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A NEW DUALITY APPROACH TO ELASTICITY

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The displacement-traction problem of three-dimensional linearized elasticity can be posed as three different minimization problems, depending on whether the displacement vector field, or the stress tensor field, or the strain tensor field, is the unknown.

The objective of this paper is to put these three different formulations of the same problem in a new perspective, by means of Legendre-Fenchel duality theory. More specifically, we show that both the displacement and strain formulations can be viewed as Legendre-Fenchel dual problems to the stress formulation. We also show that each corresponding Lagrangian has a saddle-point, thus fully justifying this new duality approach to elasticity.

Keywords: Linearized elasticity; intrinsic elasticity; constrained quadratic optimization; Legendre-Fenchel transform; duality; Lagrangians.

AMS Subject Classification: 49N10, 49N15, 74B05

1. Introduction

All notions, notations, or assumptions not defined or stated here are defined in the next sections.

Let Ω be a domain in $\mathbb{R}^3$, let its boundary Γ be partitioned as $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ with $\Gamma_0 \cap \Gamma_1 = \varnothing$ and $\mathrm{d}\Gamma\text{-meas}\,\Gamma_0 > 0$, and let $\overline{\Omega}$ be the *reference configuration* of a *linearly elastic body* with *elasticity tensor field* $\mathbf{A}$, subjected to an *applied body forces* of

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density $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and to an *applied surface forces* of density $\boldsymbol{F} \in \boldsymbol{L}^2(\Gamma_1)$, and subjected to a *boundary condition of place* $\boldsymbol{u} = \boldsymbol{0}$ on Γ_0 (only homogeneous boundary conditions of place are considered here). This problem is called a *displacement-traction problem* if $\mathrm{d}\Gamma\text{-meas}\,\Gamma_1 > 0$, or a *pure displacement problem* if $\Gamma_0 = \Gamma$. The *pure traction problem*, which corresponds to $\Gamma_1 = \Gamma$, is not considered here.

It is then well-known (see, e.g., Ref. **?**) that the unknown *displacement vector field* $\overline{\boldsymbol{v}} : \overline{\Omega} \to \mathbb{R}^3$ is the unique solution of the minimization problem

$$J(\overline{\boldsymbol{v}}) = \inf_{\boldsymbol{v} \in \boldsymbol{V}} J(\boldsymbol{v}),$$

where

$$J(\boldsymbol{v}) = \frac{1}{2} \int_\Omega \mathbf{A} \boldsymbol{\nabla}_s \boldsymbol{v} : \boldsymbol{\nabla}_s \boldsymbol{v} \, \mathrm{d}x - L(\boldsymbol{v}),$$

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma,$$

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \mathrm{tr}\, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}.$$

This minimization problem, which constitutes the modern version of the classical *principle of minimum potential energy* (for a historical perspective, see Gurtin [?] or Benvenuto [?]), will be referred to as the *displacement formulation* of the displacement-traction problem of three-dimensional linearized elasticity.

It is also well-known (see, e.g., Brezzi & Fortin [?]) that the same problem can be also formulated as another minimization problem, where the *stress tensor field* $\overline{\boldsymbol{\sigma}} = \mathbf{A} \boldsymbol{\nabla}_s \overline{\boldsymbol{v}} : \overline{\Omega} \to \mathbb{S}^3$ is the unknown, viz.,

$$g(\overline{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma} \in \mathbb{S}} g(\boldsymbol{\sigma}),$$

where

$$g(\boldsymbol{\sigma}) = \frac{1}{2} \int_\Omega \mathbf{B} \boldsymbol{\sigma} : \boldsymbol{\sigma} \, \mathrm{d}x,$$

$$\mathbb{S} = \{\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}; \Omega); \mathbf{div}\, \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \text{ in } \boldsymbol{L}^2(\Omega),$$
$$\langle \boldsymbol{\sigma} \boldsymbol{\nu} - \boldsymbol{F}, \mathrm{tr}\, \boldsymbol{v} \rangle_\Gamma = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V}\},$$

and $\mathbf{B}$ denotes the *compliance tensor*, i.e., the inverse of the tensor field $\mathbf{A}$.

This minimization problem, which constitutes the modern version of the classical principle of *minimum complementary energy* (for a historical perspective, see again Gurtin [?] or Benvenuto [?]), will be referred to as the *stress formulation* of the displacement-traction problem of three-dimensional linearized elasticity.

It is much less known that yet another approach is possible, where the *strain tensor field* $\overline{\boldsymbol{e}} = \boldsymbol{\nabla}_s \overline{\boldsymbol{v}}$ is the unknown. The *idea* of such an approach, which bears the name of *intrinsic approach*, is not new: in nonlinear three-dimensional elasticity, it was first suggested, albeit only briefly, by Antman [?] in 1977; a similar idea for shells and plates goes back even earlier, to Synge & Chien [?] (see also Chien [?]), who already in 1941 advocated using the change of metric and change of curvature

tensors as the primary unknowns. This approach was then considerably developed during the past decades, from the mechanical and computational viewpoints, by Wojciech Pietraszkiewicz and his group. See in particular Opoka & Pietraszkiewicz [?] and Pietraszkiewicz [?]. But it is only recently that the *mathematical analysis* and *numerical analysis* of the intrinsic approach to three-dimensional elasticity were undertaken, by Ciarlet & Ciarlet, Jr. [?,?], Amrouche, Ciarlet, Gratie & Kesavan [?], Ciarlet, Ciarlet, Jr., Iosifescu, Sauter & Zou [?] in the linear case, and by Ciarlet & Mardare [?] in the nonlinear case.

As shown in Ref. **?** (in the case of the pure displacement problem, but the extension to a genuine displacement-traction problem poses no difficulty), one intrinsic approach to the same problem takes the form of the following minimization problem:

$$j(\overline{\boldsymbol{e}}) = \inf_{\boldsymbol{e} \in \mathbb{M}^{\perp}} j(\boldsymbol{e}),$$

where

$$j(\boldsymbol{e}) = \frac{1}{2} \int_{\Omega} \mathbf{A}\boldsymbol{e} : \boldsymbol{e} \, \mathrm{d}x - L(\boldsymbol{\mathcal{F}}(\boldsymbol{e})),$$

$$\mathbb{M} = \{\boldsymbol{\mu} \in \mathbb{L}^2_s(\Omega); \mathbf{div}\, \boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega); \langle \boldsymbol{\mu}\boldsymbol{\nu}, \operatorname{tr} \boldsymbol{v} \rangle_{\Gamma} = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V}\},$$
$$\mathbb{M}^{\perp} = \{\boldsymbol{e} \in \mathbb{L}^2_s(\Omega); \int_{\Omega} \boldsymbol{e} : \boldsymbol{\mu} \, \mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{M}\}.$$

This minimization problem will be referred to as the *strain formulation* of the displacement-traction problem of three-dimensional linearized elasticity. Note that *another* strain formulation is possible, but it does not seem to be amenable to the present approach, however (cf. the discussion given in Section **??**).

The objective of this paper is to put these three different formulations of the same problem in a different perspective, new to the best of our knowledge, by means of *Legendre-Fenchel duality theory* (the principles of which are recalled in Section 2). More specifically, we show in Sections 6 and **??** that *both the displacement and strain formulations are nothing but Legendre-Fenchel dual problems to the stress formulation*, once this formulation has been appropriately recast within the framework of this duality theory (Section 5). We also show that each corresponding *Lagrangian* has a *saddle-point*, thus fully justifying this new duality approach to elasticity.

The results of this paper were announced in Ref. **?**.

2. Legendre-Fenchel duality

All vector spaces, matrices, etc., considered in this paper are real. The dual space of a normed vector space X is denoted X^*, and ${}_{X^*}\langle \cdot, \cdot \rangle_X$ designates the associated duality. The bidual space of X is denoted X^{**}; if X is a reflexive Banach space, X^{**} will be identified with X by means of the usual canonical isometry.

The *indicator function* I_A of a subset A of a set X is the function I_A defined by $I_A(x) := 0$ if $x \in A$ and $I_A(x) := +\infty$ if $x \notin A$. A function $g : X \to \mathbb{R} \cup \{+\infty\}$ is *proper* if $\{x \in X; g(x) < +\infty\} \neq \varnothing$.

For the reader's convenience, the notations chosen below are on purpose either identical or similar (up to, e.g., italics instead of boldface) to those used later on.

Let Σ be a normed vector space and let $g : \Sigma \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The *Legendre-Fenchel transform* of g is the function $g^* : \Sigma^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$g^* : e \in \Sigma^* \to g^*(e) := \sup_{\sigma \in \Sigma} \{{}_{\Sigma^*}\langle e, \sigma \rangle_{\Sigma} - g(\sigma)\}.$$

The next theorem summarizes some basic properties of the Legendre-Fenchel transform when the space Σ is a reflexive Banach space. For proofs, see, e.g., Ekeland & Temam [?] or Brezis [?].

Theorem 2.1. *Let Σ be a reflexive Banach space, and let $g : \Sigma \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semi-continuous function. Then the Legendre-Fenchel transform $g^* : \Sigma^* \to \mathbb{R} \cup \{+\infty\}$ of g is also proper, convex, and lower semi-continuous. Let*

$$g^{**} : \sigma \in \Sigma^{**} \to g^{**}(\sigma) := \sup_{e \in \Sigma^*} \{{}_{\Sigma^*}\langle e, \sigma \rangle_{\Sigma} - g^*(e)\}$$

denote the Legendre-Fenchel transform of g^*. Then (recall that X^{**} is here identified with X)

$$g^{**} = g.$$

The equality $g^{**} = g$ constitutes the *Fenchel-Moreau theorem*; cf. Fenchel [?] and Moreau [?].

Given a *minimization problem*

$$\inf_{\sigma \in \Sigma} G(\sigma), \tag{$\mathcal{P}$}$$

with a function $G : \Sigma \to \mathbb{R} \cup \{+\infty\}$ of the specific form given in Theorem 2.2, the following simple result will be the basis for defining *two different dual problems of problem* $(\mathcal{P})$. The functions $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ defined in the next theorem are the *Lagrangians* associated with the minimization problem $(\mathcal{P})$.

Theorem 2.2. *Let Σ and V be two reflexive Banach spaces, let $g : \Sigma \to \mathbb{R} \cup \{+\infty\}$ and $h : V^* \to \mathbb{R} \cup \{+\infty\}$ be two proper, convex, and lower semi-continuous functions, let $\Lambda : \Sigma \to V^*$ be a linear and continuous mapping, let the function $G : \Sigma \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$G : \sigma \in \Sigma \to G(\sigma) := g(\sigma) + h(\Lambda\sigma),$$

and finally, let the two functions

$$\mathcal{L} : \Sigma \times \Sigma^* \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\} \quad \text{and} \quad \widetilde{\mathcal{L}} : \Sigma \times V \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

be defined by

$$\mathcal{L} : (\sigma, e) \in \Sigma \times \Sigma^* \to \mathcal{L}(\sigma, e) := {}_{\Sigma^*}\langle e, \sigma\rangle_{\Sigma} - g^*(e) + h(\Lambda\sigma),$$
$$\widetilde{\mathcal{L}} : (\sigma, v) \in \Sigma \times V \to \widetilde{\mathcal{L}}(\sigma, v) := g(\sigma) + {}_{V^*}\langle \Lambda\sigma, v\rangle_V - h^*(v).$$

Then

$$\inf_{\sigma\in\Sigma} G(\sigma) = \inf_{\sigma\in\Sigma} \sup_{e\in\Sigma^*} \mathcal{L}(\sigma, e) = \inf_{\sigma\in\Sigma} \sup_{v\in V} \widetilde{L}(\sigma, v).$$

Proof. Since $g(\sigma) = g^{**}(\sigma)$ for each $\sigma \in \Sigma$ (Theorem 2.1), we also have

$$\begin{aligned} G(\sigma) = g^{**}(\sigma) + h(\Lambda\sigma) &= \sup_{e\in\Sigma^*} \{{}_{\Sigma^*}\langle e, \sigma\rangle_{\Sigma} - g^*(e)\} + h(\Lambda\sigma) \\ &= \sup_{e\in\Sigma^*} \mathcal{L}(\sigma, e) \text{ for each } \sigma \in \Sigma. \end{aligned}$$

Hence

$$\inf_{\sigma\in\Sigma} G(\sigma) = \inf_{\sigma\in\Sigma} \sup_{e\in\Sigma^*} \mathcal{L}(\sigma, e).$$

Since likewise $h(v^*) = h^{**}(v^*)$ for each $v^* \in V^*$ and the space V^{**} is identified with V, we have

$$h(v^*) = h^{**}(v^*) = \sup_{v\in V}\{{}_{V^*}\langle v^*, v\rangle_V - h^*(v)\} \text{ for each } v^* \in V^*,$$

so that

$$\begin{aligned} G(\sigma) &= g(\sigma) + h(\Lambda\sigma) \\ &= g(\sigma) + \sup_{v\in V}\{{}_{V^*}\langle \Lambda\sigma, v\rangle_V - h^*(v)\} = \sup_{v\in V} \widetilde{\mathcal{L}}(\sigma, v) \text{ for each } \sigma \in \Sigma. \end{aligned}$$

Hence

$$\inf_{\sigma\in\Sigma} G(\sigma) = \inf_{\sigma\in\Sigma} \sup_{v\in V} \widetilde{\mathcal{L}}(\sigma, v).$$

□

As is classical in *duality theory* (see, e.g., Chapter 6 in Ekeland & Temam [?]), the replacement of the minimization problem $(\mathcal{P})$ by an *inf-sup problem*, such as either one found in Theorem 2.2, is the basis for defining a *dual problem* of the minimization problem $(\mathcal{P})$, as the corresponding *sup-inf problem*. In our case, this means that the dual problem corresponding to the first inf-sup problem found in Theorem 2.2 is defined as:

$$\sup_{e\in\Sigma^*} G^*(e), \text{ where } G^*(e) := \inf_{\sigma\in\Sigma}\{{}_{\Sigma^*}\langle e, \sigma\rangle_{\Sigma} - h(\Lambda\sigma)\} - g^*(e) \text{ for each } e \in \Sigma^*, \quad (\mathcal{P}^*)$$

while the dual problem corresponding to the second sup-inf problem is defined as:

$$\sup_{v\in V} \widetilde{G}^*(v), \text{ where } \widetilde{G}^*(v) := \inf_{\sigma\in\Sigma}\{g(\sigma) + {}_{\Sigma^*}\langle \Lambda\sigma, v\rangle_V\} - h^*(v) \text{ for each } v \in V. \quad (\widetilde{\mathcal{P}}^*)$$

A key issue (see *ibid.*) then consists in deciding whether the infimum found in problem $(\mathcal{P})$ is equal to the supremum found in either one of its dual problems, i.e., for instance in the case of the first dual problem (to fix ideas), whether

$$\inf_{\sigma\in\Sigma} G(\sigma) = \sup_{e\in\Sigma^*} G^*(e),$$

or equivalently, whether

$$\inf_{\sigma\in\Sigma}\sup_{e\in\Sigma^*} \mathcal{L}(\sigma,e) = \sup_{e\in\Sigma^*}\inf_{\sigma\in\Sigma} \mathcal{L}(\sigma,e).$$

If this is the case, the next issue consists in deciding whether the Lagrangian $\mathcal{L}$ possesses a *saddle-point* $(\overline{\sigma},\overline{e}) \in \Sigma\times\Sigma^*$, i.e., that satisfies

$$\inf_{\sigma\in\Sigma}\sup_{e\in\Sigma^*} \mathcal{L}(\sigma,e) = \inf_{\sigma\in\Sigma} \mathcal{L}(\sigma,\overline{e}) = \mathcal{L}(\overline{\sigma},\overline{e}) = \sup_{e\in\Sigma^*} \mathcal{L}(\overline{\sigma},e) = \sup_{e\in\Sigma^*}\inf_{\sigma\in\Sigma} \mathcal{L}(\sigma,e).$$

This is precisely the type of questions that will be addressed in this paper, the point of departure $(\mathcal{P})$ being a classical quadratic minimization problem arising in three-dimensional linearized elasticity.

3. Some functional analytic preliminaries

Latin indices vary in the set $\{1,2,3\}$, save when they are used for indexing sequences, and the summation convention with respect to repeated indices is systematically used in conjunction with this rule.

In what follows, Ω is a *domain in* $\mathbb{R}^3$, i.e., a bounded, connected, open subset of $\mathbb{R}^3$ whose boundary, denoted Γ, is Lipschitz-continuous, the set Ω being locally on a single side of Γ (see, e.g., Adams [?] or Nečas [?]), and $x = (x_i)$ designates a generic point in Ω. Partial derivative operators of the first, second, and third order are then denoted $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i\partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i\partial x_j\partial x_k$. The same symbols also designate partial derivatives in the sense of distributions.

The notation $D(\Omega)$ denotes the space of functions that are infinitely differentiable in Ω and have compact supports in Ω. The notation $D'(\Omega)$ denotes the space of distributions defined over Ω. The notations $H^1(\Omega), H^1_0(\Omega), H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))^*$ designate the usual Sobolev spaces. The trace operator from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$ is denoted tr.

Spaces of functions, vector fields in $\mathbb{R}^3$, and 3×3 matrix fields, defined over Ω are respectively denoted by italic capitals, boldface Roman capitals, and special Roman capitals. The space of all symmetric matrices of order 3 is denoted $\mathbb{S}^3$. The subscript s appended to a special Roman capital denotes a space of symmetric matrix fields. Combining the above rules, we shall thus encounter spaces such as $\boldsymbol{D}(\Omega)$, $\boldsymbol{D}'(\Omega)$, $\mathbb{D}'(\Omega)$, $\mathbb{L}^2_s(\Omega)$, $\boldsymbol{H}^{1/2}(\Gamma)$, etc.

The notation $(\boldsymbol{v})_i$ designates the i-th component of a vector $\boldsymbol{v}\in\mathbb{R}^3$ and the notation $\boldsymbol{v} = (v_i)$ means that $v_i = (\boldsymbol{v})_i$. The notation $(\boldsymbol{A})_{ij}$ designates the element at the i-th row and j-th column of a square matrix $\boldsymbol{A}$ of order three and the notation $\boldsymbol{A} = (a_{ij})$ means that $a_{ij} = (\boldsymbol{A})_{ij}$. The inner-product of $\boldsymbol{a}\in\mathbb{R}^3$ and $\boldsymbol{b}\in\mathbb{R}^3$ is

denoted $\boldsymbol{a} \cdot \boldsymbol{b}$. The notation $\boldsymbol{s} : \boldsymbol{t} := s_{ij} t_{ij}$ designates the matrix inner-product of two matrices $\boldsymbol{s} := (s_{ij})$ and $\boldsymbol{t} := (t_{ij})$ of order three.

The inner product in the space $\mathbb{L}^2_s(\Omega)$ is given by

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \mathbb{L}^2_s(\Omega) \times \mathbb{L}^2_s(\Omega) \to \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \, \mathrm{d}x,$$

so that the corresponding norm is given by

$$\|\boldsymbol{\sigma}\|_{\mathbb{L}^2_s(\Omega)} = \left(\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\sigma} \, \mathrm{d}x \right)^{1/2} \text{ for any } \boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega).$$

The space $\mathbb{L}^2_s(\Omega)$ will be identified with its dual space; hence the corresponding duality bracket will be identified with the inner product of $\mathbb{L}^2_s(\Omega)$.

The norm in the space $\boldsymbol{H}^1(\Omega)$ is given by

$$\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega) \to \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} = \left(\sum_{i=1}^{3} \|v_i\|^2_{H^1(\Omega)} \right)^{1/2}.$$

For notational conciseness, the duality bracket between the space $\boldsymbol{H}^{1/2}(\Gamma)$ and its dual space will be denoted

$$\langle \cdot, \cdot \rangle_\Gamma := {}_{\boldsymbol{H}^{-1/2}(\Gamma)} \langle \cdot, \cdot \rangle_{\boldsymbol{H}^{1/2}(\Gamma)}.$$

Note in this respect that, if $\boldsymbol{F} \in \boldsymbol{L}^2(\Gamma)$, then

$$\langle \boldsymbol{F}, \operatorname{tr} \boldsymbol{v} \rangle_\Gamma = \int_\Gamma \boldsymbol{F} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma \text{ for any } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

The *matrix gradient operator* $\boldsymbol{\nabla} : \boldsymbol{D}'(\Omega) \to \mathbb{D}'(\Omega)$ is defined by

$$(\boldsymbol{\nabla} \boldsymbol{v})_{ij} := \partial_j v_i \text{ for any } \boldsymbol{v} = (v_i) \in \boldsymbol{D}'(\Omega).$$

For any vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{D}'(\Omega)$, the associated *linearized strain tensor* is the symmetric matrix field $\boldsymbol{\nabla}_s \boldsymbol{v} \in \mathbb{D}'_s(\Omega)$ defined by

$$\boldsymbol{\nabla}_s \boldsymbol{v} := \frac{1}{2} (\boldsymbol{\nabla} v^T + \boldsymbol{\nabla} v),$$

or equivalently, by

$$(\boldsymbol{\nabla}_s \boldsymbol{v})_{ij} = \frac{1}{2} (\partial_i v_j + \partial_j v_i).$$

The *vector divergence operator* $\mathbf{div} : \mathbb{D}'(\Omega) \to \mathbf{D}'(\Omega)$ is defined by

$$(\mathbf{div}\, \boldsymbol{\mu})_i := \partial_j \mu_{ij} \text{ for any } \boldsymbol{\mu} = (\mu_{ij}) \in \mathbb{D}'(\Omega).$$

We now recall some functional analytic preliminaries, due to Geymonat & Suquet [?] and Geymonat & Krasucki [?,?], which are the "matrix analogs" of results of Girault & Raviart [?] for spaces of vector fields. Given a domain Ω in $\mathbb{R}^3$, define the space

$$\mathbb{H}_s(\mathbf{div}; \Omega) := \{\boldsymbol{\mu} \in \mathbb{L}^2_s(\Omega);\ \mathbf{div}\, \boldsymbol{\mu} \in \boldsymbol{L}^2(\Omega)\}$$

(in this definition $\mathbf{div}\,\boldsymbol{\mu}$ is of course to be understood in the sense of distributions). Equipped with the norm defined by

$$\|\boldsymbol{\mu}\|_{\mathbb{H}_s(\mathbf{div};\Omega)} := \left(\|\boldsymbol{\mu}\|^2_{\mathbb{L}^2_s(\Omega)} + \|\,\mathbf{div}\,\boldsymbol{\mu}\|^2_{\boldsymbol{L}^2(\Omega)}\right)^{1/2} \quad \text{for all } \boldsymbol{\mu} \in \mathbb{H}_s(\mathbf{div};\Omega),$$

the space $\mathbb{H}_s(\mathbf{div};\Omega)$ is a Hilbert space. Let $\boldsymbol{\nu} : \Gamma \to \mathbb{R}^3$ denote the unit outer normal vector field along the boundary Γ of Ω (such a field is defined $\mathrm{d}\Gamma$-everywhere since Γ is Lipschitz-continuous). The set Ω being a domain, the density of the space $\mathbb{C}^\infty_s(\overline{\Omega})$ in the space $\mathbb{H}(\mathbf{div};\Omega)$ then implies that the mapping $\boldsymbol{\mu} \in \mathbb{C}^\infty_s(\overline{\Omega}) \to \boldsymbol{\mu}\boldsymbol{\nu}|_\Gamma$ can be extended to a continuous linear mapping from the space $\mathbb{H}_s(\mathbf{div};\Omega)$ into $\boldsymbol{H}^{-1/2}(\Gamma)$, which for convenience will be simply denoted

$$\boldsymbol{\mu} \in \mathbb{H}_s(\mathbf{div};\Omega) \to \boldsymbol{\mu}\boldsymbol{\nu} \in H^{-1/2}(\Gamma).$$

Theorem 3.1. *The Green formula*

$$\int_\Omega \boldsymbol{\mu} : \boldsymbol{\nabla}_s \boldsymbol{v}\,\mathrm{d}x + \int_\Omega (\mathbf{div}\,\boldsymbol{\mu}) \cdot \boldsymbol{v}\,\mathrm{d}x = \langle \boldsymbol{\mu}\boldsymbol{\nu}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma$$

holds for all $\boldsymbol{\mu} \in \mathbb{H}_s(\mathbf{div};\Omega)$ *and all* $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$.

The following extension of the classical *Donati theorem* (for a brief history of this result, see Section 7 in [?]) plays an essential role in the sequel. The case where $\Gamma_0 = \varnothing$ (which is thus excluded here) is considered in Refs. **?** and **?**.

Theorem 3.2. *Let* Ω *be a domain in* $\mathbb{R}^3$*, let* Γ_0 *and* Γ_1 *be two relatively open subsets of* Γ *such that*

$$\mathrm{d}\Gamma\text{-}\operatorname{meas}\Gamma_0 > 0, \quad \Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1, \quad \text{and} \quad \Gamma_0 \cap \Gamma_1 = \varnothing,$$

and let there be given a matrix field $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$*. Then there exists a vector field*

$$\boldsymbol{v} \in \boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \operatorname{tr}\boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\}$$

such that $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ *if and only if*

$$\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{M},$$

where the space $\mathbb{M}$ *is defined as*

$$\mathbb{M} := \{\boldsymbol{\mu} \in \mathbb{L}^2_s(\Omega); \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega),\ \langle \boldsymbol{\mu}\boldsymbol{\nu}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V}\}.$$

Besides, such a vector field $\boldsymbol{v} \in \boldsymbol{V}$ *is uniquely defined.*

4. Three different formulations of the displacement-traction problem of three-dimensional linearized elasticity as a minimization problem

Let Ω be a domain in $\mathbb{R}^3$ and let Γ_0 and Γ_1 be two relatively open subsets of $\Gamma := \partial\Omega$ that satisfy

$$\mathrm{d}\Gamma\text{-}\operatorname{meas}\Gamma_0 > 0, \quad \Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1, \quad \text{and} \quad \Gamma_0 \cap \Gamma_1 = \varnothing.$$

The following assumptions are made in the rest of the paper. The set $\overline{\Omega}$ is the reference configuration of a linearly elastic body, characterized by its *elasticity tensor field* $\mathbf{A} = (A_{ijk\ell})$ with components $A_{ijk\ell} \in L^\infty(\Omega)$. It is assumed as usual that these components satisfy the symmetry relations $A_{ijk\ell} = A_{jik\ell} = A_{k\ell ij}$ and that the tensor field $\mathbf{A}$ is uniformly positive-definite almost-everywhere in Ω, in the sense that there exists a constant $\alpha > 0$ such that

$$\mathbf{A}\boldsymbol{t} : \boldsymbol{t} \geq \alpha \boldsymbol{t} : \boldsymbol{t} \text{ for almost all } x \in \Omega \text{ and all matrices } \boldsymbol{t} = (t_{ij}) \in \mathbb{S}^3,$$

where $(\mathbf{A}(x)\boldsymbol{t})_{ij} := A_{ijk\ell}(x)t_{k\ell}$. The body is subjected to *applied body forces* with density $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ in its interior and to *applied surface forces* of density $\boldsymbol{F} \in \boldsymbol{L}^2(\Gamma_1)$ on the portion Γ_1 of its boundary. Finally, it is assumed that the body is subjected to a *homogeneous boundary condition of place* along Γ_0.

Then the corresponding *displacement-traction problem*, or the *pure displacement problem* if $\Gamma_0 = \Gamma$, *of three-dimensional linearized elasticity* classically takes the form of the minimization problem described in the next theorem, where the minimizer $\overline{\boldsymbol{v}} : \overline{\Omega} \to \mathbb{R}^3$ is the *unknown displacement field.*

Theorem 4.1 (the classical displacement formulation). *There exists a unique vector field*

$$\overline{\boldsymbol{v}} \in \boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \operatorname{tr} \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}$$

that satisfies

$$J(\overline{\boldsymbol{v}}) = \inf_{\boldsymbol{v} \in \boldsymbol{V}} J(\boldsymbol{v}), \text{ where } J(\boldsymbol{v}) := \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{\nabla}_s \boldsymbol{v} : \boldsymbol{\nabla}_s \boldsymbol{v}\, \mathrm{d}x - L(\boldsymbol{v}) \text{ for all } \boldsymbol{v} \in \boldsymbol{V},$$

and

$$L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{v}\, \mathrm{d}\Gamma \text{ for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

That this minimization problem has one and only one solution is well known (see, e.g., Theorem 3.4 in Duvaut & Lions [?]).

In view of describing a second formulation of the *same* displacement-traction problem, we first note that, because the elasticity tensor field $\mathbf{A} = (A_{ijk\ell})$ with components $A_{ijk\ell} \in L^\infty(\Omega)$ is uniformly positive-definite almost-everywhere in Ω, there exists a tensor field $\mathbf{B} = (B_{ijk\ell})$ that is the inverse of $\mathbf{A}$, in the sense that, for almost all $x \in \Omega$,

$$\boldsymbol{s} = \mathbf{A}(x)\boldsymbol{t} \text{ is equivalent to } \boldsymbol{t} = \mathbf{B}(x)\boldsymbol{s} \text{ for all matrices } \boldsymbol{t} \in \mathbb{S}^3.$$

Furthermore, it is easily seen that $B_{ijk\ell} \in L^\infty(\Omega)$ and that $\mathbf{B}$ is also uniformly positive-definite almost-everywhere in Ω, i.e., there exists a constant $\beta > 0$ such that

$$\mathbf{B}(x)\boldsymbol{s} : \boldsymbol{s} \geq \beta \boldsymbol{s} : \boldsymbol{s} \text{ for almost all } x \in \Omega \text{ and all matrices } \boldsymbol{s} \in \mathbb{S}^3.$$

The tensor field $\mathbf{B}$ is called the *compliance tensor field.*

It is then classical that, like the displacement field, the *stress tensor field* $\overline{\boldsymbol{\sigma}} := \mathbf{A}\boldsymbol{\nabla}_s\overline{\boldsymbol{v}} \in \mathbb{L}^2_s(\Omega)$ inside the body can be also obtained as the solution of a minimization problem. To show this, one first uses the *Babuška-Brezzi inf-sup theorem* (Babuška ? and Brezzi ?) to derive a "mixed" formulation of the elasticity problem, i.e., where *both* $\overline{\boldsymbol{v}} \in \overline{\boldsymbol{V}}$ and $\overline{\boldsymbol{\sigma}} \in \mathbb{L}^2_s(\Omega)$ are the unknowns; then, using a standard procedure in optimization theory, one constructs a *dual* problem with $\overline{\boldsymbol{\sigma}} = \mathbf{A}\boldsymbol{\nabla}_s\overline{\boldsymbol{v}}$ as the sole unknown (Brezzi & Fortin ?). The dual problem obtained in this fashion is then a *constrained quadratic minimization problem* (the constraints here are the relations $\mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0}$ in $\boldsymbol{L}^2(\Omega)$ and $\langle \boldsymbol{\sigma\nu} - \boldsymbol{F}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}$ that must be satisfied by the "admissible" stress fields $\boldsymbol{\sigma}$; cf. Theorem 4.2).

Theorem 4.2 (the classical stress formulation). *Let the space* $\boldsymbol{V}$ *be defined as in* Theorem 4.1*, i.e.,*

$$\boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \operatorname{tr}\boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\}.$$

Then there exists a unique tensor field

$$\overline{\boldsymbol{\sigma}} \in \mathbb{S} := \{\boldsymbol{\sigma} \in \mathbb{H}_s(\mathbf{div};\Omega); \mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} \text{ in } \boldsymbol{L}^2(\Omega), \\ \langle \boldsymbol{\sigma\nu} - \boldsymbol{F}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V}\},$$

that satisfies

$$g(\overline{\boldsymbol{\sigma}}) = \inf_{\sigma\in\mathbb{S}} g(\boldsymbol{\sigma}), \text{ where } g(\boldsymbol{\sigma}) := \frac{1}{2}\int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x \text{ for all } \sigma \in \mathbb{L}^2_s(\Omega).$$

Besides,

$$\overline{\boldsymbol{\sigma}} = \mathbf{A}\nabla_s\overline{\boldsymbol{v}} \text{ in } \mathbb{L}^2_s(\Omega),$$

where the vector field $\overline{\boldsymbol{v}} \in \boldsymbol{V}$ *is the unique solution to the minimization problem of* Theorem 4.1.

An *intrinsic approach* to the *same* displacement-traction problem consists in considering the *linearized strain tensor field* $\overline{\boldsymbol{e}} := \boldsymbol{\nabla}_s\overline{\boldsymbol{v}} \in \mathbb{L}^2_s(\Omega)$ inside the body as the primary unknown, instead of the displacement field itself.

Accordingly, one first needs to characterize those 3×3 matrix fields $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$ that can be written as $\boldsymbol{e} = \boldsymbol{\nabla}_s\boldsymbol{v}$, with $\boldsymbol{v} \in \boldsymbol{V}$. Theorem 3.2 provides such a characterization.

Thanks to this theorem, the displacement-traction problem of three-dimensional elasticity can then be recast as yet another *constrained quadratic minimization problem*, with $\overline{\boldsymbol{e}} = \boldsymbol{\nabla}_s\overline{\boldsymbol{v}} \in \mathbb{L}^2_s(\Omega)$ as the unknown.

Theorem 4.3 (the strain formulation, a.k.a. the intrinsic approach). *Let the space* $\mathbb{M}$ *be defined as in* Theorem 3.2*, i.e.,*

$$\mathbb{M} := \{\boldsymbol{\mu} \in \mathbb{L}^2_s(\Omega); \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega); \langle \boldsymbol{\mu\nu}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V}\}.$$

Define the Hilbert space

$$\mathbb{M}^\perp := \{\boldsymbol{e} \in \mathbb{L}^2_s(\Omega); \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{M}\},$$

and, for each $\boldsymbol{e} \in \mathbb{M}^{\perp}$*, let* $\boldsymbol{\mathcal{F}}(\boldsymbol{e})$ *denote the unique element in the space* $\boldsymbol{V}$ *that satisfies* $\boldsymbol{\nabla}_s \boldsymbol{\mathcal{F}}(\boldsymbol{e}) = \boldsymbol{e}$ (Theorem 3.2)*. Then the mapping* $\boldsymbol{\mathcal{F}} : \mathbb{M}^{\perp} \to \boldsymbol{V}$ *defined in this fashion is an isomorphism between the Hilbert spaces* $\mathbb{M}^{\perp}$ *and* $\boldsymbol{V}$*.*

The minimization problem: Find $\overline{\boldsymbol{e}} \in \mathbb{M}^{\perp}$ *such that*

$$j(\overline{\boldsymbol{e}}) = \inf_{\boldsymbol{e} \in \mathbb{M}^{\perp}} j(\boldsymbol{e}), \text{ where } j(\boldsymbol{e}) := \frac{1}{2} \int_{\Omega} \mathbf{A}\boldsymbol{e} : \boldsymbol{e} \, \mathrm{d}x - L(\boldsymbol{\mathcal{F}}(\boldsymbol{e})),$$

has one and only one solution $\overline{\boldsymbol{e}}$*. Besides,*

$$\overline{\boldsymbol{e}} = \boldsymbol{\nabla}_s \overline{\boldsymbol{v}},$$

where the vector field $\overline{\boldsymbol{v}} \in \boldsymbol{V}$ *is the unique solution to the minimization problem of* Theorem 4.1*.*

Remark 4.1. A proof similar to that of the corollary to Theorem 4.1 in Ref. **?** shows that the *Korn inequality in the space* $\boldsymbol{V}$ can then be recovered as a simple corollary to Theorem 4.3, which thus provides an entirely new proof of this classical inequality.

5. The classical stress formulation of the displacement-traction problem as a point of departure

The minimization problem found in the classical stress formulation described in Theorem 4.2 constitutes our point of departure for constructing dual problems, by means of the approach described in Section 2. Accordingly, our first task consists in verifying that this formulation can be indeed recast in the abstract framework of Theorem 2.2. Note in this respect that the spaces denoted $\mathbb{L}^2_s(\Omega)$, identified here with its dual space, $\boldsymbol{V}$, and $\boldsymbol{V}^*$, in the next theorem play the rôle of the spaces respectively denoted Σ, Σ^*, V, and V^* in Theorem 2.2.

Theorem 5.1. *Let the space* $\boldsymbol{V}$ *and the linear form* $L \in \boldsymbol{V}^*$ *be defined as in* Theorem 4.1*, let the mapping* $\Lambda : \boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega) \to \Lambda\boldsymbol{\sigma} \in \boldsymbol{V}^*$ *be defined by*

$${}_{\boldsymbol{V}^*}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v} \rangle_{\boldsymbol{V}} := \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\nabla}_s \boldsymbol{v} \, \mathrm{d}x \text{ for all } \boldsymbol{v} \in \boldsymbol{V},$$

and finally, let the functions $g : \mathbb{L}^2_s(\Omega) \to \mathbb{R}$ *and* $h : \boldsymbol{V}^* \to \mathbb{R} \cup \{+\infty\}$ *be respectively defined by*

$$\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega) \to g(\boldsymbol{\sigma}) := \frac{1}{2} \int_{\Omega} \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma} \, \mathrm{d}x,$$
$$\boldsymbol{v}^* \in \boldsymbol{V}^* \to h(\boldsymbol{v}^*) := 0 \text{ if } \boldsymbol{v}^* = L \quad \text{or} \quad h(\boldsymbol{v}^*) := +\infty \text{ if } \boldsymbol{v}^* \neq L.$$

Then $\Lambda \in \mathcal{L}(\mathbb{L}^2_s(\Omega); \boldsymbol{V}^*)$ *and the functions* g *and* h *are both proper, convex, and lower semi-continuous.*

Define the function $G : \mathbb{L}^2_s(\Omega) \to \mathbb{R} \cup \{+\infty\}$ *by*

$$G(\boldsymbol{\sigma}) := g(\boldsymbol{\sigma}) + h(\Lambda\boldsymbol{\sigma}) \text{ for all } \boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega).$$

Then

$$h(\Lambda\boldsymbol{\sigma}) = I_{\mathbb{S}}(\boldsymbol{\sigma}) \text{ for all } \boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega),$$

where the set $\mathbb{S}$ *is defined as in* Theorem 4.2*, and the minimization problem of* Theorem 4.2*, viz.,*

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma})$$

is the same as the minimization problem

$$\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} G(\boldsymbol{\sigma}). \tag{$\mathcal{P}$}$$

Proof. Given any $\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)$, the linear functional

$$\Lambda\boldsymbol{\sigma} : \boldsymbol{v} \in \boldsymbol{V} \to \Lambda\boldsymbol{\sigma}(\boldsymbol{v}) := \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla}_s \boldsymbol{v} \,\mathrm{d}x \in \mathbb{R}$$

is clearly continuous; hence $\Lambda\boldsymbol{\sigma} \in \boldsymbol{V}^*$. Besides, the mapping $\Lambda : \mathbb{L}^2_s(\Omega) \to \boldsymbol{V}^*$ defined in this fashion is continuous since

$$\|\Lambda\boldsymbol{\sigma}\|_{\boldsymbol{V}^*} = \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{|\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla}_s \boldsymbol{v} \,\mathrm{d}x|}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \le \|\boldsymbol{\sigma}\|_{\mathbb{L}^2_s(\Omega)} \text{ for all } \boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega).$$

Therefore the mapping $\Lambda : \boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega) \to \Lambda\boldsymbol{\sigma} \in \boldsymbol{V}^*$, which is clearly linear, is continuous.

The function $g : \mathbb{L}^2_s(\Omega) \to \mathbb{R}$ is convex since the compliance tensor $\mathbf{B}$ is uniformly positive-definite almost-everywhere in Ω, and lower semi-continuous since g is continuous for the norm $\|\cdot\|_{\mathbb{L}^2_s(\Omega)}$.

The function $h : \boldsymbol{V}^* \to \mathbb{R} \cup \{+\infty\}$ is the indicator function of the subset $\{L\}$ of $\boldsymbol{V}^*$. Hence it is proper, convex because $\{L\}$ is a convex subset of $\boldsymbol{V}^*$, and lower semi-continuous because $\{L\}$ is a closed subset of $\boldsymbol{V}^*$.

Clearly, solving the minimization problem

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma})$$

of Theorem 4.2 is the same as solving the minimization problem

$$\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} (g(\boldsymbol{\sigma}) + I_{\mathbb{S}}(\boldsymbol{\sigma})).$$

To prove the last assertion in the theorem therefore amounts to proving that $G(\boldsymbol{\sigma}) = g(\boldsymbol{\sigma}) + I_{\mathbb{S}}(\boldsymbol{\sigma})$ for all $\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)$, i.e., that

$$h(\Lambda\boldsymbol{\sigma}) = I_{\mathbb{S}}(\boldsymbol{\sigma}) \text{ for all } \boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega).$$

Assume that $\boldsymbol{\sigma} \in \mathbb{S}$. Then the Green formula of Theorem 3.1, and the definition of the set $\mathbb{S}$ together imply that

$$\begin{aligned} {}_{\boldsymbol{V}^*}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_V &= \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla}_s \boldsymbol{v} \,\mathrm{d}x = -\int_\Omega (\mathbf{div}\,\boldsymbol{\sigma}) \cdot \boldsymbol{v} \,\mathrm{d}x + \langle \boldsymbol{\sigma}\boldsymbol{\nu}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma \\ &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x + \langle \boldsymbol{F}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma = \boldsymbol{L}(\boldsymbol{v}) \text{ for all } \boldsymbol{v} \in \boldsymbol{V}. \end{aligned}$$

Therefore $h(\Lambda\boldsymbol{\sigma}) = h(\boldsymbol{L}) = 0$, by definition of the function h.

Conversely, assume that $\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)$ is such that $h(\Lambda\boldsymbol{\sigma}) = 0$; hence $\Lambda\boldsymbol{\sigma} = \boldsymbol{L}$, again by definition of h. Consequently, the same Green formula and the definition of the mapping Λ together imply that

$$\begin{aligned} {}_{V^*}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_V &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \langle \boldsymbol{F}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla}_s\boldsymbol{v}\,\mathrm{d}x \\ &= -\int_\Omega (\mathbf{div}\,\boldsymbol{\sigma})\cdot\boldsymbol{v}\,\mathrm{d}x + \langle \boldsymbol{\sigma\nu}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma \text{ for all } \boldsymbol{v}\in\boldsymbol{V}. \end{aligned}$$

Letting the functions $\boldsymbol{v}$ vary in the subspace $\boldsymbol{D}(\Omega)$ of $\boldsymbol{V}$ first shows that $-\mathbf{div}\,\boldsymbol{\sigma} = \boldsymbol{f} \in \mathbb{L}^2(\Omega)$; hence we are left with $\langle \boldsymbol{\sigma\nu} - \boldsymbol{F}, \operatorname{tr}\boldsymbol{v}\rangle = 0$ for all $\boldsymbol{v}\in\boldsymbol{V}$. Consequently, $\boldsymbol{\sigma}\in\mathbb{S}$.

We have thus shown that, given $\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)$, $h(\Lambda\boldsymbol{\sigma}) = 0$ if and only if $\boldsymbol{\sigma}\in\mathbb{S}$. Therefore $h(\Lambda\boldsymbol{\sigma}) = I_{\mathbb{S}}(\boldsymbol{\sigma})$ for all $\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)$, and the proof is complete. □

In view of identifying the dual problems of the minimization problem $(\mathcal{P})$ of Theorem 5.1, it remains to identify the Legendre-Fenchel transforms (Section 2) of the functions h and g introduced in this theorem. Although the next result is known (see, e.g., Ref. **?**), we nevertheless include its proof for completeness.

Theorem 5.2. *Let the functions $g : \mathbb{L}^2_s(\Omega) \to \mathbb{R}$ and $h : \boldsymbol{V}^* \to \mathbb{R}\cup\{+\infty\}$ be respectively defined by*

$$\begin{aligned} \boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega) &\to g(\boldsymbol{\sigma}) := \frac{1}{2}\int_\Omega \mathsf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x \\ \boldsymbol{v}^*\in\boldsymbol{V}^* &\to h(\boldsymbol{v}^*) := 0 \text{ if } \boldsymbol{v}^* = \boldsymbol{L} \quad\text{or}\quad h(\boldsymbol{v}^*) := +\infty \text{ if } \boldsymbol{v}^*\neq\boldsymbol{L}. \end{aligned}$$

Then their Legendre-Fenchel transforms $g^* : \mathbb{L}^2_s(\Omega) \to \mathbb{R}$ and $h^* : \boldsymbol{V} \to \mathbb{R}$ are respectively given by

$$\begin{aligned} g^*(\boldsymbol{e}) &:= \frac{1}{2}\int_\Omega \mathsf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x \text{ for all } \boldsymbol{e}\in\mathbb{L}^2_s(\Omega), \\ h^*(\boldsymbol{v}) &:= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \boldsymbol{F}\cdot\boldsymbol{v}\,\mathrm{d}\Gamma = L(\boldsymbol{v}) \text{ for all } \boldsymbol{v}\in\boldsymbol{V}. \end{aligned}$$

Proof. By definition, for each $\boldsymbol{e}\in\mathbb{L}^2_s(\Omega) = (\mathbb{L}^2_s(\Omega))^*$.

$$\begin{aligned} g^*(\boldsymbol{e}) &:= \sup_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} \left\{ \int_\Omega \boldsymbol{e} : \boldsymbol{\sigma}\,\mathrm{d}x - g(\boldsymbol{\sigma}) \right\} \\ &:= \sup_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} \left\{ \int_\Omega \boldsymbol{e} : \boldsymbol{\sigma}\,\mathrm{d}x - \frac{1}{2}\int_\Omega \mathsf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x \right\} \\ &:= -\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} \left\{ \frac{1}{2}\int_\Omega \mathsf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x - \int_\Omega \boldsymbol{e} : \boldsymbol{\sigma}\,\mathrm{d}x \right\} = \frac{1}{2}\int_\Omega \boldsymbol{e} : \overline{\boldsymbol{\sigma}}\,\mathrm{d}x, \end{aligned}$$

where $\overline{\boldsymbol{\sigma}}\in\mathbb{L}^2_s(\Omega)$ satisfies

$$\int_\Omega \mathsf{B}\overline{\boldsymbol{\sigma}} : \boldsymbol{\sigma}\,\mathrm{d}x = \int_\Omega \boldsymbol{e} : \boldsymbol{\sigma}\,\mathrm{d}x \text{ for all } \boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega).$$

Therefore $\mathbf{B}\overline{\boldsymbol{\sigma}} = \boldsymbol{e}$, or equivalently $\overline{\boldsymbol{\sigma}} = \mathbf{A}\boldsymbol{e}$. This shows that $g^*(\boldsymbol{e}) = \dfrac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x$, as announced.

Likewise, for each $\boldsymbol{v} \in \boldsymbol{V}^{**} = \boldsymbol{V}^*$,

$$h^*(\boldsymbol{v}) := \sup_{\boldsymbol{v}^* \in \boldsymbol{V}^*} \{{}_{\boldsymbol{V}^*}\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}} - h(\boldsymbol{v}^*)\}.$$

But the definition of the function h implies that ${}_{\boldsymbol{V}^*}\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}} - h(\boldsymbol{v}^*) = -\infty$ unless $\boldsymbol{v}^* = \boldsymbol{L}$. Therefore the supremum is attained for $\boldsymbol{v}^* = \boldsymbol{L}$, which means that

$$h^*(\boldsymbol{v}) = {}_{\boldsymbol{V}^*}\langle L, \boldsymbol{v}\rangle_{\boldsymbol{V}} = L(\boldsymbol{v}). \qquad \square$$

6. A first dual problem to the stress formulation

Following the approach described in Section 2, we now *identify the first dual formulation* $(\mathcal{P}^*)$ to the stress formulation of the displacement-traction problem, formulated for this purpose in the form of the equivalent minimization problem $(\mathcal{P})$ described in Theorem 5.1. In so doing, we also show that *the infimum found in* $(\mathcal{P})$ *is equal to the supremum found in* $(\mathcal{P}^*)$.

Theorem 6.1. *Consider the minimization problem*

$$\inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)} G(\boldsymbol{\sigma}),$$

where

$$G(\boldsymbol{\sigma}) := g(\boldsymbol{\sigma}) + h(\Lambda\boldsymbol{\sigma}) \text{ for each } \sigma \in \mathbb{L}^2_s(\Omega),$$

the functions $g : \mathbb{L}^2_s(\Omega) \to \mathbb{R}$ *and* $h : \boldsymbol{V}^* \to \mathbb{R} \cup \{+\infty\}$ *and the operator* $\Lambda \in \mathcal{L}(\mathbb{L}^2_s(\Omega); \boldsymbol{V}^*)$ *being defined as in* Theorem 5.1. *Let*

$$G^*(\boldsymbol{e}) := \inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)} \left\{ \int_\Omega \boldsymbol{e} : \boldsymbol{\sigma}\,\mathrm{d}x + h(\Lambda\boldsymbol{\sigma}) \right\} - g^*(\boldsymbol{e}) \text{ for each } \boldsymbol{e} \in \mathbb{L}^2_s(\Omega),$$

where $g^* : \mathbb{L}^2_s(\Omega) \to \mathbb{R}$ *is the Legendre-Fenchel transform of the function* g, *and let*

$$\sup_{\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)} G^*(\boldsymbol{e}),$$

be the corresponding dual problem. Let the space $\mathbb{M}^\perp$ *and the functional* $j : \mathbb{M}^\perp \to \mathbb{R}$ *be defined as in* Theorem 4.3.

Then the dual problem $(\mathcal{P}^*)$ *can be also written as*

$$\sup_{\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)} G^*(\boldsymbol{e}) = -\inf_{\boldsymbol{e} \in \mathbb{M}^\perp} j(\boldsymbol{e}).$$

Besides,

$$G(\overline{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)} G(\boldsymbol{\sigma}) = \sup_{\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)} G^*(\boldsymbol{e}) = G^*(\overline{\boldsymbol{e}}),$$

where $\overline{\boldsymbol{\sigma}} \in \mathbb{S} \subset \mathbb{L}^2_s(\Omega)$ *and* $\overline{\boldsymbol{e}} \in \mathbb{M}^\perp \subset \mathbb{L}^2_s(\Omega)$ *are the solutions of the minimization problems of* Theorems 4.2 *and* 4.3.

Proof. We showed in Theorem 5.1 that $h(\Lambda\boldsymbol{\sigma}) = I_{\mathbb{S}}(\boldsymbol{\sigma})$ for all $\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)$ and we showed in Theorem 5.2 that $g^*(\boldsymbol{e}) = \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x$ for all $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$. Consequently,

$$\begin{aligned} G^*(\boldsymbol{e}) &:= \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} \left\{ \int_\Omega \boldsymbol{e} : \boldsymbol{\sigma}\,\mathrm{d}x + h(\Lambda\boldsymbol{\sigma}) \right\} - g^*(\boldsymbol{e}) \\ &= \inf_{\boldsymbol{\sigma}\in\mathbb{S}} \left\{ \int_\Omega \boldsymbol{e} : \boldsymbol{\sigma}\,\mathrm{d}x \right\} - \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x \text{ for each } \boldsymbol{e} \in \mathbb{L}^2_s(\Omega). \end{aligned}$$

Let an element $\widetilde{\boldsymbol{\sigma}} \in \mathbb{S}$ be chosen and kept fixed in what follows. Then any element $\boldsymbol{\sigma} \in \mathbb{S}$ can be written as $\boldsymbol{\sigma} = \widetilde{\boldsymbol{\sigma}} + \boldsymbol{\mu}$ with $\boldsymbol{\mu} \in \mathbb{M}$, so that

$$G^*(\boldsymbol{e}) = \inf_{\boldsymbol{\mu}\in\mathbb{M}} \left\{ \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x \right\} + \int_\Omega \boldsymbol{e} : \widetilde{\boldsymbol{\sigma}}\,\mathrm{d}x - \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x \text{ for each } \boldsymbol{e} \in \mathbb{L}^2_s(\Omega).$$

Since $\inf_{\boldsymbol{\mu}\in\mathbb{M}}\{\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x\} = -\infty$ unless $\boldsymbol{e} \in \mathbb{M}^\perp$, in which case $\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0$ for all $\boldsymbol{\mu} \in \mathbb{M}$, it is clear that

$$\sup_{\boldsymbol{e}\in\mathbb{L}^2_s(\Omega)} G^*(\boldsymbol{e}) = \sup_{\boldsymbol{e}\in\mathbb{M}^\perp} G^*(\boldsymbol{e}).$$

For each $\boldsymbol{e} \in \mathbb{M}^\perp$, there exists a unique element $\boldsymbol{\mathcal{F}}(\boldsymbol{e}) \in \boldsymbol{V}$ such that $\boldsymbol{e} = \nabla_s\boldsymbol{\mathcal{F}}(\boldsymbol{e})$ (Theorem 3.2). Using the Green formula of Theorem 3.1, we can therefore re-write the inner product $\int_\Omega \boldsymbol{e} : \widetilde{\boldsymbol{\sigma}}\,\mathrm{d}x$ as

$$\begin{aligned} \int_\Omega \boldsymbol{e} : \widetilde{\boldsymbol{\sigma}}\,\mathrm{d}x &= \int_\Omega \nabla_s\boldsymbol{\mathcal{F}}(\boldsymbol{e}) \cdot \widetilde{\boldsymbol{\sigma}}\,\mathrm{d}x \\ &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{e})\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{e})\,\mathrm{d}\Gamma = L(\boldsymbol{\mathcal{F}}(\boldsymbol{e})). \end{aligned}$$

Consequently,

$$\begin{aligned} \sup_{\boldsymbol{e}\in\mathbb{L}^2_s(\Omega)} G^*(\boldsymbol{e}) &= \sup_{\boldsymbol{e}\in\mathbb{M}^\perp} \left\{ -\frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x + L(\boldsymbol{\mathcal{F}}(\boldsymbol{e})) \right\} \\ &= \sup_{\boldsymbol{e}\in\mathbb{M}^\perp} \{-j(\boldsymbol{e})\} = -\inf_{\boldsymbol{e}\in\mathbb{M}^\perp} j(\boldsymbol{e}), \end{aligned}$$

as announced. Finally

$$\begin{aligned} \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} G(\boldsymbol{\sigma}) &= \inf_{\boldsymbol{\sigma}\in\mathbb{S}} \left\{ \frac{1}{2}\int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x \right\} = \frac{1}{2}\int_\Omega \mathbf{B}\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}}\,\mathrm{d}x = G(\overline{\boldsymbol{\sigma}}) \\ &= -\frac{1}{2}\int_\Omega \mathbf{A}\nabla_s\overline{\boldsymbol{v}} : \nabla_s\overline{\boldsymbol{v}}\,\mathrm{d}x + L(\overline{\boldsymbol{v}}) = -j(\boldsymbol{\nabla}_s\overline{\boldsymbol{v}}) = -j(\overline{\boldsymbol{e}}) \\ &= -\inf_{\boldsymbol{e}\in\mathbb{M}^\perp} j(\boldsymbol{e}) = \sup_{\boldsymbol{e}\in\mathbb{M}^\perp} G^*(\boldsymbol{e}) = G^*(\overline{\boldsymbol{e}}) = \sup_{\boldsymbol{e}\in\mathbb{L}^2_s(\Omega)} G^*(\boldsymbol{e}), \end{aligned}$$

and the proof is complete. □

Theorem 6.1 thus shows that *the dual problem* $(\mathcal{P}^*)$ *of the stress formulation of the displacement-traction problem of linearized elasticity is, up to a change of sign, the strain formulation of the same problem* (Theorem 4.3).

In addition, we give a positive answer to the last question raised in Section 2, by showing that *the Lagrangian associated to* $(\mathcal{P}^*)$ *has a saddle-point*. The notations and definitions used in the next theorem are those of Theorem 6.1.

Theorem 6.2. *Define the Lagrangian*

$$\mathcal{L} : \mathbb{L}^2_s(\Omega) \times \mathbb{L}^2_s(\Omega) \to \mathbb{R} \cup \{+\infty\}$$

by

$$\mathcal{L}(\boldsymbol{\sigma}, \boldsymbol{e}) := \int_\Omega \boldsymbol{e} : \boldsymbol{\sigma} \, \mathrm{d}x - g^*(\boldsymbol{e}) + h(\Lambda \boldsymbol{\sigma}) \text{ for all } (\boldsymbol{\sigma}, \boldsymbol{e}) \in \mathbb{L}^2_s(\Omega) \times \mathbb{L}^2_s(\Omega).$$

Then

$$\inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)} \sup_{\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)} \mathcal{L}(\boldsymbol{\sigma}, \boldsymbol{e}) = \mathcal{L}(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{e}}) = \sup_{\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)} \inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)} \mathcal{L}(\boldsymbol{\sigma}, \boldsymbol{e}).$$

Proof. In view of Theorem 6.1, it suffices to prove that

$$G(\overline{\boldsymbol{\sigma}}) = \mathcal{L}(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{e}}),$$

where (recall that $\overline{\boldsymbol{\sigma}} \in \mathbb{S}$)

$$G(\overline{\boldsymbol{\sigma}}) = g(\overline{\boldsymbol{\sigma}}) = \frac{1}{2} \int_\Omega \mathbf{B} \overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}} \, \mathrm{d}x = \frac{1}{2} \int_\Omega \overline{\boldsymbol{e}} : \overline{\boldsymbol{\sigma}} \, \mathrm{d}x,$$

and

$$\mathcal{L}(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{e}}) = \int_\Omega \overline{\boldsymbol{e}} : \overline{\boldsymbol{\sigma}} \, \mathrm{d}x - g^*(\overline{\boldsymbol{e}}) + h(\Lambda \overline{\boldsymbol{\sigma}}).$$

Hence the conclusion follows since, by Theorem 5.2,

$$g^*(\overline{\boldsymbol{e}}) = \frac{1}{2} \int_\Omega \mathbf{A} \overline{\boldsymbol{e}} : \overline{\boldsymbol{e}} \, \mathrm{d}x = \frac{1}{2} \int_\Omega \overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{e}} \, \mathrm{d}x,$$

and, by Theorem 5.1, $h(\Lambda \overline{\boldsymbol{\sigma}}) = 0$ because $\overline{\boldsymbol{\sigma}} \in \mathbb{S}$. □

7. A second dual problem to the stress formulation

We now *identify the second dual formulation* $(\widetilde{\mathcal{P}}^*)$ to the stress formulation of the displacement problem, again formulated as the problem $(\mathcal{P})$ of Theorem 5.1. In so doing, we also show that the infimum found in $(\mathcal{P})$ is equal to the supremum found in $(\widetilde{\mathcal{P}}^*)$.

Theorem 7.1. *Consider the minimization problem*

$$\inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)} G(\boldsymbol{\sigma}),$$

where the function $G : \mathbb{L}^2_s(\Omega) \to \mathbb{R} \cup \{+\infty\}$ *is defined as in* Theorem 5.1. *Let*

$$\widetilde{G}^*(\boldsymbol{v}) := \inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)} \left\{ \frac{1}{2} \int_\Omega \mathbf{B} \boldsymbol{\sigma} : \boldsymbol{\sigma} \, \mathrm{d}x + {}_{\boldsymbol{V}^*}\langle \Lambda \boldsymbol{\sigma}, \boldsymbol{v} \rangle_{\boldsymbol{V}} \right\} - h^*(\boldsymbol{v}) \text{ for each } \boldsymbol{v} \in \boldsymbol{V},$$

where $h^* : \boldsymbol{V} \to \mathbb{R}$ *is the Legendre-Fenchel transform of the function* h*, and let*

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{G}^*(\boldsymbol{v}) \tag{$\widetilde{\mathcal{P}}^*$}$$

be the corresponding dual problem.

Let the functional $J : \boldsymbol{V} \to \mathbb{R}$ *be defined as in* Theorem 4.1*. Then the dual problem* $(\widetilde{\mathcal{P}}^*)$ *can be also written as*

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{G}^*(\boldsymbol{v}) = -\inf_{\boldsymbol{v}\in\boldsymbol{V}} J(\boldsymbol{v}).$$

Besides,

$$G(\overline{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} G(\boldsymbol{\sigma}) = \sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{G}^*(\boldsymbol{v}) = G^*(-\overline{\boldsymbol{v}}),$$

where $\overline{\boldsymbol{\sigma}} \in \mathbb{S} \subset \mathbb{L}^2_s(\Omega)$ *and* $\overline{\boldsymbol{v}} \in \boldsymbol{V}$ *are the solutions of the minimization problems of* Theorems 4.2 *and* 4.1*.*

Proof. By definition (Theorem 5.1), ${}_{\boldsymbol{V}^*}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\boldsymbol{V}} = \int_\Omega \boldsymbol{\sigma} : \nabla_s \boldsymbol{v}\,\mathrm{d}x$ for all $\sigma \in \mathbb{L}^2_s(\Omega)$ and all $\boldsymbol{v} \in \boldsymbol{V}$; besides, $h^*(\boldsymbol{v}) = L(\boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{V}$ (Theorem 5.2). Consequently,

$$\widetilde{G}^*(\boldsymbol{v}) = \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} \left\{ \frac{1}{2}\int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x + \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla}_s \boldsymbol{v}\,\mathrm{d}x \right\} - L(\boldsymbol{v}) \text{ for each } \boldsymbol{v} \in \boldsymbol{V}.$$

But

$$\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} \left\{ \frac{1}{2}\int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x + \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla}_s \boldsymbol{v}\,\mathrm{d}x \right\} = -\frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{\nabla}_s \boldsymbol{v} : \boldsymbol{\nabla}_s \boldsymbol{v}\,\mathrm{d}x,$$

so that

$$\widetilde{G}^*(\boldsymbol{v}) = -\frac{1}{2}\int_\Omega \mathbf{A}\nabla_s \boldsymbol{v} : \boldsymbol{\nabla}_s \boldsymbol{v}\,\mathrm{d}x - L(\boldsymbol{v}) \text{ for each } \boldsymbol{v} \in \boldsymbol{V}.$$

Noting that $\boldsymbol{V}$ is a vector space, we thus have

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{G}^*(\boldsymbol{v}) = \sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{G}^*(-\boldsymbol{v}) = -\inf_{\boldsymbol{v}\in\boldsymbol{V}} J(\boldsymbol{v}),$$

as announced. Finally,

$$\begin{aligned}
\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_s(\Omega)} G(\boldsymbol{\sigma}) &= \inf_{\boldsymbol{\sigma}\in\mathbb{S}} \left\{ \frac{1}{2}\int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x \right\} = \frac{1}{2}\int_\Omega \mathbf{B}\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}}\,\mathrm{d}x = G(\overline{\boldsymbol{\sigma}}) \\
&= -\frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{\nabla}_s \overline{\boldsymbol{v}} : \boldsymbol{\nabla}_s \overline{\boldsymbol{v}}\,\mathrm{d}x + L(\overline{\boldsymbol{v}}) = J(\overline{\boldsymbol{v}}) = \widetilde{G}^*(-\overline{\boldsymbol{v}}) \\
&= -\inf_{\boldsymbol{v}\in\boldsymbol{V}} J(\boldsymbol{v}) = \sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{G}^*(\boldsymbol{v}),
\end{aligned}$$

and the proof is complete. □

Theorem **??** thus shows that *the dual problem* $(\widetilde{\mathcal{P}}^*)$ *to the stress formulation of the displacement-traction problem of linearized elasticity is, up to a change of sign, the displacement formulation of the same problem* (Theorem 4.1).

To conclude this analysis, we show that *the Lagrangian associated to* $(\widetilde{\mathcal{P}}^*)$ *has a saddle-point* (like the Lagrangian associated to $(\mathcal{P}^*)$; cf. Theorem **??**). The notations and definitions are those of Theorem **??**.

Theorem 7.2. *Define the Lagrangian*

$$\widetilde{\mathcal{L}} : \mathbb{L}^2_s(\Omega) \times \boldsymbol{V} \to \mathbb{R}$$

by

$$\widetilde{\mathcal{L}}(\boldsymbol{\sigma}, \boldsymbol{v}) := \frac{1}{2} \int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma} \, \mathrm{d}x + {}_{\boldsymbol{V}^*}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\boldsymbol{V}} - h^*(\boldsymbol{v}) \text{ for all } (\boldsymbol{\sigma}, \boldsymbol{v}) \in \mathbb{L}^2_s(\Omega) \times \boldsymbol{V}.$$

Then

$$\inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)} \sup_{\boldsymbol{v} \in \boldsymbol{V}} = \widetilde{\mathcal{L}}(\boldsymbol{\sigma}, \boldsymbol{v}) = \widetilde{\mathcal{L}}(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{v}}) = \sup_{\boldsymbol{v} \in \boldsymbol{V}} \inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_s(\Omega)} \widetilde{\mathcal{L}}(\boldsymbol{\sigma}, \boldsymbol{v}).$$

Proof. In view of Theorem **??**, it suffices to prove that

$$G(\overline{\boldsymbol{\sigma}}) = \widetilde{\mathcal{L}}(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{v}}),$$

where

$$G(\overline{\boldsymbol{\sigma}}) = \frac{1}{2} \int_\Omega \mathbf{B}\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}} \, \mathrm{d}x$$

(see the proof of Theorem **??**), and

$$\widetilde{\mathcal{L}}(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{v}}) := \frac{1}{2} \int_\Omega \mathbf{B}\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}} \, \mathrm{d}x + {}_{\boldsymbol{V}^*}\langle \Lambda\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{v}}\rangle_{\boldsymbol{V}} - h^*(\overline{\boldsymbol{v}}).$$

Hence the conclusion follows since, by Theorem 5.1,

$${}_{\boldsymbol{V}^*}\langle \Lambda\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{v}}\rangle_{\boldsymbol{V}} = \int_\Omega \overline{\boldsymbol{\sigma}} : \boldsymbol{\nabla}_s \overline{\boldsymbol{v}} \, \mathrm{d}x = \int_\Omega \mathbf{A}\boldsymbol{\nabla}_s \overline{\boldsymbol{v}} : \boldsymbol{\nabla}_s \overline{\boldsymbol{v}} \, \mathrm{d}x = L(\overline{\boldsymbol{v}}),$$

and, by Theorem 5.2, $h^*(\overline{\boldsymbol{v}}) = L(\overline{\boldsymbol{v}})$. □

8. Concluding remarks

The strain formulation of, a.k.a. the intrinsic approach to, the displacement-traction problem described in Theorem 4.3 was derived *a priori* in [?], as a way to re-formulate this problem as a quadratic minimization problem with the strain tensor field as the sole unknown. One main conclusion to be drawn from the present analysis is thus that *this strain formulation may be also viewed as a Legendre-Fenchel dual problem to the classical stress formulation* (Theorem 6.1). This constitutes the main novelty of this paper.

Another novelty is that *the classical displacement formulation can be also viewed as a Legendre-Fenchel dual problem to the same classical stress formulation* (Theorem **??**).

For the *pure traction problem* (not considered here), i.e., when $\Gamma_0 = \varnothing$, *another* strain formulation is possible. More specifically, Ciarlet & Ciarlet, Jr. [?] have established the following *Saint Venant theorem in* $\mathbb{L}^2_s(\Omega)$: *Let* Ω *be a simply-connected domain in* $\mathbb{R}^3$ *and let* $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$ *be a matrix field that satisfies the Saint Venant compatibility conditions*

$$R_{ijk\ell}(\boldsymbol{e}) := \partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \text{ in } H^{-2}(\Omega).$$

Then there exists a vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ *such that* $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ in $\mathbb{L}^2_s(\Omega)$. Further extensions, to Sobolev spaces of weaker regularity, have since then been given, in Refs. **?** and **?**.

Furthermore, it was subsequently shown, in Ciarlet & Ciarlet, Jr. [?], that the above Saint-Venant compatibility conditions can be *exactly* satisfied in an *ad hoc* finite element subspace of $\mathbb{L}^2_s(\Omega)$ that uses *edge finite elements* in the sense of Nédélec [?,?], thus yielding an efficient way to directly approximate the strain tensor field $\overline{\boldsymbol{e}}$ or equivalently the stress tensor field $\overline{\boldsymbol{\sigma}}$ through the constitutive equation $\overline{\boldsymbol{\sigma}} = \mathbf{A}\overline{\boldsymbol{e}}$. Indeed, the first numerical simulations are very encouraging; cf. Ciarlet, Ciarlet, Jr. & Vicard [?].

Be that as it may, the latter approach suffers from *two shortcomings*. First, the domain Ω must be simply-connected; second, this approach does not seem so far to be amenable to treat displacement-traction, or even pure displacement, problems. By contrast, the *Donati-like* strain approach considered here does not suffer from such shortcomings.

It is therefore remarkable that a totally different approach, based on Legendre-Fenchel duality, leads to the same conclusion, namely that *the Donati-like approach is more natural, and more efficient, than the Saint-Venant-like approach for constructing a strain formulation of the displacement-traction problem of three-dimensional linearized elasticity.*

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