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# Quantifier Elimination and Rectilinearisation Theorem for Generalised Quasianalytic Algebras

J.-P. Rolin<sup>\*</sup> and T. Servi<sup>†</sup>

## Abstract

We consider for every  $n \in \mathbb{N}$  an algebra  $\mathcal{A}_n$  of germs at  $0 \in \mathbb{R}^n$  of continuous real-valued functions, such that we can associate to every germ  $f \in \mathcal{A}_n$  a (divergent) series  $\mathcal{T}(f)$  with nonnegative real exponents, which can be thought of as an asymptotic expansion of  $f$ . We require that the  $\mathbb{R}$ -algebra homomorphism  $f \mapsto \mathcal{T}(f)$  be injective (quasianalyticity property). In this setting we prove analogue results to Denef and van den Dries' quantifier elimination and Hironaka's rectilinearisation theorems for subanalytic sets.

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## 1 Introduction

In [Par01] the author proves a preparation theorem for subanalytic functions (whose original statement can be found in [LR97, Par94]), as a multivariable Puiseux Theorem, or as a primitive version of Hironaka's Rectilinearisation Theorem. Our aim is to extend this result to a general quasianalytic setting.

The results of this paper lie within the framework of o-minimal geometry. The notion of *o-minimal structure* was introduced as a model-theoretic concept. However, in order to render it and our results accessible to a larger audience, we give a geometric version of its definition (see [Dri98] and [DM96] for an overview of the subject).

We consider a collection  $\mathcal{F}$  of functions  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  for  $m \in \mathbb{N}$ . A set  $A \subseteq \mathbb{R}^n$  is *definable* in the *structure*  $\mathbb{R}_{\mathcal{F}} = (\mathbb{R}, <, 0, 1, +, \cdot, \mathcal{F})$  (which we call the *expansion of the real field by  $\mathcal{F}$* ) if  $A$  belongs to the smallest collection  $\mathcal{S} = (\mathcal{S}_n)_{n \in \mathbb{N}}$  of subsets of  $\mathbb{R}^n$  ( $n \in \mathbb{N}$ ) such that:

1.  $\mathcal{S}_n$  contains all semi-algebraic subsets of  $\mathbb{R}^n$ ,
2.  $\mathcal{S}$  contains the graphs of all functions in  $\mathcal{F}$ ,
3.  $\mathcal{S}$  is closed under the boolean set operations and projections.

A map  $f : A \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$  is *definable* in  $\mathbb{R}_{\mathcal{F}}$  if its graph is a definable subset of  $\mathbb{R}^{m+n}$  (this implies that  $A$  itself is definable).

The structure  $\mathbb{R}_{\mathcal{F}}$  is said to be:

1. *model-complete* if in order to generate the whole family of definable sets one can dispense with taking complements;
2. *o-minimal* if all definable sets have finitely many connected components;
3. *polynomially bounded* if the germs at infinity of definable unary functions have at most polynomial growth.

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The two following classical example of polynomially bounded model-complete o-minimal expansion of the real field show how o-minimal geometry can be seen as a generalisation of real algebraic and analytic geometry.

- $\bar{\mathbb{R}}$ , where  $\mathcal{F} = \emptyset$ , i.e. the structure whose definable sets are the semi-algebraic sets (the proof of model-completeness and o-minimality follows from the Tarski-Seidenberg *elimination* principle [Tar51, Sei54]).
- $\mathbb{R}_{\text{an}}$ , where  $\mathcal{F}$  is the set of functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  ( $n \in \mathbb{N}$ ) whose restriction to the unit cube  $[-1, 1]^n$  is real analytic and which are identically zero outside the unit cube (the proof is a consequence of Gabrielov's *Theorem of the Complement*, which states that the complement of a subanalytic subset of a real analytic manifold is again subanalytic [Gab68, Gab96, BM88]).

Besides the finiteness property mentioned in the definition, sets and functions definable in an o-minimal structure share many good geometric properties with semi-algebraic and subanalytic sets. In particular, for all  $k \in \mathbb{N}$ , a definable set admits a finite stratification with  $C^k$  definable manifolds (see for example [DM96]).

In the last two decades the list of examples has grown considerably, including several polynomially bounded structures for which o-minimality is a consequence of model-completeness. Among these examples we find:

- $\mathbb{R}_{\text{an}^*}$ , where  $\mathcal{F}$  is the collection of all so-called *convergent generalised power series*, namely all functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  whose restriction to the unit cube is given by a convergent series of monomials with positive real exponents (with a well-order condition on the support) and which are identically zero outside the unit cube (see [DS98]).
- $\mathbb{R}_G$ , where  $\mathcal{F}$  is a collection of Gevrey functions in several variables (see [DS00] for the definitions and proofs).
- $\mathbb{R}_{C(M)}$ , where  $\mathcal{F}$  is a collection of  $C^\infty$  functions, restricted to the unit cube, whose derivatives are bounded by a strictly increasing sequence of positive constants  $M = (M_n)_{n \in \mathbb{N}}$  satisfying the *Denjoy-Carleman quasianalyticity* condition (see [RSW03] for the definitions and proofs).
- $\mathbb{R}_{\text{an}, H}$ , where  $\mathcal{F}$  is a collection of functions containing all real analytic functions restricted to the unit cube and a solution  $H$  of a first order analytic differential equation which is singular at the origin (see [RSS07]).
- $\mathbb{R}_Q$ , in which certain Dulac transition maps are definable (see [Ily02] for a complete survey on Dulac's problem and [KRS09] for the proof of model-completeness and o-minimality). In this example  $\mathcal{F}$  is a collection of functions, restricted to the unit cube, whose germ at zero admits an asymptotic expansion (with positive real exponents) which is, in general, divergent (as opposed to the case of  $\mathbb{R}_{\text{an}^*}$ ).

The Tarski-Seidenberg elimination principle, which implies model-completeness, is a *quantifier elimination* result, which can be resumed by saying that the projection of a semi-algebraic set is again semi-algebraic. It is well known that the analogue result does not hold for  $\mathbb{R}_{\text{an}}$ . However, Denef and van den Dries proved in [DD88] that every relatively compact subanalytic set can be described by a system of equations and inequalities satisfied by some compositions of restricted analytic functions and quotients. In other words, the structure  $\mathbb{R}_{\text{an}}$  admits quantifier elimination in the language of restricted analytic functions expanded by the function  $D : (x, y) \mapsto x/y$  for  $|y| \geq |x|$  and zero otherwise.

Our main goal is to prove a quantifier elimination result in the spirit of [DD88] for all the structures in the above examples. In order to do this, we first define a common framework of which each of the above examples appears as a special case. Secondly, we isolate the common properties of these structures which are relevant to the proof of their o-minimality. Finally, we proceed to develop a new strategy for the proof of quantifier elimination. The originality of this work lies especially in this last part, since the main tool in Denef and van den Dries' proof, namely the Weierstrass Preparation Theorem, is not available to us in this setting, as we will explain later.

The proofs of model-completeness and o-minimality for the above examples are similar and all inspired by [Gab68, BM88]. The common strategy is to parametrise a definable set by maps whose components are in  $\mathcal{F}$ . To obtain this, the key property is the *quasianalyticity* (explained below) of the algebras generated by the germs of functions in  $\mathcal{F}$ . However, there are significant differences between these examples. In some of them (see [DS98, DS00, KRS09]), we can use the Weierstrass Preparation Theorem with respect to a certain type of variables. In this case the proof of o-minimality is very close to Gabrielov’s proof for  $\mathbb{R}_{\text{an}}$  in [Gab68]. In the other cases, one uses instead some form of *resolution of singularities*, which allows to get to the required parametrisation result in a more indirect way.

Our first goal is to unify all these different proofs, so that all the examples above appear as particular cases of a general o-minimality statement. We consider, for all  $n \in \mathbb{N}$ , an algebra  $\mathcal{A}_n$  of continuous functions such that to each germ at  $0 \in \mathbb{R}^n$  of  $f \in \mathcal{A}_n$  we can associate a (divergent) series  $\mathcal{T}(f)$  with nonnegative real exponents. This series can be thought of as an *asymptotic expansion* of  $f$ . The algebra of all germs at 0 of functions in  $\mathcal{A}_n$  is *quasianalytic* if the  $\mathbb{R}$ -algebra homomorphism  $\mathcal{T} : f \mapsto \mathcal{T}(f)$  is injective. The link between quasianalyticity and o-minimality appears already in [Dri99, DS00]. The quasianalyticity property can be obtained de facto (as in [DS98], where the functions under consideration are *equal* to the sum of their convergent expansion), or it has to be proved using analysis techniques (resummation methods in [DS00] and [RSS07], Denjoy-Carleman’s theorem in [RSW03], Ilyashenko’s method for Dulac’s problem in [KRS09]).

It is not our purpose here to prove quasianalyticity. We assume rather to have been given a collection of quasianalytic algebras. We show, then, that the structure  $\mathbb{R}_{\mathcal{A}}$  generated by the algebras  $\mathcal{A}_n$  is model-complete, o-minimal and polynomially bounded. Given the level of generality we aim to keep, we cannot make use of the Weierstrass Preparation Theorem. Hence, we show how to prove the parametrisation result for definable sets by using an appropriate *blow-up* procedure. This latter takes inspiration from the methods in [BM88], adapted to series with real exponents in [DS98].

Once we have proved o-minimality for the structures which fall into this general framework, we can proceed towards our quantifier elimination result. In [DD88], by a clever use of the Weierstrass Preparation Theorem and of the Tarski-Seidenberg elimination principle, Denef and van den Dries prove their result without needing to solve explicitly any system of analytic equations.

In our framework, recall that we have already shown that a bounded  $\mathbb{R}_{\mathcal{A}}$ -definable set can be parametrised by maps whose components are in the algebras  $\mathcal{A}_n$ . Our aim is to eliminate the parameters. Since we cannot use the Weierstrass Preparation Theorem and hence reduce to the polynomial situation, our strategy is, instead, to solve explicitly the parametrising equations with respect to the parameters. In order to do that, we use o-minimality of the structure  $\mathbb{R}_{\mathcal{A}}$ , established in the first part of the paper, and apply an *o-minimal Preparation Theorem* for functions definable in a polynomially bounded o-minimal structure [DS02]. This latter result, whose proof uses valuation and model-theoretic methods, allows to find, piecewise, a “principal part” of a definable function. This is the starting point for a Newton-Puiseux solving method for quasianalytic equations.

Finally, Denef and van den Dries deduce Hironaka’s *Rectilinearisation Theorem* [Hir73] as a corollary of their main result. This result states that every subanalytic set  $A \subseteq \mathbb{R}^n$  can be transformed into a finite union of quadrants of dimension at most  $\dim(A)$ , via a finite sequence of blow-ups of the ambient space  $\mathbb{R}^n$ . In the same spirit, we prove a Rectilinearisation Theorem for bounded  $\mathbb{R}_{\mathcal{A}}$ -definable sets.

The plan of the paper is the following. In Section 2 we introduce formally the setting we are working in and, in Subsection 2.3, we give the two main statements we prove, namely o-minimality of  $\mathbb{R}_{\mathcal{A}}$  (Theorem A) and quantifier elimination (Theorem B). Section 3 is dedicated to a monomialisation, or desingularisation, algorithm. In Section 4 we prove the parametrisation result mentioned above. The proof of Theorem A is completed in Subsection 4.1, following a traditional

approach à la Gabrielov, thanks to Proposition 4.12. Section 5 is dedicated to the proof of Theorem B. The key result is a monomialisation theorem for  $\mathbb{R}_{\mathcal{A}}$ -definable functions (Theorem 5.2), from which we deduce the Rectilinearisation Theorem 5 and Theorem B. The proof of Theorem 5.2 is obtained by a significant modification of the monomialisation process described in Subsection 3.2. We develop a *vertical monomialisation algorithm* which allows to solve explicitly a system of quasianalytic equations (Theorem 5.11, part B), inverse a parametrisation (Theorem 5.11, part C) and finally monomialise definable functions (Theorem 5.11, part A). The first step for solving explicitly quasianalytic equations is to “weakly monomialise” the solutions. This is done in Lemma 5.13, where we use the o-minimal Preparation Theorem in [DS02] mentioned above.

## 2 Setting and main results

### 2.1 Generalised power series

**2.1 Definition.** Let  $m \in \mathbb{N}$ . A set  $S \subset [0, \infty)^m$  is called *good* if  $S$  is contained in a cartesian product  $S_1 \times \dots \times S_m$  of well ordered subsets of  $[0, \infty)$ . If  $S$  is a good set, define  $S_{\min}$  as the set of minimal elements of  $S$ . By [DS98, Lemma 4.2],  $S_{\min}$  is finite. For  $\alpha = (\alpha_1, \dots, \alpha_m), \beta = (\beta_1, \dots, \beta_m) \in S$  we write  $\alpha \leq \beta$  if  $\alpha_i \leq \beta_i$  for all  $i = 1, \dots, m$ . So if  $S$  is good,  $\forall \alpha \in S \exists \beta \in S_{\min}$  such that  $\alpha \geq \beta$ .

Denote by  $\Sigma(S)$  the set of all finite sums (done component-wise) of elements of  $S$ . By [DS98, Lemma 4.3], if  $S$  is good then  $\Sigma(S)$  is also good.

We recall the definition of generalised formal power series, originally due to [DS98].

**2.2 Definition.** Let  $\mathbb{A}$  be a commutative ring,  $m \in \mathbb{N}$  and  $X = (X_1, \dots, X_m)$  be a tuple of variables. We consider formal series

$$F(X) = \sum_{\alpha} c_{\alpha} X^{\alpha},$$

where  $\alpha = (\alpha_1, \dots, \alpha_m) \in [0, \infty)^m$ ,  $c_{\alpha} \in \mathbb{A}$  and  $X^{\alpha}$  denotes the formal monomial  $X_1^{\alpha_1} \cdot \dots \cdot X_m^{\alpha_m}$ , and the set

$$\text{Supp}(F) := \{\alpha \in [0, \infty)^m : c_{\alpha} \neq 0\} \text{ (the support of } F\text{)}$$

is a good set. These series are added the usual way and form a ring denoted by  $\mathbb{A}[[X^*]]$ .

The ring of usual power series  $\mathbb{A}[[X]]$  can be seen as the subring of  $\mathbb{A}[[X^*]]$  consisting of the series  $F$  with  $\text{Supp}(F) \subset \mathbb{N}^m$ .

**2.3 Definition.** Let  $\mathcal{F} \subset \mathbb{A}[[X^*]]$  be a (possibly infinite) family of series such that the *total support of  $\mathcal{F}$*

$$\text{Supp}(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} \text{Supp}(F)$$

is a good set. Then  $\text{Supp}(\mathcal{F})_{\min}$  is finite and we define the *set of minimal monomials of  $\mathcal{F}$*

$$\mathcal{F}_{\min} := \{X^{\alpha} : \alpha \in \text{Supp}(\mathcal{F})_{\min}\}.$$

**2.4 Definition.** We fix  $m, n \in \mathbb{N}$ . Let  $(X, Y) = (X_1, \dots, X_m, Y_1, \dots, Y_n)$ . We define  $\mathbb{A}[[X^*, Y]]$  as the subring of  $\mathbb{A}[[X, Y]^*]$  consisting of those series  $F$  such that  $\text{Supp}(F) \subset [0, \infty)^m \times \mathbb{N}^n$ . Moreover, for  $F \in \mathbb{A}[[X^*, Y]]$ , we define

$$\text{Supp}_X(F) := \{\alpha \in [0, \infty)^m : \exists N \in \mathbb{N}^n \text{ s.t. } (\alpha, N) \in \text{Supp}(F)\}.$$

Notice that  $\text{Supp}(F)$  is good if and only if  $\text{Supp}_X(F)$  is good.

**2.5 Notation.** Let  $F \in \mathbb{R}[[X^*, Y]]$ . For  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, n\}$ , let

$$\hat{X} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_m) \text{ and } \hat{Y} = (Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_n).$$

We can write  $F(X, Y) = \sum_{\alpha \in S} G_{\alpha}(\hat{X}, Y) X_i^{\alpha} = \sum_{k \in \mathbb{N}} H_k(X, \hat{Y}) Y_j^k$ , where  $S$  is the  $i^{\text{th}}$  projection of  $\text{Supp}(F)$  and  $G_{\alpha} \in \mathbb{R}[[\hat{X}, Y]]$ ,  $H_k \in \mathbb{R}[[X^*, \hat{Y}]]$ . We denote by  $\partial_i F$  the series  $\sum_{\alpha \in S} \alpha G_{\alpha} X_i^{\alpha}$  and by  $\frac{\partial F}{\partial Y_j}$  the series  $\sum_{k \geq 1} k H_k Y_j^{k-1}$ .

**2.6 Notation.** Let  $\lambda, \alpha > 0$ . We denote by  $(Y + \lambda)^\alpha$  the power series  $\lambda^\alpha \sum_{i \in \mathbb{N}} \binom{\alpha}{i} \left(\frac{Y}{\lambda}\right)^i \in \mathbb{R}[[Y]]$ .

## 2.2 Quasianalytic algebras

In this subsection we define the basic object of our interest, namely we fix a family  $\mathcal{A}$  of real functions satisfying the properties in 2.8 and 2.10. Moreover, we require the collection of all germs at zero of the functions in  $\mathcal{A}$  to satisfy the properties in 2.11 and 2.14.

**2.7 Notation.** Let  $m, n \in \mathbb{N}$ . A *polyradius* of type  $(m, n)$  in  $\mathbb{R}^{m+n}$  is a tuple of the form  $r = (s, t) = (s_1, \dots, s_m, t_1, \dots, t_n) \in [0, \infty)^{m+n}$ . If  $r, r'$  are polyradii in  $\mathbb{R}^{m+n}$ , we write  $r' \leq r$  if  $s'_i \leq s_i$  for all  $i = 1, \dots, m$  and  $t'_j \leq t_j$  for all  $j = 1, \dots, n$ . If  $r'$  is of type  $(m, n-1)$  and  $r$  is of type  $(m, n)$ , we write  $r' \leq r$  if  $(r', 0) \leq r$ . We also define:

$$I_{m,n,r} := (0, s_1) \times \dots \times (0, s_m) \times (-t_1, t_1) \times \dots \times (-t_n, t_n),$$

$$\hat{I}_{m,n,r} := [0, s_1] \times \dots \times [0, s_m] \times (-t_1, t_1) \times \dots \times (-t_n, t_n).$$

We also denote by  $\hat{I}_{m,n,\infty}$  the set  $[0, +\infty)^m \times \mathbb{R}^n$ .

**2.8.** For every  $m, n \in \mathbb{N}$  and  $r \in (0, \infty)^{m+n}$ , we let  $\mathcal{A}_{m,n,r}$  be an algebra of real functions, which are  $\mathcal{C}^0$  on  $\hat{I}_{m,n,r}$  and  $\mathcal{C}^1$  on  $I_{m,n,r}$ . We require that the algebras  $\mathcal{A}_{m,n,r}$  satisfy the following list of conditions. Let  $x = (x_1, \dots, x_m)$  and  $y = (y_1, \dots, y_n)$ .

- The coordinate functions of  $\mathbb{R}^{m+n}$  are in  $\mathcal{A}_{m,n,r}$ .
- If  $r' \leq r$  and  $f \in \mathcal{A}_{m,n,r}$ , then  $f \upharpoonright \hat{I}_{m,n,r'} \in \mathcal{A}_{m,n,r'}$ .
- If  $f \in \mathcal{A}_{m,n,r}$  then there exists  $r' > r$  and  $g \in \mathcal{A}_{m,n,r'}$  such that  $g \upharpoonright \hat{I}_{m,n,r} = f$ .
- If  $f \in \mathcal{A}_{m,n,r}$ ,  $s \in (0, \infty)$  and  $r' = (s_1, \dots, s_m, s, t_1, \dots, t_n)$  then the function

$$F: \begin{array}{ccc} \hat{I}_{m+1,n,r'} & \longrightarrow & \mathbb{R} \\ (x_1, \dots, x_m, z, y) & \longmapsto & f(x, y) \end{array}$$

is in  $\mathcal{A}_{m+1,n,r'}$ .

- $\mathcal{A}_{m,n,r} \subset \mathcal{A}_{m+n,0,r}$ , in the sense that we identify  $f(x_1, \dots, x_m, y_1, \dots, y_n) \in \mathcal{A}_{m,n,r}$  with  $f(x_1, \dots, x_{m+n}) \in \mathcal{A}_{m+n,0,r}$ .
- Let  $\sigma \in \Sigma_m$  be a permutation and let  $\sigma(x) = (x_{\sigma(1)}, \dots, x_{\sigma(m)})$ . If  $f \in \mathcal{A}_{m,n,r}$  then there exists  $f_\sigma \in \mathcal{A}_{m,n,r}$  such that  $f_\sigma(x, y) = f(\sigma(x), y)$ .
- If  $f \in \mathcal{A}_{m,n,r}$  then there exists  $g \in \mathcal{A}_{m-1,n,r}$  such that

$$g(x_1, \dots, x_{m-1}, y) = f(x_1, \dots, x_{m-1}, 0, y).$$

- If  $f \in \mathcal{A}_{m,n,r}$  then  $f(x_1/s_1, \dots, x_m/s_m, y_1/t_1, \dots, y_n/t_n) \in \mathcal{A}_{m,n,1}$ .

**2.9 Notation.** We denote by  $\mathcal{A}_{m,n}$  the algebra of germs at the origin of the elements of  $\mathcal{A}_{m,n,r}$ , for  $r$  a polyradius in  $(0, \infty)^{m+n}$ . When  $n = 0$  we write  $\mathcal{A}_m$  for  $\mathcal{A}_{m,0}$ . We will often denote the germ and a representative by the same letter.

We require all the functions in  $\mathcal{A}$  to satisfy the property described in the next definition, which mimics the property of real analytic germs of being analytic on a whole neighbourhood of the origin.

**2.10 Definition.** Let  $f(x, y) \in \mathcal{A}_{m,n,r}$ , for some  $m, n \in \mathbb{N}$  and  $r \in (0, \infty)^{m+n}$ . For  $a \in \hat{I}_{m,n,r}$ , put  $m' = |\{i : 1 \leq i \leq m, a_i = 0\}|$  and choose a permutation  $\sigma$  of  $\{1, \dots, m\}$  such that  $\sigma(\{i : 1 \leq i \leq m, a_i = 0\}) = \{1, \dots, m'\}$ . We say that  $f$  is  $\mathcal{A}$ -analytic if for all  $a \in \hat{I}_{m,n,r}$  there exists a germ  $g_a \in \mathcal{A}_{m',m+n-m'}$  such that the germ at  $a$  of  $f$  is equal to the germ at  $a$  of

$$(x, y) \mapsto g_a(x_{\sigma(1)} - a_{\sigma(1)}, \dots, x_{\sigma(m)} - a_{\sigma(m)}, y_1 - a_{m+1}, \dots, y_n - a_{m+n}).$$

The next definition establishes a relevant analogy with the behaviour of analytic germs, namely the fact that a germ in the collection  $\{\mathcal{A}_{m,n} : m, n \in \mathbb{N}\}$  is uniquely determined by its “generalised Taylor expansion”.

**2.11 Definition.** We say that  $\{\mathcal{A}_{m,n} : m, n \in \mathbb{N}\}$  is a collection of *quasianalytic algebras* if, for all  $m, n \in \mathbb{N}$ , there exists an **injective**  $\mathbb{R}$ -algebra morphism

$$\mathcal{T}_{m,n} : \mathcal{A}_{m,n} \rightarrow \mathbb{R}[[X^*, Y]],$$

where  $X = (X_1, \dots, X_m)$ ,  $Y = (Y_1, \dots, Y_n)$ . Moreover, for all  $m' \geq m$ ,  $n' \geq n$  we require that the morphism  $\mathcal{T}_{m',n'}$  extend  $\mathcal{T}_{m,n}$ , hence, from now on we will write  $\mathcal{T}$  for  $\mathcal{T}_{m,n}$ .

We require  $\{\mathcal{A}_{m,n} : m, n \in \mathbb{N}\}$  to be a collection of quasianalytic algebras which, together with the morphism  $\mathcal{T}$ , satisfies the list of closure and compatibility properties in 2.14 below. First, we need some definitions.

**2.12 Definition.** A number  $\alpha \in [0, \infty)$  is an *admissible exponent* if there are  $m, n \in \mathbb{N}$ ,  $f \in \mathcal{A}_{m,n}$ ,  $\beta \in \text{Supp}(\mathcal{T}(f)) \subset \mathbb{R}^m \times \mathbb{N}^n$  such that  $\alpha$  is a component of  $\beta$ . We denote by  $\mathbb{A}$  semi-ring generated by all admissible exponents and by  $\mathbb{K}$  the field of fractions of  $\mathbb{A}$ .

**2.13 Definition.** Let  $m, n \in \mathbb{N}$ ,  $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_n)$ . For  $m', n' \in \mathbb{N}$  with  $m' + n' = m + n$ , we set  $(x', y') = (x'_1, \dots, x'_{m'}, y'_1, \dots, y'_{n'})$ . Let  $r, r'$  be polyradii in  $\mathbb{R}^{m+n}$ . A *blow-up chart* is a map

$$\begin{aligned} \pi : \hat{I}_{m',n',r'} &\longrightarrow \hat{I}_{m,n,r} \\ (x', y') &\longmapsto (x, y) \end{aligned}$$

of either of the following forms:

- For  $1 \leq j < i \leq m$  and  $\lambda \in (0, \infty)$ , let  $m' = m - 1$  and  $n' = n + 1$  and define

$$\pi_{i,j}^\lambda(x', y') = (x, y), \quad \text{where} \quad \begin{cases} x_k = x'_k & 1 \leq k < i \\ x_i = x'_j(\lambda + y'_1) \\ x_k = x'_{k-1} & i < k \leq m \\ y_k = y'_{k+1} & 1 \leq k \leq n \end{cases}.$$

- For  $1 \leq j, i \leq m$ , let  $m' = m$  and  $n' = n$  and define

$$\pi_{i,j}^0(x', y') = (x, y), \quad \text{where} \quad \begin{cases} x_k = x'_k & 1 \leq k \leq m, \quad k \neq i \\ x_i = x'_j x'_i \\ y_k = y'_k & 1 \leq k \leq n \end{cases}$$

$$\text{and } \pi_{i,j}^\infty = \pi_{j,i}^0;$$

- For  $1 \leq i \leq n$ ,  $1 \leq j \leq m$  and  $\lambda \in \mathbb{R}$ , let  $m' = m$  and  $n' = n$  and define

$$\pi_{m+i,j}^\lambda(x', y') = (x, y), \quad \text{where} \quad \begin{cases} x_k = x'_k & 1 \leq k \leq m \\ y_i = x'_j(\lambda + y'_i) \\ y_k = y'_k & 1 \leq k \leq n, \quad k \neq i \end{cases}.$$

- For  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ , let  $m' = m + 1$  and  $n' = n - 1$  and define

$$\pi_{m+i,j}^{\pm\infty}(x', y') = (x, y), \quad \text{where} \quad \begin{cases} x_k = x'_k & 1 \leq k \leq m, \quad k \neq j \\ x_j = x'_{m+1} x'_j \\ y_k = y'_k & 1 \leq k < i \\ y_i = \pm x'_{m+1} \\ y_k = y'_{k-1} & i < k \leq n \end{cases}.$$



- For  $1 \leq i, j \leq n$  and  $\lambda \in \mathbb{R}$ , let  $m' = m$  and  $n' = n$  and define

$$\pi_{m+i, m+j}^\lambda(x', y') = (x, y), \quad \text{where} \quad \begin{cases} x_k = x'_k & 1 \leq k \leq m \\ y_i = y'_j(\lambda + y'_i) \\ y_k = y'_k & 1 \leq k \leq n, k \neq i, j \end{cases}$$

and  $\pi_{m+i, m+j}^\infty = \pi_{m+j, m+i}^0$ .

We also define the following collections:

$$\begin{aligned} &\text{for } 1 \leq i, j \leq m, \quad \pi_{i,j}^\lambda := \{\pi_{i,j}^\lambda : \lambda \in [0, \infty]\}, \\ &\text{for } 1 \leq i \leq n, \quad 1 \leq j \leq m, \quad \pi_{m+i,j}^\lambda := \{\pi_{m+i,j}^\lambda : \lambda \in \mathbb{R} \cup \{\pm\infty\}\}, \\ &\text{for } 1 \leq i, j \leq n, \quad \pi_{m+i, m+j}^\lambda := \{\pi_{i,j}^\lambda : \lambda \in \mathbb{R} \cup \{\infty\}\}. \end{aligned}$$

**2.14.** We require that the family of algebras of germs  $\{\mathcal{A}_{m,n} : m, n \in \mathbb{N}\}$  satisfy the following properties:

1. *Monomials and ramifications.* If  $\mathbb{A} \neq \mathbb{N}$ , then for every  $\alpha \in \mathbb{K}^{\geq 0}$ , the germ  $x_1 \mapsto x_1^\alpha$  is in  $\mathcal{A}_1$  and  $\mathcal{T}(x_1^\alpha) = X_1^\alpha$ . Moreover, if  $f \in \mathcal{A}_{m,n}$ , then  $g(x, y) := f(x_1^\alpha, x_2, \dots, x_m, y) \in \mathcal{A}_{m,n}$  and  $\mathcal{T}(g) = \mathcal{T}(f)(X_1^\alpha, X_2, \dots, X_m, Y)$ .
2. *Monomial division.* Let  $f \in \mathcal{A}_{m,n}$  and suppose that there exist  $\alpha \in \mathbb{K}$ ,  $n \in \mathbb{N}$  and  $G \in \mathbb{R}[[X^*, Y]]$  such that  $\mathcal{T}(f)(X, Y) = X_1^\alpha Y_1^n G(X, Y)$ . Then there exists  $g \in \mathcal{A}_{m,n}$  such that  $f(x, y) = x_1^\alpha y_1^n g(x, y)$ . It follows that  $\mathcal{T}(g) = G$ : in fact,  $\mathcal{T}(f) = \mathcal{T}(x_1^\alpha y_1^n g) = X_1^\alpha Y_1^n \mathcal{T}(g)$  and  $\mathcal{T}(f)$  is also equal to  $X_1^\alpha Y_1^n G$ .
3. *Permutations of the  $x$ -variables.* Let  $\sigma \in \Sigma_m$  be a permutation and  $f \in \mathcal{A}_{m,n}$ . Then  $\mathcal{T}(f_\sigma) = \mathcal{T}(f)_\sigma$ .
4. *Setting a variable equal to zero.* For  $f \in \mathcal{A}_{m,n}$ , we have

$$\mathcal{T}(f(x_1, \dots, x_{m-1}, 0, y)) = \mathcal{T}(f)(X_1, \dots, X_{m-1}, 0, Y).$$

5. *Composition in the  $y$ -variables.* Let  $g_1, \dots, g_n \in \mathcal{A}_{m', n'}$  with  $g_i(0) = 0$  and let  $f \in \mathcal{A}_{m,n}$ . Then  $h := f(x, g_1, \dots, g_n) \in \mathcal{A}_{m+m', n'}$  and  $\mathcal{T}(h) = \mathcal{T}(f)(X, \mathcal{T}(g_1), \dots, \mathcal{T}(g_n))$ .
6. *Implicit functions in the  $y$ -variables.* Let  $f \in \mathcal{A}_{m,n}$  and suppose that  $\frac{\partial f}{\partial y_n}(0)$  exists and is nonzero. Then there exists  $g \in \mathcal{A}_{m, n-1}$  such that  $f(x, y_1, \dots, y_{n-1}, g(x, y_1, \dots, y_{n-1})) = 0$ . It follows that

$$\mathcal{T}(f)(X, Y_1, \dots, Y_{n-1}, \mathcal{T}(g)(X, Y_1, \dots, Y_{n-1})) = 0.$$

7. *Blow-ups.* Let  $f \in \mathcal{A}_{m,n}$  and  $\pi : \hat{I}_{m', n', \infty} \rightarrow \hat{I}_{m, n, \infty}$  be a blow-up chart (see Definition 2.13). Then  $f \circ \pi \in \mathcal{A}_{m', n'}$  and  $\mathcal{T}(f \circ \pi) = \mathcal{T}(f) \circ \pi$ .

**2.15 Remarks.** Here is a list of consequences of the above properties.

*Closure under differentiation.*  $\mathcal{A}_{m,n}$  is closed under partial derivatives with respect to the  $y$ -variables and  $\mathcal{T}$  is compatible with this operation:  $f \in \mathcal{A}_{m,n} \Rightarrow \frac{\partial f}{\partial y_i} \in \mathcal{A}_{m,n}$  and  $\mathcal{T}\left(\frac{\partial f}{\partial y_i}\right) = \frac{\partial \mathcal{T}(f)}{\partial y_i}$ . In fact, consider the germ  $g(x, y, z_1) := f(x, \dots, y_i + z_1, \dots) - f(x, y) \in \mathcal{A}_{m, n+1}$ . Then  $g(x, y, 0) = 0$ , so  $\mathcal{T}(g)(x, y, 0) = 0$ ; hence  $\mathcal{T}(g)(x, y, z_1) = z_1 H(x, y, z_1)$ , for some series  $H$ . By monomial division, there exists  $h \in \mathcal{A}_{m, n+1}$  such that  $g(x, y, z_1) = z_1 h(x, y, z_1)$  and hence  $\frac{\partial f}{\partial y_i}(x, y) = h(x, y, 0) \in \mathcal{A}_{m,n}$ .

*Taylor expansion.* If  $f \in \mathcal{A}_{m,1}$ , then  $\mathcal{T}(f)(0, Y)$  is the Taylor expansion of  $f(0, y)$  with respect to  $y$ . In fact,  $\mathcal{T}(f) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i \mathcal{T}(f)}{\partial Y^i}(X, 0) Y^i = \sum_{i=0}^{\infty} \frac{1}{i!} \mathcal{T}\left(\frac{\partial^i f}{\partial y^i}(x, 0)\right) Y^i$ .

*Closure under  $\partial_i$ .* Let  $f \in \mathcal{A}_{m,n}$  and let  $\partial_i f$  be the germ at zero of  $x_i \frac{\partial f}{\partial x_i}$  (extended by continuity at zero), for  $i = 1, \dots, m$ . Then  $\partial_i f \in \mathcal{A}_{m,n}$  and  $\mathcal{T}(\partial_i f) = \partial_i \mathcal{T}(f)$ . In fact, consider

$$g(x, y, z) := f(x_1, \dots, x_{i-1}, x_i(1+z), x_{i+1}, \dots, x_m, y) \in \mathcal{A}_{m, n+1}.$$



Then,  $\frac{\partial g}{\partial z}(x, y, 0) \in \mathcal{A}_{m,n}$ . Notice that, for some  $r \in (0, \infty)^{m+n+1}$ , there is a representative of  $g$  (still denoted by  $g$ ), such that

$$\frac{\partial g}{\partial z}(x, y, z) = x_i \frac{\partial f}{\partial x_i}(x_1, \dots, x_i(1+z), \dots, x_m, y) \text{ on } I_{m,n+1,r}.$$

Hence,  $\partial_i f = \frac{\partial g}{\partial z}(x, y, 0)$ . The compatibility with  $\mathcal{T}$  follows easily.

*Closure under homothety.* Let  $f \in \mathcal{A}_m$  and  $\lambda \in (0, \infty)$ . Then  $g(x) = f(\lambda x_1, x_2, \dots, x_m) \in \mathcal{A}_m$ . In fact, let  $F(x_1, z, x_2, \dots, x_m) = f(x) \in \mathcal{A}_{m+1}$ . Using the appropriate blow-up chart involving  $x_1$  and  $z$ , we obtain  $G(x_1, z, x_2, \dots, x_m) = F(x_1, x_1(\lambda + z), x_2, \dots, x_m) \in \mathcal{A}_{m+1}$  and finally  $g(x) = G(x_1, 0, x_2, \dots, x_m) \in \mathcal{A}_m$ .

*$\mathcal{C}^\infty$  germs.* Let  $f \in \mathcal{A}_1$  be  $\mathcal{C}^\infty$  at zero and suppose that  $f(0) = 0$ . Then  $f \in \mathcal{A}_{0,1}$ , i.e. the exponents appearing in  $\mathcal{T}(f)$  are natural numbers. In fact, let  $\alpha \in (0, \infty) \setminus \mathbb{N}$  be the smallest exponent of  $\mathcal{T}(f)$  which is not natural. Then there exists  $g \in \mathcal{A}_1$  such that  $f(x) - c_1 x^{n_1} - \dots - c_k x^{n_k} = x^\alpha g(x)$  and  $g(0) \neq 0$ . Since  $f$  is  $\mathcal{C}^\infty$  at zero, it must be  $\alpha \in \mathbb{N}$ .

*Binomial formula.* Let  $\alpha \in \mathbb{K}$ . Then  $(1+y)^\alpha \in \mathcal{A}_{0,1}$ . In fact, suppose first that  $\alpha > 0$ . Notice that  $(1+y)^\alpha$  is an analytic germ and its Taylor expansion is  $\sum_{n \in \mathbb{N}} \binom{\alpha}{n} y^n$ ; let  $g(x, y) = y^\alpha \in \mathcal{A}_2$ . By composing a ramification with a suitable blow-up chart, we obtain  $g_1(x, y) = x^\alpha (1+y)^\alpha \in \mathcal{A}_2$  and  $\mathcal{T}(g_1)(X, Y) = X^\alpha (1+Y)^\alpha$ . By monomial division, there exists  $h \in \mathcal{A}_1$  such that  $g_1(x, y) = x^\alpha h(y)$  and  $\mathcal{T}(h)(Y) = \sum_{n \in \mathbb{N}} \binom{\alpha}{n} Y^n$ . Hence,  $h(y) = (1+y)^\alpha \in \mathcal{A}_{0,1}$ , by the previous remark. Now notice that  $(1+y)^{-\alpha} - 1$  is the solution of the implicit function problem  $(1+y)^\alpha (1+z) - 1 = 0$ , and hence  $(1+y)^{-\alpha} \in \mathcal{A}_{0,1}$ .

*Units.* A *unit* of  $\mathcal{A}_m$  is an invertible element or, equivalently, a germ  $U \in \mathcal{A}_m$  such that  $U(0) \neq 0$ . We claim that, for all  $\alpha \in \mathbb{K}$ ,  $U(x)^\alpha \in \mathcal{A}_m$ . In fact, we may suppose  $U(0) = 1$  and write  $U(x) = 1 + \varepsilon(x)$ , where  $\varepsilon \in \mathcal{A}_m$  and  $\varepsilon(0) = 0$ . By composition and the previous remark,  $(1 + \varepsilon(x))^\alpha \in \mathcal{A}_m$ .

**2.16 Definition.** Let  $f \in \mathcal{A}_m$ . We say that  $f$  is *normal* if there exist  $\alpha \in \mathbb{A}^m$  and  $u \in \mathcal{A}_m$  such that  $f(x) = x^\alpha u(x)$  and  $u(0) \neq 0$ .

**2.17 Lemma.** Let  $f \in \mathcal{A}_N$  and  $G = (g_1, \dots, g_N) \in \mathcal{A}_M^N$ , and suppose that the  $g_i$  are all normal. Then  $f \circ G \in \mathcal{A}_M$ .

*Proof.* We will only treat the case  $N = 1$ ,  $M = 2$ , the proof in the general case being a straightforward generalisation. Let  $g(x, y) = x^\alpha y^\beta U(x, y)$ , where  $U \in \mathcal{A}_2$  is a unit.

First, suppose that  $\alpha = \beta = 0$ . Up to homothety, we may assume  $U(0, 0) = 1$  and write  $U(x, y) = 1 + \varepsilon(x, y)$ , with  $\varepsilon \in \mathcal{A}_2$  and  $\varepsilon(0, 0) = 0$ . By  $\mathcal{A}$ -analyticity of  $f$  (see Definition 2.10), there exists  $\varphi \in \mathcal{A}_{0,1}$  such that  $\varphi(x) = f(x+1)$ . Since  $\mathcal{T}(\varphi)$  has only integer exponents, we may apply Axiom 5 of 2.14 and deduce that the germ of  $x \mapsto \varphi(\varepsilon(x, y)) \in \mathcal{A}_2$ .

Now suppose that  $\alpha > 0$ . Let  $F(u, x) = f(u) \in \mathcal{A}_2$ . By Axioms 1 and 7 of 2.14,  $F(x^\alpha(u+1), x)$  belongs to  $\mathcal{A}_2$ . By Axiom 4 of 2.14, the germ  $f_1 : x \mapsto F(x^\alpha, x) \in \mathcal{A}_1$ . This shows that for every germ  $\tilde{f} \in \mathcal{A}_{n+1}$ , we have  $\tilde{f}(x^\alpha, y_1, \dots, y_n) \in \mathcal{A}_{n+1}$ . Let  $\beta > 0$  and  $F_1(x, y) = f_1(x) \in \mathcal{A}_2$ . By Axioms 1 and 7 of 2.14, the germ of  $h : (x, y) \mapsto F_1(y^\beta x, x) \in \mathcal{A}_2$ . Now, by what we have just proved, the germ  $h_1 : (x, y) \mapsto h(x, y^{1/\alpha}) \in \mathcal{A}_2$ . Notice that  $h_1(x, y) = f(x^\alpha y^\beta)$ . Recall that, by Remarks 2.15,  $U^{1/\beta} \in \mathcal{A}_2$ . We may assume  $U^{1/\beta}(0, 0) = 1$  and write  $U^{1/\beta}(x, y) = 1 + \varepsilon(x, y)$ , with  $\varepsilon \in \mathcal{A}_2$  and  $\varepsilon(0, 0) = 0$ . Let  $H(x, y, z) = h_1(x, z) \in \mathcal{A}_3$ . By Axiom 7 of 2.14, the germ  $H_1 : (x, y, z) \mapsto H(x, y, y(1+z)) \in \mathcal{A}_{2,1}$ . By Axiom 5 of 2.15, the germ  $h_2 : (x, y) \mapsto H_1(x, y, \varepsilon(x, y)) \in \mathcal{A}_2$ . Since  $h_2(x, y) = f(x^\alpha y^\beta U(x, y))$ , we are done.  $\square$

## 2.3 The theorems

**2.18 Proviso.** For the rest of the paper  $\mathcal{A} = \{\mathcal{A}_{m,n,r} : m, n \in \mathbb{N}, r \in (0, \infty)^{m+n}\}$  will be a collection of algebras of functions as in the previous subsection, i.e. such that:

- $\mathcal{A}$  satisfies the conditions in 2.8.
- Every function in the collection in  $\mathcal{A}$ -analytic (see Definition 2.10).

- $\{\mathcal{A}_{m,n} : m, n \in \mathbb{N}\}$  is a collection of quasianalytic algebras of germs (see Definition 2.11).
- The collection  $\{\mathcal{A}_{m,n} : m, n \in \mathbb{N}\}$  and the morphism  $\mathcal{T}$  satisfy the closure and compatibility properties in 2.14.

**2.19 Definition.** Let  $\mathcal{A}$  be as above.

For  $f \in \mathcal{A}_{m,n,1}$  we define  $\tilde{f} : \mathbb{R}^{m+n} \rightarrow \mathbb{R}$  as

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in I_{m,n,1} \\ 0 & \text{otherwise} \end{cases}.$$

Let  $\mathcal{L}_{\mathcal{A}}$  be the language of ordered rings  $\{<, 0, 1, +, -, \cdot\}$  augmented by a new function symbol for each function  $\tilde{f}$ . Let  $\mathbb{R}_{\mathcal{A}}$  be the real ordered field with its natural  $\mathcal{L}_{\mathcal{A}}$ -structure. Let  $^{-1}$  be a function symbol for  $x \mapsto \frac{1}{x}$ , where  $0^{-1} = 0$  by convention, and, for  $n \in \mathbb{N}$ , let  $\sqrt[n]{\phantom{x}}$  be a function symbol for the function

$$x \mapsto \begin{cases} x^{1/n} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

**Theorem A.** *The structure  $\mathbb{R}_{\mathcal{A}}$  is model-complete, o-minimal and polynomially bounded. Its field of exponents<sup>1</sup> is the field  $\mathbb{K}$ , defined in Definition 2.12.*

**Theorem B.** *The (natural) expansion of the structure  $\mathbb{R}_{\mathcal{A}}$  to the language  $\mathcal{L}_{\mathcal{A}} \cup \{^{-1}\} \cup \{\sqrt[n]{\phantom{x}} : n \in \mathbb{N}\}$  admits quantifier elimination.*

*2.20 Remark.* The choice of putting in  $\mathcal{L}_{\mathcal{A}}$  a function symbol only for representatives on the unit box is not binding; we could have put a function symbol for every function in  $\mathcal{A}_{m,n,r}$  and dispensed with the last condition in 2.8, and the two above theorems would still be true.

## 2.4 Examples

Most known examples of o-minimal polynomially bounded expansions of the real field can be generated by a family  $\mathcal{A}$  of algebras satisfying the requirements of Proviso 2.18. In particular, this is true for all the structures mentioned in the introduction.

a) In [DS98] the authors consider, for every polyradius  $r$ , the sub-algebra  $\mathbb{R}\{X^*, Y\}_r$  of  $\mathbb{R}[[X^*, Y]]$  consisting of all generalised power series which converge on  $\hat{I}_{m,n,r}$  (see [DS98, p. 4377]). The morphism  $\mathcal{T}$  in this case is the inclusion  $\mathbb{R}\{X^*, Y\}_r \hookrightarrow \mathbb{R}[[X^*, Y]]$ , so the quasianalyticity property is obvious, and  $\mathcal{A}$ -analyticity is proved in [DS98, Corollary 6.7].

b) In [DS00] the authors consider a family of algebras  $\mathcal{G}_m$  of Gevrey functions in  $m$  variables (see [DS00, Definition 2.20]). The morphism  $\mathcal{T}$  is the Taylor map at zero, the quasianalyticity property follows from a fundamental result in multisummability theory (see [DS00, Proposition 2.18]) and  $\mathcal{A}$ -analyticity is proved in [DS00, Lemma 4.8].

c) In [RSW03] the authors consider a family of quasianalytic Denjoy-Carleman algebras  $\mathcal{C}_B(M)$ , where  $B$  is a box in  $\mathbb{R}^n$  and  $M = (M_1, M_2, \dots)$  is an increasing sequence of positive constants (see [RSW03, p.751]). The morphism  $\mathcal{T}$  is the Taylor map at zero, the quasianalyticity property is equivalent to the condition  $\sum_{i=0}^{\infty} \frac{M_i}{M_{i+1}} = \infty$  and  $\mathcal{A}$ -analyticity is automatically verified, since these algebras are closed under translation.

d) In [RSS07] the authors consider a solution  $H = (H_1, \dots, H_r) : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^r$  of a first order analytic differential equation which is singular at the origin and they construct the smallest collection  $\mathcal{A}_H$  of algebras of germs containing the germ of  $H$  and all the analytic germs, and closed under (a subset of) the operations implicit in 2.14. They consider the family of algebras of functions consisting of all  $\mathcal{A}$ -analytic representatives of the germs in  $\mathcal{A}_H$  (it is proved in [RSS07, Lemma 3.4] that every germ in  $\mathcal{A}_H$  has an  $\mathcal{A}$ -analytic representative) and let the morphism  $\mathcal{T}$  be the Taylor expansion at zero. The quasianalyticity property is proved in [RSS07, Theorem 3.5] and  $\mathcal{A}$ -analyticity is automatic by construction.

<sup>1</sup>Recall that the field of exponents of a polynomially bounded o-minimal structure is the set of all  $\alpha \in \mathbb{R}$  such that the function  $x \mapsto x^\alpha$  is definable. It is indeed a field (see for example [Mil96]).

e) Finally, in [KRS09] the authors consider a family of algebras  $\mathcal{Q}_m^{m+n}(U)$  of real functions which have a holomorphic extension to a so-called “quadratic domain”  $U \subseteq \mathbf{L}^{m+n}$ , where  $\mathbf{L}$  is the Riemann surface of the logarithm (see [KRS09, Definition 5.1]). These algebras contain the *Dulac transition maps* of real analytic planar vector fields in a neighbourhood of hyperbolic non-resonant singular points. The morphism  $\mathcal{T}$  defined in [KRS09, Definition 2.6] associates to the germ  $f$  of a function  $\mathcal{Q}_m^{m+n}$  an *asymptotic expansion*  $\mathcal{T}(f) \in \mathbb{R}[[X^*, Y]]$ . The quasianalyticity property is proved in [KRS09, Proposition 2.8] and  $\mathcal{A}$ -analyticity is proved in [KRS09, Corollary 3.7]. For all the above examples the closure and compatibility properties in 2.14 are also proven in the mentioned papers.

**2.21 Remark.** Let  $f : \hat{I}_{m_0, n_0, r_0} \rightarrow \mathbb{R}$  be a *weak- $\mathcal{C}^\infty$*  function (in the sense of [Mil95]), i.e. for every point  $a \in \hat{I}_{m_0, n_0, r_0}$  and for every  $k \in \mathbb{N}$ , the germ of  $f$  at  $a$  has a  $\mathcal{C}^k$  representative. Suppose furthermore that  $f$  is definable in a polynomially bounded expansion of the real field. Let  $\{\mathcal{A}_{m,n} : m, n \in \mathbb{N}\}$  be smallest collection of  $\mathbb{R}$ -algebras of germs containing the germs at zero of the functions  $f_a(x) := f(x+a)$  (for all  $a \in \hat{I}_{m_0, n_0, r_0}$ ) and of the coordinate functions, and satisfying the closure properties in 2.14. Let  $\mathcal{A} = \{\mathcal{A}_{m,n,r}\}$  be the collection of the  $\mathbb{R}$ -algebras made of all  $\mathcal{A}$ -analytic representatives of the germs in  $\{\mathcal{A}_{m,n} : m, n \in \mathbb{N}\}$ . It is easy to check (see for example [RSS07, Lemma 3.4]) that every germ in  $\mathcal{A}_{m,n}$  has an  $\mathcal{A}$ -analytic representative. Clearly,  $\mathcal{A}$  satisfies the first (see Remark 2.20) and the last two conditions in Proviso 2.18. Finally, it follows from [Mil95, Corollary 2] that  $\{\mathcal{A}_{m,n} : m, n \in \mathbb{N}\}$  is a collection of quasianalytic algebras. Hence, the structure  $\mathbb{R}_{\mathcal{A}}$  satisfies Theorems A and B.

In analogy with [Gab96] and [RSW03], we deduce from Theorem A the following explicit model-completeness result.

**2.22 Corollary.** *Let  $f : \hat{I}_{m_0, n_0, r_0} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^\infty$  function definable in some polynomially bounded expansion of the real field. Let  $\mathcal{L}_f$  be the language of ordered rings  $\{<, 0, 1, +, -, \cdot\}$  augmented by function symbols for  $f$  and for each partial derivative of  $f$  and by a constant symbol for every real number. Then the natural  $\mathcal{L}_f$ -expansion  $\mathbb{R}_f$  of the real field is model-complete.*

*Proof.* Let us construct  $\mathcal{A}$  as in Remark 2.21. It is easy to see that the structures  $\mathbb{R}_{\mathcal{A}}$  and  $\mathbb{R}_f$  have the same 0-definable sets. We claim that the primitives of  $\mathbb{R}_{\mathcal{A}}$  are existentially definable. To see this, the only non-trivial observation to make is that, if  $g \in \mathcal{A}_{m,n,r}$ , and  $h \in \mathcal{A}_{m,n,r}$  is obtained from  $g$  by monomial division, e.g.  $g(x) = x_1 \cdot h(x)$  (where  $x = (x_1, \dots, x_{m+n})$ ), then the graph of  $h$  is the set

$$\left\{ (x, y) : (x_1 \neq 0 \wedge x_1 \cdot y = g(x)) \vee \left( x = 0 \wedge y = \frac{\partial g}{\partial x_1}(x) \right) \right\},$$

which is existentially definable from the graph of  $g$ . □

## 2.5 Model-completeness of Euler’s Gamma function

The following application was obtained in collaboration with Gareth Jones.

**2.23 Corollary.** *Let  $\mathcal{L}_\Gamma$  be the language of ordered rings  $\{<, 0, 1, +, -, \cdot\}$  augmented by function symbols for  $\Gamma \upharpoonright (0, \infty)$  and for each derivative of  $\Gamma \upharpoonright (0, \infty)$  and by a constant symbol for every real number. Then the natural  $\mathcal{L}_\Gamma$ -expansion  $\mathbb{R}_\Gamma$  of the real field is model-complete.*

*Proof.* Let  $\psi$  be defined as in [DS00, Example 8.1]. In particular, recall that, by Binet’s second formula, we have, for all  $x \in (1, \infty)$ ,

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right) \log x + \frac{1}{2} \log(2\pi) + \psi\left(\frac{1}{x}\right).$$

As remarked in [DS00, Example 8.1], the functions  $\psi \upharpoonright (0, 1)$  and  $\Gamma \upharpoonright (0, 1)$  are both definable in the polynomially bounded o-minimal structure  $\mathbb{R}_{\mathcal{G}}$ . Moreover,  $\psi \upharpoonright (0, 1)$  is  $\mathcal{C}^\infty$  at zero and  $1/\Gamma \upharpoonright (0, 1)$  is analytic at zero, hence by Corollary 2.22, the expansion  $\mathcal{R}$  of the real field by the functions  $\exp \upharpoonright (0, 1)$ ,  $\psi \upharpoonright (0, 1)$  and  $\Gamma \upharpoonright (0, 1)$  and their derivatives is model-complete. By [DS00, Theorem

B], the structure  $\langle \mathcal{R}, \exp \rangle$  is model-complete. Now, using Legendre's Duplication Formula (see for example [EMOT81, p.5])

$$\Gamma(x) \Gamma\left(x + \frac{1}{2}\right) = 2^{1-2x} \sqrt{\pi} \Gamma(2x),$$

it is easy to show that the structures  $\langle \mathcal{R}, \exp \rangle$  and  $\mathbb{R}_\Gamma$  have the same 0-definable sets and that the primitives of  $\langle \mathcal{R}, \exp \rangle$  are existentially definable in  $\mathbb{R}_\Gamma$ .  $\square$

### 3 Monomialisation

The aim of this section is to define a class of transformations of  $\mathbb{R}^{m+n}$ , which we call admissible, which are bijective outside a set with empty interior, and modulo which we may assume that the germs in  $\mathcal{A}_{m,n}$  are normal (see Definition 2.16). More precisely, in Subsection 3.2 we develop a monomialisation algorithm for generalised power series and in Subsection 3.3 we use quasianalyticity and the compatibility properties of the morphism  $\mathcal{T}$  in 2.14 to deduce a monomialisation result for germs.

#### 3.1 Admissible transformations and admissible trees

**3.1 Definition.** Let  $m, n \in \mathbb{N}$ ,  $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_n)$ . For  $m', n' \in \mathbb{N}$  with  $m' + n' = m + n$ , we set  $(x', y') = (x'_1, \dots, x'_{m'}, y'_1, \dots, y'_{n'})$ . Let  $r, r'$  be polyradii in  $\mathbb{R}^{m+n}$ . An *elementary transformations* of  $\mathbb{R}^{m+n}$  is a map  $\nu : \hat{I}_{m', n', r'} \rightarrow \hat{I}_{m, n, r}$  of either of the following types.

- A *blow-up chart* (see Definition 2.13), i.e. an element  $\pi : \hat{I}_{m', n', r'} \rightarrow \hat{I}_{m, n, r}$  of either of the collections  $\pi_{i,j}, \pi_{m+i,j}, \pi_{m+i,m+j}$ .
- A *Tschirnhausen translation*: let  $m = m', n = n'$ , let  $r'' = (s'_1, \dots, s'_m, t'_1, \dots, t'_{n-1})$  and  $h \in \mathcal{A}_{m, n-1, r''}$  with  $h(0) = 0$  and set

$$\tau_h(x', y') = (x, y), \quad \text{where} \quad \begin{cases} x_k = x'_k & 1 \leq k \leq m \\ y_n = y'_n + h(x'_1, \dots, x'_m, y'_1, \dots, y'_{n-1}) \\ y_k = y'_k & 1 \leq k \leq n-1 \end{cases}.$$

- A *linear transformation*: let  $m = m', n = n'$ , let  $1 \leq i \leq n$  and  $c \in \mathbb{R}$ , and set

$$L_{i,c}(x', y') = (x, y), \quad \text{where} \quad \begin{cases} x_k = x'_k & 1 \leq k \leq m \\ y_i = y'_i \\ y_k = y'_k + cy'_i & 1 \leq k \leq n, k \neq i \end{cases}.$$

- A *ramification* is either of the following maps: let  $m = m', n = n'$ ,

for  $1 \leq i \leq m$  and  $\gamma \in \mathbb{K}^{>0}$  (see Definition 2.12),

$$r_i^\gamma(x', y') = (x, y), \quad \text{where} \quad \begin{cases} x_k = x'_k & 1 \leq k \leq m, k \neq i \\ x_i = x_i'^\gamma \\ y_k = y_k & 1 \leq k \leq n \end{cases}$$

and for  $1 \leq i \leq n$  and  $d \in \mathbb{N}$ ,

$$r_{m+i}^{d,\pm}(x', y') = (x, y), \quad \text{where} \quad \begin{cases} x_k = x'_k & 1 \leq k \leq m \\ y_i = \pm y_i'^d \\ y_k = y_k & 1 \leq k \leq n, k \neq i \end{cases}.$$

*3.2 Remark.* Notice that, by 2.14 and Lemma 2.17, the components of an elementary transformation  $\nu : \hat{I}_{m',n',r'} \rightarrow \hat{I}_{m,n,r}$  are elements of  $\mathcal{A}_{m',n',r'}$ . Moreover, it follows from the axioms in 2.14 that, if  $h \in \mathcal{A}_{m,n}$ , then  $h \circ \nu \in \mathcal{A}_{m',n'}$ .

**3.3 Definition.** Let  $m, n, m', n', N \in \mathbb{N}$  with  $m' + n' = m + n$  and let  $\nu_i : \hat{I}_{m'_i, n'_i, r'_i} \rightarrow \hat{I}_{m_i, n_i, r_i}$  be elementary transformations (for  $i = 1, \dots, N$ ) with  $m_1 = m$ ,  $n_1 = n$ ,  $m'_N = m'$ ,  $n'_N = n'$  and  $m_i + n_i = m'_i + n'_i = m + n$ . If  $N > 1$ , then in order for the composition  $\nu_1 \circ \dots \circ \nu_N$  to be well-defined it is enough that  $m_i \geq m'_{i-1}$ ,  $n_i \leq n'_{i-1}$  and  $r_i \leq r'_{i-1}$  for all  $i = 1, \dots, N$ . A map  $\rho : \hat{I}_{m',n',r'} \rightarrow \hat{I}_{m,n,r}$  is called an *admissible transformation* if  $\rho = \nu_1 \circ \dots \circ \nu_N$  and moreover, if  $N > 1$ , then  $m_i = m'_{i-1}$  and  $n_i = n'_{i-1}$  for all  $i = 1, \dots, N$ . The number  $N$  is called the *length* of the admissible transformation  $\rho$ .

*3.4 Remark.* By Remark 3.2 and by induction on the length of  $\rho$ , it is easy to see that the components of the admissible transformation  $\rho : \hat{I}_{m',n',r'} \rightarrow \hat{I}_{m,n,r}$  are elements of  $\mathcal{A}_{m',n',r'}$  and that  $\rho$  induces an algebra morphism

$$\begin{aligned} \rho : \mathcal{A}_{m,n} &\longrightarrow \mathcal{A}_{m',n'} . \\ f &\longmapsto f \circ \rho \end{aligned}$$

**3.5 Lemma.** An elementary transformation  $\nu : \hat{I}_{m',n',r'} \rightarrow \hat{I}_{m,n,r}$  induces an injective algebra homomorphism

$$\begin{aligned} T_\nu : \mathbb{R}[[X^*, Y]] &\longrightarrow \mathbb{R}[[X'^*, Y']] , \\ F &\longmapsto F \circ \nu \end{aligned}$$

where we set

$$F \circ \tau_h(X', Y') := F(X', Y'_1, \dots, Y'_{n-1}, Y'_n + \mathcal{T}(h)(X', Y'_1, \dots, Y'_{n-1})) .$$

Moreover, if  $F \in \mathbb{R}[[X^*, Y]] \cap \text{Im}(\mathcal{T})$ , then  $F \circ \nu \in \mathbb{R}[[X'^*, Y']] \cap \text{Im}(\mathcal{T})$ .

*Proof.* It is clear that  $T_\nu$  is a homomorphism, which preserves being a member of  $\text{Im}(\mathcal{T})$  by the properties in 2.14. If  $\nu$  is either a linear transformation, or a Tschirnhausen translation or a ramification, then  $\nu : (X', Y') \mapsto (X, Y)$  is a bijective change of coordinates, hence  $T_\nu$  is injective. It remains to show injectivity when  $\nu$  is a blow-up chart. In order to do this, let  $u, v, w_1, \dots, w_l$  be variables, with  $\bar{w} = (w_1, \dots, w_l)$ , and, for  $\lambda \geq 0$ , consider the map  $\pi^\lambda : (u, v, \bar{w}) \mapsto (u, u(\lambda + v), \bar{w})$ . For  $F \in \mathbb{R}[[u, v, \bar{w}]^*] \setminus \{0\}$ , we prove that  $F \circ \pi^\lambda \neq 0$ . Write  $F(u, v, \bar{w}) = \sum_{\alpha, \beta} a_{\alpha, \beta}(\bar{w}) u^\alpha v^\beta$ , where  $a_{\alpha, \beta} \in \mathbb{R}[[\bar{w}^*]]$ . Let us regroup the homogeneous terms as follows:

$$F(u, v, \bar{w}) = \sum_{\gamma} Q_{\gamma}(u, v, \bar{w}) \text{ where } Q_{\gamma}(u, v, \bar{w}) = \sum_{\alpha + \beta = \gamma} a_{\alpha, \beta}(\bar{w}) u^\alpha v^\beta .$$

It follows from the well-order properties of the support (see for example [DS98, Lemma 4.2 (2)]) that  $Q_{\gamma}$  is a finite sum, hence let us rewrite  $Q_{\gamma} = \sum_{i=1}^q c_{\beta_i} u^{\gamma - \beta_i} v^{\beta_i}$ , where  $c_{\beta_i} = a_{\gamma - \beta_i, \beta_i}(\bar{w})$  and  $\beta_1 < \dots < \beta_q$ .

Let us first consider the case  $\lambda = 0$ . Suppose that  $0 \equiv F \circ \pi^0(u, v, \bar{w}) = \sum_{\gamma} u^{\gamma} \sum_{\alpha + \beta = \gamma} a_{\alpha, \beta}(\bar{w}) v^{\beta}$ .

Then for every  $\gamma$  we have that  $a_{\alpha, \beta} \equiv 0$  whenever  $\alpha + \beta = \gamma$ , hence  $F \equiv 0$ .

Now suppose  $\lambda > 0$  and assume that

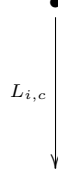
$$0 \equiv F \circ \pi^{\lambda}(u, v, \bar{w}) = \sum_{\gamma} u^{\gamma} \sum_{\alpha + \beta = \gamma} a_{\alpha, \beta}(\bar{w}) \sum_{k=0}^{\infty} \binom{\beta}{k} \lambda^{\beta-k} v^k .$$

Then for every  $\gamma$  the series

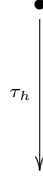
$$Q_{\gamma}(1, \lambda + v, \bar{w}) = \sum_{k=0}^{\infty} \left( \sum_{i=1}^q c_{\beta_i} \binom{\beta_i}{k} \lambda^{\beta_i - k} \right) v^k \in \mathbb{R}[[\bar{w}^*, v]]$$

is identically zero. It follows that  $\sum_{i=1}^q c_{\beta_i} \lambda^{\beta_i} \binom{\beta_i}{k} = 0 \forall k \in \mathbb{N}$ . Now, it is not difficult to see that there exist  $j_1, \dots, j_q \in \mathbb{N}$  such that the determinant of the linear system  $\left\{ \sum_{i=1}^q \binom{\beta_i}{j_s} Z_i = 0 \text{ for } s = 1, \dots, q \right\}$  is nonzero and hence the only solution is  $Z_1 = \dots = Z_q = 0$ . It follows that for every  $\gamma$  we have that  $a_{\alpha, \beta} \equiv 0$  whenever  $\alpha + \beta = \gamma$ , hence  $F \equiv 0$ .  $\square$

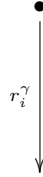
**3.6 Definition.** An *elementary tree* has either of the following forms: a vertex  $\bullet$ , or



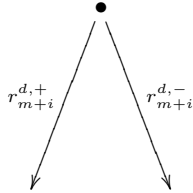
where  $L_{i,c}$  is a linear transformation, or



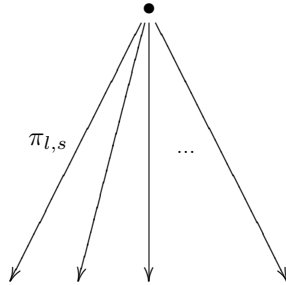
where  $\tau_h$  is a Tschirnhausen transformation, or



where  $r_i^\gamma$  is a ramification of the first type, or



where  $r_{m+i}^{d,+}$ ,  $r_{m+i}^{d,-}$  are ramifications of the second type, or



where the pair  $(l, s)$  is of the form  $(i, j)$ ,  $(m + i, j)$  or  $(m + i, m + j)$  (corresponding to the three types of blow-up transformation) and each branch corresponds to an element of the collection  $\pi_{l,s}$ .

A *tree* is defined inductively as follows: a tree of *height* zero is a vertex  $\bullet$ ; a tree of height  $\leq N$  is obtained from a tree  $T$  of height  $\leq N - 1$  by adjoining an elementary tree to the end of each branch of  $T$ . The height of  $T$  will be denoted by  $h(T)$ . A branch  $\mathfrak{b}$  of a tree  $T$  can be represented as an ordered tuple (from the vertex to the end of the branch) of elementary transformations  $(\nu_1, \dots, \nu_N)$ .

An *admissible tree* is a tree  $T$  such that for each branch  $\mathfrak{b} = (\nu_1, \dots, \nu_N)$  of  $T$ , the map  $\rho_{\mathfrak{b}} = \nu_1 \circ \dots \circ \nu_N$  is an admissible transformation. It follows that  $\mathfrak{b}$  induces a homomorphism  $T_{\mathfrak{b}} : \mathbb{R}[[X^*, Y]] \rightarrow \mathbb{R}[[X'^*, Y']]$ , by setting  $T_{\mathfrak{b}}(F) = F \circ \nu_1 \circ \dots \circ \nu_N$ .

*3.7 Remarks.*

1. Let  $T$  be an admissible tree and  $\mathfrak{b}$  be one of its branches. Since  $T_{\mathfrak{b}}$  is a homomorphism, if  $U$  is a unit, then  $T_{\mathfrak{b}}(U)$  is also a unit.
2. Let  $X = (X_1, \dots, X_m)$ ,  $Y = (Y_1, \dots, Y_n)$ . It is easy to see that, if  $T$  is an admissible tree and  $\mathcal{F} \subset \mathbb{R}[[X^*, Y]]$  is a family with good total support, then, for every branch  $\mathfrak{b}$  of  $T$ , the family  $T_{\mathfrak{b}}(\mathcal{F})$  still has good total support. More precisely, let  $F \in \mathbb{R}[[X^*, Y]]$  and  $H \in \mathbb{R}[[X^*, \hat{Y}]]$ , where  $Y = (Y_1, \dots, Y_n)$  and  $\hat{Y} = (Y_1, \dots, Y_{n-1})$ ; suppose  $\text{Supp}_X(F) \subseteq S_1 \times \dots \times S_m$  and  $\text{Supp}_X(H) \subseteq S'_1 \times \dots \times S'_m$ , where  $S_i, S'_i \subset [0, \infty)$  are well ordered sets. Then we have  $\text{Supp}_X(L_{i,c}(F)) = \text{Supp}_X(r_{m+i}^{\pm}(F)) = \text{Supp}_X(F)$  and  $\text{Supp}_X(\tau_h(F)) \subseteq \tilde{S}_1 \times \dots \times \tilde{S}_m$ , with  $\tilde{S}_k = \{a + nb : a \in S_k, b \in S'_k, n \in \mathbb{N}\}$ . Moreover,  $\text{Supp}_X(r_i^{\gamma}(F)) \subseteq \tilde{S}_1 \times \dots \times \tilde{S}_m$ , with  $\tilde{S}_i = \{\gamma a : a \in S_i\}$  and  $\tilde{S}_k = S_k$  for  $k \neq i$ . Finally, for  $1 \leq i, j \leq m$  and  $\lambda \in [0, \infty)$ , we have  $\text{Supp}_X(\pi_{i,j}^{\lambda}(F)) \subseteq \tilde{S}_1 \times \dots \times \tilde{S}_m$ , with  $\tilde{S}_j = \{a + b : a \in S_j, b \in S_i\}$  and  $\tilde{S}_k = S_k$  for  $k \neq j$  (the argument for the other types of blow-up transformation is similar).
3. Let  $T$  be an admissible tree and  $\mathfrak{b}$  be one of its branches, inducing a homomorphism  $T_{\mathfrak{b}} : \mathbb{R}[[X^*, Y]] \rightarrow \mathbb{R}[[X'^*, Y']]$ . It follows from Lemma 3.5 that if  $F \in \mathbb{R}[[X^*, Y]] \setminus \{0\} \cap \text{Im}(\mathcal{T})$ , then  $T_{\mathfrak{b}}(F) \in \mathbb{R}[[X'^*, Y']] \setminus \{0\} \cap \text{Im}(\mathcal{T})$ .

## 3.2 Monomialisation of generalised power series

**3.8 Definition.** A series  $F \in \mathbb{R}[[X^*]] \setminus \{0\}$  is *normal* if there exist  $\alpha \in \mathbb{A}^m$  and an invertible series  $U \in \mathbb{R}[[X^*]]$  such that  $F(X) = X^{\alpha}U(X)$ .

In analogy with [BM88, Lemma 4.7] and [RSW03, Lemma 2.2], we have the following result.

**3.9 Lemma.**

1. The series  $F_1, \dots, F_k \in \mathbb{R}[[X^*]] \setminus \{0\}$  are all normal if and only if the series  $\prod_{i=1}^k F_i$  is normal.
2. If  $F_1, F_2$  and  $F_1 - F_2$  are normal, then either  $F_1|F_2$  or  $F_2|F_1$ .

*Proof.* Regarding 1., suppose that  $\prod F_i(X)$  is normal. Then there exist two disjoint subsets  $Z$  and  $W$  of the set  $X$  of all the variables, and multi-indices  $\alpha, \beta$ , and series  $U(X), \tilde{F}_1(X), \dots, \tilde{F}_k(X)$  such that

$$Z^{\alpha} \prod \tilde{F}_i(X) = W^{\beta} U(X),$$

where all the components of  $\alpha$  and  $\beta$  are strictly positive, no power of any of the variables divides any of the  $\tilde{F}_i$  and  $U(0) \neq 0$ .

If  $W = \emptyset$ , then the  $\tilde{F}_i$  are all units, and the statement is proved. Otherwise, suppose that  $X_1 \in W$ ; then  $\prod \tilde{F}_i(0, X_2, \dots, X_{m+n}) = 0$ . But this is impossible, because no power of  $X_1$  divides any of the  $\tilde{F}_i$ .

Regarding 2., there are multi-indices  $\alpha_1, \alpha_2$  and units  $U_1, U_2$  such that  $F_1 = X^{\alpha_1}U_1$  and  $F_2 = X^{\alpha_2}U_2$ . Hence, the minimal elements of  $\text{Supp}(F_1 - F_2)$  are contained in the set  $\{\alpha_1, \alpha_2\}$ . Since  $F_1 - F_2$  is normal, either  $\alpha_1 \leq \alpha_2$  or  $\alpha_2 \leq \alpha_1$ .  $\square$



**3.10 Notation.** We fix  $m, n, m', n' \in \mathbb{N}$  such that  $m + n = m' + n'$ . Let  $X = (X_1, \dots, X_m)$ ,  $Y = (Y_1, \dots, Y_n)$  and  $X' = (X'_1, \dots, X'_{m'})$ ,  $Y' = (Y'_1, \dots, Y'_{n'})$ . If  $T$  is an admissible tree and  $\mathfrak{b}$  is one of its branches, we will always implicitly assume that  $T_{\mathfrak{b}} : \mathbb{R}[[X^*, Y^*]] \rightarrow \mathbb{R}[[X'^*, Y'^*]]$ . Let  $\hat{Y} = (Y_1, \dots, Y_{n-1})$  and  $\hat{Y}' = (Y'_1, \dots, Y'_{n'-1})$ .

The main result of this subsection is the following monomialisation algorithm for generalised power series. The proof methods take inspiration from [BM88] and [RSW03].

**3.11 Theorem.** *Let  $F_1, \dots, F_p \in \mathbb{R}[[X^*, Y]] \setminus \{0\} \cap \text{Im}(\mathcal{T})$ . There exists an admissible tree  $T$  such that for each branch  $\mathfrak{b}$  of  $T$  the series  $T_{\mathfrak{b}}(F_1), \dots, T_{\mathfrak{b}}(F_p) \in \mathbb{R}[[X'^*, Y'^*]] \setminus \{0\} \cap \text{Im}(\mathcal{T})$  are normal and linearly ordered by division.*

**3.12 Remark.** In the proof of Theorem 3.11 we will often compose branches of different admissible trees. We leave it to the reader to check at each stage that such compositions are allowed, i.e. the composition defines a branch of some admissible tree.

**3.13 Lemma.** *Let  $\mathcal{F} = \{F_k : k \in \mathbb{N}\} \subset \mathbb{R}[[X^*, Y]]$  be a family such that  $\text{Supp}_X(\mathcal{F})$  is good. Then there is an admissible tree  $T$  (consisting of ramifications and blow-up transformations) such that, for every branch  $\mathfrak{b}$  of  $T$ , we have  $T_{\mathfrak{b}} : \mathbb{R}[[X^*, Y]] \rightarrow \mathbb{R}[[X'^*, Y'^*]]$  with  $m' \leq m$  and there exist  $\alpha \in [0, \infty)^{m'}$  and series  $G_k \in \mathbb{R}[[X'^*, Y'^*]]$  ( $k \in \mathbb{N}$ ) such that, for every  $k \in \mathbb{N}$ ,  $T_{\mathfrak{b}}(F_k)(X', Y') = X'^{\alpha} G_k(X', Y')$  and for some  $k_0 \in \mathbb{N}$ ,  $G_{k_0}(0, Y') \neq 0$ .*

*Proof.* Here we view  $\mathcal{F}$  as a subset of  $\mathbb{A}[[X^*]]$ , with  $\mathbb{A} = \mathbb{R}[[Y]]$ . In [DS98, 4.11] the authors define the *blow-up height* of a finite set of monomials, denoted by  $b_X$ . It follows from the definition of  $b_X$  that if  $b_X(\mathcal{F}_{\min}) = (0, 0)$ , then there exists  $\alpha \in [0, \infty)^m$  such that  $\mathcal{F}_{\min} = \{X^{\alpha}\}$ . The proof is by induction on the pairs  $(m, b_X(\mathcal{F}_{\min}))$ , ordered lexicographically. If  $m = 0$ , there is nothing to prove. If  $m = 1$ , then  $b_X(\mathcal{F}_{\min}) = (0, 0)$ . In this case, for every  $k \in \mathbb{N}$ , there are a series  $G_k$  such that  $F_k = X^{\alpha} G_k$  and for some  $k_0 \in \mathbb{N}$  we have  $G_{k_0}(0, Y) \neq 0$ .

Hence we may assume that  $m > 1$  and  $b_X(\mathcal{F}_{\min}) \neq (0, 0)$ . It follows from the proof of [DS98, Proposition 4.14] that there are  $i, j \in \{1, \dots, m\}$  and suitable ramifications  $r_i^{\gamma}, r_j^{\delta}$  of the variables  $X_i$  and  $X_j$  such that  $b_X(r_i^{\gamma} \circ r_j^{\delta} \circ \pi_{i,j}^0(\mathcal{F}_{\min})) < b_X(\mathcal{F}_{\min})$  and  $b_X(r_i^{\gamma} \circ r_j^{\delta} \circ \pi_{i,j}^{\infty}(\mathcal{F}_{\min})) < b_X(\mathcal{F}_{\min})$  for all  $k \in \mathbb{N}$ . Notice that  $r_i^{\gamma} \circ r_j^{\delta} \circ \pi_{i,j}^0(\mathcal{F}_{\min})$  and  $r_i^{\gamma} \circ r_j^{\delta} \circ \pi_{i,j}^{\infty}(\mathcal{F}_{\min})$  are finite collections of monomials and they are in fact the collections of the minimal monomials of the families  $\mathcal{F}_0 = \{r_i^{\gamma} \circ r_j^{\delta} \circ \pi_{i,j}^0(F) : F \in \mathcal{F}\}$  and  $\mathcal{F}_{\infty} = \{r_i^{\gamma} \circ r_j^{\delta} \circ \pi_{i,j}^{\infty}(F) : F \in \mathcal{F}\}$ , respectively. Moreover, for every series and every  $\lambda \in (0, \infty)$ , the series of the family  $\mathcal{F}_{\lambda} = \{r_i^{\gamma} \circ r_j^{\delta} \circ \pi_{i,j}^{\lambda}(F) : F \in \mathcal{F}\}$  belong to  $\mathbb{R}[[X'^*, Y'^*]]$ , where  $m' = m - 1$ . By Remark 3.7 the families  $\mathcal{F}_{\lambda}$  ( $\lambda \in [0, \infty]$ ) have good total support, so the inductive hypothesis applies and we obtain the required conclusion.  $\square$

**3.14 Remark.** Let  $F = X^{\alpha} U(X) \in \mathbb{R}[[X^*]] \cap \text{Im}(\mathcal{T})$  be a normal series and let  $k \in \mathbb{N}$ . Then  $F^{1/k}$  is a well defined series and belongs also to  $\text{Im}(\mathcal{T})$ . In fact,  $U^{1/k} - 1$  can be viewed as the solution  $Y$  to the implicit function problem  $(1 + Y)^k - U(X) = 0$ .

*Proof of Theorem 3.11.* The proof is by induction on  $m + n$ . In view of Lemma 3.9, it is enough to prove the theorem for one series  $F \in \mathbb{R}[[X^*, Y]]$ . By Lemma 3.13, we may assume that  $n > 0$  and there exist  $\alpha \in [0, \infty)^m$  and a series  $G \in \mathbb{R}[[X^*, Y]]$  such that  $F(X, Y) = X^{\alpha} G(X, Y)$  and  $G(0, Y) \neq 0$ . It is well known (see for example [DS98, 6.1]) that, after performing some linear transformation of the form  $L_{n,c}$ , the series  $G$  becomes regular in the variable  $Y_n$ , of some order  $d \in \mathbb{N}$ .

Suppose that  $d = 1$ . Then  $\frac{\partial G}{\partial Y_n}(0) \neq 0$ . Hence, by the Implicit Function Theorem, there exists a series  $A(X, \hat{Y})$  such that  $G(X, \hat{Y}, A(X, \hat{Y})) = 0$  and  $A(0, 0) = 0$ . By Axiom 6 of 2.14,  $A \in \text{Im}(\mathcal{T})$ , so  $A(X, \hat{Y}) = \mathcal{T}(a(x, \hat{y}))$ , for some germ  $a \in \mathcal{A}_{m, n-1}$ . Then,  $\tau_a(G) = G(X, \hat{Y}, A(X, \hat{Y})) + Y_n U(X, Y)$ , for some unit  $U(X, Y) \in \text{Im}(\mathcal{T})$ , and we are done.

Suppose next that  $d > 1$ . Since  $\frac{\partial^d G}{\partial Y_n^d}(0) \neq 0$ , by the Implicit Function Theorem, there exists a series  $B(X, \hat{Y}) \in \text{Im}(\mathcal{T})$  (hence  $B(X, \hat{Y}) = \mathcal{T}(b(x, \hat{y}))$ , for some germ  $b \in \mathcal{A}_{m, n-1}$ ) such that

$\frac{\partial^{d-1}G}{\partial Y_n^{d-1}}(X, \hat{Y}, B(X, \hat{Y})) = 0$  and  $B(0, 0) = 0$ . Hence by Taylor's Theorem,

$$\tau_b(G)(X, Y) = U(X, Y)Y_n^d + G_1(X, \hat{Y})Y_n^{d-1} + \dots + G_d(X, \hat{Y}),$$

where  $U$  is a unit and  $G_i = \frac{\partial^{d-i}G}{\partial Y_n^{d-i}}(X, \hat{Y}, B(X, \hat{Y}))$ . Hence  $G_1(X, \hat{Y}) = 0$ .

By the inductive hypothesis, there is an admissible tree  $\tilde{T}$  (acting as the identity on  $Y_n$ ) such that, for every branch  $\tilde{b}$  of  $\tilde{T}$ , the series in the set  $\{\tilde{T}_{\tilde{b}}(G_i) : i = 2, \dots, d\} \cup \{\tilde{T}_{\tilde{b}}(X^\alpha)\}$  are normal. Let  $r^\pm := r_{m+1}^{d1, \pm} \circ \dots \circ r_{m+n}^{d1, \pm}$  and let  $\tilde{G}_i := \tilde{T}_{\tilde{b}} \circ r^+(G_i)$  (an argument analogous to the one which follows will hold for  $r^-$ ). By the inductive hypothesis and Remark 3.14, there is an admissible tree  $\tilde{T}$  (acting as the identity on  $Y_n$ ) such that, for every branch  $\tilde{b}$  of  $\tilde{T}$ , the series in the set  $\{\tilde{T}_{\tilde{b}}[\tilde{G}_i^{1/i}] : i = 2, \dots, d\} \cup \{\tilde{T}_{\tilde{b}} \circ r^+ \circ \tilde{T}_{\tilde{b}}(X^\alpha)\}$  are normal and linearly ordered by division i.e.  $\tilde{T}_{\tilde{b}}[\tilde{G}_i^{1/i}] = (X'Y')^{\frac{\alpha_i}{i}} U_i(X', Y')$ , where  $U_i$  are units and  $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,m+n-1}) \in [0, \infty)^{m'} \times \mathbb{N}^{n'-1}$ , with  $i|\alpha_{i,m'+j}$  for all  $j = 1, \dots, n'$ . Since  $\tilde{T}_{\tilde{b}}$  is a ring homomorphism, if we put  $T_b := \tau_B \circ \tilde{T}_{\tilde{b}} \circ r^+ \circ \tilde{T}_{\tilde{b}}$ , we obtain

$$T_b(G) = \hat{U}(X', Y')Y_n'^d + \sum_{i=2}^d (X'Y')^{\alpha_i} \hat{U}_i(X', Y')Y_n'^{d-i},$$

where  $\hat{U} = \tilde{T}_{\tilde{b}} \circ r^+ \circ \tilde{T}_{\tilde{b}}(U)$  and  $\hat{U}_i = U_i^i$ .

Let us rename  $X' = X$  and  $Y' = Y$ . We are going to perform a series of ramifications and blow-ups with the aim of decreasing the order of  $T_b(G)$  in the variable  $Y_n$ , possibly after factoring out a monomial in the variables  $(X, \hat{Y})$ .

Let  $l \in \{2, \dots, d\}$  be maximal such that  $\frac{\alpha_i}{l} \leq \frac{\alpha_i}{i}$  for all  $i = 2, \dots, d$ .

We first consider the variables in the tuple  $X$ . Suppose that  $X_1$  does appear in the monomial  $(X\hat{Y})^{\alpha_i}$  and let  $\gamma = \frac{\alpha_{1,1}}{l} > 0$ . Notice that  $d-i + \frac{\alpha_{1,i}}{\gamma} \geq d$  for all  $i = 2, \dots, d$ . Let  $\hat{X} = (X_2, \dots, X_m)$  and  $\alpha'_i = (\alpha_{i,2}, \dots, \alpha_{i,m+n-1})$ . We perform the ramification  $r_1^{1/\gamma}$ , so

$$T_b \circ r_1^{1/\gamma}(G) = \bar{U}(X, Y)Y_n^d + \sum_{i=2}^d (\hat{X}\hat{Y})^{\alpha'_i} \bar{U}_i(X, \hat{Y})X_1^{\alpha_{1,i}/\gamma}Y_n^{d-i},$$

where  $\bar{U} = r_1^{1/\gamma}(\hat{U})$  and  $\bar{U}_i = r_1^{1/\gamma}(\hat{U}_i)$ .

We consider the blow-up charts in the collection  $\pi_{m+n,1}$ . For a better readability, in what follows we will still denote the variables as  $(X, Y)$  after blow-up transformation.

**Case 1.**  $\lambda = \pm\infty$

We will only treat the case  $\lambda = +\infty$ , the other case is analogous. The transformation  $\pi_{m+n,1}^{+\infty}$  maps  $(X, Y)$  to  $(X_1Y_n, X_2, \dots, X_m, Y)$ . Then,

$$\begin{aligned} T_b \circ r_1^{1/\gamma} \circ \pi_{m+n,1}^{+\infty}(G) &= \tilde{U}(X, Y)Y_n^d + \sum_{i=2}^d (\hat{X}\hat{Y})^{\alpha'_i} \tilde{U}_i(X, Y)X_1^{\alpha_{1,i}/\gamma}Y_n^{d-i + \frac{\alpha_{1,i}}{\gamma}} \\ &= Y_n^d \left[ \tilde{U}(X, Y) + \sum_{i=2}^d (\hat{X}\hat{Y})^{\alpha'_i} \tilde{U}_i(X, Y)X_1^{\alpha_{1,i}/\gamma}Y_n^{-i + \frac{\alpha_{1,i}}{\gamma}} \right] \\ &= Y_n^d V(X, Y), \end{aligned}$$

where  $\tilde{U} = \pi_{m+n,1}^{+\infty}(\bar{U})$ ,  $\tilde{U}_i = \pi_{m+n,1}^{+\infty}(\bar{U}_i)$  and  $V$  has the form  $\tilde{U} + X_1^\beta B(X, Y)$ , for some  $\beta \in \mathbb{K}^{>0}$  and  $B \in \mathbb{R}[[X^*, Y]]$ , and hence is a unit.

**Case 2.**  $\lambda = 0$

The transformation  $\pi_{m+n,1}^0$  maps  $(X, Y)$  to  $(X, \hat{Y}, X_1 Y_n)$ . Then,

$$\begin{aligned} T_b \circ r_1^{1/\gamma} \circ \pi_{m+n,1}^0 (G) &= \tilde{U}(X, Y) X_1^d Y_n^d + \sum_{i=2}^d \left( \hat{X} \hat{Y} \right)^{\alpha'_i} \tilde{U}_i(X, Y) X_1^{d-i+\frac{\alpha_{1,i}}{\gamma}} Y_n^{d-i} \\ &= X_1^d \left[ \tilde{U}(X, Y) Y_n^d + \left( \hat{X} \hat{Y} \right)^{\alpha'_i} \tilde{U}_i(X, Y) Y_n^{d-i} \right. \\ &\quad \left. + \sum_{i=2, i \neq l}^d \left( \hat{X} \hat{Y} \right)^{\alpha'_i} \tilde{U}_i(X, Y) X_1^{-i+\frac{\alpha_{1,i}}{\gamma}} Y_n^{d-i} \right] \\ &= X_1^d G_0(X, Y), \end{aligned}$$

where  $\tilde{U} = \pi_{m+n,1}^0(\bar{U})$  and  $\tilde{U}_i = \pi_{m+n,1}^0(\bar{U}_i)$ . Notice that  $X_1$  does not appear any more in the coefficient of  $Y_n^{d-l}$  in  $G_0$ .

**Case 3.**  $\lambda \neq 0, \pm\infty$

The transformation  $\pi_{m+n,1}^\lambda$  maps  $(X, Y)$  to  $(X, \hat{Y}, X_1(\lambda + Y_n))$ . Then,

$$\begin{aligned} T_b \circ r_1^{1/\gamma} \circ \pi_{m+n,1}^\lambda (G) &= \tilde{U}(X, Y) X_1^d (\lambda + Y_n)^d + \sum_{i=2}^d \left( X' \hat{Y} \right)^{\alpha'_i} \tilde{U}_i(X, Y) X_1^{d-i+\frac{\alpha_{1,i}}{\gamma}} (\lambda + Y_n)^{d-i} \\ &= X_1^d \left[ \tilde{U}(X, Y) Y_n^d + d\lambda \tilde{U}(X, Y) Y_n^{d-1} + \sum_{i=2}^d B_i(X, \hat{Y}) Y_n^{d-i} \right] \\ &= X_1^d G_\lambda(X, Y), \end{aligned}$$

where  $\tilde{U} = \pi_{m+n,1}^\lambda(\bar{U})$ ,  $\tilde{U}_i = \pi_{m+n,1}^\lambda(\bar{U}_i)$  and  $B_i \in \mathbb{R}[[X^*, \hat{Y}]]$ . Notice that, thanks to the initial Tschirnhausen translation  $\tau_b$ ,  $G_\lambda$  is regular in  $Y_n$  of order at most  $d-1$ .

Arguing in the same way for every variable of the tuple  $X$  which actually appears in the monomial  $(X\hat{Y})^{\alpha_i}$ , after factoring out a monomial in the variables  $X$ , either we monomialise  $G$  (case 1), or we reduce the order of regularity in  $Y_n$  (case 3), or we eliminate the variables  $X$  from the monomial  $(X\hat{Y})^{\alpha_i}$  (case 2).

Next we perform a blow-up transformation to obtain similar results in the variables  $\hat{Y}$ . If the variable  $Y_1$  appears in the monomial  $(X\hat{Y})^{\alpha_i}$ , we perform the blow-up transformation  $\pi_{m+n,m+1}$ . For the charts  $\pi_{m+n,m+1}^\infty$  and  $\pi_{m+n,m+1}^\lambda$  ( $\lambda \in \mathbb{R}$ ), we obtain the same result as in cases 1 and 3 respectively, namely either  $G$  has become normal or, after factoring out a power of  $Y_1$ , the order of regularity in  $Y_n$  has decreased. For the chart  $\pi_{m+n,m+1}^0$ , we obtain, after factoring out a power of  $Y_1$ , that the exponents  $\alpha_{i,m+1}$  of  $Y_1$  each decrease by the quantity  $i$ , and hence by repeating the process we can reduce to the case  $\alpha_{l,1} = 0$ . Hence, as in case 2, we have eliminated the variable  $Y_1$  from the monomial  $(X\hat{Y})^{\alpha_i}$ .

Summing up, there is an admissible tree  $\hat{T}$  such that, for every branch  $\hat{\mathbf{b}}$  of  $\hat{T}$ , we have that  $T_b \circ \hat{T}_{\hat{\mathbf{b}}}(X^\alpha)$  is a monomial (in the variables  $(X, Y)$ ) and  $T_b \circ \hat{T}_{\hat{\mathbf{b}}}(G) = (X\hat{Y})^{\alpha_{\hat{\mathbf{b}}}} V_{\hat{\mathbf{b}}}(X, Y) G_{\hat{\mathbf{b}}}(X, Y)$ , where  $\alpha_{\hat{\mathbf{b}}} \in [0, \infty)^m \times \mathbb{N}^{n-1}$ ,  $V_{\hat{\mathbf{b}}}$  is a unit and  $G_{\hat{\mathbf{b}}}$  is either a monomial in  $Y_n$  (case 1), or it is regular in  $Y_n$  of order 1 (repeated use of cases 2 and 3). In this latter case, we compose with a further Tschirnhausen translation in order to render  $G_{\hat{\mathbf{b}}}(X, Y)$  normal. This concludes the proof of Theorem 3.11.  $\square$

### 3.3 Monomialisation of germs

Recall that  $f \in \mathcal{A}_{m,n}$  is *normal* if there exist  $\alpha \in \mathbb{A}^m$ ,  $N \in \mathbb{N}^n$  and  $u \in \mathcal{A}_{m,n}$  such that  $f(x, y) = x^\alpha y^N u(x, y)$  and  $u(0, 0) \neq 0$ .

**3.15 Remark.** If  $f \in \mathcal{A}_{m,n}$  then it follows from the axioms in 2.14 that  $f(x, y)$  is normal if and only if  $\mathcal{T}(f)(X, Y)$  is normal.

**3.16 Theorem.** Let  $f_1, \dots, f_p \in \mathcal{A}_{m,n}$ . Then there exist a polyradius  $r'$  such that the  $f_j$  have a representative on  $\hat{I}_{m,n,r'}$ , and a finite family

$$\mathcal{F} = \left\{ \rho_i : \hat{I}_{m_i, n_i, r_i} \rightarrow \hat{I}_{m,n,r} \right\}_{i=1, \dots, N}$$

of admissible transformations such that, for all  $i = 1, \dots, N$ , the germs  $f_1 \circ \rho_i, \dots, f_p \circ \rho_i$  are normal and linearly ordered by division, and

$$\hat{I}_{m,n,r'} \subseteq \bigcup_{i=1}^N \rho_i \left( \hat{I}_{m_i, n_i, r_i} \right).$$

**3.17 Remark.** Let  $\nu : \hat{I}_{m', n', r'} \rightarrow \hat{I}_{m,n,r}$  be an elementary transformation. Notice that for every polyradius  $r''' \leq r'$  there exists a polyradius  $r'' \leq r$  such that:

- if  $\nu$  is either  $\tau_h$ , or  $L_{i,c}$ , or  $r_i^\gamma$ , then  $\hat{I}_{m,n,r''} \subseteq \nu \left( \hat{I}_{m,n,r'''} \right)$ ;
- $\hat{I}_{m,n,r''} \subseteq r_{m+i}^{d,+} \left( \hat{I}_{m,n,r'''} \right) \cup r_{m+i}^{d,-} \left( \hat{I}_{m,n,r'''} \right)$ ;
- $\hat{I}_{m,n,r''} \subseteq \bigcup_{\lambda \in (0, \infty)} \pi_{i,j}^\lambda \left( \hat{I}_{m-1, n+1, r'''} \right) \cup \pi_{i,j}^0 \left( \hat{I}_{m,n,r'''} \right) \cup \pi_{i,j}^\infty \left( \hat{I}_{m,n,r'''} \right)$ ;
- $\hat{I}_{m,n,r''} \subseteq \bigcup_{\lambda \in \mathbb{R}} \pi_{m+i,j}^\lambda \left( \hat{I}_{m,n,r'''} \right) \cup \pi_{m+i,j}^{+\infty} \left( \hat{I}_{m+1, n-1, r'''} \right) \cup \pi_{m+i,j}^{-\infty} \left( \hat{I}_{m+1, n-1, r'''} \right)$ ;
- $\hat{I}_{m,n,r''} \subseteq \bigcup_{\lambda \in \mathbb{R} \cup \{\infty\}} \pi_{m+i, m+j}^\lambda \left( \hat{I}_{m,n,r'''} \right)$ .

Moreover, by a compactness argument as in [DS98, p. 4406], it is easy to see that the three last inclusions remain true for suitable finite families of parameters  $\lambda$ .

*Proof of Theorem 3.16.* Notice first that, using Lemma 3.9 and Remark 3.15, we only need to prove the statement for one germ  $f \in \mathcal{A}_{m,n}$ . By Theorem 3.11 and by compatibility of  $\mathcal{T}$  with the operations in 2.14, there exists an admissible tree  $T$  such that for each branch  $\mathfrak{b}$  of  $T$ , the germ  $f \circ \rho_{\mathfrak{b}}$  is normal. We prove a stronger statement, namely that the admissible transformations in the statement are in fact induced by branches of  $T$ . The proof is by induction on the height of  $T$ . If the height of  $T$  is zero, then  $f$  is normal. Hence, let us assume  $T$  has positive height, consider the initial vertex of  $T$  and the elementary tree  $T_0$  immediately attached to it.

Let  $\nu : \hat{I}_{m_\nu, n_\nu, r_\nu} \rightarrow \hat{I}_{m,n,r}$  be an elementary transformation induced by a branch of  $T_0$  (i.e.  $\nu$  is either a Tschirnhausen translation, or a linear transformation, or a ramification, or it is a blow-up chart). The germ  $g^\nu = f \circ \nu$  belongs to  $\mathcal{A}_{m_\nu, n_\nu}$  and can be monomialised by a tree  $T'$  with the property that, for every branch  $\mathfrak{b}'$  of  $T'$ , there exists a unique branch  $\mathfrak{b}$  of  $T$  such that  $\rho_{\mathfrak{b}} = \nu \circ \rho_{\mathfrak{b}'}$ . Since the height of  $T'$  is smaller than the height of  $T$ , the inductive hypothesis applies and there are a polyradius  $r'_\nu$  and a finite family  $\mathcal{F}^\nu = \left\{ \rho'_i : \hat{I}_{m'_i, n'_i, r'_i} \rightarrow \hat{I}_{m_\nu, n_\nu, r_\nu} \right\}_{i=1, \dots, N_\nu}$  as in the statement of the theorem such that the germs  $g^\nu \circ \rho'_i$  are normal and  $\hat{I}_{m_\nu, n_\nu, r'_\nu} \subseteq \bigcup_{i=1}^{N_\nu} \rho'_i \left( \hat{I}_{m'_i, n'_i, r'_i} \right)$ . Since the admissible transformations  $\nu \circ \rho'_i$  are induced by branches of  $T$ , we have that the germs  $f \circ \nu \circ \rho'_i$  are normal and  $\nu \left( \hat{I}_{m_\nu, n_\nu, r'_\nu} \right) \subseteq \bigcup_{i=1}^{N_\nu} \nu \circ \rho'_i \left( \hat{I}_{m'_i, n'_i, r'_i} \right)$ .

Now, by Remark 3.2, there are a polyradius  $r'$  and a finite collection  $\mathcal{G}$  of branches of  $T_0$  such that  $\hat{I}_{m,n,r'} \subseteq \bigcup_{\nu \in \mathcal{G}} \nu \left( \hat{I}_{m_\nu, n_\nu, r'_\nu} \right)$ , and we are done.  $\square$

## 4 A parametrisation theorem

The main result of this section is Theorem 4.13, which essentially provides a way to parametrise every bounded  $\mathbb{R}_{\mathcal{A}}$ -definable set by means of maps whose components are in  $\mathcal{A}$ . This provides the main step in the proof of Theorem A, which will be completed at the end of this section, and a key tool for proving Theorem B, in the next section.

**4.1 Definition.** A set  $A \subset \hat{I}_{m,n,r}$  is said to be  $\mathcal{A}_{m,n}$ -basic if it is a finite union of sets of the form

$$\{(x, y) \in \hat{I}_{m,n,r} : g_0(x, y) = 0, g_1(x, y) > 0, \dots, g_k(x, y) > 0\},$$

where  $g_0, \dots, g_k \in \mathcal{A}_{m,n,r}$ .

A set  $A \subset \mathbb{R}^{m+n}$  is said to be  $\mathcal{A}_{m,n}$ -semianalytic if for every point  $a \in \mathbb{R}^{m+n}$  there exists  $r_a \in (0, \infty)^{m+n}$  such that for every choice of signs  $\sigma = (\sigma_1, \dots, \sigma_m) \in \{-1, 1\}^m$ , there exists an  $\mathcal{A}_{m,n}$ -basic set  $A_{a,\sigma} \subset \hat{I}_{m,n,r_a}$  with

$$A \cap (h_{a,\sigma}(\hat{I}_{m,n,r_a})) = h_{a,\sigma}(A_{a,\sigma}),$$

where  $h_{a,\sigma}(x, y) = (\sigma_1 x_1 + a_1, \dots, \sigma_m x_m + a_m, y_1 + a_{m+1}, \dots, y_n + a_{m+n})$ .

We will simply say that a set is  $\mathcal{A}$ -basic or  $\mathcal{A}$ -semianalytic when the indices  $m, n$  are either obvious from the context or irrelevant.

**4.2 Remark.** Notice that since all functions in  $\mathcal{A}$  are  $\mathcal{A}$ -analytic (see Definition 2.10), an  $\mathcal{A}$ -basic set is also  $\mathcal{A}$ -semianalytic.

**4.3 Definition.** Let  $r = (r_1, \dots, r_{m+n}) \in (0, \infty)^{m+n}$  be a polyradius. A set  $Q \subseteq \hat{I}_{m,n,r}$  of the form  $B_1 \times \dots \times B_m$ , where  $B_i$  is either  $\{0\}$ , or  $(-r_i, 0)$ , or  $(0, r_i)$ , is called a *sub-quadrant* of  $\mathbb{R}^{m+n}$ . The cardinality of the set  $C := \{i : B_i \neq \{0\}\}$ , denoted by  $\dim(Q)$ , is called the *dimension* of  $Q$ . Notice that  $\hat{I}_{m,n,r}$  is a finite union of sub-quadrants.

The following proposition states that the germ at zero of an  $\mathcal{A}$ -basic subset of  $\mathbb{R}^{m+n}$  can be transformed into a finite union of sub-quadrants of  $\mathbb{R}^{m+n}$  by means of a finite family of admissible transformations.

**4.4 Proposition.** Let  $A \subseteq \hat{I}_{m,n,r}$  be an  $\mathcal{A}_{m,n}$ -basic set. Then there exist a neighbourhood  $W$  of 0 in  $\mathbb{R}^{m+n}$  and a finite family  $\mathcal{F} = \{(\rho_i, Q_i) : i = 1, \dots, N\}$ , where  $\rho_i : \hat{I}_{m'_i, n'_i, r'_i} \rightarrow \hat{I}_{m,n,r}$  is an admissible transformation and  $Q_i \subseteq \bar{Q}_i \subseteq \hat{I}_{m'_i, n'_i, r'_i}$  is a sub-quadrant, such that  $\rho_i \upharpoonright Q_i : Q_i \rightarrow \rho_i(Q_i)$  is a diffeomorphism and

$$W \cap A = \bigcup_{i=1}^N \rho_i(Q_i).$$

*Proof.* The proof is by induction on the dimension  $m+n$  of the ambient space. Let  $S := \{x_1 = 0\} \cup \dots \cup \{x_m = 0\} \cup \{y_1 = 0\} \cup \dots \cup \{y_n = 0\}$ . The set  $A \cap S$  is an  $\mathcal{A}$ -basic sets contained in an ambient space of dimension strictly smaller than  $m+n$ , hence, by the inductive hypothesis, it is sufficient to prove the proposition for the set  $A' := A \setminus S$ . We may assume that  $A'$  is of the following form:

$$A' = \{(x, y) \in \hat{I}_{m,n,r} : f_0(x, y) = 0 \wedge f_1(x, y) > 0 \wedge \dots \wedge f_p(x, y) > 0\},$$

where  $f_i \in \mathcal{A}_{m,n}$ .

By Theorem 3.11, there is an admissible tree  $T$  such that for each branch  $\mathfrak{b}$  of  $T$ , the germs  $f_i \circ \rho_{\mathfrak{b}}$  are all normal. We prove by induction on the height of  $T$  that the admissible transformations which parametrise  $A'$  are in fact induced by branches of  $T$ . If the height of  $T$  is zero, then all the  $f_i$  are normal and  $A'$  is a sub-quadrant of  $\hat{I}_{m,n,r}$ , so for some box around zero  $W$ , the closure of the sub-quadrant  $W \cap A'$  is contained in  $\hat{I}_{m,n,r}$ . Hence, let us assume  $T$  has positive height, consider the initial vertex of  $T$  and the elementary tree  $T_0$  immediately attached to it.

Let  $\nu : \hat{I}_{m_\nu, n_\nu, r_\nu} \rightarrow \hat{I}_{m,n,r}$  be an elementary transformation induced by a branch of  $T_0$ . Consider the germs  $g'_i = f_i \circ \nu \in \mathcal{A}_{m_\nu, n_\nu}$ . The set

$$A_\nu = \{(x, y) \in \hat{I}_{m_\nu, n_\nu, r_\nu} : g'_0(x, y) = 0 \wedge g'_1(x, y) > 0 \wedge \dots \wedge g'_p(x, y) > 0\}$$

is  $\mathcal{A}_{m_\nu, n_\nu}$ -basic and the germs  $g_i^\nu$  can be monomialised by a tree  $T'$  with the property that, for every branch  $\mathfrak{b}'$  of  $T'$ , there exists a unique branch  $\mathfrak{b}$  of  $T$  such that  $\rho_{\mathfrak{b}} = \nu \circ \rho_{\mathfrak{b}'}$ . Since the height of  $T'$  is smaller than the height of  $T$ , the inductive hypothesis applies and there are a neighbourhood  $W_\nu$  of 0 in  $\mathbb{R}^{m+n}$  and a finite family  $\mathcal{F}_\nu = \{(\rho_i^\nu, Q_i^\nu) : i = 1, \dots, N_\nu\}$  as in the statement such that the  $\rho_i^\nu$  are induced by branches of  $T'$  and  $W_\nu \cap A_\nu = \bigcup_{i=1}^{N_\nu} \rho_i^\nu(Q_i^\nu)$ . By Remark 3.2, there are a polyradius  $r'$  and a finite collection  $\mathcal{G}$  of branches of  $T_0$  such that  $\hat{I}_{m, n, r'} \subseteq \bigcup_{\nu \in \mathcal{G}} \nu \left( \hat{I}_{m_\nu, n_\nu, r_\nu} \cap W_\nu \right) \subseteq \hat{I}_{m, n, r}$ .

Hence, there is a neighbourhood  $W$  of zero such that  $\bigcup_{\nu \in \mathcal{G}} \nu \left( \hat{I}_{m_\nu, n_\nu, r_\nu} \cap W_\nu \right) = \hat{I}_{m, n, r} \cap W$ . It follows that

$$A' \cap W = A' \cap \bigcup_{\nu \in \mathcal{G}} \nu \left( \hat{I}_{m_\nu, n_\nu, r_\nu} \cap W_\nu \right) = \bigcup_{\nu \in \mathcal{G}} \nu (A_\nu \cap W_\nu) = \bigcup_{\nu \in \mathcal{G}} \bigcup_{i=1}^{N_\nu} \nu \circ \rho_i^\nu (Q_i^\nu).$$

We claim that  $\nu \upharpoonright A_\nu$  is a diffeomorphism onto its image (and hence so is  $\nu \circ \rho_i^\nu \upharpoonright Q_i^\nu$ ). This is clear if  $\nu$  is either of type  $L_{i,c}$ , or  $\tau_h$ , or  $r_{m+i}^{d,\pm}$ . If  $\nu = r_i^\gamma$ , then  $A_\nu \cap \{x'_i = 0\} = \emptyset$  (because  $A' \cap S = \emptyset$ ). For the same reason, if  $\nu = \pi_{m+i,j}^\lambda$  then  $A_\nu \cap \{x'_j = 0\} = \emptyset$ , and similarly for the other blow-up charts. Summing up,  $A_\nu$  does not meet the subset of  $\hat{I}_{m_\nu, n_\nu, r_\nu}$  on which  $\nu$  is either not bijective or not differentiable.  $\square$

**4.5 Corollary.** *Let  $A \subseteq \mathbb{R}^{m+n}$  be a bounded  $\mathcal{A}_{m,n}$ -semianalytic set. Then there exists a finite family  $\mathcal{F} = \{(\rho_i, Q_i, a_i, \sigma_i) : i = 1, \dots, N\}$ , where  $a_i \in \mathbb{R}^{m+n}$ ,  $\sigma_i \in \{-1, 1\}^m$ ,  $\rho_i : \hat{I}_{m'_i, n'_i, r'_i} \rightarrow \hat{I}_{m, n, r}$  is an admissible transformation and  $Q_i \subseteq \hat{I}_{m'_i, n'_i, r'_i}$  is a sub-quadrant, such that  $h_{a_i, \sigma_i} \circ \rho_i \upharpoonright Q_i : Q_i \rightarrow h_{a_i, \sigma_i} \circ \rho_i(Q_i)$  is a diffeomorphism and*

$$A = \bigcup_{i=1}^N h_{a_i, \sigma_i} \circ \rho_i(Q_i),$$

where  $h_{a_i, \sigma_i}$  is as in Definition 4.1.

**4.6 Remark.** Since  $h_{a_i, \sigma_i} \circ \rho_i \upharpoonright Q_i$  is a diffeomorphism onto its image, it follows that  $A$  has dimension (in the sense of [DS98, 8.2]). Notice also that the components of  $h_{a_i, \sigma_i} \circ \rho_i$  belong to  $\mathcal{A}_{m'_i, n'_i, r'_i}$ .

*Proof.* The set  $A$  is a finite union of translates of reflections of  $\mathcal{A}_{m,n}$ -basic sets, by a compactness argument. Hence the parametrisation required is obtained from the parametrisation in Proposition 4.4 composed with suitable translations and reflections.  $\square$

The next few definitions and statements are inspired by the approach to model-completeness developed in [BM88], [DS98] and [RSW03]. The proofs follow the usual pattern, with some minor differences dictated by the generality of our setting. It is however worth noticing that in order to prove the Fibre Cutting Lemma 4.11, we need to use both quasianalyticity (see Definition 2.11) and  $\mathcal{A}$ -analyticity (see Definition 2.10).

**4.7 Definition.** Let  $F = (F_1, \dots, F_{m+n}) : \hat{I}_{m, n, r} \rightarrow I_{0, m+n, r'}$  be a map such that  $F_i \in \mathcal{A}_{m, n, r}$ , and let  $Q \subseteq \bar{Q} \subseteq \hat{I}_{m, n, r}$  be a sub-quadrant of dimension  $q \leq m+n$ . The set

$$\Gamma(F \upharpoonright Q) = \{(u, w) : u \in Q, w = F(u)\} \subseteq I_{m, m+2n, (r, r')}$$

is called a *trivial manifold*.

A trivial manifold  $M$  is clearly a  $\mathcal{C}^1$  manifold of dimension  $q$  and an  $\mathcal{A}$ -basic set. The frontier  $\text{fr}(M) = \bar{M} \setminus M$  is the set  $\{(u, w) : u \in \text{fr}(Q), w = F(u)\}$ , which, by  $\mathcal{A}$ -analyticity of the components of  $F$  (see Definition 2.10), is an  $\mathcal{A}$ -semianalytic set. Moreover,  $\dim(\text{fr}(M)) < \dim(M)$ .

**4.8 Notation.** Let  $k, l \in \mathbb{N}$  with  $k \leq l$ . Denote by  $\Pi_k^l : \mathbb{R}^l \rightarrow \mathbb{R}^k$  the projection onto the **last**  $k$  coordinates, i.e.  $\Pi_k^l(z_1, \dots, z_l) = (z_{l-k+1}, \dots, z_l)$ .

**4.9 Corollary.** Let  $A \subseteq \mathbb{R}^{m+n}$  be a bounded  $\mathcal{A}_{m,n}$ -semianalytic set. Then there exist finitely many trivial manifolds  $M_1, \dots, M_N \subseteq \mathbb{R}^{2(m+n)}$  such that  $A = \bigcup_{i=1}^N \Pi_{m+n}^{2(m+n)}(M_i)$  and  $\Pi_{m+n}^{2(m+n)} \upharpoonright M_i$  is an immersion.

*Proof.* Apply Corollary 4.5 to  $A$  and let  $M_i = \Gamma(h_{a_i, \sigma_i} \circ \rho_i \upharpoonright Q_i)$ . Notice that  $\dim(M_i) = \dim(Q_i) \leq \dim(A)$ .  $\square$

**4.10 Definition.** An  $\mathcal{A}_{m,n}$ -basic set  $M \subseteq I_{m,n,r}$  is called an  $\mathcal{A}_{m,n}$ -manifold if  $M$  is a  $\mathcal{C}^1$ -manifold of dimension  $d \leq m+n$  and there are  $h_1, \dots, h_{m+n-d} \in \mathcal{A}_{m,n}$  such that  $h_i \upharpoonright M \equiv 0$  and for all  $z \in M$  the vectors  $\nabla h_1(z), \dots, \nabla h_{m+n-d}(z)$  are linearly independent (see [DS98, Def. 8.3]). Notice that a trivial manifold is an  $\mathcal{A}_{m,n}$ -manifold.

Let  $\iota : \{1, \dots, d\} \rightarrow \{1, \dots, m+n\}$  be a strictly increasing sequence and let  $\Pi_\iota(z_1, \dots, z_{m+n}) = (z_{\iota(1)}, \dots, z_{\iota(d)})$ . Let  $M_\iota = \{z \in M : \Pi_\iota \upharpoonright M \text{ has rank } d \text{ at } z\}$ . Then  $M_\iota = \{z \in M : f_\iota(z) \neq 0\}$  for some  $f_\iota \in \mathcal{A}_{m,n,r}$ , which is a polynomial in the partial derivatives and modified derivatives  $\partial_i$  of  $h_1, \dots, h_{m+n-d}$ . In particular,  $M_\iota$  is an  $\mathcal{A}_{m,n}$ -manifold of dimension  $d$  and  $M$  is the union of all the  $M_\iota$ .

Let  $k \leq m+n$  and for every increasing sequence  $\iota$ , define  $m_\iota(k)$  as the dimension of the vector space  $\Pi_\iota(\mathbb{R}^{m+n}) \cap \Pi_k^{m+n}(\mathbb{R}^{m+n})$ . Notice that  $m_\iota(k) \in \{0, \dots, d\}$ . Suppose that  $\Pi_k^{m+n} \upharpoonright M_\iota$  has constant rank  $m_\iota(k)$ . Then, arguing as in [DS98, 8.11], it is easy to see that for every  $a \in \mathbb{R}^k$ , the fibre  $M_a := (\Pi_k^{m+n})^{-1}(a) \cap M_\iota$  is either empty or an  $\mathcal{A}_{m,n}$ -manifold of dimension  $d - m_\iota(k)$  and every connected component of the fibre has non-empty frontier.

**4.11. Fibre Cutting Lemma.** Let  $M_\iota \subseteq I_{m,n,r}$  be as in the above definition and let  $k \leq m+n$ . Suppose that  $\Pi_k^{m+n} \upharpoonright M_\iota$  has constant rank  $m_\iota(k)$  and that  $m_\iota(k) < d = \dim(M_\iota)$ . Then there is an  $\mathcal{A}_{m,n,r}$ -basic set  $A \subseteq M_\iota$  such that  $\dim(A) < d$  and  $\Pi_k^{m+n}(A) = \Pi_k^{m+n}(M_\iota)$ .

*Proof.* Let  $f, g_1, \dots, g_q \in \mathcal{A}_{m,n,r}$  be such that

$$M_\iota = \{(x, y) \in I_{m,n,r} : f(x, y) = 0, g_1(x, y) = 0, \dots, g_q(x, y) = 0\}.$$

Recall that for every  $a \in \Pi_k^{m+n}(M_\iota)$ , the fibre  $M_a := (\Pi_k^{m+n})^{-1}(a) \cap M_\iota$  is an  $\mathcal{A}_{m,n}$ -manifold of dimension  $d - m_\iota(k) > 0$  and every connected component of the fibre has non-empty frontier. Let  $r = (s, t)$ . The function  $g(x, y) = \prod_{i=1}^m x_i(s_i - x_i) \cdot \prod_{i=1}^n (t_i^2 - y_i^2) \cdot \prod_{i=1}^q g_i(x, y) \in \mathcal{A}_{m,n,r}$  is positive on  $M_a$  and vanishes on  $\text{fr}(M_a)$ , hence  $g \upharpoonright M_a$  has a critical point on every connected component of  $M_a$ . The set

$$A = \{(x, y) \in M_\iota : g \upharpoonright M_a \text{ is critical at } (x, y), \text{ where } a := \Pi_k^{m+n}(x, y)\}$$

is clearly  $\mathcal{A}_{m,n,r}$ -basic and  $\Pi_k^{m+n}(A) = \Pi_k^{m+n}(M_\iota)$ . We claim that  $A$  has empty interior in  $M_\iota$ , and hence  $\dim(A) < d$ .

Let  $a \in \Pi_k^{m+n}(M_\iota)$  and let  $h_1, \dots, h_{m+n-d} \in \mathcal{A}_{m,n}$  such that  $h_i \upharpoonright M_a \equiv 0$  and for all  $z = (x, y) \in M_a$  the vectors  $\nabla h_1(z), \dots, \nabla h_{m+n-d}(z)$  are linearly independent. If  $C$  is a connected component of  $M_a$ , let

$$S = \{z \in C : g \upharpoonright M_a \text{ is critical at } z\} = \{z \in C : dg(z) \wedge dh_1(z) \wedge \dots \wedge dh_{m+n-d}(z) = 0\}.$$

It suffices to prove that  $S$  has empty interior on  $C$ . Clearly,  $S$  is closed in  $C$ . We prove that, if  $S$  has non-empty interior in  $C$ , then  $S$  is also open in  $C$ , which contradicts the fact that  $g$  is not constant on the fibres. Let  $z_0 \in S \setminus \overset{\circ}{S}$  and define  $\bar{g}(z) := g(z + z_0)$ ,  $\bar{h}_i(z) := h_i(z + z_0)$  for  $i = 1, \dots, m+n-d$ . By  $\mathcal{A}$ -analyticity (see Definition 2.10), the germs at zero of these functions belong to  $\mathcal{A}_{0,m+n}$ . By the Implicit Function Theorem and Axiom 6 in 2.14, there are a polyradius  $r' \in \mathbb{R}^d$  and a map  $\varphi = (\varphi_1, \dots, \varphi_{m+n}) \in (\mathcal{A}_{0,d,r'})$  such that  $\varphi(0) = z_0$  and  $\varphi$  is a local parametrisation of  $C$  around  $z_0$ . Let  $F \in \mathcal{A}_{0,d,r'}$  be such that

$$\begin{aligned} \varphi^{-1}(S) &= \{w \in I_{0,d,r'} : d(\bar{g} \circ \varphi)(w) \wedge d(\bar{h}_1 \circ \varphi)(w) \wedge \dots \wedge d(\bar{h}_{m+n-d} \circ \varphi)(w) = 0\} \\ &= \{w \in I_{0,d,r'} : F(w) = 0\}. \end{aligned}$$



Since  $\varphi$  is a homeomorphism,  $\varphi^{-1}\left(\overset{\circ}{S}\right) \subseteq I_{0,d,r'}$  is an open set on which  $F$  vanishes, and  $0 \in \text{cl}\left(\varphi^{-1}\left(\overset{\circ}{S}\right)\right)$ . By quasianalyticity, there exists  $r'' \leq r'$  such that  $F$  vanishes on  $I_{0,d,r''}$ , which contradicts the fact that  $z_0 \notin \overset{\circ}{S}$ .  $\square$

**4.12 Proposition.** *Let  $A \subseteq \mathbb{R}^{m+n}$  be a bounded  $\mathcal{A}_{m,n}$ -semianalytic set and  $k \leq m+n$ . Then there exist finitely many trivial manifolds  $M_1, \dots, M_N \subseteq \mathbb{R}^{2(m+n)}$ , with  $\dim(M_i) \leq k$ , such that  $\Pi_k^{m+n}(A) = \bigcup_{i=1}^N \Pi_k^{2(m+n)}(M_i)$  and for every  $i = 1, \dots, N$  there is a strictly increasing sequence  $\iota : \{1, \dots, \dim(M_i)\} \rightarrow \{m+n-k+1, \dots, m+n\}$  such that  $\Pi_\iota \upharpoonright M_i$  is an immersion.*

*Proof.* The proof is by induction on  $d = \dim(A)$ , for all  $m, n, k \in \mathbb{N}$ . If  $d = 0$ , then by Corollary 4.5  $A$  is a finite set and there is nothing to prove. Let  $d > 0$ . By Corollary 4.9 and the inductive hypothesis, it is enough to prove the proposition for an  $\mathcal{A}_{m,n}$ -manifold  $M$  of dimension  $d$ , instead of  $A$ . We argue by induction on  $r = \max\{\text{rk}_z \Pi_k^{m+n} \upharpoonright M : z \in M\}$ . If  $r = 0$ , then  $\Pi_k^{m+n}(M)$  is a finite set and there is nothing to prove, so let us assume that  $r > 0$ . Let  $M_1 = \{z \in M : \text{rk}_z \Pi_k^{m+n} \upharpoonright M = r\}$  and  $M_2 = \{z \in M : \text{rk}_z \Pi_k^{m+n} \upharpoonright M < r\}$ . Then  $M_1$  and  $M_2$  are  $\mathcal{A}_{m,n}$ -semianalytic and  $M_1$  is open in  $M$ , and hence is an  $\mathcal{A}_{m,n}$ -manifold. By Corollary 4.9 and the inductive hypothesis on the rank, we obtain the statement of the proposition for  $M_2$ . Hence we may assume that  $\Pi_k^{m+n} \upharpoonright M$  has constant rank  $r$ .

Suppose first that  $r = d$ . Notice that in this case  $d \leq k$  and  $M$  is the union of all  $\mathcal{A}_{m,n}$ -manifolds  $M_\iota$  (as in Definition 4.10) such that  $\iota : \{1, \dots, d\} \rightarrow \{m+n-k+1, \dots, m+n\}$  is a strictly increasing sequence. By Corollary 4.9 and the inductive hypothesis on the dimension, the proposition holds then trivially for  $M_\iota$ .

Hence we may assume that  $r < d$ . For  $\iota : \{1, \dots, d\} \rightarrow \{1, \dots, m+n\}$  an increasing sequence, consider the  $\mathcal{A}_{m,n}$ -manifold  $M_\iota$ . If  $r = m_\iota(k)$ , then by the Fibre Cutting Lemma 4.11, Corollary 4.9 and the inductive hypothesis on the dimension, the proposition holds for  $M_\iota$ . Now, one can check that for every  $z \in M_3 := M \setminus \bigcup\{M_\iota : m_\iota(k) = r\}$  we have  $\text{rk}_z \Pi_k^{m+n} \upharpoonright M < r$ , hence  $M_3 = \emptyset$  and we are done.  $\square$

We are now ready to prove the parametrisation result which was announced at the beginning of this section. In Subsection 4.1 we will show that every bounded  $\mathbb{R}_\mathcal{A}$ -definable set is a projection of an  $\mathcal{A}$ -semianalytic set, hence the result below provides a parametrisation theorem for all bounded  $\mathbb{R}_\mathcal{A}$ -definable sets.

**4.13 Theorem.** *Let  $A \subseteq \mathbb{R}^{m+n}$  be a bounded  $\mathcal{A}_{m,n}$ -semianalytic set and let  $k \leq m+n$ . Then there exists  $N \in \mathbb{N}$  and for all  $i = 1, \dots, N$ , there exist  $m'_i, n'_i \in \mathbb{N}$ , with  $m'_i + n'_i = m+n$ , a polyradius  $r_i$ , a sub-quadrant  $Q_i \subseteq \hat{I}_{m'_i, n'_i, r_i}$  and a map  $H_i : \hat{I}_{m'_i, n'_i, r_i} \rightarrow \mathbb{R}^k$ , whose components are in  $\mathcal{A}_{m'_i, n'_i, r_i}$ , such that  $H_i \upharpoonright Q_i : Q_i \rightarrow H_i(Q_i)$  is a diffeomorphism and*

$$\Pi_k^{m+n}(A) = \bigcup_{i=1}^N H_i(Q_i).$$

*Proof.* By Proposition 4.12, there are trivial manifolds  $M_i \subseteq \mathbb{R}^{2(m+n)}$ , with  $\dim(M_i) \leq k$ , such that  $\Pi_k^{m+n}(A) = \bigcup_{i=1}^N \Pi_k^{2(m+n)}(M_i)$  and  $\Pi_k^{2(m+n)} \upharpoonright M_i$  is an immersion. By definition of trivial manifold, there exist maps  $F_i : \hat{I}_{m'_i, n'_i, r_i} \rightarrow I_{0, m+n, r'_i}$  with components in  $\mathcal{A}_{m'_i, n'_i, r_i}$ , and sub-quadrants  $Q_i \subseteq \bar{Q}_i \subseteq \hat{I}_{m'_i, n'_i, r_i}$  of dimension  $q_i \leq k$  such that  $M_i = \Gamma(F_i \upharpoonright Q_i)$ . Let  $G_i : \hat{I}_{m'_i, n'_i, r_i} \rightarrow I_{0, 2(m+n), (r_i, r'_i)}$  be the map defined as  $G_i(z) = (z, F_i(z))$ , whose components are in  $\mathcal{A}_{m'_i, n'_i, r_i}$ . Then the maps  $H_i := \Pi_k^{2(m+n)} \circ G_i$  and the quadrants  $Q_i$  satisfy the required properties.  $\square$

## 4.1 Proof of Theorem A

We use Gabrielov's approach, as illustrated in [DS98, 2.1-2.9].

**4.14 Definition.** Let  $\Lambda_n$  be the collection of all  $\mathcal{A}$ -semianalytic subsets of the unit cube  $[-1, 1]^n$ . Following [DS98, Definition 2.1], we say that a set  $B \subseteq \mathbb{R}^n$  is a *sub- $\Lambda$ -set* if there exist  $m \geq n$  and a set  $A \in \Lambda_n$  such that  $B = \Pi_n^m(A)$ .

The collection  $\Lambda = (\Lambda_n)_{n \in \mathbb{N}}$  clearly satisfies conditions (I),(II) and (III) of [DS98, 2.3]. Condition (IV), namely the  $\Lambda$ -Gabrielov property, is also satisfied by Proposition 4.12, since, up to composing with a homothety, we may assume that the trivial manifolds in the proposition are contained in the unit cube. It follows that  $\Lambda$  satisfies the Theorem of the Complement [DS98, 2.7] and [DS98, Corollary 2.8].

**4.15 Definition.** Let  $n \in \mathbb{N}$  and  $x = (x_1, \dots, x_n)$ . For  $u \in \mathbb{R}$ , define

$$\tau(u) = \begin{cases} u & \text{if } |u| \leq 1 \\ 1/u & \text{if } |u| > 1 \end{cases}.$$

Let  $\iota \subseteq \{1, \dots, n\}$ . Define:

$$R_\iota^n := \{x \in \mathbb{R}^n : |x_i| \leq 1 \text{ for } i \in \iota, |x_i| > 1 \text{ for } i \notin \iota\},$$

$$I_\iota^n := \{x \in \mathbb{R}^n : |x_i| \leq 1 \text{ for } i \in \iota, |x_i| < 1 \text{ for } i \notin \iota\},$$

$$\begin{array}{ccc} \tau_\iota^n : R_\iota^n & \xrightarrow{\quad} & I_\iota^n \\ x & \longmapsto & (\tau(x_1), \dots, \tau(x_n)) \end{array}.$$

**4.16 Remark.** Notice that  $\tau_\iota^n : R_\iota^n \rightarrow I_\iota^n$  is a bijection, hence it commutes with the boolean set operations. In particular, for  $A \subseteq R_\iota^n$ , we have  $\tau_\iota^n(R_\iota^n \setminus A) = I_\iota^n \setminus \tau_\iota^n(A)$ . Let  $\kappa \subseteq \{1, \dots, n\}$  and  $\pi$  be the projection of  $\mathbb{R}^n$  onto the vector subspace spanned by the coordinates  $\{x_i : i \in \kappa\}$ . Let  $\iota^* = \iota \cap \kappa$ . Since  $\tau_\iota^n$  acts coordinate-wise, we have that  $\tau_{\iota^*}^{|\kappa|}(\pi(A)) = \pi(\tau_\iota^n(A))$ . Finally, note that the sets  $R_\iota^n$  partition  $\mathbb{R}^n$  and that the union of all the sets  $I_\iota^n$  is the closed unit cube.

**4.17.** We now prove the model-completeness and o-minimality of  $\mathbb{R}_{\mathcal{A}}$ .

Let  $\tilde{f}_1, \dots, \tilde{f}_k$  be as in Definition 2.19, i.e.  $\tilde{f}_i$  coincides with some  $f_i \in \mathcal{A}_{n', n'', 1}$  (with  $n' + n'' = n$ ) inside the unit cube, and is zero outside the cube. Let  $P(x, y) \in \mathbb{Q}[x, y]$  be a polynomial in  $n + k$  variables. It is easy to see that, if  $A = \{x \in R_\iota^n : P(x, \tilde{f}_1(x), \dots, \tilde{f}_k(x)) = 0\}$ , then the set  $\tau_\iota^n(A)$  is an  $\mathcal{A}$ -basic set (this follows from the fact that the unbounded variables appear only semi-algebraically in the formula defining  $A$ ). Now, routine manipulations show that every set  $A \subseteq \mathbb{R}^n$  definable in the structure  $\mathbb{R}_{\mathcal{A}}$  is of the form

$$A = \bigcup_{\iota \subseteq \{1, \dots, n\}} \bigcup_{\iota^* \subseteq \{1, \dots, m\}} A_{\iota, \iota^*}, \text{ with}$$

$$A_{\iota, \iota^*} = \left\{ x \in R_\iota^n : \mathcal{Q}_1 y_1 \dots \mathcal{Q}_m y_m \ y \in R_{\iota^*}^m \wedge P(x, y, \tilde{f}_1(x, y), \dots, \tilde{f}_k(x, y)) = 0 \right\},$$

where  $y = (y_1, \dots, y_m)$ ,  $\mathcal{Q}_i$  is either the existential or the universal quantifier,  $P$  is a polynomial in  $n + m + k$  variables and  $\tilde{f}_i$  are as in Definition 2.19. Using Remark 4.16, we immediately see that  $\tau_\iota^n(A_{\iota, \iota^*})$  is definable in the structure  $(I, \Lambda)$  (in particular it is a sub- $\Lambda$ -set, by [DS98, Corollary 2.8]) and that  $A_{\iota, \iota^*}$  is existentially definable and has finitely many connected components.

**4.18 Remark.** Let  $A \subseteq [-1, 1]^n$  be an  $\mathbb{R}_{\mathcal{A}}$ -definable set. Using the above decomposition of  $A$  and the substitution  $y_i \mapsto 1/y_i$  whenever  $y_i > 1$ , one can see that  $A$  is actually a sub- $\Lambda$ -set. In particular, the Parametrisation Theorem 4.13 applies to  $A$ .

**4.19.** We now prove polynomial boundedness.

*4.20 Remark.* Let  $f \in \mathcal{A}_1$ . Then, since the support of  $\mathcal{T}(f)$  is a well ordered set,  $f$  is normal. It follows from Lemma 2.17 that if  $g$  is a function of one variable, which is obtained as a finite composition of functions in  $\mathcal{A}$  and vanishing at 0, then  $g \in \mathcal{A}$ .

Let  $\varepsilon > 0$  and let  $f : (0, \varepsilon) \rightarrow \mathbb{R}$  be definable in  $\mathbb{R}_{\mathcal{A}}$ . We proceed as in [DS98, Theorem B, pag. 4419], using Theorem 4.13 and Remark 4.20 instead of 9.6 and [KRS09, Lemma 7.10]<sup>2</sup> instead of 9.9. We conclude that there exists  $h \in \mathcal{A}_1$  such that  $f(t) = h(t)$  for all sufficiently small  $t > 0$ . In particular, there exist  $\alpha \in \mathbb{K}$  and a unit  $u(t) \in \mathcal{A}_1$  such that  $f(t) = t^\alpha u(t)$ . This finishes the proof of Theorem A.

## 5 Vertical monomialisation

In this section we prove Theorem B.

**5.1 Proviso.** Consider the ramifications  $r_{m+i}^{d,\pm} : \hat{I}_{m,n,r} \rightarrow \hat{I}_{m,n,r}$  in Definition 3.1 and let  $\sigma_{m+i}^\pm$  be the restriction of  $r_{m+i}^{1,\pm}$  to the half-space  $\{y'_i \geq 0\}$ . In this section we will consider the transformations  $\sigma_{m+i}^\pm$  as elementary transformations (and extend accordingly the notion of admissible transformation in Definition 3.3). Notice that  $\sigma_{m+i}^+(\hat{I}_{m+1,n-1,r}) \cup \sigma_{m+i}^-(\hat{I}_{m+1,n-1,r}) = \hat{I}_{m,n,r}$ .

The main result of this section is the following.

**5.2 Theorem.** Let  $D \subseteq \mathbb{R}^N$  and  $\eta : D \rightarrow \mathbb{R}$  be an  $\mathbb{R}_{\mathcal{A}}$ -definable function such that the graph of  $\eta$  is a sub- $\Lambda$ -set. Then there exist a polyradius  $r$  and a finite family  $\mathcal{F} = \left\{ \rho_i : \hat{I}_{m_i,n_i,r_i} \rightarrow \mathbb{R}^N \right\}_{i=1,\dots,M}$  (with  $m_i + n_i = N$ ) of admissible transformations such that for all  $i = 1, \dots, M$  the function  $\eta \circ \rho_i$  belongs to  $\mathcal{A}_{m_i,n_i,r_i}$  and its germ at zero is normal, and  $D \cap \hat{I}_{0,N,r} \subseteq \bigcup_{i=1}^M \rho_i \left( \hat{I}_{m_i,n_i,r_i} \right)$ .

Before proving the above theorem, we give a list of consequences.

**5.3 Corollary.** Let  $D \subseteq \mathbb{R}^N$  and  $\eta : D \rightarrow \mathbb{R}$  be an  $\mathbb{R}_{\mathcal{A}}$ -definable function. Then there exist finitely many terms  $t_1, \dots, t_M$  of the language  $\mathcal{L} := \mathcal{L}_{\mathcal{A}} \cup \{-1\} \cup \left\{ \sqrt[n]{\phantom{x}} : n \in \mathbb{N} \right\}$  such that

$$\forall x \in D \exists i \in \{1, \dots, M\} \quad \eta(x) = t_i(x).$$

*Proof.* The proof is by induction on  $N$ . If  $N = 1$ , we can conclude by 4.19, after using the substitution  $x \mapsto -x$  on  $D \cap \mathbb{R}^{<0}$ , the substitution  $x \mapsto 1/x$  on  $D \cap \mathbb{R} \setminus [-1, 1]$  and arguing with  $1/\eta$  on  $\{x \in D : |\eta(x)| > 1\}$ . Hence we may suppose  $N > 1$ . Notice that if  $\dim(D) < N$ , then by cell decomposition we may assume that, up to a permutation of the variables,  $D$  is the graph of some definable function  $\eta' : D' \rightarrow \mathbb{R}$ , where  $D' \subseteq \mathbb{R}^{N-1}$ . Hence, the function  $\eta(x_1, \dots, x_N)$  coincides with the function  $\eta(x_1, \dots, x_{N-1}, \eta'(x_1, \dots, x_{N-1}))$  and we can conclude by the inductive hypothesis.

Let  $\Gamma(\eta)$  be the graph of  $\eta$ . Let  $A = D \times \mathbb{R}$  and consider the partition of  $A$  given by the sets  $A_{\ell,\ell^*}$  as in Subsection 4.1. Then each set  $\Gamma(\eta) \cap A_{\ell,\ell^*}$  is either empty or the graph of an  $\mathbb{R}_{\mathcal{A}}$ -definable function and  $\tau_\ell^{N+1}(\Gamma(\eta) \cap A_{\ell,\ell^*})$  is a sub- $\Lambda$ -set. Moreover,

$$(x, y) \in \Gamma(\eta) \cap A_{\ell,\ell^*} \Leftrightarrow \tau_\ell^{N+1}(x, y) \in \Gamma\left(\tau_\ell^{N+1} \circ \eta \circ (\tau_\ell^{N+1})^{-1}\right) \cap \tau_\ell^{N+1}(A_{\ell,\ell^*}),$$

hence it suffices to prove the corollary for the function  $\tilde{\eta} := \tau_\ell^{N+1} \circ \eta \circ (\tau_\ell^{N+1})^{-1}$ , whose graph is a sub- $\Lambda$ -set. We apply Theorem 5.2 to the function  $\tilde{\eta} : \tilde{D} \rightarrow \mathbb{R}$  and obtain in particular that the function  $\tilde{\eta} \circ \rho_i : \hat{I}_{m_i,n_i,r_i} \rightarrow \mathbb{R}$  is an  $\mathcal{L}$ -term  $t_i$ . Arguing by induction on the length of  $\rho_i$ , it is easy to see that there is a closed sub- $\Lambda$ -set  $S \subseteq \hat{I}_{m_i,n_i,r_i}$  of dimension strictly smaller than  $N$  such that  $\rho_i \upharpoonright \hat{I}_{m_i,n_i,r_i} \setminus S$  is a diffeomorphism onto its image and that the components of the map  $\left(\rho_i \upharpoonright \hat{I}_{m_i,n_i,r_i} \setminus S\right)^{-1}$  are  $\mathcal{L}$ -terms. Hence, for  $x \in \rho_i \left( \hat{I}_{m_i,n_i,r_i} \setminus S \right)$  we have that  $\tilde{\eta}(x) = t_i \circ \rho_i^{-1}(x)$

<sup>2</sup>Notice that in [KRS09, Lemma 7.10] one should replace  $\lambda$  by  $\lambda^{-1/\alpha}$  in the blow-up  $\mathbf{r}^{\rho,\lambda}$  and in the definition of  $g$ .

is an  $\mathcal{L}$ -term. Notice that the function  $\tilde{\eta} \upharpoonright \rho_i(S)$  can be dealt with by using the inductive hypothesis. Hence we have proved the corollary for  $\tilde{\eta} \upharpoonright \tilde{D} \cap \hat{I}_{0,N,r}$ , for some polyradius  $r$ .

By applying the same argument to the function  $\tilde{\eta}(x+a)$  for every point  $a$  of the closed unit cube, we obtain the full statement of the corollary by a compactness argument.  $\square$

As an immediate consequence of the above corollary we obtain:

*Proof of Theorem B.* Let  $A \subseteq \mathbb{R}^N$  be an  $\mathbb{R}_A$ -definable set. We prove by induction on  $N$  that  $A$  is quantifier free definable in the language  $\mathcal{L} := \mathcal{L}_A \cup \{-1\} \cup \{\sqrt[n]{\phantom{x}} : n \in \mathbb{N}\}$ . This is clear if  $N = 1$ . Hence, we may suppose that  $N > 1$  and that, by cell decomposition,  $A$  is a cell; in particular, there exist a cell  $C \subseteq \mathbb{R}^{N-1}$  and  $\mathbb{R}_A$ -definable functions  $f, g : C \rightarrow \mathbb{R}$  such that  $A$  is either the graph of  $f \upharpoonright C$ , or the epigraph of  $f \upharpoonright C$ , or the set  $\{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in C \wedge g(x) < y < f(x)\}$ . By Corollary 5.3, there is a finite partition of  $C$  into definable sets such that on every set of the partition the functions  $f$  and  $g$  coincides with some  $\mathcal{L}$ -terms  $t_1$  and  $t_2$ , respectively. By the inductive hypothesis, each set of the partition is quantifier free definable in the language  $\mathcal{L}$ , hence so is  $A$ , in each of the above cases.  $\square$

Finally, we obtain the following Rectilinearisation Theorem, in the spirit of [Hir73].

**Rectilinearisation Theorem.** *Let  $D \subseteq \mathbb{R}^N$  be a sub- $\Lambda$ -set. Then there exist a neighbourhood  $W$  of 0 in  $\mathbb{R}^N$  and a finite family  $\mathcal{F} = \{(\rho_i, Q_i) : i = 1, \dots, M\}$ , where  $\rho_i : \hat{I}_{m_i, n_i, r_i} \rightarrow \mathbb{R}^N$  is an admissible transformation (with  $m_i + n_i = N$ ) and  $Q_i \subseteq \bar{Q}_i \subseteq \hat{I}_{m_i, n_i, r_i}$  is a sub-quadrant, such that  $\rho_i \upharpoonright Q_i : Q_i \rightarrow \rho_i(Q_i)$  is a diffeomorphism and*

$$W \cap D = \bigcup_{i=1}^M \rho_i(Q_i).$$

The Rectilinearisation Theorem is a consequence of Proposition 5.12 below. We need some definitions.

**5.4 Notation.** Let  $N \in \mathbb{N}$ . For the rest of the section, whenever we write an admissible transformation  $\rho : \hat{I}_{m_\rho, n_\rho, r_\rho} \rightarrow \hat{I}_{m, n, r}$ , we will implicitly assume that  $m + n = m_\rho + n_\rho = N$  and that  $r = (r', r'') \in (0, \infty)^{m_\rho} \times (0, \infty)^{n_\rho}$ ,  $r_\rho = (r'_\rho, r''_\rho) \in (0, \infty)^{m_\rho} \times (0, \infty)^{n_\rho}$  are polyradii in  $\mathbb{R}^N$ .

Recall that, if  $f \in \mathcal{A}_{m, n}$  and  $\rho : \hat{I}_{m_\rho, n_\rho, r_\rho} \rightarrow \hat{I}_{m, n, r}$  is an admissible transformation, then  $f \circ \rho \in \mathcal{A}_{m_\rho, n_\rho}$ . The next definition is intended to extend this property to the case when the arity of  $f$  is bigger than  $N$ .

**5.5 Definition.** Let  $N, l \in \mathbb{N}$  and  $\hat{m}, \hat{n}, k_1, k_2 \in \mathbb{N}$  with  $\hat{m} + \hat{n} = N$  and  $k_1 + k_2 = l$ . Let  $\rho : \hat{I}_{m_\rho, n_\rho, r_\rho} \rightarrow \hat{I}_{m, n, r}$  be an admissible transformation. We say that  $\rho$  *respects* the elements of  $\mathcal{A}_{\hat{m}+k_1, \hat{n}+k_2}$  if  $m \geq \hat{m}$ .

Let  $s = (s_1, s_2) \in (0, \infty)^{k_1} \times (0, \infty)^{k_2}$  be a polyradius and  $\hat{r} := (r', s_1, r'', s_2)$ ,  $\hat{r}_\rho := (r'_\rho, s_1, r''_\rho, s_2)$ . Let  $x = (x_1, \dots, x_N)$ ,  $x' = (x'_1, \dots, x'_N)$ ,  $u = (u_1, \dots, u_l)$ ,  $u' = (u'_1, \dots, u'_l)$ .

Define the map

$$\begin{aligned} \hat{\rho} : \hat{I}_{m_\rho+k_1, n_\rho+k_2, \hat{r}_\rho} &\longrightarrow \hat{I}_{m+k_1, n+k_2, \hat{r}}, \\ \langle x', u' \rangle &\longmapsto \langle x, u \rangle \end{aligned}$$

where  $\langle x', u' \rangle$  is the ordered tuple  $(x'_1, \dots, x'_{m_\rho}, u'_1, \dots, u'_{k_1}, x'_{m_\rho+1}, \dots, x'_N, u'_{k_1+1}, \dots, u'_l)$ ,  $\langle x, u \rangle$  is the ordered tuple  $(x_1, \dots, x_m, u_1, \dots, u_{k_1}, x_{m+1}, \dots, x_N, u_{k_1+1}, \dots, u_l)$ , and  $x = \rho(x')$ ,  $u = u'$ . The map  $\hat{\rho}$  is an admissible transformation, which we call a *trivial extension* of  $\rho$ .

Notice that  $\rho$  respects  $f \in \mathcal{A}_{\hat{m}+k_1, \hat{n}+k_2}$  if and only if  $f \circ \hat{\rho} \in \mathcal{A}_{m_\rho+k_1, n_\rho+k_2, \hat{r}_\rho}$ .

In the next definition we will consider two sets of variables, the “horizontal variables”, usually denoted by  $x$ , and the “vertical variables”, usually denoted by  $u$ . We will define a special type of admissible transformations, the “vertical admissible transformations”, which respect in some way the partition of the set of variables into horizontal and vertical. The aim is to obtain a class of admissible transformations  $\rho : (x', u') \mapsto (x, u)$  with the property that, given  $f \in \mathcal{A}$ , if we can solve explicitly the equation  $f \circ \rho(x', u') = 0$  with respect to  $u'$ , then we can solve explicitly the equation  $f(x, u) = 0$  with respect to  $u$ .

**5.6 Definition.** Let  $r = (s, t)$ ,  $r_\rho = (s_\rho, t_\rho) \in (0, \infty)^N \times (0, \infty)^l$  be polyradii. Consider a map

$$\begin{aligned} \rho: \quad \hat{I}_{r_\rho} &\longrightarrow \hat{I}_r, \\ \langle x', u' \rangle &\longmapsto \langle x, u \rangle \end{aligned}$$

where  $\hat{I}_{r_\rho}$  is either  $\hat{I}_{m_\rho, n_\rho + l, r_\rho}$  (type 1) or  $\hat{I}_{m_\rho + l, n_\rho, r_\rho}$  (type 2) and  $\hat{I}_r$  is either  $\hat{I}_{m, n + l, r}$  (type 1) or  $\hat{I}_{m + l, n, r}$  (type 2). If  $\hat{I}_{r_\rho}$  is of type 1, then let  $\langle x', u' \rangle$  be the ordered pair  $(x', u')$ , whereas if  $\hat{I}_{r_\rho}$  is of type 2, then let  $\langle x', u' \rangle$  be the ordered pair  $(x'_1, \dots, x'_{m_\rho}, u'_1, \dots, u'_l, x'_{m_\rho + 1}, \dots, x'_N)$ . If  $\hat{I}_r$  is of type 1, then let  $\langle x, u \rangle$  be the ordered pair  $(x, u)$  and  $\rho' = (\rho_1, \dots, \rho_N)$ ,  $\rho'' = (\rho_{N+1}, \dots, \rho_{N+l})$ , whereas if  $\hat{I}_r$  is of type 2, then let  $\langle x, u \rangle$  be the ordered pair  $(x_1, \dots, x_m, u_1, \dots, u_l, x_{m+1}, \dots, x_N)$  and  $\rho' = (\rho_1, \dots, \rho_m, \rho_{m+l+1}, \dots, \rho_{N+l})$ ,  $\rho'' = (\rho_{m+1}, \dots, \rho_{m+l})$ .

We say that  $\rho = \langle \rho', \rho'' \rangle$  is a *vertical admissible transformation* (of type  $(i, j) \in \{1, 2\}^2$ ) if the following conditions are satisfied:

- $\hat{I}_{r_\rho}$  is of type  $i$  and  $\hat{I}_r$  is of type  $j$ ;
- $\rho$  is an admissible transformation;
- $\rho'$  does not depend on  $u'$ , hence we may write  $x = \rho'(x')$ ;
- there exists a closed sub- $\Lambda$ -set  $S_\rho \subseteq \hat{I}_{m_\rho, n_\rho, s_\rho}$  such that  $\dim(S_\rho) < N$  and  $\rho' \upharpoonright \hat{I}_{m_\rho, n_\rho, s_\rho} \setminus S_\rho$  is a diffeomorphism onto its image. Moreover, for every  $x' \in \hat{I}_{m_\rho, n_\rho, s_\rho} \setminus S_\rho$ , the map  $u' \mapsto \rho''(\langle x', u' \rangle)$  is a diffeomorphism onto its image, and we denote by  $\gamma_\rho$  the map  $(x', u) \mapsto u'$ .

Notice that  $\rho \upharpoonright \hat{I}_{r_\rho} \setminus S_\rho \times \mathbb{R}^l$  is a diffeomorphism onto its image.

Let  $D \subseteq \mathbb{R}^N$  and  $\Phi : D \rightarrow \mathbb{R}^l$  be a map whose graph is a sub- $\Lambda$ -set and let  $D_\rho := (\rho')^{-1}(D) \setminus S_\rho$ . Suppose that  $\Phi(\rho'(D_\rho)) \subseteq \rho''(\hat{I}_{r_\rho})$ . We define the map  $\Phi_\rho : D_\rho \rightarrow \mathbb{R}^l$  as follows: for every  $x' \in D_\rho$ ,  $\Phi_\rho(x') := \gamma_\rho(x', \Phi \circ \rho'(x'))$ .

Examples of admissible transformations which are not vertical are the blow-up chart  $(x, u) \mapsto (xu, u)$  and the linear transformation  $(x, u) \mapsto (x + cu, u)$ , because in these cases the first component of the transformation depends on the vertical variable  $u$ . Another example is the blow-up chart  $(x, u_1, u_2) \mapsto (x, u_1 u_2, u_2)$ , because the second component of the transformation is not a bijection.

Our aim in this section is to give a monomialisation algorithm which only uses vertical admissible transformations. We will do so at the expenses of covering a proper subset of the domain of the functions. Such a subset will turn out to be the graph of a definable map. This motivates the next definition.

**5.7 Definition.** Let  $N, l \in \mathbb{N}$  with  $N \geq l$ . Let  $S \subseteq D \subseteq \mathbb{R}^N$  be sub- $\Lambda$ -sets, with  $\dim(S) < N$ .

1. Let  $\mathcal{F}$  be a finite family of admissible transformations

$$\rho : \hat{I}_{m_\rho, n_\rho, r_\rho} \rightarrow \hat{I}_{m, n, r}.$$

We say that  $\mathcal{F}$  *satisfies the covering property with respect to  $D$*  if for every choice of polyradii  $r'_\rho \leq r_\rho$  ( $\rho \in \mathcal{F}$ ) there exists a polyradius  $r^* \leq r$  such that

$$\hat{I}_{m, n, r^*} \cap D \cap \rho(\hat{I}_{m_\rho, n_\rho, r'_\rho}) \neq \emptyset \text{ and } \hat{I}_{m, n, r^*} \cap D \subseteq \bigcup_{\rho \in \mathcal{F}} \rho(\hat{I}_{m_\rho, n_\rho, r'_\rho}).$$

2. Let  $\Phi : D \rightarrow \mathbb{R}^l$  be a map whose graph is a sub- $\Lambda$ -set and let  $\mathcal{F}$  be a finite family of vertical admissible transformations (of the same type)

$$\rho : \hat{I}_{r_\rho} \rightarrow \hat{I}_r.$$

We say that  $\mathcal{F}$  *satisfies the covering property with respect to  $(\Phi, S)$*  if for every choice of polyradii  $r'_\rho \leq r_\rho$  ( $\rho \in \mathcal{F}$ ) there exists a polyradius  $r^* \leq r$  such that

$$\hat{I}_{r^*} \cap \Gamma(\Phi \upharpoonright D \setminus S) \cap \rho(\hat{I}_{r'_\rho}) \neq \emptyset \text{ and } \hat{I}_{r^*} \cap \Gamma(\Phi \upharpoonright D \setminus S) \subseteq \bigcup_{\rho \in \mathcal{F}} \rho(\hat{I}_{r'_\rho}).$$

The following remarks will be used several times throughout the rest of the paper.

**5.8 Remarks.** Let  $D$  and  $\Phi$  be as above.

1. Let  $\mathcal{F}$  be a finite family of admissible transformations which satisfies the covering property with respect to  $D$ . Suppose that for every  $\rho \in \mathcal{F}$  there is a finite family  $\tilde{\mathcal{F}}_\rho$  of admissible transformations such that  $\tilde{\mathcal{F}}_\rho$  satisfies the covering property with respect to  $\rho^{-1}(D)$ . Then  $\mathcal{G}$  satisfies the covering property with respect to  $D$ , where  $\mathcal{G} = \{\rho \circ \tilde{\rho} : \rho \in \mathcal{F}, \tilde{\rho} \in \tilde{\mathcal{F}}_\rho\}$ .
2. Let  $\mathcal{F}$  be a finite family of admissible transformations which satisfies the covering property with respect to  $D$ . In the notation of Definition 5.5, let  $\pi$  be the projection  $\langle x, u \rangle \mapsto x$  and  $\hat{D} \subseteq \mathbb{R}^{N+l}$  be a sub- $\Lambda$ -set such that  $\pi(\hat{D}) = D$ . Then  $\hat{\mathcal{F}}$  satisfies the covering property with respect to  $\hat{D}$ , where  $\hat{\mathcal{F}}$  is the family of all admissible transformations  $\hat{\rho}$  obtained by extending trivially  $\rho \in \mathcal{F}$ .
3. Let  $\mathcal{F}$  be a finite family of admissible transformations and suppose that  $\mathcal{F}$  satisfies the covering property with respect to  $D$ . Then  $\hat{\mathcal{F}}$  satisfies the covering property with respect to  $(\Phi, \emptyset)$ , where  $\hat{\mathcal{F}}$  is a finite family of vertical admissible transformations (of the same type) obtained by extending trivially each  $\rho \in \mathcal{F}$  to  $\hat{\rho} : \hat{I}_{r_\rho} \rightarrow \hat{I}_r$ , where  $\hat{I}_{r_\rho}, \hat{I}_r$  are of either of the two types. Conversely, if  $\mathcal{F}$  is a family of vertical admissible transformations which satisfies the covering property with respect to  $(\Phi, \emptyset)$ , then the family  $\mathcal{F}' = \{\rho' : \rho \in \mathcal{F}\}$  of admissible transformations satisfies the covering property with respect to  $D$ .
4. Let  $\mathcal{F}$  be a finite family of vertical admissible transformations (of the same type). Suppose that  $S \supseteq \bigcup_{\rho \in \mathcal{F}} \rho(S_\rho)$  and that  $\mathcal{F}$  satisfies the covering property with respect to  $(\Phi, S)$ . Suppose furthermore that for every  $\rho \in \mathcal{F}$  there are a finite family  $\tilde{\mathcal{F}}_\rho$  of vertical admissible transformations (of the same type) and a sub- $\Lambda$ -set  $S'_\rho \supseteq \bigcup_{\tilde{\rho} \in \tilde{\mathcal{F}}_\rho} \tilde{\rho}(S_{\tilde{\rho}})$  of dimension strictly smaller than  $N$  such that  $\tilde{\mathcal{F}}_\rho$  satisfies the covering property with respect to  $(\Phi_\rho, S'_\rho)$ . Then  $\mathcal{G}$  satisfies the covering property with respect to  $(\Phi, S')$ , where  $\mathcal{G} = \{\rho \circ \tilde{\rho} : \rho \in \mathcal{F}, \tilde{\rho} \in \tilde{\mathcal{F}}_\rho\}$  and  $S' = S \cup \bigcup_{\rho \in \mathcal{F}} \rho(S'_\rho)$ .

The above definitions and remarks allow us to revisit the statement of Theorem 3.16.

**5.9 Remark.** Let  $f \in \mathcal{A}_{\hat{m}+k_1, \hat{n}+k_2}$  and  $g \in \mathcal{A}_{\hat{m}, \hat{n}}$ . Define  $m := \max\{\hat{m}, \check{m}\}$  and  $n := N - m$ . Then we can apply Theorem 3.16 to  $g$ , seen as an element of  $\mathcal{A}_{m, n}$  and obtain that the admissible transformations in the statement respect  $f$ . Moreover, using Remark 3.2, the conclusion of Theorem 3.16 can be strengthened by saying that  $\mathcal{F}$  satisfies the covering property with respect to  $\hat{I}_{m, n, r'}$ .

The purpose of statements **(B)** and **(C)** in the next theorem is to solve a given system of equations with respect to the vertical variables. Statement **(A)** implies directly Theorem 5.2.

**5.10 Notation.** Let  $\Phi = (\varphi_1, \dots, \varphi_l) : D \rightarrow \mathbb{R}^l$  be a map whose graph is a sub- $\Lambda$ -set. We denote by  $j(\Phi)$  the cardinality of the set  $\{i : 1 \leq i \leq l, \varphi_i \text{ is not identically } 0\}$ . Up to a permutation, we may always assume that the first  $j(\Phi)$  coordinates of  $\Phi$  are not identically zero. We set  $\hat{\Phi} = \emptyset$  if  $j(\Phi) = 0$  and  $\hat{\Phi} = (\varphi_1, \dots, \varphi_{j(\Phi)})$  if  $j(\Phi) > 0$ . For  $F(x, u) \in \mathcal{A}_{\hat{m}, \hat{n}+l}$ , we let  $F_0(x, u_1, \dots, u_{j(\Phi)}) := F(x, u_1, \dots, u_{j(\Phi)}, 0) \in \mathcal{A}_{\hat{m}, \hat{n}+j(\Phi)}$ .

**5.11 Theorem.** Let  $N, l, \hat{m}, \hat{n}, k_1, k_2 \in \mathbb{N}$  with  $l \leq N = \hat{m} + \hat{n}$  and let  $f \in \mathcal{A}_{\hat{m}+k_1, \hat{n}+k_2}$ . Let  $D \subseteq [-1, 1]^N$  be a sub- $\Lambda$ -set and suppose that for every sufficiently small polyradius  $r'$  in  $\mathbb{R}^N$ , the intersection  $\hat{I}_{\hat{m}, \hat{n}, r'} \cap D$  is not empty.

**(A)<sub>N</sub>** Let  $\eta : D \rightarrow \mathbb{R}$  be a function whose graph is a sub- $\Lambda$ -set. Then there exists a finite family  $\mathcal{F}$  of admissible transformations

$$\rho : \hat{I}_{m_\rho, n_\rho, r_\rho} \rightarrow \hat{I}_{m, n, r}$$

such that  $\mathcal{F}$  satisfies the covering property with respect to  $D$  and for every  $\rho \in \mathcal{F}$ ,

- $m = \hat{m}$ , hence  $\rho$  respects  $f$ ;
- $\eta \circ \rho \in \mathcal{A}_{m_\rho, n_\rho, r_\rho}$  and is normal.

(B)<sub>N,l</sub> Let  $\tilde{m}, \tilde{n} \in \mathbb{N}$  with  $\tilde{m} + \tilde{n} = N$  and let  $F(x, u) \in \mathcal{A}_{\tilde{m}, \tilde{n}+l}$ . Let  $\dim(D) = N$  and  $\Phi : D \rightarrow \mathbb{R}^l$  be a map whose graph is a sub- $\Lambda$ -set.

Then there exist a finite family  $\mathcal{F}$  of vertical admissible transformations

$$\rho : \hat{I}_{m_\rho, n_\rho + j(\Phi), r_\rho} \rightarrow \hat{I}_{m, n + j(\Phi), r}$$

and a sub- $\Lambda$ -set  $S \subseteq D$  of dimension strictly smaller than  $N$  such that  $\mathcal{F}$  satisfies the covering property with respect to  $(\hat{\Phi}, S)$  and for every  $\rho \in \mathcal{F}$ ,

- $m = \max\{\hat{m}, \tilde{m}\}$ , hence  $\rho'$  respects  $f$  and  $\rho$  respects  $F$ ;
- either  $j(\hat{\Phi}_\rho) < j(\Phi)$  or  $F_0 \circ \rho$  is normal.

(C)<sub>N,l</sub> Let  $\tilde{m}, \tilde{n} \in \mathbb{N}$  with  $\tilde{m} + \tilde{n} = N$  and let  $f_1(x, u), \dots, f_N(x, u) \in \mathcal{A}_{\tilde{m}, \tilde{n}+N}$ . Let  $\dim(D) = N$  and  $\Phi : D \rightarrow \mathbb{R}^N$  be a map whose graph is a sub- $\Lambda$ -set.

Suppose that  $j(\Phi) = l$  and that

$$\forall (x, u) \in D \times \mathbb{R}^N \quad \begin{cases} f_1(x, u) = 0 \\ \vdots \\ f_N(x, u) = 0 \end{cases} \Leftrightarrow u = \Phi(x).$$

Then there exist a finite family  $\mathcal{F}$  of vertical admissible transformations

$$\rho : \hat{I}_{m_\rho, n_\rho + N, r_\rho} \rightarrow \hat{I}_{m, n + N, r}$$

and a sub- $\Lambda$ -set  $S \subseteq D$  of dimension strictly smaller than  $N$  such that  $\mathcal{F}$  satisfies the covering property with respect to  $(\Phi, S)$  and for every  $\rho \in \mathcal{F}$ ,

- $m = \max\{\hat{m}, \tilde{m}\}$ , hence  $\rho'$  respects  $f$  and  $\rho$  respects  $f_1, \dots, f_N$ ;
- $\forall (x, u) \in D_\rho \setminus (\rho')^{-1}(S) \times \mathbb{R}^N \quad \begin{cases} f_1 \circ \rho(x, u) = 0 \\ \vdots \\ f_N \circ \rho(x, u) = 0 \end{cases} \Leftrightarrow u = 0.$

Before proving Theorem 5.11, we show how it implies the Rectilinearisation Theorem. We prove the following stronger statement.

**5.12 Proposition.** Let  $m, n, k_1, k_2 \in \mathbb{N}$ , with  $N = m + n$ , and  $f \in \mathcal{A}_{m+k_1, n+k_2}$ . Let  $D \subseteq \mathbb{R}^N$  be a sub- $\Lambda$ -set. Then there exist a neighbourhood  $W$  of 0 in  $\mathbb{R}^N$  and a finite family  $\mathcal{F} = \{(\rho_i, Q_i) : i = 1, \dots, M\}$ , where  $\rho_i : \hat{I}_{m_i, n_i, r_i} \rightarrow \hat{I}_{m, n, r}$  is an admissible transformation (which respects  $f$ ) and  $Q_i \subseteq \bar{Q}_i \subseteq \hat{I}_{m_i, n_i, r_i}$  is a sub-quadrant, such that  $\rho_i \upharpoonright Q_i : Q_i \rightarrow \rho_i(Q_i)$  is a diffeomorphism and

$$W \cap \hat{I}_{m, n, r} \cap D = \bigcup_{i=1}^M \rho_i(Q_i).$$

*Proof.* Notice that, by Proposition 4.4 and Remark 5.9, the proposition has already been proved whenever  $D$  is an  $\mathcal{A}$ -basic set. We aim to show that we can reduce to this situation.

The proof is by induction on the pairs  $(N, d)$ , where  $d = \dim(D)$ , ordered lexicographically. The basic cases  $(0, 0)$  and  $(N, 0)$  are straightforward. By a cell decomposition argument, we may assume that  $D$  is a cell and, without loss of generality, that

$$D = \{(x, y) : x \in C, \theta(x, y)\},$$

where, either

- (Case 1)  $d = N$ ,  $x = (x_1, \dots, x_{N-1})$ ,  $y = x_N$ ,  $C \subseteq \mathbb{R}^{N-1}$  is a sub- $\Lambda$ -set and a cell of dimension  $N - 1$  and  $\theta(x, y)$  is  $y > \varphi_0(x)$ , where the graph of the function  $\varphi_0 : C \rightarrow \mathbb{R}$  is a sub- $\Lambda$ -set, or



- (Case 2)  $d < N$ ,  $x = (x_1, \dots, x_d)$ ,  $y = (y_1, \dots, y_{N-d}) = (x_{d+1}, \dots, x_N)$ ,  $C \subseteq \mathbb{R}^d$  is a sub- $\Lambda$ -set and a cell of dimension  $d$  and  $\theta(x, y)$  is  $\bigwedge_{i=1}^{N-d} y_i = \varphi_i(x)$ , where the graphs of the functions  $\varphi_i : C \rightarrow \mathbb{R}$  are sub- $\Lambda$ -sets.

We will treat the two cases simultaneously. Notice that in both cases  $c := \dim(C) \leq N - 1$ .

We first show that we can assume that  $\varphi_i \in \mathcal{A}$ . In fact, by the statement **(A)**<sub>c</sub>, there exists a finite family  $\mathcal{F}$  of admissible transformations  $\rho : \hat{I}_{l'_1, l'_2, r'} \rightarrow \hat{I}_{l_1, l_2, r}$ , where  $l_1 + l_2 = l'_1 + l'_2 = c$ , such that

$$\hat{I}_{l_1, l_2, r} \cap C \subseteq \bigcup_{\rho \in \mathcal{F}} \rho(\hat{I}_{l'_1, l'_2, r'})$$

and for all  $\rho \in \mathcal{F}$  we have that  $\rho$  respects  $f$  and  $\varphi_i \circ \rho \in \mathcal{A}_{l'_1, l'_2}$ . Arguing by induction on the length of  $\rho$  it is easy to see that there exists a closed sub- $\Lambda$ -set  $S_\rho \subseteq \mathbb{R}^c$  of dimension  $< c$  such that  $\rho \upharpoonright \hat{I}_{l'_1, l'_2, r'} \setminus S_\rho$  is a diffeomorphism onto its image. Let  $B_\rho = \rho^{-1}(\hat{I}_{l_1, l_2, r} \cap C)$ ,  $B'_\rho = B_\rho \setminus S_\rho$  and  $B''_\rho = B_\rho \cap S_\rho$ . Let  $\hat{\rho} : \hat{I}_{m', n', \hat{r}'} \rightarrow \hat{I}_{m, n, \hat{r}}$  be the trivial extension of  $\rho$  and  $\hat{\mathcal{F}} = \{\hat{\rho} : \rho \in \mathcal{F}\}$ . Let

$$\begin{aligned} D'_\rho &= \{(x, y) : x \in B'_\rho, \theta_\rho(x, y)\}, \\ D''_\rho &= \{(x, y) : x \in B''_\rho, \theta_\rho(x, y)\}, \end{aligned}$$

where  $\theta_\rho(x, y)$  is  $y > \varphi_0 \circ \rho(x)$  in case 1 and  $\bigwedge_{i=1}^{N-d} y_i = \varphi_i \circ \rho(x)$  in case 2. Notice that there exists a neighbourhood  $W \subseteq \mathbb{R}^N$  of 0 such that

$$\bigcup_{\hat{\rho} \in \hat{\mathcal{F}}} \hat{\rho}(D'_\rho) \cup \hat{\rho}(D''_\rho) = W \cap \hat{I}_{m, n, \hat{r}} \cap D.$$

In both cases  $\dim(\hat{\rho}(D''_\rho)) \leq \dim(D''_\rho) < d$ , so by the inductive hypothesis on the dimension  $d$  of  $D$ , and by the fact that  $\hat{\rho} \upharpoonright D'_\rho$  is a diffeomorphism onto its image, we have reduced to the situation where the  $\varphi_i$  are in  $\mathcal{A}$ .

Next we show that we can reduce to the case when  $C$  is a sub-quadrant (and hence  $D$  is an  $\mathcal{A}$ -basic set). In order to see this, since  $c < N$ , by the inductive hypothesis on the dimension  $N$  of the ambient space, we can apply the proposition to  $C$ . Hence there are a neighbourhood  $\tilde{W} \subseteq \mathbb{R}^c$  of 0 and a finite family  $\mathcal{F}$  of admissible transformations  $\rho : \hat{I}_{l'_1, l'_2, r'} \rightarrow \hat{I}_{l_1, l_2, r}$  (respecting  $f, \varphi_0, \dots, \varphi_{N-l}$ ), where  $l_1 + l_2 = l'_1 + l'_2 = c$ , and sub-quadrants  $Q \subseteq \tilde{Q} \subseteq \hat{I}_{l'_1, l'_2, r'}$  such that  $\rho \upharpoonright Q$  is a diffeomorphism onto its image and  $\tilde{W} \cap \hat{I}_{l_1, l_2, r} \cap C = \bigcup_{(\rho, Q) \in \mathcal{F}} \rho(Q)$ . Let  $\hat{\rho} : \hat{I}_{m', n', \hat{r}'} \rightarrow \hat{I}_{m, n, \hat{r}}$

be the trivial extension of  $\rho$  and  $\hat{\mathcal{F}} = \{\hat{\rho} : \rho \in \mathcal{F}\}$ . Let

$$D_\rho = \{(x, y) : x \in Q, \theta_\rho(x, y)\},$$

where  $\theta_\rho(x, y)$  is  $y > \varphi_0 \circ \rho(x)$  in case 1 and  $\bigwedge_{i=1}^{N-d} y_i = \varphi_i \circ \rho(x)$  in case 2. Notice that there exists a neighbourhood  $W \subseteq \mathbb{R}^N$  of 0 such that

$$\bigcup_{\hat{\rho} \in \hat{\mathcal{F}}} \hat{\rho}(D_\rho) = W \cap \hat{I}_{m, n, \hat{r}} \cap D,$$

and since  $\hat{\rho} \upharpoonright D_\rho$  is a diffeomorphism onto its image, we have reduced to the situation where  $D$  is an  $\mathcal{A}$ -basic set, and we are done.  $\square$

The rest of the section is devoted to the proof of Theorem 5.11, which is by induction on the pairs  $(N, l)$ , ordered lexicographically.

If  $N = 0$ , then statements **(A)**, **(B)** and **(C)** are trivial.

For any  $N$ , we have that **(B)**<sub>N,0</sub> follows from Theorem 3.16 and Remark 5.9 and **(C)**<sub>N,0</sub> is trivially true. Hence let us assume that  $N, l \geq 1$ .

**5.13 Lemma.** Let  $N \geq 1$  and suppose that  $(\mathbf{A})_{N-1}$  holds. Let  $l, \hat{m}, \hat{n}, k_1, k_2 \in \mathbb{N}$  with  $l \leq N = \hat{m} + \hat{n}$  and let  $f \in \mathcal{A}_{\hat{m}+k_1, \hat{n}+k_2}$ . Let  $D \subseteq [-1, 1]^N$  be a sub- $\Lambda$ -set and suppose that for every sufficiently small polyradius  $r'$  in  $\mathbb{R}^N$ , the intersection  $\hat{I}_{\hat{m}, \hat{n}, r'} \cap D$  is not empty. Let  $\dim(D) = N$  and  $\Phi : D \rightarrow \mathbb{R}^l$  be a map whose graph is a sub- $\Lambda$ -set such that  $j(\Phi) = l$ . Then there are constants  $K_1, K_2 \in \mathbb{R}^{>0}$  and a finite family  $\mathcal{F}$  of admissible transformations

$$\rho : \hat{I}_{m_\rho, n_\rho, r_\rho} \rightarrow \hat{I}_{m, n, r}$$

such that  $\mathcal{F}$  satisfies the covering property with respect to  $D$  and for every  $\rho \in \mathcal{F}$ ,

- $m = \hat{m}$ , hence  $\rho$  respects  $f$ ;
- for every  $i = 1, \dots, l$  there are exponents  $\alpha_i \in \mathbb{A}^n$  and functions  $U_i : D \rightarrow \mathbb{R}$  (whose graph is a sub- $\Lambda$ -set) with  $K_1 \leq |U_i| \leq K_2$  such that  $\varphi_i \circ \rho(x) = x^{\alpha_i} U_i(x)$ .

**5.14 Remark.** Notice that we do not require at this stage that  $U_i$  be an element of  $\mathcal{A}$ , hence we say that  $\varphi_i \circ \rho$  is weakly normal.

*Proof.* Let us first consider the case  $N = 1$ . Notice that, by polynomial boundedness, for all  $x \in D \cap \mathbb{R}^{\geq 0}$  we have  $\varphi_i(x) = x^{\alpha_i/\beta_i} U_i(x)$ , for some  $U_i$  as in the statement of the lemma and  $\alpha_i, \beta_i \in \mathbb{A}$ . Let  $\beta = \beta_1 \cdot \dots \cdot \beta_l$ . If  $\hat{m} = 1$  then  $\mathcal{F} = \{\sigma_1^+ \circ r_1^\beta\}$  satisfies the conclusion of the lemma. If  $\hat{n} = 1$  then  $\mathcal{F} = \{\sigma_1^+ \circ r_1^\beta, \sigma_1^- \circ r_1^\beta\}$  satisfies the conclusion of the lemma.

Let  $N > 1$ . Let  $\hat{x} = (x_1, \dots, x_{N-1})$ ,  $y = x_N$  and  $\hat{D}$  be the projection of  $D$  onto the coordinate space spanned by  $\hat{x}$ . By cell decomposition and by [DS02, Thm. 2.1], we may assume that there are  $\beta_1, \dots, \beta_l \in \mathbb{K}$ ,  $\mathbb{R}_{\mathcal{A}}$ -definable functions  $a_0, a_1, \dots, a_l : \hat{D} \rightarrow \mathbb{R}$  and  $U_1, \dots, U_l : D \rightarrow \mathbb{R}$  such that  $\forall x = (\hat{x}, y) \in D$   $y > a_0(\hat{x})$  and  $\forall i = 1, \dots, l$

$$\forall x = (\hat{x}, y) \in D \quad \varphi_i(\hat{x}, y) = (y - a_0(\hat{x}))^{\beta_i} a_i(\hat{x}) U_i(\hat{x}, y) \quad \text{and} \quad \frac{1}{2} < |U_i| < \frac{3}{2}.$$

By a further cell decomposition we may assume that all the units  $U_i$  are positive on  $D$  and that for all  $i = 0, 1, \dots, l$  either  $\forall \hat{x} \in \hat{D}$   $a_i(\hat{x}) \leq 1$  or  $\forall \hat{x} \in \hat{D}$   $a_i(\hat{x}) > 1$ . For  $i = 0, \dots, l$  let

$$\hat{a}_i(\hat{x}) := \begin{cases} a_i(\hat{x}) & \text{if } a_i \leq 1 \text{ on } \hat{D} \\ 1/a_i(\hat{x}) & \text{if } a_i > 1 \text{ on } \hat{D} \end{cases}$$

and let  $\hat{a}(\hat{x}) = \prod_{i=0}^l \hat{a}_i(\hat{x})$ . By Remark 4.18 the graph of  $\hat{a}$  is a sub- $\Lambda$ -set. Hence, by  $(\mathbf{A})_{N-1}$ , there is a finite family  $\mathcal{F}$  of admissible transformations such that every  $\rho \in \mathcal{F}$  extends trivially to  $\hat{\rho} : \hat{I}_{m_\rho, n_\rho, r_\rho} \rightarrow \hat{I}_{\hat{m}, \hat{n}, r}$  (hence  $\hat{\rho}$  respects  $f$ ),  $\hat{\mathcal{F}} : \{\hat{\rho} : \rho \in \mathcal{F}\}$  satisfies the covering property with respect to  $D$  and for all  $\hat{\rho} \in \hat{\mathcal{F}}$ , for all  $i = 0, \dots, l$  we have  $\hat{a}_i \circ \hat{\rho} \in \mathcal{A}_{m_\rho, n_\rho, r_\rho}$ . Let

$$g_\rho(\hat{x}, y) = \begin{cases} y - \hat{a}_0 \circ \rho(\hat{x}) & \text{if } a_0 \leq 1 \text{ on } \hat{D} \\ y \cdot \hat{a}_0 \circ \rho(\hat{x}) - 1 & \text{if } a_0 > 1 \text{ on } \hat{D} \end{cases}$$

and  $h_\rho(\hat{x}, y) := g_\rho(\hat{x}, y) \cdot \prod_{i=0}^l \hat{a}_i \circ \rho(\hat{x}) \in \mathcal{A}_{m_\rho, n_\rho, r_\rho}$ . By Theorem 3.16 and Remark 5.9, there is a finite family  $\tilde{\mathcal{F}}_\rho$  of admissible transformations such that  $\tilde{\mathcal{F}}_\rho$  satisfies the covering property with respect to  $\hat{I}_{m_\rho, n_\rho, r_\rho}$  and for every  $\tilde{\rho} : \hat{I}_{m_{\tilde{\rho}}, n_{\tilde{\rho}}, r_{\tilde{\rho}}} \rightarrow \hat{I}_{m_\rho, n_\rho, r_\rho} \in \tilde{\mathcal{F}}_\rho$  we have that  $\tilde{\rho}$  respects  $f \circ \hat{\rho}$  and finally there are  $\gamma_0, \dots, \gamma_{l+1} \in \mathbb{A}^N$  and units  $v_0(x), \dots, v_{l+1}(x) \in \mathcal{A}_{m_{\tilde{\rho}}, n_{\tilde{\rho}}, r_{\tilde{\rho}}}$  such that

$$\hat{a}_i \circ \rho \circ \tilde{\rho}(x) = x^{\gamma_i} v_i(x) \quad \text{for } i = 0, \dots, l, \text{ and } g_\rho \circ \tilde{\rho}(x) = x^{\gamma_{l+1}} v_{l+1}(x).$$

After a suitable sequence  $\sigma$  of sign-changing transformations as in 5.1, we may assume that  $x \in (\mathbb{R}^{\geq 0})^N$ , hence,

$$\varphi_i \circ \hat{\rho} \circ \tilde{\rho} \circ \sigma(x) = \begin{cases} x^{\gamma_{l+1}\beta_i + \gamma_i} U_i(x) & \text{if } a_0 \leq 1 \text{ and } a_i \leq 1 \text{ on } \hat{D} \\ x^{\gamma_{l+1}\beta_i - \gamma_i} U_i(x) & \text{if } a_0 \leq 1 \text{ and } a_i > 1 \text{ on } \hat{D} \\ x^{-\gamma_0 + \gamma_{l+1}\beta_i + \gamma_i} U_i(x) & \text{if } a_0 > 1 \text{ and } a_i \leq 1 \text{ on } \hat{D} \\ x^{-\gamma_0 + \gamma_{l+1}\beta_i - \gamma_i} U_i(x) & \text{if } a_0 > 1 \text{ and } a_i > 1 \text{ on } \hat{D} \end{cases},$$

for some  $U_i$  as in the statement of the lemma. Notice that all the multi-exponents in the expression above belong to  $(\mathbb{K}^{\geq 0})^N$ , because  $\Phi$  is bounded. Hence, after an appropriate series of ramifications, the above multi-exponents belong to  $\mathbb{A}^N$  and we can conclude by the first remark in 5.8.  $\square$

### 5.1 (A) $_{N-1}$ and (B) $_{N,l'}$ ( $\forall l' \leq l-1$ ) imply (B) $_{N,l}$

Let  $\hat{u} = (u_1, \dots, u_{j(\Phi)-1})$  and  $v = u_{j(\Phi)}$ . Let  $\Phi' = (\varphi_1, \dots, \varphi_{j(\Phi)-1})$ .

We first show the following reduction.

**5.15.** We can assume that  $F_0(x, \hat{u}, v)$  is regular of some order  $d > 0$  in the variable  $v$ , after factoring out a monomial in the variables  $x, \hat{u}$ .

It is enough to show that the series  $\mathcal{T}(F_0)(X, \hat{U}, V)$  is regular in the variable  $V$ . Clearly, we cannot simply perform a linear transformation in the variables  $x_{\tilde{m}+1}, \dots, x_N, \hat{u}$  as we did in Section 3.2, because that would not be a vertical transformation. We will follow a different approach, inspired by the methods in commutative algebra to prove that the ring of formal power series is Noetherian. Clearly the ring of generalised power series is not Noetherian. However, Lemma 5.18 below gives us a finiteness result which is enough for our purposes.

**5.16 Notation.** Let  $m, n, m', n', l \in \mathbb{N}$ , with  $m + n = m' + n' = N$ . Let  $Z = (Z_1, \dots, Z_m)$ ,  $Y = (Y_1, \dots, Y_n)$ ,  $U = (U_1, \dots, U_l)$  and  $Z' = (Z'_1, \dots, Z'_{m'})$ ,  $Y' = (Y'_1, \dots, Y'_{n'})$ ,  $U' = (U'_1, \dots, U'_l)$ . If  $V \in \{Y_1, \dots, Y_n, U_1, \dots, U_l\}$ , we denote by  $\widehat{(Y, U)}$  the set  $\{Y_1, \dots, Y_n, U_1, \dots, U_l\} \setminus \{V\}$ .

**5.17 (Formal Weierstrass division).** Let  $F, G \in \mathbb{R}[[Z^*, Y, U]]$  and suppose that  $G$  is *regular of order*  $d \in \mathbb{N}$  in the variable  $V$ , i.e.  $G(0, V) = V^d W(V)$ , where  $W(V) \in \mathbb{R}[[V]]$  and  $W(0) \neq 0$ . Then, by [DS98, 4.17], there exist  $Q \in \mathbb{R}[[Z^*, Y, U]]$  and  $R = \sum_{i=0}^{d-1} B_i(Z, \widehat{(Y, U)}) V^i \in \mathbb{R}[[Z^*, \widehat{(Y, U)}]] [V]$  such that  $F = G \cdot Q + R$ . A careful analysis of the proof of [DS98, 4.17] shows that  $\text{Supp}_Z(Q)$  and  $\text{Supp}_Z(R)$  are contained in the good set  $\Sigma(\text{Supp}_Z(F) \cup \text{Supp}_Z(G))$ .

**5.18 Lemma.** Let  $\mathcal{G} = \{F_i = (F_{i,0}, \dots, F_{i,d-1}) : i \in \mathbb{N}\} \subset (\mathbb{R}[[Z^*, Y, U]])^d$  be a family with good total support. Then there exists an admissible tree  $T$  such that, for every branch  $\mathbf{b}$  of  $T$ , acting as  $T_{\mathbf{b}} : \mathbb{R}[[Z^*, Y, U]] \rightarrow \mathbb{R}[[Z'^*, Y', U']]$ , the  $\mathbb{R}[[Z'^*, Y', U']]$ -module generated by the set

$$\{(T_{\mathbf{b}}(F_{i,1}), \dots, T_{\mathbf{b}}(F_{i,d})) : i \in \mathbb{N}\}$$

is finitely generated. Moreover,  $T_{\mathbf{b}}$  induces an admissible transformation which is vertical (with respect to the variables  $U$ ).

In particular, for some  $p \in \mathbb{N}$ ,  $\{(T_{\mathbf{b}}(F_{i,0}), \dots, T_{\mathbf{b}}(F_{i,d})) : i = 0, \dots, p\}$  is a set of generators.

*Proof.* The proof is by induction on the pairs  $(N + l, d)$ , ordered lexicographically.

Let us first examine the case  $d = 1$ . If  $N + l = 1$ , then there exist  $\alpha \in [0, \infty)$  and  $i_0 \in \mathbb{N}$  such that  $\forall i \in \mathbb{N} F_i = Z'^{\alpha} G_i$  and  $G_{i_0}(0) \neq 0$ . Therefore the ideal generated by the set  $\{F_i : i \in \mathbb{N}\}$  is principal, generated by  $F_{i_0}$ .

Let us suppose  $N + l > 1$ . By Lemma 3.13, there exists an admissible tree  $T$  (acting as the identity on the variables  $U$ ) such that, for every branch  $\mathbf{b}$  of  $T$ , acting as  $T_{\mathbf{b}} : \mathbb{R}[[Z^*, Y, U]] \rightarrow \mathbb{R}[[Z'^*, Y', U']]$ , there exist  $\alpha \in [0, \infty)^m$  and  $G_i \in \mathbb{R}[[Z'^*, Y', U']]$  such that  $T_{\mathbf{b}}(F_i) = Z'^{\alpha} G_i$  and for some  $i_0 \in \mathbb{N}$ ,  $G_{i_0}(0, Y', U') \neq 0$ .

If  $Y' = \emptyset$ , then there is a linear change of coordinates  $L_{l,c}$  in the  $U'$ -variables such that  $G_{i_0}$  is regular of order  $d' \in \mathbb{N}$  in the variable  $U'_l$ .

If  $Y' \neq \emptyset$ , then there is a linear change of coordinates  $L_{n',c}$  in the  $(Y', U')$ -variables such that  $G_{i_0}$  is regular of order  $d' \in \mathbb{N}$  in the variable  $Y'_{n'}$ . Let  $V$  be either  $U'_l$  (in the first case) or  $Y'_{n'}$  (in the second case) and let  $L_c$  be either  $L_{l,c}$  (in the first case) or  $L_{n',c}$  (in the second case), and let us rename  $Z' = Z$ ,  $Y' = Y$  and  $U' = U$ . By 5.17, there are  $Q_i \in \mathbb{R}[[Z^*, Y, U]]$  and  $B_{i,0}, \dots, B_{i,d'-1} \in \mathbb{R}[[Z^*, \widehat{(Y, U)}]]$  such that

$$L_c(G_i) = L_c(G_{i_0}) \cdot Q_i + R_i,$$

where  $R_i = \sum_{j=0}^{d'-1} B_{i,j} \left( Z, \widehat{(Y, U)} \right) V^j$ .

By Remark 3.7, the family  $\{L_c(G_i) : i \in \mathbb{N}\}$  has good total support, and hence, by 5.17, so does the family  $\mathcal{B} = \{B_i = (B_{i,0}, \dots, B_{i,d'-1}) : i \in \mathbb{N}\} \subset \left( \mathbb{R} \llbracket Z^*, \widehat{(Y, U)} \rrbracket \right)^{d'}$ .

By the inductive hypothesis, there is an admissible tree  $T'$  (acting as the identity on the variable  $V$ ) such that, for every branch  $\mathfrak{b}'$  of  $T'$ , we have  $T'_{\mathfrak{b}'} : \mathbb{R} \llbracket Z^*, Y, U \rrbracket \rightarrow \mathbb{R} \llbracket Z'^*, Y', U' \rrbracket$ , the  $\mathbb{R} \llbracket Z'^*, \widehat{(Y', U')} \rrbracket$ -module generated by the set  $T'_{\mathfrak{b}'}(\mathcal{B})$  is finitely generated and  $T'_{\mathfrak{b}'}$  acts vertically with respect to the variables  $U$ . Let us again rename  $Z' = Z$ ,  $Y' = Y$  and  $U' = U$ . We may suppose that, for some  $k \in \mathbb{N}$ , the generators are  $T'_{\mathfrak{b}'}(B_0), \dots, T'_{\mathfrak{b}'}(B_k)$ . Hence,  $\forall i \in \mathbb{N}$ , there exist series  $C_{i,0}, \dots, C_{i,k} \in \mathbb{R} \llbracket Z^*, \widehat{(Y, U)} \rrbracket$  such that  $T'_{\mathfrak{b}'}(B_i) = \sum_{s=0}^k C_{i,s} T'_{\mathfrak{b}'}(B_s)$ .

Notice that

$$T'_{\mathfrak{b}'}(R_i) = \sum_{j=0}^{d'-1} T'_{\mathfrak{b}'}(B_{i,j}) V^j = \sum_{s=0}^k C_{i,s} \sum_{j=0}^{d'-1} T'_{\mathfrak{b}'}(B_s) V^j = \sum_{s=0}^k C_{i,s} T'_{\mathfrak{b}'}(R_s).$$

Finally,

$$\begin{aligned} T_{\mathfrak{b}} \circ L_c \circ T'_{\mathfrak{b}'}(F_i) &= T'_{\mathfrak{b}'}(Z^\alpha \cdot L_c(G_{i_0}) \cdot Q_i) + \sum_{s=0}^k C_{i,s} T'_{\mathfrak{b}'}(Z^\alpha \cdot R_s) = \\ &= [L_c \circ T'_{\mathfrak{b}'}(Z^\alpha \cdot G_{i_0})] \cdot T'_{\mathfrak{b}'}(Q_i) + \sum_{s=0}^k C_{i,s} \cdot [L_c \circ T'_{\mathfrak{b}'}(Z^\alpha \cdot G_i) - L_c \circ T'_{\mathfrak{b}'}(Z^\alpha \cdot G_{i_0}) \cdot T'_{\mathfrak{b}'}(Q_i)]. \end{aligned}$$

The series within square brackets in the last line of above formula are indeed elements of the ideal generated by the set  $S = \{T_{\mathfrak{b}} \circ L_c \circ T'_{\mathfrak{b}'}(F_i) : i \in \mathbb{N}\}$ . In particular, we can choose a finite set of generators within the set  $S$  itself. This concludes the proof of the case  $d = 1$ .

As for the general case, consider the family  $\mathcal{G}' = \{(F_{i,1}, \dots, F_{i,d-1}) : i \in \mathbb{N}\} \subset (\mathbb{R} \llbracket Z^*, Y, U \rrbracket)^{d-1}$ . Since the total support of  $\mathcal{G}'$  is good, we can apply the inductive hypothesis and find an admissible tree  $T$  such that, for every branch  $\mathfrak{b}$  of  $T$ , the module generated by  $T_{\mathfrak{b}}(\mathcal{G}')$  is finitely generated.

Let  $\{(T_{\mathfrak{b}}(F_{i,1}), \dots, T_{\mathfrak{b}}(F_{i,d-1})) : i \leq p\}$  be a set of generators, for some  $p \in \mathbb{N}$ . Now consider the family  $\mathcal{G}'' = \{T_{\mathfrak{b}}(F_{i,0}) : i \in \mathbb{N}\} \subset \mathbb{R} \llbracket Z'^*, Y', U' \rrbracket$ . By Remark 3.7,  $\mathcal{G}''$  has good total support, hence there exists an admissible tree  $T'$  such that, for every branch  $\mathfrak{b}'$  of  $T'$ , the ideal generated by  $T'_{\mathfrak{b}'}(\mathcal{G}'')$  is finitely generated. Let  $\{T_{\mathfrak{b}} \circ T'_{\mathfrak{b}'}(F_{i,0}) : i \leq s\}$  be a set of generators, for some  $s \in \mathbb{N}$ . It is then clear that the module generated by  $T'_{\mathfrak{b}'} \circ T_{\mathfrak{b}}(\mathcal{G})$  is generated by the set  $\{(T_{\mathfrak{b}} \circ T'_{\mathfrak{b}'}(F_{i,0}), 0) : i \leq s\} \cup \{(0, T_{\mathfrak{b}} \circ T'_{\mathfrak{b}'}(F_{i,1}), \dots, T_{\mathfrak{b}} \circ T'_{\mathfrak{b}'}(F_{i,d-1})) : i \leq p\}$ .  $\square$

Going back to the proof of 5.15, let  $F_0(x, \hat{u}, v) = \sum_{i \geq 0} f_i(x, \hat{u}) v^i$ , where  $f_i(x, \hat{u}) = \frac{1}{i!} \frac{\partial^i F_0}{\partial v^i}(x, \hat{u}, 0)$ , which, by a remark in 2.15, belongs to  $\mathcal{A}_{\tilde{m}, \tilde{n}+j(\Phi)-1}$ . The family  $\mathcal{G} = \{f_i : i \in \mathbb{N}\}$  has good total support, hence, by Lemma 5.18 and Remark 3.2, there is a finite family  $\mathcal{F}$  of vertical admissible transformations  $\rho : \hat{I}_{m, n, n_\rho+j(\Phi)-1, r_\rho} \rightarrow \hat{I}_{m, n+j(\Phi)-1, r}$ , where  $m = \max\{\hat{m}, \tilde{m}\}$  (hence they all respect  $f$ ), such that  $\mathcal{F}$  satisfies the covering property with respect to  $\hat{I}_{m, n+j(\Phi)-1, r}$  (and hence with respect to  $\Phi'$ ) and for every  $\rho \in \mathcal{F}$ , either  $j(\Phi'_\rho) < j(\Phi')$ , or  $j(\Phi'_\rho) = j(\Phi') = j(\Phi) - 1 \leq l - 1$  and the ideal generated by the family  $\{f_i \circ \rho : i \in \mathbb{N}\}$  is generated by  $f_0 \circ \rho, \dots, f_p \circ \rho$ , for some  $p \in \mathbb{N}$ , i.e. there are series  $Q_{i,n}(X, \hat{U})$ , not necessarily in  $\text{Im}(\mathcal{T})$ , such that for all  $n \in \mathbb{N}$ ,  $f_n \circ \rho = \sum_{i=0}^p Q_{i,n} \cdot f_i \circ \rho$ . Hence we can write

$$\mathcal{T}(F_0 \circ \rho) = \sum_{i=0}^p \mathcal{T}(f_i \circ \rho) \left( X, \hat{U} \right) V^i W_i \left( X, \hat{U}, V \right),$$

where the series  $W_i = 1 + \sum_{n > p} Q_{i,n}(X, \hat{U}) V^{n-i}$  are units, not necessarily in  $\text{Im}(\mathcal{T})$ . We can apply the inductive hypothesis  $(\mathbf{B})_{N, j(\Phi)-1}$  to  $\tilde{F}(x, \hat{u}) := \prod_{0 \leq i, j \leq p, i \neq j} f_i \circ \rho (f_i \circ \rho - f_j \circ \rho)$  and

$\tilde{\Phi} := \Phi'_\rho : D_\rho \rightarrow \mathbb{R}^{j(\Phi')}$  and obtain that there exist a sub- $\Lambda$ -set  $\tilde{S} \subseteq D_\rho$  of dimension  $\leq N - 1$  and

a finite family  $\tilde{\mathcal{F}}$  of vertical admissible transformations  $\tilde{\rho} : \hat{I}_{m_{\tilde{\rho}}, n_{\tilde{\rho}}+j(\Phi'), r_{\tilde{\rho}}} \rightarrow \hat{I}_{m_{\rho}, n_{\rho}+j(\Phi'), r_{\rho}}$  such that  $\tilde{\mathcal{F}}$  satisfies the covering property with respect to  $(\tilde{\Phi}, \tilde{S})$  and for every  $\tilde{\rho} \in \tilde{\mathcal{F}}$ ,  $\tilde{\rho}$  respects  $f \circ \rho$  and either  $j(\tilde{\Phi}_{\tilde{\rho}}) < j(\tilde{\Phi})$  or  $f_0 \circ \rho \circ \tilde{\rho}, \dots, f_p \circ \rho \circ \tilde{\rho}$  are all normal and (by Lemma 3.9) linearly ordered by division. In this latter case, after factoring out a monomial, we obtain that  $F_0 \circ \rho \circ \tilde{\rho}$  is regular in  $v$ . By Remarks 5.8, we have proved 5.15.

The next step is to show the following reduction.

**5.19.** We may assume that  $F_0$  is regular of order 1 in the variable  $v$ .

Since we may assume that  $F_0$  is regular of some order  $d > 1$  in the variable  $v$ , after a Tschirnhausen translation, we can write

$$F_0(x, \hat{u}, v) = W(x, \hat{u}, v) v^d + a_2(x, \hat{u}) v^{d-2} + \dots + a_d(x, \hat{u}),$$

where  $a_i \in \mathcal{A}_{\tilde{m}, \tilde{n}+j(\Phi)-1}$  and  $W \in \mathcal{A}_{\tilde{m}, \tilde{n}+j(\Phi)}$  is a unit. By Lemma 5.13, we may assume that there are constants  $K_1, K_2 > 0$ , a function  $U : D \rightarrow \mathbb{R}$  (whose graph is a sub- $\Lambda$ -set) and a multi-exponent  $\alpha \in \mathbb{A}^N$  such that  $\varphi_{j(\Phi)}(x) = x^\alpha U(x)$  and  $K_1 < |U| < K_2$ . Without loss of generality we may assume  $U$  is strictly positive on  $D$ .

Let  $a_0(x, \hat{u}) = 1$ ,  $a_1(x, \hat{u}) = 0$  and consider the family  $\{x^{\alpha(d-i)a_i}(x, \hat{u})\}_{i=0, \dots, d}$ . We apply (B) $_{N, j(\Phi)-1}$  simultaneously to the members of this family, i.e. to the function  $A(x, \hat{u}) = \prod_{0 \leq i, j \leq d, i \neq j} x^{\alpha(d-i)a_i}(x, \hat{u}) (x^{\alpha(d-i)a_i}(x, \hat{u}) - x^{\alpha(d-j)a_j}(x, \hat{u}))$ , and to  $\Phi'$ . Hence there are a sub- $\Lambda$ -set  $\tilde{S} \subseteq \tilde{D}$  of dimension  $< N$  and a finite family of vertical admissible transformations which extends trivially to a finite family  $\mathcal{F}$  of vertical admissible transformations  $\rho : \hat{I}_{m_{\rho}, n_{\rho}+j(\Phi), r_{\rho}} \rightarrow \hat{I}_{m, n+j(\Phi), r}$ , where  $m = \max\{\tilde{m}, \hat{m}\}$  and  $n = N - m$  (hence  $\rho'$  respects  $f$ ), such that  $\mathcal{F}$  satisfies the covering property with respect to  $(\hat{\Phi}, \tilde{S})$  and for every  $\rho \in \mathcal{F}$ , either  $j(\hat{\Phi}_{\rho}) < j(\hat{\Phi})$  or, for all  $i = 2, \dots, d$ , there exist  $\gamma_i \in \mathbb{A}^N$ ,  $\delta_i \in \mathbb{N}^{j(\Phi)-1}$  and units  $W_i \in \mathcal{A}_{m_{\rho}, n_{\rho}+j(\Phi)-1}$  such that  $a_i \circ \rho(x, \hat{u}) = x^{\gamma_i} \hat{u}^{\delta_i} W_i(x, \hat{u})$ . Moreover, by Lemma 3.9, the family  $\{(x^{\alpha(d-i)a_i}(x, \hat{u})) \circ \rho\}_{i=0, \dots, d}$  is linearly ordered by division, hence the set of  $(N + j(\varphi))$ -tuples  $\mathcal{E} = \{(\beta(d-i) + \gamma_i, \delta_i)\}_{i=0, \dots, d}$  is totally ordered. Notice that  $\varphi_{j(\Phi)} \circ \rho'(x)$  is still weakly normal, i.e. there exists  $\beta \in \mathbb{A}^N$  such that  $\varphi_{j(\Phi)} \circ \rho'(x) = x^\beta U \circ \rho'(x)$ , and, after suitable ramifications of the variables  $x_1, \dots, x_{\tilde{m}}$ , we may suppose that  $\beta \in \mathbb{N}^N$ .

Let  $N_0 = \max\{M \leq N : \beta_M \neq 0\}$ . Define, for  $j = 1, \dots, N_0 - 1$ ,  $\rho_{\lambda, j}(x, \hat{u}, v) = x_j^{\beta_j} v$  and  $\rho_{\lambda, N_0}(x, \hat{u}, v) = x_{N_0}^{\beta_{N_0}} (\lambda + v)$ . Notice that the function  $(x', \hat{u}', v') \mapsto v = \rho_{\lambda, 1} \circ \dots \circ \rho_{\lambda, N_0-1} \circ \rho_{\lambda, N_0}(x', \hat{u}', v')$  is a finite composition of blow-up charts and extends trivially to a vertical admissible transformation  $\rho_\lambda : \hat{I}_{m_{\rho}, n_{\rho}+j(\Phi), r_\lambda} \rightarrow \hat{I}_{m_{\rho}, n_{\rho}+j(\Phi), r_\rho}$ .

Let

$$\begin{aligned} \varepsilon : \mathbb{R}^{>0} &\longrightarrow \mathbb{R}^{>0} \\ \lambda &\longmapsto \varepsilon_\lambda \end{aligned}$$

be any fixed function and let  $\mathcal{G}$  be a finite family of positive real numbers such that  $[K_1, K_2] \subseteq \bigcup_{\lambda \in \mathcal{G}} (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)$ .

Let  $\mathcal{F}_{\mathcal{G}} = \{\rho_\lambda : \lambda \in \mathcal{G}\}$ . Since for all  $x \in D$  there exists  $\lambda \in \mathcal{G}$  such that  $x^\beta (\lambda - \varepsilon_\lambda) < \varphi_{j(\Phi)}(x) < x^\beta (\lambda + \varepsilon_\lambda)$ , the family  $\mathcal{F}_{\mathcal{G}}$  satisfies the covering property with respect to  $(\hat{\Phi}_\rho, S_\rho)$ .

We show that, for every  $\rho_\lambda \in \mathcal{F}_{\mathcal{G}}$ , either  $j\left(\left(\hat{\Phi}_\rho\right)_{\rho_\lambda}\right) < j(\hat{\Phi}_\rho)$ , or, possibly after factoring out a monomial in the variables  $x$ ,  $F_0 \circ \rho \circ \rho_\lambda$  is regular of order  $d' < d$  in the variable  $v$ . Let  $\tilde{F} = F_0 \circ \rho \circ \rho_\lambda$  and  $\tilde{W}(x, \hat{u}, v) = W \circ \rho \circ \rho_\lambda(x, \hat{u}, v)$ , which is still a unit. Then,

$$\tilde{F}(x, \hat{u}, v) = \tilde{W}(x, \hat{u}, v) (\lambda + v)^d x^{\beta d} + (\lambda + v)^{d-2} x^{\beta(d-2)+\gamma_2} \hat{u}^{\delta_2} W_2(x, \hat{u}) + \dots + x^{\gamma_d} \hat{u}^{\delta_d} W_d(x, \hat{u}).$$

Let  $i_0 = \min\{i : 0 \leq i \leq d, (\beta(d-i) + \gamma_i, \delta_i) = \min(\mathcal{E})\}$ . Notice that necessarily  $\delta_{i_0} = 0$ . After factoring  $\tilde{F}$  out by the monomial  $x^{\beta(d-i_0)+\gamma_{i_0}}$ , we obtain that, either  $i_0 = 0$  and, thanks to the Tschirnhausen translation we did at the beginning, the coefficient of the term  $v^{d-1}$  is a unit, or  $i_0 > 0$  and the coefficient of the term  $v^{d-i_0}$  is a unit. In either of the two cases, after factorisation

$\tilde{F}$  has become regular of order  $d' < d$  in  $v$ . Hence we can start again with the procedure just described until we reduce to  $d' = 1$ . By Remarks 5.8, this concludes the proof of 5.19.

Since  $F_0$  is regular of order 1 in the variable  $v$ , after performing a last Tschirnhausen translation  $\tau_h$  as in the proof of Theorem 3.11, we have that, either  $j(\hat{\Phi}_{\tau_h}) < j(\hat{\Phi})$ , or  $F_0 \circ \tau_h(x, \hat{u}, v) = x^\alpha \hat{u}^\beta v^\gamma W(x, \hat{u}, v)$  for some  $\alpha \in \mathbb{A}^N$ ,  $(\beta, \gamma) \in \mathbb{N}^{j(\Phi)}$  and a unit  $W \in \mathcal{A}_{m, n+j(\Phi)}$  (where  $m = \max\{\tilde{m}, \hat{m}\}$ ). Let  $r$  be a polyradius in  $\mathbb{R}^{N+j(\Phi)}$  such that  $W$  has a representative which does not vanish on  $I_{m, n+j(\Phi), r}$ . Notice that the size of the last coordinate of  $r$  determines the choice of  $\varepsilon_\lambda$  in the proof of 5.19.

Hence we can conclude the proof of  $(\mathbf{B})_{N, l}$  by Remarks 5.8.

## 5.2 $(\mathbf{B})_{N, N}$ and $(\mathbf{C})_{N, l'}$ ( $\forall l' \leq l - 1$ ) imply $(\mathbf{C})_{N, l}$

We apply  $(\mathbf{B})_{N, N}$  to  $F(x, u) := \prod_{i=1}^N f_i(x, u)$  and  $\Phi$  and obtain that there exist a finite family  $\mathcal{F}$  of vertical admissible transformations  $\rho : \hat{I}_{m_\rho, n_\rho+l, r_\rho} \rightarrow \hat{I}_{m, n+l, r}$ , where  $m = \max\{\hat{m}, \tilde{m}\}$  (hence  $\rho'$  respects  $f$ ), and a sub- $\Lambda$ -set  $S \subseteq D$  of dimension strictly smaller than  $N$  such that  $\mathcal{F}$  satisfies the covering property with respect to  $(\hat{\Phi}, S)$  and for every  $\rho \in \mathcal{F}$ , either  $j(\hat{\Phi}_\rho) < j(\hat{\Phi})$  or  $F_0 \circ \rho$  is normal. Let  $\hat{\mathcal{F}}$  be the family obtained by extending trivially each  $\rho \in \mathcal{F}$  to  $\hat{\rho} : \hat{I}_{m_\rho, n_\rho+N, \hat{r}_\rho} \rightarrow \hat{I}_{m, n+N, \hat{r}}$ . By Remarks 5.8,  $\hat{\mathcal{F}}$  satisfies the covering property with respect to  $(\Phi, S)$ .

Suppose first that  $j(\hat{\Phi}_\rho) = l' < l$ . Let  $\tilde{f}_i(x, u) = f_i \circ \hat{\rho}(x, u)$ ,  $\tilde{D} = D_\rho$  and  $\tilde{\Phi} = (\hat{\Phi}_\rho, 0) : \tilde{D} \rightarrow \mathbb{R}^N$ . Then by  $(\mathbf{C})_{N, l'}$  there exist a finite family  $\tilde{\mathcal{F}}$  of vertical admissible transformations  $\tilde{\rho} : \hat{I}_{m_{\tilde{\rho}}, n_{\tilde{\rho}}+N, \tilde{r}_{\tilde{\rho}}} \rightarrow \hat{I}_{m_\rho, n_\rho+N, \tilde{r}_\rho}$  (respecting  $f \circ \rho'$ ) and a sub- $\Lambda$ -set  $\tilde{S} \subseteq \tilde{D}$  of dimension strictly smaller than  $N$  such that  $\tilde{\mathcal{F}}$  satisfies the covering property with respect to  $(\tilde{\Phi}, \tilde{S})$  and for every  $\tilde{\rho} \in \tilde{\mathcal{F}}$ ,

$$\forall (x, u) \in \tilde{D}_\rho \times \mathbb{R}^N \quad \begin{cases} \tilde{f}_1 \circ \tilde{\rho}(x, u) = 0 \\ \vdots \\ \tilde{f}_N \circ \tilde{\rho}(x, u) = 0 \end{cases} \Leftrightarrow u = 0.$$

Hence, by Remarks 5.8, we are done in this case.

Now suppose that  $F_0 \circ \rho$  is normal. Let  $\hat{u} = (u_1, \dots, u_l)$ . Then there are  $(\alpha_i, \beta_i) \in \mathbb{A}^N \times \mathbb{N}^l$  and units  $U_i \in \mathcal{A}_{m_\rho, n_\rho+l}$  such that  $\tilde{f}_i(x, \hat{u}) := (f_i)_0 \circ \rho(x, \hat{u}) = x^{\alpha_i} u^{\beta_i} U_i(x, \hat{u})$ . Let  $\tilde{D} = D_\rho$  and  $\tilde{\Phi} = \hat{\Phi}_\rho$ . Notice that

$$\forall (x, \hat{u}) \in \tilde{D} \times \mathbb{R}^l \quad \begin{cases} \tilde{f}_1(x, \hat{u}) = 0 \\ \vdots \\ \tilde{f}_N(x, \hat{u}) = 0 \end{cases} \Leftrightarrow \hat{u} = \tilde{\Phi}(x). \quad (*)$$

Let  $\tilde{S}$  be the sub- $\Lambda$ -set (of dimension strictly smaller than  $N$ )  $\{x_1 = 0\} \cup \dots \cup \{x_N = 0\}$ . Clearly we may assume that  $\rho(\tilde{S}) \subseteq S$ . Then we have

$$\forall (x, \hat{u}) \in \tilde{D} \setminus \tilde{S} \times \mathbb{R}^l \quad \begin{cases} \tilde{f}_1(x, \hat{u}) = 0 \\ \vdots \\ \tilde{f}_N(x, \hat{u}) = 0 \end{cases} \Leftrightarrow \begin{cases} \hat{u}^{\beta_1} = 0 \\ \vdots \\ \hat{u}^{\beta_N} = 0 \end{cases}. \quad (**)$$

We claim that  $\forall x \in \tilde{D} \setminus \tilde{S} \quad \tilde{\Phi}(x) = 0$ . In fact if this is not the case then, without loss of generality, for some  $x \in \tilde{D} \setminus \tilde{S}$  we have  $\tilde{\varphi}_1(x) \neq 0$ . This means that for every  $a \in \mathbb{R}$ , the tuple  $(x, a, \tilde{\varphi}_2(x), \dots, \tilde{\varphi}_l(x))$  satisfies the system of equations on the right side of (\*\*). But this contradicts the equivalence in (\*).

To conclude, notice that

$$\forall (x, u) \in \tilde{D} \setminus \tilde{S} \times \mathbb{R}^N \quad \begin{cases} f_1 \circ \hat{\rho}(x, u) = 0 \\ \vdots \\ f_N \circ \hat{\rho}(x, u) = 0 \end{cases} \Leftrightarrow \hat{u} = \tilde{\Phi}(x), u_{l+1} = \dots = u_N = 0.$$

### 5.3 (A)<sub>N-1</sub> and (C)<sub>N,N</sub> imply (A)<sub>N</sub>

Notice that, if  $N = 1$ , then by 4.19, for all  $x \in D \cap \mathbb{R}^{\geq 0}$  we have  $\eta(x) = x^{\alpha/\beta} U(x)$ , for some unit  $U \in \mathcal{A}_1$  and  $\alpha, \beta \in \mathbb{A}$ . If  $\hat{m} = 1$  then take  $\mathcal{F} = \{\sigma_1^+ \circ r_1^\beta\}$ , if  $\hat{n} = 1$  then take  $\mathcal{F} = \{\sigma_1^+ \circ r_1^\beta, \sigma_1^- \circ r_1^\beta\}$ . In both cases  $\mathcal{F}$  satisfies the conclusion of (A)<sub>1</sub>.

Arguing as in the first paragraph of the proof of Corollary 5.3 and by the inductive hypothesis (A)<sub>N-1</sub>, we may assume that  $\dim(D) = N$ . By the same argument, it is enough to prove the statement for  $\eta \upharpoonright D \setminus S$ , where  $S$  is any sub- $\Lambda$ -set of dimension  $< N$ .

Since  $\Gamma(\eta)$  is a sub- $\Lambda$ -set, we can apply the Parametrisation Theorem 4.13 and obtain that  $\Gamma(\eta)$  is the finite union of the diffeomorphic images of sub-quadrants under maps whose components are in  $\mathcal{A}$ . Without loss of generality, we may assume that  $\Gamma(\eta) = H(Q)$ , where  $Q \subseteq (\mathbb{R}^{>0})^N$  is a sub-quadrant of dimension  $N$  and  $H = (g_1, \dots, g_N, h) \in (\mathcal{A}_{N,0})^{N+1}$ . Moreover,

$$\forall (x, z) \in \mathbb{R}^{N+1} \quad x \in D \text{ and } z = \eta(x) \Leftrightarrow \exists! u \in Q \text{ s.t. } \begin{cases} x_1 = g_1(u) \\ \vdots \\ x_N = g_N(u) \\ z = h(u) \end{cases}.$$

In particular, there is a map  $\Phi : D \rightarrow Q$ , whose graph is a sub- $\Lambda$ -set, such that

$$\forall (x, u) \in D \times Q \quad \begin{cases} f_1(x, u) = 0 \\ \vdots \\ f_N(x, u) = 0 \end{cases} \Leftrightarrow u = \Phi(x),$$

where  $f_i(x, u) = x_i - g_i(u)$ . Note that the  $f_i$  might not satisfy the hypotheses of (C)<sub>N,N</sub>, because the variables  $u$  may appear with real exponents in  $\mathcal{T}(f_i)$ . Our next task is to reduce to the case when  $f_i \in \mathcal{A}_{\tilde{m}, \tilde{n}+N}$ . In order to do this, we first apply Lemma 5.13 in order to reduce to the case when all the components of  $\Phi$  are weakly normal (notice that  $j(\Phi) = N$ ), i.e.  $\varphi_i(x) = x^{\alpha_i} U_i(x)$ , where  $\alpha_i \in \mathbb{A}$  and  $\mathbb{R}_{\mathcal{A}}$ -definable functions  $U_i$  which are bounded away from zero. Secondly, we argue as in the proof of 5.19 and produce a finite family  $\mathcal{F}_{\mathcal{G}} = \{\rho_\lambda : \in \mathcal{G}\}$  of vertical admissible transformations

$$\rho_\lambda : \hat{I}_{m, n+N, r_\lambda} \rightarrow \hat{I}_{m+N, n, r}$$

of type (1,2), obtained by performing a series of ramifications and blow-up charts, such that  $\rho_\lambda(x, u) = \langle x, x^{\alpha_1}(\lambda + u_1), \dots, x^{\alpha_N}(\lambda + u_N) \rangle$ . Notice that  $f_i \circ \rho_\lambda \in \mathcal{A}_{m, n+N}$ .

Now we can apply (C)<sub>N,N</sub> to the  $f_i \circ \rho_\lambda$  and  $\Phi_{\rho_\lambda}$  and obtain that there exist a finite family  $\tilde{\mathcal{F}}$  of vertical admissible transformations

$$\tilde{\rho} : \hat{I}_{m_{\tilde{\rho}}, n_{\tilde{\rho}}+N, r_{\tilde{\rho}}} \rightarrow \hat{I}_{m, n+N, r}$$

and a sub- $\Lambda$ -set  $\tilde{S} \subseteq D$  of dimension strictly smaller than  $N$  such that  $\mathcal{F}$  satisfies the covering property with respect to  $(\Phi, S)$  and for every  $\tilde{\rho} \in \tilde{\mathcal{F}}$ ,

$$\forall (x, u) \in D_\rho \setminus (\tilde{\rho}')^{-1}(\tilde{S}) \times \mathbb{R}^N \quad \begin{cases} f_1 \circ \rho_\lambda \circ \tilde{\rho}(x, u) = 0 \\ \vdots \\ f_N \circ \rho_\lambda \circ \tilde{\rho}(x, u) = 0 \end{cases} \Leftrightarrow u = 0.$$



Summing up, there are a finite family  $\mathcal{F}$  of vertical admissible transformations

$$\rho : \hat{I}_{m_\rho, n_\rho + N, r_\rho} \rightarrow \hat{I}_{m+N, n, r}$$

of type (1, 2) such that  $\rho'$  respects  $f$  and  $\rho''$  respects  $h$ , and a sub- $\Lambda$ -set  $S \subseteq D$  of dimension strictly smaller than  $N$  such that, by Remarks 5.8, the family  $\mathcal{F}' = \{\rho' : \rho \in \mathcal{F}\}$  satisfies the covering property with respect to  $D \setminus S$  and for every  $\rho \in \mathcal{F}$ ,

$$\eta \circ \rho'(x) = h \circ \rho''(x, 0) \in \mathcal{A}_{m_\rho, n_\rho}.$$

We conclude the proof of  $(\mathbf{A})_N$  by Theorem 3.16 and Remark 5.9. As in the proof of 5.19, the size of the domain on which  $\eta$  has become normal determines the choice of  $\varepsilon_\lambda$ .

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