Bistable travelling waves for nonlocal reaction diffusion equations

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Abstract

We are concerned with travelling wave solutions arising in a reaction diffusion equation with bistable and nonlocal nonlinearity, for which the comparison principle does not hold. Stability of the equilibrium $u \equiv 1$ is not assumed. We construct a travelling wave solution connecting 0 to an unknown steady state, which is "above and away" from the intermediate equilibrium. For focusing kernels we prove that, as expected, the wave connects 0 to 1. Our results also apply readily to the nonlocal ignition case.

Key Words: travelling waves, nonlocal reaction-diffusion equation, bistable case, ignition case.

AMS Subject Classifications: 45K05, 35C07.

1 Introduction

We consider the nonlocal bistable reaction diffusion equation

$$\partial_t u = \partial_{xx} u + u(u-\theta)(1-\phi * u) \quad \text{in } (0,\infty)\times\mathbb{R}, \tag{1}$$

where $0<\theta<1$. Here $\phi * u(x) := \displaystyle\int_{\mathbb{R}} u(x-y)\phi(y)\,dy$, with ϕ a given bounded kernel such that

$$\phi \geq 0, \quad \phi(0) > 0, \quad \int_{\mathbb{R}} \phi = 1. \tag{2}$$

We are looking for travelling waves solutions supported by the integro-differential equation (1), that is a speed $c^* \in \mathbb{R}$ and a smooth U such that

$$-U'' - c^* U' = U(U-\theta)(1-\phi * U) \quad \text{in } \mathbb{R}, \tag{3}$$

supplemented with the boundary conditions

$$\liminf_{x\to-\infty} U(x) > \theta, \quad \lim_{x\to+\infty} U(x) = 0. \tag{4}$$

In this work we construct such a travelling wave solution, and show that the behavior on the left is improved to $\lim_{x\to-\infty} U(x) = 1$ for *focusing kernels*. Our results also apply readily to the nonlocal *ignition case* (see below).

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Equation (1) is a nonlocal version of the well known reaction-diffusion equation

$$\partial_t u = \partial_{xx} u + f(u) \quad \text{in } (0,\infty)\times\mathbb{R}, \tag{5}$$

with the bistable nonlinearity $f(s) := s(s-\theta)(1-s)$. Homogeneous reaction diffusion equations have been extensively studied in the literature (see [14], [3, 4], [10], [6], [18] among others) and are known to support the existence of monotone travelling fronts for three classes of nonlinearity: bistable, ignition and monostable. Moreover, for bistable and ignition nonlinearities, there exists a unique front speed c^* whereas, for monostable nonlinearities, there exists a critical speed c^* such that all speeds $c \geq c^*$ are admissible. In this local context, many techniques based on the comparison principle — such as some monotone iterative schemes or the sliding method [7]— can be used to get *a priori* bounds, existence and monotonicity properties of the fronts.

Recently, much attention was devoted to the introduction of a nonlocal effect into the nonlinear reaction term. From the mathematical point of view, the analysis is quite involved since integro-differential equations with a nonlocal competition term generally do not satisfy the comparison principle. In [5], Berestycki, Nadin, Perthame and Ryzhik have considered the following non-local version of the Fisher-KPP equation

$$\partial_t u = \partial_{xx} u + u(1-\phi * u) \quad \text{in } (0,\infty)\times\mathbb{R}. \tag{6}$$

They prove that equation (6) admits a critical speed c^* so that, for all $c \geq c^*$, there exists a travelling wave (c, U) solution of

$$\begin{cases} -U'' - cU' = U(1-\phi * U) \quad \text{in } \mathbb{R}. \\ \liminf_{x\to-\infty} U(x) > 0, \quad \lim_{x\to+\infty} U(x) = 0. \end{cases}$$

In favorable situations, namely when the steady state $u \equiv 1$ remains linearly stable, they further obtain $\lim_{x\to-\infty} U(x) = 1$. Nevertheless, the positive steady state $u \equiv 1$ may present, for some kernels, a Turing instability (see e.g. [11], [5], [1]). In such situations, it was proved in [9] and in [1] that the waves with large speeds actually connect the two unstable states 0 and 1. Notice that the former work considers kernels with exponential decay and uses monotonicity arguments inspired by [13], whereas the latter uses more direct arguments which allow kernels with algebraic decay. Concerning this issue of the behavior of the wave on the left, we also refer the reader to [17], [16]. In a related framework, the authors of the present work have constructed curved fronts for nonlocal reaction diffusion equations [2] of the form

$$\begin{aligned} \partial_t u(t,x,y) =& \Delta_{xx} u(t,x,y) + \partial_{yy} u(t,x,y) + \\ & u(t,x,y)\left(r(v - Bx.e) - \int_{\mathbb{R}} k(y - Bx.e, y' - Bx.e) u(t,x,y')\, dy' \right) \end{aligned}$$

for $t > 0$, $x \in \mathbb{R}^d$ (spatial variable), $y \in \mathbb{R}$ (phenotypical trait). In population dynamics, such equations serve as prototypes of models for structured populations evolving in a environmental cline.

In view of the existence of fronts for both the nonlocal Fisher-KPP equation (6) and the local bistable equation (5), it is then expected that the nonlocal bistable equation (1) supports the existence of travelling waves. In this work, we shall construct such a solution. It is worth being mentioned that, among other things, nonlinearities such as $u(\phi * u - \theta)(1-u)$ are treated in [19]. Notice that our equation does not fall into [19, equation (1.6)] since $g(u,v) = u(1-u)(1-v)$ does not satisfy [19, hypothesis (H1)], which actually provides the stability of both $u \equiv 0$ and $\equiv 1$.

Let us now state our main result on the existence of a travelling wave solution.

Theorem 1.1 (A bistable travelling wave) *There exist a speed $c^* \in \mathbb{R}$ and a positive profile $U \in C^2(\mathbb{R})$ solution of*

$$-U'' - c^* U' = U(U-\theta)(1-\phi * U) \quad \text{on } \mathbb{R}, \tag{7}$$

such that, for some $\varepsilon > 0$,

$$U(x) \geq \theta + \varepsilon \quad \textit{for all } x \in (-\infty, -1/\varepsilon), \tag{8}$$

U *is decreasing on* $[\bar{x}, +\infty)$ *for some* $\bar{x} > 0$, *and*

$$\lim_{x \to +\infty} U(x) = 0. \tag{9}$$

Now, if the kernel tends to the Dirac mass, we expect the above travelling wave to be a perturbation of the underlying wave for the local case, namely (c_0^*, U_0) the unique solution of

$$\begin{cases} U_0'' + c_0^* U_0' + U_0(U_0 - \theta)(1 - U_0) = 0, \\ \lim_{x \to -\infty} U_0(x) = 1, \quad U_0(0) = \theta, \quad \lim_{x \to +\infty} U_0(x) = 0, \end{cases} \tag{10}$$

and so to satisfy $\lim_{x \to -\infty} U(x) = 1$. Our next result states such a behavior assuming $c_0^* \neq 0$, which is equivalent to $\theta \neq \frac{1}{2}$. To make this perturbation analysis precise, we take $\sigma > 0$ as a focusing parameter, define

$$\phi_\sigma(x) := \frac{1}{\sigma} \phi\left(\frac{x}{\sigma}\right), \tag{11}$$

and are interested in the asymptotics $\sigma \to 0$.

Proposition 1.2 (Focusing kernels) *Denote by* (c_σ^*, U_σ) *the travelling wave associated with the kernel* ϕ_σ, *as constructed in Theorem 1.1.*

(i) Assume $\int_{\mathbb{R}} |z| \phi(z)\, dz < \infty$. *Then* $c_\sigma^* \to c_0^*$, *as* $\sigma \to 0$.

(ii) Assume $\theta \neq \frac{1}{2}$ *and* $\int_{\mathbb{R}} z^2 \phi(z)\, dz < \infty$. *Then there is* $\sigma_0 > 0$ *such that, for all* $0 < \sigma < \sigma_0$,

$$\lim_{x \to -\infty} U_\sigma(x) = 1.$$

Remark 1.3 (Ignition case) *While proving the above results for the bistable case, it will become clear that the same (with the additional information* $c^* > 0$*) holds for the ignition case, that is*

$$-U'' - c^* U' = \begin{cases} 0 & \textit{where } U < \theta \\ (U - \theta)(1 - \phi * U) & \textit{where } U \geq \theta, \end{cases}$$

for which proofs are simpler. This will be clarified in Section 6.

Let us comment on the main result, Theorem 1.1. Due to the lack of comparison principle, the construction of a travelling wave solution is based on a topological degree argument, a method introduced initially in [6]. After establishing a series of *a priori* estimates, it enables to construct a solution in a bounded box. Then we let the size of the box tend to infinity to construct a solution on the whole line. In contrast with [5] and because of the intermediate equilibrium $u \equiv \theta$, it is far from clear that the constructed wave is non trivial — or, equivalently, that it "visits" both $(0, \theta)$ and $(\theta, 1)$. Such an additional difficulty also arises in the construction of bistable waves in cylinders [8], where the authors use energy arguments to exclude the possibility of triviality. This seems not to be applicable to our nonlocal case. Our arguments are rather direct and are based on the sharp property of Proposition 3.1 and the construction of bump-like sub-solutions in Lemma 3.2.

The organization of the paper is as follows. In Section 2, we construct a solution u on a bounded interval thanks to a Leray-Schauder topological degree argument. In Section 3, we show that, when we let the bounded interval tend to the whole line, the limit profile U is non trivial. The behaviors (8) and (9) are then proved in Section 4. We investigate the case of the focusing kernels, that is Proposition 1.2, in Section 5. Last, in Section 6 we indicate how to handle the ignition case.

2 Construction of a solution u in a box

Notice that the methods used in this section are inspired by [5].

For $a > 0$ and $0 \le \tau \le 1$, we consider the problem of finding a speed $c = c_\tau^a \in \mathbb{R}$ and a profile $u = u_\tau^a : [-a, a] \to \mathbb{R}$ such that

$$P_\tau(a) \quad \begin{cases} -u'' - cu' = \tau \mathbf{1}_{\{u \ge 0\}} u(u-\theta)(1-\phi * \bar{u}) & \text{in } (-a,a) \\ u(-a) = 1, \qquad u(0) = \theta, \qquad u(a) = 0, \end{cases}$$

where $\bar{u}$ denotes the extension of u equal to 1 on $(-\infty, -a)$ and 0 on (a, ∞) (in the sequel, for ease of notation, we always write u in place of $\bar{u}$). This realizes a homotopy from a local problem ($\tau = 0$) to our nonlocal problem ($\tau = 1$) in the box $(-a, a)$. We shall construct a solution to $P_1(a)$ by using a Leray-Schauder topological degree argument.

If (c, u) is a solution achieving a negative minimum at x_m then $x_m \in (-a, a)$ and $-u'' - cu' = 0$ on a neighborhood of x_m. The maximum principle thus implies $u \equiv u(x_m)$, which cannot be. Therefore any solution of $P_\tau(a)$ satisfies $u \ge 0$ and, by the strong maximum principle,

$$u > 0 \quad \text{and} \quad -u'' - cu' = \tau u(u-\theta)(1-\phi * u) \quad \text{in } (-a,a). \tag{12}$$

2.1 A priori bounds of solutions in the box

The following lemma provides *a priori* bounds for u.

Lemma 2.1 (A priori bounds for u) *There exist $M > 1$ (depending only on the kernel ϕ) and $a_0 > 0$ such that, for all $a \ge a_0$ and all $0 \le \tau \le 1$, any solution (c, u) of $P_\tau(a)$ satisfies*

$$0 \le u(x) \le M, \quad \forall x \in [-a, a].$$

Proof. If $\tau = 0$ we directly get $0 \le u \le 1$ for the local problem. Now, for $0 < \tau \le 1$, assume $M := \max_{x \in [-a,a]} u(x) > 1$ (otherwise there is nothing to prove). In view of the boundary conditions, there is a $x_m \in (-a, a)$ such that $M = u(x_m)$. Evaluating (12) at x_m we see that $\phi * u(x_m) \le 1$.

Now since $u \ge 0$, we also have

$$-u'' - cu' = \tau u(u-\theta)(1-\phi*u) \le u^2 + \theta u(\phi * u) \le (1+\theta)M^2 \le 2M^2. \tag{13}$$

Let us first assume that $c < 0$. For $x \in [-a, x_m]$ it follows from (13) that

$$\int_x^{x_m} \left(u'(z) e^{-|c|z} \right)' dz \ge -\int_x^{x_m} 2M^2 e^{-|c|z}\, dz.$$

Using $u'(x_m) = 0$, isolating $u'(x)$ and integrating again from x to x_m, we discover

$$\int_x^{x_m} u'(z)\, dz \le -\frac{2M^2}{|c|} \int_x^{x_m} (e^{-|c|(x_m - z)} - 1)\, dz.$$

Using $u(x_m) = M$ and isolating $u(x)$, we get after elementary computations

$$u(x) \ge M \left[1 - 2M(x - x_m)^2 B(|c|(x_m - x)) \right],$$

where $B(y) := \frac{e^{-y} + y - 1}{y^2}$. Observe that $B(y) \le \frac{1}{2}$ for $y > 0$ so that

$$u(x) \ge M(1 - M(x - x_m)^2), \quad \forall x \in [-a, x_m], \tag{14}$$

and in particular, for $x = -a$,

$$1 \ge M(1 - M(a + x_m)^2). \tag{15}$$

Now we define

$$x_0 := \frac{1}{2\sqrt{M}}. \tag{16}$$

If $x_m \in (-a, -a + x_0]$, then (15) shows that $M \le \left(1 - Mx_0{}^2\right)^{-1} = \frac{4}{3}$. If $x_m \in [-a + x_0, a)$, then

$$1 \ge \phi * u(x_m) \ge \int_0^{x_0} \phi(z) u(x_m - z)\, dz \ge M \int_0^{x_0} \phi(z)(1 - Mz^2)\, dz,$$

where we have used (14). From the definition of x_0 we deduce that

$$1 \ge \frac{3}{4} M \int_0^{1/2\sqrt{M}} \phi(z)\, dz \ge \frac{3}{4} M \left(\int_0^{1/2\sqrt{M}} (\phi(0) - \|\phi'\|_{L^\infty(-1,1)} z)\, dz \right),$$

so that

$$M \le \left(\frac{8}{3} \frac{1 + \frac{3}{32}\|\phi'\|_{L^\infty(-1,1)}}{\phi(0)} \right)^2. \tag{17}$$

This concludes the proof in the case $c < 0$. The case $c > 0$ can be treated in a similar way by working on $[x_m, a]$ rather than on $[-a, x_m]$.

Last if $c = 0$, by integrating twice the inequality $-u'' \le 2M^2$ on $[x, x_m]$ we directly obtain (14) and we can repeat the above arguments. This completes the proof of the lemma. □

We now provide *a priori* bounds for the speed c.

Lemma 2.2 (A priori upper bound for c) *There exists $a_0 > 0$ such that, for all $a \ge a_0$ and all $0 \le \tau \le 1$, any solution (c, u) of $P_\tau(a)$ satisfies $c \le 2\sqrt{2M} =: c_{max}$, where M is the upper bound for u defined in Lemma 2.1.*

Proof. Since $-u'' - cu' \le u^2 + \theta u(\phi * u) \le (1 + \theta)Mu \le 2Mu$, we can reproduce the proof of [5, Lemma 3.2] with $\mu \leftarrow 2M$. □

We now provide *a priori* bounds for the speed c. We will prove two separate estimates.

Lemma 2.3 (A priori lower bound for c, uniform w.r.t. τ) *For any $a > 0$, there exists $\tilde{c}_{min}(a) > 0$ such that, for all $0 \le \tau \le 1$, any solution (c, u) of $P_\tau(a)$ satisfies $c \ge -\tilde{c}_{min}(a)$.*

Proof. Let $a > 0$ be given. We consider a solution (c, u) of $P_\tau(a)$. It satisfies:

$$-u'' - cu' + (M^2 + 1)u \ge 0,$$

as well as $u(-a) = 1$ and $u(a) = 0$. Since $M^2 + 1 \ge 0$, the comparison principle applies and $u \ge v$, where v is the solution of $-v'' - cv' + (M^2 + 1)v = 0$ such that $v(-a) = 1$ and $v(a) = 0$. Explicitly computing v, we get

$$v(0) = \frac{1 - e^{(\lambda_+ - \lambda_-)a}}{e^{-\lambda_+ a} - e^{(\lambda_+ - 2\lambda_-)a}}, \quad \lambda_\pm := \frac{-c \pm \sqrt{c^2 + 4(M^2 + 1)}}{2}.$$

We see that $v(0) \to 1$ as $c \to -\infty$. It follows that, for any $a > 0$, there exists $\tilde{c}_{min}(a) > 0$ such that $c \le -\tilde{c}_{min}(a)$ implies $\theta < v(0) \le u(0)$, so that u cannot solve $P_\tau(a)$. Hence, all solutions (c, u) of $P_\tau(a)$ with $0 \le \tau \le 1$ are such that $c \ge -\tilde{c}_{min}(a)$. □

Lemma 2.4 (A priori lower bound for c for $\tau = 1$, uniform w.r.t. a) *There exist $c_{min} > 0$ and $a_0 > 0$ such that, for all $a \ge a_0$, any solution (c, u) of $P_1(a)$ satisfies $c \ge -c_{min}$.*

Proof. We assume that $c \le -1$ (otherwise there is nothing to prove), and define $M > 0$ and $a_0 > 0$ as in Lemma 2.1.

The first step of the proof is to find uniform bounds on u', following ideas from [5, Lemma 3.3]. We first notice that $(e^{cx}u'(x))' = e^{cx}\left(u''(x) + cu'(x)\right)$, and an integration of this expression provides for $x > y$:

$$e^{cx}u'(x) - e^{cy}u'(y) = \int_y^x e^{cz}u(z)(u(z)-\theta)(1-\phi * u(z))\,dz.$$

Thank to Lemma 2.1, we have $|u(u-\theta)(1-\phi*u)| \le M(M+\theta)(1+M) =: Q$, so that

$$u'(y)e^{|c|(x-y)} - \frac{Q}{|c|}e^{|c|(x-y)} \le u'(x) \le u'(y)e^{|c|(x-y)} + \frac{Q}{|c|}e^{|c|(x-y)}, \quad \forall x, y \in [-a,a],\ x > y, \quad (18)$$

$$u'(y) \le \frac{2Q}{|c|}, \quad \forall y \in (-a,a), \quad (19)$$

where we have chosen $x = a$, and used the fact that $u'(a) \le 0$ to obtain this last estimate.

Next, define

$$K_0 := 2\max_{c\le -1}\frac{1}{|c|}\ln\left(\frac{Mc^2}{Q}+1\right).$$

We claim that, for all $c \le -1$, all $a \ge a_0$,

$$-\frac{2Q}{|c|} \le u'(x), \quad \forall x \in (-a, a-K_0]. \quad (20)$$

Indeed, assume by contradiction that there are some $c \le -1$, $a \ge a_0$, $y \in (-a, a-K_0]$ such that $u'(y) < -\frac{2Q}{|c|}$. From (18) we deduce that $u'(x) \le -\frac{Q}{|c|}e^{|c|(x-y)}$ for $x > y$. Integrating this from y to a and using $u(a) = 0$ we see that

$$M \ge u(y) \ge \frac{Q}{c^2}(e^{|c|(a-y)} - 1) \ge \frac{Q}{c^2}(e^{|c|K_0} - 1),$$

which contradicts the definition of K_0. This proves (20).

Next, since $\phi \in L^1(\mathbb{R})$, there exists $R > 0$ such that $M\int_{[-R,R]^c}\phi \le \frac{1-\theta}{8}$. Thanks to the conditions $u(-a) = 1$ and $u(0) = \theta$ in $P_1(a)$, we can define $x_0 < 0$ as the largest negative real such that $u(x_0) = \theta + \frac{1-\theta}{2}$. We can use (20) to estimate $u(x)$ from below for $x \in [x_0 - R, x_0 + 2R] \cap [-a, a]$:

$$u(x) \ge \theta + \frac{1-\theta}{2} - \frac{2Q}{|c|}2R \ge \theta + \frac{1-\theta}{4}, \quad (21)$$

as soon as $c \le -\frac{16QR}{1-\theta}$. Similarly, using (19),

$$u(y) \le \theta + \frac{1-\theta}{2} + \frac{2Q}{|c|}2R \le \theta + \frac{3(1-\theta)}{4}, \quad (22)$$

as soon as $c \le -\frac{16QR}{1-\theta}$. In particular (21) and (22) imply that $[x_0 - R, x_0 + 2R] \subset (-a, 0)$ if $-c$ is large enough. We then estimate $\phi * u(x)$ for $x \in [x_0, x_0 + R]$:

$$\begin{aligned}
\phi * u(x) &\le \int_{[-R,R]}\phi(y)u(x-y)\,dy + \int_{[-R,R]^c}\phi(y)u(x-y)\,dy \\
&\le \max_{[x_0-R,x_0+2R]} u + M\int_{[-R,R]^c}\phi \\
&\le \theta + \frac{1-\theta}{2} + \frac{2Q}{|c|}2R + \frac{1-\theta}{8} \\
&\le 1 - \frac{1-\theta}{8}, \quad (23)
\end{aligned}$$

as soon as $c \leq -\frac{16QR}{1-\theta}$.

If u is not non-increasing on $[x_0, x_0+R]$, the definition of x_0 implies the existence of a local minimum $\bar{x} \in (x_0, x_0+R)$. An evaluation (12) in $\bar{x}$ then shows that $1 \leq \phi * u(\bar{x})$, which is only possible if $-c$ is not too large, thanks to (23).

If on the contrary, u is non-increasing on $[x_0, x_0+R]$ and $c \leq 0$, then, for $[x_0, x_0+R]$,

$$u''(x) \leq u''(x) + cu'(x) = -u(x)(u(x)-\theta)(1-\phi * u(x)) \leq -\theta\frac{(1-\theta)^2}{32}.$$

It follows that $u'(x_0) - u'(x_0+R) \geq \frac{\theta(1-\theta)^2}{32}R$, which, combined to (20) and (19) implies

$$c \geq -\frac{128QR}{\theta(1-\theta)^2},$$

so that in any case, $c_{min} := -\frac{128QR}{\theta(1-\theta)^2}$ is an explicit lower bound for c. □

2.2 Construction of a solution in the box

Equipped with the above *a priori* estimates, we now use a Leray-Schauder topological degree argument (see e.g. [6], [5] or [2] for related arguments) to construct a solution (c, u) to $P_1(a)$.

Proposition 2.5 (A solution in the box) *There exist $K > 0$ and $a_0 > 0$ such that, for all $a \geq a_0$, Problem $P_1(a)$ admits a solution (c, u), that is*

$$\begin{cases} -u'' - cu' = u(u-\theta)(1-\phi * u) & \text{in } (-a, a) \\ u(-a) = 1, \qquad u(0) = \theta, \qquad u(a) = 0, \\ u > 0 \quad \text{in } (-a, a), \end{cases}$$

which is such that

$$\|u\|_{C^2(-a,a)} \leq K, \quad -c_{min} \leq c \leq c_{max}.$$

Proof. For a given nonnegative function v defined on $(-a, a)$ and satisfying the Dirichlet boundary conditions as requested in $P_\tau(a)$ — that is $v(-a) = 1$ and $v(a) = 0$— consider the family $0 \leq \tau \leq 1$ of linear problems

$$P_\tau^c(a) \begin{cases} -u'' - cu' = \tau v(v-\theta)(1-\phi * v) & \text{in } (-a, a) \\ u(-a) = 1, \quad u(a) = 0. \end{cases} \tag{24}$$

Denote by $\mathcal{K}_\tau$ the mapping of the Banach space $X := \mathbb{R} \times C^{1,\alpha}(Q)$ — equipped with the norm $\|(c, v)\|_X := \max(|c|, \|v\|_{C^{1,\alpha}})$— onto itself defined by

$$\mathcal{K}_\tau : (c, v) \mapsto (\theta - v(0) + c, u_\tau^c := \text{ the solution of } P_\tau^c(a)).$$

Constructing a solution (c, u) of $P_1(a)$ is equivalent to showing that the kernel of $\mathrm{Id} - \mathcal{K}_1$ is nontrivial. The operator $\mathcal{K}_\tau$ is compact and depends continuously on the parameter $0 \leq \tau \leq 1$. Thus the Leray-Schauder topological argument can be applied. Define the open set

$$S := \{(c, v) : -\tilde{c}_{min}(a) - 1 < c < c_{max} + 1,\ v > 0,\ \|v\|_{C^{1,\alpha}} < M + 1\} \subset X.$$

It follows from the *a priori* estimates Lemma 2.1, Lemma 2.2 and Lemma 2.3, that there exists $a_0 > 0$ such that, for any $a \geq a_0$, any $0 \leq \tau \leq 1$, the operator $\mathrm{Id} - \mathcal{K}_\tau$ cannot vanish on the boundary ∂S. By the homotopy invariance of the degree we thus have $\deg(\mathrm{Id} - \mathcal{K}_1, S, 0) = \deg(\mathrm{Id} - \mathcal{K}_0, S, 0)$.

To conclude, observe that we can compute

$$u_0^c(x) = \frac{e^{-cx} - e^{-ca}}{e^{ca} - e^{-ca}} \quad \text{if } c \neq 0, \quad u_0^c(x) = -\frac{1}{2a}x + \frac{1}{2} \quad \text{if } c = 0, \tag{25}$$

and that $u_0^c(0)$ is decreasing with respect to c (in particular there is a unique c_0 such that $u_0^{c_0}(0) = \theta$). Hence by using two additional homotopies (see [5] or [2] for details) we can compute $\deg(\mathrm{Id} - \mathcal{K}_0, S, 0) = -1$ so that $\deg(\mathrm{Id} - \mathcal{K}_1, S, 0) = -1$ and there is a $(c, u) \in S$ solution of $P_1(a)$. Finally, Lemma 2.4 provides a lower bound $c \geq -c_{min}$, uniform in $a \geq a_0$. This concludes the proof of the proposition. □

A solution on $\mathbb{R}$. Equipped with the solution (c, u) of $P_1(a)$ of Proposition 2.5, we now let $a \to +\infty$. This enables to construct — passing to a subsequence $a_n \to +\infty$— a speed $-c_{min} \leq c^* \leq c_{max}$ and a function $U : \mathbb{R} \to (0, M)$ in $C_b^2(\mathbb{R})$ such that

$$-U'' - c^* U' = U(U - \theta)(1 - \phi * U) \quad \text{on } \mathbb{R}, \tag{26}$$

$$U(0) = \theta. \tag{27}$$

In contrast with the nonlocal Fisher-KPP equation considered in [5] we need additional arguments to show that the constructed U is non trivial, i.e. that U "visits" both $(0, \theta)$ and $(\theta, 1)$. This is the purpose of the next section.

3 Non triviality of U the solution on $\mathbb{R}$

In this section, we provide additional *a priori* estimates on the solution u in the box $(-a, a)$, which in turn will imply the non triviality of the solution U on $\mathbb{R}$.

First, using the homotopy of the previous section, we show that the solution in the box cannot attain θ elsewhere that at $x = 0$.

Proposition 3.1 (θ is attained only at $x = 0$) *For all $a \geq a_0$, the solution (c, u) of Proposition 2.5 satisfies*

$$u(x) = \theta \ \text{ if and only if } \ x = 0.$$

Proof. From Proposition 2.5, we know that there is a solution (c_τ, u_τ) of

$$\begin{cases} -u_\tau'' - c_\tau u_\tau' = \tau u_\tau (u_\tau - \theta)(1 - \phi * u_\tau) & \text{in } (-a, a) \\ u_\tau(-a) = 1, \qquad u_\tau(0) = \theta, \qquad u_\tau(a) = 0, \end{cases} \tag{28}$$

and that (c_τ, u_τ) depends continuously upon $0 \leq \tau \leq 1$. For $\tau = 0$, in view of (25), the solution u_0 satisfies $u_0(x) = \theta$ if and only if $x = 0$. We can therefore define

$$\tau^* := \sup \left\{ 0 \leq \tau \leq 1, \forall \sigma \in [0, \tau], u_\sigma(x) = \theta \ \text{ iff } \ x = 0 \right\}.$$

Assume by contradiction that there is a $x^* \neq 0$ such that $u_{\tau^*}(x^*) = \theta$. Without loss of generality, we can assume $x^* < 0$ and $u_{\tau^*} > \theta$ on $(x^*, 0)$. By the definition of τ^* as a supremum, one must have $u_{\tau^*} \geq \theta$ on $(-a, 0)$, which in turn enforces $u'_{\tau^*}(x^*) = 0$. Hence $v := u_{\tau^*} - \theta$ is positive on $(x^*, 0)$, zero at x^* and satisfies the linear elliptic equation

$$-v'' - c_{\tau^*} v' = \left[\tau^* u_{\tau^*} (1 - \phi * u_{\tau^*}) \right] v \quad \text{on } (x^*, 0).$$

It then follows — see e.g. [12, Lemma 3.4]— that $v'(x^*) > 0$, which is a contradiction. Hence u_{τ^*} attains θ only at $x = 0$. To conclude let us prove that $\tau^* = 1$.

Assume by contradiction that $0 \leq \tau^* < 1$. By the definition of τ^*, there exists a sequence (τ_n, x_n) such that $\tau_n \downarrow \tau^*$, $x_n \neq 0$, and $u_{\tau_n}(x_n) = 0$. Up to an extraction, the sequence x_n converges to a limit x^*, which implies, thanks to the continuity of $(\tau, x) \mapsto u_\tau(x)$ with respect to τ and x, that $u_{\tau^*}(x^*) = 0$. As seen above one must have $x^* = 0$, and then $x_n \to 0$. Then, for some $-1 \leq C_n \leq 1$, we have $0 = u_{\tau_n}(x_n) = u_{\tau_n}(0) + u'_{\tau_n}(0) x_n + C_n \|u_{\tau_n}\|_{C^{1,\alpha}} |x_n|^{1+\alpha}$, that is

$$|u'_{\tau_n}(0)| \leq |C_n| \, \|u_{\tau_n}\|_{C^{1,\alpha}} |x_n|^\alpha \leq C |x_n|^\alpha \to_{n \to \infty} 0.$$

The continuity of (u'_τ) with respect to τ then implies that $u'_{\tau^*}(0) = 0$. Since $u_{\tau^*} > \theta$ on $(-a, 0)$ we derive a contradiction as above. As a result $\tau^* = 1$ and the proposition is proved. □

We now construct a subsolution of a linear equation, having the form of a bump, that will be very useful in the following.

Lemma 3.2 (A bump as a sub-solution) *For any $\kappa > 0$, there exists $A > 0$ such that for all $c > -2\sqrt{\kappa}$, there exist $0 < \tilde{x} < A$, $X > \tilde{x}$ and $\psi : [0, X] \to [0, 1]$, satisfying $\psi(0) = 0$, $\psi(\tilde{x}) = 1$ and*

$$-\psi'' - c\psi' \le \kappa\psi \quad \text{in } (0, X). \tag{29}$$

Proof. If $-2\sqrt{\kappa} < c < 2\sqrt{\kappa}$, we define $\psi(x) := e^{-\frac{c}{2}x} \sin\left(\frac{\sqrt{4\kappa - c^2}}{2}x\right)$ which solves $-\psi'' - c\psi' = \kappa\psi$. Also on $[0, 2\pi/\sqrt{4\kappa - c^2}]$ we have $\psi(0) = \psi(2\pi/\sqrt{4\kappa - c^2}) = 0$, $\psi \ge 0$ and maximal at point

$$\tilde{x} = \tilde{x}(c) = \begin{cases} \frac{2}{\sqrt{4\kappa - c^2}} \tan^{-1}\left(\frac{\sqrt{4\kappa - c^2}}{c}\right) & \text{if } 0 < c < 2\sqrt{\kappa} \\ \frac{\pi}{\sqrt{4\kappa}} & \text{if } c = 0 \\ \frac{2}{\sqrt{4\kappa - c^2}} \left(\tan^{-1}\left(\frac{\sqrt{4\kappa - c^2}}{c}\right) + \pi\right) & \text{if } -2\sqrt{\kappa} < c < 0. \end{cases}$$

$\tilde{x}(c)$ is then uniformly bounded for $c \in (-2\sqrt{\kappa}, 2\sqrt{\kappa})$, so that the renormalized function $\psi/\psi(\tilde{x})$ is as requested.

If $c \ge 2\sqrt{\kappa}$, we define $\psi(x) := e^{\frac{-\sqrt{\kappa}}{2}x} \sin\left(\frac{\sqrt{\kappa}}{2}x\right)$ and $\tilde{x} = \pi/\sqrt{4\kappa}$, so that ψ increases on $[0, \tilde{x}]$ and starts to decreases after $\tilde{x}$. On $[0, \tilde{x}]$, we have $\tan\left(\frac{\sqrt{\kappa}}{2}x\right) \le 1$ so that

$$\begin{aligned} -\psi''(x) - c\psi'(x) - \kappa\psi(x) &= \sqrt{\kappa} e^{\frac{-\sqrt{\kappa}}{2}x} \cos\left(\frac{\sqrt{\kappa}}{2}x\right) \left(\left(\frac{\sqrt{\kappa}}{2} - \frac{c}{2}\right) + \left(\frac{c}{2} - \sqrt{\kappa}\right) \tan\left(\frac{\sqrt{\kappa}}{2}x\right)\right) \\ &\le \frac{\sqrt{\kappa}}{2} e^{\frac{-\sqrt{\kappa}}{2}x} \cos\left(\frac{\sqrt{\kappa}}{2}x\right) (\sqrt{\kappa} - c + c - 2\sqrt{\kappa}) \\ &\le 0. \end{aligned}$$

Observe that $-\psi''(\tilde{x}) - c\psi'(\tilde{x}) - \kappa\psi(\tilde{x}) \le -\frac{\kappa e^{-\pi/4} \cos(\pi/4)}{2} < 0$, so that there is $X > \tilde{x}$ such that $-\psi'' - c\psi' - \kappa\psi \le 0$ on $[0, X]$. Hence $\psi/\psi(\tilde{x})$ is as requested. □

We will also use the elementary following lemma.

Lemma 3.3 (An auxiliary solution) *Let $\rho > 0$ and $b > 0$ be given. Then, for all $c \in \mathbb{R}$, there is a decreasing function $\chi = \chi_c : \mathbb{R} \to \mathbb{R}$ such that $\chi(0) = 1$, $\chi(b) = 0$ and*

$$-\chi'' - c\chi' = -\rho\chi \quad \text{in } \mathbb{R}. \tag{30}$$

Proof. One solves the linear ODE and sees that the function

$$\chi(x) := \left(1 - \frac{1}{1 - e^{-\sqrt{c^2 + 4\rho}\, b}}\right) e^{\frac{-c + \sqrt{c^2 + 4\rho}}{2}x} + \frac{1}{1 - e^{-\sqrt{c^2 + 4\rho}\, b}} e^{\frac{-c - \sqrt{c^2 + 4\rho}}{2}x}$$

is as requested. □

We now show that u can be uniformly (with respect to a) bounded away from θ far on the right or the left, depending on the sign of the speed c.

Proposition 3.4 (Moving away from θ) *There exist $\varepsilon > 0$ and $a_0 > 0$ such that, for all $a \ge a_0$, any solution (c, u) of*

$$\begin{cases} -u'' - cu' = u(u - \theta)(1 - \phi * u) & \text{in } (-a, a) \\ u(-a) = 1, \qquad u(0) = \theta, \qquad u(a) = 0, \\ u(x) = \theta \ \text{ if and only if } \ x = 0, \end{cases} \tag{31}$$

satisfies, if we define $\kappa := \theta(1 - \theta)/8 > 0$,

(i) $c > -2\sqrt{\kappa} \Longrightarrow u \le \theta/2$ *on* $[1/\varepsilon, a]$

(ii) $c < 2\sqrt{\kappa} \Longrightarrow u \ge \theta + \varepsilon$ *on* $[-a, -1/\varepsilon]$.

Proof. For $\kappa = \theta(1-\theta)/8$, let ψ be the bump of Lemma 3.2.

Assume $c > -2\sqrt{\kappa}$ and let us prove (i). Since $\phi \in L^1$, there exists $R > 0$ such that $\int_R^\infty \phi \le \frac{1-\theta}{2M}$, where M is the L^∞ bound we have on u. We turn upside down the bump and make it slide from the right towards the left until touching the solution u. Precisely one can define

$$\alpha_0 := \min\left\{\alpha \ge 0 : \forall x - \alpha \in [0, X],\, u(x) < \theta - \frac{\theta}{2}\psi(x-\alpha)\right\} \in [0, a).$$

We aim at proving that $\alpha_0 \le R$ uniformly with respect to large a. Assume by contradiction that $\alpha_0 > R$. The function $v := u - \theta + \frac{\theta}{2}\psi(\cdot - \alpha_0)$ has a zero maximum at some point x_0. Notice that since $\psi(\tilde{x}) = \max\psi$, the definition of α_0 implies that $x_0 - \alpha_0 \in (0, \tilde{x}]$, so that ψ is a subsolution of (29) around $x_0 - \alpha_0$. Thus $0 \ge v''(x_0) + cv'(x_0)$ implies

$$\begin{aligned} 0 &\ge (u'' + cu')(x_0) + \frac{\theta}{2}(\psi'' + c\psi')(x_0 - \alpha_0) \\ &\ge -u(x_0)(u(x_0) - \theta)(1 - \phi * u(x_0)) - \frac{\kappa\theta}{2}\psi(x_0 - \alpha_0) \\ &\ge (\theta - u(x_0))\left[u(x_0)(1 - \phi * u(x_0)) - \kappa\right]. \end{aligned} \tag{32}$$

Now, since $\alpha_0 \ge R$, we have $x_0 \ge R$ and we can estimate the nonlocal term by

$$\begin{aligned} \phi * u(x_0) &\le \int_{-\infty}^0 \phi(x_0 - y)u(y)\,dy + \int_0^\infty \phi(x_0 - y)u(y)\,dy \\ &\le M\int_R^\infty \phi + \theta\int_{\mathbb{R}} \phi \le \frac{1+\theta}{2}. \end{aligned} \tag{33}$$

Since $\frac{\theta}{2} \le u(x_0) < \theta$, it follows from (32) that $0 \ge \frac{\theta}{2}\frac{1-\theta}{2} - \kappa$, which contradicts the definition of κ. As a result $\alpha_0 \le R$, which means that the minimum $\theta/2$ of the reversed bump can slide to the left at least until $R + \tilde{x}$. In other words we have $u \le \theta/2$ on $[R + \tilde{x}, a]$ which concludes the proof of (i).

Assume $c < 2\sqrt{\kappa}$ and let us prove (ii). Since $\phi \in L^1$, we can choose $R > 0$ such that $\int_{[-R,R]^c} \phi \le \frac{1-\theta}{4M}$, where M is the L^∞ bound we have on u. Before using the bump we need a preliminary result via the function χ of Lemma 3.3.

For $\rho := 2M$ and $b := 2R + 1$ define χ as in Lemma 3.3. Provided that $a > 2b$, Proposition 3.1 shows that, for $\lambda > 0$ small enough, $\theta + \lambda\chi(\cdot + a) < u$ on $[-a, -a + b]$. We can therefore define

$$\lambda_0 := \max\left\{\lambda > 0 : \forall x \in [-a, -a + b],\, \theta + \lambda\,\chi(x + a) < u(x)\right\} \in (0, 1 - \theta].$$

The function $v := u - \theta - \lambda_0\chi(\cdot + a)$ thus has a zero minimum at a point x_0. Assume by contradiction that $\lambda_0 < 1 - \theta$, which in turn implies $x_0 \ne -a$. Also Proposition 3.1 implies $u(b) > \theta$ so that $x_0 \ne -a + b$. Thus $0 \le v''(x_0) + cv'(x_0)$ so that

$$\begin{aligned} 0 &\le (u'' + cu')(x_0) - \lambda_0\,(\chi'' + c\chi')(x_0 + a) \\ &= -u(x_0)(\theta - u(x_0))(1 - \phi * u(x_0)) - \lambda_0\,\rho\,\chi(x_0 + a) \\ &= (u(x_0) - \theta)\left[u(x_0)(1 - \phi * u(x_0)) - \rho\right] \\ &\le (u(x_0) - \theta)(M - \rho) < 0, \end{aligned}$$

which is absurd. Hence $\lambda_0 = 1 - \theta$ and thus

$$u(x) \ge \theta + (1 - \theta)\chi(x + a),\ \forall x \in [-a, -a + b] = [-a, -a + 2R + 1]. \tag{34}$$

Let us now define, for $\varepsilon > 0$ to be selected,

$$\alpha_0 := \max\{\alpha \le 0 : \forall \alpha - x \in [0, X],\, u(x) > \theta + \varepsilon\psi(\alpha - x)\} \in (-a, 0].$$

The estimate (34) shows that it is enough to choose $\varepsilon < (1-\theta)\min_{[-a,-a+2R]}\chi(\cdot + a)$ to get the lower bound $\alpha_0 \ge -a + 2R$. We aim at proving that $\alpha_0 \ge -2R$ uniformly with respect to large a. Assume by contradiction that $\alpha_0 < -2R$. The function $v := u - \theta - \varepsilon\psi(\alpha_0 - \cdot)$ has a zero minimum at some point x_0. Notice that since $\psi(\tilde{x}) = \max\psi$, the definition of α_0 implies that $\alpha_0 - x_0 \in (0, \tilde{x}]$, so that ψ is a subsolution of (29) around $\alpha_0 - x_0$. Thus, we have

$$\begin{aligned} 0 &\le (u'' + cu')(x_0) - \varepsilon(\psi'' + c\psi')(\alpha_0 - x_0) \\ &\le -u(x_0)(u(x_0) - \theta)(1 - \phi * u(x_0)) + \varepsilon\kappa\psi(\alpha_0 - x_0) \\ &\le (u(x_0) - \theta)\left[\kappa - u(x_0)(1 - \phi * u(x_0))\right]. \end{aligned} \tag{35}$$

Now observe that $-a + 2R \le \alpha_0 < -2R$ implies $-a + 2R \le x_0 \le -2R$, so that $[x_0 - R, x_0 + R] \subset [-a + R, -R]$. Therefore the Harnack inequality applied to $u - \theta$ provides a constant $C > 0$, independent of a, such that

$$0 < u(x) - \theta \le C(u(x_0) - \theta),\ \forall x \in [x_0 - R, x_0 + R].$$

This allows to estimate the nonlocal term by

$$\begin{aligned} \phi * u(x_0) &\le \int_{[-R,R]} \phi(y)u(x_0 - y)\,dy + \int_{[-R,R]^c} \phi(y)u(x_0 - y)\,dy \\ &\le \theta + C(u(x_0) - \theta) + \frac{1-\theta}{4} \le \frac{1+\theta}{2}, \end{aligned} \tag{36}$$

provided that $u(x_0) \le \theta + \frac{1-\theta}{4C}$, which is satisfied if we choose $\varepsilon > 0$ small enough (we recall that $u(x_0) \le \theta + \varepsilon\|\psi\|_\infty = \theta + \varepsilon$). It follows that $\kappa - u(x_0)(1 - \phi * u(x_0)) \le -\frac{\theta(1-\theta)}{4} < 0$, which contradicts (35). As a result $\alpha_0 \ge -2R$, which concludes the proof. □

Non triviality of U. Let us recall that (c^*, U) is constructed as the limit of (c_a, u_a) as $a \to \infty$. By extraction if necessary we can assume that the (c_a, u_a)' satisfy either (i) or (ii) of Proposition 3.4, and so does (c^*, U). As a result, the constructed wave (c^*, U) is non trivial.

4 Behaviors of U in $\pm\infty$

We now prove the behavior (8) as $x \to -\infty$, the limit (9) as $x \to \infty$, and that the constructed front is decreasing for $x > 0$ large enough. This will complete the proof of Theorem 1.1.

Proposition 4.1 (Behaviors of U at infinity) *Let (c^*, U) be the solution of (7) constructed in the end of Section 2. Then, for some $\varepsilon > 0$,*

$$U(x) \ge \theta + \varepsilon \ \text{ for all } x \in (-\infty, -1/\varepsilon),$$

and

$$\lim_{x \to +\infty} U(x) = 0.$$

Moreover, there exists $\bar{x} > 0$ such that U is decreasing on $[\bar{x}, \infty)$.

Proof. *Step 1: We show that $U > \theta$ on $(-\infty, 0)$, and $U < \theta$ on $(0, \infty)$.*

Thanks to Lemma 3.1, for any $a > 0$, the solution (c, u) in the box of Proposition 2.5 satisfies $u \ge \theta$ on $[-a, 0]$, and $u \le \theta$ on $[0, a]$. Since (c^*, U) is, on any compact interval, the uniform limit of such solutions, it satisfies $U \ge \theta$ on $(-\infty, 0]$ and $U \le \theta$ on $[0, \infty)$. Thus, any $x \ne 0$ such that $U(x) = \theta$ is a local extremum and $U'(x) = 0$; this is impossible, since U is a solution of (7) and $U \not\equiv \theta$ thanks to Section 3.

Step 2: We show that there exists $\varepsilon > 0$ such that $|U - \theta| \geq \varepsilon$ on $[-1/\varepsilon, 1/\varepsilon]^c$.

Consider first the case where $c^* \geq 0$. Since U is a limit of solutions of Proposition 2.5, the Proposition 3.4 shows that there exists $\varepsilon > 0$ such that $U \leq \theta/2$ on $(1/\varepsilon, \infty)$. To investigate the left side, as in the proof of Proposition 3.4 (ii), we choose $R > 0$ such that $\int_{[-R,R]^c} \phi \leq \frac{1-\theta}{4M}$, where M is the L^∞ bound we have on U. By the Harnack inequality applied to $U - \theta$, there exists $C > 0$ such that, for all $x_0 \leq -2R$,

$$0 < U(x) - \theta \leq C(U(x_0) - \theta),\ \forall x \in [x_0 - R, x_0 + R]. \tag{37}$$

Let us now define, for $\eta \geq 0$,

$$\psi_\eta(x) := \theta + \gamma(1 - \eta(-2R - x)), \quad \gamma := \min\left(\gamma_1 := \frac{1}{2}(U(-2R) - \theta), \gamma_2 := \frac{1-\theta}{4C}\right) > 0.$$

Since $\psi_\eta \leq U$ on $(-\infty, -2R]$ for $\eta > 0$ large enough, we can define

$$\eta_0 := \min\left\{\eta \geq 0 :\ \forall x \leq -2R,\ \psi_\eta(x) \leq U(x)\right\}.$$

Let us assume by contradiction that $\eta_0 > 0$. The function $U - \psi_{\eta_0}$ then attains a zero minimum at a point $x_0 < -2R$ (notice that $\gamma \leq \gamma_1$ prevents $x_0 = -2R$). Hence

$$\begin{aligned} 0 &\geq -(U - \psi_{\eta_0})''(x_0) - c^*(U - \psi_{\eta_0})'(x_0) \\ &\geq c^*\psi'_{\eta_0}(x_0) + U(x_0)(U(x_0) - \theta)(1 - \phi * U(x_0)) \\ &\geq c^*\gamma\eta_0 + U(x_0)(U(x_0) - \theta)\frac{1-\theta}{2} > 0, \end{aligned}$$

where we have used the estimate (36) for U (notice that this is possible since we have the two ingredients (37) and $U(x_0) = \psi_{\eta_0}(x_0) \leq \theta + \frac{1-\theta}{4C}$). This is a contradiction which proves that $\eta_0 = 0$, and then $U \geq \theta + \gamma$ on $(-\infty, -2R)$. This concludes the case $c^* \geq 0$.

Consider next the case where $c^* \leq 0$. Since U is a limit of solutions of Proposition 2.5, the Proposition 3.4 shows that there exists $\varepsilon > 0$ such that $U \geq \theta + \varepsilon$ on $(-\infty, 1/\varepsilon)$. To investigate the right side, as in the proof of Proposition 3.4 (i), we choose $R > 0$ such that $\int_R^\infty \phi \leq \frac{1-\theta}{2M}$, were M is the L^∞ bound we have on U. We define

$$\psi_\eta(x) := \theta + \gamma(-1 + \eta(x - 2R)), \quad \gamma := \frac{1}{2}(\theta - U(2R)) > 0,$$

which satisfies $\psi_\eta \geq U$ on $[2R, \infty)$ for $\eta > 0$ large enough. We can then define

$$\eta_0 := \min\left\{\eta \geq 0 : \forall x \geq 2R,\ \psi_\eta(x) \geq U(x)\right\}.$$

Let us assume by contradiction that $\eta_0 > 0$. The function $U - \psi_{\eta_0}$ then attains a zero maximum at a point $x_0 > 2R$, and therefore

$$\begin{aligned} 0 &\leq -(U - \psi_{\eta_0})''(x_0) - c^*(U - \psi_{\eta_0})'(x_0) \\ &\leq c^*\psi'_{\eta_0}(x_0) + U(x_0)(U(x_0) - \theta)(1 - \phi * U(x_0)) \\ &< c^*\gamma\eta_0 + U(x_0)(U(x_0) - \theta)\frac{1-\theta}{2} < 0, \end{aligned}$$

where we have used the estimate (33) for U. This is a contradiction which proves that $\eta_0 = 0$, and then $U \leq \theta - \gamma$ on $(2R, \infty)$. This concludes the case $c^* \leq 0$.

Step 3: We show that U decreases to 0 on some interval $(\bar{x}, \infty)$.

Choose $R > 0$ large enough so that $\int_R^\infty \phi \leq \frac{1-\theta}{2M}$. Assume by contradiction that U admits a local maximum at some point $x_m \geq R$. Since $U(x_m) < \theta$, by evaluating the equation (7) we see that $1 \leq \phi * U(x_m)$. But on the other hand

$$\phi * U(x_m) \leq \int_{-\infty}^0 \phi(x_m - y)u(y)\,dy + \int_0^\infty \phi(x_m - y)u(y)\,dy \leq M\int_R^\infty \phi + \theta \leq \frac{1+\theta}{2} < 1,$$

which is a contradiction. Hence U cannot attain a maximum on (R,∞), which in turn implies that there is $\bar{x} > 0$ such that U is monotonic (increasing or decreasing) on $[\bar{x},\infty)$. Hence, as $x \to \infty$, $U(x) \to l$ and, by Step 2, $0 \le l \le \theta - \varepsilon$.

We define next $v_n(x) := U(x+n)$, which solves

$$-{v_n}'' - c^* {v_n}' = v_n(v_n - \theta)(1 - \phi * v_n) \quad \text{on } \mathbb{R}.$$

Since the L^∞ norm of the right hand side member is uniformly bounded with respect to n, the interior elliptic estimates imply that, for all $R > 0$, all $1 < p < \infty$, the sequence (v_n) is bounded in $W^{2,p}([-R,R])$. From Sobolev embedding theorem, one can extract $v_{\varphi(n)} \to v$ strongly in $C^{1,\beta}_{loc}(\mathbb{R})$ and weakly in $W^{2,p}_{loc}(\mathbb{R})$. Since $v_n(x) = U(x+n) \to l$ we have $v \equiv l$ and $v' \equiv 0$. Combining this with the fact that v solves

$$-v'' - c^* v' = v(v-\theta)(1 - \phi * v) \quad \text{on } \mathbb{R},$$

we have $l(l-\theta)(1-l) = 0$, which implies $l = 0$ and the decrease of U on $[\bar{x},\infty)$. □

5 Focusing kernels

In this section we consider (c^*_σ, U_σ) the constructed waves for the focusing kernels

$$\phi_\sigma(x) = \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right), \quad \sigma > 0.$$

We prove Proposition 1.2. Item (i) consists in a perturbation analysis, and item (ii) will follow from the L^2 analysis performed in [1].

Proof of (i)**.** Assume $m_1 := \int_{\mathbb{R}} |z|\phi(z)\,dz < \infty$. We have

$$-{U_\sigma}'' - c^*_\sigma {U_\sigma}' = U_\sigma(U_\sigma - \theta)(1 - \phi_\sigma * U_\sigma) \quad \text{on } \mathbb{R}, \tag{38}$$

and $0 \le U_\sigma \le M_\sigma$, $c_{min,\sigma} \le c^*_\sigma \le c_{max,\sigma}$, with M_σ, $c_{min,\sigma}$, $c_{max,\sigma}$ depending *a priori* on $\sigma > 0$. The following lemma improves the bounds for the travelling waves: as $\sigma \to 0$ solutions (c^*_σ, U_σ) are uniformly bounded.

Lemma 5.1 (Uniform bounds for (c^*_σ, U_σ)**)** *Let $\sigma_0 > 0$ be arbitrary. Then there is $M > 0$, $c_{min} \in \mathbb{R}$, $c_{max} \in \mathbb{R}$ such that, for all $\sigma \in (0,\sigma_0)$,*

$$0 \le U_\sigma \le M, \quad \text{and} \quad c_{min} \le c^*_\sigma \le c_{max}.$$

Proof. It is sufficient to work on the solutions (c_σ, u_σ) in the box. Define $M_\sigma := \max_{x\in[-a,a]} u_\sigma(x)$. A first lecture of Lemma 2.1 yields the rough bound (17) with the kernel ϕ_σ in place of ϕ. Since $\|\phi'_\sigma\|_{L^\infty(-1,1)} \le \frac{1}{\sigma^2}\|\phi'\|_{L^\infty(\mathbb{R})}$ and $\phi_\sigma(0) = \frac{1}{\sigma}\phi(0)$, we infer from (17) that there is a constant $b > 0$, such that $M_\sigma \le b^2/\sigma^2$. Equipped with this rough bound, we go back to the proof of Lemma 2.1 but rather than (16) we select the improvement

$$x_0 := \frac{1}{2b}\sigma.$$

Hence, going further into the proof, we discover

$$1 \ge \frac{3}{4} M_\sigma \int_0^{\frac{\sigma}{2b}} \phi_\sigma = \frac{3}{4} M_\sigma \int_0^{\frac{1}{2b}} \phi,$$

so that $M_\sigma \le M := \frac{4}{3}\left(\int_0^{\frac{1}{2b}} \phi\right)^{-1}$, that is a uniform bound M for M_σ as $\sigma \to 0$.

In view of Lemma 2.2 and of the proof of Lemma 2.4, the uniform bound M yields uniform bounds c_{max} and c_{min} for the speed c_σ. The lemma is proved. □

Hence, the coefficients and the right hand side member of the elliptic equation (38) are uniformly bounded w.r.t. $\sigma \in (0, \sigma_0)$. Therefore Schauder's elliptic estimates — see [15, (1.11)] for instance— imply that, $\|U_\sigma\|_{C^{2,\beta}} \le C_0$ with $C_0 > 0$ not depending on σ. It follows that

$$|U_\sigma - \phi_\sigma * U_\sigma|(x) \le \int_{\mathbb{R}} \phi_\sigma(x-y)|U_\sigma(x) - U_\sigma(y)|\,dy \le \|U_\sigma{}'\|_\infty \int_{\mathbb{R}} \phi_\sigma(x-y)|x-y|\,dy \le C_0 m_1 \sigma.$$

Hence writing $1 - \phi_\sigma * U_\sigma = 1 - U_\sigma + U_\sigma - \phi_\sigma * U_\sigma$ in (38), we get, for some $C > 0$,

$$f_\sigma^+(U_\sigma) \ge -U_\sigma{}'' - c_\sigma^* U_\sigma{}' \ge f_\sigma^-(U_\sigma) \quad \text{on } \mathbb{R},$$

where

$$f_\sigma^\pm(s) := s(s-\theta)(1-s) \pm C\sigma.$$

Hence, by the comparison principle, $\psi_\sigma^+(x,t) \ge U_\sigma(x - c_\sigma^* t) \ge \psi_\sigma^-(x,t)$, with $\psi_\sigma^\pm$ the solutions of the Cauchy parabolic problems

$$\begin{cases} \partial_t \psi = \partial_{xx}\psi + f_\sigma^\pm(\psi) & \text{in } (0,\infty)\times\mathbb{R}, \\ \psi(x,0) = U_\sigma(x) & \text{in } \mathbb{R}. \end{cases}$$

Observe that, for $\sigma > 0$ small enough, the functions $f_\sigma^\pm$ are still of the bistable type with three zeros $\alpha_\sigma^\pm = \mathcal{O}(\sigma)$, $\beta_\sigma^\pm = \theta + \mathcal{O}(\sigma)$, $\gamma_\sigma^\pm = 1 + \mathcal{O}(\sigma)$. It is therefore well-known [10, Theorem 3.1] that, for a given small $\sigma > 0$, the solutions $\psi_\sigma^\pm$ approach $U_\sigma^\pm(x - c_\sigma^\pm t - x_0^\pm)$, for two given $x_0^\pm \in \mathbb{R}$, uniformly in x as $t \to \infty$. Here $(c_\sigma^\pm, U_\sigma^\pm)$ denotes the bistable wave

$$\begin{cases} U_\sigma^\pm{}'' + c_\sigma^\pm U_\sigma^\pm{}' + f_\sigma^\pm(U_\sigma^\pm) = 0, \\ \lim_{x\to-\infty} U_\sigma^\pm(x) = \gamma_\sigma^\pm, \quad U_\sigma^\pm(0) = \beta_\sigma^\pm, \quad \lim_{x\to+\infty} U_\sigma^\pm(x) = \alpha_\sigma^\pm. \end{cases}$$

This enforces

$$c_\sigma^+ \ge c_\sigma^* \ge c_\sigma^-.$$

Since, as $\sigma \to 0$, $c_\sigma^\pm$ converge to c_0^* the speed of the wave (10), this concludes the proof of (i).

Proof of (ii)**.** Assume $\theta \neq \frac{1}{2}$, which in turn implies $c_0^* \neq 0$, and $m_2 := \int_{\mathbb{R}} z^2\phi(z)\,dz < \infty$. Observe that $\int_{\mathbb{R}} z^2\phi_\sigma(z)\,dz = \sigma^2 m_2$ so that, in virtue of [1, Lemma 5], to get $\lim_{x\to-\infty} U_\sigma(x) = 1$ it is enough to have

$$\sigma\sqrt{m_2}{M_\sigma}^2 < |c_\sigma^*|, \tag{39}$$

which is clear, for small enough $\sigma > 0$, since $M_\sigma \le M$, and $|c_\sigma^*| \to |c_0^*| \neq 0$. □

6 The ignition case

Here we explain briefly how to use similar arguments to handle the case of the ignition case.

The typical local ignition case is given by $-U'' - c^*U' = \mathbf{1}_{\{U\ge\theta\}}(U-\theta)(1-U)$, and the corresponding nonlocal problem we consider is written as

$$-U'' - c^*U' = \begin{cases} 0 & \text{where } U < \theta \\ (U-\theta)(1-\phi * U) & \text{where } U \ge \theta. \end{cases} \tag{40}$$

Then, one can construct a solution $(c,u) = (c_a, u_a)$ in a bounded box $[-a,a]$ exactly as in Section 2, and thus a solution (c^*, U) of (40) as a limit of solutions (c,u). One can also readily get Proposition 3.1, which in turn implies that the solution u solves $-u'' - cu' = 0$ on $(0,a)$, $u(0) = \theta$, $u(a) = 0$ and therefore becomes explicit on this interval:

$$u(x) = \frac{-\theta}{e^{ca} - 1} + \frac{\theta e^{-cx}}{1 - e^{-ca}}, \quad \text{for } 0 \le x \le a. \tag{41}$$

Assume by contradiction that $c^* \leq 0$. Then Proposition 3.4 (ii), which also directly applies to the ignition case, implies that there exists $\varepsilon > 0$ such that, for $a > 0$ large enough, $u \geq \theta + \varepsilon$ on $(-\infty, -1/\varepsilon)$, which in turn implies the non triviality of U. If $c^* < 0$ then, as $a \to \infty$,

$$u'(0) = \frac{-c\theta}{1 - e^{-ca}} \to 0,$$

since $c \to c^* < 0$. Hence $U'(0) = 0$ and then $U \equiv \theta$, a contradiction. If $c^* = 0$ then U is a bounded solution of $-U'' = 0$ on $(0, \infty)$ such that $U(0) = \theta$, that is $U \equiv \theta$, a contradiction. As a result $c^* > 0$. Letting $a \to \infty$ in (41) yields

$$U(x) = \theta e^{-c^* x}, \quad \text{for all } x \geq 0.$$

To conclude, the behavior (8) as $x \to -\infty$ is proved as in Proposition 4.1.

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