# On quas-monotonous graphs

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**Abstract**

A *dominating coloring* by $k$ colors is a proper $k$ coloring where every color $i$ has a representative vertex $x_i$ adjacent to at least one vertex in each of the other classes. The *b-chromatic number*, $b(G)$, of a graph $G$ is the largest integer $k$ such that $G$ admits a dominating coloring by $k$ colors.

A graph $G = (V, E)$ is said *b-monotonous* if $b(H_1) \geq b(H_2)$ for every induced subgraph $H_1$ of $G$ and every subgraph $H_2$ of $H_1$.

Here we say that a graph $G$ is *quasi b-monotonous* , or simply quasi-monotonous, if for every vertex $v \in V$, $b(G - v) \leq b(G) + 1$.

We show study the quasi-monotonicity of several classes. We show in particular that chordal graphs are not quasi-monotonous in general,whereas chordal graphs with large b-chromatic number, and $(P, coP, chair, cochair)$-free graphs are quasi-monotonous; $(P_5, coP_5, P)$-free graphs are monotonous.Finally we give new bounds for the b-chromatic number of any vertex deleted subgraph of a chordal graph.

*Key words:*

## 1 Introduction

All graphs considered here are simple and undirected. We denote by $P_n$ (respectively $C_n$) an elementary path (resp. an elementary cycle) with $n$ vertices. Let $G$ be a graph with a proper coloring. For any two disjoints subsets $A$ and $B$, let $E(A, B)$ be the set of edges of $G$ with one extremity in $A$ and the other in $B$. Let $u_i$ be any vertex $u$ of color $i$. Let us denote by $\mathcal{C}_i$ the class of color $i$. If $y$ is a vertex of the graph $G$, let $N_i(y)$ be the set of neighbours of $y$ of color $i$; while for any integer $p$ non zero, $N^p(y)$ is the set of vertices at distance exactly $p$ from $y$.

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In a proper coloring, a vertex $x_i$ of color $i$ is said a *dominating vertex* if $x_i$ adjacent to at least one vertex in each of the other classes. The vertex $x_i$ is also called a dominant. The color $i$ is said *dominating* if there exists at least a vertex of color $i$ which is dominating. A *dominating coloring* by $k$ colors is a proper $k$ coloring where every color $i$ has a at least a dominating vertex.

The *b-chromatic number*, $b(G)$, of a graph $G$ is the largest integer $k$ such that $G$ admits a dominating coloring by $k$ colors. A dominating coloring with $b(G)$ colors will be called a b-coloring.

This parameter was defined by Irving and Manlove [6]. They proved that determining $b(G)$ for an arbitrary graph $G$ is an NP-complete problem.

For a given graph $G$, it may be easily remarked that $\chi(G) \leq b(G) \leq \Delta(G)+1$. If we are limited to regular graphs, Kratochvil et al. proved in [7] that for a $d$-regular graph $G$ with at least $d^4$ vertices, $b(G) = d + 1$. Kouider and El Sahili ([4]) proved that for every regular graph of girth 5 and no induced cycle $C_6$ the same equality holds.

Let $v$ be any vertex of a graph $G$. It is known that $\chi(G - v) \leq \chi(G)$. The function $\chi$ is said monotonous. This is not the case for the $b$ chromatic number.

A graph $G = (V, E)$ is called *b-monotonous* if $b(H_1) \geq b(H_2)$ for every induced subgraph $H_1$ of $G$ and every subgraph $H_2$ of $H_1$. This was a definition of Bonomo et al.

Here we say that a graph $G$ is *quasi b-monotonous* , or simply quasi-monotonous, if for every vertex $v \in V$, $b(G - v) \leq b(G) + 1$.

A *chordal* graph is a graph where every cycle of length at least 4 has at least one chord. A *quasi-line* graph is a graph where the neighborhood of every vertex has a partition into at most 2 cliques. A *$P_4$-sparse graph* is a graph where every 5-vertex subset contains at most one induced $P_4$.

It was shown by Bonomo and al.([2]) that $P_4$-sparse graphs are $b$ monotonous. On the other hand, we showed in [5] that every graph of girth at least 5 is b-monotonous.

*Clique-width* A parameter used in complexity of graphs is the clique-width. Many problems of optimisation which are NP Hard can be solved efficiently on graphs with bounded clique-width. Some of the classes with bounded clique-width are defined by forbidden subgraphs on 5 vertices, among $P_5$,$coP_5$, $P$ , chair, cochair, coP (see fig. 1)([3]).

The class of $(P_5, coP_5, cochair)$-free graphs was defined, by Giakoumakis and

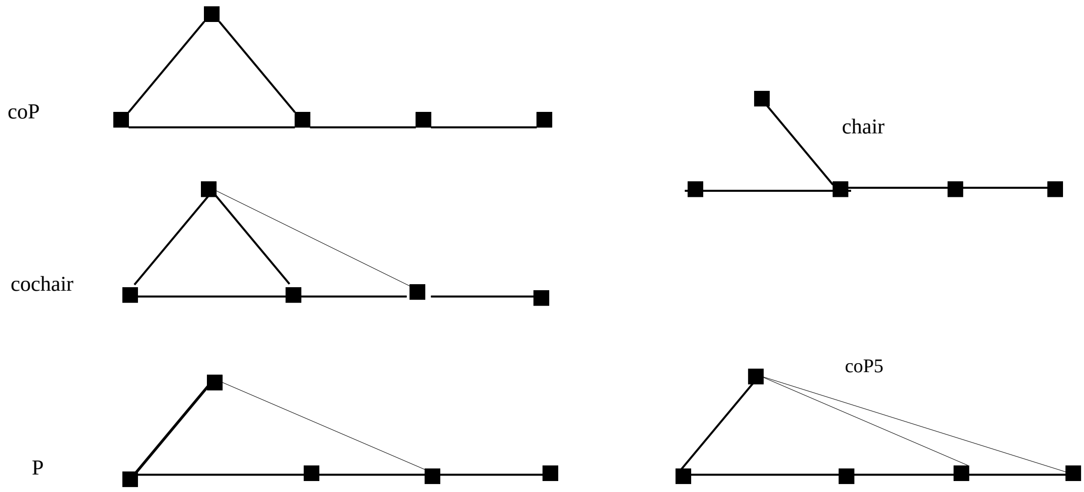


Fig. 1. Extensions of P4

Fouquet,as the class of semi-$P_4$-sparse graphs, a superclass of $P_4$-sparse graphs.The class $(P_5, coP, chair, cochair)$-free deserves the name of semi $P_4$-sparse.

For general graphs S.F.Raj and R.Balakrishnan proved that

**Theorem 1** *[1] For every connected graph of order $n \geq 5$, and for every vertex $v \in V(G)$,*

$b(G) - (\lceil n/2 \rceil - 2) \leq b(G - v) \leq b(G) + (\lfloor n/2 \rfloor - 2)$

The upper bound is attained.

## 2 Quasi-monotonous graphs

Our main results are the following.

**Theorem 2** *Let $G = (V, E)$ be a graph.*

*1) If each vertex is contained in at most two cycles of length 4. Then $G$ is quasi-b-monotonous.*

*2) If $G$ is $(P, coP, Chair, Cochair)$-free, $G$ is quasi-b-monotonous.*

**Corollary 1** *[5] Every graph of girth at least 5 is quasi-b-monotonous.*

**Theorem 3** *Let $G = (V, E)$ be a graph.*

*1) If $G$ is $(P_5, P, Cochair)$-free, $G$ is b-monotonous.*

*2) If $G$ is $(P_5, coP_5, P)$-free, $G$ is b-monotonous.*

**Theorem 4** *Let $G = (V, E)$ be quasi-line-graph. Then for each vertex $x$, $b(G-x) \leq b(G) + 2$*

**Theorem 5** *Let $G = (V, E)$ be a chordal graph of clique-number $\omega$.*

*Then, for each vertex $x$ of $G$, of degree $d(x)$,*

$$b(G-x) \leq b(G) + 1 \ + \ \frac{d(x)-1}{b(G-x)} \quad (1)$$

$$b(G-x) \leq b(G) + 1 + \ \frac{\omega - 1}{b(G-x) - \omega} \quad (2).$$

$$b(G-x) \leq b(G) + 1 + \frac{(\omega-1)^{3/4}}{(b(G)-\omega)^{1/2}} \quad (3)$$

From the preceding theorem, we deduce

**Corollary 2** *Let $G = (V, E)$ be a chordal graph of clique-number $\omega$ and b-chromatic number $b(G)$. Then, for each vertex $x$,*

$$b(G-x) \leq b(G) + 1 + \sqrt{d(x) - 1}$$

$$b(G-x) \leq b(G) + 1 + \sqrt{\omega - 1}.$$

**Corollary 3** *Let $G = (V, E)$ be a chordal graph of clique-number $\omega$ and b-chromatic number $b$ such that $b \geq 2\omega - 3$.*

*Then $G$ is quasi b-monotonous.*

There exist chordal graphs and quasi-line graphs not b-monotonous.

**Examples** 1) Let $k \geq 3$ be an integer .

Let $\omega$ be an even integer and $2k$ be a divisor of $\omega$, furthermore we suppose $\omega \geq 4.k^2$. We give an example of a chordal graph $G_1$ with dominating number $b = \omega + \dfrac{\omega}{2k} - k + 1$ and not quasi-b-monotonous. The gap $b(G_1 - x) - b(G_1)$ in this example is of the order of $\sqrt{\omega}$.

Consider the following graph $H$ composed by 4 vertex-disjoint cliques $A_1, A_2, A_3, A_4$ such the order of $A_1$ (resp. of $A_4$) is $\omega - \omega/2k$. Furthermore, $A_1 \cup A_2$ is a clique of order $\omega$ as well as $A_4 \cup A_3$, and, $A_3 \cup A_2$ is a clique of order $\omega/k$ (see fig.2).Let us call $H'$ the graph $H - A_4$.

The graph $G_1$ is a graph composed by $k$ disjoint copies of $H$ and a copy of $H'$, and, with a vertex $x$, external to the copies of $H$ and $H'$, joined to every vertex of any copy of $A_1$ and to every vertex of $H'$.

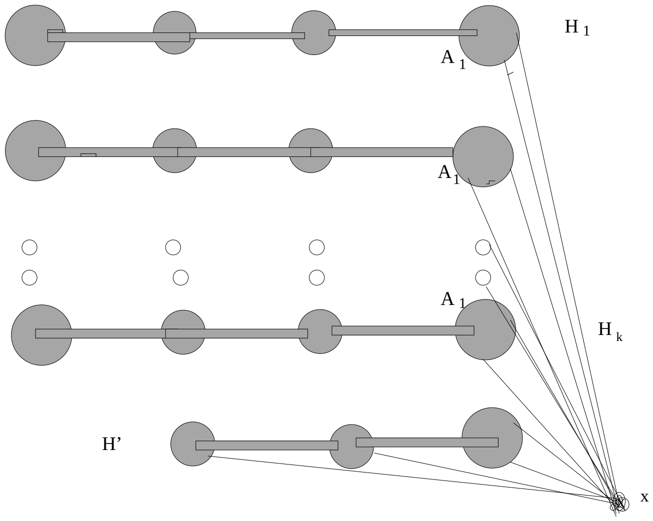


Fig. 2. Chordal graph not quasi-monotonous

We observe that

$$d(v) \leq \omega \ +1$$

for every vertex $v$ of $H'$;

$$d(x) = k.\omega \ + \ \omega/2 + \omega/2k;$$

and,

$$d(u) = \omega \ + \ \omega/2k \ -1$$

for every vertex $u$ of a copy of $A_2$ in $G_1$ or every vertex of a copy of $A_3$ in $G_1 \ - \ H'$. The other vertices have degree at most $\omega$. The number of vertices of degree at least $\omega + \omega/2k \ -k-1$ is

$$\omega + \omega/2k \ +1 \qquad (a)$$

The graph $G_1 - x$ has a b-chromatic number equal to $\omega + \omega/2k$, the set of dominating vertices is the set of vertices of maximum degree in$G_1 - x$ . We show that the graph $G_1$ has a b-chromatic number equal to $\omega+\omega/2k \ -k+1$.

Indeed, suppose $b(G_1) \geq \omega + \omega/2k \ -k$, so $b(G_1) \geq \omega + 3$. Given a $b$ coloring of $G$, at least $\omega/2k \ -k-1$ colors have a dominating vertex in $H'$; then $x$ is neighbour of $b-1$ colors, and is dominating. Every dominating vertex outside $H'$ must be neighbour of the color $c(x)$. And as $\omega/2k \ -(k+1) > 0$, each copy of $A_2$ and each copy of $A_3$ outside $H'$ must contains a dominating vertex; as $x$ is neighbour of each copy of $A_1$, this implies that each copy of $A_2 \cup A_3$ outside

$H'$ must contain a vertex of color $c(x)$. Then by (a), $b(G_1) \leq \omega + \omega/2k \ - k + 1$. One can verify easily the equality $b(G_1) \leq \omega + \omega/2k \ - k + 1$. So $b(G_1 - x) = b(G_1) \ + \ k - 1$.

In that example, if $\omega = 4k^2$, then

$$b(G_1 - x) = b(G_1) - 1 + \frac{\sqrt{w}}{2}.$$

2) With the notations of the precedent example, we consider a graph $G_2$ composed by two vertex disjoint copies of $H$, and an external vertex $x$ joined to each copy of $A_1$. We have $|A_1| = |A_4| = 2\omega/3$ and $|A_2| = |A_3| = \omega/3$.

Then $G_2$ is a quasi-line graph and $b(G_2 - x) = b(G_2) + 1$.

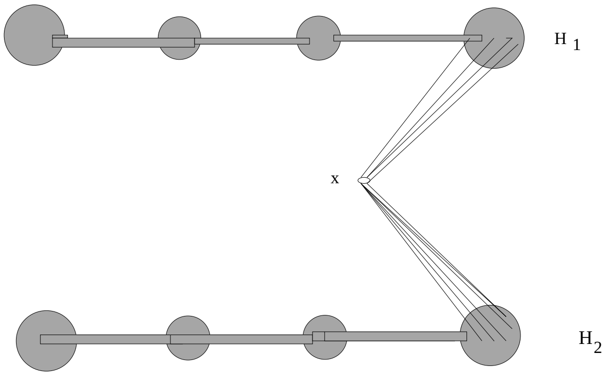


Fig. 3. Quasi-line graph not monotonous

**Remark:** There exist chordal graphs with b-chromatic number at least $2\omega$. Let $\omega \geq 2$, $p = 5(\omega - 1)$. Let $S(u)$ be the graph composed by 5 cliques sharing exactly one vertex $u$. Consider the graph $H$ composed by an elementary path $P(u_1, ..., u_p)$ and a family of $p$ graphs $S(u_i)$ for $1 \leq i \leq p$. It is easy to see that $H$ is chordal and $b(H) \geq p \geq 4\omega$.

## 3 Proofs

*Remarks* For each of our results the following remarks are valid. The proofs are by contradiction. Let $x$ be a fixed vertex of $G$. Let $H = G - x$. We may suppose first that $q = b(H) \geq \omega \ + \ 2$ otherwise we have $b(H) \leq b(G) + 1$. Suppose the b-coloring of $H$ is not extendable to $G$. Then, necessarily,

R1) All the colors appear in $N(x)$.

R2) $x$ is not neighbour to $q$ dominating vertices s of different colors, so there exists an integer $r$ such that for every color $i \geq r + 1$, $N_i(x)$ contains no dominanting vertex.

R3)Let us consider $r$ minimum. For each $i \geq r+1$, $N(N_i(x)) \cap N^2(x)$ contains all the dominating vertices of some color $j$ and no one of these dominating vertices has a neighbour of color $i$ outside $N_i(x)$; otherwise for each $y \in N_i(x)$, we change the color $i$ of the vertex $y$ into a color missing in the neighborhood of $y$ and we give color $i$ to the vertex $x$ and we obtain a b-coloring of $G$ with $q$ colors. Let us call $I$ the set colors $i, i \geq r+1$.

Note that, as the coloring is proper and $j > r$,we get $q - r \geq 2$. From now we say that a set $A \in N_i(x)$ *covers* a color $j$ if $N(A)) \cap N^2(x)$ contains all the dominating vertices of some color $j$ and no one of these dominating vertices has a neighbour of color $i$ outside $A$. Note that $j \in I$ necessarily.

Now we give first the proof of theorem2.

**Proof of Theorem 2**

*Proof of the first part of the theorem2*

*Case 1*:*There exists* $i \geq r+1$ *such that* $N_i(x)$ *covers at most one color, say* $j$

For each $y \in N_i(x)$, we change the color $i$ of $y$ into a color missing in the neighborhood of $y$.

We $x$ by $i$. Either there remains a dominating vertex of the color $j$, we keep color $j$. Or the color $j$ has no dominating vertex.For each vertex of color $j$ we give a color missing in its neighborhood. In any case, we get a $b$ coloring of $G$ with at least $q - 1$ colors..

*Case 2:For each* $i \geq r+1$, $N_i(x)$ *covers at least 2 different colors.*

Let $b'$ the number of colors with all dominating vertices contained in $N^2(x)$ and neighbours of $\cup_{i \geq r+1} N_i(x)$. As the coloring is proper, and $q - r \geq 2$, then $b' \geq 3$ by definition of case 2; so $b - r \geq b' \geq 3$. Let $S$ be a system of $b'$ dominating vertices of the $b'$ different colors.

There exist at least $2(q - r)$ edges between $\cup_{i \geq r+1} N_i(x)$ and the set $S$, by definition of case 2.Then either there exists a vertex $s_0$ which sends at least 3 edges to $\cup_{i \geq r+1} N_i(x)$, or each vertex of $S$ sends exactly 2 edges to $\cup_{i \geq r+1} N_i(x)$. As $|S| \geq 3$, there are at least 3 cycles of length 4 containing $x$. A contradiction to the hypothesis on $G$.

This ends the proof of part1 of theorem 2.

*Proof of part 2 of Theorem 2*

We use the proof of the first part. And we may suppose we are in case 2. For each color $j, j \geq r+1$, $N_j(x)$ covers at least 2 colors. Let $i, i \geq r+1$ be a color such that a dominating vertex $x_i$ is in $N^2(x)$, neighbour of $u_t \in N(x)$. Let us note that, by definition of the covering,$x_i$ is independent of the dominants of each color covered by $N_i(x)$.

There exists a vertex $y_i \in N_i$ covering a color $k$, otherwise, there are $y_i$ and $y'_i$ in $N_i(x)$, $x_k$ and $x'_k$ dominants of the same color such that the path $Q = \{x_k, y_i, x, y'_i, x'_k\}$ is induced and forms with $u_k \in N_k(x)$ a chair,a coP or a cochair; and we get a contradiction with the hypothesis.

Furthermore,there is a second color, say $s$, covered by $N_i(x)$; either $s$ is covered by $y_i$, then as there is no chair the dominants $x_k$ and $x_s$ are adjacent; or$|N_i(x)| \geq 2$, and there exists $y'_i \in N_i(x)$ neighbour of a dominant $x_s, s \neq k$ independent of $y_i$; $x_k$ is independent of $y'_i$ otherwise $x_i, y'_i, x, y_i, x_k$ should be a $P$.

*Case $\alpha$: $x_k, u_t$ adjacent*

The set $u_t, x, y_i, x_i, x_k$ forms $P$ if $y_i, u_t$ are independent, or a cochair if $y_i, u_t$ are adjacent.

*Case $\beta$: $x_k, u_t$ independent*

Case $\alpha$ being excluded, we may suppose that $x_s, u_t$ are independent too. (a) If $y_i, u_t$ are independent,

we have a chair or coP composed by $u_t, y_i, x_s, x_k$, or if $y'_i$ exists, a chair or a cochair composed by $u_t, x, y_i, y'_i, x_k$.

(b) If $y_i, u_t$ are adjacent,

Either $y_i$ is adjacent to $x_k, x_s$, we have coP composed by $x_i, u_t, y_i, x_s, x_k$.

Or, $y_i$ is adjacent to $x_k$ and not to $x_s$, and $y'_i$ is adjacent to $x_s$ and not to $x_k$ otherwise we are in some precedent case. Then $y_i, u_t, x, y'_i, x_s$ forms either a coP (if $[y'_i, u_t] \notin E(G)$) or a cochair (if $[y'_i, u_t] \in E(G)$). As the subgraphs $P$,coP,chair,cochair are excluded, in any subcase, there is a contradiction. Case2 cannot occur.We get part2 of the theorem. □

**Proof of theorem 3**

1) Each class considered here is hereditary. It is sufficient to show by contradiction that for any graph $G$ of the class and any vertex $x$ of $G$ a dominating coloring of $G - x$ extends to $G$, so $b(G - x) \leq b(G)$.

We keep the notations of the proof of the second part of Theorem 2. There exists at least a vertex $y_i$ such that $N(y) \cap N^2(x)$ contains all the dominants of at least a color $k$, otherwise we know that there is at least a color $k$ with all dominants in $N(N_i(x)) \cap N^2(x)$ and we get a $P_5$ composed by $xy_iy'_ix_kx'_k$ where $y_i, y'_i$ are in $N_i(x)$ and $x_k, x'_k$ are 2 dominants of color $k$, a contradiction with the hypothesis on $G$ . By remark R3, no dominating vertex of the color $k$ has a neighbour of color $i$ outside $N_i(x)$.

1) *Case $\alpha$: $y_i, u_t$ independent*

We have a $P_5$ or $P$ composed by $x_k, y_i, x, u_t, x_i$.

*Case $\beta$: $y_i, u_t$ adjacent*

As $u_t$ is not dominant, there is a color $s$ and a vertex $z_s \in N(x)$ not neighbour of $u_t$. As $x_i$ is dominant,there exists a vertex $v_s$ neighbour of $x_i$. We may suppose $x_k, u_t$ independent otherwise we have a cochair $x_k, y_i, u_t, x_i$.

<u>subcase:$z_s$ neighbour of $x_i$</u>:

(a) Either $x_k$ is not neighbour of $z_s$, then $x_k, y_i, u_t, x_i, z_s$ gives either $P_5$ or $P$;

(b) Or, $x, u_k, z_s, x_i, x_k$ form $P$.

<u>subcase:$z_s, x_i$ independent </u>:

As $u_t$ and $v_s$ are independent, then $x, u_t, v_s, x_i$ and $z_s$ form either $P_5$ or $P$. As the graph is $(P_5, P, cochair)$ free, there is a contradiction in any case. So there exists a color $j$ such that $N(N_j(x)) \cap N^2(x)$ contains no dominant. 2) The proof is similar to that of part1 . The only difference is that in the subcase where $x_k$ and $y_i$ are adjacent to $u_t$, we have a $coP_5$ composed by $y_i, u_t, x_i, v_s, x$ □

**Proof of theorem4**

Let $i$ be a color in $I$. As $G$ is a quasi-line graph,there are at most two neighbours $u_i$ and $u'_i$ of $x$ which are of color $i$; $N^2(x) \cap N(u_i)$ is a clique, the same holds for $N^2(x) \cap N(u'_i)$ if $u'_i$ exists. By remarque R3, there exists at least a color $j \in I$ with all dominating vertices in $N^2(x) \cap (N(u_i) \cup N(u'_i))$.

We do the following operation:

1) We change the color of $u_i$ into a missing color $p(u_i)$ and that of $u'_i$ into a missing color $p(u'_i)$. We color $x$ by $i$.

2)We choose one color, say $s$, which is no more dominating

a) either the initial dominating vertex $w_s$ was unique and in $N^2(x) \cap N(u)$ ,where $u \in \{u_i, u'_i\}$, we recolor $w_s$ by the color $i$.

Furthermore, if there is a color $s'$ which the initial dominating vertex $w_{s'}$ was unique and in $N^2(x) \cap N(v)$ where $v \in \{u_i, u'_i\}, v \neq u$, we choose one such color and we do the same operation as precedently on $w_{s'}$.

b) case (a) being excluded, the color $s$ had exactly 2 initial dominating vertices and they were in $N^2(x) \cap (N(u_i) \cup (N(u'_i))$, we recolor them by $i$.

3) For each color, $s$ or $s'$, we recolor each vertex of the corresponding class by a missing color.

Thus, after steps(1) and (2) of this operation, we get a proper coloring; the vertex $x$ is so a dominant vertex of the color $i$;furthermore, each neighbour of $u_i$ (resp. of $u'_i$) which is not of color $i$ is neighbour of a vertex of color $i$. At most 2 colors of the initial coloring are not used, namely $s$ and $s'$. After step (3), we have a dominating coloring of $G$ with at least $q - 2$ colors.

This finishes the proof of theorem4 $\square$

**Proof of theorem 5** We suppose we have a dominating coloring of $H = G - x$. We use notations and remarks R1, R2,R3 given upper, in the proof of Theorem2. For each color $i$ in $I$, we choose a dominating vertex $w_i$. Let $W$ be the set of these dominating vertices. Let $t, t \geq r + 1$ be a fixed color. Let $y$ in $N_t(x)$, and let $p(y) \neq t$ a color not neighbour of $y$. If no ambiguity, we shall write simply $p$ instead of $p(y)$. We call $\mathcal{C}_{t,p}(y)$ the set of vertices joined to $y$ by a path with vertices in $\mathcal{C}_t \cup \mathcal{C}_p \cup W$ such that no 2 vertices of $W$ are consecutive.

Let us first describe the operation $\mathcal{O}$ on $N(x)$.

We fix a color $t$.

$O_1$) Process $O_1(y)$: If $y \in N_t(x)$ is not neighbour of $N_p(x)$ for some $p$, we choose such a color and we color $y$ by $p$ . We exchange the two colors $t$ and $p$ in the component $\mathcal{C}_{t,p}(y)$.
We do this operation successively for each vertex $y$ of $N_t(x)$.
$O_2$) Finally, we give the color $t$ to $x$.

We remark that:

As $G$ is a chordal graph, two vertices of $N_t(x)$ have no common neighbour outside $x \cup N(x)$; furthermore, there is no path $P(y, N_t(x))$ with internal vertices in $\mathcal{C}_{(t,p)}(y)$, and, no path $P(y, N_p(x))$ in $\mathcal{C}_{(t,p)}(y)$ ;so $\mathcal{C}_{(t,p)}(y)$ does not meet $x \cup N(x)$ outside $y$.

The operation $\mathcal{O}$ is possible for every color $t$ with no dominating vertex in $N(x)$.

**Lemma 1** *Let $t \geq r+1$ fixed.*

*1)After an application of operation $O_1(y)$, we obtain a proper coloring of $G$; at most one element of $W$, $w_t$ or $w_p$, is no more dominant. If the color $p(y)$ is no more dominating, then $w_p \in \mathcal{C}_{t,p(y)}(y)$*

*Furthermore, if $y, y' \in N_t(x)$, $y \neq y'$, then $\mathcal{C}_{t,p(y)}(y) \cap \mathcal{C}_{t,p(y')}(y') = \emptyset$.*

*2)After operation $\mathcal{O}$, the vertex $x$ is a dominating vertex of color $t$. The colors which have no dominating vertex are among the chosen missing colors.*

*Proof of Lemma 1*

1)If for some $j \neq p(y), j \neq t$, $w_j \in \mathcal{C}_{t,p(y)}(y)$, then necessarily its neighbours of colors $p(y)$ and $t$ are also in $\mathcal{C}_{t,p(y)}(y)$; then by operation $O_1(y)$, there is a permutation of the colors $p(y)$ and $t$ in $\mathcal{C}_{t,p(y)}(y)$;so the vertex $w_j$ remains dominating of color $j$. We remark that if $w_j \notin \mathcal{C}_{t,p(y)}(y)$, then its neighbours of colors $p(y)$ and $t$ are not in $\mathcal{C}_{t,p(y)}(y)$; their colors are not changed after operation $O_1(y)$, the same conclusion holds for $w_j$; so vertex $w_j$ remains dominating of color $j$.

If $w_t \in \mathcal{C}_{t,p(y)}(y)$, then after operation $O_1(y)$, the vertex $w_t$ becomes dominating of color $p$. Analogously, if $w_p \in \mathcal{C}_{t,p(y)}(y)$, after operation $O_1(y)$, $w_p$ becomes a dominating vertex of color $t$.

Suppose $u \in \mathcal{C}_{t,p(y)}(y) \cap \mathcal{C}_{t,p(y')}(y')$. Then there exists a path from $u$ to $y$ and another one from $u$ to $y'$, so there is a cycle $C$ containing the induced path $[y, x, y']$ and no vertex of $C - \{y, y'\}$ is neighbour of $x$. We may suppose $C$ is a shortest cycle with the latest properties . $C$ is of length at least 4 and has no chord. As $G$ is chordal we get a contradiction.

2) After operation $\mathcal{O}$, the vertex $x$ is neighbour of every color except $t$; so $x$ is dominating vertex of color $t$. As $\mathcal{C}_{t,p(y)}(y) \cap \mathcal{C}_{t,p(y')}(y') = \emptyset$, any vertex is recolored at most one time and by 1) of the Lemma, the possible non dominating colors are in the set $(p(y))_{y \in N_t(x)}$ •

Let $R_t$ be the set of vertices of $\mathcal{C}_t$ such that for each vertex $y$ of $R_t$ for any missing color $p(y)$, after the operation $\mathcal{O}$, the color $p(y)$ has no dominating vertex. Furthermore, if $\mathcal{C}_t = \mathcal{C}$ , let us denote $R_t$ simply by $R$.

**Lemma 2** *Let $G$ be a chordal graph. Then, for each $t \in I$,*

*1) $b(G-x) \leq b(G) + |R_t|$*

*2)For any vertices $y, y'$ of $R_t$, then $p(y) \neq p(y')$. And, if $d^-(y)$ is the number of colors which do not appear in $N(x) \cap N(y)$, we have*

$$\sum_{y \in R_t} d^-(y) \leq (q-1)$$

.

*Proof of Lemma 2*

1) The first part is a consequence of (2) of Lemma1.

2)If $y$ and $y'$ are elements of $R_t$, there exists a path $P(y, w_{p(y)})$ in $\mathcal{C}_{(t,p(y))}$, and a path $P(y, w_{p(y')})$ $\mathcal{C}_{(t,p(y'))}$ in by Lemma 1. As $\mathcal{C}_{(t,p(y))} \cap \mathcal{C}_{(t,p(y'))} = \emptyset$, then by Lemma1, they have no common missing color $j$, so we get the inequality •

*End of the proof of theorem 5*

We consider a b-dominating coloring of $G-x$ by $q = b(G-x)$ colors. After operation $O_1(v)$ applied on a vertex $v$, the color $p(v)$ remains dominating if $N_t(x) = v$ was missing only one color in $G$ and $v$ is a reprentative of $p(v)$ .

In $N(x)$, let $\mathcal{C}$ be a class of colors such that by the operation $O$ the minimum number of colors are no more dominating.

In a chordal graph, if $S_1$ and $S_2$ are two vertex disjoint stable sets, the induced vertex graph of vertex- set $S_1 \cup S_2$ is a forest. Then given a coloring of $N(x) \setminus \mathcal{C}$ by a minimum number of colors, we have a partition $X_1, ..., X_s$ of $N(x) \setminus \mathcal{C}$. Then, as there is no induced $C_4$, the set of edges $E(R, X_i)$ is a matching between $R$ and $X_i$. So

$$e(R, X_i\ ) \leq min(|R|,\ |X_i|)$$

for each $i \leq s$.

As $N(x) - \mathcal{C}$ is chordal of clique-number at most $(\omega - 1)$, we get by summing

$$e(R, N(x) - \mathcal{C}) \leq min((\omega-1).|R|, (d(x) - |R|)) \qquad (5)$$

Each vertex $y$ of $\mathcal{C}$ is not neighbour in $N(x)$ of at least $(q-1) - d_{N(x)}(y))$ colors. So

$$(q-1)|R| - \ e(R, N(x) - \mathcal{C}) \leq \sum_{y \in R} d^-(y).$$

On the other hand, as $G$ is chordal,by Lemma 2,

$$\sum_{y\in R} d^-(y) \leq (q-1),$$

we get, using inequality (5), that

$$(q-1)|R| - (d(x) - |R|] \leq q-1$$
$$and, (q-1)|R| - (\omega-1)|R| \leq q-1$$

So $|R| \leq 1 \ + \ min(\frac{d(x)-1}{q}, \frac{(\omega-1)}{(q-\omega)})$

As by Lemma2, $b(G-x) \leq b(G) + |R|$, we get the inequalities 1 and 2 of the theorem.

Let us set $\theta = b(G-x) - b(G) - 1$, and $a = q - \omega$. From inequality (2), we get

$$\theta^2 + a.\theta - (\omega-1) \leq 0.$$

It follows that

$$\theta \leq \frac{(\omega-1)}{\sqrt{a^2/4 + (\omega-1)} + a/2} \qquad (6)$$

As $\sqrt{a^2/4 + (\omega-1)} \geq \sqrt{a}.(\omega-1)^{1/4}$, we get the third inequality of the theorem □

**Proof of corollary 2** From inequality 1 of the last theorem, we have $(\theta+1)^2 \leq d(x) - 1$. Whereas inequality (6) of the last proof gives

$$\theta \leq \frac{(\omega-1)}{\sqrt{a^2/4 + (\omega-1)} + a/2} \qquad (6)$$

**Proof of corollary 3** Let us set $\theta = b(G-x) - b(G)$. The inequality (2) of the last theorem gives

$$(\theta-1)(b(G) - \omega + 3) \leq \omega - 1$$

Suppose $\theta \geq 2$. Then we get $b(G) \leq 2\omega - 4$. The corollary 2 follows •

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