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# SIGN CHANGES IN SHORT INTERVALS OF COEFFICIENTS OF SPINOR ZETA FUNCTION OF A SIEGEL CUSP FORM OF GENUS 2

EMMANUEL ROYER, JYOTI SENGUPTA, AND JIE WU

ABSTRACT. In this paper, we establish a Voronoi formula for the spinor zeta function of a Siegel cusp form of genus 2. We deduce from this formula quantitative results on the number of its positive (resp. negative) coefficients in some short intervals.

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## 1. INTRODUCTION

Let  $S_k$  be the space of Siegel cusp forms of integral weight  $k$  on the group  $\mathrm{Sp}_4(\mathbb{Z}) \subset \mathrm{GL}_4(\mathbb{Q})$  and let  $F \in S_k$  be an eigenfunction of all the Hecke operators. Let

$$Z_F(s) := \prod_{p \in \mathcal{P}} Z_{F,p}(p^{-s}) \quad (\mathrm{Re} s > 1)$$

be the spinor zeta function of  $F$ . Here  $\mathcal{P}$  is the set of prime numbers and if  $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$  are the Satake  $p$ -parameters attached to  $F$  then

$$Z_{F,p}(t)^{-1} := (1 - \alpha_{0,p}t)(1 - \alpha_{0,p}\alpha_{1,p}t)(1 - \alpha_{0,p}\alpha_{2,p}t)(1 - \alpha_{0,p}\alpha_{1,p}\alpha_{2,p}t).$$

They satisfy

$$\alpha_{0,p}^2 \alpha_{1,p} \alpha_{2,p} = 1$$

for all  $p$ . A Siegel form is in the Maass subspace  $S_k^M$  of  $S_k$  if it is a linear combination of Siegel forms  $F$  that are eigenvectors of all the Hecke operators and for which there

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exists a primitive modular form,  $f$ , of weight  $2k - 2$  such that

$$Z_F(s) = \zeta\left(s - \frac{1}{2}\right)\zeta\left(s + \frac{1}{2}\right)L(f, s).$$

Here  $L(f, s)$  is the L-function of  $f$  (note that we normalise all the L-functions so that the critical strip is  $0 \leq \text{Res} \leq 1$  and the functional equation relates the value at  $s$  to the value at  $1 - s$ ). This happens only if  $k$  is even. The bijective linear application between  $S_k^M$  and the space of modular forms of weight  $2k - 2$  is called the Saito-Kurokawa lifting [Zag81]. The Ramanujan-Petersson conjecture says that

$$(1) \quad |\alpha_{j,p}| = 1 \text{ for } j = 0, 1, 2 \text{ and all primes } p.$$

It is not true for Siegel Hecke-eigenforms in  $S_k^M$ . But, if  $k$  is odd or, if  $k$  is even and the form is in the orthogonal complement of  $S_k^M$ , then it has been established by Weis-sauer [Wei09]. We denote by  $H_k^*$  the set of Siegel cuspidal Hecke-eigenforms of weight  $k$  and genus 2 that, if  $k$  is even, are in the orthogonal complement of  $S_k^M$ . The forms we consider in this paper all belong to  $H_k^*$ . According to Breulmann [Bre99], a Siegel Hecke-eigenform is in  $S_k^M$  if and only if all its Hecke eigenvalues are positive.

According to [And74, Evd80], the function

$$\Lambda_F(s) := (2\pi)^{-s}\Gamma\left(s + k - \frac{3}{2}\right)\Gamma\left(s + \frac{1}{2}\right)Z_F(s)$$

has an entire continuation to  $\mathbb{C}$  since  $F \in H_k^*$ . Further it satisfies the functional equation

$$(2) \quad \Lambda_F(s) = (-1)^k \Lambda_F(1 - s)$$

on  $\mathbb{C}$ . The spinor zeta function of  $F$  has the Dirichlet expansion:

$$Z_F(s) = \sum_{n \geq 1} a_F(n) n^{-s}$$

for  $\text{Res} > 1$ . By using (1), one sees that

$$(3) \quad |a_F(n)| \leq d_4(n)$$

for all  $n \geq 1$ , where  $d_4(n)$  is the number of solutions in positive integers  $a, b, c, d$  of  $n = abcd$ .

In this paper, we investigate the problem of sign changes for the sequence  $(a_F(n))_{n \geq 1}$  in short intervals. Define

$$\mathcal{N}_F^+(x) := \sum_{\substack{n \leq x \\ a_F(n) > 0}} 1 \quad \text{and} \quad \mathcal{N}_F^-(x) := \sum_{\substack{n \leq x \\ a_F(n) < 0}} 1.$$

We apply a method due to Lau & Tsang [LT02] to establish the following Theorem. Convergence issues however appear and we have to deal with them.

**Theorem**— *Let  $F$  be in  $H_k^*$  and  $\varepsilon > 0$ . There are constants  $c > 0$  absolute and  $x_0(F)$  depending only on  $F$  such that for all  $x \geq x_0(F)$ , we have*

$$\mathcal{N}_F^+(x + cx^{3/4}) - \mathcal{N}_F^+(x) \gg x^{3/8 - \varepsilon},$$

and

$$\mathcal{N}_F^-(x + cx^{3/4}) - \mathcal{N}_F^-(x) \gg x^{3/8-\varepsilon},$$

where the implied constants in  $\gg$  depends only on  $\varepsilon$ .

**Remark** - An ingredient of our proof is the inequality

$$(4) \quad \sum_{n \leq x} a_F(n) \ll_{F,\varepsilon} x^{3/5+\varepsilon} \quad (x \geq 2).$$

(see Lemma 1). We also prove, and use an Omega-result:

$$\sum_{n \leq x} a_F(n) = \Omega_{\pm}(x^{3/8})$$

(see Lemma 2).

Two related problems have already been studied. Denote by  $\lambda_F(n)$  the  $n$ -th normalised Hecke eigenvalue of  $F$ . Then we have

$$(5) \quad \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s} = \frac{Z_F(s)}{\zeta(2s+1)} \quad (\text{Res} > 1).$$

In [Koh07], Kohnen proved that

$$\#\{n \leq x: \lambda_F(n) > 0\} \rightarrow \infty \quad (x \rightarrow \infty).$$

and

$$\#\{n \leq x: \lambda_F(n) < 0\} \rightarrow \infty \quad (x \rightarrow \infty).$$

Then, Das [Das13] proved that, as  $x$  tends to  $+\infty$ , the quantities

$$\frac{1}{\#\{p \in \mathcal{P}: p \leq x\}} \#\{p \in \mathcal{P} \cap [1, x]: \lambda_F(p) > 0\}$$

and

$$\frac{1}{\#\{p \in \mathcal{P}: p \leq x\}} \#\{p \in \mathcal{P} \cap [1, x]: \lambda_F(p) < 0\}$$

are bounded from below (and naturally also bounded from above). In [KS07], Kohnen & Sengupta proved that under the same assumption there is an integer  $n \ll k^2(\log k)^{20}$  such that  $\lambda_F(n) < 0$ . Their result has been generalised to higher levels by Brown [Bro10]. An interesting study of sign changes is also due to Pitale & Schmidt [PS08]. They prove that if  $F$  is not in the Maass subspace, there exists an infinite set of prime numbers  $p$  not dividing the level so that there are infinitely many  $r$  with  $\lambda_F(p^r) > 0$  and infinitely many  $r$  with  $\lambda_F(p^r) < 0$ .

**Remark** - Das' result is on the counting function of the Hecke eigenvalues. It implies that, as  $x$  tends to  $+\infty$ , the quantities

$$\frac{1}{\#\{p \in \mathcal{P}: p \leq x\}} \#\{p \in \mathcal{P} \cap [1, x]: a_F(p) > 0\}$$

and

$$\frac{1}{\#\{p \in \mathcal{P}: p \leq x\}} \#\{p \in \mathcal{P} \cap [1, x]: a_F(p) < 0\}$$

are bounded from below. The reason is that (5) implies

$$a_{\mathbb{F}}(n) = \sum_{\substack{(d,m) \in \mathbb{N}^2 \\ d^2 m = n}} \frac{\lambda_{\mathbb{F}}(m)}{d}.$$

Thus  $a_{\mathbb{F}}(n) = \lambda_{\mathbb{F}}(n)$  for  $n$  squarefree and in particular for  $n$  a prime. Moreover, the proof of Kohnen & Sengupta can be adapted to prove that there is an integer  $n \ll k^2(\log k)^{20}$  such that  $a_{\mathbb{F}}(n) < 0$ .

To end this introduction, we give a very short amount on what is known in the case of classical modular forms, referring to [LLW13] for a more complete survey. Let  $f$  be a primitive modular form of weight  $k$  on the congruence subgroup  $\Gamma_0(N)$ . Lau & Wu [LW09] proved that, as  $x$  tends to  $+\infty$ , the quantities

$$(6) \quad \frac{1}{\#\{n \in \mathbb{N}^*: n \leq x\}} \#\{n \in \mathbb{N} \cap [1, x]: \lambda_f(n) > 0\}$$

and

$$\frac{1}{\#\{n \in \mathbb{N}^*: n \leq x\}} \#\{n \in \mathbb{N} \cap [1, x]: \lambda_f(n) < 0\}$$

are bounded from below. Even though we know by the Sato-Tate Theorem [BLGHT11] that

$$\lim_{x \rightarrow \infty} \frac{1}{\#\{p \in \mathcal{P}: p \leq x\}} \#\{p \in \mathcal{P} \cap [1, x]: \lambda_f(p) > 0\} = \frac{1}{2}$$

it does not seem easy to deduce a similar limit for (6). Lau & Wu proved also the following result on intervals. There exists  $C > 0$  such that, for any  $\varepsilon > 0$ , there exists  $K > 0$  such that for any even integer  $k \geq 4$ , for any integer  $N \geq 1$  we have

$$\#\{n \in [x, x + CE_N x^{1/2}]: \lambda_f(n) > 0\} \geq K(Nx)^{1/4-\varepsilon}$$

as soon as  $x \geq N^2 x_0(k)$  where  $x_0(k)$  is a positive real number only depending on  $k$ . Here,

$$E_N = N^{1/2} \left( \sum_{d|N} d^{-1/2} \log(2d) \right)^3.$$

An important ingredient used by Lau & Wu is the following result by Serre [Ser81]. Let  $f$  be a primitive modular form of weight  $k$  on the congruence subgroup  $\Gamma_0(N)$ . Let  $\delta < \frac{1}{2}$ , there exists  $C > 0$  such that, for any  $x \geq 2$ , we have

$$\frac{1}{\#\{p \in \mathcal{P}: p \leq x\}} \#\{p \in \mathcal{P} \cap [1, x]: \lambda_f(p) = 0\} \leq \frac{C}{\log(x)^\delta}.$$

Such an inequality is missing in the case of Siegel modular forms.

## 2. TRUNCATED VORONOI FORMULA

The aim of this section is to establish the following truncated Voronoi formula, which will be needed in the proof of the Theorem.

**Lemma 1**—Let  $F$  be in  $H_k^*$ . Then for any  $A > 0$  and  $\varepsilon > 0$ , we have

$$(7) \quad \sum_{n \leq x} a_F(n) = \frac{x^{3/8}}{(2\pi)^{3/4}} \sum_{n \leq M} \frac{a_F(n)}{n^{5/8}} \cos\left(4\sqrt{2\pi}(nx)^{1/4} + \frac{\pi}{4}\right) \\ + O_{A,F,\varepsilon}\left((x^3 M^{-1})^{1/4+\varepsilon} + (xM)^{1/4+\varepsilon}\right)$$

uniformly for  $x \geq 2$  and  $1 \leq M \leq x^A$ , where the implied constant depends on  $A$ ,  $F$  and  $\varepsilon$  only. In particular

$$(8) \quad \sum_{n \leq x} a_F(n) \ll_{F,\varepsilon} x^{3/5+\varepsilon} \quad (x \geq 2).$$

*Proof.* Without loss of generality, we assume that  $M \in \mathbb{N}$ . Let  $\kappa := 1 + \varepsilon$  and

$$(9) \quad T^4 = 4\pi^2(M + \frac{1}{2})x.$$

By the Perron formula (see [Ten95, Corollary II.2.4]) we have

$$(10) \quad \sum_{n \leq x} a_F(n) = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} Z_F(s) \frac{x^s}{s} ds + O_{F,\varepsilon}\left(x^{3/4+\varepsilon} M^{-1/4} + x^\varepsilon\right).$$

We shift the line of integration horizontally to  $\text{Re } s = -\varepsilon$ , the main term gives

$$\frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} Z_F(s) \frac{x^s}{s} ds = Z_F(0) + \frac{1}{2\pi i} \int_{\mathcal{L}} Z_F(s) \frac{x^s}{s} ds,$$

where  $\mathcal{L}$  is the contour joining the points  $\kappa \pm iT$  and  $-\varepsilon \pm iT$ . Using the convexity bound [Mic07, §1.3]

$$Z_F(\sigma + it) \ll_{F,\varepsilon} (|t| + 1)^{\max\{2(1-\sigma), 0\} + \varepsilon} \quad (-\varepsilon \leq \sigma \leq \kappa),$$

the integrals over the horizontal segments and the term  $Z_F(0)$  can be absorbed in  $O_{F,\varepsilon}\left((Tx)^\varepsilon(T + T^{-1}x)\right) = O_{F,\varepsilon}\left(x^{1/4+\varepsilon}M^{1/4} + x^{3/4+\varepsilon}M^{-1/4}\right)$ .

To handle the integral over the vertical segment  $\mathcal{L}_v := [-\varepsilon - iT, -\varepsilon + iT]$ , we invoke the functional equation (2). We deduce that

$$(11) \quad \frac{1}{2\pi i} \int_{\mathcal{L}_v} Z_F(s) \frac{x^s}{s} ds = (-1)^k \sum_{n \geq 1} \frac{a_F(n)}{n} I_{\mathcal{L}_v}(nx),$$

where

$$I_{\mathcal{L}_v}(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_v} (2\pi)^{2s-1} \frac{\Gamma(k - \frac{1}{2} - s)\Gamma(\frac{3}{2} - s)}{\Gamma(s + k - \frac{3}{2})\Gamma(s + \frac{1}{2})} \frac{y^s}{s} ds.$$

By using the Stirling formula

$$\Gamma(\sigma + it) = \sqrt{2\pi}|t|^{\sigma-1/2} e^{-\pi|t|/2+i(t \log|t|-t)+i \operatorname{sgn}(t)(\pi/2)(\sigma-1/2)} \left\{1 + O\left(t^{-1}\right)\right\}$$

uniformly for  $\sigma_1 \leq \sigma \leq \sigma_2$  and  $|t| \geq 1$ , the quotient of the four gamma factors is

$$(12) \quad |t|^{2-4\sigma} e^{-4i(t \log|t|-t) + i \operatorname{sgn}(t)\pi(1-k)} \{1 + O(t^{-1})\}$$

for bounded  $\sigma$  and any  $|t| \geq 1$ , where the implied constant depends on  $\sigma$  and  $k$ . Together with the second mean value theorem for integrals [Ten95, Theorem I.0.3], we obtain

$$(13) \quad \begin{aligned} I_{\mathcal{L}_v}(nx) &\ll (nx)^{-\varepsilon} \left( \left| \int_1^T t^{1+4\varepsilon} e^{-ig(t)} dt \right| + T^{1+4\varepsilon} \right) \\ &\ll T \left( \frac{T^4}{nx} \right)^\varepsilon \left( \left| \int_a^T e^{-ig(t)} dt \right| + 1 \right) \end{aligned}$$

for some  $1 \leq a \leq T$ , where  $g(t) := t \log(t^4/(4\pi^2 nx)) - 4t$ . In view of (9), we have

$$g'(t) = -\log(4\pi^2 nx/t^4) < 0 \quad \text{and} \quad |g'(t)| \geq |\log(n/(M + \frac{1}{2}))|$$

for  $n \geq M+1$  and  $1 \leq t \leq T$ . Using (3) and [Ten95, Theorem I.6.2], we infer that

$$(14) \quad \begin{aligned} \sum_{n>M} \frac{a_F(n)}{n} I_{\mathcal{L}_v}(nx) &\ll T \left( \frac{T^4}{x} \right)^\varepsilon \sum_{n>M} \frac{d_4(n)}{n^{1+\varepsilon}} \left( \left| \log \frac{n}{M + \frac{1}{2}} \right|^{-1} + 1 \right) \\ &\ll T \left( \frac{T^4}{x} \right)^\varepsilon \left\{ \sum_{M < n \leq 2M} \frac{d_4(n)(M + \frac{1}{2})}{n^{1+\varepsilon} |n - M - \frac{1}{2}|} + \frac{1}{M^{\varepsilon/2}} \right\} \\ &\ll T \left( \frac{T^4}{\sqrt{M}x} \right)^\varepsilon \\ &\ll Tx^\varepsilon. \end{aligned}$$

For  $n \leq M$ , we extend the segment of integration  $\mathcal{L}_v$  to an infinite line  $\mathcal{L}_v^*$  in order to apply Lemma 1 in [CN63]. Write

$$\mathcal{L}_v^\pm := [\frac{1}{2} + \varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm i\infty), \quad \mathcal{L}_h^\pm := [-\varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm iT]$$

and define  $\mathcal{L}_v^*$  to be the positively oriented contour consisting of  $\mathcal{L}_v$ ,  $\mathcal{L}_v^\pm$  and  $\mathcal{L}_h^\pm$ . In view of (12), the contribution over the horizontal segments  $\mathcal{L}_h^\pm$  is

$$\begin{aligned} I_{\mathcal{L}_h^\pm}(nx) &\ll \int_{-\varepsilon}^{1/2-\varepsilon} (2\pi)^{2\sigma-1} T^{2-4\sigma} \frac{(nx)^\sigma}{T} d\sigma \\ &\ll T \int_{-\varepsilon}^{1/2-\varepsilon} \left( \frac{nx}{T^4} \right)^\sigma d\sigma \\ &\ll Tx^\varepsilon. \end{aligned}$$

As in (13), for  $n \leq M$  we get that

$$\begin{aligned} I_{\mathcal{L}_v^\pm}(nx) &\ll (nx)^{1/2+\varepsilon} \left( \int_T^\infty t^{-1-4\varepsilon} e^{-ig(t)} dt + \frac{1}{T^{1+4\varepsilon}} \right) \\ &\ll T \left( \frac{nx}{T^4} \right)^{1/2+\varepsilon} \left( \left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right) \\ &\ll T \left( \left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right). \end{aligned}$$

So

$$\begin{aligned} \sum_{n \leq M} \frac{a_F(n)}{n} (I_{\mathcal{L}_v^\pm}(nx) + I_{\mathcal{L}_h^\pm}(nx)) &\ll Tx^{\varepsilon/2} \sum_{n \leq M} \frac{d_4(n)}{n} \left( \left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right) \\ (15) \qquad \qquad \qquad &\ll Tx^{\varepsilon/2} \sum_{n \leq M} \frac{d_4(n)(M + \frac{1}{2})}{n|n - M - \frac{1}{2}|} + Tx^\varepsilon \\ &\ll Tx^\varepsilon. \end{aligned}$$

Define

$$I_{\mathcal{L}_v^*}(y) = \frac{1}{4\pi^2 i} \int_{\mathcal{L}_v^*} \frac{\Gamma(k - \frac{1}{2} - s) \Gamma(\frac{3}{2} - s) \Gamma(s)}{\Gamma(s + k - \frac{3}{2}) \Gamma(s + \frac{1}{2}) \Gamma(1 + s)} (4\pi^2 y)^s ds.$$

After a change of variable  $s$  into  $1 - s$ , we see that

$$I_{\mathcal{L}_v^*}(y) = \frac{I_0(4\pi^2 y)}{2\pi},$$

with

$$I_0(t) := \frac{1}{2\pi i} \int_{\mathcal{L}_\varepsilon} \frac{\Gamma(s + k - \frac{3}{2}) \Gamma(s + \frac{1}{2}) \Gamma(1 - s)}{\Gamma(k - \frac{1}{2} - s) \Gamma(\frac{3}{2} - s) \Gamma(2 - s)} t^{1-s} ds.$$

Here  $\mathcal{L}_\varepsilon$  consists of the line  $s = \frac{1}{2} - \varepsilon + i\tau$  with  $|\tau| \geq T$ , together with three sides of the rectangle whose vertices are  $\frac{1}{2} - \varepsilon - iT$ ,  $1 + \varepsilon - iT$ ,  $1 + \varepsilon + iT$  and  $\frac{1}{2} - \varepsilon + iT$ . Note that all the poles of the integrand in  $I_0(t)$  lie on the left of the line  $\mathcal{L}_\varepsilon$ .

Using a result due to Chandrasekharan and Narasimhan [CN63, Lemma 1] generalised by Lau & Tsang [LT02, Lemma 2.2] we obtain (note that a factor  $\sqrt{2}$  is missing for the definition of  $e_0$  in both references)

$$I_0(t) = \frac{(-1)^k}{\sqrt{2\pi}} t^{3/8} \cos\left(4t^{1/4} + \frac{\pi}{4}\right) + O(t^{1/8}).$$

It hence follows that

$$(16) \qquad I_{\mathcal{L}_v^*}(nx) = (-1)^k \frac{(nx)^{3/8}}{(2i)^{3/4}} \cos\left(4\sqrt{2\pi}(nx)^{1/4} + \frac{\pi}{4}\right) + O((nx)^{1/8}).$$

We conclude

$$(17) \qquad \sum_{n \leq M} \frac{a_F(n)}{n} I_{\mathcal{L}_v^*}(nx) = \frac{(-1)^k}{(2\pi)^{3/4}} x^{3/8} \sum_{n \leq M} \frac{a_F(n)}{n^{5/8}} \cos\left(4\sqrt{2\pi}(nx)^{1/4} + \frac{\pi}{4}\right) + O(x^{1/4+\varepsilon} M^{1/4})$$



from (15) and (16). Finally the asymptotic formula (7) by (10)-(11), (14) and (17).

Since

$$x^{3/8} \sum_{n \leq M} \frac{a_F(n)}{n^{5/8}} \cos\left(4\sqrt{2\pi}(nx)^{1/4} + \frac{\pi}{4}\right) \ll (xM)^{3/8+\varepsilon},$$

the choice of  $M = x^{3/5}$  in (7) gives (8).  $\square$

### 3. PROOF OF THE THEOREM

We establish a lemma that has a similar statement as a one due to Lau & Wu [LW09, Lemma 3.2]. However, due to convergence issue, the proof is more delicate.

**Lemma 2**— Let  $F$  be in  $H_k^*$ . Define

$$S_F(x) := \sum_{n \leq x} a_F(n).$$

There exist positive absolute constants  $C, c_1, c_2$  and  $X_0(F)$  depending only on  $F$  such that for all  $X \geq X_0(F)$ , we can find  $x_1, x_2 \in [X, X + CX^{3/4}]$  for which

$$S_F(x_1) > c_1 X^{3/8} \quad \text{and} \quad S_F(x_2) < -c_2 X^{3/8}.$$

*Proof.* We begin the proof with Theorem C of Hafner [Haf81]. In order to use this result, it is more convenient to introduce the notion of  $(C, \ell)$ -summability and to present related simple facts (see [Moo66] for more details). Let  $\{g_n(t)\}_{n \geq 0}$  be a sequence of functions. We write

$$s(g; n) := \sum_{0 \leq \nu \leq n} g_\nu(t), \quad \sigma(g; n) := \frac{1}{C_n^{(\ell+1)}} \sum_{\nu=0}^n C_{n-\nu}^{(\ell)} s(g; \nu),$$

where  $C_n^{(\ell)} := \binom{\ell+n-1}{n}$ . We say that the series of general term  $g_n(t)$  is uniformly  $(C, \ell)$ -summable to the sum  $G(t)$  if  $\sigma(g; n)$  converges uniformly to  $G(t)$  as  $n \rightarrow \infty$ . We have  $C_0^{(\ell)} + \dots + C_n^{(\ell)} = C_n^{(\ell+1)}$  and if the series  $\sum_n \int g_n(t) dt$  converges then the series of general term  $\int g_n(t) dt$  is also  $(C, \ell)$ -summable and their limits are the same.

As in [Haf81, page 151], for  $\rho > -1$  and  $x \notin 2\pi\mathbb{N}$ , define

$$A_\rho(x) := \frac{1}{\Gamma(\rho+1)} \sum_{2\pi n \leq x} a_F(n) (x - 2\pi n)^\rho.$$

Now let  $\mathcal{C}$  be the rectangle with vertices  $c \pm iR$  and  $1 - b \pm iR$  (taken in the counter-clockwise direction), where  $b > c > \max\{1, |k - \frac{3}{2}|\}$  and  $R > |k - \frac{3}{2}|$  are real numbers. Let

$$Q_\rho(x) := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s)(2\pi)^{-s} Z_F(s)}{\Gamma(s+\rho+1)} x^{\rho+s} ds.$$

Denote by  $\mathcal{C}_{0,b}$  the oriented polygonal path with vertices  $-i\infty, -iR, b-iR, b+iR, iR$  and  $+i\infty$ . Let

$$f_\rho(x) := \frac{1}{2\pi i} \int_{\mathcal{C}_{0,b}} \frac{\Gamma(1-s)\Delta(s)}{\Gamma(2+\rho-s)\Delta(1-s)} x^{1+\rho-s} ds$$

where

$$\Delta(s) = \Gamma(s+k-\frac{3}{2})\Gamma(s+\frac{1}{2}).$$

By [Haf81, Theorem C], the series of general term  $(-1)^k(2\pi n)^{-1-\rho}a_F(n)f_\rho(2\pi nx)$  is uniformly  $(C, \ell)$ -summable for  $\ell > \max\{\frac{1}{2} - \rho, 0\}$  on any finite closed interval in  $(0, \infty)$  only under the condition  $\rho > -1$  and the sum is  $A_\rho(x) - Q_\rho(x)$ . In particular, we can fix  $\ell = 1$  and  $\rho = 0$ . We shall say  $C$ -summable for  $(C, 1)$ -summable.

The only pole of the integrand of  $Q_0(x)$  is 0, it is encircled by  $\mathcal{C}$  hence

$$Q_0(x) \ll_F 1 \quad (x \geq 1).$$

To estimate  $f_0(x)$ , we use again the result by Lau & Tsang [LT02, Lemma 2.2] already used to establish Voronoi formula. We get

$$(18) \quad f_0(y) = \frac{(-1)^k}{\sqrt{2\pi}} y^{3/8} \cos\left(4y^{1/4} + \frac{\pi}{4}\right) + (-1)^k e_1 y^{1/8} \cos\left(4y^{1/4} + \frac{3\pi}{4}\right) + O\left(\frac{1}{y^{1/8}}\right),$$

where  $e_1$  is a absolute constant.

Let

$$\begin{aligned} \Phi(v) &:= (2\pi)^{3/4} \frac{A_0(2\pi v^4)}{v^{3/2}}, \\ g_n(v) &:= \frac{a_F(n)}{n^{5/8}} \cos\left(4\sqrt{2\pi}n^{1/4}v + \frac{\pi}{4}\right), \\ g_n^*(v) &:= \frac{e_1}{v} \frac{a_F(n)}{n^{7/8}} \sin\left(4\sqrt{2\pi}n^{1/4}v + \frac{\pi}{4}\right). \end{aligned}$$

Then the series of general term  $g_n(v) - g_n^*(v)$  is uniformly  $C$ -summable on any finite closed interval in  $(0, \infty)$  and the sum is  $\Phi(v) + O(v^{-3/2})$  (here the term  $O(v^{-3/2})$  comes from  $Q_0(2\pi v^4)$  and the  $O$ -term of (18)). In view of (4), a simple partial integration shows that the series of general term  $g_n^*(v)$  converges to the sum  $\sum_n g_n^*(v)$  uniformly on any finite closed interval in  $(0, \infty)$ . Thus the series of general term  $g_n(v)$  is uniformly  $C$ -summable on any finite closed interval in  $(0, \infty)$  and the sum is  $\Phi(v) + \sum_n g_n^*(v) + O(v^{-3/2})$ .

Let  $t$  be any large natural number,  $\kappa > 1$  a large parameter that will be fixed later. Write

$$K_\tau(u) = (1 - |u|)(1 + \tau \cos(4\sqrt{2\pi}\kappa u))$$

with  $\tau = \pm 1$ . We consider the integral

$$J_\tau = \int_{-1}^1 \Phi(t + \kappa u) K_\tau(u) du.$$

We have

$$\begin{aligned} \int_{-1}^1 g_n(t + \kappa u) K_\tau(u) du &= r_\beta \frac{a_F(n)}{n^{5/8}}, \\ \int_{-1}^1 g_n^*(t + \kappa u) K_\tau(u) du &= s_\beta e_1 \frac{a_F(n)}{n^{7/8}}, \end{aligned}$$

where

$$r_\beta := \int_{-1}^1 K_\tau(u) \cos\left(4\sqrt{2\pi}\beta(t + \kappa u) + \frac{\pi}{4}\right) du,$$

$$s_\beta := \int_{-1}^1 \frac{K_\tau(u)}{t + \kappa u} \sin\left(4\sqrt{2\pi}\beta(t + \kappa u) + \frac{\pi}{4}\right) du.$$

As in [LW09, (3.13)], we have

$$r_\beta = \delta_{\beta=1} \frac{\tau}{2} + O\left(\frac{1}{\kappa^2 \beta^2} + \delta_{\beta \neq 1} \frac{1}{\kappa^2 (\beta - 1)^2}\right)$$

and

$$s_\beta \ll (t\beta\kappa)^{-1}.$$

It follows that

$$\int_{-1}^1 g_1(t + \kappa u) K_\tau(u) du = \frac{\tau}{2} + O\left(\frac{1}{\kappa^2}\right),$$

$$\int_{-1}^1 g_n(t + \kappa u) K_\tau(u) du \ll \frac{d_4(n)}{\kappa^2 n^{9/8}} \quad (n \geq 2),$$

$$\int_{-1}^1 g_n^*(t + \kappa u) K_\tau(u) du \ll \frac{d_4(n)}{\kappa t n^{9/8}},$$

where all the implied constants are absolute. These estimates show that

$$\sum_{n \geq 1} \int_{-1}^1 g_n(t + \kappa u) K_\tau(u) du = \frac{\tau}{2} + O\left(\frac{1}{\kappa^2}\right),$$

$$\sum_{n \geq 1} \int_{-1}^1 g_n^*(t + \kappa u) K_\tau(u) du \ll \frac{1}{\kappa t}.$$

In view of the remark about C-summability, we obtain

$$J_\tau = \frac{\tau}{2} + O\left(\frac{1}{\kappa t} + \frac{1}{t^{3/2}}\right).$$

We fix  $\kappa$  large enough. When  $X \geq \kappa^4$ , we take  $t = \lfloor X^{1/4} \rfloor$ . So  $t > 2\kappa$  and the O-term in  $J_\tau$  is  $\ll \kappa^{-2}$ , so the main term dominates if  $\kappa$  has been chosen sufficiently large. Therefore

$$J_{-1} < -\frac{1}{4} \quad \text{and} \quad J_1 > \frac{1}{4}.$$

Since  $S_F(x) = A_0(2\pi x)$ , we rewrite this as

$$\int_{-1}^1 \frac{S_F(t + \kappa u)}{(t + \kappa u)^{3/2}} K_{-1}(u) du < -\frac{1}{4(2\pi)^{3/4}} \quad \text{and} \quad \int_{-1}^1 \frac{S_F(t + \kappa u)}{(t + \kappa u)^{3/2}} K_1(u) du > \frac{1}{4(2\pi)^{3/4}}.$$

The kernel function  $K_\tau(u)$  is nonnegative and satisfies

$$1 - (3\pi\kappa)^{-2} \leq \int_{-1}^1 K_\tau(u) du \leq 2 \quad (\tau = \pm 1).$$

As a consequence, we have

$$\frac{S_F((t + \kappa\eta_+)^4)}{(t + \kappa\eta_+)^{3/2}} \geq \frac{1}{2(2\pi)^{3/4}}$$

and

$$\frac{S_F((t + \kappa\eta_-)^4)}{(t + \kappa\eta_-)^{3/2}} \leq -\frac{1}{4(1 - (3\pi\kappa)^{-2})(2\pi)^{3/4}}$$

for some  $\eta_{\pm} \in [-1, 1]$ . These two points deviate from  $X$  by a distance  $\ll X^{3/4}$ , since the difference between  $(t \pm \kappa)^4$  is  $\ll \kappa t^3 \asymp X^{3/4}$ .

This implies the result of Lemma 2.  $\square$

Now we are ready to prove the Theorem.

By Lemma 2, for any  $x \geq X_0(F)$  we can pick three points  $x < x_1 < x_2 < x_3 < x + 3Cx^{3/4}$  such that  $S_F(x_i) < -cx^{3/8}$  ( $i = 1, 3$ ) and  $S_F(x_2) > cx^{3/8}$  for some absolute constant  $c > 0$ . (Note that  $y + Cy^{3/4} \leq x + 3Cx^{3/4}$  for  $y = x + Cx^{3/4}$ .) Hence we deduce that

$$\sum_{\substack{x_1 < n < x_2 \\ a_F(n) > 0}} a_F(n) \geq S_F(x_2) - S_F(x_1) > 2cx^{3/8}$$

and

$$\sum_{\substack{x_2 < n < x_3 \\ a_F(n) < 0}} (-a_F(n)) \geq -(S_F(x_3) - S_F(x_2)) > 2cx^{3/8}.$$

Thus, the Theorem follows as each term in the two sums are positive and  $\ll_{\epsilon} n^{\epsilon}$ .

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