma 8.4 is proved. □

Proof of Lemma 8.5. Let us fix a sequence $(R_\varepsilon^{"})$ satisfying (1.7). By Lemma 5.8 (i), the estimates deduced from (5.34) by substituting $R_\varepsilon^{"}$ for R'_ε are satisfied. Hence the sequence $\left(\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S, R_\varepsilon^{"}D;\pm 1)}{\varepsilon^2}\right)$ is bounded from above and below by positive reals. After possibly extracting a subsequence we can assume that besides (8.30), the following estimate is satisfied:

$$\lim_{\varepsilon\to 0}\frac{\mathrm{cap}^{f^{\infty,2}}(r_\varepsilon S, R_\varepsilon^{"}D;\pm 1)}{\varepsilon^2}=\gamma^{(2)}c^{"}(\pm 1), \tag{8.42}$$

for some positive reals $c^{"}(\pm 1)$. We just have to prove that $c^{"}(\pm 1)=c(\pm 1)$. To this purpose, we repeat the argument of the proof of Lemma 8.4, substituting (8.42) for (8.30): we find that the conclusions of propositions 7.1 and 7.2 also hold true for the last mentioned subsequence if we replace Φ_c by $\Phi_{c^{"}}$. It follows that $\Phi_c=\Phi_{c^{"}}$, hence $c(\pm 1)=c^{"}(\pm 1)$. □

Acknowledgment

During this work, Somsak is supported by a research fund from Commission on Higher Education and the Thailand Research Fund (MRG5080422).

9 References

1. E. Acerbi, G. Buttazzo, D. Percivale, *A variational definition for the strain energy of an elastic string,* J. Elasticity, **25** (1991), pp. 137–148.
2. D. R. Adams, L. I. Hedberg, *Function Spaces and Potential Theory,* Grundlehren der mathematischen Wissenschaften, Springer, **314** (1999).
3. C. Amrouche, V. Girault, J. Giroire, *Weighted Sobolev spaces for Laplace's equation in $\mathbb{R}^n$,* J. Math. Pures Appl., **73** (1994), pp. 579–606.
4. H. Attouch. *Variational Convergence for Functions and Operators.* Applicable Mathematics Series. Pitman Advanced Publishing Program, (1985).

5. H. Attouch, G. Buttazzo, G. Michaille, *Variational Analysis in Sobolev and BV Spaces: Application to PDEs and Optimization,* MPS-SIAM Book Series on Optimization, (2005).
6. M. Bellieud: *A notion of capacity related to linear elasticity. Applications to homogenization.* Arch. Ration. Mech. Anal., vol. 203, no1, (2012), pp. 137–187.
7. M. Bellieud and G. Bouchitté, *Homogenization of elliptic problems in a fiber reinforced structure. Nonlocal effect, Ann. Scuola Norm. Sup. Pisa Cl. Sci.,* (4) **26**(3) (1998), pp. 407–436.
8. M. Bellieud, I. Gruais: *Homogenization of an elastic material reinforced by very stiff or heavy fibers. Non local effects. Memory effects.* J. Math. Pures Appl. 84, (2005) pp. 55–96.
9. G. Bouchitte, D. Felbacq : *Homogenization of a wire photonic crystal: the case of small volume fraction.* SIAM J. Appl. Math., 66(6) (2006), pp. 2061–2084.
10. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations* Universitext, Springer, (2010).
11. M. Briane: *A new approach for the homogenization of high-conductivity periodic problems. Application to a general distribution of one directional fibers .* SIAM J. Math. Anal., 35 (1) (2003), pp. 33–60.
12. M. Briane: *Homogenization of the Stokes equations with high-contrast viscosity.* J. Math. Pures Appl., 82 (7) (2003), p.p. 843–876.
13. M. Briane: *Nonlocal effects in two-dimensional conductivity,* Arch. Rat. Mech. Anal., 182 (2) (2006), pp. 255-267.
14. M. Briane, J. Casado-Díaz: *Asymptotic behaviour of equicoercive diffusion energies in two dimension,* Calc. Var. Part. Diff. Equa., 29 (4) (2007), pp. 455-479.
15. M. Briane, J. Casado-Díaz: *Compactness of sequences of two-dimensional energies with a zero-order term. Application to three-dimensional nonlocal effects,* Calc. Var. Part. Diff. Equa., 33 (2008), pp. 463-492.
16. D. Caillerie, B. Dinari: *A perturbation problem with two small parameters in the framework of the heat conduction of a fiber reinforced body.* Partial Differential Equations Banach Center Publications, Warsaw (1987).
17. J. Casado-Díaz: *The capacity for pseudomontone operators.* Potential Analysis 14 (2001), pp. 73–91.
18. G. Dal Maso. *An introduction to* Γ*-convergence.* Birkäuser, Boston, (1993).
19. G. Dal Maso: Γ*-convergence and* μ*-capacities.* Ann. Scuola Norm. Sup. Pisa, 14 (1988), p.p. 423–464.
20. G. Dal Maso, A. Garroni: *New results on the asymptotic behavior of Dirichlet problems in perforated domains,* Math. Mod. Meth. Appl. Sci. 4(3) (1994), pp. 373–407.
21. G. Dal Maso, F. Murat: *Asymptotic behaviour and correctors for linear Dirichlet problems with simultaneously varying operators and domains.* Annales de l'Institut Henri Poincaré. Analyse non linaire, vol. 21, n°4, (2004), pp. 445–486.
22. G. Dal Maso, I.V. Skrypnik: *Capacity theory for monotone operators.* Potential Analysis 7 (1997), pp. 765– 803.
23. B. Dacorogna, *Direct method in the calculus of variations,* Springer Verlag, Berlin, (1989).
24. E. De Giorgi, *Sulla convergenza di alcune successioni d'integrali del tipo dell'aera, Rend. Mathematica,* **8** (1975), pp. 277–294.
25. L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*, CRC Press, (1992).
26. V. N. Fenchenko, E. Y. Khruslov: *Asymptotic behavior of solutions of differential equations with a strongly oscillating coefficient matrix that does not satisfy a uniform boundedness condition.* (Russian) Dokl. Akad. Nauk Ukrain. SSR Ser. A 4 (1981), pp. 24–27.

27. J. Frehse, *Capacity methods in the theory of partial differential equations, J. ber. d. Dt. Math.-Verein.,* **84** (1982), pp. 1–44.

28. E. Y. Khruslov: *Homogenized models of composite media,* Progress in Nonlinear Differential Equations and their Application, Birkhäuser (1991).

29. C. Licht and G. Michaille, A non local energy functional in pseudo-plasticity, *Asymptotic Analysis,* **45** (2005), 313–339.

30. V. Maz'ja: *Sobolev Spaces, Springer Verlag, New York,* (1985).

31. G.G. Stokes: *On the effect of the internal friction of fluids on the motion of pendulums, Trans. Camb. Phil. Soc.* **9**, Part II (1851), pp.8-106.

32. W. Ziemer: *Weakly Differentiable functions, Springer Verlag, New York,* (1989).