A Semi Number-Theoretic Approach and Proposed proof that CH is false.

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Abstract
We introduce number theoretic systems for use in the study of subsets associated with the reals via its association with fractional portions of real numbers. Our techniques will then highlight properties associated with such fractionals that render these as part of the reals. We will then show that subsets of such fractionals can be formed, and finally we will then show that such sets-subsets possess different cardinalities, whilst having cardinality greater than that of the naturals.

INTRODUCTION.

In mathematics, especially with new mathematics, there is a huge difficulty in drawing an entire picture of its usefulness in solving a particular problem or its effectiveness in the way of a theory in just a few lines, yet this is necessary, as very few will want to undertake the understanding of a new concept if one does not see its potential immediately!
So in the next few lines, we will aim to draw a picture of the usefulness of number theory in the resolution of CH. The time is ripe for many such pictures as was expressed in a Lecture by Matt Foreman, University of California, Irvine. Therein was described all mainstream set-theoretic attempts at resolving CH with concluding remarks surrounding other possible un-ventured possibilities which the writer writes "There are viable alternatives to the Woodin 'Solution' of the CH and these should be considered and explored before we rush to celebrate."
Many such attempts pioneered by W.H.Woodin involve a logical concept of models, as our approach is far from set theoretic we will not explore this concept beyond this mention [W1]-[W4].

Aside from many other similar beliefs we also hold that mathematic should not be just about finding an answer but more about understanding it. More specifically instead of some *reductio ad absurdum* resolution to CH, a more appealing answer would express in conjunction with such an argument, (much like the original diagonalization argument of Cantor) what specific property possessed by numbers disallows/allows a grouping of numbers so as to form a cardinality $\aleph_1 | \aleph_0 < \aleph_1 < 2^{\aleph_0}$.

Inclusion of elements into a set is informally dependent on how well the elements we wish to associate is understood. When dealing with any notion of cardinality, we require precision in our definition of $r \in \mathbb{R}$. A precise definition

enables us to assign properties to the possible distributions of units forming the fractional portions of real numbers. This in turn will surely enable us to precisely ask the questions, can one form a subset of the reals? If so what is its cardinality with respect to the reals? Or do all subsets of the reals have the same cardinality?
Having this in mind, a precise understanding of real numbers will be the task of the section that follows.

We choose from here-on consider fractionals and its associated numbers in base two, the reasons for this stems from an insight providing the basis for this paper and will be made clear in the sections that follow.

Let us fix some notation before we continue. Fractional portions of numbers we denote via use of $\{x\}$. The letter d with subscript i will be used to denote the i^{th} unit of a fractional. $n\{x\}$ will denote the number associated with the fractional represented by $\{x\}$, thus for instance in $11.1001, n = 11$.

We will concern ourselves in the next few sections with the understanding of what exactly constitutes an irrational number? The question can be made much less philosophical in nature if we ask instead, what are the structures associated with the ordering of units associated with fractionals of irrational numbers?
In the case of rational numbers there are a few means of knowing whether a number is a rational number or not, for instance, the size of the fractional portion of a number being finite in unit-measure[1] is a clear indicator that the number is rational one. What then if the fractional portion is not finite in its units?[2] The definition k is rational if and only if k is expressible as a quotient $k = \frac{a}{b} | a, b \in \mathbb{N}$ is too facile to be of any use in this instance.
However an extrapolated property of the above definition, if examined, yields an insight into the structure associated with the ordering of units forming the fractional portion of such raional numbers.
Keeping in mind that the fractional associated with k is non zero and rewriting the expression $k = \frac{a}{b}$ as $bk = a$ highlights the fact that the fractional portion of k can be reduced to zero or be made into $frac(k) = 111111...$[3] by summing k to itself b number of times. Surely some structural ordering of units associated with $frac(k)$ that enables this transformation via addition.

we might understand what it is that is required of the structure associated with such fractionals that renders itself as one of the rationals by observing the mechanical processes involved in the addition of numbers in base two.

Following two simple rules, one can perform addition in base two, both of which pertain to the alignment of symbols associated with the numbers involved in

[1]The number of units in the fractional portion of the number is finite.
[2]examples of these are $\frac{1}{3}$ or $\frac{1}{11}$ in base two
[3]An argument can be made that as $(frac(0.1111...))$ increases in one units, 0.1111... approaches 1.

the addition.

Given two numbers (with fractional portions) in base two, that are being added.

a) (0) and (0) in alignment remains (0).

a) (1) and (1) in alignment at some position p associated with the units leaves (0) at p and introduces (1) at $p - 1$ (as part of some 'third' number in the addition process).

Using this, let us observe the role of addition in the fractional portion of numbers by considering a simple sequence of one units spaced evenly apart by a series of zero units, forming the fractional portion of some arbitrary number.

$$d_u = (000010000100001) \tag{1}$$

It is easy to see that sufficient action[4]Refer to definition 2.0, action is the process of summing a number to itself as many number of times as is required.) on d_u (sufficient number of additions of d_u to itself) will transform the above sequence into:

$$(000100001000010)$$

and with further action into

$$(001000010000100)$$

With sufficiently many more all zeros inevitably begin to fill with one symbols

$$(110001100011000)$$
$$(111001110011100)$$

Until eventually:

$$(111111111111111)$$

Logic dictates that even in the case of a fractional with infinitely many such pairs of one symbols spaced **evenly** by finitely many zeros, via finite action, the chained fractional can surely be rendered into d_{one}.

When then and for which schema of zero and one units forming a fractional is such a reduction impossible?

One logically intuitive idea seems to be the forming of a schema via the targeting of the measure of zero symbols alone that lie between pairs of one symbols. From the initial portion of this discourse, it seems clear that it is such spacing of one symbols that seems intuitively responsible for whether such reductions are possible or not. More clearly, the number of such periodically recursive intervals is irrelevant, as each interval is periodically equivalent to every other:

10001.....10001

[4] (

and upon action (the number summed to itself), the result of the addition is that every respective interval involved in the addition process is transformed equivalently.

10001.....10001
10001.....10001
00010.....00010

If each such interval were to differ in the magnitude of zeros between pairs of one units, one would easily find that with continuous action, the smaller such interval would initially fill out with one units before other larger such intervals. For instance given the following schema:

000100000100000001......

the above with sufficient action would transform into:

111100111100001111......

which illustrates how a smaller interval of zeros between ones would fill out with one symbols before other larger such intervals with sufficient action.
Naturally this argument can be generalized via induction. Specifically for ever growing such intervals, the smaller interval of any one set of growing intervals would naturally fill out in one symbols via finite action earlier than all others exceeding in measure of zero symbols, thus by this logic, if there is always one interval I_i exceeding in measure of zeros than every other, then by induction if it takes d summations to fill interval I_{i-1} then more summations than d is required to fill out I_i in this manner, since I is arbitrary, it is impossible to fill every interval in this manner.

Let us formalize the means by which we define and describe such numbers and the sets to which these belong, with the following definitions.

Definition 0.0 (Stream)
We define a **stream** to be an arbitrary series of non-terminating zero and one units forming the fractional portion of a number.
An instance of this would be : 0001000101010001001.... in 101.0001000101010001001....
We will from hereon denote such streams with the symbol $\mathbb{S}$.

Definition 1.0 (Interval)
Given a stream S, a series of uninterrupted zero units in a stream we refer to as an **interval**. We will denote these via use of the symbol $\mathcal{I}$.

Definition 2.0 (Action)
Given a stream $\mathbb{S}$ the summing of a stream to itself we define to be an **action** on $\mathbb{S}$. We will denote such an action on a stream $\mathbb{S}$ via use of the symbol $\mathcal{A}(\mathbb{S})$

Definition 3.0 (Packet)
Given a stream $\mathbb{S}$, the $(n, n+1, n+2, .., n+t)|n, t \in \mathbb{N}$ units associated with $\mathbb{S}$ of a stream we will refer to as a packet. We will denote such packets via use of the symbol $\mathcal{P}$
For instance: 100101 in 1000000**100101**000000001, will be referred to as a packet.
Remarks
The units compiled into a packet vary in accordance with those occupying the $(n, n+1, n+2, .., n+t)$ 'positions' within the stream to which the packet is associated. In instances where zeros associated with packets maybe confused with those belonging to intervals, for instance the final zero of the packet **100101010**0000. However we will show in the definition (packet-measures) that when defining a stream **characteristic unless the stream is said to be interval-extended**, a clear distinction exists in the way of segregating units of packets and intervals respectively.

Definition 4.0 (Packet/Interval-Measure)
We define a **packet/Interval-measure** to be the number of units associated with any one packet/Interval. We will denote such measures via use of the symbol $\mathcal{M}$ and $\mathcal{M}(\mathcal{I}), \mathcal{M}(\mathcal{P})$ will mean the measure associated with an interval, packet respectively.
In the above case the number (6) would be the measure of the packet.

Definition 4.1 (Interval/Packet-Structure)
Given a stream $\mathbb{S}(\mathcal{P}_i, \mathcal{I}_i)|\forall i \in \mathbb{N}$, the **interval/packet structure** associated with the stream is the association of finite interval/packet-measures with each interval $\mathcal{I}_i$.

Definition 5.0 (To Fill-Out a Packet)
Given a stream $\mathbb{S}$ and a packet $\mathcal{P}$ forming part of $\mathbb{S}$. We say that $\mathcal{P}$ is filled out if via finite action on $\mathbb{S}$, the resultant units of $\mathcal{P}$ consist only of ones.
For instance if **1000000**100101000000001 is reduced to **1111111**......0001, we say that the targeted packet has been filled out.

Theorem 1.0
Given a stream $\mathbb{S}$ with filled-out packets spaced by ever-growing intervals, specifically each proceeding interval $\mathcal{I}_i$ is greater in measure than its preceding interval $\mathcal{I}_{i-1}$, are irreducible to d_{one} via finite action on $\mathbb{S}$.
proof
If there is always one interval $\mathcal{I}_i$ exceeding in measure of zeros than every other, then by induction if it takes d summations to fill interval $\mathcal{I}_{i-1}$ then more summations than d is required to fill out $\mathcal{I}_i$ in this manner, since $\mathcal{I}$ is arbitrary, it is impossible to fill every interval in this manner. The nature of the packet following the interval is arbitrary in the way of affecting the nature of how fast such intervals are filled. □

Definition 6.0 (Real Stream)
$\mathbb{S}_R := \{\forall t \in \mathbb{N},\ \mathcal{M}(\mathcal{I}_t) = \mathcal{M}(\mathcal{I}_{t-1}) + l$ and $\mathcal{M}(\mathcal{P}_i) = g | l, g \in \mathbb{N}\}$, for finitely countable $i \mapsto \mathbb{N}$.

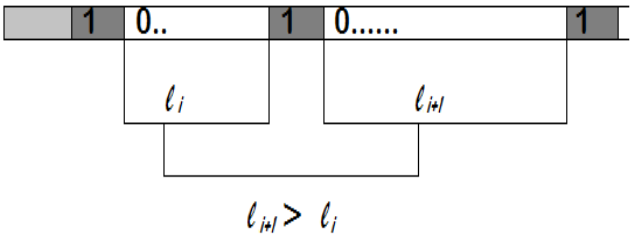

Figure 1: Real Stream: One symbols with growing number of zeros between them.

The Continuum Hypothesis
Given the preceding constructions, specifically those involving the formation of real streams $\mathbb{S}_R$ and a subset $/\mathbb{S}_R$ of such streams.
A natural question that follows is, how can one compare the cardinalities associated with $\mathbb{S}, /\mathbb{S}$ respectively? It is the resolution of this question that is the main result of this article, and also that which we will be occupying ourselves with through to the end of this section.
The former case, within the context of the continuum hypothesis, involved the comparison of cardinalities associated with the sets $\mathbb{N}$ and $\mathbb{R}$ respectively. The resolution involving a unique diagonalization argument by Cantor[C1], paving the inspiration for many other works (See. [B1][A1]), resolved the matter by showing that the cardinality of $\mathbb{R}$ is strictly greater than that of $\mathbb{N}$.
We give a brief overview of this argument along with some minor amendments in the way of transcribing this result, formed specifically to pave the way toward extending the technique to sets of our own creation.

Simply put, the cardinality of a set with respect to another depends on whether one can place elements of both sets being compared in one to one corrospondence with each other or not. [5]
We remind the reader that, if one attempts to do so with $\mathbb{N}$ and $\mathbb{R}$, for instance by the ordering:

[5] The details surrounding various methods involved in doing this for various sets can be found in.[H1].

$\mathbb{N}$	$\mathbb{R}$
1	098402970940807....
2	353674745635266....
3	809887097098084....
4	989848709479869....
5	798401640819247....
6	284457433535688....
7	097856587568787....
8	242570870980928....
9	879769869986999....
10	809809809809800....
11	236364346538485....
12	745374475648568....
13	368374849067900....
14	336374589697067....
15	242356346346367....
16*	059807579838967....

then, Cantor pointed out that, given any number of such associations, it is always possible to construct a number belonging to the reals which is different from all others included in the association by forming the number using the first unit of the first real being associated, the second unit of the second real being associated and so on. For instance 16* in the above table is formed along these lines and is distinct from all the other fifteen reals. Since this number fifteen is arbitrary and can in fact be any number, if it were possible to associate all real numbers with the naturals in this manner, then it should not be possible to find a number that remains unassociated. However by the argument above, regardless of the number of associations made, one can always be found that remains unassociated and as such, such a one-to-one correspondence cannot be made.[6]

The arguments that follow are mostly procedural in nature, and to facilitate the transcribing of these, we will now introduce a few concepts along with some terminology.

Definition 7.0 (Interval Removed Stream)
Given a stream $\mathbb{S}(\mathcal{P}_i, \mathcal{I}_i)|\forall i \in \mathbb{N}$. The stream $\partial\mathbb{S}$ is a stream formed with units of each packet $\mathcal{P}_i$ ordered left to right.

Definition 8.0 (Interval Integrated Stream)
Given a stream $\mathbb{S}(\mathcal{P}_i)$ [7], the stream $\int \mathbb{S}[\mathcal{I}_i]$ is a stream formed via the introduction of interval $\mathcal{I}_1$ between packets $\mathcal{P}_1, \mathcal{P}_2$ and interval $\mathcal{I}_2$ between packets $\mathcal{P}_2, \mathcal{P}_3$ and in general, the introduction of $\mathcal{I}_i$ between packets $\mathcal{P}_i, \mathcal{P}_{i+1}$ $\forall i \in \mathbb{N}$, for a given interval structure associated with $\mathcal{I}_i|\forall i \in \mathbb{N}$.

[6] Let us make a quick philosophical remark surrounding numbers.
For sake of convenience, if one chose to represent all numbers from 100 onward as $100 - \alpha_0, 101 - \alpha_1, 110 - \alpha_2$, one can simply associate the symbol with a real or a number with a real plus/minus such numbers in terms of α_i.

[7] $\mathbb{S}(\mathcal{P}_i)$ is a stream with structured packets $\mathcal{P}_i$

Definition 9.0 (Interval Extended Stream)
Given a stream $\mathbb{S}(\mathcal{P}_i, \mathcal{I}_i)|\forall i \in \mathbb{N}$, an interval extension on $\mathbb{S}$ is a stream $/\mathbb{S}$, specifically the resulting stream formed via the replacement of a series of units (ordered left to right) of $\mathcal{P}_i$ with zeros, for a series of packets.

For sake of simplicity, should we express a reduction or integration on a set S of streams, then the result is a set of the reduced/integrated streams.

In the proof that follows, we will be making references to a few new constructions involving the above definitions, and as such we will give some attention toward illustrating these constructions in this sub-section.

Remarks
We will make references to the sets of streams having elements with the quality (interval extended stream $/\mathbb{S}_R$ formed via replacement of the first half[8] (ordered left to right) of the units of each packet $\mathcal{P}_i$ with zeros.). The best way to understand the formation of such streams is to think of each packet in any one given stream as having an ordered set of positions representing from left to right, the location within the packet of any specific unit associated with the packet. Thus for instance, within the packet $\{100101\}$, position one is associated with the unit 1 and position two with 0 and finally position six with 1. By doing this the statement within brackets can be understood to be the replacement of the first half of the units associated with every packet of every stream element of some set of such elements with zeros forming new ones. This is illustrated in the following figure. The set ∂G_R will have stream elements formed by the removal of all intervals appropriated to some set of stream elements. To understand this, we simply note that upon defining a stream, the measures associated with every packet and interval forming the characteristic of the stream is predefined, and as such should we choose to form streams involving only the units of every packet forming part of the stream, then we simply include units (left to right) of every packet associated with any one stream into a new one. We conclude this article with a proposed proof of the fallacy of CH.
Proof **(CH is False)** Given the set S_R of elements generated by the real stream $\mathbb{S}_R(\mathcal{P}_i, \mathcal{I}_i)|\forall i \in \mathbb{N}$ having packet measure g, and a set of elements $/S_R$ generated by the interval extended stream $/\mathbb{S}_R$ formed via replacement of the first half[9] (ordered left to right) of the units of each packet $\mathcal{P}_i$ with zeros. Noting simply that $/S_R \subset S_R$ and representing elements of $/S_R$ as $\alpha_1, .., \alpha_i$[10], one can write elements of $E_z \in S_R$ as $e_i + \alpha_j$. Using the set $G_R := \{e_i|(e_i + \alpha_j) \in S_R\}$ using

[8] or first half minus one in the case of an odd measure
[9] or first half minus one in the case of an odd measure
[10] *We make no assumptions as to whether all $/S_R$ can be enumerated in this manner (we know that this is not possible), we will use such enumeration only in the way of simplifying an argument, the crux of which, is focused on the properties of a set formed by the difference $E_z - \alpha_i|E_z \in S_R$.

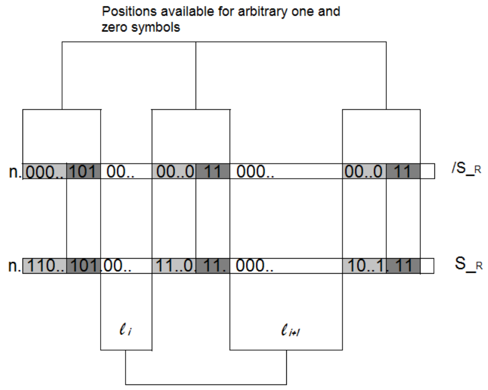

Figure 2: Stream elements of $/S_R$ formed by the replacement of the first half of the units associated with every packet of every stream element of S_R, with zeros forming new ones associated with $/S_R$

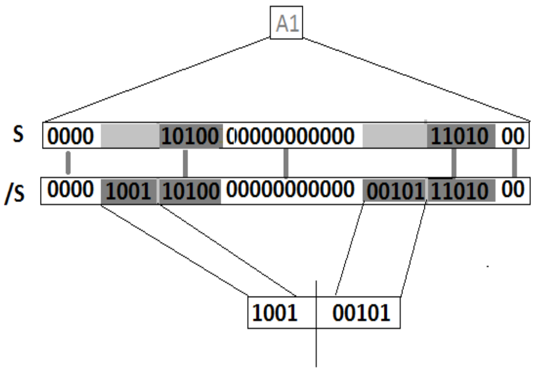

Figure 3:

$\alpha_i \in /S_R$, such that $w_z \in G_R | \forall z$ are associated with interval extended streams formed via replacement of the second half[11] (ordered left to right) of the units of each packet $\mathcal{P}_i$(packets associated with streams of the set S_R) with zeros[12], we construct ∂G_R.We note here that, ordering elements of $/S_R$ with those of S_R is the same as ordering α_i with $e_j + \alpha_k$ $\forall i, j, k$. As α_i, α_k is in one to one correspondence, the question of whether this is true of $/S_R$ and S_R comes down

[11]or first half plus one in the case of an odd measure

[12]Accomplishing this is the same as saying that $n \in \mathbb{N}$ always exists that when subtracted from $r \in \mathbb{R}$ results in r having the integral portion zero.

to whether this is possible via the ordering α_i:∂G_R. It is easy to see that this is impossible for the same reasons that the cardinality of $\mathbb{R}$ exceeds that of $\mathbb{N}$.

units within brackets formed by interval reduction

a_1 : [001010101111110111010....] + a_t

a_2 : [111110010101000001111....] + a_q

a_3 : [000011110100101010111....] + a_k

......

......

......

Figure 4:

Remarks

It might be tempting to think that it is in the introduction of α_i enumeration of $/S_R$ that some flaw has been introduced, which upon removal of such enumeration would be exposed. Let us then elaborate on some ideas by reflecting on this prospect for the present.

There is surely no question of whether $u_i \in S_U$ and $v_i \in S_V$ can be put into one-to-one correspondence or not should $S_U = S_V$, as both sets clearly have the same number of elements as the other. The question is whether the pairing $\alpha_i \in /S_R$ and $e_j \in G_R$ if $k_l = (e_j + \alpha_i) \in S_R$ possible or not. Careful reflection will show that given any element α_i, when subtracted from k_l does produce an element, and one that is interval-extended with all units to the right of the half-way point of the packet measures associated with elements of S_R replaced with zeros. There is no question of any assumptions here of the necessity in any way for α_i, e_j to be in one-to-one correspondence with each other in order to perform this type of subtraction and association. The above statement is the same as saying, for $h_i \in \mathbb{N}$ and $f_j \in \mathbb{R}$ $\exists w_l \in \mathbb{R}$ such that $w_l = (f_j + h_i)|\forall i, j, l$, and this statement makes no inferences about the cardinalities of h_i, f_j respectively.

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