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MULTIFRACTAL ANALYSIS OF FUNCTIONS ON THE HEISENBERG GROUP

S. SEURET AND F. VIGNERON

ABSTRACT. In this article, we investigate the pointwise behaviors of functions on the Heisenberg group. We find wavelet characterizations for the global and local Hölder exponents. Then we prove some a priori upper bounds for the multifractal spectrum of all functions in a given Hölder, Sobolev or Besov space. These upper bounds turn out to be optimal, since in all cases they are reached by typical functions in the corresponding functional spaces.

1. INTRODUCTION

In this article, we draw the first results for a multifractal analysis for functions defined on the Heisenberg group \mathbb{H} . Multifractal analysis is now a widespread issue in analysis. Its objective is to provide a description of the variety of local behaviors of a given function or a given measure. The local behaviors are measured thanks to the pointwise Hölder exponent and one aims at describing the distribution of the iso-Hölder sets *i.e.* the sets of points $x \in \mathbb{H}$ with same pointwise exponent. What makes the Heisenberg group interesting for our dimensional considerations is that its Hausdorff dimension is $\dim_H(\mathbb{H}) = 4$ while it is defined using only three topological coordinates. This is due to the special form of the metric in the “vertical” direction. This induces surprising properties from the geometric measure theoretic standpoint which are currently being investigated; for instance, Besicovitch’s covering theorem and Marstrand’s projection theorem are not true, see [1, 2, 25, 17, 20]. In this paper, we pursue this investigation by studying the multifractal properties of functions defined on \mathbb{H} . We find an *a priori* upper bound for the Hausdorff dimensions of iso-Hölder sets for all functions in a given Hölder and Besov space and we prove that these bounds are optimal, since they are reached for *generic* functions (in the sense of Baire’s categories) in these function spaces. To do so, we develop methods based on wavelets on \mathbb{H} [18].

Let us start by some basic facts on \mathbb{H} . The first Heisenberg group \mathbb{H} consists [22, p. 530] of the set \mathbb{R}^3 equipped with a non-commutative group law

$$(p, q, r) * (p', q', r') = (p + p', q + q', r + r' + 2(qp' - pq'))$$

which is also denoted by the absence of multiplicative symbol when the context is clear. The inverse of $x = (p, q, r)$ is $x^{-1} = (-p, -q, -r)$. The Haar measure is $dx = dp \wedge dq \wedge dr$ and is also denoted by ℓ . It is a homogeneous group [22, p. 618] whose dilations are defined by:

$$\lambda \circ (p, q, r) = (\lambda p, \lambda q, \lambda^2 r).$$

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A left-invariant distance $\delta(x, y) = \|x^{-1} * y\|_{\mathbb{H}}$ is given by the homogeneous pseudo-norm:

$$(1) \quad \|x\|_{\mathbb{H}} = \{(p^2 + q^2)^2 + r^2\}^{1/4}.$$

The Lie Algebra \mathfrak{h} is the vector-space of left invariant vector fields on \mathbb{H} . It is nilpotent of step 2 (see [22, p. 544]) *i.e.* $\mathfrak{h} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where \mathfrak{n}_1 is spanned by

$$(2) \quad X = \frac{\partial}{\partial p} + 2q \frac{\partial}{\partial r} \quad \text{and} \quad Y = \frac{\partial}{\partial q} - 2p \frac{\partial}{\partial r}$$

and \mathfrak{n}_2 is spanned by $Z = \frac{\partial}{\partial r}$. All commutators vanish except $[X, Y] = -4Z$. Hence $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{n}_2$ and $[\mathfrak{n}_2, \mathfrak{h}] = 0$. The homogeneous structure of \mathbb{H} induces dilations of \mathfrak{h} :

$$\lambda \circ (\alpha X + \beta Y + \gamma Z) = \lambda(\alpha X + \beta Y) + \lambda^2 \gamma Z,$$

which satisfy $\lambda \circ [U, V] = [\lambda \circ U, \lambda \circ V]$.

The positive self-adjoint hypoelliptic Laplace operator on \mathbb{H} is (see [11]):

$$(3) \quad \mathcal{L} = -(X^2 + Y^2).$$

Sobolev spaces of regularity index $s \geq 0$ can be defined by functional calculus:

$$(4) \quad H^s(\mathbb{H}) = \{u \in L^2(\mathbb{H}) : \mathcal{L}^{s/2} u \in L^2(\mathbb{H})\}.$$

Throughout the article, $Q = 4$ denotes the homogeneous dimension of \mathbb{H} . In order to state the results quickly, we postpone the classical definitions and notations (horizontal paths, Carnot balls, polynomials, Hausdorff dimension, Besov and Hölder spaces) to §2.

Let us define the pointwise Hölder regularity of a function.

Definition 1. *Let $f : \mathbb{H} \rightarrow \mathbb{R}$ be a function belonging to $L_{loc}^\infty(\mathbb{H})$. For $s > 0$ and $x_0 \in \mathbb{H}$, f is said to belong to $C^s(x_0)$ if there exist constants $C > 0$, $\delta > 0$ and a polynomial P with homogeneous degree $\deg_{\mathbb{H}}(P) < s$ such that*

$$(5) \quad \forall x \in \mathbb{H}, \quad \|x\| < \delta \implies |f(x_0 x) - P(x)| \leq C \|x\|_{\mathbb{H}}^s.$$

One says that $f \in C_{log}^s(x_0)$ if, instead of (5), the following holds:

$$(6) \quad |f(x_0 x) - P(x)| \leq C \|x\|_{\mathbb{H}}^s \cdot |\log \|x\|_{\mathbb{H}}|$$

Observe that this definition is left-invariant: $f \in C^s(x_0)$ if and only if $f_y \in C^s(y^{-1}x_0)$ with $f_y : x \mapsto f(yx)$.

The following quantities are crucial in multifractal analysis.

Definition 2. *Let $f : \mathbb{H} \rightarrow \mathbb{R}$ be a function belonging to $L_{loc}^\infty(\mathbb{H})$.*

The pointwise regularity exponent of f at x_0 is

$$(7) \quad h_f(x_0) = \sup\{s > 0 : f \in C^s(x_0)\}$$

with the convention that $h_f(x_0) = 0$ if $f \notin C^s(x_0)$ for any $s > 0$.

The multifractal spectrum of f is the mapping $d_f : [0, \infty] \rightarrow \{-\infty\} \cup [0, Q]$

$$d_f(h) = \dim_H(E_f(h)) \quad \text{where} \quad E_f(h) = \{x \in \mathbb{H} : h_f(x) = h\},$$

where \dim_H stands for the Hausdorff dimension on \mathbb{H} . By convention, $\dim_H \emptyset = -\infty$.

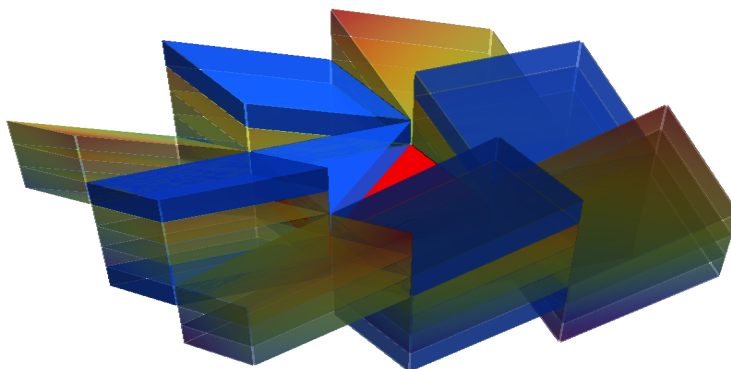


FIGURE 1. The cube C_0 admits 34 closest neighbors in \mathbb{H} contrary to euclidian cubes of \mathbb{R}^3 that admit only 26 neighbors.

The multifractal spectrum of f describes the geometrical distribution of the singularities of f over \mathbb{H} . The Hausdorff dimension is the right notion to use here, since (at least intuitively, but also for generic functions) the iso-Hölder sets $E_f(h)$ are dense over the support of f and the Minkowski dimension does not distinguish dense sets.

Wavelets are a key tool in our analysis. The construction of wavelets on stratified Lie groups has been achieved in [18]. A convenient observation is that $\mathcal{Z} = \mathbb{Z}^3$ is a sub-group of \mathbb{H} . For $j \in \mathbb{Z}$ and $k = (k_p, k_q, k_r) \in \mathcal{Z}$, one defines:

$$x_{j,k} = 2^{-j} \circ k = (2^{-j}k_p, 2^{-j}k_q, 2^{-2j}k_r).$$

Note that $x_{j,k}^{-1} = x_{j,-k}$. The dyadic cubes are defined in the following way:

$$C_0 = \{(p, q, r) \in \mathbb{H} : 0 \leq p, q, r < 1\} \quad \text{and} \quad C_{j,k} = x_{j,k} * (2^{-j} \circ C_0).$$

The left-multiplication by $x_{j,k}$ maps affine planes of \mathbb{R}^3 on affine planes, thus the shape of $C_{j,k}$ is a regular parallelogram with vertices on $2^{-j} \circ \mathcal{Z}$. But one shall observe that two different cubes $C_{j,k}$ and $C_{j,k'}$ are not in general euclidian translates of each other. A neighborhood $\Lambda_{j,k}$ of $C_{j,k}$ is given by

$$(8) \quad \Lambda_{j,k} = \bigcup_{k' \in \Xi} C_{j,k*k'},$$

where Ξ is the set of 35 multi-integers $k' = (k'_p, k'_q, k'_r)$ given by (see Figure 1):

k'_p	k'_q	k'_r	k'_p	k'_q	k'_r	k'_p	k'_q	k'_r
0	0	-1, 0, 1	0	1	-1, 0, 1, 2, 3	-1	-1	-1, 0, 1
1	0	-3, -2, -1, 0, 1	-1	1	1, 2, 3	0	-1	-3, -2, -1, 0, 1
1	1	-1, 0, 1	-1	0	-1, 0, 1, 2, 3	1	-1	-3, -2, -1.

Given $x \in \mathbb{H}$ and $j \in \mathbb{Z}$, there exists a unique $k \in \mathcal{K}$ such that $x \in C_{j,k}$. For this choice of k , it is convenient to write

$$(9) \quad C_j(x) = C_{j,k} \quad \text{and} \quad \Lambda_j(x) = \Lambda_{j,k}.$$

The diameter of $C_{j,k}$ is $13^{1/4} \times 2^{-j} < 2^{1-j}$ (because the diameter of C_0 is $13^{1/4}$). In particular, if $\delta(x, y) < 2^{-j}$, then x, y belong simultaneously to at least one $\Lambda_{j,k}$.

Let us recall now the construction of wavelets on \mathbb{H} by Lemarié [18]. For any integer $M > Q/2$, there exist $2^Q - 1 = 15$ functions $(\vartheta_\varepsilon)_{1 \leq \varepsilon \leq 15}$ in $H^{4M}(\mathbb{H})$ such that:

- There exist $C_0, r_0 > 0$ such that for any multi-index α of length $|\alpha| < 4M - Q$:

$$(10) \quad \forall x \in \mathbb{H}, \quad |\nabla_{\mathbb{H}}^\alpha \vartheta_\varepsilon(x)| \leq C_0 \exp(-\|x\|_{\mathbb{H}}/r_0).$$

- Each function $\Psi_\varepsilon = \mathcal{L}^M \vartheta_\varepsilon$ has $2M$ vanishing moments, i.e. for every polynomial function P of homogeneous degree $\deg_{\mathbb{H}} P < 2M$, then

$$\int_{\mathbb{H}} \Psi_\varepsilon(x) P(x) dx = 0.$$

Moreover, $|\Psi^\varepsilon(x)| \leq C_0 \exp(-\|x\|_{\mathbb{H}}/r_0)$ and $\Psi^\varepsilon \in H^\sigma(\mathbb{H})$ for $\sigma < 2M - Q$.

- The family of functions $(2^{jQ/2} \Psi_{j,k}^\varepsilon)_{j \in \mathbb{Z}, k \in \mathbb{Z}, 1 \leq \varepsilon \leq 15}$, where

$$\Psi_{j,k}^\varepsilon(x) = \Psi_\varepsilon \left(2^j \circ (x_{j,k}^{-1} * x) \right),$$

forms a Hilbert basis of $L^2(\mathbb{H})$, i.e.:

$$(11) \quad f \stackrel{L^2}{=} \sum_{\varepsilon, j, k} d_{j,k}^\varepsilon(f) \Psi_{j,k}^\varepsilon \quad \text{with} \quad d_{j,k}^\varepsilon(f) = 2^{jQ} \int_{\mathbb{H}} f(x) \Psi_{j,k}^\varepsilon(x) dx.$$

The real numbers $d_{j,k}^\varepsilon(f)$ are called the *wavelet coefficients* of f . Note that we use an L^∞ normalization for the wavelet in (11) and that our choice implies that the family $(2^{-jQ/2} d_{j,k}^\varepsilon(f))$ belongs to ℓ^2 and thus tends to 0 when $j \rightarrow \pm\infty$ and $\|k\|_{\mathbb{H}} \rightarrow \infty$.

We can now state our main theorems. For non-integer regularity, Hölder classes can be totally described with wavelets coefficients, as in the Euclidian case.

Theorem 1. *For $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $[s] < 2M$, a function f belongs to $C^s(\mathbb{H})$ if and only if there exists a constant $C > 0$ such that*

$$(12) \quad \forall (\varepsilon, j, k) \in \{1, \dots, 2^Q - 1\} \times \mathbb{Z} \times \mathbb{Z}, \quad |d_{j,k}^\varepsilon(f)| \leq C 2^{-js}.$$

Theorem 1 is essentially proved in [13, 14], we give another proof here.

Up to a logarithmic factor, the pointwise regularity class $C^s(x_0)$ can also be described with wavelets coefficients.

Theorem 2. *Given $f \in L^2(\mathbb{H})$, the following properties hold.*

- If $f \in C^s(x_0)$, then there is $R > 0$ such that for any indices ε, j, k :

$$(13) \quad \delta(x_{j,k}, x_0) < R \implies |d_{j,k}^\varepsilon(f)| \leq C 2^{-js} (1 + 2^j \delta(x_{j,k}, x_0))^s.$$

- Conversely, if f satisfies (13) and belongs to $C^\sigma(\mathbb{H})$ for an arbitrary small $\sigma > 0$, then f belongs to $C_{\log}^s(x_0)$.

Remark 1. *The important information contained in (13) does not just lie in the coefficients closest (at each dyadic scale) to x_0 :*

$$|d_{j,k}^\varepsilon(f)| \leq \begin{cases} C 2^{-js} & \text{if } \delta(x_{j,k}, x_0) \lesssim 2^{-j} \\ C \delta(x_{j,k}, x_0)^s & \text{if } 2^{-j} \lesssim \delta(x_{j,k}, x_0) < R. \end{cases}$$

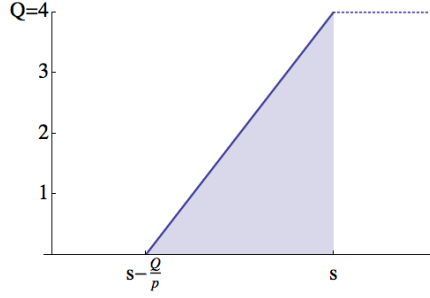


FIGURE 2. Upper bound for the multifractal spectrum of functions in $B_{p,q}^s(\mathbb{H})$.

Remark 2. In the Euclidian case, wavelet leaders [16] are more stable numerically. The wavelet leaders of a function $f \in L^2(\mathbb{H})$ is the sequence

$$D_j(f, x) = \sup \{ |d_{j',k'}^\varepsilon(f)| : j' \geq j \text{ and } C_{j',k'} \subset \Lambda_j(x) \}$$

with $\Lambda_j(x)$ defined by (9). One checks easily that another statement equivalent to (13) is:

$$f \in C^s(x_0) \implies \forall j \geq 0, \quad D_j(f, x_0) \leq C2^{-js}.$$

It is also obvious from the last two theorems that $f \in C^s(\mathbb{H})$ implies $h_f(x) \geq s$ for every $x \in \mathbb{H}$. The optimality of this result is asserted by the following theorem. Recall that a property \mathcal{P} is *generic* in a complete metric space E when it holds on a *residual* set *i.e.* a set with a complement of first Baire category. A set is of first Baire category if it is the union of countably many nowhere dense sets. As it is often the case, it is enough to build a residual set which is a countable intersection of dense open sets in E .

Theorem 3. *There exists a dense open set (hence a generic set) \mathcal{R} of functions in $C^s(\mathbb{H})$ such that for every $f \in \mathcal{R}$ and every $x \in \mathbb{H}$, $h_f(x) = s$.*

In particular, generic functions in $C^s(\mathbb{H})$ are monofractal *i.e.* $E_f(h) = \emptyset$ if $h \neq s$.

One can also obtain *a priori* upper bounds for the multifractal spectrum of functions belonging to Besov and Sobolev spaces on \mathbb{H} (see Section 2.6 for precise definitions).

Theorem 4. *For $s > Q/p$, every $f \in B_{p,q}^s(\mathbb{H})$ satisfies:*

$$(14) \quad d_f(h) \leq \min(Q, p(h - s + Q/p)).$$

This theorem has many remarkable consequences. For instance, it illustrates the optimality of the Sobolev inclusion $B_{p,q}^s(\mathbb{H}) \hookrightarrow C^{s-Q/p}(\mathbb{H})$: the sets of points with the least possible pointwise Hölder exponent $s - Q/p$ has Hausdorff dimension at most 0. Similarly, as a consequence of the proof, the set of points whose pointwise Hölder exponent is at least s and has full Haar measure in \mathbb{H} . The main difference with $C^s(\mathbb{H})$ is that functions in $B_{p,q}^s(\mathbb{H})$ may really be multifractal, meaning that many iso-Hölder sets $E_f(h)$ are non-empty with a non-trivial Hausdorff dimension. This is the case for generic functions in $B_{p,q}^s(\mathbb{H})$.

Theorem 5. *For $s > Q/p$, there is a residual set $\tilde{\mathcal{R}} \subset B_{p,q}^s(\mathbb{H})$ such that for all $f \in \tilde{\mathcal{R}}$,*

$$(15) \quad \forall h \in [s - Q/p, s], \quad d_f(h) = p(h - s + Q/p),$$

and $E_f(h) = \emptyset$ for all other exponents.

In particular, for generic functions $f \in B_{p,q}^s(\mathbb{H})$, Haar-almost every point has a pointwise Hölder exponent equal to s .

This paper is organized as follows. Section 2 contains the definitions and previous results that we use in the sequel. In Sections 3 and 4 respectively, we deal with global and pointwise Hölder regularity (Theorems 1 and 2). In particular, the monofractality of generic functions (Theorem 3) in $C^s(\mathbb{H})$ is proved in Section 4.3. The multifractal properties of functions in a Besov space are then investigated in Sections 5 and 6. Finally, we explain how to extend our results to general stratified nilpotent groups in Section 7.

Let us finish with a question. It would be very interesting to be able to represent the functions on \mathbb{H} , or at least the traces of such functions on affine subspaces of \mathbb{R}^3 . Indeed, the natural anisotropy induced by the metric on \mathbb{H} should create some anisotropic pictures, and it is actually a challenge in image processing to create natural and simple models for anisotropic textures. Of course the starting point would be to understand how to draw a wavelet (a Lemarié wavelet, or another one!) on \mathbb{H} . We believe that this is a very promising research direction.

2. DEFINITIONS AND RECALLS

2.1. Balls on \mathbb{H} . As the shape of balls is rather counter-intuitive on \mathbb{H} , a few geometric statements will be useful in the following. The volume of the gauge balls

$$\mathcal{B}(x, r) = \{y \in \mathbb{H} : \delta(x, y) < r\}$$

is denoted by $\ell(\mathcal{B}(x, \rho))$ and is equal to $\frac{\pi^2}{2}\rho^Q$ with $Q = 4$. In particular, the Haar measure ℓ has the doubling property: $\ell(\mathcal{B}(x, 2r)) \leq C\ell(\mathcal{B}(x, r))$ for some universal constant C .

For any $x, x' \in \mathbb{H}$ and $r, r' > 0$, one has $x' * (r' \circ \mathcal{B}(x, r)) = \mathcal{B}(x' * (r' \circ x), rr')$ and $\mathcal{B}(x, r) = x * (r \circ \mathcal{B}(0, 1))$.

The triangular inequality holds in general with a constant depending on the metric.

Proposition 6 (Folland, Stein, prop.1.6). *There exists a constant $\gamma_1 > 0$ such that*

$$(16) \quad \forall x, y \in \mathbb{H}, \quad \|xy\|_{\mathbb{H}} \leq \gamma_1 (\|x\|_{\mathbb{H}} + \|y\|_{\mathbb{H}}).$$

In particular, the diameter of a gauge ball $\mathcal{B}(x, r)$ does not exceed $2\gamma_1 r$. One will use this property later in the following form:

Corollary 7. *There exists $C > 0$ such that for any $\eta > 0$ and $x \in \mathcal{B}(0, \eta/C)$, one has*

$$\mathcal{B}(0, \eta/C) \subset \mathcal{B}(x, \eta).$$

2.2. Polynomial functions. A polynomial function P on \mathbb{H} is a polynomial function of the coordinates (p, q, r) ; its homogeneous degree is defined by

$$\deg_{\mathbb{H}} P = \deg P(t, t, t^2)$$

where the right-hand side is computed in $\mathbb{R}[t]$. Given $x = (p, q, r)$ and $\alpha \in \{1, 2, 3\}^m$, one defines $x^\alpha = x(\alpha_1) \dots x(\alpha_m)$ with $x(1) = p$, $x(2) = q$ and $x(3) = r$. A polynomial

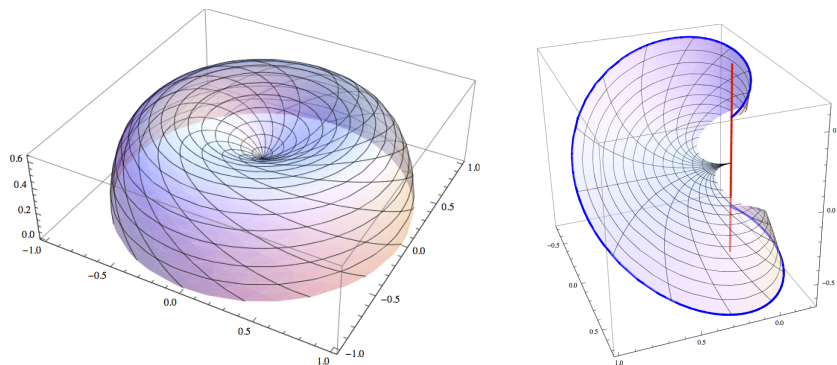


FIGURE 3. Left : Upper half of the unit Carnot ball of \mathbb{H} . Right: one meridian arc of the unit ball and the unitary “horizontal” geodesics joining the origin to each point of this meridian arc.

function \mathbb{H} of homogeneous degree at most N is thus a function of the form:

$$P(x) = \sum_{|\alpha| \leq N} c_\alpha x^\alpha$$

with $c_\alpha \in \mathbb{R}$ and $|\alpha| = \sum \omega(\alpha_i)$ with $\omega(1) = \omega(2) = 1$ and $\omega(3) = 2$.

2.3. The operator $\nabla_{\mathbb{H}}$. Let us denote by $\nabla_{\mathbb{H}} = (X, Y)$ the basis (equation (2)) of horizontal derivatives. Given a multi-index $\alpha \in \{1, 2, 3\}^m$ one will denote by

$$(17) \quad \nabla_{\mathbb{H}}^\alpha f = V_{\alpha_1} \dots V_{\alpha_m} f$$

where $V_1 = X$, $V_2 = Y$ and $V_3 = Z$. As $Z = -\frac{1}{4}[X, Y]$, one may reduce $\nabla_{\mathbb{H}}^\alpha f$ to a linear combination of erms that contain exactly $|\alpha|$ powers of X and Y . One says that $\nabla_{\mathbb{H}}^\alpha f$ is a *horizontal derivative* of f of order $|\alpha|$.

2.4. Horizontal paths and Taylor formula. Two points $x, y \in \mathbb{H}$ can always be joined by a sub-unitary horizontal path, i.e. a piecewise Lipschitz arc $\gamma : [0, L] \rightarrow \mathbb{H}$ such that for almost every t , the tangent vector can be decomposed as

$$\gamma'(t) = \alpha(t)X(\gamma(t)) + \beta(t)Y(\gamma(t))$$

with $\alpha^2(t) + \beta^2(t) \leq 1$. The so-called Carnot-length $d_C(x, y) = \inf_\gamma \left(\int_0^L \alpha^2(t) + \beta^2(t) dt \right)^{1/2}$ is uniformly equivalent to $\delta(x, y)$. Integrating along such an arc provides the first order Taylor formula:

$$f(y) = f(x) + \int_0^L \nabla_{\mathbb{H}} f(\gamma(t)) \gamma'(t) dt.$$

In turn, this identity provides a Lipschitz estimate.

Proposition 8 ([12], Theorem 1.41). *There exists $C > 0$ and $\gamma_2 > 0$ such that for all $f \in C^1(\mathbb{H})$ and $x, y \in \mathbb{H}$,*

$$|f(y) - f(x)| \leq C \delta(x, y) \sup_{\|z\| \leq \gamma_2 \delta(x, y)} |\nabla_{\mathbb{H}} f(xz)|.$$

The left-invariant Taylor expansion of a function is given by the next definition.

Definition 3 ([12]). *The right Taylor polynomial of homogeneous degree k of a smooth function f at $x_0 \in \mathbb{H}$ is the unique polynomial P_{x_0} of homogeneous degree $\leq k$ such that*

$$\forall \alpha \in \bigcup_{m \in \mathbb{N}} \{1, 2, 3\}^m, \quad |\alpha| \leq k \quad \implies \quad \nabla_{\mathbb{H}}^{\alpha} f(x_0) = \nabla_{\mathbb{H}}^{\alpha} P_{x_0}(0).$$

To proceed with the subsequent calculations we will need to write down the Taylor expansion *explicitly*. It must be done carefully for various reasons. The most obvious one is that $XYf \neq YXf$ but $pq = qp$. The second “problem” induced by the anisotropy of the Heisenberg structure is that the traditional match between the index of the derivative and the index of the polynomial will break down. For example, at the 2nd order near the origin, $f(p, q, 0)$ will be computed using only the first 2 powers of p and q but, contrary to the euclidian setting, it will involve vertical derivatives at the origin, through $Zf(0)$.

With this in mind, a good way to write the Taylor polynomial of order N down is:

$$(18) \quad P_{x_0}(y) = \sum_{k=0, \dots, N} \sum_{|\alpha|=k} y^{\alpha} \left(\sum_{|\beta|=|\alpha|} c_{\alpha, \beta} \nabla_{\mathbb{H}}^{\beta} f(x_0) \right) = \sum_{|\alpha|=|\beta| \leq N} c_{\alpha, \beta} \nabla_{\mathbb{H}}^{\beta} f(x_0) y^{\alpha}.$$

Beyond order 2, even though the polynomial P_{x_0} remains unique, the coefficients $c_{\alpha, \beta}$ in (18) are not and thus a choice has to be done once for all before starting a computation. For example, one possible writing of the polynomial of order 3 at the origin is:

$$\begin{aligned} P_0(p, q, r) &= f(0) + pXf(0) + qYf(0) \\ &+ \frac{1}{2} (p^2 X^2 f(0) + 2pqXYf(0) + q^2 Y^2 f(0)) + (2pq + r) \cdot Zf(0) \\ &+ \frac{1}{3!} (p^3 X^3 f(0) + 3p^2 q X^2 Y f(0) + 3pq^2 XY^2 f(0) + q^3 Y^3 f(0)) \\ &+ (2pq + r) \cdot (pXZf(0) + qYZf(0)). \end{aligned}$$

The actual choice between the possible expressions is irrelevant. Two points are key:

- For each term in the Taylor expansion (18), the homogeneous degree $|\alpha|$ of each monomial y^{α} matches the order $|\beta|$ of the derivative $\nabla_{\mathbb{H}}^{\beta} f(x_0)$ associated to it.
- Since monomials are commutative and $Z = -\frac{1}{4}[X, Y]$ one can assume from now on that the formula is reduced to indices $\beta \in \bigcup_{m \in \mathbb{N}} \{1, 2\}^m$.

Further results and explicit Taylor formulas on homogenous groups can be found in [5].

As expected, the right Taylor polynomial approximates $f(x_0 y)$ for y small enough.

Theorem 9 (Folland, Stein, corr. 1.44). *If $f \in C^{k+1}(\mathbb{H})$, then the following estimate holds for some universal constant C_k :*

$$|f(x_0 y) - P_{x_0}(y)| \leq C_k \|y\|_{\mathbb{H}}^{k+1} \sup_{|\alpha|=k+1} \left(\sup_{\|z\| \leq \gamma_2^{k+1}} |\nabla_{\mathbb{H}}^{\alpha} f(x_0 z)| \right)$$

2.5. Hausdorff dimension on \mathbb{H} . The diameter of a set $E \subset \mathbb{H}$ will be denoted by

$$|E| = \sup\{\delta(x, y) : x, y \in E\}.$$

Let us recall the definition of the Hausdorff measures and dimension. Let $s > 0$ and $\eta > 0$ be two positive real numbers. For any set $E \subset \mathbb{H}$, one defines

$$(19) \quad \mathcal{H}_{\eta}^s(E) = \inf_{\mathcal{R}} \sum_{B \in \mathcal{R}} |B|^s \in [0, +\infty],$$

where the infimum is taken over all possible coverings \mathcal{R} of E by gauge balls of radii less than η . Recall that a covering of E is a family $\mathcal{R} = \{B_i\}_{i \in I}$ of balls satisfying

$$E \subset \bigcup_{i \in I} B_i.$$

The mapping $\eta \mapsto \mathcal{H}_\eta^s(E)$ is decreasing with η , hence one can define

$$\mathcal{H}^s(E) = \lim_{\eta \rightarrow 0^+} \mathcal{H}_\eta^s(E) \in [0, +\infty].$$

From this definition, it is standard to see that $s \mapsto \mathcal{H}^s(E)$ is a decreasing function that jumps from infinity to zero at a unique real number called the Hausdorff dimension of E :

$$\dim_H E = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = +\infty\}.$$

2.6. Hölder, Sobolev and Besov regularity.

Definition 4. For $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in]0, 1[$, $C^s(\mathbb{H})$ is the set of functions such that for any multi-index of length $|\alpha| \leq k$, the function $\nabla_{\mathbb{H}}^\alpha f$ is continuous and:

$$\sup_{|\alpha|=k} \frac{|\nabla_{\mathbb{H}}^\alpha f(x) - \nabla_{\mathbb{H}}^\alpha f(y)|}{\delta(x, y)^\sigma} < \infty.$$

Hence, Hölder classes are defined as in the Euclidian case (see [7, 8]). Two equivalent Banach norms on $C^s(\mathbb{H})$, denoted by $\|f\|_{C^s(\mathbb{H})}$, are given by

$$\sup_{|\alpha|=s} \frac{|\nabla_{\mathbb{H}}^\alpha f(x) - \nabla_{\mathbb{H}}^\alpha f(y)|}{\delta(x, y)^{s-[\alpha]}} \quad \text{and} \quad \sup_{\varepsilon, j, k} (2^{js} |d_{j,k}^\varepsilon(f)|).$$

Sobolev spaces, that we introduced before in (4), can also be described using the horizontal derivatives (17) (see [19] and references therein).

Proposition 10. For $k \in \mathbb{N}$, a function f belongs to $H^k(\mathbb{H})$ if and only if for any multi-index α of length $|\alpha| \leq k$, $\nabla_{\mathbb{H}}^\alpha f \in L^2(\mathbb{H})$.

For $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in]0, 1[$, one has $f \in H^s(\mathbb{H})$ if and only if $f \in H^k(\mathbb{H})$ and for $|\alpha| = k$ (with $Q = 4$, the homogeneous dimension of \mathbb{H}):

$$(20) \quad \iint_{\mathbb{H} \times \mathbb{H}} \frac{|\nabla_{\mathbb{H}}^\alpha f(x) - \nabla_{\mathbb{H}}^\alpha f(y)|^2}{\delta(x, y)^{Q+2\sigma}} dx dy < \infty.$$

One has the continuous inclusion $H^s(\mathbb{H}) \subset C^{s-Q/2}(\mathbb{H})$, which holds if $s > Q/2$ and $s - Q/2 \notin \mathbb{N}$. Namely, for any multi-index α such that $s = Q/2 + |\alpha| + \sigma$ and $\sigma \in]0, 1[$:

$$|\nabla_{\mathbb{H}}^\alpha f(x) - \nabla_{\mathbb{H}}^\alpha f(y)| \leq C_s \|f\|_{H^s(\mathbb{H})} \delta(x, y)^{s-Q/2}.$$

As in the euclidian case, this inclusion fails if $s - Q/2 \in \mathbb{N}$, because one can then find $f \in H^s(\mathbb{H})$ and $|\alpha| = s - Q/2$ such that $\nabla_{\mathbb{H}}^\alpha f$ is not a continuous function (note that $Q = 4$ is even). However, the corresponding inclusion in BMO holds (see [8]).

On \mathbb{R}^n , Besov spaces can be defined in various ways and the equivalence between all those definitions is part of the folklore. For nilpotent Lie groups, the situation is less straightforward and all the equivalences should be checked carefully. For multifractal analysis, the most convenient definition of Besov spaces is:

Definition 5. *The Besov space $B_{p,q}^s(\mathbb{H})$ of [24] consists of functions f on \mathbb{H} such that:*

$$(21) \quad a_j = \left\| 2^{j(s-Q/p)} d_{j,k}^\varepsilon(f) \right\|_{\ell^p(k)} \in \ell^q(j).$$

Other definition of Besov spaces involve Littlewood-Paley theory, but all definitions coincide [23, 13]. In particular, $B_{2,2}^s(\mathbb{H}) = H^s(\mathbb{H})$. Depending on the applications, other definitions have proved useful. Trace theory [9, 27] is better understood with the geometric norm (20), real interpolation norms with operator theory [4, 24], while complex interpolation norms relate to microlocal analysis and Weyl calculus on \mathbb{H} [6, 7, 19].

3. GLOBAL HÖLDER REGULARITY WITH WAVELETS COEFFICIENTS: THEOREM 1

3.1. Upper bound for the wavelets coefficients. Assume that $f \in C^s(\mathbb{H})$. Let $s = [s] + \sigma$ with $[s] \in \mathbb{N}$ and $0 < \sigma < 1$. The change of variables $y = 2^j \circ (x_{j,k}^{-1}x)$ reads:

$$d_{j,k}^\varepsilon = \int_{\mathbb{H}} f(x_{j,k}(2^{-j} \circ y)) \Psi_\varepsilon(y) dy.$$

When $[s] = 0$, one infers from the vanishing moment of Ψ_ε (i.e. $\int_{\mathbb{H}} \Psi_\varepsilon(x) dx = 0$) that:

$$\begin{aligned} |d_{j,k}^\varepsilon| &= \left| \int_{\mathbb{H}} (f(x_{j,k}(2^{-j} \circ y)) - f(x_{j,k})) \Psi_\varepsilon(y) dy \right| \leq \int_{\mathbb{H}} |f(x_{j,k}(2^{-j} \circ y)) - f(x_{j,k})| |\Psi_\varepsilon(y)| dy \\ &\leq 2^{-js} \|f\|_{C^s(\mathbb{H})} \int_{\mathbb{H}} \|y\|_{\mathbb{H}}^s |\Psi_\varepsilon(y)| dy = C 2^{-js}. \end{aligned}$$

When $[s] \geq 1$, one uses $\Psi^\varepsilon = \mathcal{L}^M \vartheta_\varepsilon$ and proceeds with $[s]$ integrations by part against the function $g_{j,k}(y) = f(x_{j,k}(2^{-j} \circ y))$.

Observe that the homogeneity of the horizontal derivatives yields that for every α ,

$$(22) \quad \nabla_{\mathbb{H}}^\alpha g_{j,k}(y) = 2^{-j|\alpha|} \times [\nabla_{\mathbb{H}}^\alpha f]((x_{j,k} * (2^{-j} \circ y)).$$

When $[s] = 2m$ is even ($m \geq 1$), one writes $d_{j,k}^\varepsilon = \int_{\mathbb{H}} (\mathcal{L}^m g_{j,k})(x) \tilde{\vartheta}_\varepsilon(x) dx$, with $\tilde{\vartheta}_\varepsilon = \mathcal{L}^{M-m} \vartheta_\varepsilon$. The term $\mathcal{L}^m g_{j,k} = (-X^2 - Y^2)^m (g_{j,k})$ can be developed, and one gets

$$\mathcal{L}^m g_{j,k} = \sum_{|\alpha|=[s]} l_\alpha \nabla_{\mathbb{H}}^\alpha (g_{j,k}),$$

for some coefficients l_α independent of the problem. Recalling (10), $\tilde{\vartheta}_\varepsilon$ is a well-localized function. One can use the vanishing moments of ϑ_ε (and thus of $\tilde{\vartheta}_\varepsilon$) to get

$$\begin{aligned} d_{j,k}^\varepsilon &= \sum_{|\alpha|=[s]} l_\alpha \int_{\mathbb{H}} (\nabla_{\mathbb{H}}^\alpha g_{j,k}(y) - \nabla_{\mathbb{H}}^\alpha g_{j,k}(0)) \tilde{\vartheta}_\varepsilon(y) dy, \\ &= 2^{-j[s]} \sum_{|\alpha|=[s]} l_\alpha \int_{\mathbb{H}} (\nabla_{\mathbb{H}}^\alpha f((x_{j,k} * (2^{-j} \circ y)) - \nabla_{\mathbb{H}}^\alpha f(x_{j,k})) \tilde{\vartheta}_\varepsilon(y) dy, \end{aligned}$$

The assumption $f \in C^s(\mathbb{H})$ implies $\nabla_{\mathbb{H}}^\alpha f \in C^\sigma(\mathbb{H})$, thus ultimately providing:

$$|d_{j,k}^\varepsilon| \leq 2^{-j([s]+\sigma)} \|f\|_{C^s(\mathbb{H})} \sum_{|\alpha|=[s]} |l_\alpha| \int_{\mathbb{H}} \|y\|_{\mathbb{H}}^\sigma |\tilde{\vartheta}_\varepsilon(y)| dy = C 2^{-js}.$$

When $[s] = 2m - 1$ is odd ($m \geq 1$), one has:

$$d_{j,k}^\varepsilon = \int_{\mathbb{H}} (X\mathcal{L}^{m-1}g_{j,k})(X\tilde{\vartheta}_\varepsilon) + \int_{\mathbb{H}} (Y\mathcal{L}^{m-1}g_{j,k})(Y\tilde{\vartheta}_\varepsilon).$$

Again, $X\tilde{\vartheta}_\varepsilon$ and $Y\tilde{\vartheta}_\varepsilon$ are well-localized functions, with at least one vanishing moment. Using the same arguments as above,

$$d_{j,k}^\varepsilon = \sum_{|\alpha|=[s]} l'_\alpha \int_{\mathbb{H}} (\nabla_{\mathbb{H}}^\alpha g_{j,k}(y) - \nabla_{\mathbb{H}}^\alpha g_{j,k}(0)) \tilde{\Psi}_{\varepsilon,\alpha}(y) dy$$

with $\tilde{\Psi}_{\varepsilon,\alpha} = X\tilde{\vartheta}_\varepsilon$ or $Y\tilde{\vartheta}_\varepsilon$ (depending on whether the first slot in α codes for X or Y) and some other coefficients l'_α . The rest of the proof is the same, giving finally $|d_{j,k}^\varepsilon| \leq C2^{-js}$.

3.2. Hölder estimate derived from wavelets coefficients. Let us now focus on the converse assertion in Theorem 1 and assume that (12) holds. The normal convergence of the series (11) up to the $[s]^{\text{th}}$ derivatives ensures that, for any multi-index α such that $|\alpha| \leq [s]$ the function $\nabla_{\mathbb{H}}^\alpha f$ is continuous and that the following identity holds:

$$\nabla_{\mathbb{H}}^\alpha f(x) = \sum_{\varepsilon,j,k} 2^{j|\alpha|} d_{j,k}^\varepsilon(f) \cdot (\nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon) \left(2^j \circ (x_{j,k}^{-1}x) \right).$$

Let us estimate the $[s]^{\text{th}}$ derivatives. As before $s = [s] + \sigma$ and for $|\alpha| = [s]$, one gets:

$$|\nabla_{\mathbb{H}}^\alpha f(x) - \nabla_{\mathbb{H}}^\alpha f(y)| \leq \sum_{\varepsilon,j,k} 2^{-j\sigma} \left| (\nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon) \left(2^j \circ (x_{j,k}^{-1}x) \right) - (\nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon) \left(2^j \circ (x_{j,k}^{-1}y) \right) \right|.$$

Let $j_0 \in \mathbb{Z}$ such that $2^{-j_0-1} \leq \delta(x, y) < 2^{-j_0}$. There exists $\tilde{k} = (k_1, k_2, k_3) \in \mathcal{Z}$ such that x and y both belong to the dyadic neighborhood of $x_{j_0, \tilde{k}}$, namely $x, y \in \Lambda_{j_0, \tilde{k}}$, where $\Lambda_{j_0, \tilde{k}}$ has been defined by (8) and the remarks that follow.

For $j \leq j_0$, one uses that $\nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon$ is Lipschitz:

$$\left| (\nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon) \left(2^j \circ (x_{j,k}^{-1}x) \right) - (\nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon) \left(2^j \circ (x_{j,k}^{-1}y) \right) \right| \leq C2^j \delta(x, y) \times \xi_{j,k},$$

where $\xi_{j,k}$ satisfies the following estimate (Proposition 8)

$$\begin{aligned} \xi_{j,k} &= \sup_{\|z\|_{\mathbb{H}} \leq \gamma_2 2^{j-j_0}} |\nabla_{\mathbb{H}} \nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon((2^j \circ (x_{j,k}^{-1}x_{j_0, \tilde{k}})) * z)| \\ &\leq \sup_{z' \in k^{-1} * (2^{j-j_0} \circ \mathcal{B}(\tilde{k}, \gamma_2))} (C_0 \exp(-\|z'\|_{\mathbb{H}}/r_0)) \leq C'_0 \exp(-\delta(k, 2^{j-j_0} \circ \tilde{k})/r'_0). \end{aligned}$$

For $j > j_0$, one uses simply the boundedness of $\nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon$:

$$\begin{aligned} &\left| (\nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon) \left(2^j \circ (x_{j,k}^{-1}x) \right) - (\nabla_{\mathbb{H}}^\alpha \Psi_\varepsilon) \left(2^j \circ (x_{j,k}^{-1}y) \right) \right| \\ &\leq C_0 \{ \exp(-\delta(k, 2^j \circ x)/r_0) + \exp(-\delta(k, 2^j \circ y)/r_0) \}. \end{aligned}$$

Each right-hand side is obviously summable in the variable $k \in \mathcal{K}$ because there exists a constant C such that

$$\forall z \in \mathbb{H}, \quad \sum_{k \in \mathcal{Z}} \exp(-\delta(k, z)/r_0) \leq C.$$

Combining the previous estimates and the fact that the ε variable belongs to a finite set $\{1, \dots, 2^Q - 1\}$, one gets the following upper bound for $|\nabla_{\mathbb{H}}^\alpha f(x) - \nabla_{\mathbb{H}}^\alpha f(y)|$:

$$\sum_{j \leq j_0} 2^{j(1-\sigma)} \delta(x, y) + \sum_{j > j_0} 2^{-j\sigma} \leq C(2^{j_0(1-\sigma)} \delta(x, y) + 2^{-j_0\sigma}) \leq C' \delta(x, y)^\sigma$$

i.e. $f \in C^s(\mathbb{H})$.

4. POINTWISE HÖLDER REGULARITY

4.1. Upper bound for the wavelets coefficients. Assume first that $f \in C^s(x_0)$. Let P be the unique polynomial of degree $\deg_{\mathbb{H}}(P) < s$ such that (5) holds on a small neighborhood of the origin $\mathcal{B}(0, \eta)$. Using the vanishing moments of Ψ^ε , one has

$$d_{j,k}^\varepsilon(f) = 2^{jQ} \int_{\mathbb{H}} (f(x_0x) - P(x)) \Psi_{j,k}^\varepsilon(x_0x) dx$$

thus

$$\begin{aligned} |d_{j,k}^\varepsilon(f)| &\leq C 2^{jQ} \int_{\mathcal{B}(0, \eta)} \|x\|_{\mathbb{H}}^s |\Psi_{j,k}^\varepsilon(x_0x)| dx + 2^{jQ} \int_{\mathbb{H} \setminus \mathcal{B}(x_0, \eta)} |f(x)| |\Psi_{j,k}^\varepsilon(x)| dx \\ &\quad + 2^{jQ} \int_{\mathbb{H} \setminus \mathcal{B}(x_0, \eta)} |P(x_0^{-1}x)| |\Psi_{j,k}^\varepsilon(x)| dx. \end{aligned}$$

We denote the three integrals above by respectively I_1 , I_2 and I_3 . In the following, we assume (as in the statement of Theorem 2) that

$$(23) \quad \delta(x_{j,k}, x_0) < R$$

for some constant $R \leq 1$ that we will adjust on the way.

The change of variables $y = 2^j \circ (x_{j,k}^{-1} x_0 x)$, Hölder's inequality $\|ab\|_{\mathbb{H}}^s \leq C_s (\|a\|_{\mathbb{H}}^s + \|b\|_{\mathbb{H}}^s)$ and the exponential decay (10) of the mother wavelet yield:

$$\begin{aligned} I_1 &\leq C \int_{\mathcal{B}(2^j(x_{j,k}^{-1}x_0), 2^j\eta)} \|x_0^{-1}x_{j,k}(2^{-j} \circ x)\|_{\mathbb{H}}^s |\Psi_\varepsilon(x)| dx \\ &\leq C \int_{\mathbb{H}} C_s (\delta(x_0, x_{j,k})^s + \|2^{-j} \circ x\|_{\mathbb{H}}^s) \times C_0 \exp(-\|x\|_{\mathbb{H}}/r_0) dx \\ &\leq C (\delta(x_0, x_{j,k})^s + 2^{-js}) \end{aligned}$$

for some other constant C , independent of x_0 , j and k .

For the second integral, one uses the Cauchy-Schwarz inequality:

$$I_2 \leq \int_{\mathbb{H} \setminus \mathcal{B}(x_0, \eta)} |f(x)| \times 2^{jQ} |\Psi_{j,k}^\varepsilon(x)| dx \leq \|f\|_{L^2(\mathbb{H})} \left(\int_{\mathbb{H} \setminus \mathcal{B}(x_0, \eta)} 2^{2jQ} |\Psi_{j,k}^\varepsilon(x)|^2 dx \right)^{1/2}.$$

The weight of the tail of the wavelet depends on how $\delta(x_{j,k}, x_0)$ compares to η . Let us assume that $R = \eta/C$ in (23) with the constant C given by Corollary 7, thus

$$\mathcal{B}(0, 2^j \eta/C) \subset \mathcal{B}(2^j \circ (x_{j,k}^{-1} x_0), 2^j \eta).$$

Then the usual change of variable $y = 2^j \circ (x_{j,k}^{-1}x)$ reads

$$\begin{aligned} \int_{\mathbb{H} \setminus \mathcal{B}(x_0, \eta)} 2^{2jQ} |\Psi_{j,k}^\varepsilon(x)|^2 dx &= 2^{jQ} \int_{\mathbb{H} \setminus \mathcal{B}(2^j \circ (x_{j,k}^{-1}x_0), 2^j \eta)} |\Psi_\varepsilon(y)|^2 dy \\ &\leq 2^{jQ} \int_{\{y \in \mathbb{H} : \|y\|_{\mathbb{H}} \geq 2^j \eta / C\}} \left(\frac{C \|y\|}{2^j \eta} \right)^{Q+2s} |\Psi_\varepsilon(y)|^2 dy \\ &\leq C 2^{-2js}. \end{aligned}$$

The last inequality uses the decay property (10) of Ψ^ε . Finally, one gets $I_2 \leq C 2^{-js}$.

For I_3 , the idea is similar except that one has to compensate for $P \notin L^2(\mathbb{H})$ by adding an extra weight. For example, one choses an integer $N > s$ such that $(1 + \|x\|_{\mathbb{H}})^{-N} P(x)$ is bounded. Then, as previously, one has:

$$\begin{aligned} I_3 &\leq 2^{jQ} \|(1 + \|\cdot\|_{\mathbb{H}})^{-N} P\|_{L^\infty(\mathbb{H})} \times \int_{\mathbb{H} \setminus \mathcal{B}(x_0, \eta)} (1 + \|x_0^{-1}x\|_{\mathbb{H}})^N |\Psi_{j,k}^\varepsilon(x)| dx \\ &\leq C \int_{\|y\|_{\mathbb{H}} \geq 2^j \eta / C} (1 + \|x_0^{-1}x_{j,k}\| + 2^{-j}\|y\|_{\mathbb{H}})^N |\Psi^\varepsilon(y)| dy \end{aligned}$$

One sees that when $\|y\|_{\mathbb{H}} \geq 2^j \eta / C$, $1 \leq 2^{-j} \|y\|_{\mathbb{H}} C / \eta$. Moreover, one has $\|x_0^{-1}x_{j,k}\|_{\mathbb{H}} \leq R = \eta / C \leq 2^{-j} \|y\|_{\mathbb{H}}$. Hence, the estimates boil down to:

$$I_3 \leq C \int_{\|y\|_{\mathbb{H}} \geq 2^j \eta / C} (2^{-j} \|y\|_{\mathbb{H}})^N |\Psi^\varepsilon(y)| dy \leq C 2^{-jN} \int_{\mathbb{H}} \|y\|_{\mathbb{H}}^N \exp\left(-\frac{\|y\|_{\mathbb{H}}}{r_0}\right) dy$$

Finally, one gets $I_3 \leq C 2^{-jN} \leq C 2^{-js}$.

Putting together the estimates for I_1 , I_2 and I_3 , one gets:

$$|d_{j,k}^\varepsilon(f)| \leq C (2^{-js} + \delta(x_{j,k}, x_0)^s)$$

when $\delta(x_{j,k}, x_0) < \eta / C = R$ which is equivalent to (13). This concludes the proof of the first half of Theorem 2. \blacksquare

4.2. Pointwise Hölder estimate derived from wavelets coefficients. Let us move to the second part of Theorem 2 and prove the converse property. One assumes that $f \in C^\sigma(\mathbb{H})$ for some $\sigma > 0$ and that (13) holds for all triplets (ε, j, k) such that $\delta(x_{j,k}, x_0) \leq R$ for some $R > 0$. Let us fix x such that $\delta(x, x_0) \leq R$ and let j_0 and j_1 be the unique integers such that

$$(24) \quad 2^{-j_0-1} \leq \delta(x, x_0) < 2^{-j_0} \quad \text{and} \quad j_1 = \left\lceil \frac{s}{\sigma} \cdot j_0 \right\rceil.$$

We aim at proving (6) for x close enough to x_0 *i.e.* for j_0 large enough.

The wavelet decomposition of f is $f = \sum_{j \in \mathbb{Z}} f_j(x)$, where for every $j \in \mathbb{Z}$,

$$f_j(x) = \sum_{\varepsilon \in \{1, \dots, 15\}} \sum_{k \in \mathcal{Z}} d_{j,k}^\varepsilon(f) \Psi_{j,k}^\varepsilon(x).$$

For subsequent use, let us notice immediately that the low frequency term

$$f^b(x) = \sum_{j=-\infty}^0 f_j(x)$$

is as regular as the wavelet itself. In particular at least $C^{[s]+2}(\mathbb{H})$.

Assumption (13) reads

$$(25) \quad |d_{j,k}^\varepsilon(f)| \leq C(2^{-js} + \|x_{j,k}^{-1}x_0\|_{\mathbb{H}}^s) \leq C(2^{-js} + \|x_{j,k}^{-1}x\|_{\mathbb{H}}^s + \|x^{-1}x_0\|_{\mathbb{H}}^s).$$

For every $n \in \{0, \dots, [s] + 1\}$ and any multi-index α with $|\alpha| = n$, one has:

$$(26) \quad \nabla_{\mathbb{H}}^\alpha f_j(x) = \sum_{\varepsilon} \sum_{k \in \mathcal{Z}} d_{j,k}^\varepsilon(f) \cdot (\nabla_{\mathbb{H}}^\alpha \Psi_{j,k}^\varepsilon)(x).$$

As $\Psi_{j,k}^\varepsilon(x) = \Psi_\varepsilon(2^j \circ (x_{j,k}^{-1}x))$ and using (10), a computation similar to (22) gives:

$$(27) \quad |\nabla_{\mathbb{H}}^\alpha \Psi_{j,k}^\varepsilon(x)| \leq C2^{j|\alpha|} \exp\left(-\|2^j \circ (x_{j,k}^{-1}x)\|_{\mathbb{H}}/r_0\right) \leq \frac{C2^{j|\alpha|}}{(1 + \|2^j \circ (x_{j,k}^{-1}x)\|_{\mathbb{H}})^{Q+1+s}}.$$

Next, let us notice that for a constant that does not depends on $j \in \mathbb{Z}$ or $x \in \mathbb{H}$,

$$(28) \quad \forall \gamma > 0, \quad \exists C > 0, \quad \sum_{k \in \mathcal{Z}} \frac{\|x_{j,k}^{-1}x\|_{\mathbb{H}}^\gamma}{(1 + \|2^j \circ (x_{j,k}^{-1}x)\|_{\mathbb{H}})^{Q+1+\gamma}} \leq C2^{-j\gamma}.$$

Combining (27), (25) and (28) provides, in a neighborhood of x_0 :

$$(29) \quad |\nabla_{\mathbb{H}}^\alpha f_j(x)| \leq C2^{j|\alpha|} \sum_{k \in \mathcal{Z}} \frac{(2^{-js} + \|x_{j,k}^{-1}x\|_{\mathbb{H}}^s + \|x^{-1}x_0\|_{\mathbb{H}}^s)}{(1 + \|2^j \circ (x_{j,k}^{-1}x)\|_{\mathbb{H}})^{Q+1+s}} \leq C2^{j|\alpha|} (2^{-js} + \|x^{-1}x_0\|_{\mathbb{H}}^s).$$

In particular $|\nabla_{\mathbb{H}}^\alpha f_j(x_0)| \leq C2^{j(|\alpha|-s)}$ and the series (used subsequently) $\sum_{j=0}^{\infty} \nabla_{\mathbb{H}}^\alpha f_j(x_0)$ converges absolutely for every α such that $|\alpha| \leq [s]$.

Let us now introduce the (right)-Taylor polynomial P_j of f_j at x_0 . According to (18), it can be written

$$(30) \quad P_j(y) = \sum_{|\alpha|=|\beta| \leq [s]} c_{\alpha,\beta} \nabla_{\mathbb{H}}^\beta f_j(x_0) y^\alpha.$$

The coefficients $c_{\alpha,\beta}$ are chosen once and for all for the rest of this computation. Let also $P^b(y)$ stand for the (right)-Taylor polynomial of the low frequency part f^b at x_0 .

The polynomial P that we are going to use to prove (6) is defined by:

$$P(y) = P^b(y) + \sum_{j=0}^{\infty} P_j(y).$$

The previous estimates ensure that P is indeed well defined and of degree at most $[s]$. One gets the following decomposition:

$$(31) \quad |f(x) - P(x_0^{-1}x)| \leq |f^b(x) - P^b(x_0^{-1}x)| + \sum_{j=0}^{j_0} |f_j(x) - P_j(x_0^{-1}x)| + R_1(x) + R_2(x)$$

with two remainders:

$$R_1(x) = \sum_{j=j_0}^{\infty} |f_j(x)| \quad \text{and} \quad R_2(x) = \sum_{j=j_0}^{\infty} |P_j(x_0^{-1}x)|.$$

The low frequency is instantaneously dealt with by Theorem 9:

$$(32) \quad |f^b(x) - P^b(x_0^{-1}x)| \leq C \|x_0^{-1}x\|_{\mathbb{H}}^{[s]+1}.$$

Let us now focus on the three other terms. One uses the Taylor development of the wavelet at x_0 and the unicity of the Taylor expansion to recover the polynomial P_j . Let us thus write:

$$\Psi_{j,k}^{\varepsilon}(x) = \sum_{|\alpha|=|\beta|\leq[s]} c_{\alpha,\beta} \nabla_{\mathbb{H}}^{\beta} \Psi_{j,k}^{\varepsilon}(x_0) (x_0^{-1}x)^{\alpha} + R_{j,k}^{\varepsilon}(x).$$

Theorem 9 ensures that, for some constant $r_1 > 0$ and x in the neighborhood of x_0 :

$$(33) \quad |R_{j,k}^{\varepsilon}(x)| \leq C \|x_0^{-1}x\|_{\mathbb{H}}^{[s]+1} \sup_{|\alpha|=[s]+1, \|z\|\leq r_1} |\nabla_{\mathbb{H}}^{\alpha} \Psi_{j,k}^{\varepsilon}(x_0 z)|.$$

Substitution in the definition of f_j reads:

$$f_j(x) = \sum_{\varepsilon,k} d_{j,k}^{\varepsilon}(f) \left(\sum_{|\alpha|=|\beta|\leq[s]} c_{\alpha,\beta} \nabla_{\mathbb{H}}^{\beta} \Psi_{j,k}^{\varepsilon}(x_0) (x_0^{-1}x)^{\alpha} + R_{j,k}^{\varepsilon}(x) \right).$$

In the first double sum, by combining (26) and (30), one recognizes $P_j(x_0^{-1}x)$ and thus

$$f_j(x) - P_j(x_0^{-1}x) = \sum_{\varepsilon,k} d_{j,k}^{\varepsilon}(f) R_{j,k}^{\varepsilon}(x).$$

Combining (25), (33) and the definition (24) of j_0 gives:

$$\begin{aligned} |f_j(x) - P_j(x_0^{-1}x)| &\leq \sum_{\varepsilon,k} |d_{j,k}^{\varepsilon}(f)| |R_{j,k}^{\varepsilon}(x)| \\ &\leq C 2^{-j_0([s]+1)} \sum_{\varepsilon,k} (2^{-js} + 2^{-j_0s} + \|x_{j,k}^{-1}x\|_{\mathbb{H}}^s) \sup_{\substack{|\alpha|=[s]+1 \\ \|z\|\leq r_1}} |\nabla_{\mathbb{H}}^{\alpha} \Psi_{j,k}^{\varepsilon}(x_0 z)|. \end{aligned}$$

To deal with the summation in k , one uses (27) and (28): for all $\varepsilon = 1, \dots, 2^Q - 1$,

$$\sum_k \sup_{\|z\|\leq r_1} |\nabla_{\mathbb{H}}^{\alpha} \Psi_{j,k}^{\varepsilon}(x_0 z)| \leq C \sum_k \frac{2^{j|\alpha|}}{(1 + \|2^j \circ (x_{j,k}^{-1}x_0)\|_{\mathbb{H}})^{Q+1+s}} \leq C 2^{j|\alpha|}$$

and similarly

$$\sum_k \|x_{j,k}^{-1}x\|_{\mathbb{H}}^s \sup_{\|z\|\leq r_1} |\nabla_{\mathbb{H}}^{\alpha} \Psi_{j,k}^{\varepsilon}(x_0 z)| \leq C \sum_k \frac{2^{j|\alpha|} \|x_{j,k}^{-1}x_0\|_{\mathbb{H}}^s}{(1 + \|2^j \circ (x_{j,k}^{-1}x_0)\|_{\mathbb{H}})^{Q+1+s}} \leq C 2^{j(|\alpha|-s)}.$$

The summation in ε plays no role. Putting it all together, one gets:

$$|f_j(x) - P_j(x_0^{-1}x)| \leq C 2^{-(j_0-j)([s]+1)} (2^{-js} + 2^{-j_0s}).$$

Finally, the sum over $j \in \{0, \dots, j_0\}$ boils down to:

$$(34) \quad \begin{aligned} \sum_{j=0}^{j_0} |f_j(x) - P_j(x_0^{-1}x)| &\leq C2^{-j_0([s]+1)} \sum_{j=0}^{j_0} 2^{j([s]+1-s)} + 2^{-j_0([s]+1+s)} \sum_{j=0}^{j_0} 2^{j([s]+1)} \\ &\leq C2^{-j_0s} \leq C \|x_0^{-1}x\|_{\mathbb{H}}^s. \end{aligned}$$

The term R_1 contains the high-frequency components of the Littlewood-Paley decomposition of f and is responsible for the logarithmic correction in (6). By (24) and (29),

$$\forall j \geq j_0, \quad |f_j(x)| \leq C(2^{-js} + \|x^{-1}x_0\|_{\mathbb{H}}^s) \leq C(2^{-js} + 2^{-j_0s}) \leq C\|x^{-1}x_0\|_{\mathbb{H}}^s.$$

Let us split this remainder depending on whether $j_0 \leq j < j_1$ or $j \geq j_1$

$$R_1(x) \leq \sum_{j=j_0}^{j_1} |f_j(x)| + \sum_{j=j_1}^{\infty} |f_j(x)|.$$

Our choice (24) for j_1 and j_0 gives $j_1 \sim sj_0/\sigma \sim s/\sigma \cdot |\log \|x^{-1}x_0\|_{\mathbb{H}}^s|$. Hence

$$(35) \quad \sum_{j=j_0}^{j_1} |f_j(x)| \leq j_1 \cdot C\|x^{-1}x_0\|_{\mathbb{H}}^s \leq C\|x^{-1}x_0\|_{\mathbb{H}}^s \cdot |\log \|x^{-1}x_0\|_{\mathbb{H}}^s|.$$

When $j \geq j_1$, one uses $f \in C^\sigma(\mathbb{H})$ instead and Theorem 1 which gives $|d_{j,k}^\varepsilon(f)| \leq C2^{-j\sigma}$. Combined with (10), one deduces that

$$|f_j(x)| \leq \sum_{\varepsilon} \sum_{k \in \mathcal{Z}} C2^{-j\sigma} e^{-\delta(x_{j,k},x)/r_0} \leq C2^{-j\sigma}.$$

Using (24) one last time yields to the expected conclusion:

$$(36) \quad \sum_{j=j_1}^{\infty} |f_j(x)| \leq C \sum_{j=j_1}^{\infty} 2^{-j\sigma} \leq C2^{-j_1\sigma} \leq C \|x_0^{-1}x\|_{\mathbb{H}}^s.$$

Let us move to R_2 which contains the Taylor expansions of the high-frequency components of the Littlewood-Paley decomposition of f . Intuitively, it is small because of the natural spectral separation between polynomials and highly-oscillatory functions.

Using (24) and (29), each term of the sum boils down to:

$$|P_j(x_0^{-1}x)| \leq \sum_{|\alpha|=|\beta| \leq [s]} |c_{\alpha,\beta}| |\nabla_{\mathbb{H}}^\beta f_j(x_0)| \| (x_0^{-1}x)^\alpha \|_{\mathbb{H}} \leq C \sum_{n=0}^{[s]} 2^{j(n-s)-j_0n}$$

and thus

$$(37) \quad \sum_{j=j_0}^{\infty} |P_j(x_0^{-1}x)| \leq C \sum_{n=0}^{[s]} 2^{-j_0n} \left(\sum_{j \geq j_0} 2^{-j(s-n)} \right) \leq C2^{-j_0s} \leq C \|x_0^{-1}x\|_{\mathbb{H}}^s.$$

Substituting (32), (34), (35), (36) and (37) back in the original question (31) proves that (6) holds in a neighborhood of x_0 and concludes the proof of Theorem 2. \blacksquare

4.3. Generic monofractality of functions in $C^s(\mathbb{H})$. The proof of Theorem 3 is classical in the Euclidian context [15] and can be adapted quickly to ours.

Let us recall that for any $f \in C^s(\mathbb{H})$, Theorem 1 gives a constant $C > 0$ such that

$$(38) \quad f = \sum_{\varepsilon, j, k} d_{j,k}^\varepsilon(f) \Psi_{j,k}^\varepsilon \quad \text{with} \quad |d_{j,k}^\varepsilon(f)| \leq C 2^{-js}$$

and $\|f\|_{C^s} = \inf\{C > 0 : (38) \text{ is satisfied for all } \varepsilon, j, k\}$ is a Banach norm on $C^s(\mathbb{H})$. For each integer N , let us define:

$$(39) \quad \begin{aligned} E_N &= \{f \in C^s(\mathbb{H}) : \forall (\varepsilon, j, k), 2^{js+N} d_{j,k}^\varepsilon(f) \in \mathbb{Z}^*\} \\ F_N &= \{g \in C^s(\mathbb{H}) : \exists f \in E_N, \|f - g\|_{C^s(\mathbb{H})} < 2^{-N-2}\}. \end{aligned}$$

Lemma 11. *For every $N \geq 1$, all functions in F_N are monofractal of exponent s :*

$$\forall g \in F_N, \quad \forall x \in \mathbb{H}, \quad h_g(x) = s.$$

Proof . This simply follows from the fact that, given $f \in E_N$, all the wavelet coefficients of f satisfy

$$2^{-N-js} \leq |d_{j,k}^\varepsilon(f)| \leq \|f\|_{C^s} 2^{-js}.$$

Thus for any function $g \in F_N$ and its associated $f \in E_N$:

$$2^{-N-js} - 2^{-N-2-2js} \leq |d_{j,k}^\varepsilon(g)| \leq \|f\|_{C^s} 2^{-js} + 2^{-N-2-2js}$$

i.e.

$$2^{-N-1-2js} \leq |d_{j,k}^\varepsilon(g)| \leq (\|f\|_{C^s} + 2^{-N-2}) 2^{-js}.$$

In particular, $g \in C^{s'}(x)$ for any $x \in \mathbb{H}$ and there is no $x_0 \in \mathbb{H}$ and $s' > s$ such that $g \in C^{s'}(x_0)$. Indeed, (13) with $s' > s$ is not compatible when j tends to infinity with the left hand-side of the above inequality. \blacksquare

Lemma 12. *The set $\mathcal{R} = \bigcup_{N \geq 1} F_N$ is a dense open set in $C^s(\mathbb{H})$ containing only monofractal functions with exponent s .*

Proof . The preceding lemma ensures that \mathcal{R} is composed of monofractal functions. According to (39), F_N is an open set and thus, so is \mathcal{R} . Let us check the density. Given $f \in C^s(\mathbb{H})$ and $\eta > 0$, let us choose $N \in \mathbb{N}$ so that $2^{-N} < \eta$. Let us define the “non-zero integer part” function

$$E^*(x) = \begin{cases} 1 & \text{if } 0 \leq x < 2, \\ [x] & \text{else.} \end{cases}$$

Obviously $E^* : \mathbb{R} \rightarrow \mathbb{Z}^*$ and $|x - E^*(x)| \leq 1$. Let us finally define a function $g \in F_N$ by its wavelets coefficients:

$$d_{j,k}^\varepsilon(g) = 2^{-js-N} E^*(2^{js+N} d_{j,k}^\varepsilon(f)).$$

By construction,

$$2^{js} |d_{j,k}^\varepsilon(f) - d_{j,k}^\varepsilon(g)| = 2^{-N} |2^{js+N} d_{j,k}^\varepsilon(f) - E^*(2^{js+N} d_{j,k}^\varepsilon(f))| \leq 2^{-N} < \eta$$

thus $\|f - g\|_{C^s} < \eta$. This proves the density of \mathcal{R} in $C^s(\mathbb{H})$. \blacksquare

5. UPPER BOUND FOR THE MULTIFRACTAL SPECTRUM IN A BESOV SPACE

The classical Sobolev embedding $B_{p,q}^s(\mathbb{H}) \hookrightarrow C^{s-Q/p}(\mathbb{H})$ can be retrieved easily using wavelets. Indeed, the definition (21) reads

$$\left\| 2^{j(s-Q/p)} d_{j,k}^\varepsilon(f) \right\|_{\ell^p(k)} \in \ell^q(j)$$

and implies the existence of a constant $C_0 > 0$ such that for every triplet (ε, j, k) :

$$(40) \quad |d_{j,k}^\varepsilon(f)| \leq C_0 2^{-j(s-Q/p)}.$$

Thus (12) holds and Theorem 1 ensures that $f \in C^{s-Q/p}(\mathbb{H})$. In particular, (13) is satisfied around any point $x_0 \in \mathbb{H}$. and thus by Theorem 2, one has

$$\forall x \in \mathbb{H}, \quad h_f(x) \geq s - Q/p.$$

It is worth mentioning that the index q of the Besov space $B_{p,q}^s(\mathbb{H})$ does not play any role in the Sobolev embedding and neither does it in the multifractal analysis of f .

Let us now establish Theorem 4, *i.e.* that for any $h \geq s - Q/p$, the iso-Hölder set of regularity h is of Hausdorff dimension

$$d_f(h) \leq \min(Q, p(h - s + Q/p)).$$

The inequality is obvious as soon as $h \geq s$, since the upper bound reduces to Q which is the Hausdorff dimension of \mathbb{H} itself. Thus one can now assume that

$$(41) \quad s - Q/p \leq h < s.$$

and in particular, that $1 \leq p < +\infty$.

By Theorem 2, heuristically, the wavelet coefficients that might give rise to an exponent $h_f(x_0) \leq h$ for some $x_0 \in \mathbb{H}$ satisfy $|d_{j,k}^\varepsilon| \geq 2^{-jh}$. Hence we focus on

$$N_f(j, h) = \left\{ k \in \mathcal{Z} : \exists \varepsilon \in \{1, 2, \dots, 2^Q - 1\}, \quad |d_{j,k}^\varepsilon(f)| \geq C_0 2^{-jh} \right\}.$$

For a technical reason, the constant C_0 is the one from the Sobolev embedding (40).

Lemma 13. *There exists $C > 0$ such that for every $j \geq 1$, for every $h' \in (s - Q/p, s]$,*

$$\#N_f(j, h') \leq C 2^{jp(h' - s + Q/p)}.$$

Proof . Obviously from (21), $B_{p,q}^s(\mathbb{H}) \subset B_{p,\infty}^s(\mathbb{H})$ and thus there is a constant $C > 0$ such that for any $j \in \mathbb{Z}$, one has $\sum_{k \in \mathcal{Z}} 2^{j(ps-Q)} |d_{j,k}^\varepsilon(f)|^p \leq C$. Hence,

$$C 2^{-j(ps-Q)} \geq \sum_{\varepsilon} \sum_{k \in \mathcal{Z}} |d_{j,k}^\varepsilon(f)|^p \geq \sum_{k \in N_f(j, h')} \sum_{\varepsilon} |d_{j,k}^\varepsilon(f)|^p \geq C_0^p \times (\#N_f(j, h')) \times 2^{-jph'},$$

which yields the result. Observe that it also holds when $h' \geq s$, but it is useless. \blacksquare

Lemma 14. *The set*

$$E_f^{\leq}(h) = \{x \in \mathbb{H} : h_f(x) \leq h\}.$$

has the following property:

$$(42) \quad \dim_H E_f^{\leq}(h) \leq p(h - s + Q/p).$$

Estimate (42) is stronger than (14) since $E_f(h) \subset E_f^{\leq}(h) = \bigcup_{h' \leq h} E_f(h')$. In particular, $\dim_H E_f(h) \leq \dim_H E_f^{\leq}(h)$ and Theorem 4 follows immediately.

Proof . The definition (7) of h_f as a supremum implies that

$$E_f^{\leq}(h) = \left\{ x \in \mathbb{H} : \forall h' > h, f \notin C^{h'}(x) \right\}.$$

Joint with Theorem 2, this observation provides:

$$(43) \quad E_f^{\leq}(h) = \left\{ x \in \mathbb{H} : \forall h' > h, \sup_{\varepsilon, j, k} [2^{-j} + \delta(x, x_{j,k})]^{-h'} |d_{j,k}^\varepsilon(f)| = +\infty \right\}.$$

Note that as soon as $f \in L^2(\mathbb{H})$, one has $|d_{j,k}^\varepsilon| \leq C2^{jQ/2}$ and thus the only real constraint contained in (43) concerns the regime $j \rightarrow +\infty$ and $h' \in [h, \alpha h]$ for an arbitrary $\alpha > 1$.

We fix $\alpha > 1$, and let T be a large integer. As said above, the only interesting wavelet coefficients are those satisfying $|d_{j,k}^\varepsilon| \geq C_0 2^{-jh'}$ (it is not enough to consider only those greater than $C_0 2^{-jh}$). One splits $[s - Q/p, \alpha h]$ into intervals of length

$$\eta = (\alpha h - s + Q/p) / T,$$

namely the intervals $I_m = [h_{m-1}, h_m]$ for $m \in \{1, \dots, T\}$ and $h_m = s - Q/p + m\eta$. One choses T large enough so that $h_0 - \eta = s - Q/p - \eta > 0$. The next idea is that if $x \in \mathbb{H}$ is far from the dyadic set $(x_{j,k})_{j \in \mathbb{Z}, k \in \mathcal{K}}$ and simultaneously the wavelet coefficient $|d_{j,k}^\varepsilon|$ is too small, then it cannot contribute to (43). More precisely, if simultaneously

$$(44) \quad |d_{j,k}^\varepsilon(f)| \leq C_0 2^{-jh\alpha} \quad \text{or} \quad \exists m \in \{0, \dots, n_1 - 1\}, \quad \begin{cases} |d_{j,k}^\varepsilon(f)| \leq C_0 2^{-jh_m} \\ \delta(x, x_{j,k}) \geq 2^{-j(h_m - \eta)/\alpha h} \end{cases}$$

with the same constant $C_0 > 0$ as in the Sobolev embedding (40), then

$$[2^{-j} + \delta(x, x_{j,k})]^{-h'} |d_{j,k}^\varepsilon(f)| \leq \begin{cases} C_0 2^{-j(\alpha h - h')} & \text{in the first case,} \\ C_0 2^{j(h_m - \eta)(h'/\alpha h)} 2^{-jh_m} & \text{in the others.} \end{cases}$$

In the range of $1 < h'/h \leq \alpha$ and for large j , one infers in both cases that:

$$[2^{-j} + \delta(x, x_{j,k})]^{-h'} |d_{j,k}^\varepsilon(f)| \leq C_0.$$

Therefore, if (44) holds for *any* family $(\varepsilon_n, j_n, k_n)$ with $j_n \rightarrow +\infty$, then $x \notin E_f^{\leq}(h)$.

By contraposition, for any $x \in E_f^{\leq}(h)$, there exists a family $(\varepsilon_n, j_n, k_n)$ with $j_n \rightarrow +\infty$ contradicting (44). By the Sobolev embedding (40), each wavelet coefficient is bounded above by $C_0 2^{-jh_0}$. Hence, for each n , there exists necessarily $m \in \{1, \dots, T\}$ such that

$$C_0 2^{-j_n h_m} < |d_{j_n, k_n}^{\varepsilon_n}(f)| \leq C_0 2^{-j_n h_{m-1}} \quad \text{and} \quad x \in \mathcal{B}(x_{j_n, k_n}, 2^{-j_n(h_{m-1} - \eta)/\alpha h}).$$

The previous statement can be expressed more easily in term of lim-sup sets:

$$(45) \quad E_f^{\leq}(h) \subset \bigcup_{m=1}^T S_{m,\eta}, \quad \text{with} \quad S_{m,\eta} = \bigcap_{J \geq 1} \bigcup_{j \geq J, k \in N_f(j, h_m)} \mathcal{B}(x_{j,k}, 2^{-j(h_m - 2\eta)/\alpha h}).$$

Let us now establish an upper bound for the Hausdorff dimension of each set $S_{m,\eta}$. Given $\xi > 0$, one chooses an integer J_ξ so large that $2\gamma_1 \times 2^{-J_\xi(s - Q/p - \eta)/\alpha h} \leq \xi$ (with

the constant γ_1 from (16)). A covering of $S_{m,\eta}$ by balls of diameter less than ξ is thus provided by:

$$S_{m,\eta} \subset \bigcup_{j \geq J_\xi, k \in N_f(j, h_m)} \mathcal{B}(x_{j,k}, 2^{-j(h_m-2\eta)/\alpha h}).$$

For any $d \geq 0$, the \mathcal{H}_ξ^d -premeasure of $S_{m,\eta}$ can then be estimated easily:

$$\mathcal{H}_\xi^d(S_{m,\eta}) \leq \sum_{j \geq J_\xi} \sum_{k \in N_f(j, h_m)} \left(C 2^{-j(h_m-2\eta)/\alpha h} \right)^d.$$

Using Lemma 13 to estimate $\#N_f(j, h_m) \leq C 2^{jp(h_m-s+Q/p)}$ gives:

$$\mathcal{H}_\xi^d(S_{m,\eta}) \leq C \sum_{j \geq J_\xi} 2^{j[p(h_m-s+Q/p)-d(h_m-2\eta)/\alpha h]}.$$

This series converges when

$$(46) \quad d > \frac{\alpha p h}{h_m - 2\eta} (h_m - s + Q/p)$$

and in that case

$$\mathcal{H}_\xi^d(S_{m,\eta}) \leq C 2^{-J_\xi[d(h_m-2\eta)/\alpha h - p(h_m-s+Q/p)]}.$$

As ξ tends to zero, the d -Hausdorff measure of $S_{m,\eta}$ is 0, which in turn implies that $\dim_H(S_{m,\eta}) \leq d$. Finally, optimizing for any d that satisfies (46) provides

$$\dim_H(S_{m,\eta}) \leq \frac{\alpha p h}{h_m - 2\eta} (h_m - s + Q/p) = \alpha p h \left(1 - \frac{s - Q/p - 2\eta}{h_m - 2\eta} \right).$$

Looking back at (45) one deduces that

$$\dim_H E_f^{\leq}(h) \leq \max_{m=1, \dots, T} \alpha p h \left(1 - \frac{s - Q/p - 2\eta}{h_m - 2\eta} \right) \leq \alpha p h \left(1 - \frac{s - Q/p - 2\eta}{\alpha h - 2\eta} \right).$$

The limit $\eta \rightarrow 0$ provides

$$\dim_H E_f^{\leq}(h) \leq \alpha p h \left(1 - \frac{s - Q/p}{\alpha h} \right) = p(\alpha h - s + Q/p).$$

Finally, letting $\alpha \rightarrow 1$ gives (42) and Theorem 4. \blacksquare

6. ALMOST ALL FUNCTIONS IN $B_{p,q}^s(\mathbb{H})$ ARE MULTIFRACTAL

To prove Theorem 5, one will explicitly construct a G_δ set of functions in $B_{p,q}^s(\mathbb{H})$ that satisfies (15). The proof is adapted from the one of [15], modifications are due to the metric on \mathbb{H} . We first construct a subset \mathcal{R}_0 of $B_{p,q}^s(\mathbb{H})$ whose restriction to $[0, 1]^3$ is generic in $B_{p,q}^s([0, 1]^3)$ and satisfies (15). Next, we define:

$$\forall k \in \mathcal{K}, \quad \mathcal{R}_k = \{f(k^{-1}x) : f \in \mathcal{R}_0\}.$$

Finally, the intersection $\mathcal{R} = \bigcap_{k \in \mathcal{K}} \mathcal{R}_k$ is generic in $B_{p,q}^s(\mathbb{H})$ because it is a countable intersection of G_δ sets and thus still a G_δ set. By construction, it will still satisfy (15).

The actual proof of Theorem 5 is contained in §6.3. To build up for it, one needs to recall a few classical definitions and results on dyadic approximation in §6.1. One then

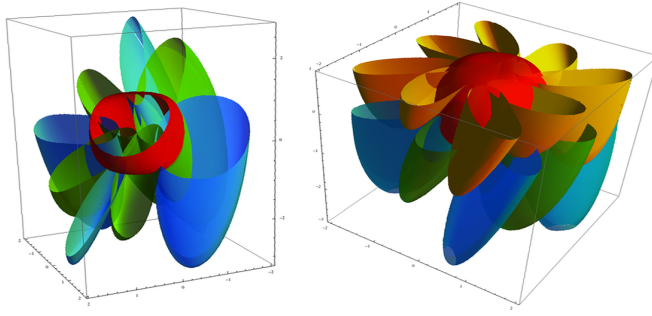


FIGURE 4. Left : section of the unit ball centered at the origin (red) and the balls of radius 1 centered at $(k_1, k_2, 0)$ with $k_1^2 + k_2^2 = 1$ (green) or $k_1^2 + k_2^2 = 2$ (blue). Right : another section of the same balls (same colors) and the balls centered in $(k_1, k_2, 2)$ (orange).

constructs a single function that satisfies (15) in §6.2, which is the starting point for growing the set \mathcal{R}_0 in §6.3.

6.1. Dyadic approximation in \mathbb{H} . For any $j \in \mathbb{N}$, one considers the subset of indices

$$\mathcal{L}_0(j) = \{k \in \mathcal{K} : x_{j,k} = 2^{-j} \circ k \in [0, 1]^3\}$$

For later use, let us observe immediately that

$$(47) \quad \#(\mathcal{L}_0(j)) = 2^{Qj}.$$

Definition 6. A dyadic point $x_{J,K}$ is called *irreducible* if $K = (K_p, K_q, K_r)$ and at least one of the three fractions $2^{-J}K_p$, $2^{-J}K_q$ or $2^{-2J}K_r$ is irreducible. A point $x_{J,K}$ is called the *irreducible version* of $x_{j,k}$ if $x_{J,K}$ is irreducible and $x_{j,k} = x_{J,K}$.

One can check immediately that for a given couple (j, k) the corresponding irreducible couple (J, K) is unique. Note that one may have $(j, k) = (J, K)$ but that one always has $J \leq j$. Conversely, given an irreducible $x_{J,K}$ and $j \geq J$, there exists a unique $k \in \mathcal{K}$ such that $x_{j,k} = x_{J,K}$, namely $k = 2^{j-J} \circ K$.

Given an integer $J \in \mathbb{N}$, the number of irreducible elements $x_{J,K} \in [0, 1]^3$ is:

$$(48) \quad \#\{K \in \mathcal{L}_0(J) : x_{J,K} \text{ irreducible}\} = (2^Q - 1) \times 2^{Q(J-1)}.$$

Indeed, $K = (K_1, K_2, K_3)$ provides a non irreducible $x_{J,K} \in [0, 1]^3$ if and only if $0 \leq K_1, K_2 < 2^J$, $0 \leq K_3 < 2^{2J}$ and

$$K_1 \equiv 0 \pmod{2} \quad \text{and} \quad K_2 \equiv 0 \pmod{2} \quad \text{and} \quad K_3 \equiv 0 \pmod{4}.$$

The complementary set in $\mathcal{L}_0(J)$ is thus of cardinal $2^{J-1} \times 2^{J-1} \times 2^{2J-2} = 2^{Q(J-1)}$ and (48) follows from (47).

Recall that $\mathcal{B}(x, r) = \{y \in \mathbb{H} : \|x^{-1}y\|_{\mathbb{H}} < r\}$ denotes the open gauge ball of radius r . For fixed $j \in \mathbb{Z}$, the dyadic elements $\{x_{j,k} : k \in \mathcal{Z}\}$ are well-distributed in \mathbb{H} , in the sense that the open balls $\{\mathcal{B}(x_{j,k}, 2^{-j}) : k \in \mathcal{Z}\}$ do not intersect too much. One can check easily the following lemma (see Figure 4).

Lemma 15. *For a given $j \in \mathbb{N}$ and $k \in \mathcal{Z}$, the only parameters $k' \in \mathcal{K}$ such that $\mathcal{B}(x_{j,kk'}, 2^{-j}) \cap \mathcal{B}(x_{j,k}, 2^{-j}) \neq \emptyset$ are the 43 cubes defined by $k' = (k'_1, k'_2, k'_3)$, with*

$$\begin{cases} k'_1 = k'_2 = 0 \\ |k'_3| \leq 1 \end{cases} \quad \text{or} \quad \begin{cases} 1 \leq k'_1{}^2 + k'_2{}^2 \leq 2 \\ |k'_3| \leq 2. \end{cases}$$

Proof . By scale invariance of the pseudo-norm $\|\cdot\|_{\mathbb{H}}$, it is sufficient to investigate $j = 0$ and $k = (0, 0, 0)$. Then a counting argument applies. \blacksquare

Observe that if $r_0^2 = \frac{\sqrt{3}}{2} < 1$, then the cylinder $\Gamma_0 = \{(p, q, r) \in \mathbb{H} : p^2 + q^2 < r_0^2\}$ is included in $\bigcup_{k_3 \in \mathbb{Z}} \mathcal{B}((0, 0, k_3), 1)$. Since for any $k = (k_1, k_2, k_3) \in \mathcal{K}$, the left translation of Γ_0 is another vertical cylinder, one can choose a constant $C > 0$ such that for each $j \in \mathbb{N}$, the family of balls

$$\{\mathcal{B}(x_{j,k}, C2^{-j}) : k \in \mathcal{Z}\}$$

covers the whole space \mathbb{H} and that for any strictly increasing sequence $(j_m)_{m \geq 1} \in \mathbb{N}^{\mathbb{N}^*}$:

$$(49) \quad [0, 1]^3 = \limsup_{m \rightarrow +\infty} \bigcup_{k \in \mathcal{L}_0(j_m)} \mathcal{B}(x_{j_m, k}, C2^{-j_m}).$$

Since each point $x \in [0, 1]^3$ belongs to an infinite number of balls $\mathcal{B}(x_{j_m, k}, C2^{-j_m})$, one can wonder of the exact proximity of x to the dyadic elements of \mathbb{H} .

Definition 7. *Let $\mathcal{J} = (j_m)_{m \geq 1}$ be a strictly increasing sequence of integers. For $\xi > 0$, an element $x \in \mathbb{H}$ is said to be ξ -approximable with respect to \mathcal{J} when the inequality*

$$(50) \quad \delta(x, x_{j_m, k}) \leq C2^{-j_m \xi}$$

holds true for an infinite number of couples (m, k) and the same constant C that appears in (49). For a given $\xi > 0$, one defines also:

$$S_{\xi}(\mathcal{J}) = \{x \in [0, 1]^3 : x \text{ is } \xi\text{-approximable with respect to } \mathcal{J}\}.$$

The dyadic approximation rate of x with respect to \mathcal{J} is the real number:

$$\xi_x(\mathcal{J}) = \sup\{\xi > 0 : x \text{ is } \xi\text{-approximable with respect to } \mathcal{J}\}.$$

Finally, the iso-approximable set of rate ξ is:

$$\tilde{S}_{\xi}(\mathcal{J}) = \{x \in [0, 1]^3 : \xi_x(\mathcal{J}) = \xi\}.$$

When $\mathcal{J} = \mathbb{N}$ one simply writes S_{ξ} , \tilde{S}_{ξ} and ξ_x and in that case it is sufficient to restrict oneself in (50) to irreducible dyadic elements $x_{j,k}$. Let us observe that, because of (49):

$$\forall x \in \mathbb{H}, \quad \xi_x(\mathcal{J}) \geq 1.$$

The size of the sets $S_{\xi}(\mathcal{J})$ and $\tilde{S}_{\xi}(\mathcal{J})$ in terms of Hausdorff dimension and measures can be described thanks to the so-called *mass transference principle* by V. Beresnevich, D. Dickinson and S. Velani [3].

Proposition 16. *For every $\xi \geq 1$, one has:*

$$(51) \quad \dim_H \tilde{S}_{\xi}(\mathcal{J}) = \dim_H S_{\xi}(\mathcal{J}) = Q/\xi.$$

Proof . It is quite easy to obtain that for every $\xi \geq 1$,

$$(52) \quad \dim_H S_\xi(\mathcal{J}) \leq Q/\xi \quad \text{and} \quad \dim_H \tilde{S}_\xi(\mathcal{J}) \leq Q/\xi$$

Indeed, by definition, one has:

$$S_\xi(\mathcal{J}) \subset \bigcap_{n \geq 1} \bigcup_{j \geq n, k \in \mathcal{L}_0(j)} \mathcal{B}(x_{j,k}, 2^{-j\xi}).$$

For $d > Q/\xi$ and an arbitrary $\eta > 0$, one choses n large enough such that $2^{-n\xi} < \eta$. The previous inclusion provides a covering of $S_\xi(\mathcal{J})$ by balls of radius smaller than η , thus:

$$\mathcal{H}_\eta^d(S_\xi(\mathcal{J})) \leq C \sum_{j \geq n} 2^{jQ} (2^{-j\xi})^d \leq C \eta^{Q/\xi-d} \xrightarrow{\eta \rightarrow \infty} 0,$$

which proves the first half. The second half follows by noticing that, as $S_{\xi'}(\mathcal{J})$ is a decreasing family (for inclusion) when ξ' increases:

$$\tilde{S}_\xi(\mathcal{J}) \subset \bigcap_{\xi' < \xi} S_{\xi'}(\mathcal{J}).$$

The converse inequality to (52) is very difficult, but is contained in [3]. Their main theorem (stated as Theorem 17 below) holds at a great level of generality. It holds in particular on the Heisenberg group since \mathbb{H} endowed with the metric (1) and the Haar measure $\ell = dp dq dr$ satisfies the following conditions :

- ℓ is translation-invariant,
- ℓ has a scaling behavior *i.e.* there exists a constant $C > 1$ such that:

$$\forall x \in \mathbb{H}, \quad \forall r > 0, \quad C^{-1}r^Q \leq \ell(\mathcal{B}(x, r)) \leq Cr^Q$$

and in particular ℓ is doubling *i.e.*:

$$\forall x \in \mathbb{H}, \quad \forall r > 0, \quad \ell(\mathcal{B}(x, 2r)) \leq C\ell(\mathcal{B}(x, r)).$$

- The dyadic set $\{x_{j,k} : j \in \mathcal{J}, k \in \mathcal{K}\}$ is discrete.

Joint with the covering property (49), the main result of [3], called *mass transference principle*, implies the following:

Theorem 17 ([3], Theorem 2, p. 15). *For every $\xi \geq 1$, one has*

$$(53) \quad \mathcal{H}^{Q/\xi}(S_\xi(\mathcal{J})) = +\infty \quad \text{and} \quad \mathcal{H}^{Q/\xi}(\tilde{S}_\xi(\mathcal{J})) = +\infty.$$

The statement (53) implies that both the Hausdorff dimension of $S_\xi(\mathcal{J})$ and of $\tilde{S}_\xi(\mathcal{J})$ are greater or equal to Q/ξ . Combined with (52), this proves (51). \blacksquare

6.2. Example of a function with the maximal possible spectrum. Recall that for $x_{j,k} \in [0, 1]^3$, we denote by $x_{J,K}$ its irreducible version.

Proposition 18. *Let $\beta = 1/p + 2/q$ and F be the function whose wavelet coefficients are*

$$(54) \quad F_{j,k}^\varepsilon := d_{j,k}^\varepsilon(F) = \begin{cases} \frac{2^{-j(s-Q/p)-JQ/p}}{j^\beta} & \text{if } x_{j,k} \in [0, 1]^3 \\ 0 & \text{otherwise.} \end{cases}$$

The function F belongs to $B_{p,q}^s(\mathbb{H})$ and it satisfies (15).

Observe that, by construction, F is essentially supported in $C_0 = [0, 1]^3$. Outside C_0 , F is as smooth as the mother wavelet and decays rapidly at infinity.

Proof. Let us fix a generation $j \geq 1$ and consider the sequence $a_j = \|2^{j(s-Q/p)} F_{j,k}^\varepsilon\|_{\ell^p(k)}$. For a given integer j , one has

$$\begin{aligned} 2^{-j(ps-Q)} a_j^p &= \sum_{k \in \mathcal{L}_0(j)} \sum_{1 \leq \varepsilon < 2^Q} |F_{j,k}^\varepsilon|^p = (2^Q - 1) \sum_{k \in \mathcal{L}_0(j)} |F_{j,k}^1|^p \\ &\leq (2^Q - 1) \sum_{J=0}^j \frac{2^{-j(ps-Q)-QJ}}{j^{p\beta}} \#\{K : x_{J,K} \in \mathcal{L}_0(J) \text{ is irreducible}\}. \end{aligned}$$

Consequently, using (48):

$$a_j \leq \frac{(2^Q - 1)^{1/p}}{j^\beta} \left(1 + (2^Q - 1) \sum_{J=1}^j 2^{Q(J-1)-QJ}\right)^{1/p} \leq \frac{(2^Q - 1)^{2/p}}{j^\beta} (1 + j2^{-Q})^{1/p}.$$

Finally, the choice (54) for β provides $a_j \leq (2^Q - 1)^{2/p}/j^{2/q}$, thus the sequence $(a_j)_{j \geq 1}$ belongs to ℓ^q and $F \in B_{p,q}^s(\mathbb{H})$.

In order to compute the multifractal spectrum of F one uses the following lemma.

Lemma 19. For each $x \in [0, 1]^3$, $h_F(x) = s - \frac{Q}{p} + \frac{Q}{p\xi_x}$.

Here $\xi_x = \xi_x(\mathbb{N})$ is the approximation rate of x by all the dyadic elements. Assume for a while that Lemma 19 holds true ; let us explain how to conclude from there. Since $\xi_x \in [1, +\infty]$ for each x one first observes that $h_F(x)$ belongs necessarily to the interval $[s - Q/p, s]$. If $h \in (s - Q/p, s]$, one can observe further that

$$E_F(h) = \{x \in [0, 1]^3 : h_f(x) = h\} = \left\{x \in [0, 1]^3 : \xi_x = \frac{Q}{ph - ps + Q}\right\}.$$

Applying (51), one deduces that

$$d_F(h) = \dim_H E_F(h) = \dim_H \tilde{S}_{\frac{Q}{ph-ps+Q}} = \frac{Q}{\frac{Q}{ph-ps+Q}} = ph - ps + Q$$

which is the expected result. If $h = s - Q/p$, the dimension cannot exceed 0 by the general upper bound given by Theorem 4. The dimension is exactly 0 because one can find $x \in \mathbb{H}$ such that $\xi_x = \infty$. Such points x are analogues of Liouville numbers in \mathbb{H} . ■

Proof of Lemma 19. Consider $x \in [0, 1]^3$ with $1 \leq \xi_x < +\infty$. By definition, for every $\varepsilon > 0$, one has the following properties:

(i) There exists $J_x > 0$ such that for every $j \geq J_x$, for every k ,

$$\delta(x, x_{j,k}) \geq 2^{-j(\xi_x + \varepsilon)}.$$

(ii) There exists a strictly increasing sequence of integers $(j_n)_{n \geq 1}$ and a sequence $(k_n)_{n \geq 1} \in \mathbb{Z}^{\mathbb{N}}$ such that $2^{-j_n} \circ k_n$ is irreducible and

$$\delta(x, x_{j_n, k_n}) \leq 2^{-j_n(\xi_x - \varepsilon)}.$$

When $\xi_x = 1$ one may take $\varepsilon = 0$ in the last inequality.

To get the lower bound for the Hölder exponent, consider dyadic elements $x_{j,k}$ such that their associated irreducible element $x_{J,K}$ satisfy $J \geq J_x$. By item (i), one necessarily has $\delta(x, x_{j,k}) = \delta(x, x_{J,K}) \geq 2^{-J(\xi_x + \varepsilon)}$. By using that 2^{-j} and $\delta(x, x_{j,k})$ are bounded by above by their sum $2^{-j} + \delta(x, x_{j,k})$, we get that

$$\begin{aligned} F_{j,k}^\varepsilon &= \frac{1}{j^\beta} 2^{-j(s-Q/p)-JQ/p} \leq (2^{-j} + \delta(x, x_{j,k}))^{s-Q/p} (\delta(x, x_{j,k})^{\frac{1}{\xi_x + \varepsilon}})^{Q/p} \\ &\leq (2^{-j} + \delta(x, x_{j,k}))^{s-Q/p + \frac{Q}{p(\xi_x + \varepsilon)}}. \end{aligned}$$

This is equivalent to (13), hence $h_F(x) \geq s - \frac{Q}{p} + \frac{Q}{p(\xi_x + \varepsilon)}$. Letting ε tend to zero yields the lower bound in Lemma 19.

Let us bound by above the Hölder exponent of F at x , by using item (ii). Assume that $1 < \xi_x < +\infty$ and fix $\varepsilon > 0$ such that $\xi_x - \varepsilon > 1$. For any integer $n \geq 1$, let $\tilde{j}_n = [j_n(\xi_x - \varepsilon)]$. Consider the unique dyadic element $x_{\tilde{j}_n, \tilde{k}_n}$ such that $x_{\tilde{j}_n, \tilde{k}_n} = x_{j_n, k_n}$. Using that $2^{-\tilde{j}_n} \circ k_n$ is irreducible, one sees that

$$F_{\tilde{j}_n, \tilde{k}_n}^\varepsilon = \frac{1}{(\tilde{j}_n)^\beta} 2^{-\tilde{j}_n(s-Q/p)-\tilde{j}_n Q/p} \geq \frac{1}{(\xi_x - \varepsilon)^\beta j_n^\beta} 2^{-\tilde{j}_n(\xi_x - \varepsilon)(s - \frac{Q}{p} + \frac{Q}{p(\xi_x - \varepsilon)})}.$$

Hence, since $\log(j_n(\xi_x - \varepsilon))$ is negligible with respect to j_n when $n \rightarrow +\infty$, one has

$$F_{\tilde{j}_n, \tilde{k}_n}^\varepsilon \geq 2^{-\tilde{j}_n(\xi_x - \varepsilon)(s - \frac{Q}{p} + \frac{Q}{p(\xi_x - \varepsilon)} + \varepsilon)} \geq d(x, x_{j_n, k_n})^{s - \frac{Q}{p} + \frac{Q}{p(\xi_x - \varepsilon)} + \varepsilon} = d(x, x_{\tilde{j}_n, \tilde{k}_n})^{s - \frac{Q}{p} + \frac{Q}{p(\xi_x - \varepsilon)} + \varepsilon}.$$

This proves that $h_F(x) \leq s - \frac{Q}{p} + \frac{Q}{p(\xi_x - \varepsilon)} + \varepsilon$, for every $\varepsilon > 0$. Letting ε tend to zero yields the upper bound in Lemma 19. The cases $\xi_x = 1$ and $\xi_x = +\infty$ are dealt with similarly. \blacksquare

6.3. The residual set in $B_{p,q}^s(\mathbb{H})$. Let us define the wavelet version of local spaces.

Definition 8. The space $B_{p,q}^s([0,1]^3)$ is the closed subspace of $B_{p,q}^s(\mathbb{H})$ defined by

$$k \notin \mathcal{L}_0(j) \implies d_{j,k}^\varepsilon(f) = 0.$$

It is equipped with norm $\|f\|_{B_{p,q}^s(\mathbb{H})} = \|f\|_\infty + \|(a_j)_{j \geq 1}\|_{l^q}$, where a_j is given by (21).

Since $B_{p,q}^s([0,1]^3)$ is separable, let $(f_n)_{n \geq 1}$ be a countable dense sequence in $B_{p,q}^s([0,1]^3)$. Let us consider the sequence $(g_n)_{n \geq 1}$ built as follows.

Definition 9. For every $n \geq 1$, the wavelet coefficients of g_n up to the generation $j = n - 1$ are those of f_n ; for $j \geq n$, the wavelet coefficients of generation j of g_n are those of the function F .

Since $\|f_n - g_n\|_{B_{p,q}^s(\mathbb{H})}$ tends to zero when $n \rightarrow +\infty$, $(g_n)_{n \geq 1}$ is also dense in $B_{p,q}^s([0,1]^3)$.

Definition 10. Let $r_n = n^{-\beta} 2^{-nQ/p} / 2$ with β given by (54). One defines the set $\tilde{\mathcal{R}}$

$$\tilde{\mathcal{R}} = \bigcap_{N \geq 1} \bigcup_{n \geq N} \mathcal{B}(g_n, r_n)$$

where $\mathcal{B}(g, r) = \{f \in B_{p,q}^s([0,1]^3) : \|f - g\|_{B_{p,q}^s(\mathbb{H})} < r\}$.

The set $\tilde{\mathcal{R}}$ is an intersection of dense open set, hence a residual set in $B_{p,q}^s([0,1]^3)$. The choice for the radius r_n is small enough to ensure that any function f in $\mathcal{B}(g_n, r_n)$ has its wavelet coefficients at generation n close to those of g_n (and thus to those of F).

Lemma 20. *If $f \in \mathcal{B}(g_n, r_n)$, then $|d_{n,k}^\varepsilon(f) - d_{n,k}^\varepsilon(g_n)| \geq |d_{n,k}^\varepsilon(g_n)|/2$.*

Proof . By definition, one has $d_{n,k}^\varepsilon(g_n) = F_{n,k}^\varepsilon, \forall k$. Hence, by definition of the Besov norm and the inclusion $\ell^q \subset \ell^\infty$:

$$\left(\sum_k 2^{pn(s-Q/p)} |d_{n,k}^\varepsilon(f) - F_{n,k}^\varepsilon|^p \right)^{1/p} < r_n.$$

In particular, for any ε and k ,

$$|d_{n,k}^\varepsilon(f) - F_{n,k}^\varepsilon| \leq r_n 2^{-n(s-Q/p)} \leq 2^{-ns} n^{-\beta} / 2.$$

The inequality $J \leq j$ in (54) reads $|F_{j,k}^\varepsilon| \geq 2^{-js}/j^\beta$. Combining both inequalities ensures the result. \blacksquare

Lemma 21. *If $f \in \tilde{\mathcal{R}}$, then its multifractal spectrum d_f satisfies (15).*

Proof . Given a function $f \in \tilde{\mathcal{R}}$, there exists a strictly increasing sequence $(n_m)_{m \geq 1}$ of integers such that $f \in \mathcal{B}(g_{n_m}, r_{n_m})$. Lemma 20 provides a precise estimate of the wavelet coefficients of f , namely for any $m \geq 1$:

$$\frac{1}{2} F_{n_m,k}^\varepsilon \leq |d_{n_m,k}^\varepsilon(f)| \leq \frac{3}{2} F_{n_m,k}^\varepsilon.$$

The same proof as the one developed for Lemma 19 ensures that for any $x \in [0,1]^3$:

$$s - Q/p \leq h_f(x) \leq s - Q/p + Q/(p\xi_x(\mathcal{J})) \leq s,$$

where $\xi_x(\mathcal{J})$ is the approximation rate by the family $\mathcal{J} = (n_m)_{m \geq 1}$.

Given $h \in [s - Q/p, s]$ and the unique ξ such that $h = s - Q/p + Q/(p\xi)$, one introduces the set (see Definition 7 and Lemma 14):

$$\mathcal{E} = S_\xi(\mathcal{J}) \setminus \bigcup_{i=1}^{+\infty} E_f^{\leq}(h - 1/i).$$

By (42) one knows that $\dim_H E_f^{\leq}(h') \leq p(h' - s - Q/p)$ for any $h' < h$. In particular, for every $i \geq 1$, one has:

$$\dim_H E_f^{\leq}(h - 1/i) \leq p(h - 1/i - s - Q/p) < p(h - s - Q/p) = Q/\xi.$$

But according to (53), one has $\mathcal{H}^{Q/\xi}(S_\xi(\mathcal{J})) = +\infty$, thus $\mathcal{H}^{Q/\xi}(\mathcal{E}) = +\infty$ and

$$\dim_H \mathcal{E} \geq Q/\xi.$$

Next, one observes that $\mathcal{E} \subset E_f(h)$, since every $x \in S_\xi(\mathcal{J})$ satisfies $h_f(x) \leq h$ and, by definition, \mathcal{E} does not contains those elements x which have a local exponent strictly smaller than h . One can thus finally infer that:

$$\dim_H E_f(h) \geq \dim_H \mathcal{E} \geq Q/\xi = p(h - s - Q/p).$$

The converse inequality is provided by Theorem 4 because $f \in B_{p,q}^s([0,1]^3)$. Consequently, the identity (15) is satisfied. \blacksquare

To conclude the proof of Theorem 5, let us go back to the initial remarks of §6. The subset \mathcal{R}_0 of $B_{p,q}^s(\mathbb{H})$ whose (wavelet) restriction to $[0,1]^3$ satisfies (15) and is generic in $B_{p,q}^s([0,1]^3)$ is simply

$$\mathcal{R}_0 = \pi^{-1}(\tilde{\mathcal{R}})$$

where $\pi : B_{p,q}^s(\mathbb{H}) \rightarrow B_{p,q}^s([0,1]^3)$ is the projection defined in wavelet coefficients by $d_{j,k}^\varepsilon(\pi(f)) = d_{j,k}(f) \cdot \mathbf{1}_{\mathcal{L}_0(j)}(k)$, $\mathbf{1}_A(x)$ being equal to 1 if $x \in A$, 0 otherwise.

7. GENERALIZATION TO STRATIFIED NILPOTENT GROUPS

A Carnot group G is a connected, simply connected and nilpotent Lie group whose Lie algebra \mathfrak{g} admits a stratification, *i.e.*

$$\mathfrak{g} = \bigoplus_{k=1}^N \mathfrak{n}_k \quad \text{where} \quad [\mathfrak{n}_1, \mathfrak{n}_k] = \mathfrak{n}_{k+1}$$

with $\mathfrak{n}_N \neq \{0\}$ but $\mathfrak{n}_{N+1} = \{0\}$. Let us denote the dimensions $q_k = \dim \mathfrak{n}_k$,

$$d = \sum_{k=1}^N q_k \quad \text{and} \quad Q = \sum_{k=1}^N kq_k.$$

Given a basis $(X_i)_{i=1,\dots,d}$ of \mathfrak{g} adapted to the stratification, each index $i \in \{1, \dots, d\}$ can be associated to a unique $\sigma_i = j \in \{1, \dots, N\}$ such that $X_i \in \mathfrak{n}_j$.

Similarly to (17), the horizontal derivatives are the derivatives of the first layer:

$$\nabla_G f = (X_1 f, \dots, X_{q_1} f).$$

The stratification hypothesis ensures that each derivative $X_i f$ can be expressed as at most $\sigma_i - 1$ commutators of horizontal derivatives.

A Carnot group is naturally endowed with a family of algebra homomorphisms called dilations $\{D_\lambda\}_{\lambda>0}$ that are defined by:

$$\forall i \in \{1, \dots, d\}, \quad D_\lambda(X_i) = \lambda^{\sigma_i} X_i.$$

The exponential map $\exp : \mathfrak{g} \rightarrow G$ is a global analytic diffeomorphism and one can identify G and \mathfrak{g} equipped with the (non commutative if $N \geq 2$) law:

$$X * Y = \exp^{-1}(\exp(X) \cdot \exp(Y)).$$

Finally, one can identify \mathfrak{g} to \mathbb{R}^d through the basis $(X_i)_{i=1,\dots,d}$. The gauge distance is then defined by

$$\delta(x, y) = \|x^{-1} * y\|_G, \quad \text{where} \quad \|x\|_G = \left(\sum_{i=1}^d |x_i|^{2\sigma/\sigma_i} \right)^{1/2\sigma}$$

and $\sigma = \text{lcm}\{\sigma_1, \dots, \sigma_d\}$. The distance is left-invariant and homogeneous of degree 1 with respect to the dilations. The Haar measure coincides with the Lebesgue measure on \mathbb{R}^d and the volume of the ball $\mathcal{B}(x, r)$ is $r^Q \text{Vol } \mathcal{B}(0, 1)$. Hausdorff measures can be defined in a similar fashion as in \mathbb{H} . As it was the case for \mathbb{H} , the group G is defined on \mathbb{R}^d but its Hausdorff dimension is $Q > d$. We refer to [21] for further references.

One can wonder whether the results of multifractal analysis obtained in the present paper still hold in any Carnot group. Though caution is always necessary, we claim that the answer is positive because each of the main tools that we used is already available in the literature for at least Carnot groups and sometimes even more general structures. Here is a list of references:

- Wavelets have been constructed by Lemarié [18] on any Carnot group. When the integer lattice $\mathcal{K} = \mathbb{Z}^d$ identifies to a subgroup of $(\mathfrak{g}, *)$, the same properties as those we used hold with just an obvious adaptation of notations. In that case, wavelets of generation j are indeed the push-forward of a finite family $(\Psi^\varepsilon)_{1 \leq \varepsilon < 2^Q}$ of mother wavelets through the dilation D_{2^j} and group translations. When \mathbb{Z}^d is not a subgroup, wavelets can still be constructed but the exact scaling property to a **finite** family of mother wavelets might be lost. However, the wavelets will be of the form

$$\Psi_{j,k}^\varepsilon(x) = \Psi_{\varepsilon,j,k}^\# \left(2^j \circ (x_{j,k}^{-1}x) \right)$$

with $\Psi_{\varepsilon,j,k}^\# = \mathcal{L}^M \vartheta_{\varepsilon,j,k}^\#$ and functions $\vartheta_{\varepsilon,j,k}^\#$ that satisfy (10) with uniform constants with respect to $\varepsilon \in \{1, \dots, 2^Q - 1\}$, $j \in \mathbb{Z}$ and $k \in \mathcal{K}$. Throughout the proofs, the estimates on the tail of the wavelets will therefore be unchanged.

- Taylor polynomial and estimates of the error term in a Taylor expansion (Theorem 9) hold on stratified groups (see [12]). For general homogeneous groups, a weaker estimate is given in [12] and explicit Taylor formulas with various remainder terms can also be found in [5]. For example, in the same spirit as (18) around $x_0 = 0$, a derivative $(X_1, \dots, X_d)^\alpha f(0)$ will be associated to homogeneous polynomial of weight $|\alpha| = \sum \sigma_i \alpha_i$. As expected, it will not just be $(x_1, \dots, x_d)^\alpha$ but the α^{th} power of some fixed polynomials that are the exponential coordinates of x (see [5, Lemma 4]).

- Throughout the generalization, one will substitute the new value of the homogeneous dimension Q . The numerical value of the constants related to the numeration of neighboring cubes or balls will also have to be modified.

- For the diophantine approximation, one should note that the $\mathcal{K} = \mathbb{Z}^d$ will not always be a subgroup of \mathfrak{g} . However, this does not affect the analysis of the dyadic set $\bigcup_{j \geq 0} D_{2^{-j}}(\mathcal{K})$. For example, the results of V. Beresnevich, D. Dickinson and S. Velani [3] used in the proof of the Hausdorff dimension of iso-approximable sets (51) hold actually on metric spaces endowed with a translation invariant homogeneous measure and (in our case) a discrete dyadic set.

Taking those remarks in consideration, one can assert that Theorems 1 to 5 remain valid on any Carnot group.

Further generalization (*e.g.* to the realm of homogeneous groups) are not as straightforward. For example, even though the metric structure of homogeneous groups is still defined in a similar fashion to the gauge distance on Carnot groups, the notion of horizontal derivatives cease to exist which changes deeply the nature of the Taylor formula and its remainder [5] and thus the subsequent analysis. The construction and analysis of wavelets in such a general setting is also an active area of mathematics. We refer to [13] and the references therein for a general review of continuous and discrete wavelets on Carnot groups and their relations with homogeneous Besov spaces.

The reader might also be interested in the following works concerning wavelets on compact Lie groups [23], on general Lie groups [26], on homogeneous spaces [10] and even riemannian manifolds [14].

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