



**HAL**  
open science

# The infinite associahedron and R.J. Thompson's group T

Ariadna Fossas

► To cite this version:

| Ariadna Fossas. The infinite associahedron and R.J. Thompson's group T. 2013. hal-00801808v1

**HAL Id: hal-00801808**

**<https://hal.science/hal-00801808v1>**

Preprint submitted on 18 Mar 2013 (v1), last revised 20 Mar 2013 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The infinite associahedron and R.J. Thompson's group $T$

Ariadna Fossas Tenas  
Institut Fourier  
Université de Grenoble I  
38402 St. Martin d'Hères. France  
ariadna.fossas@ujf-grenoble.fr

## Abstract

In this paper we construct a cellular complex which is an infinite analogue to Stasheff's associahedra. We prove that it is contractible and state that its (combinatorial) automorphism group is isomorphic to a semi-direct product of R.J. Thompson's group  $T$  with  $\mathbb{Z}/2\mathbb{Z}$ .

**MSC classification:** 20F65, 20F38, 57Q05, 57M07.

**Keywords:** Stasheff's associahedra, cellular complexes, automorphism groups, R.J. Thompson's groups, contractible spaces.

## 1 Introduction

Although Stasheff's associahedra were first described combinatorially in 1951 by Dov Tamari [33] in his thesis as realizations of his poset lattice of bracketings of a word of length  $n$ , they are named after Jim Stasheff's construction [30, 31] as crucial ingredients to his homotopy theoretic characterization of based loop spaces. Associahedra were proved to be polytopes by John Milnor (unpublished) and have been realized as convex polytopes many times [26, 7, 8, 34, 19]. The vertices of the associahedron are in bijection with all ways to put brackets in an expression of  $n$  non-associative variables (avoiding the bracket containing all the expression and brackets containing a single variable), but also with all rooted binary trees with  $n + 1$  leaves, and all the minimal triangulations of a convex polygon with  $n + 2$  sides. The name *associahedra* comes from the bracketing viewpoint, where edges are obtained by replacing a sub-word  $t(su)$  by  $(ts)u$  (associativity relation). The associahedron can also be constructed as the dual of the arc complex of a polygon [25].

Stasheff associahedra play a role in different domains of mathematics such as combinatorics, homotopy theory, cluster algebras and topology (see [29]). In this paper we construct an infinite dimensional cellular complex  $C$  that can be seen as a generalisation of the associahedron for an infinitely sided convex polygon. To give an idea, the vertices of  $C$  are in bijection with all possible triangulations of the circle which differ from a given one only in finitely many diagonals. We also prove that  $C$  is contractible. Finally, we study the group of combinatorial automorphisms of  $C$  and we prove that it is isomorphic to the semi-direct product of Thompson's group  $T$  with  $\mathbb{Z}/2\mathbb{Z}$ .

A relation between associahedra and Thompson's group  $F$  was first established by Greenberg [20]. Thompson's group  $F$ , which is the smallest of the three classical Thompson groups  $F, T, V$  (first introduced by McKenzie and Thompson [28], see [6] for an introduction), is usually seen as the group of all piecewise-linear order-preserving self-homeomorphisms of  $[0, 1]$  with only finitely many breakpoints, each of which has dyadic rational coefficients, and where every slope is an integral power of 2. Thompson's group  $T$  is the analogue of  $F$  when considering self-homeomorphisms of the circle  $S^1$  seen as the unit interval with

identified endpoints. In this case one must include the condition of self-homeomorphisms preserving set-wise the dyadic rational numbers (which for  $F$  is a consequence of the other conditions).

The two dimensional skeleton of  $C$  was introduced by Funar, Kapoudjian and Sergiescu [15, 16, 17, 18] as a 2-dimensional complex where vertices are isotopy classes of decompositions of an infinite type surface, edges are elementary moves and faces can be seen as relations between this elementary moves. This is inspired from their version of Thompson's group  $T$  as an asymptotically rigid mapping class group of a connected, non compact surface of genre 0 with infinitely many ends. To give a hint on what an asymptotically rigid mapping class group is, one can think about it as homotopy classes of homeomorphisms which preserve a given tessellation of the surface outside a compact subsurface.

Finally, it is worth to mention a few other cellular complexes where  $T$  was proved to act nicely. Brown [4], Brown and Geoghegan [5], and Stein [32] use the action of  $T$  in complexes of basis of Jonsson-Tarski algebras [1] to obtain, respectively, finiteness properties and homological properties. Farley uses diagram groups to construct CAT(0) cubical complexes where Thompson's groups act [10, 11, 9, 12]. Greenberg [21] and Martin [27] constructed contractible complexes where  $T$  acts. Unfortunately, none of the automorphism groups of these complexes is known.

## Acknowledgements.

The author wishes to thank Ross Geoghegan and Louis Funar for asking about the contractibility of the infinite associahedron, and Jim Stasheff for his comments and suggestions. This work was partially supported by "Fundación Caja Madrid" Postgraduated Fellowship and the ANR 2011 BS 01 020 01 ModGroup.

## 2 Stasheff's associahedra

We adapt Greenberg's construction of Stasheff's associahedra [20] to the language of minimal tessellations of a convex polygon. The original idea of Greenberg's construction in terms of planar trees is due to Boardman and Vogt [2].

Let  $P_n$  ( $n \geq 3$ ) be a convex polygon with  $n$  vertices,  $v_1, \dots, v_n$ , where the vertices  $v_i$  and  $v_j$  are adjacent if and only if  $|i - j| \equiv 1 \pmod n$ . Let  $D_n$  be the set of interior diagonals of  $P_n$ , i.e.

$$D_n = \{(v_i, v_j) \in V_n^2 : |i - j| > 1 \pmod n\}.$$

Let  $\mathcal{T}(P_n)$  be the maximal subset of  $\mathcal{P}(D_n)$  containing only subsets of  $D_n$  without crossing diagonals. The empty set belongs to  $\mathcal{T}(P_n)$  and is denoted  $\emptyset_n$ . The set  $\mathcal{T}(P_n)$  can be seen as the set of *minimal* tessellations of  $P_n$  (minimal in the sense that there are no interior vertices).

We can define a partial order in  $\mathcal{T}(P_n)$  by saying that  $\alpha < \beta$  if  $\beta \subset \alpha$ . Greenberg's method consists to associate a closed cell  $f_\alpha$  to every  $\alpha \in \mathcal{T}(P_n) - \emptyset_n$ . The dimension of  $f_\alpha$  is  $n - 3 - |\alpha|$ . Furthermore, if  $\alpha < \beta$ , then  $f_\alpha$  is included into  $f_\beta$ . Stasheff's associahedron  $\mathcal{A}(P_n)$  is the union of all these cells, preserving the inclusions.

We use induction over  $n$  to define Stasheff's associahedra. The first two cases to consider are:

1. The triangular case. Note that  $\mathcal{T}(P_3) = \{\emptyset_3\}$ . Thus,  $\mathcal{A}(P_3)$  is a point (the 0-cell associated to the triangle itself as a tessellation).
2. The square case. Note that, given a square with vertices 1,2,3,4 as before, it admits only two interior diagonals: (1,3) and (2,4). Furthermore, the two diagonals cross each other. Thus,  $\mathcal{T}(P_4) = \{(1,3), (2,4), \emptyset_4\}$ . The faces associated to  $\{(1,3)\}$  and  $\{(2,4)\}$  are the two endpoints of a closed segment, which is itself the 1-cell associated to  $\emptyset_4$ , and coincides with  $\mathcal{A}(P_4)$ .

Take  $n > 4$ . Suppose that:

1. The cells  $f_\alpha$  are defined for all  $\alpha \in \mathcal{T}(P_j)$  and for all  $j$  satisfying  $3 \leq j \leq n - 1$ .
2. The inclusions  $i_{\beta\alpha} : f_\alpha \rightarrow f_\beta$  are defined for all  $\alpha, \beta \in \mathcal{T}(P_j)$  such that  $\alpha < \beta$ , and all  $j$  satisfying  $3 \leq j \leq n - 1$ .
3. The cellular complex  $(\mathcal{A}(P_j), \partial\mathcal{A}(P_j))$  is topologically equivalent to  $(B^{j-3}, S^{j-4})$  for all  $j$  satisfying  $3 \leq j \leq n - 1$ , where  $B^k, S^k$  respectively are the closed ball and the sphere of dimension  $k$ .
4. For all  $j$  satisfying  $3 \leq j \leq n - 1$  and for all  $\alpha \in \mathcal{T}(P_j)$ , the cell  $f_\alpha$  is isomorphic to

$$\prod_{i=1}^{|\alpha|+1} \mathcal{A}(P_{n_i}),$$

where  $p_1, p_2, \dots, p_{|\alpha|}, p_{|\alpha|+1}$  (respectively  $n_1, \dots, n_{|\alpha|+1}$ ) be the polygons obtained by cutting  $P_n$  along the diagonals of  $\alpha$  (respectively their number of sides).

Let  $\alpha \in \mathcal{T}(P_n) - \emptyset_n$ . Let  $p_1, p_2, \dots, p_{|\alpha|}, p_{|\alpha|+1}$  (respectively  $n_1, \dots, n_{|\alpha|+1}$ ) be the polygons obtained from  $P_n$  by cutting along the diagonals of  $\alpha$  (respectively their number of sides) as before. Define  $f_\alpha$  as the product space

$$f_\alpha \equiv \prod_{i=1}^{|\alpha|+1} \mathcal{A}(P_{n_i}).$$

Let  $\alpha, \beta \in \mathcal{T}(P_n) - \emptyset_n$  such that  $\alpha < \beta$ . To define the inclusion  $i_{\beta\alpha} : f_\alpha \rightarrow f_\beta$ , first suppose that  $1 \leq |\beta| = |\alpha| - 1$ . Then, there exist a unique diagonal  $d$  which belongs to  $\alpha$  and does not belong to  $\beta$ . Enumerate the polygons obtained by cutting  $P_n$  along the diagonals of  $\alpha$  starting by the two polygons containing the diagonal  $d$  on the border. By construction,

$$f_\alpha \equiv \mathcal{A}(P_{n_1}) \times \mathcal{A}(P_{n_2}) \times \prod_{i=3}^{|\alpha|+1} \mathcal{A}(P_{n_i}),$$

where the third element of the product is non-empty (note that  $|\alpha| \geq 2$ ). Furthermore,

$$f_\beta \equiv \mathcal{A}(P_{n_1+n_2-2}) \times \prod_{i=3}^{|\alpha|+1} \mathcal{A}(P_{n_i}),$$

where  $p_{n_1+n_2-2}$  is obtained by glueing  $p_{n_1}$  and  $p_{n_2}$  along  $d$ . Consider  $\gamma = \{(1, n_1)\} \in \mathcal{A}(P_{n_1+n_2-2})$ . By induction hypothesis  $f_\gamma \equiv \mathcal{A}(P_{n_1}) \times \mathcal{A}(P_{n_2})$  and the inclusion  $\tau : f_\gamma \rightarrow \mathcal{A}(P_{n_1+n_2-2})$  is defined. Hence,  $i_{\beta\alpha} : f_\alpha \rightarrow f_\beta$  can be defined using  $\tau$  and taking the identity over the factor  $\prod_{i=3}^{|\alpha|+1} \mathcal{A}(P_{n_i})$ .

When  $|\alpha| > |\beta| + 1$ , it suffices to consider  $\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_k = \beta$  such that  $|\alpha_{i+1}| = |\alpha_i| + 1$ . The composition of the applications  $f_{\alpha_{i+1}\alpha_i}$  gives  $f_{\beta\alpha}$ .

Finally, define  $\partial\mathcal{A}(P_n)$  as

$$\left( \bigsqcup_{\alpha \in \mathcal{T}(P_n) - \emptyset_n} f_\alpha \right) / \sim,$$

where  $f_\alpha \sim i_{\beta\alpha}(f_\alpha)$  for all  $\alpha < \beta$ . Stasheff [30] proved that  $\partial\mathcal{A}(P_n)$  is homeomorphic to the sphere  $S^{n-4}$  of dimension  $n - 4$ . Thus,  $\mathcal{A}(P_n)$  is defined by filling the interior of  $\partial\mathcal{A}(P_n)$  by a ball of dimension  $n - 3$ .

**Example:**  $\mathcal{A}(P_5)$ . There are five different triangulations of a pentagon, and there are also five tessellations cutting the pentagon into one quadrilateral and one triangle, each of them being contained in two different triangulations. Hence, the associahedron of a pentagon is itself a pentagon.

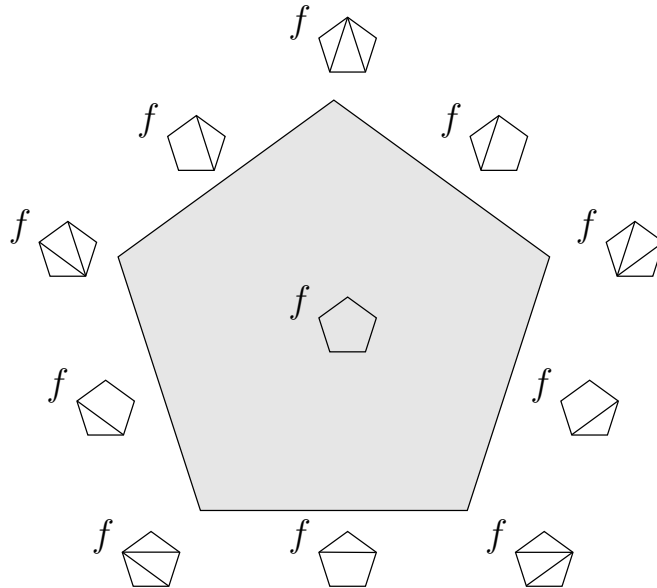


Figure 1: The associahedron  $\mathcal{A}(P_5)$ .

**Remark:** one can pass from our version to Greenberg's by taking the dual graph of each tessellation, rooted with respect to a marked side of the polygon  $P_n$ . In Greenberg's construction, the poset indexing the faces of the  $n$ -th associahedron is the set of rooted trees with  $n$  leaves, and a tree  $t_1$  is smaller than a tree  $t_2$  if  $t_1$  can be obtained from  $t_2$  by a sequence of collapsing edges. Note that our  $\mathcal{A}(P_n)$  corresponds to Greenberg's  $A_{n-1}$  (the dual graph of a tessellation of an  $n$ -sided polygon has  $n$  vertices with valence 1, i.e. the root and  $n - 1$  leaves).

### 3 The infinite associahedron

As in the finite case, the infinite associahedron is constructed as the union of closed cells, each one indexed by an element of a given partially ordered set, modulo inclusions following the order relation. The elements of the partially ordered set can be seen geometrically as all possible tessellations of the circle which differ from a given triangulation only in finitely many diagonals. It is worth to mention that, although the objects are described geometrically, only their combinatorial properties are used.

#### 3.1 Construction

Let  $D$  be the open disk in  $\mathbb{R}^2$  of center  $(0, 0)$  and perimeter 1. The boundary of  $D$  is denoted  $\partial D$ . Let  $\gamma : [0, 1] \rightarrow \partial D$  be the arc-parametrization of  $\partial D$  with  $\gamma(0) = (\frac{1}{2\pi}, 0)$ . Let  $\mathcal{A}$  be the set of geodesic segments with (different) extremal points in  $\gamma(\mathbb{Z}[1/2] \cap [0, 1])$ , where

$$\mathbb{Z}[1/2] \cap [0, 1] = \left\{ \frac{m}{2^n} \in \mathbb{R} : m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}, m \leq 2^n \right\}.$$

The elements of  $\mathcal{A}$  are called *dyadic arcs* and, for  $x, y \in \mathbb{Z}[1/2] \cap [0, 1]$ , the pair  $(x, y)$  denotes the dyadic arc with extremal points  $\gamma(x)$  and  $\gamma(y)$ .

Consider the following subset of  $\mathcal{A}$ :

$$A_F = \left\{ \left(0, \frac{1}{2}\right) \right\} \cup \left\{ \left(\frac{m}{2^n}, \frac{m+1}{2^n}\right) : m, n \in \mathbb{Z}, m \in \{0, \dots, 2^n - 1\}, n > 1 \right\}.$$

Note that  $A_F$  defines a triangulation<sup>1</sup> of  $D$ , meaning that  $D - A_F$  is a disjoint union of open triangles. Furthermore, the triangulation is minimal in the sense that, for every dyadic arc  $a$  of  $A_F$ , the set  $A_F - \{a\}$  no longer defines a triangulation of  $D$ .

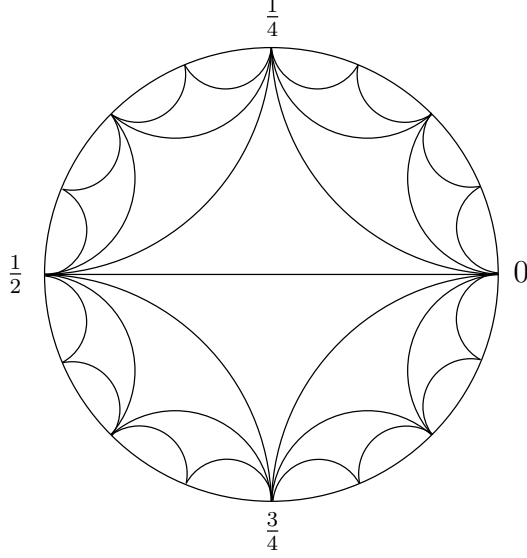


Figure 2: Triangulation of  $D$  defined by  $A_F$ .

A subset  $A$  of  $\mathcal{A}$  is an  $F$ -triangulation if the following conditions are satisfied:

1.  $D - A$  is a disjoint union of open triangles,
2. for every dyadic arc  $a$  of  $A$ , the set  $A - \{a\}$  no longer defines a triangulation of  $D$ , and
3. the subsets  $A$  and  $A_F$  differ only on finitely many dyadic arcs, meaning that their symmetric difference is a finite set.

In particular, any two different dyadic arcs  $a_1, a_2$  of  $A$  do not cross each other in  $D$  (but they can have an endpoint in common).

A subset  $B$  of  $\mathcal{A}$  is an  $F$ -tessellation if there exists an  $F$ -triangulation  $A$  and  $a_1, \dots, a_k$  dyadic arcs of  $A$  such that  $B = A - \{a_1, \dots, a_k\}$ . The number  $k$  is the *rank* of  $B$  and it is well-defined because all the  $F$ -tessellations are minimal. In particular, an  $F$ -triangulation is an  $F$ -tessellation of rank 0. Note that the  $F$ -triangulation  $A$  and the dyadic  $a_1, \dots, a_k$  defining  $B$  are not unique, but there are finitely many possibilities.

Let  $\mathcal{I}$  be the set of  $F$ -tessellations of  $D$ . Let  $A \in \mathcal{I}$  of rank  $k$ . By definition,  $D - A$  is a disjoint union of infinitely many triangles and finitely many non-triangles. Let  $n_1, \dots, n_m$  be the number of sides of the non-triangular polygons of  $D - A$ . Then, we associate to  $A$  the following  $k$ -cell:

$$f_A \equiv \prod_{i=1}^m \mathcal{A}(P_{n_i}).$$

<sup>1</sup>The pictures are in hyperbolic geometry for clarity. Furthermore, Thurston noticed that Thompson's group  $T$  can be seen as the group of homeomorphisms of the real projective line which are piecewise  $PSL(2, \mathbb{Z})$  [22, 13].

If  $k = 0$  the product is empty and we associate to  $A$  a point. Note that one could also take as a definition the infinite product of all associahedra since only finitely may be different from a point.

As in the case of Stasheff's associahedra, we can define a partial order on  $\mathcal{I}$ : if  $A, B \in \mathcal{I}$  are such that  $B \subset A$ , we say that  $A < B$ . Thus, if  $A < B$ , then there is an injective map  $\iota_{BA} : f_A \rightarrow f_B$  defined as in the case of Stasheff's associahedra. To do so, consider the smallest polygon  $P_m$  inscribed in  $A \cap B$  containing all non-triangular polygons of  $D - B$ . Let  $\alpha$  (respectively  $\beta$ ) be the tessellation of  $P_m$  obtained by the restriction of  $A$  (respectively  $B$ ) on  $P_m$ . Furthermore,  $\alpha < \beta$ . Then,  $f_\alpha \equiv f_A$ ,  $f_\beta \equiv f_B$  and there is an inclusion  $i_{\beta\alpha} : f_\alpha \rightarrow f_\beta$  on  $\mathcal{A}(P_m)$  defining  $\iota_{BA}$  by composition with the two previous isomorphisms. Note that, the vertices of an  $F$ -tessellation being indexed by all dyadic rational numbers of the unit interval, the injections  $\iota_{BA}$  are well determined by the inclusions  $B \subset A$ . Note also that the boundary  $\partial f_B$  of a  $k$ -cell is isomorphic to the union of all  $j$ -cells  $f_A$  such that  $A < B$  (hence  $j < k$ ).

The *infinite associahedra* is the cellular complex

$$C = \left( \bigsqcup_{A \in \mathcal{I}} f_A \right) / \sim,$$

where  $\sim$  is the equivalence relation generated by  $\{x \sim f_{BA}(x) : (A, B) \in \mathcal{I}^2, x \in A, A < B\}$ . Remark that  $C$  is a *regular CW complex* in the sense that the attaching maps are homeomorphisms.

### 3.2 Low dimensional cells

It can be useful to describe explicitly all kinds of cells up to dimension 3. The set of vertices of  $C$  is the set of  $F$ -triangulations, and two  $F$ -triangulations  $A_1, A_2$  are joined by an edge in  $C^1$  if and only if their intersection  $A_1 \cap A_2$  is an  $F$ -tessellation of rank 1. Equivalently, there exists a unique dyadic arc  $a_1 \in A_1$  such that  $a_1 \notin A_2$ , and there exists a unique dyadic arc  $a_2 \in A_2$  satisfying  $a_2 \notin A_1$ . Furthermore,  $a_1, a_2$  are the two possible diagonals of the single non-triangular component of  $D - (A_1 \cap A_2)$  (this component is a square).

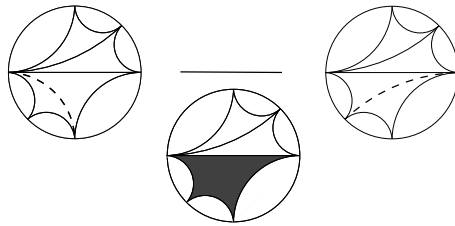


Figure 3: Edge of  $C^1$  with its associated  $F$ -tessellations of rank 0 and 1.

The 2-skeleton  $C^2$  is constructed from  $C^1$  by attaching 2-cells as follows: let  $B$  be an  $F$ -tessellation of rank 2. Consider the non-triangular components of  $D - B$ . One of the following situations is satisfied:

1. There are exactly two non-triangular components and both of them are squares. Let  $a_1, a_2$  be the two dyadic arcs which are interior diagonals of the first squared component, and  $a_3, a_4$  the diagonals of the second square. Define  $A_1 = B \cup \{a_1, a_3\}$ ,  $A_2 = B \cup \{a_1, a_4\}$ ,  $A_3 = B \cup \{a_2, a_3\}$  and  $A_4 = B \cup \{a_2, a_4\}$ . For  $i \in \{1, 2, 3, 4\}$ ,  $A_i$  is an  $F$ -triangulation. Furthermore,  $\{A_1, A_2, A_3, A_4\}$  is the set of all  $F$ -triangulations containing  $B$ . The sub-graph of  $C^1$  induced by the set of vertices  $\{A_1, A_2, A_3, A_4\}$  is a closed path of length 4. Thus, it can be seen as the boundary of a square. One glues a 2-cell along this closed path such that the attaching map is an homeomorphism.

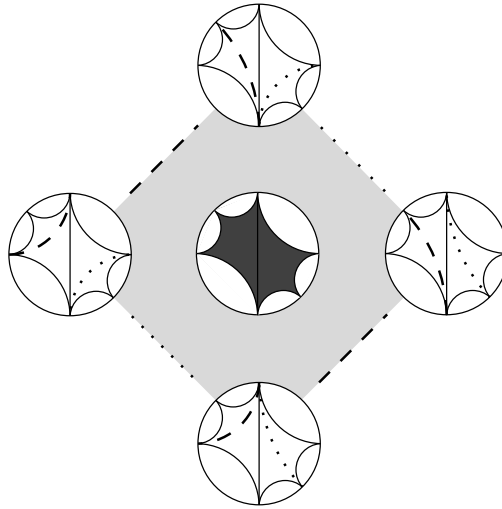


Figure 4: Squared 2-cell with its associated  $F$ -tessellations of rank 0 and 2.

2. There is a unique non-triangular component and it is a pentagon. Let  $\gamma(v_1), \gamma(v_2), \gamma(v_3), \gamma(v_4), \gamma(v_5)$  be its vertices, where  $0 \leq v_1 < v_2 < v_3 < v_4 < v_5 < 1$ . Define  $A_1 = B \cup \{(v_1, v_3), (v_1, v_4)\}$ ,  $A_2 = B \cup \{(v_1, v_3), (v_3, v_5)\}$ ,  $A_3 = B \cup \{(v_1, v_4), (v_2, v_4)\}$ ,  $A_4 = B \cup \{(v_2, v_4), (v_2, v_5)\}$  and  $A_5 = B \cup \{(v_2, v_5), (v_3, v_5)\}$ . For  $i \in \{1, 2, 3, 4, 5\}$ ,  $A_i$  is an  $F$ -triangulation. Furthermore,  $\{A_1, A_2, A_3, A_4, A_5\}$  is the set of all  $F$ -triangulations containing  $B$ . The sub-graph of  $C^1$  induced by the set of vertices  $\{A_1, A_2, A_3, A_4, A_5, A_1\}$  is a closed path of length 5. Thus, it can be seen as the boundary of a pentagon. One glues a 2-cell along this closed path in a way that the attaching map is a homeomorphism.

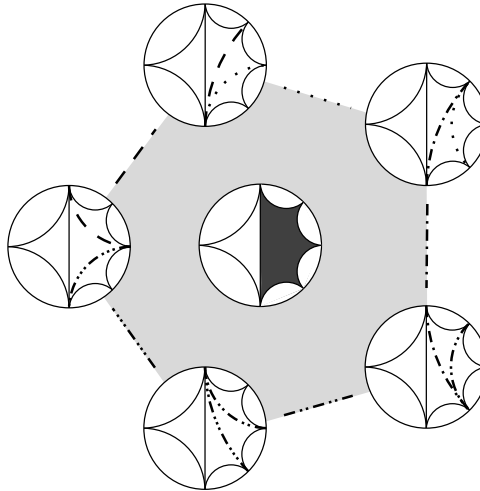


Figure 5: Pentagonal 2-cell with its associated  $F$ -tessellation of rank 0 and 2.

Let  $B$  be an  $F$ -tessellation of rank 3. One of the following is satisfied:

- There are exactly 3 non-triangular components of  $D - B$ , all of them being squares. In this case, the set of all  $F$ -triangulations containing  $B$  has 8 elements  $\{A_1, \dots, A_8\}$  (each one obtained by adding to  $B$  one diagonal of each squared component of  $D - B$ ). The sub-complex of  $C^2$  induced by the



set of vertices  $\{A_1, \dots, A_8\}$  is the boundary of a cube, so it is topologically a sphere of dimension 2. One glues a 3-cell inside this cube by *choosing* the attaching map to be a homeomorphism.

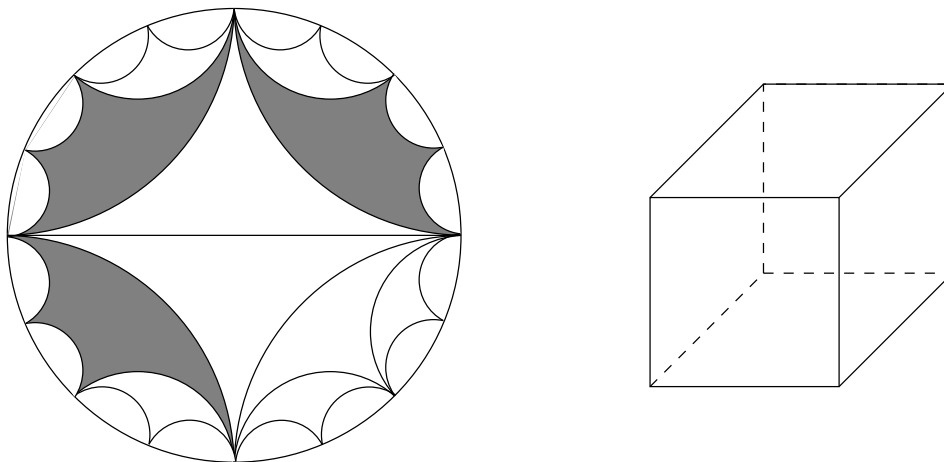


Figure 6: Cubical 3-cell with its associated  $F$ -tessellation of rank 3.

- There are exactly 2 non-triangular components, one of them being a square and the other being a pentagon. To get an  $F$ -triangulation from  $B$ , one chooses independently a triangulation of the pentagonal component and a triangulation of the square. Thus, the set of all  $F$ -triangulations containing  $B$  has 10 elements and the sub-complex of  $C^2$  they induce is the boundary of the polyhedron obtained by the cross product of a pentagon and an interval, which is topologically a sphere of dimension 2. One glues a 3-cell inside this polyhedron by choosing the attaching map to be a homeomorphism.

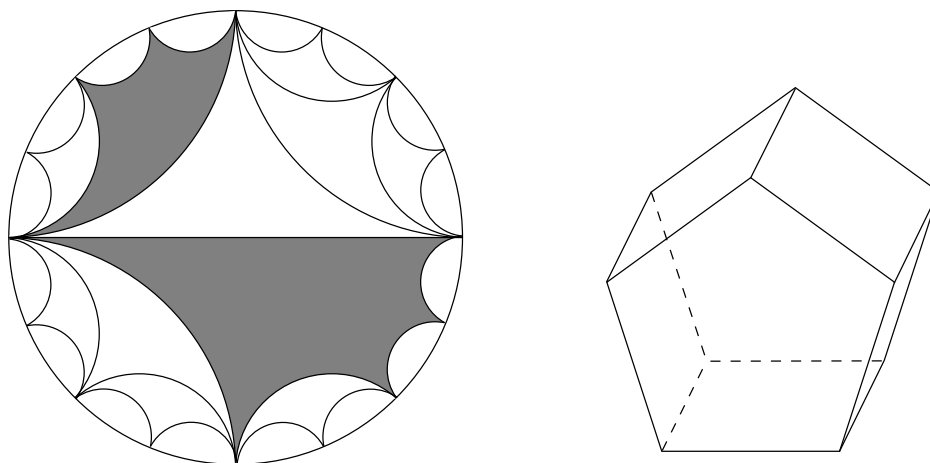


Figure 7: Prism 3-cell with its associated  $F$ -tessellation of rank 3.

- There is a unique non-triangular component, and it is an hexagon. Then, the sub-complex of  $C^2$  induced by the set of  $F$ -triangulations containing  $B$  is isomorphic to the boundary of Stasheff's associahedron of an hexagon, which is topologically a sphere of dimension 2. One glues a 3-cell inside by choosing the attaching map to be an homeomorphism.

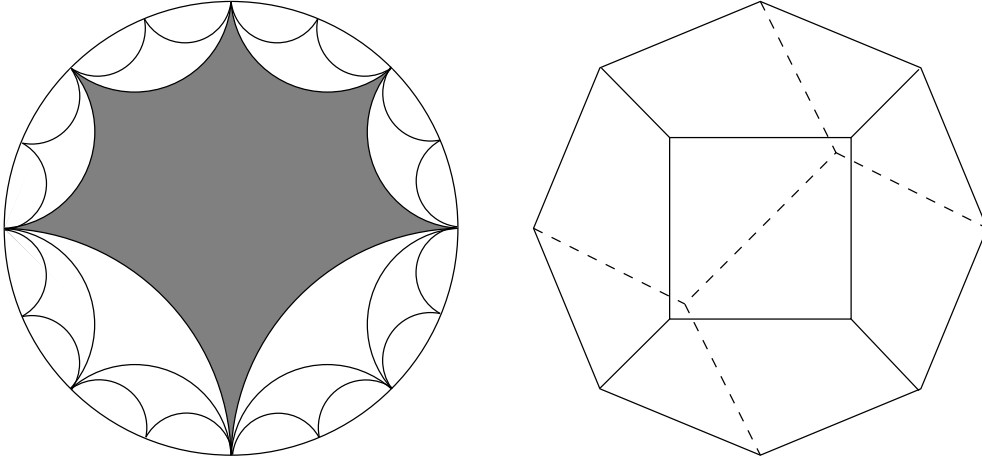


Figure 8: Associahedral 3-cell with its associated  $F$ -tessellation of rank 3.

### 3.3 The complex $C$ is aspherical and contractible

**Lemma 1.** *The complex  $C$  is path-connected.*

*Proof.* Let  $x$  and  $y$  be two points of  $C$ . Then, there exist  $A, B \in \mathcal{I}$  such that  $x \in f_A$  and  $y \in f_B$ . Consider  $D = A \cap B \in \mathcal{I}$ . Note that  $A < D$  and  $B < D$ , thus  $f_{DA}(x)$  and  $f_{DB}(y)$  are points of  $f_D$ , which is a cell of  $C$  of dimension  $\text{rank}(D)$ , thus  $f_D$  is path-connected.  $\square$

Whitehead's theorem [23, 24] states that every continuous map between connected CW-complexes which induces isomorphisms on all homotopy groups is a homotopy equivalence. We have already seen that  $C$  is a CW-complex. In particular, we can prove that  $C$  is contractible by showing that all homotopy groups  $\pi_n(C)$  ( $n \geq 1$ ) are trivial.

**Lemma 2.** *The homotopy groups  $\pi_n(C)$  of  $C$  are trivial for all  $n \geq 1$ . In particular,  $C$  is aspherical ( $\pi_n$  are trivial for  $n \geq 2$ ).*

*Proof.* Let  $g : S^n \rightarrow C$  a continuous map ( $n \geq 1$ ). The image  $g(S^n)$  intersects finitely many open cells of  $C$ ,  $f_{A_1}, \dots, f_{A_k}$ , because  $S^n$  is compact and  $C$  is a CW-complex. By open cell we mean the image of an open ball by its attaching map. Suppose that  $g(S^n)$  is not a single point of  $C$ . If there exists  $g(x) \in g(S^n)$  such that  $g(x) \notin f_{A_1} \cup \dots \cup f_{A_k}$ , then  $g(x)$  is a vertex of  $C$ . Furthermore, every open neighbourhood  $U$  of  $x$  contains a point  $y \in U$  such that  $g(y) \neq g(x)$ ,  $g(y) \in f_{A_1} \cup \dots \cup f_{A_k}$  (the 0-skeleton of  $C$  is totally disconnected). Hence,  $g(x)$  belongs to the closure of  $f_{A_1} \cup \dots \cup f_{A_k}$ , which is included into  $f_{A_1} \cup \dots \cup f_{A_k}$ , proving that  $g(S^n) \subset f_{A_1} \cup \dots \cup f_{A_k}$ .

Let  $A_1, \dots, A_k$  be the elements of  $I$  corresponding to the cells  $f_{A_1}, \dots, f_{A_k}$ . Define  $A = A_1 \cap \dots \cap A_k$ . By definition  $f_A$  is a cell of  $C$  containing  $f_{A_1} \cup \dots \cup f_{A_k}$ . Hence,  $f_A$  is contractible and contains  $g(S^n)$ . Thus,  $g(S^n)$  is homotopy-equivalent to a point.

The argument holds for all maps  $g : S^n \rightarrow C$  ( $n \geq 1$ ). Thus  $\pi_n(C)$  is trivial for all  $n \geq 1$ .  $\square$

Note that the main argument of the proof can be stated as follows: for every couple of cells of  $C$ , there exist a third cell which contains both of them. The following result is a direct consequence of Whitehead's theorem and the previous lemma.

**Corollary 1.** *The infinite associahedron is contractible.*

## 4 The automorphism group $C$ and isometries of $C^1$

### 4.1 Non-oriented Thompson's group $T$

The non-oriented Thompson's group  $T$  is the group of piecewise linear homeomorphisms of the circle  $S^1$ , thought as the unit interval  $[0, 1]$  with identified endpoints, which:

1. map the set of dyadic rational numbers to itself,
2. are differentiable except at finitely many points, all of them being dyadic rational numbers, and
3. on intervals of differentiability, the derivatives are powers of 2.

We denote the non-oriented Thompson's group  $T$  by  $T^{no}$ . Note that  $T^{no} \simeq T \rtimes \mathbb{Z}/2\mathbb{Z}$  because the following short sequence

$$\begin{array}{ccccccc} 1 & \longrightarrow & T & \longrightarrow & T^{no} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \\ & & & & t & \longmapsto & \begin{cases} 0, & \text{if } t \text{ preserves the orientation of } [0, 1] \\ 1, & \text{if } t \text{ reverses the orientation of } [0, 1]. \end{cases} \end{array}$$

is exact and admit the section  $\mathbb{Z}/2\mathbb{Z} \simeq \langle -\text{id} \rangle$ .

The elements of Thompson's group  $T$  can be seen as pairs of standard dyadic partitions of the unit interval  $[0, 1]$ . One can easily adapt this result to non-oriented  $T$ . This characterization will be used to define the action of both groups on the set of  $F$ -tessellations.

A subinterval  $[x_1, x_2]$  of the unit interval  $[0, 1]$  is called a *standard dyadic interval* if it is of the form  $\left[ \frac{m}{2^n}, \frac{m+1}{2^n} \right]$  for some positive integers  $m$  and  $n$  satisfying  $0 \leq m \leq 2^n - 1$ .

A partition of the unit interval given by  $x_0 = 0 < x_1 < x_2 < \dots < x_{k-1} < x_k = 1$  is a *standard dyadic partition* if, for all  $i \in \{1, \dots, k-1\}$ , the subinterval  $[x_i, x_{i+1}]$  is a standard dyadic interval.

**Lemma 3.** (analogue to [6], Lemma 2.2) *Let  $t$  be an element of  $T^{no}$ . Then, there exists a standard dyadic partition of the unit interval  $0 = x_0 < x_1 < \dots < x_k = 1$  such that:*

1.  $t$  is affine on every subinterval of the partition, and
2. the induced partition on the  $y$  axis, which is either

$$0 = t(x_i) < t(x_{i+1}) < \dots < t(x_k) = t(x_0) < \dots < t(x_{i-1}) = 1$$

or

$$1 = t(x_i) > t(x_{i+1}) > \dots > t(x_k) = t(x_0) > \dots > t(x_{i-1}) = 0$$

is also a standard dyadic partition of the unit interval.

*Proof.* Consider the  $x$  axis partition associated to  $t$ ,  $0 = z_0 < z_1 < \dots < z_k = 1$ . As  $t \in T^{no}$ ,  $z_0, \dots, z_k$  are dyadic rational numbers and  $t$  is affine on each interval of the partition. Let  $[z_i, z_{i+1}]$  be an interval of this partition and suppose that  $t'(x) = \pm 2^{-r}$ , if  $x \in [z_i, z_{i+1}]$ . Let  $n$  be an integer such that  $2^n z_i$ ,  $2^n z_{i+1}$ ,  $2^{n+r} t(z_i)$  and  $2^{n+r} t(z_{i+1})$  are integers. Then,

$$z_i < z_i + \frac{1}{2^n} < z_i + \frac{2}{2^n} < z_i + \frac{3}{2^n} < \dots < z_{i+1}$$

is a standard dyadic partition of the interval  $[z_i, z_{i+1}]$ , and its image

$$\begin{cases} t(z_i) < t(z_i) + \frac{1}{2^{n+r}} < t(z_i) + \frac{2}{2^{n+r}} < t(z_i) + \frac{3}{2^{n+r}} < \dots < t(z_{i+1}) & \text{if } t'(x) > 0, \\ t(z_i) > t(z_i) + \frac{1}{2^{n+r}} > t(z_i) + \frac{2}{2^{n+r}} > t(z_i) + \frac{3}{2^{n+r}} > \dots > t(z_{i+1}) & \text{if } t'(x) < 0, \end{cases}$$

is a standard dyadic partition of the interval  $[t(z_i), t(z_{i+1})]$  as well. One can repeat the previous procedure for every interval of the  $x$ -axis partition associated to  $t$ .  $\square$

Since any standard dyadic interval can be split into two standard dyadic intervals by taking the midpoint, one can not expect the partitions of Lemma 3 to be unique. However, there exists a standard dyadic partition satisfying Proposition 3 with a minimum number of standard dyadic intervals, which is called the *minimal standard dyadic partition*. It can be proved that the minimal standard dyadic partition exists and every partition fulfilling Lemma 3 is a sub-partition of this minimal standard dyadic partition [6].

## 4.2 Action of $T^{no}$ on the set of $\mathcal{I}$

Let  $A$  be an  $F$ -tessellation and  $t$  an element of  $T^{no}$ . Recall that  $t$  induces a bijection into the set of dyadic numbers of the interval. Let  $a$  be a dyadic arc of  $A$ , with dyadic endpoints  $d_1, d_2$ . Then the  $F$  tessellation  $t \cdot A$  contains the dyadic arc with endpoints  $t(d_1), t(d_2)$ .

Note that the dyadic arcs of  $A_F$  correspond to standard dyadic intervals of length less than (or equal to)  $1/2$ , where  $[0, 1/2]$  and  $[1/2, 1]$  have been identified. In particular, standard dyadic partitions of the unit interval with at least three pieces are in one to one correspondence with inscribed polygons of  $A_F$  containing (eventually on the boundary) the center  $(0, 0)$  of  $D$ .

**Lemma 4.** *The action of  $T^{no}$  on the set  $\mathcal{I}$  of  $F$ -tessellations given before is well defined.*

*Proof.* Let  $A$  be an  $F$ -tessellation and let  $t$  be an element of  $T^{no}$ . Let  $P$  be the smallest polygon inscribed in  $F$  containing all non-triangular polygons of the  $F$ -tessellation  $A \cap A_F$ , and the center of  $D$ . Let  $\bar{p}$  be the standard dyadic partition associated to  $P$ , and let  $\bar{x}$  be the minimal standard dyadic partition of  $t$ . Let  $\bar{z}$  be a common sub-partition of  $\bar{x}$  and  $\bar{p}$ . By Lemma 3,  $t$  is affine in every interval of  $\bar{z}$ , which means that the tessellation  $t \cdot A$  coincides exactly with  $A_F$  outside the image of the polygon  $P$ . Inside  $P$  there are finitely many dyadic arcs, thus  $t \cdot A$  contains finitely many dyadic arcs different from those on  $A_F$  and  $t \cdot A \in \mathcal{I}$ . Remark that the rank of  $t \cdot A$  coincides with the rank of  $A$ .  $\square$

Recall that the infinite associahedron has been defined by associating to each  $F$ -tessellation a closed cell. It is thus natural to ask if the action of  $T^{no}$  on  $\mathcal{I}$  induces an action on  $C$ .

A combinatorial automorphism of  $C$  (automorphism for short) is a bijection between the set of closed cells  $\{f_A : A \in \mathcal{I}\}$  of  $C$  to itself preserving dimensions, inclusions and boundaries. Let  $f_1, f_2$  be two different cells of  $C$  of the same dimension. Note that, by construction, the boundaries  $\partial f_1, \partial f_2$  are different. Thus, for all  $k \in \mathbb{N}$ ,

$$\text{Aut}(C^k) \subseteq \text{Aut}(C^{k-1}).$$

**Proposition 1.** *Non-oriented  $T^{no}$  acts faithfully on  $C$  by automorphisms. The action is given by  $t \cdot f_A = f_{t \cdot A}$ . Furthermore, the automorphism group of  $C$  is isomorphic to  $T^{no}$ .*

*Proof.* The action of  $T^{no}$  on the set of  $F$ -tessellations preserves the rank and the partial order. Thus, the action is well-defined. Furthermore, it coincides with the action of  $T^{no}$  given in [14], which is faithful. Finally, we know that  $\text{Aut}(C)$  is a subgroup of  $\text{Aut}(C^2)$ , and  $\text{Aut}(C^2) \simeq T^{no}$  (see [14]). Hence, the two groups are isomorphic.  $\square$

**Corollary 2.** *For all  $2 \leq n \in \mathbb{N}$ , the automorphism group of the  $n$ -skeleton of  $C$  is isomorphic to non-oriented  $T^{no}$ .*

### 4.3 Isometries of $C^1$

The 1-skeleton of  $C$  can be easily realised as a metric space by identifying each edge with the unit euclidean segment. Then, the group of isometries of the metric realisation of  $C^1$  coincides with the automorphism group of the graph  $C^1$ .

**Proposition 2.** *The group of isometries of the metric realisation of  $C^1$  is isomorphic to  $T^{no}$ .*

*Proof.* Let  $e_1, e_2$  be two consecutive edges of  $C^1$ . By definition, there exist  $A, B$  two different  $F$ -tessellations of rank 1 such that  $e_1 = f_A$  and  $e_2 = f_B$ . Furthermore,  $A \cap B$  is a  $F$ -tessellation of rank 2,  $e_1, e_2 \in \partial f_{A \cap B} \subset C^1$ , and  $\partial f_{A \cap B}$  is the unique minimal closed path of  $C^1$  containing  $e_1, e_2$ . The length of this path is either 4 or 5, and it depends only on the number of non-triangular polygons of  $D - A \cap B$ .

Let  $\varphi$  be an automorphism of  $C^1$ , and let  $C$  and  $D$  be  $F$ -tessellations of rank 1 such that  $\varphi(e_1) = f_C$  and  $\varphi(e_2) = f_D$ . Then,  $\varphi(\partial f_{A \cap B})$  is the unique minimal closed path containing  $\varphi(e_1), \varphi(e_2)$ . Thus, it coincides with  $\partial f_{C \cap D}$ , forcing  $\partial f_{C \cap D}$  and  $\partial f_{A \cap B}$  to have the same length. Hence,  $\varphi$  can be extended to a unique automorphism of  $C^2$ .  $\square$

If  $g$  is an isometry of a metric space  $X$ , then its translation length is

$$|g| = \inf\{d(x, g(x)) : x \in X\}.$$

We say that  $g$  is *semi-simple* when the infimum in the definition of  $|g|$  is realised as a minimum. Note that  $C^1$  can be seen as a cubical complex of dimension 1 (by identifying each edge with the unit interval). Bridson [3] proved that all isometries of a polyhedral complex where the number of isometry types of polygons is finite are semi-simple. In particular, this result can be applied to the metric realisation of  $C^1$ .

**Corollary 3.** *All isometries of  $C^1$  are semi-simple.*

In References [3] poly... in [6]notes on?? [7] full data needed in 11 check if published no in 13 Publ Math = ?? in [20] and [25] never published;? url? in [26] why not reference the Thompson article specifically

## References

- [1] B.Jónsson and A. Tarski. On two properties of free algebras. *Math. Scand.*, 9:95–101, 1961.
- [2] J.M. Boardman and R.M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Springer Verlag, L.N.M 347, 1973.
- [3] M. R. Bridson. On the semisimplicity of polyhedral isometries. *Proc. Am. Math. Soc.*, 127 (7):2143–2146, 1999.
- [4] K. S. Brown. Finiteness properties of groups. *Journal of Pure and Applied Algebra*, 44:45–75, 1987.
- [5] K. S. Brown and R. Geoghegan. An infinite-dimensional torsion free FP infinity group. *Invent. Math.*, 77:367–381, 1984.

- [6] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math.*, 42:215–256, 1996.
- [7] F. Chapoton, S. Fomin, and A. Zelevinsky. Polytopal realizations of generalized associahedra. *Canad. Math. Bull.*, 45(4):537–566, 2002.
- [8] S. L. Devadoss. Tessellations of moduli spaces and the mosaic operad. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 91–114. Amer. Math. Soc., Providence, RI, 1999.
- [9] D. S. Farley. Proper isometric actions of Thompson's groups on Hilbert spaces. *Int. Math. Res. Not.*, 45:2409–2414, 2003.
- [10] D. S. Farley. Actions of picture groups on CAT(0) cubical complexes. *Geom. Dedicata*, 110:221–242, 2005.
- [11] D. S. Farley. The actions of Thompson's group on a CAT(0) boundary. *Groups Geom. Dyn.*, 2:185–222, 2008.
- [12] D. S. Farley. A proof that Thompson's groups have infinitely many relative ends. *Journal of Group Theory*, 14:649—656, 2011.
- [13] A. Fossas.  $PSL_2(\mathbb{Z})$  as a non distorted subgroup of Thompson's group  $T$ . *Indiana Univ. Math. J.*, to appear.
- [14] A. Fossas and M. Nguyen. Thompson's group  $T$  is the orientation-preserving automorphism group of a cellular complex. *Publ. Mat.*, 56 (2):305–326, 2012.
- [15] L. Funar and C. Kapoudjian. On a universal mapping class group of genus zero. *Geometric and Functional Analysis*, 14:965–1012, 2004.
- [16] L. Funar and C. Kapoudjian. The braided Ptolemy-Thompson group is finitely presented. *Geom. Topol.*, 12:475–530, 2008.
- [17] L. Funar and C. Kapoudjian. The braided Ptolemy-Thompson group is asynchronously combable. *Comment. Math. Helv.*, 86(3):707–768, 2011.
- [18] L. Funar, C. Kapoudjian, and V. Sergiescu. Asymptotically rigid mapping class groups and Thompson's. In A. Papadopoulos, editor, *Handbook of Teichmüller theory*, volume IV, pages 595–664. European Mathematical Society Publishing House, 2012.
- [19] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.
- [20] P. Greenberg. Les espaces de bracelets, les complexes de Stasheff et le groupe de Thompson. *Boletín Soc. Mat. Mexicana*, 37:189–201, 1992.
- [21] P. Greenberg. Primer on CPP geometry. *preprint in Strukturen in der Mathematik*, 92-006, 1992.
- [22] M. Imbert. Sur l'isomorphisme du groupe de Richard Thompson avec le groupe de Ptolémée. In *Geometric Galois actions 2*, pages 313–324. L.M.S. Lecture Notes Series, 243, Cambridge, 1997.
- [23] J.H.C.Whitehead. Combinatorial homotopy I. *Bull. Amer. Math. Soc.*, 55:213–245, 1949.
- [24] J.H.C.Whitehead. Combinatorial homotopy II. *Bull. Amer. Math. Soc.*, 55:453–496, 1949.

- [25] C. W. Lee. The associahedron and triangulations of the  $n$ -gon. *European J. Combin.*, 10:551–560, 1989.
- [26] J.L. Loday. Realization of the Stasheff polytope. *Arch. Math.*, 83:267–278, 2004.
- [27] X. Martin. Sur la géométrie du groupe de Thompson. *PhD Thesis. Institut Fourier. Grenoble.*, 2002.
- [28] R. McKenzie and R. J. Thompson. An elementary construction of unsolvable word problems in group theory. In F.B. Cannonito W.W. Boone and R.C. Lyndon, editors, *Studies in Logic and the Foundations of Mathematics*, volume 71, pages 457–478. Word Problems, North Holland, Amsterdam, 1973.
- [29] F. Muller-Hoissen, J.M. Pallo, and J. Stasheff. *Associahedra, Tamari Lattices and Related Structures*, volume 299 of *Progress in Mathematics*. Springer, Basel, 2012.
- [30] J. Stasheff. Homotopy associativity of H-spaces I. *Trans. Amer. Math. Soc.*, 108:275–292, 1963.
- [31] J. Stasheff. Homotopy associativity of H-spaces II. *Trans. Amer. Math. Soc.*, 108:293–312, 1963.
- [32] M. Stein. Groups of piecewise linear homeomorphisms. *Trans. Amer. Math. Soc.*, 332:477–514, 1992.
- [33] D. Tamari. Monoides préordonnés et chaines de Malcev. *PhD Thesis. Paris.*, 1951.
- [34] A. Tonks. Relating the associahedron and the permutohedron. In *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, volume 202 of *Contemp. Math.*, pages 33–36. Amer. Math. Soc., Providence, RI, 1997.