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Initial trace of solutions of semilinear heat equation with absorption

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Abstract We study the initial trace problem for positive solutions of semilinear heat equations with strong absorption. We show that in general this initial trace is an outer regular Borel measure. We emphasize in particular the case where u satisfies (E) $\partial_t u - \Delta u + t^\alpha |u|^{q-1}u = 0$, with $q > 1$ and $\alpha > -1$ and prove that in the subcritical case $1 < q < q_{\alpha,N} := 1 + 2(1+\alpha)/N$ the initial trace establishes a one to one correspondence between the set of outer regular Borel in measures $\mathbb{R}^N$ and the set of positive solutions of (E) in $\mathbb{R}^N \times \mathbb{R}_+$.

Keywords: Heat equation; Singularities; Harnack inequalities; Radon measures; Borel measures; barrier; initial trace.

MSC2010: 35K58; 35K15

1 Introduction

In this paper we study the initial trace problem for positive solutions of

$$\partial_t u - \Delta u + g(x,t,u) = 0 \text{ dans } \; Q_T^\Omega := \Omega \times (0,T) \tag{1.1}$$

where Ω is an open domain in $\mathbb{R}^N$, $g \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ such that $g(x,t,.)$ is nondecreasing $\forall (x,t) \in \Omega \times \mathbb{R}$ and $rg(x,t,r) \geq 0$ for all $(x,t,r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}$. Our first result establishes the existence of an initial trace.

Theorem A *Assume g satisfies the above conditions and that equation* (1.1) *possesses a barrier at any $z \in \Omega$. If $u \in C^1(Q_T^\Omega)$ is a positive solution of (1.1), it admits an initial trace which belongs to the class of outer regular positive Borel measure in Ω.*

The barrier assumption will be made precise later on in full generality. It is fulfilled if $g(x,t,r) = h(x)t^\alpha |r|^{q-1} r$ with $\alpha > -1$, $q > 1$ and $h \in L^\infty_{loc}(\Omega)$ satisfies $\inf \operatorname{ess} h > 0$ for any compact subset $K \subset \Omega$, or if g satisfies the Keller-Osserman condition, that is $g(x,t,r) \geq h(r) \geq 0$ where h is nondecreasing and there exists a such that

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} \qquad \text{where } H(s) = \int_0^s h(t)dt. \tag{1.2}$$

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The initial trace of positive solutions of (1.1) exists in the following sense: *there exists a relatively closed set $\mathcal{S} \subset \Omega$ and a positive Radon measure μ on $\mathcal{R} := \Omega \setminus \mathcal{S}$ with the following properties:*

(i) for any $x_0 \in \mathcal{S}$ and any $\epsilon > 0$

$$\lim_{t \to 0} \int_{B_\epsilon(x_0) \cap \Omega} u(x,t) dx = \infty, \tag{1.3}$$

(ii) for any $\zeta \in C_c(\mathcal{R})$

$$\lim_{t \to 0} \int_\Omega u(x,t) \zeta(x) dx = \int_\Omega \zeta d\mu. \tag{1.4}$$

The couple $(\mathcal{S}, \mu)$ is unique and characterizes a unique positive outer regular Borel measure ν on Ω.

A similar notion of boundary trace has been introduced by Marcus and Véron [10] in the study of positive solutions of

$$-\Delta u + g(x,u) = 0 \text{ in } \Omega. \tag{1.5}$$

This notion in itself has turned out to be a very usefull tool for classifying the positive solutions of (1.5).

In the second part we concentrate on the particular case of equation

$$\partial_t u - \Delta u + t^\alpha |u|^{q-1} u = 0 \text{ dans } Q_T^\Omega \tag{1.6}$$

where $T > 0$, $\alpha > -1$ and $q > 1$. Among the most useful tools, we point out the description of positive solutions with an isolated singularity at $(a, 0)$ for $a \in \Omega$, whenever they exist: they are solutions u of (1.6) in Q_T^Ω, which belong to $C^{2,1}\left(Q_T^\Omega\right) \cap C\left(\Omega \times [0,T) \setminus \{(a,0)\}\right)$ and satisfy

$$u(x,0) = 0 \text{ in } \Omega \setminus \{a\}. \tag{1.7}$$

When $\alpha = 0$, Brezis and Friedman prove in [2] that if $B_{2R}(a) \subset \Omega$, then any such solution satisfies

$$u(x,t) \leq \frac{C(N,q,R)}{\left(|x-a|^2 + t\right)^{\frac{1}{q-1}}} \quad \forall (x,t) \in B_R(a) \setminus \{0\} \times [0,T]. \tag{1.8}$$

They also prove that if $1 < q < q_N := 1 + \frac{2}{N}$ and $k > 0$ there exist singular solutions with initial data $u(.,0) = k\delta_a$, unique if u vanishes on $\partial\Omega \times [0,T]$. In this range of exponents, Brezis, Peletier, Terman obtain in [1] the existence and uniqueness of a *very singular solution* of (1.6), always with $\alpha = 0$: it is a positive solution in $Q_\infty := Q_\infty^{\mathbb{R}^N}$ under the form

$$v_0(x,t) = t^{-1/(q-1)} V_0\left(\frac{x}{\sqrt{t}}\right),$$

where $V_0 > 0$ is C^2 and satisfies

$$\begin{array}{c} -\Delta V_0 - \frac{1}{2}\eta.\nabla V_0 - \frac{1}{q-1} V_0 + V_0^q = 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|\eta| \to \infty} |\eta|^{\frac{2}{q-1}} V_0(\eta) = 0. \end{array} \tag{1.9}$$

Actually, Kamin and Peletier show that v_0 is the limit of the solutions u_k of (1.6) in Q_∞ which satisfy $u(.,0) = k\delta_0$. The very singular singular solution plays a fundamental role in Marcus and Véron's description [7] of the initial trace of positive solutions of (1.6) with $\alpha = 0$. In [9], Marcus et Véron study this equation when $\alpha \geq 0$ and $1 < q < q_{\alpha,N} = 1 + \frac{2(1+\alpha)}{N}$. They obtain the existence of a self-similar solution of (1.6) in Q_∞ under the form

$$v_\alpha(x,t) = t^{-\frac{1+\alpha}{q-1}} V_\alpha\left(\frac{x}{\sqrt{t}}\right),$$

which satisfies

$$\lim_{t\to 0} v_\alpha(x,t) = 0 \quad \forall x \neq 0$$

and

$$\lim_{t\to 0} \int_{B_\epsilon} v_\alpha(x,t)dx = \infty \quad \forall \epsilon > 0.$$

The function V_α is nonnegative and verifies

$$-\Delta V_\alpha - \frac{1}{2}\eta.\nabla V_\alpha - \frac{1+\alpha}{q-1}V_\alpha + V_\alpha^q = 0 \text{ in } \mathbb{R}^N. \tag{1.10}$$

Furthermore

$$V_\alpha(\eta) = C|\eta|^{\frac{2(1+\alpha)}{q-1}-N} e^{\frac{-|\eta|^2}{4}} (1+o(1)) \text{ as } |\eta| \to \infty. \tag{1.11}$$

If $1 < q < q_{\alpha,N}$, they show that for every $k > 0$ there exists a unique solution $u_{k\delta_a}$ of (1.6) in Q_∞ with initial data $k\delta_a$. Furthermore $\lim_{k\to\infty} u_{k\delta_a} = v_\alpha$. Actually the limitation $\alpha \geq 0$ can be relaxed to $\alpha > -1$ has we will see it later on. Furthermore Q_∞ can be replaced by Q_∞^Ω provided $\partial\Omega$ is compact and smooth enough and $u_{k\delta_a}$ vanishes on $\partial\Omega \times [0,\infty)$.

In this article we extend Brezis-Friedman removability result to equation (1.6). We also extend Oswald's classification of positive isolated singularities [12]. The starting point of our study is the following extension of estimate (1.8) valid for any $\alpha > -1$ and $q > 1$.

$$u(x,t) \leq \frac{C(N,q,\alpha,R)}{\left(|x-a|^2 + t\right)^{\frac{1+\alpha}{q-1}}} \quad \forall (x,t) \in B_R \backslash \{a\} \times [0,T]. \tag{1.12}$$

The obstacle for obtaining such an estimate arises when $\alpha > 0$ and the absorption term $t^\alpha u^q$ is degenerate near $t = 0$. We overcome this difficulty by a delicate construction of 1-dim self-similar supersolutions. Thanks to this estimate, we obtain that the following classification result holds.

Theorem B *Assume $1 < q < q_{\alpha,N}$ and $u \in C^1(Q_T^\Omega) \cap C(\Omega \times [0,T] \backslash \{(a,0))\})$ is a solution of (1.6) which vanishes on $\Omega \setminus \{a\}$ at $t = 0$. Then*

(i) either there exists $k \geq 0$ such that $u(.,0) = k\delta_a$ and

$$u(x,t) \sim kE(x-a,t) \qquad \text{as } (x,t) \to (a,0), \tag{1.13}$$

where $E(x,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$,

(ii) or

$$u(x,t) \sim v_\alpha(x-a,t) \qquad \text{as } (x,t) \to (a,0). \tag{1.14}$$

In the supercritical case the following removability statement holds.

Theorem C *Assume $q \geq q_{\alpha,N}$ and $u \in C^1(Q_T^\Omega) \cap C(\Omega \times [0,T] \backslash \{(a,0))\})$ is a solution of (1.6) which vanishes on $\Omega \backslash \{a\}$ at $t=0$. Then u can be extended by continuity as a function in $C(\Omega \times [0,T])$.*

We prove that equation (1.6) admits a barrier at any $z \in \Omega$. More precisely we construct a positive solution w_{B_R} of (1.6) in $Q_\infty^{B_R}$ which tends to 0 locally uniformly in B_R when $t \to 0$ and which blows-up uniformly on $\partial B_R \times [\tau, \infty)$, for any $\tau > 0$. Applying Theorem A, we infer that any positive solution admits an initial trace which is an outer regular Borel measure $\nu \approx (\mathcal{S}, \mu)$. Using sharp parabolic Harnack inequality and a concentration principle, we prove the following result which is the key-stone for analyzing the behaviour of u on the set $\mathcal{S}$.

Theorem D *Assume $1 < q < q_{\alpha,N}$ and $u \in C^{2,1}(Q_T^\Omega)$ is a positive solution of (1.6) with initial trace $(\mathcal{S}, \mu)$. Then for any $a \in \mathcal{S}$ there holds*

$$u(x,t) \geq u_{\infty,a}(x,t) \qquad \forall (x,t) \in Q_T^\Omega \tag{1.15}$$

where $u_{\infty,a} = \lim_{k\to\infty} u_{k\delta_a}$, $u_{k\delta_a}$ being the solution of (1.6) in Q_T^Ω with initial trace $k\delta_a$ and which vanishes on $\partial\Omega \times [0,T]$.

It is important to notice that the behaviour of $u_{\infty,a}$ near $(a,0)$ is given by (1.14) and (1.11). Using (1.15), (1.12) and sharp asymptotics of the function V_α, we are able to prove the following result which extends Theorem A.

Theorem E *Assume $1 < q < q_{\alpha,N}$ and $\Omega \subset \mathbb{R}^N$ is open with a C^2 compact boundary, eventually empty. Then for any couple $(\mathcal{S}, \mu)$ where $\mathcal{S} \subset \Omega$ is relatively closed and $\mu \in \mathfrak{M}_+(\Omega \setminus \mathcal{S})$, there exists a maximal positive solution $\overline{u}$ and a minimal positive solution $\underline{u}$ of (1.6), which belong to $C^{2,1}(Q_T^\Omega) \cap C(\overline{\Omega} \times (0,T])$, satisfy (1.3) and (1.4) and vanish on $\partial\Omega \times (0,T]$. If $\inf\{|z-z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$ and μ is bounded in a neighborhood of $\partial\Omega$, then $\overline{u} = \underline{u}$.*

2 Initial trace

In this section $\Omega \subset \mathbb{R}^N$ is an open domain, $Q_T^\Omega = \Omega \times (0,T)$, $\partial_\ell Q_T^\Omega = \Omega \times \{0\} \cup \partial\Omega \times [0,T)$ and $g \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R})$. If u is defined in $\Omega \times \mathbb{R}_+$, we denote by $g \circ u$ the function $(x,t) \mapsto g(x,t,u(x,t))$. We say that g belongs to $\mathcal{H}$ (resp $\mathcal{H}_0$) if

$$\begin{aligned} &g(x,t,r) \geq 0 \quad \forall (x,t,r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \\ &\text{(resp. } g \in \mathcal{H} \text{ and } r \mapsto g(x,t,r) \text{ is nondecreasing).} \end{aligned} \tag{2.1}$$

We denote by $\mathfrak{M}(\Omega)$ the set of Radon measures in Ω, and by $\mathfrak{M}^b(\Omega)$ (resp. $\mathfrak{M}^{b,\rho}(\Omega)$) the subset of Radon measures such that

$$\int_\Omega d\,|\mu| < \infty \quad \left(\text{resp. } \int_\Omega \rho d\,|\mu| < \infty\right),$$

where $\rho(x) := \mathrm{dist}\,(x, \partial\Omega)$. Their positive cones are respectively $\mathfrak{M}_+(\Omega)$, $\mathfrak{M}_+^b(\Omega)$ and $\mathfrak{M}_+^{b,\rho}(\Omega)$.

Definition 2.1 *Let $\mathcal{S}$ be a relatively closed subset of Ω and μ a Radon measure on $\mathcal{R} := \Omega \setminus \mathcal{S}$. We say that a nonnegative function $u \in C(Q_T^\Omega)$ admits the couple $(\mathcal{S}, \mu)$ for initial trace if*

$$\lim_{t\to 0} \int_{\mathcal{R}} u(x,t)\zeta(x)dx = \int_{\mathcal{R}} \zeta d\mu \qquad \forall \zeta \in C_c(\mathcal{R}),$$

and

$$\lim_{t\to 0} \int_U u(x,t)dx = \infty \qquad \forall U \subset \Omega, U \text{ open}, U \cap \mathcal{S} \neq \emptyset$$

The set $\mathcal{S}$ is the set of singular initial points of u and its complement $\mathcal{R}$ the set of regular initial points. We write $tr_\Omega(u) = (\mathcal{S}, \mu)$.

Let $\tilde{\mu}$ be the extension of μ as a locally bounded Borel measure. To the couple $(\mathcal{S}, \tilde{\mu})$ we can associate a unique outer regular Borel measure ν defined by

$$\nu(E) = \begin{cases} \infty & \forall E \subset \Omega : E \text{ Borel}, E \cap \mathcal{S} \neq \emptyset \\ \tilde{\mu}(E) & \forall E \subset \Omega : E \text{ Borel}, E \subset \mathcal{R}. \end{cases} \tag{2.2}$$

Lemma 2.1 *Assume Ω is a bounded open domain with a C^2 boundary, $T > 0$, $g \in \mathcal{H}$, and let $u \in C(\overline{\Omega} \times (0,T])$ be a positive solution of*

$$\partial_t u - \Delta u + g \circ u = 0 \qquad \text{in } Q_T^\Omega. \tag{2.3}$$

If $g \circ u \in L^1_\rho(Q_T^\Omega)$, then $u \in L^\infty\left(0, T, L^1_\rho(\Omega)\right)$ and there exists $\mu \in \mathfrak{M}_+^\rho(\Omega)$ such that

$$\lim_{t\to 0} \int_\Omega u(x,t)\zeta(x)dx = \int_\Omega \zeta d\mu \qquad \forall \zeta \in C_c(\Omega). \tag{2.4}$$

Proof. If $\phi_1 > 0$ is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ and λ_1 is the corresponding eigenvalue, we have

$$\frac{d}{dt}\int_\Omega u\phi_1 dx + \lambda_1 \int_\Omega u\phi_1 dx + \int_\Omega g\circ u\, \phi_1 dx + \int_{\partial\Omega} u\frac{\partial \phi_1}{\partial \nu} dS = 0,$$

where ν is the normal vector. Set $X = \displaystyle\int_\Omega u\phi_1 dx$, then by Hopf Lemma,

$$X' + \lambda_1 X + \int_\Omega g\circ u\, \phi_1 dx \geq 0$$

which yields to

$$\frac{d}{dt}\left(e^{\lambda_1 t} X\right) + e^{\lambda_1 t}\int_\Omega g\circ u\, \phi_1 dx \geq 0.$$

For $s \in (t,T)$

$$\frac{d}{dt}\left(e^{\lambda_1 t} X - \int_t^T e^{\lambda_1 s}\int_\Omega g\circ u\, \phi_1 dx ds\right) \geq 0,$$

which means that the mapping

$$t \to e^{\lambda_1 t} X - \int_t^T e^{\lambda_1 s} \int_\Omega g \circ u \, \phi_1 dx ds$$

is nondecreasing. Therefore

$$e^{\lambda_1 t} X - \int_t^T e^{\lambda_1 s} \int_\Omega g \circ u \, \phi_1 dx ds \leq e^{\lambda_1 T} X.$$

and finally

$$X \leq e^{\lambda_1 (T-t)} X + e^{-\lambda_1 t} \int_t^T e^{\lambda_1 s} \int_\Omega g \circ u \, \phi_1 dx ds.$$

Since $\rho^{-1}\phi_1$ is positively bounded from above and from below, $u \in L^\infty\left(0, T, L^1_\rho(\Omega)\right)$ and there exists a sequence $\{t_n\}$ decreasing to 0 and a measure $\mu \in \mathfrak{M}^\rho_+(\Omega)$ such that

$$\lim_{t_n \to 0} \int_\Omega u(x, t_n) \zeta dx = \int_\Omega \zeta d\mu \qquad \forall \zeta \in C_c(\Omega).$$

If $\zeta \in C^2_c(\Omega)$ there holds

$$\int_\Omega u(x, t_n) \zeta dx = \int_{t_n}^T \int_\Omega \left(g \circ u \, \zeta - u \Delta \zeta\right) dx dt + \int_\Omega u(x, T) \zeta dx,$$

thus

$$\int_\Omega \zeta d\mu = \iint_{Q^\Omega_T} \left(g \circ u \, \zeta - u \Delta \zeta\right) dx dt + \int_\Omega u(x, T) \zeta dx.$$

This implies that μ is uniquely determined and $u(., t)$ converges to μ in the weak sense of measures. □

Corollary 2.1 *Assume Ω is an open domain, $g \in \mathcal{H}$ and $u \in C^{2,1}(Q^\Omega_T)$ is a positive solution of (2.4). Suppose that for any $z \in \Omega$ there exists an open neighborhood $U \subset \Omega$ such that*

$$\int_0^T \int_U g \circ u dx dt < \infty.$$

Then $u(x, t) \in L^\infty\left(0, T, L^1_{loc}(\Omega)\right)$ and there exists a positive Radon measure μ on U such that

$$\lim_{t \to 0} \int_{\mathcal{R}} u(x, t) f(x) dx = \int_{\mathcal{R}} f d\mu \quad \forall f \in C_c(\mathcal{R}).$$

Proof. We apply the previous lemma in replacing U by a ball $B_\epsilon(z)$ and conclude by a partition of unity. □

The following class of nonlinearity has been introduced by Marcus and Véron [10] in order to study the boundary trace of solutions of elliptic equations.

Definition 2.2 *A function $g \in \mathcal{H}$ is a coercive nonlinearity in Q_T^Ω if, for every subdomain Ω' of Ω and every $\epsilon \in (0,T)$, the set of positive solutions of (1.1) in $Q_{\epsilon,T}^{\Omega'} := \Omega' \times (\epsilon, T)$ is uniformly bounded in compact subsets of $Q_{\epsilon,T}^{\Omega'}$.*

Definition 2.3 *Let $z \in \Omega$. We say that equation (1.1) possesses a strong barrier at z if there exists a number $r_0 \in (0, \rho(z))$ such that, for every $r \in (0, r_0)$, there exists a positive supersolution $w = w_{r,z}$ of (1.1) in $B_r(z) \times (0,T)$ such that*

$$w \in C\left(B_r(z) \times [0,T)\right), \lim_{|x-z| \to r} w(x,t) = \infty \text{ locally uniformly if } t \in (0,T). \tag{2.5}$$

Lemma 2.1 *Assume $g \in \mathcal{H}$ is a coercive nonlinearity in Q_T^Ω, then the set of solutions of (1.1) in Q_T^Ω is uniformly bounded from above in every compact subset of Q_T^Ω. Furthermore, if $g \in \mathcal{H}_0$, $A \subset \Omega$ is open and (1.1) possesses a strong barrier at every point of $z \in A$, then the set of solutions u of (1.1) such that $u \in C\left(A \times [0,T)\right)$ and $u(x,0) = 0$ on A is uniformly bounded from above in every compact subset of $A \times [0,T)$.*

Proof. Let K be a compact subset of Q_T^Ω and let Ω' be a smooth, bounded domain of Ω and $\epsilon > 0$ such that $K \subset Q_{\epsilon,T}^{\Omega'}$ Let $U = U_{Q_\infty^{\Omega'}}$ be the minimal large solution of (1.1) in $Q_T^{\Omega'}$, i.e. the limit, when $k \to \infty$, of solutions with Cauchy-Dirichlet data k on $\partial_\ell Q_{\epsilon,T}^{\Omega'} := \Omega' \times \{\epsilon\} \cup \partial\Omega' \times [\epsilon, T)$. By the maximum principle, if $u \in C(Q_T^\Omega)$ is a solution of (1.1), then $u \leq U$ in Ω'.
For the second statement, let K be a compact subset of A. For any $z \in K$ there exists $r_z > 0$ such that for any $r \in (0, r_z)$ there exists a positive supersolution of (1.1) in $Q_T^{B_r(z)}$ which satisfies (2.5). Since K is compact, there exist $z_1, ..., z_p$ such that $K \subset \cup_{j=1}^p B_{r_{z_j}/2}(z_j)$. For any $j \in \{1, ..., p\}$ we denote by w_j the supersolution in $Q_T^{B_{2r_{z_j}/3}(z_j)}$. By comparison principle, there holds

$$u(x,t) \leq \sup\{w_j(x,t) : (x,t) \in B_{r_{z_j}/2}(z_j) \times (0,T)\} := M_j, \tag{2.6}$$

for $(x,t) \in Q_T^{B_{r_{z_j}/2}(z_j)}$. Therefore $u \leq M = \max_{j=1,...,p} M_j$ in $K \times (0,T)$. □

Lemma 2.2 *Let $g \in \mathcal{H}$ and $u \in C^{2,1}(Q_T^\Omega)$ be a positive solution of (1.1) and suppose $z \in \Omega$ is such that*

$$\int_0^T \int_{B_\epsilon(z) \cap \Omega} g \circ u dx dt = \infty \qquad \forall \epsilon > 0. \tag{2.7}$$

Suppose that at least one of the following sets (i) or (ii) of conditions holds:
(i) There exists an open neighborhood $U' \subset \Omega$ of z such that $u \in L^1(U' \times (0,T))$.
(ii) The following hold:
1- $g \in \mathcal{H}_0$,
2- (1.1) possesses a strong barrier at z.
Then,

$$\lim_{t \to 0} \int_{B_\epsilon(z) \cap \Omega} u(x,t) dx = \infty \qquad \forall \epsilon > 0. \tag{2.8}$$

Proof. Assume that Ω is bounded. First consider the case when condition (i). holds. Let $\phi \in C^{2,1}(U' \times [0,T))$ with compact support in $U' \times [0,T)$ and such that $\phi(x,0) = 1$ in a neighborhood of z. Then

$$\int_t^T \int_{U'} (u(-\phi_t - \Delta\phi) + g \circ u\, \phi)\, dxdt = \int_{U'} u(t)\phi dx - \int_{U'} u(T)\phi dx. \tag{2.9}$$

By assumption $\displaystyle\int_t^T \int_{U'} u(\phi_t + \Delta\phi) dxdt$ is bounded. We let t tend to 0, the result follows from (2.7).

Next we assume that condition (ii) holds, $u \notin L^1(U' \times (0,T))$ for any neighborhood U' of z and that the conclusion is not valid. Thus there exist $r^* > 0$, such that $\overline{B}_{r^*}(z) \subset U_z$ and a sequence $\{t_n\}$ decreasing to 0 such that

$$\int_{B_{r^*}(z)} u(x,t_n) dx \leq M$$

for some $M > 0$. Furthermore g is coercive in $B_{r^*}(z) \times (0,T)$. Let $\{h_{n,k}\} \subset C^\infty(Q_T^\Omega)$ an increasing sequence with respect to k and n of nonnegative functions such that $h_{n,k} = 0$ on $B_{r^*}(z) \times \{0\}$, $0 \leq h_{n,k} \leq k$ and $h_{n,k} = k$ on $(t_n,T) \times \partial B_{r^*}(z)$. Let $w_{h_{n,k}}$ be the solution of (1.1) in $B_{r^*}(z) \times (0,T)$ such that $w_{h_{n,k}} = h_{n,k}$ on $\partial_\ell Q_T^{B_{r^*}(z)}$. By the maximum principle and condition (ii)-1, the sequence $\{w_{h_{n,k}}\}$ is monotone increasing with respect to k and n. Condition (ii)-2 implies that, for every $r < r^*$ and $\beta < T$, the sequence is bounded in $\overline{B}_r(z) \times [0,\beta]$, and since u is locally bounded in Q_T^Ω there exists $k = k(n)$ such that $k \geq u$ on $(t_n,T) \times \partial B_r(z)$ and $k(n) \to \infty$ when $n \to \infty$. Then $w = \lim_{n\to\infty} w_{h_{n,k}}$ is a solution of (1.1) which blows up on $\partial B_{r^*}(z) \times (0,T)$ and vanishes on $B_{r^*}(z) \times \{0\}$. Let v_n be the solution of the heat equation in $B_{r^*}(z) \times (t_n,T)$ such that $v_n(.,t_n) = u(.,t_n)$ in $B_{r^*}(z)$ and $v_n = 0$ on $\partial B_{r^*}(z) \times (t_n,T)$. Then $w_{h_{n,k(n)}} + v_n$ is a supersolution of (1.1) in $B_{r^*}(z) \times (t_n,T)$ which dominates u on $\partial_\ell Q_{t_n,T}^{B_{r^*}(z)}$. By the maximum principle,

$$u \leq w_{h_{n,k(n)}} + v_n \qquad \text{in } Q_{t_n,T}^{B_{r^*}(z)}.$$

And we have in particular

$$\int_{B_r(z)} u(x,t) dx \leq \int_{B_r(z)} \left(w_{h_{n,k(n)}} + v_n\right)(x,t) dx \leq M + \int_{B_r(z)} w(x,t) dx \qquad \forall t \in (t_n,T).$$

Since it holds for any n, it implies $u \in L^1(Q_T^{B_r(z)})$, which leads to a contradiction. □

Example 1. If $g(x,t,r) = b(x,t)h(r)$ where b is a Borel function defined in Q_T^Ω which satisfies $\inf \operatorname{ess}\{b(x)x \in K\} = b_K > 0$, h is continuous, nondecreasing and $h(0) \geq 0$, then

$$u_t - \Delta u + h(u) = 0 \tag{2.10}$$

possesses a strong barrier at any $z \in \Omega$ if and only if h satisfies the Keller-Osserman condition, that is there exists some $a \geq 0$ such that

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \text{ where } H(s) = \int_0^s h(\tau) d\tau. \tag{2.11}$$

The supersolution can be chosen to be the maximal solution ϕ_r of the elliptic equation

$$-\Delta\phi + b_{\overline{B}_r(z)}h(\phi) = 0 \qquad \text{in } B_r(z). \tag{2.12}$$

If we assume moreover that h is super-additive, i.e. $h(a+b) \geq h(a) + h(b)$ for all $a, b \geq 0$, then there holds

$$\int_a^\infty \frac{ds}{h(s)} < \infty, \tag{2.13}$$

and any solution u of (2.10) is dominated in $Q_T^{B_r(z)}$ by $\phi_r(x) + \psi(t)$ where ψ is defined by inversion from

$$\int_{\psi(t)}^\infty \frac{ds}{h(s)} = b_{\overline{B}_r(z)}t \quad \forall t > 0.$$

Example 2. If $g(x,t,r) = a(x)b(t)h(r)$ where $a \in C(\Omega)$, $b \in C((0,T))$, $a, b > 0$, then g is a coercive nonlinearity if h is super-additive and satisfies the Keller-Osserman condition. This is not sufficient for the existence of a barrier as it is shown in [9] with $h(r) = r^q$ $(q > 1)$ $a \equiv 1$ and $b(t) = e^{-\frac{1}{t}}$.

Proposition 2.2 *Let $g \in \mathcal{H}_0$ such that at any $z \in \Omega$ there exists a strong barrier. We assume also*

$$g(x,t,a) + g(x,t,b) \leq g(x,t,a+b) \qquad \forall (x,t,a,b) \in Q_T^\Omega \times \mathbb{R}_+ \times \mathbb{R}_+. \tag{2.14}$$

Let $\{u_n\}$ be a sequence of positive solutions of (1.1) which converges to u locally uniformly in Q_T^Ω. Denote by $tr_\Omega(u_n) = (\mathcal{S}_n, \mu_n)$ and $tr_\Omega(u) = (\mathcal{S}, \mu)$ their respective initial trace. If $\mathcal{A} \subset \cap_n \mathcal{R}_n$ is open and if $\mu_n(\mathcal{A}_n)$ remains bounded independently of $n \in \mathbb{N}$, then $\mathcal{A} \subset \mathcal{R} := \Omega \setminus \mathcal{S}$.

Proof. Let $z \in \mathcal{A}$ and $\tilde{r} \in (0, \rho(z))$ such that for any $r \in (0, \tilde{r}]$ there exists a positive supersolution $w_{r,z}$ satisfying (2.5) and $\overline{B}_{\tilde{r}}(z) \subset \mathcal{A}$. For any $n \in \mathbb{N}$ and $\tau \in (0,T)$, we denote by $u_{\tau,\chi_{B_{\tilde{r}}(z)}\mu_n}$ the solution of

$$\begin{array}{rll} \partial_t u - \Delta u + g \circ u = 0 & & \text{in } B_{\tilde{r}}(z) \times (\tau, T) \\ u(.,\tau) = \chi_{B_{\tilde{r}}(z)} u_n(.,\tau) & & \text{in } B_{\tilde{r}}(z) \\ u = 0 & & \text{in } \partial B_{\tilde{r}}(z) \times (\tau, T). \end{array} \tag{2.15}$$

By the maximum principle $u_{\tau,\chi_{B_{\tilde{r}}(z)}\mu_n} \leq u_n$ in $B_{\tilde{r}}(z) \times (\tau, T)$, and $g \circ u_{\tau,\chi_{B_{\tilde{r}}(z)}\mu_n} \leq g \circ u_n$ Furthermore, if $\zeta \in C^{1,1;1}(Q_T^{\overline{B}_{\tilde{r}}(z)})$ vanishes on $\partial B_{\tilde{r}}(z) \times [0,T)$ and for $t = T$, there holds

$$\iint_{B_{\tilde{r}}(z)\times(\tau,T)} \left(-u_{\tau,\chi_{B_{\tilde{r}}(z)}\mu_n}(\partial_t\zeta + \Delta\zeta) + \zeta g \circ u_{\tau,\chi_{B_{\tilde{r}}(z)}\mu_n}\right) dxdt = \int_{B_{\tilde{r}}(z)} u_n(x,\tau)\zeta(x,\tau)dx. \tag{2.16}$$

Since $u_{\tau,\chi_{B_{\tilde{r}}(z)}\mu_n}$ and $g \circ u_{\tau,\chi_{B_{\tilde{r}}(z)}\mu_n}$ are bounded independently of τ, standard regularity theory for parabolic equations implies that they converge a.e. in $B_{\tilde{r}}(z) \times (0,T)$ when $\tau \to 0$ to $u_{\chi_{B_{\tilde{r}}(z)}\mu_n}$ and $g \circ u_{\chi_{B_{\tilde{r}}(z)}\mu_n}$. Furthermore

$$\lim_{\tau\to 0} \int_{B_{\tilde{r}}(z)} u_n(x,\tau)\zeta(x,\tau)dx = \int_{B_{\tilde{r}}(z)} \zeta(x,0)d\mu_n(x).$$

Using the dominated convergence theorem, it follows from (2.15) that

$$\iint_{Q_T^{B_{\tilde r}(z)}} \left(-u_{\chi_{B_{\tilde r}(z)}\mu_n}(\partial_t\zeta+\Delta\zeta)+\zeta g\circ u_{\chi_{B_{\tilde r}(z)}\mu_n}\right)dxdt=\int_{B_{\tilde r}(z)}\zeta(x,0)d\mu_n(x), \tag{2.17}$$

and $u_{\chi_{B_{\tilde r}(z)}\mu_n}$ is the (unique) solution of

$$\begin{array}{rl} \partial_t u-\Delta u+g\circ u=0 & \text{in } Q_T^{B_{\tilde r}(z)}\\ u(.,0)=\chi_{B_{\tilde r}(z)}\mu_n & \text{in } B_{\tilde r}(z)\\ u=0 & \text{in } \partial B_{\tilde r}(z)\times(0,T). \end{array} \tag{2.18}$$

Furthermore, if η is the solution of the backward problem

$$\begin{array}{rl} \partial_t\eta+\Delta\eta=-1 & \text{in } Q_T^{B_{\tilde r}(z)}\\ \eta(.,T)=0 & \text{in } B_{\tilde r}(z)\\ \eta=0 & \text{in } \partial B_{\tilde r}(z)\times(0,T), \end{array} \tag{2.19}$$

there holds

$$\iint_{Q_T^{B_{\tilde r}(z)}} \left(u_{\chi_{B_{\tilde r}(z)}\mu_n}+\eta g\circ u_{\chi_{B_{\tilde r}(z)}\mu_n}\right)dxdt=\int_{B_{\tilde r}(z)}\eta(x,0)d\mu_n(x)\le M, \tag{2.20}$$

for some $M>0$ independent of n. Next we set $Z_{\tau,n}:=u_{\tau\chi_{B_{\tilde r}(z)}\mu_n}+w_{\tilde r,z}$. It is a supersolution of (2.10) in $(\tau,T)\times B_{\tilde r}(z)$ which is infinite on $\partial B_{\tilde r}(z)\times[\tau,T)$ and dominates u_n in $B_{\tilde r}(z)$ at $t=\tau$. Thus $Z_{\tau,n}\ge u_n$ in $(\tau,T)\times B_{\tilde r}(z)$. Letting $\tau\to 0$ we finally obtain

$$u_{\chi_{B_{\tilde r}(z)}\mu_n}(x,t)\le u_n(x,t)\le u_{\chi_{B_{\tilde r}(z)}\mu_n}(x,t)+w_{\tilde r,z}(x,t)\qquad \forall (x,t)\in Q_T^{B_{\tilde r}(z)}. \tag{2.21}$$

For any $r<\tilde r$ and $T'<T$, there exists $\delta,\sigma>0$ such that $\eta(x,t)\ge\delta$ and $w_{\tilde r,z}(x,t)\le\sigma$ for all $(x,t)\in Q_{T'}^{B_r(z)}$. It follows from (2.20), (2.21) and Fatou's lemma that u and $g\circ u$ are integrable in $L^1(Q_{T'}^{B_r(z)})$. By Lemma 2.1, $B_{r'}(z)\subset\mathcal{R}$. Since it holds for any $z\in\mathcal{A}$, the result is proved. □

3 Construction of a barrier

In the next results we construct the barrier function

Lemma 3.1 *Assume $\alpha>-1$ and $q>1$, then there exists a unique positive function $W_\alpha\in C^2([0,\infty))$ satisfying*

$$\begin{array}{c} W''+\dfrac{r}{2}W'+\dfrac{1+\alpha}{q-1}W-W^q=0\quad \text{in } (0,\infty)\\ \lim_{r\to 0}W(r)=\infty\\ \lim\limits_{r\to\infty} r^{\frac{2}{q-1}}W(r)=0. \end{array} \tag{3.1}$$

Furthermore W_α is decreasing and

$$W_\alpha(r)=Cr^{\frac{2(1+\alpha)}{q-1}-1}e^{-\frac{r^2}{4}}(1+\circ(1))\quad \text{as } r\to\infty. \tag{3.2}$$

Proof. Consider the functional

$$J(\phi) := \frac{1}{2}\int_0^\infty \left(\phi'^2 - \frac{1+\alpha}{q-1}\phi^2 + \frac{2}{q+1}|\phi|^{q+1}\right) e^{\frac{r^2}{4}} dr \tag{3.3}$$

defined over the convex set

$$H_k := \{\phi \in W_2^1(0,\infty; e^{\frac{r^2}{4}} dr) \cap L^{q+1}(0,\infty; e^{\frac{r^2}{4}} dr) : \phi(0) = k\}.$$

Note that if $\phi \in H_k$,

$$\begin{aligned} e^{\frac{r^2}{4}}\phi^2(r) &= \int_r^\infty (e^{\frac{s^2}{4}}\phi^2(s))' ds \\ &= 2\int_r^\infty e^{\frac{s^2}{4}}\phi\phi'(s)ds + \frac{1}{2}\int_r^\infty s e^{\frac{s^2}{4}}\phi^2(s)ds. \end{aligned}$$

In this set J admits a positive minimizer w_k which is the unique solution of

$$\begin{aligned} w'' + \frac{r}{2}w + \frac{1+\alpha}{q-1}w' - w^q = 0 \quad \text{in } (0,\infty) \\ w(0) = k. \end{aligned} \tag{3.4}$$

Furthermore, $w_k = \lim_{n\to\infty} w_{k,n}$ where $w_{k,n}$ is the unique positive solution of

$$\begin{aligned} w'' + \frac{r}{2}w' + \frac{1+\alpha}{q-1}w - w^q = 0 \quad \text{in } (0,n) \\ w(0) = k \\ w(n) = 0. \end{aligned} \tag{3.5}$$

and, by the maximum principle, $(k,n) \mapsto w_{k,n}$ is increasing. If we consider the linear equation

$$z'' + \frac{r}{2}z' + \frac{1+\alpha}{q-1}z = 0 \quad \text{in } (0,\infty), \tag{3.6}$$

it admits two linearly independent positive solutions z_1 and z_2 with the following asymptotic behaviour as $r \to \infty$

$$z_1(r) = r^{-\frac{2(1+\alpha)}{q-1}}(1 + \circ(1)) \tag{3.7}$$

and

$$z_2(r) = r^{\frac{2(1+\alpha)}{q-1}-1} e^{-\frac{r^2}{4}}(1 + \circ(1)) \tag{3.8}$$

(see [8, Appendix]. Since any solution of 3.4, and 3.5 as well, satisfies an a priori estimate of Keller-Osserman type (see [16])

$$w(r) \le C r^{-\frac{2}{q-1}} \qquad \text{for } 0 < r < 1, \tag{3.9}$$

there holds

$$w_{k,n} \le C z_2(r) \qquad \text{for } k \ge r \ge 1.$$

Letting n and k go to infinity successively, it follows that $W_\alpha = \lim_{k,n\to\infty} w_{k,n}$ exists. It is a positive solution of problem (3.1) and it satisfies

$$W_\alpha(r) \le C\left(r^{-\frac{2}{q-1}} + z_2(r)\right) \qquad \text{for } r > 0.$$

The singular behaviour at $r = 0$ is standard (see e.g. [16]) and yields to

$$W_\alpha(r) = \frac{2(q+1)}{(q-1)^2} r^{-\frac{2}{q-1}}(1 + \circ(1)) \qquad \text{as } r \to 0. \tag{3.10}$$

Thus uniqueness follows by the maximum principle and estimates (3.3) is obtained via standard linearization, using the upper estimate at infinity. □

In the sequel we set

$$w_\alpha(s,t) = t^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\frac{s}{\sqrt{t}}\right) \qquad \forall s > 0,\ t > 0. \tag{3.11}$$

Proposition 3.2 *Assume $\alpha > -1$ and $q > 1$. Then for any $R > 0$, there exists $C = C(q,\alpha,R) > 0$ such that any solution u of (1.6) in $Q_\infty^{B_R}$ which vanishes on $B_R \times \{0\}$satisfies*

$$u(x,t) \le 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\tfrac{R-|x|}{\sqrt{t}}\right) \quad \forall (x,t) \in Q_\infty^{B_R}. \tag{3.12}$$

Proof. For $m > 0$, set $S_m = \{x = (x_1, ..., x_N) : |x_j| < m,\ \forall j = 1, ..., N\}$. For $R' < R$

$$\tilde{w}_{R'}(x,t) = t^{-\frac{1+\alpha}{q-1}} \sum_{j=1}^{N} \left(W_\alpha\left(\tfrac{R'-x_j}{\sqrt{t}}\right) + W_\alpha\left(\tfrac{R'+x_j}{\sqrt{t}}\right)\right) \quad \forall (x,t) \in Q_\infty^{S_{R'}}. \tag{3.13}$$

Then $\tilde{w}_{R'}$ is a supersolution of (1.6) in $Q_\infty^{S_{R'}}$ which is infinite on $\partial S_{R'} \times (0,\infty)$, thus $u \le \tilde{w}_{R'}$. Letting $R' \to R$ yields to $u \le \tilde{w}_R$ in $Q_\infty^{S_R}$. Since the equation is invariant by rotation, for any $x \in B_R$, there is a rotation $\mathcal{R}$ such that $\mathcal{R}(x)$ has only a positive x_1- coordinate. Thus

$$\begin{aligned} u(x,t) &\le \tilde{w}_R(|x|, t) \\ &\le t^{-\frac{1+\alpha}{q-1}} \left(W_\alpha\left(\tfrac{R-x_1}{\sqrt{t}}\right) + (2N-1) W_\alpha\left(\tfrac{R}{\sqrt{t}}\right)\right) \\ &\le 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\tfrac{R-x_1}{\sqrt{t}}\right), \end{aligned} \tag{3.14}$$

which is (3.12) since $x_1 = |x|$. □

Proposition 3.3 *Assume $\alpha > -1$, $q > 1$ and $R > 0$. Then there exists a unique positive solution w_{B_R} of (1.6) in $Q_\infty^{B_R}$, continuous in $B_R \times [0,\infty)$, which vanishes on $B_R \times \{0\}$ and satisfies $\lim_{|x|\to R} w_{B_R}(x,t) = \infty$, locally uniformly in $(0,\infty)$. In particular*

$$t^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\tfrac{R-|x|}{\sqrt{t}}\right) \le w_{B_R}(x,t) \le 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\tfrac{R-|x|}{\sqrt{t}}\right) \quad \forall (x,t) \in Q_\infty^{B_R}. \tag{3.15}$$

Proof. For $k>0$, let $w^k_{B_R}$ be the solution of

$$\begin{array}{rl} \partial_t u-\Delta u+t^\alpha u^q=0 & \text{in } Q^{B_R}_\infty \\ u=k & \text{in } \partial_\ell Q^{B_R}_\infty \\ u(.,0)=0 & \text{in } B_R. \end{array} \tag{3.16}$$

By (3.12), $w^k_{B_R}(x,t)\le 2Nt^{-\frac{1+\alpha}{q-1}}W_\alpha\left(\frac{R-|x|}{\sqrt{t}}\right)$. There there exists $w_{B_R}=\lim_{k\to\infty}w^k_{B_R}$ and w_{B_R} is a solution of (1.6) in $Q^{B_R}_\infty$ which vanishes on $B_R\times\{0\}$ and is infinite on $\partial B_R\times(0,\infty)$. Consider the similarity transformation T_m which leaves equation (1.6) invariant

$$T_m[u](x,t)=m^{\frac{1+\alpha}{q-1}}u(\sqrt{m}x,mt)\qquad \forall m>0,$$

then $T_m[w^k_{B_R}]=w^{m^{\frac{1+\alpha}{q-1}}k}_{B_{\frac{R}{\sqrt{m}}}}$ which implies

$$T_m[w_{B_R}]=w_{B_{\frac{R}{\sqrt{m}}}}\qquad \forall m>0.$$

If $u\in C(B_R\times[0,\infty))$ is any positive solution of problem

$$\begin{array}{rl} \partial_t u-\Delta u+t^\alpha u^q=0 & \text{in } Q^{B_R}_\infty \\ \lim\limits_{|x|\to R}u(x,t)=\infty & \text{locally uniformly on } (0,\infty) \\ u(.,0)=0 & \text{in } B_R, \end{array} \tag{3.17}$$

then for any $m>1$ and $\epsilon>0$, there exists $\tau_\epsilon>0$ such that $u(x,t)\le\epsilon$ in $B_{\frac{R}{\sqrt{m}}}$ for $0\le t\le\tau_\epsilon$. Therefore, for any $T>0$,

$$u\le\epsilon+w_{B_{\frac{R}{\sqrt{m}}}}\qquad \text{on } \partial_\ell Q_T^{B_{\frac{R}{\sqrt{m}}}}\cup\overline{B_{\frac{R}{\sqrt{m}}}}\times\{0\}$$

Since $\epsilon+w_{B_{\frac{R}{\sqrt{m}}}}$ is a supersolution $u\le\epsilon+w_{B_{\frac{R}{\sqrt{m}}}}$ in $Q_T^{B_{\frac{R}{\sqrt{m}}}}$. Letting $\epsilon\to0$, $m\to1$ and $T\to\infty$ yields to $u\le w_{B_R}$. In the same way, with $0<m<1$, we obtain $u\ge w_{B_R}$. □

The next estimate is an immediate consequence of Proposition 3.2.

Proposition 3.4 *Assume $\alpha>-1$, $q>1$ and $K\subset\Omega$ is compact. Let u be any solution u of (1.6) in Q^Ω_∞ which vanishes on $\Omega\setminus\{K\}\times\{0\}$ and on $\partial\Omega\times[0,\infty)$, then*

$$u(x,t)\le 2Nt^{-\frac{1+\alpha}{q-1}}W_\alpha\left(\frac{\mathrm{dist}\,(x,K)}{\sqrt{t}}\right). \tag{3.18}$$

4 Upper estimates

We start with the following upper estimate already obtained by Shishkov and Véron [13] in the case $\alpha\ge0$.

Proposition 4.1 *Let $q > 1$ and $\alpha > -1$. If u is a solution of (1.6) vanishing on $\partial\Omega \times [0, T)$, there holds*

$$u(x,t) \leq c_\alpha t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x,t) \in Q_T^\Omega, \tag{4.1}$$

with $c_\alpha = \left(\frac{\alpha+1}{q-1}\right)^{\frac{1}{q-1}}$.

Proof. Let $\phi(t) = c_\alpha t^{-\frac{\alpha+1}{q-1}}$ be the maximal solution of

$$\begin{aligned} \phi' + t^\alpha \phi^q &= 0 \\ \phi(0) &= \infty. \end{aligned}$$

with $c_\alpha = \left(\frac{\alpha+1}{q-1}\right)^{\frac{1}{q-1}}$.

Case $\alpha \geq 0$. For $\tau > 0$, we denote by $\Phi_{1,\tau}$ the solution of

$$\begin{aligned} -\Delta\Phi_{1,\tau} + \tau^\alpha \Phi_{1,\tau}^q &= 0 \quad \text{in } B_1 \\ \lim_{|x|\to 1} \Phi_{1,\tau}(x) &= \infty, \end{aligned} \tag{4.2}$$

and for $R > 0$

$$\Phi_{R,\tau}(x) = R^{\frac{-2}{q-1}} \Phi_{1,\tau}\left(\frac{x}{R}\right).$$

Note that $\Phi_{R,\tau}(x)$ is the solution of the problem (4.2) in the ball B_R. The function $\Phi_{R,\tau}$ tends to 0 uniformly on every compact set of $\mathbb{R}^N$ when $R \to \infty$. Set

$$\tilde{v}(x,t) = \phi(t-\tau) + \Phi_{R,\tau}(x),$$

thus $\tilde{v}$ is a supersolution of (1.6) in $B_R \times [\tau, T)$ which is infinite on $\partial B_R \times [\tau, T) \cup B_R \times \{0\}$. Then $u(x,t) \leq \tilde{v}(x,t)$. Letting $R \to \infty$ and $\tau \to 0$, we obtain

$$u(x,t) \leq c_\alpha t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x,t) \in Q_T.$$

Case $-1 < \alpha < 0$. Let $\tau > 0$ and $\phi_\tau(t) = c_\alpha \left(t^{\alpha+1} - \tau^{\alpha+1}\right)^{-\frac{1}{q-1}}$ be the solution of

$$\begin{aligned} \phi_\tau' + t^\alpha \phi_\tau^q &= 0 \quad \text{on } (\tau, \infty) \\ \phi_\tau(\tau) &= \infty \end{aligned}$$

If $\Phi_{1,T}$ is the solution of (4.2) with $\tau = T$, we set

$$\Phi_{R,T}(x) = R^{\frac{-2}{q-1}} \Phi_{1,T}\left(\frac{x}{R}\right).$$

Clearly $\Phi_{R,T}$ tends to 0 uniformly on every compact of $\mathbb{R}^N$ when $R \to \infty$. Set

$$\hat{v}(x,t) = \phi_\tau(t) + \Phi_{R,T}(x),$$

$\hat{v}$ is a supersolution of (1.6) in $B_R \times (\tau, T)$, thus $u(x,t) \leq \hat{v}(x,t)$, as in the first case. Letting $R \to \infty$ and $\tau \to 0$, we obtain the desired estimate. □

Combining Proposition 3.2 and Proposition 4.1 we obtain,

Corollary 4.2 *Assume $q > 1$, $\alpha > -1$ and $K \subset \Omega$ is compact. If $u \in C^{2,1}(Q_T^\Omega) \cap C(\overline{Q}_T^\Omega \setminus K \times \{0\})$ is a solution of (1.6) which vanishes on $\partial\Omega \times [0,T) \cup \{(\Omega \setminus K) \times \{0\}\}$, there holds*

$$u(x,t) \leq \min\left\{2NW_\alpha\left(\frac{\operatorname{dist}(x,K)}{\sqrt{t}}\right), c_\alpha\right\} t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x,t) \in Q_T^\Omega. \tag{4.3}$$

In the particular case where $K = \{O\}$, (4.3) yields to

$$u(x,t) \leq \min\left\{2NW_\alpha\left(\frac{|x|}{\sqrt{t}}\right), c_\alpha\right\} t^{-\frac{\alpha+1}{q-1}} \leq \frac{c_1}{\left(|x|^2 + t\right)^{\frac{1+\alpha}{q-1}}} \quad \text{for all } (x,t) \in Q_T^\Omega, \tag{4.4}$$

for some $c_1 = c_1(\alpha, q) > 0$.

Remark. If Ω is replaced by $\mathbb{R}^N$, the previous estimates (4.1), (4.3) and (4.4) remain valid. Furthermore, K needs only to be closed.

5 Isolated singularities

In this section we present the results of classification of isolated singularities of positive solutions of (1.6), always in the range $q > 1$ and $\alpha > -1$. Since some proofs are somewhat similar to the ones of [2] for the removability of isolated singularities, or [8] for the classification of positive isolated boundary singularities of solutions of

$$\partial_t u - \Delta u + |u|^{q-1} u = 0 \quad \text{in } Q_T^\Omega, \tag{5.1}$$

we will essentially indicate their main ideas. If we look for solution of (1.6) in $Q_T := Q_T^{\mathbb{R}^N}$ under the form

$$u(x,t) = t^\gamma V\left(\frac{x}{\sqrt{t}}\right)$$

it is immediate that $\gamma = -\frac{1+\alpha}{q-1}$ and V is a solution of

$$-\Delta V - \frac{\eta}{2}.\nabla V - \frac{1+\alpha}{q-1} V + V^q = 0 \quad \text{in } \mathbb{R}^N. \tag{5.2}$$

It is proved by Escobedo and Kavian [4] that if $\frac{1+\alpha}{q-1} > \frac{N}{2}$, or equivalently if

$$1 < q < q_{c,\alpha} := 1 + \frac{2(1+\alpha)}{N}, \tag{5.3}$$

there exists a positive solution of (5.2) which minimizes of the functional

$$\omega \mapsto J(\omega) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla\omega|^2 - \frac{1+\alpha}{q-1}\omega^2 + \frac{2}{q+1}|\omega|^{q+1}\right) e^{\frac{|\eta|^2}{4}} d\eta, \tag{5.4}$$

over the space $W^{1,2}(\mathbb{R}^N; Qd\eta)$ where $Q(\eta) = e^{\frac{|\eta|^2}{4}}$. The minimizer V_α is unique, radial and satisfies (5.2). Furthermore, by adapting the results of [1] , it is easy to show that V_α is the unique positive C^2 function which satisfies (5.2) and

$$\lim_{|\eta|\to\infty} |\eta|^{\frac{2(1+\alpha)}{q-1}} V(\eta) = 0, \tag{5.5}$$

and that there holds (see [8, Th 2.1])

$$V_\alpha(\eta) = C\,|\eta|^{2\frac{1+\alpha}{q-1}-N}\,e^{-\frac{|\eta|^2}{4}}(1+\circ(1)) \quad \text{as } |\eta|\to\infty. \tag{5.6}$$

The function

$$v_\alpha(x,t) = t^{-\frac{1+\alpha}{q-1}}V_\alpha\left(\frac{x}{\sqrt{t}}\right) \tag{5.7}$$

is a positive solution of (1.6) in Q_∞, continuous in $\overline{Q}_\infty\setminus\{(0,0)\}$; it vanishes on $\mathbb{R}^N\times\{0\}\setminus\{(0,0)\}$ and satisfies

$$\lim_{t\to 0}\int_{B_\epsilon} v_\alpha(x,t)dx = \infty \qquad \forall\epsilon>0. \tag{5.8}$$

It is called the *very singular solution of (1.6).*

When $\mathbb{R}^N$ is replaced by a a proper open domain Ω with a compact C^2 boundary there exists no self-similar solution to (1.6). For any $k>0$ and $a\in\Omega$ there exists a unique solution $u := u_{k\delta_a}$ to the initial value problem

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^\Omega \\ u &= 0 && \text{on } \partial\Omega\times[0,\infty) \\ u(.,0) &= k\delta_a && \text{in } \Omega. \end{aligned} \tag{5.9}$$

(see e.g. [9]). The function u belongs to $L^q(Q_T^\Omega;t^\alpha dxdt)\cap L^1(Q_T^\Omega)$, $T>0$ arbitrary, and satisfies

$$\iint_{Q_T^\Omega}\left(-u\left(\partial_t\zeta+\Delta\zeta\right)+t^\alpha u^q\zeta\right)dxdt = k\zeta(a) \tag{5.10}$$

for all $\zeta\in C^{1,1;1}(\overline{Q}_T^\Omega)$ which vanishes on $\partial\Omega\times[0,T]$ and on $\Omega\times\{T\}$. It is unique in the class of of weak solutions, i.e. the functions belonging to $L^q(Q_T^\Omega;t^\alpha dxdt)\cap L^1(Q_T^\Omega)$ and satisfying the above identity. When $k\to\infty$, $u_{k\delta_a}\uparrow u_{\infty\delta_a}$, where $u_{\infty\delta_a} := u_{\infty,a}$ is a solution of (1.6) in Q_∞^Ω which vanishes on $\partial\Omega\times[0,\infty)$ and on $\overline{\Omega}\times\{0\}\setminus\{(a,0)\}$ and satisfies (5.8). Finally, if $E(x,t) = (4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$ denotes the heat kernel in $\mathbb{R}^N$,

$$u_{k\delta_a}\sim kE(x-a,t) \qquad \text{when } (x,t)\to(a,0) \tag{5.11}$$

and

$$u_{\infty,a}\sim v_\alpha(x-a,t) \qquad \text{when } (x,t)\to(a,0). \tag{5.12}$$

The following classification of isolated singularities holds.

Theorem 5.1 *Assume $\alpha>-1$, $1<q<q_{c,\alpha}$ and $a\in\Omega$. If $u\in\overline{Q}_T^\Omega\setminus\{(a,0)\}$ is a positive solution of (1.6) which vanishes on $\Omega\times\{0\}\setminus\{(a,0)\}$, then*

(i) either there exists $k\geq 0$ such that

$$u(x,t)\sim kE(x-a,t) \qquad \textit{when } (x,t)\to(a,0), \tag{5.13}$$

and u is a solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 \qquad && \text{in } Q_T^\Omega \\ u(.,0) &= k\delta_a && \text{in } \Omega, \end{aligned} \tag{5.14}$$

(ii) or

$$u(x,t) \sim v_\alpha(x-a,t) \qquad \text{when } (x,t) \to (a,0), \tag{5.15}$$

and u satisfies (5.8).

Proof. The initial trace $tr_\Omega(u)$ is a Borel measure concentrated at a and either it is of the form $(\{\emptyset\}, k\delta_a)$ for some $k \geq 0$ or of the form $(\{a\}, 0)$. In the first case $t^\alpha u^q \in L^1(Q_T^{B_r(a)})$ for any $0 < r < \rho(a)$. For $t > 0$ we set $m_r(t) = \max\{u(x,\tau) : (x,\tau) \in \partial_\ell Q_t^{B_r(a)}\}$, $e_r(t) = \max\{E(x,\tau) : (x,\tau) \in \partial_\ell Q_t^{B_r(a)}\}$. We denote by Θ_r the solution of

$$\begin{aligned} \partial_\tau \Theta_r - \Delta\Theta_r &= 0 \qquad \text{in } Q_\infty^{B_r(a)} \\ \Theta_r(x,0) &= 0 \qquad \text{in } B_r(a) \\ \Theta_r(x,t) &= 1 \qquad \text{in } \partial_\ell Q_\infty^{B_r(a)} \end{aligned}$$

and

$$\Psi(x,t) = \int_0^t \int_{B_r(a)} E(x-y,t-s) s^\alpha u^q(y,s) dyds.$$

Then

$$kE(x-a,t) - \Psi(x,t) - e_r(t)\Theta_r(x,t) \leq u(x,t) \leq kE(x-a,t) + m_r(t)\Theta_r(x,t). \tag{5.16}$$

Using the explicit expression of the Cauchy-Dirichlet heat kernel in B_r, one can easily check that $\lim_{t\to 0} \Theta_r(x,t) = 0$, uniformly on compact subsets of $B_r(a)$. Furthermore

$$\begin{aligned} \Psi(x,t) &\leq 2^{q-1} \int_0^t \int_{B_r(a)} E(x-y,t-s) s^\alpha \left(k^q E^q(y-a,s) + m_r^q(s)\Theta_r^q(y,s)\right) dyds \\ &\leq 2^{q-1} k^q \int_0^t \int_{B_r(a)} E(x-y,t-s) s^\alpha E^q(y-a,s) dyds + o(1), \end{aligned} \tag{5.17}$$

since the second term in the integral is bounded. Furthermore the first term of the right-hand side of (5.17) converges to 0 in $L^1(B_r(a))$ when $t \to 0$. Since $y \mapsto E^q(y-a,s)$ is radially decreasing with respect to a, it implies that $\displaystyle\int_0^t \int_{B_r(a)} E(x-y,t-s) s^\alpha E^q(y-a,s) dyds$ is maximal at $x = 0$ and therefore $\Psi(x,t) \to 0$, uniformly in $B_r(a)$. It follows from (5.16) that

$$|u(x,t) - kE(x-a,t)| \to 0 \qquad \text{when } t \to 0, \tag{5.18}$$

uniformly on compact subsets of $B_r(a)$, for any $r < \rho(a)$.

Next we assume that $tr_\Omega(u) = (\{a\}, 0)$, and without loss of generality, we can suppose that $a = 0$ and set $B_r = B_r(0)$. Then

$$u_{\infty,0} - m_r(t)\Theta_r(x,t) \leq u(x,t) \qquad \forall (x,t) \in Q_\infty^{B_r}, \tag{5.19}$$

where $u_{\infty,0} = \lim_{k\to\infty} u_{k\delta_0}$, and $u_{k\delta_0}$ is the solution of (1.6) in Q_∞ with initial data $k\delta_0$. Although estimate (5.19) is proved in Theorem 6.2, in next section, its proof does not require any element of the proof of the present theorem. Moreover, for any $0 < \epsilon < r$,

$$u(x,t) \le V_\epsilon + m_r(t)\Theta_r(x,t) \qquad \forall (x,t) \in Q_\infty^{B_r}, \tag{5.20}$$

where V_ϵ is the limit, when $\ell \to \infty$ of the solutions $V_{\ell,\epsilon}$ of (1.6) in Q_∞ which has initial data $m\chi_{\overline{B}_\epsilon}$. Consider the similarity transformation $T_m[\phi](x,t) = m^{-\frac{1+\alpha}{q-1}}\phi(\frac{x}{\sqrt{m}}, \frac{t}{m})$ $(m > 0)$ which leaves (1.6) and Q_∞ invariant, then, by uniqueness

$$T_m[u_{k,0}] = u_{m^{\frac{N}{2}-\frac{1+\alpha}{q-1}}k,0} \tag{5.21}$$

and

$$T_m[V_{\ell,\epsilon}] = V_{m^{\frac{N}{2}-\frac{1+\alpha}{q-1}}\ell,\sqrt{m}\epsilon}. \tag{5.22}$$

Letting $k, \ell \to \infty$ and $\epsilon \to 0$, we obtain that, for any $m > 0$,

$$T_m[u_{\infty,0}] = u_{\infty,0} \tag{5.23}$$

and

$$T_m[V_0] = V_0. \tag{5.24}$$

Therefore $u_{\infty,0}$ and V_0 are positive self-similar solutions of (1.6) in Q_∞ with initial trace $(\{0\}, 0)$. Then they coincide with the function v_α defined in (5.7). The result follows from this equality since (5.19), jointly with (5.20), implies

$$v_\alpha(x,t) - m_r(t)\Theta_r(x,t) \le u(x,t) \le v_\alpha(x,t) + m_r(t)\Theta_r(x,t) \qquad \forall (x,t) \in Q_\infty^{B_r}. \tag{5.25}$$

□

If $q \ge q_{c,\alpha}$ there holds the following result which extends Brezis and Friedman's classical one.

Theorem 5.2 *Assume $\alpha > -1$, $q \ge q_{c,\alpha}$ and $a \in \Omega$. If $u \in \overline{Q}_T^\Omega \setminus \{(a,0)\}$ is a positive solution of (1.6) which vanishes on $\Omega \times \{0\} \setminus \{(a,0)\}$, then it can be extended as a continuous function $\tilde{u}$ which vanishes on $\Omega \times \{0\}$.*

Proof. Up to modifying a few parameters the proof is similar to Brezis-Friedman's construction. The first step is to prove that $u \in L^q(Q_T^{B_R})$ for some $R > 0$. This is done by using (4.4) and the same test function used in [2]. As a consequence we obtain that u satisfies

$$\lim_{t\to 0} \int_\Omega u(x,t)\zeta(x)dx = 0 \qquad \forall \zeta \in C_0^\infty(\Omega).$$

Finally the extension of u by zero at $t = 0$ satisfies the equation in $\Omega \times [0,T)$. □

Remark. We recall that $E(x,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$. Then if $1 \le r < q_{c,\alpha}$, there holds

$$\iint_{Q_T} E^r(x,t)t^\alpha dxdt < \infty, \tag{5.26}$$

while if $r \ge q_{c,\alpha}$

$$\iint_{Q_T} E^r(x,t)t^\alpha dxdt = \infty. \tag{5.27}$$

6 The trace theorem

In all this section we assume that $\Omega \subset \mathbb{R}^N$ is an open domain with a compact C^2 boundary, $\alpha > -1$ and $1 < q < q_{c,\alpha}$, and let $u \in C(\overline{\Omega} \times (0,T])$ be a positive solution of (1.6) in Q_T^Ω which vanishes on $\partial\Omega \times (0,T]$. By Section 2 u possesses an initial trace $tr_\Omega(u) = (\mathcal{S}, \mu)$ where $\mathcal{S}$ is a relatively closed subset of Ω and μ is a Radon measure on $\mathcal{R} := \Omega \setminus \mathcal{S}$. To this couple we can associate a unique outer regular Borel measure ν defined by

$$\nu(E) = \begin{cases} \mu(E) & \text{if } E \subset \mathcal{R} \\ \infty & \text{if } E \cap \mathcal{S} \neq \emptyset \end{cases} \tag{6.1}$$

for any Borel subset E of Ω. Conversely, to any outer Borel measure ν on Ω we can associate the regular set $\mathcal{R} \subset \Omega$ which is the set of points $y \in \Omega$ which possess an open neighborhood $\mathcal{O}_y$ such that $\nu(\mathcal{O}_y) < \infty$. Clearly $\mathcal{R}$ is open and the restriction of ν to $\mathcal{R}$ is a positive Radon measure. The set $\mathcal{S} = \Omega \setminus \mathcal{R}$ is relatively closed and it is the singular part of ν. It has the property that $\nu(E) = \infty$ for any Borel set E such that $E \cap \mathcal{S} \neq \emptyset$. We shall denote by $\mathfrak{B}^{reg}(\Omega)$ the set of outer regular Borel measures in Ω and by $\mathfrak{B}_c^{reg}(\Omega)$ the subset of $\mathfrak{B}^{reg}(\Omega)$ for which $\mathcal{S}$ is a compact subset of Ω. Thus u is a solution of the following problem,

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q = 0 & \qquad \text{in } Q_T^\Omega \\ u \geq 0,\ u = 0 & \qquad \text{on } \partial\Omega \times [0,T) \\ tr_\Omega(u) = \nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega). & \end{aligned} \tag{6.2}$$

Definition 6.1 *We denote by $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$ the set of solutions of problem (6.2).*

The first step in the characterization of the singular set is the following delicate lower estimate.

Theorem 6.2 *Let $u \in \mathcal{U}_{\mathcal{S},\mu}(\Omega)$ and $a \in \mathcal{S}$, then*

$$u(x,t) \geq u_{\infty,a}(x,t). \tag{6.3}$$

Furthermore, if $\mathcal{S}$ has a nonempty interior $\mathcal{A}$, there holds

$$\lim_{t \to 0} t^{\frac{1+\alpha}{q-1}} u(x,t) = c_\alpha, \tag{6.4}$$

uniformly on compact subsets of $\mathcal{A}$.

We first give a proof of (6.3) in the case where either $-1 < \alpha \leq 0$, or $\alpha > 0$ and $1 < q < q_{c,0}$.

Proposition 6.3 *Assume either $-1 < \alpha \leq 0$ and $1 < q < q_{c,\alpha}$, or $\alpha > 0$ and $1 < q < q_{c,0}$, then inequality (6.3) holds.*

Proof. For any $\epsilon > 0$, there holds

$$\lim_{t \to 0} \int_{B_\epsilon(a)} u(x,t)dx = \infty. \tag{6.5}$$

If $k > 0$ is fixed, and $\{t_n\}$ is a sequence decreasing to 0. There exists t_{n_1} such that

$$\int_{B_{2^{-1}}(a)} u(x, t_{n_1})dx > k,$$

and there exists $m = m_1(k)$ such that

$$\int_{B_{2^{-1}}(a)} m_1(k) \wedge u(x, t_{n_1})dx = k,$$

where $A \wedge B = \inf\{A, B\}$. Assume we have constructed $t_{n_j} < t_{n_{j-1}}$ and $m_j(k) > 0$ such that

$$\int_{B_{2^{-j}}(a)} m_j(k) \wedge u(x, t_{n_j})dx = k.$$

Since (6.5) holds with $\epsilon = 2^{-j-1}$, there exists $t_{n_{j+1}} < t_{n_j}$ such that

$$\int_{B_{2^{-j-1}}(a)} u(x, t_{n_j})dx > k.$$

and thus $m_{j+1}(k) > 0$ such that

$$\int_{B_{2^{-j-1}}(a)} m_{j+1}(k) \wedge u(x, t_{n_{j+1}})dx = k.$$

Next we denote by u_j the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (t_{n_j}, T) \\ u &= 0 && \text{on } \partial\Omega \times [t_{n_j}, T) \\ u(., t_{n_j}) &= m_{j+1}(k) \wedge u(., t_{n_{j+1}}) && \text{in } \Omega. \end{aligned} \tag{6.6}$$

By the maximum principle $u_j \leq u$ in $\Omega \times [t_{n_j}, T)$, or equivalently $u_j(x, t+t_{n_j}) \leq u(x, t+t_{n_j})$ in $Q^\Omega_{T-t_{n_j}}$. Clearly

$$u_{j,0} := u(., t_{n_j}) \rightharpoonup k\delta_a \quad \text{as } j \to \infty$$

in the weak sense of measures. In order to prove that u_j converges to $u_{k\delta_a}$ we notice that $u_j(x, t + t_{n_j}) \leq E * u_{j,0}(x, t)$ in Q^Ω_∞. If $-1 < \alpha \leq 0$ and $r \in (1, q_{c,\alpha})$

$$\begin{aligned} \iint_{Q^\Omega_T} u_j^r(x, t + t_{n_j})(t + t_{n_j})^\alpha dxdt &\leq \iint_{Q_T} (E * u_{j,0})^r(x, t)(t + t_{n_j})^\alpha dxdt \\ &\leq \|u_{j,0}\|^r_{L^1} \iint_{Q_T} E^r(x, t)t^\alpha dxdt, \end{aligned}$$

using Young's inequality and (5.26). If $\alpha \geq 0$ and $r \in (1, q_{c,0})$

$$\begin{aligned} \iint_{Q^\Omega_T} u_j^r(x, t + t_{n_j})(t + t_{n_j})^\alpha dxdt &\leq 2^\alpha T^\alpha \iint_{Q_T} (E * u_{j,0})^r(x, t)dxdt \\ &\leq 2^\alpha T^\alpha \|u_{j,0}\|^r_{L^1} \iint_{Q_T} E^r(x, t)dxdt. \end{aligned}$$

Furthermore, for $s \in (1, q_{c,0})$

$$\iint_{Q_T^\Omega} u_j^s(x, t + t_{n_j}) dxdt \leq \iint_{Q_T} (E * u_{j,0})^s(x, t) dxdt$$
$$\leq \|v_{j,0}\|_{L^1}^s \iint_{Q_T} E^s(x, t) dxdt.$$

Thus the sets of functions $\{u_j^q(., .+t_{n_j})(.+t_{n_j})^\alpha\}$ and $\{u_j(., .+t_{n_j})\}$ are uniformly integrable in $L^1(Q_T^\Omega)$. Since u_j satisfies

$$\iint_{Q_T^\Omega} \left(-u_j(x, t + t_{n_j}) \left(\partial_t \zeta + \Delta\zeta\right) + (t + t_{n_j})^\alpha u_j^q(x, t + t_{n_j})\zeta\right) dxdt = \int_\Omega u_{j,0}\zeta dx \tag{6.7}$$

for any test function $\zeta \in C^{1,1;1}(\overline{Q_T^\Omega})$ vanishing on $\partial\Omega \times [0, T] \cup \Omega \times \{T\}$ and converges a.e. in Q_T^Ω to some u^* when $j \to \infty$, it follows by Vitali's theorem that

$$\iint_{Q_T^\Omega} \left(-v \left(\partial_t \zeta + \Delta\zeta\right) + t^\alpha v^q \zeta\right) dxdt = k\zeta(a) \tag{6.8}$$

thus $u^* = u_{k\delta_a}$ by uniqueness, which implies the claim since $u \geq u^*$. □

When $\alpha > 0$ and $q_{c,0} \leq q < q_{c,\alpha}$, this argument cannot work since the sequence $u_{j,0}$ could concentrate too quickly with respect to t to a Dirac mass and isolated singularities are removable for solutions of

$$\partial_t u - \Delta u + cu^q = 0.$$

We develop below a proof valid for any $\alpha > -1$ and $1 < q < q_{c,\alpha}$, which is based upon the parabolic Harnack inequality and shows that such a fast concentration never occurs.

Lemma 6.4 *Assume $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Let $\{(x_n, t_n)\} \subset Q_T^\Omega$ be a sequence converging to $(a, 0)$ and $\ell > 0$. Put $V_n = B_{\ell\sqrt{t_n}}(x_n)$ and suppose that there exist nonnegative functions $h_n \in L^\infty(\mathbb{R}^N)$ with support in V_n such that $0 \leq h_n \leq c_1 t_n^{-\frac{N}{2}}$ and*

$$h_n \rightharpoonup k\delta_0. \tag{6.9}$$

Then the solutions u_n of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (t_n, T) \\ u &= 0 && \text{on } \partial\Omega \times [t_n, T) \\ u(., t_n) &= h_n && \text{in } \Omega, \end{aligned} \tag{6.10}$$

satisfy $u_n \to u_{k\delta_a}$ when $n \to \infty$.

Proof. The estimate $h_n \leq ct_n^{-\frac{N}{2}}\chi_{V_n}$ can be written under the form

$$h_n(x) \leq c_2 E(x - x_n, t_n)\chi_{V_n}$$

where $c_2 = (4\pi)^{\frac{N}{2}} e^{\frac{\ell}{4}} c_1$. By the maximum principle

$$u_n(x, t) \leq c_2 E(x - x_n, t) \quad \text{in } \Omega \times (t_n, \infty).$$

By (5.26), (5.27), the sequences $\{E^q(.-x_n,.)t^\alpha\}$ and $\{E(.-x_n,.)\}$ are uniformly integrable in Q_T^Ω, therefore, if we extend u_n by zero in $Q_{t_n}^\Omega$ and denote by $\tilde{u}_n$ the extended function defined in Q_T^Ω, we infer that the sequences $\{t^\alpha \tilde{u}_n^q\}$ and $\{\tilde{u}_n\}$ are uniformly integrable in Q_T^Ω. Using standard regularity estimates there exists a function u^* defined in Q_T^Ω and a subsequence u_{n_j} such that $\tilde{u}_{n_j} \to u^*$ locally uniformly in Q_T^Ω. It follows by uniform integrability and Vitali's convergence theorem that

$$\tilde{u}_{n_j} \to u^* \qquad \text{in } L^q(Q_T^\Omega; t^\alpha dxdt) \cap L^1(Q_T^\Omega).$$

Let $\zeta \in C^{1,1;1}(\overline{Q}_T^\Omega)$ vanishing on $\partial\Omega \times [0,T] \cup \Omega \times \{T\}$, then

$$\iint_{Q_T^\Omega} \left(-\tilde{u}_{n_j}(\partial_t \zeta + \Delta\zeta) + t^\alpha \tilde{u}_{n_j}^q \zeta\right) dxdt = \int_\Omega \zeta(.,t_{n_j}) h_n dx.$$

Using the previous convergence results and the assumption (6.9), we derive

$$\iint_{Q_T^\Omega} (-u^*(\partial_t \zeta + \Delta\zeta) + t^\alpha u^{*q} \zeta)\, dxdt = k\zeta(a).$$

Thus $u^* = u_{k\delta_a}$ and $\tilde{u}_n \to u_{k\delta_a}$ locally uniformly in Q_T^Ω. □

Lemma 6.5 *Let u be a positive solution of (1.6) in Q_T^Ω vanishing on $\partial\Omega \times [0,T)$. Then for any $\Omega' \subset \overline{\Omega} \subset \Omega$ there exists a constant $C = C(N,q,\alpha,\Omega') > 0$ such that*

$$u(y,s) \le u(x,t) e^{C\left(\frac{|x-y|^2}{t-s} + \frac{t}{s} + 1\right)} \qquad \forall (x,t),(y,s) \in Q_T^{\Omega'},\ s < t. \tag{6.11}$$

Proof. By (4.1), $V(x,t) := t^\alpha u^{q-1} \le c_\alpha^{q-1} t^{-1}$. If we write (1.6) under the form

$$\partial_t u - \Delta u + V(x,t)u = 0 \quad \text{in } Q_T^\Omega \tag{6.12}$$

it follows (6.11) from parabolic Harnack inequality (see e.g. [5, Lemma 3.16] although the result is much older). □

If $G \subset \mathbb{R}^N$ is a bounded open subset, we denote by ϕ_G is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ normalized by $\sup \phi_G = 1$ with corresponding eigenvalue λ_G.

Lemma 6.6 *Let $G \subset \mathbb{R}^N$ be a bounded open subset with a smooth boundary $\omega \in C^1(Q_T^G)$, $\omega \ge 0$, such that*

$$\int_0^T \|\omega(.,t)\|_{L^\infty(G)}\, dt < \infty. \tag{6.13}$$

If $v \in C^{2;1}(\overline{G} \times (0,T])$ is a positive solution of

$$\partial_t v - \Delta v + \omega v = 0 \quad \text{in } Q_T^G, \tag{6.14}$$

then $v\phi_G \in L^1(Q_T^G)$, $\omega v \phi_G^3 \in L^1(Q_T^G)$ and there exists $\mu \in \mathfrak{M}_+(G)$ such that

$$\lim_{t\to 0} \int_G v(x,t)\zeta(x)dx = \int_G \zeta d\mu \qquad \forall \zeta \in C_c(\overline{G}). \tag{6.15}$$

Proof. Set $\gamma(t) = \|\omega(.,t)\|_{L^\infty(G)}$. Denote $X(t) = \int_G v(x,t)\phi_G(x)dx$. Then, from (6.14),

$$X' + \lambda_G X + \gamma(t)X \geq 0$$

This implies that the function

$$t \mapsto e^{t\lambda_G + \int_0^t \gamma(s)ds} X(t)$$

(which exists thanks to (6.13)) is nondecreasing. Therefore there exists $X(0) = \lim_{t\to 0} X(t)$ and $v\phi_G \in L^1(Q_T^G)$. Furthermore, if we set $Y(t) = \int_G v(x,t)\phi_G^3(x)dx$

$$\frac{d}{dt}(e^{3t\lambda_G}X(t)) + e^{3t\lambda_1}\int_\Omega (\omega\phi_G^3 - 6\phi_G\,|\nabla\phi_G|^2)vdx = 0.$$

Since $\nabla\phi_G$ is bounded and $v\phi_G \in L^1(Q_T^G)$, it implies

$$e^{3t\lambda_G}Y(T) + \int_0^T e^{3t\lambda_G}\int_\Omega \omega(x,t)v(x,t)\phi_G^2 dxdt = Y(0) + 6\int_0^T e^{3t\lambda_G}\int_\Omega \phi_G\,|\nabla\phi_G|^2)vdxdt,$$

which implies that $\omega v\phi_G^3 \in L^1(Q_T^G)$. The argument given in the proof of Lemma 2.1 shows that v admits an initial trace which belongs to $\mathfrak{M}_+(G)$. □

Proof of Theorem 6.2. We define the parabolic distance in $\mathbb{R}^N \times \mathbb{R}$ by

$$d_P((x,t),(y,s)) = \sqrt{|x-y|^2 + |t-s|}.$$

Step 1. We first prove that if u satisfies

$$\limsup_{d_P((x,t))(a,0))\to 0} t^{\frac{N}{2}}u(x,t) < \infty, \tag{6.16}$$

then $a \in \mathcal{R}(u)$. If (6.16) holds there exists $\epsilon, c > 0$ such that

$$u(x,t) \leq ct^{-\frac{N}{2}} \qquad \forall (x,t) \text{ s.t. } \sqrt{|x-a|^2 + t} \leq \epsilon.$$

If we set $\omega(x,t) = t^\alpha u^{q-1}(x,t)$, then

$$\omega(x,t) \leq c^{q-1}t^{-(\frac{N(q-1)}{2}-\alpha)} \qquad \forall (x,t) \in B_{\frac{\epsilon}{\sqrt{2}}}(a) \times (0, \frac{\epsilon^2}{2}].$$

Since $q < q_{c,\alpha}$, then $\frac{N(q-1)}{2}) - \alpha < 1$; thus the assumptions of Lemma 6.6 are fulfilled and there exists a positive Radon measure μ in $B_{\frac{\epsilon}{\sqrt{2}}}(a)$ such that

$$\int_\Omega u(x,t)\zeta(x)dx \to \int_\Omega \zeta d\mu \text{ when } t \to 0, \quad \forall \zeta \in C_c^\infty(B_{\frac{\epsilon}{\sqrt{2}}}(a)).$$

Furthermore $t^\alpha u^q \in L^1(B_{\frac{\epsilon}{\sqrt{3}}}(a) \times (0,T))$, which is the claim.

Step 2. Since $a \in \mathcal{S}(u)$, there holds

$$\limsup_{d_P((x,t))(a,0))\to 0} t^{\frac{N}{2}} u(x,t) = \infty. \tag{6.17}$$

Then there exists a sequence $\{(x_n, s_n)\}$ converging to $(a, 0)$ such that

$$u(x_n, t_n) \geq n t_n^{-\frac{N}{2}}. \tag{6.18}$$

We apply Lemma 6.5 with $s = s_n$, $t = 2s_n := t_n$, $y = x_n$ $|x - x_n| \leq \sqrt{s_n}$. Then

$$u(x, t_n) \geq C n t_n^{-\frac{N}{2}} \qquad \forall x \in V_n := B_{\frac{\sqrt{t_n}}{2}}(x_n).$$

This implies

$$\int_{V_n} u(x, t_n) dx \geq C_N n.$$

For $k < n$ fixed, we denote by $v := v_{n,k}$ the solution of

$$\begin{aligned} \partial_t v - \Delta v + t^\alpha v^q &= 0 && \text{in } \Omega \times (t_n, \infty) \\ v &= 0 && \text{in } \partial\Omega \times (t_n, \infty) \\ v(., t_n) &= C k t_n^{-\frac{N}{2}} \chi_{V_n} && \text{in } \Omega. \end{aligned} \tag{6.19}$$

By the maximum principle $u \geq v_{n,k}$ in $\Omega \times (t_n, \infty)$. Furthermore

$$\int_{V_n} v(., t_n) dx = C_N k.$$

Thus $v(., t_n) \rightharpoonup C_N k \delta_a$ in the weak sense of measures. It follows from Lemma 6.4 that $v_{n,k} \to u_{C_N k \delta_a}$ locally uniformly in Q_T^Ω. Therefore $u \geq u_{C_N k \delta_a}$ in Q_T^Ω. Since k is arbitrary, we obtain (6.3).

Step 3. Formula (6.4) holds. Denote by $S_m(a) = \{x \in \mathbb{R}^N : |x_j| < m\}$. If $\overline{S}_R(a) \subset \mathcal{S}$, the function

$$(x,t) \mapsto c_\alpha (t - \tau)^{-\frac{1+\alpha}{q-1}} + w_{B_R}(x - a, t - \tau)$$

is a supersolution of (1.6) in $S_R(a) \times (\tau, \infty)$ which is infinite on $S_R(a) \times \{\tau\} \cup \partial S_R(a) \times [\tau, \infty)$ by Proposition 3.3, while u is finite, thus it dominates u in $S_R(a) \times (\tau, \infty)$. Letting $\tau \to 0$ yields to

$$u(x,t) \leq c_\alpha t^{-\frac{1+\alpha}{q-1}} + w_{B_R}(x - a, t) \qquad \forall (x,t) \in Q_T^{S_R(a)}. \tag{6.20}$$

Conversely, the function

$$(x,t) \mapsto u(x, t - \tau) + w_{B_R}(x - a, t - \tau)$$

is a supersolution in $S_R(a) \times (\tau, \infty)$ which dominates $c_\alpha t^{-\frac{1+\alpha}{q-1}}$ on $S_R(a) \times \{\tau\} \cup \partial S_R(a) \times [\tau, \infty)$, thus as above, we obtain (6.20). Since

$$\lim_{t \to 0} w_{B_R}(x - a, t) = 0$$

uniformly on $B_{R'}(a)$ for any $R' < R$, we derive (6.4). □

Proposition 6.7 *For any relatively closed $\mathcal{S} \in \Omega$, the set $\mathcal{U}_{\mathcal{S},0}(\Omega)$ is not empty and it admits a minimal element $\underline{u}_{\mathcal{S},0}$ and a maximal element $\overline{u}_{\mathcal{S},0}$.*

Proof. Step 1: Existence of a maximal solution. The maximal solution is constructed by thickening Ω and $\mathcal{S}$ in defining for $0 < \sigma$

$$\Omega_\sigma = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, \Omega) < \sigma\}, \qquad \mathcal{S}_\sigma = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, \overline{\mathcal{S}}) \leq \sigma\}$$

If $z \in \partial\Omega$, we denote by $\mathbf{n}_z$ the outward unit normal vector to Ω at z. Since $\partial\Omega$ is compact and C^2, there exists $\sigma_0 > 0$ such that for any $(z, \sigma) \in \partial\Omega \times [0\sigma_0]$, the mapping $\Pi : (z, \sigma) \mapsto z + \sigma\mathbf{n}_z$ is a C^2 diffeomorphism from $\partial\Omega \times [0, \sigma_0]$ to $\Theta'_{\sigma_0} := \overline{\Omega}_{\sigma_0} \setminus \Omega$. The mapping Π defines the flow coordinates near $\partial\Omega$.

If $0 < \delta < \sigma$, there exists a unique solution $u = u_{n,\sigma,\delta}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{\Omega_\sigma} \\ u &= 0 && \text{in } \partial\Omega_\sigma \times (0, \infty) \\ u(., 0) &= n\chi_{\mathcal{S}_\delta} && \text{in } \Omega_\sigma. \end{aligned} \tag{6.21}$$

Notice that $\mathcal{S}_\delta$ is closed in Ω_σ and $\inf\{|z - z'| : z \in \mathcal{S}_\delta, z' \in \Omega_\delta^c\} = \delta - \sigma$. Existence is standard as well as uniqueness in the case where Ω is bounded. If Ω^c is bounded the proof goes as in the uniqueness proof in Proposition 6.10. When $n \to \infty$, $\{u_{n,\sigma,\delta}\} \uparrow u_{\sigma,\delta}$ which is a solution of (1.6) in $Q_\infty^{\Omega_\sigma}$. Since $u_{n,\sigma,\delta}$ satisfies (4.1), for any $r, \tau > 0$ and any $a \in \partial\Omega_\sigma$, $u_{n,\sigma,\delta}$ remains uniformly continuous with respect to n in $\overline{Q}_\infty^{\Omega_\sigma} \cap \big(\overline{B}_r(a) \times [2^{-2}\tau, 2\tau]\big)$. Consequently $u_{\sigma,\delta}(x, t) = 0$ on $\partial\Omega_\sigma \times (0, \infty) \cap \big(\overline{B}_r(a) \times [2^{-2}\tau, 2\tau]\big)$. Therefore $u_{\sigma,\delta}$ vanishes on $\partial\Omega_\sigma \times (0, \infty)$. If $a \in \Omega_\sigma$ and $\mathrm{dist}\,(a, \mathcal{S}_\delta) = r > 0$, $u_{n,\sigma,\delta}(x, t) \leq w_{B_r}(x - a, t)$ in $Q^\infty_{B_r(a)}$. This implies that $u_{n,\sigma,\delta}$ remains uniformly continuous with respect to n in $\overline{B}_{r'}(a) \times [0, T)$ for any $0 < r' < r$ and $T > 0$. Since $u_{n,\sigma,\delta}(x, t) \to 0$ in $\overline{B}_{r'}(a)$, $u_{\sigma,\delta}$ inherits the same property. Consequently $u_{\sigma,\delta}$ has initial trace $(\mathcal{S}_\delta, 0)$ in Ω_σ. By the maximum principle the mapping $(n, \delta) \mapsto u_{n,\sigma,\delta}$ is increasing with respect to n and decreasing with respect to δ. Furthermore, if $0 < \delta' < \sigma' < \sigma$ and $0 < \delta < \sigma$, there holds $u_{n,\sigma',\delta'} < u_{n,\sigma,\delta}$ in $Q_\infty^{\Omega_{\sigma'}}$, which implies $u_{\sigma',\delta'} < u_{\sigma,\delta}$. We define

$$\overline{u}_{\mathcal{S},0} = \lim_{\sigma\to 0} \lim_{\delta\to 0} \lim_{n\to\infty} u_{n,\sigma,\delta}. \tag{6.22}$$

Then $\overline{u}_{\mathcal{S},0}$ is a solution of (1.6) in Q_∞^Ω. Since $\lim_{t\to 0} u_{\sigma,\delta}(x, t) = 0$ uniformly on any compact subset $K \subset \mathcal{R} = \Omega \setminus \mathcal{S}$, $\overline{u}_{\mathcal{S},0}$ has initial trace 0 on $\mathcal{R}$. If $a \in \mathcal{S}$, we denote by $\tilde{u}_{\infty,a}$ the function defined in $Q_\infty^{\Omega_\sigma}$ by

$$\tilde{u}_{\infty,a}(x, t) = \begin{cases} u_{\infty,a}(x, t) & \text{if } (x, t) \in Q_\infty^\Omega \\ 0 & \text{if } (x, t) \in Q_\infty^{\Omega_\sigma} \setminus Q_\infty^\Omega \end{cases}$$

Then $\tilde{u}_{\infty,a}$ is a subsolution of (1.6) in $Q_\infty^{\Omega_\sigma}$ which is smaller than $u_{\infty,a,\sigma}$ which is the limit, when $k \to \infty$ of the solution $u_{k\delta_a,\sigma}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{\Omega_\sigma} \\ u &= 0 && \text{in } \partial\Omega_\sigma \times (0, \infty) \\ u(., 0) &= k\delta_a && \text{in } \Omega_\sigma. \end{aligned} \tag{6.23}$$

There holds, by Theorem 6.2,

$$u_{\sigma,\delta}(x,t) \geq u_{\infty,a,\sigma} \geq \tilde{u}_{\infty,a}(x,t) \qquad \forall (x,t) \in Q_\infty^{\Omega_\sigma}.$$

Letting successively $\delta \to 0$ and $\sigma \to 0$ yields to $\overline{u}_{\mathcal{S},0} \geq \tilde{u}_{\infty,a} = u_{\infty,a}$ in $Q_\infty^{\Omega_\sigma}$. Therefore any $a \in \mathcal{S}$ is a singular initial point of $\overline{u}_{\mathcal{S},0}$. Since $\mathcal{S} \cup \mathcal{R} = \Omega$, it follows that $tr_\Omega(u) = (\mathcal{S}, 0)$. Since $u_{\sigma,\delta}$ satisfies (4.1) and $\partial\Omega_\sigma$ has bounded curvature, independent of σ, there holds classicaly

$$|\nabla u_{\sigma,\delta}(x,t)| \leq ct^{-\frac{q+\alpha}{q-1}} \qquad \forall (x,t) \in \partial\Omega_\sigma \times (0,\infty). \tag{6.24}$$

If $z \in \partial\Omega$ and then by the mean value theorem there exists $\theta \in (0,1)$ such that,

$$0 \leq u_{\sigma,\delta}(z,t) = u_{\sigma,\delta}(z + \sigma\mathbf{n}_z, t) - \sigma\nabla u_{\sigma,\delta}(z + \theta\sigma\mathbf{n}_z, t).\mathbf{n}_z \leq c\sigma t^{-\frac{q+\alpha}{q-1}}.$$

This implies that $\overline{u}_{\mathcal{S},0}$ vanishes on $\partial\Omega_\sigma \times (0,\infty)$.

Let u be any positive solution of (1.6) in Q_∞^Ω, vanishing on $\partial\Omega \times (0,\infty)$, with initial trace $(\mathcal{S}, 0)$. For $0 < \delta < \sigma$ fixed and for $R, \epsilon > 0$, there exists $\tau_\epsilon > 0$ such that, for any $\tau \in (0, \tau_\epsilon]$,

$$u(x,\tau) \leq \epsilon \qquad \forall x \in \overline{B}_R \cap \Omega \setminus \mathcal{S}_\sigma.$$

This is due to the fact that $u(x,\tau) \to 0$ when $\tau \to 0$, uniformly on compact subset of $\overline{B}_R \cap \mathcal{R}$. Assume that Ω is unbounded (the case where Ω is bounded is simpler since it does not require to introduce the barrier w_{B_R}) and let $R > 0$ large enough so $\Omega^c \subset B_R$. By (4.1)-(4.3) there exists $0 < \tau_1 \leq \tau_0$ such that for any $\tau \in (0, \tau_1]$,

$$u(x,\tau) \leq w_{B_R}(x,\tau) + u_{\sigma,\delta}(x,\tau) \qquad \forall x \in \overline{B}_R \cap \Omega \cap \mathcal{S}_\sigma.$$

Furthermore $u(x,t) < w_{B_R}(x,t)$ for all $t > 0$ and $x \in \partial B_R \cap \Omega$. Since $\epsilon + w_{B_R} + u_{\sigma,\delta}$ is a supersolution for (1.6) in $B_R \cap \Omega \times (0,\infty)$, it follows that

$$u(x,t) \leq \epsilon + w_{B_R}(x,t) + u_{\sigma,\delta}(x,t) \qquad \forall (x,t) \in B_R \cap \Omega \times (0,\infty).$$

Letting successively $\delta \to 0$, $\sigma \to 0$, $R \to \infty$ (here we use the fact that $w_{B_R}(x,t) \to 0$ when $R \to \infty$ by Proposition 3.4) and $\epsilon \to 0$ yields to $u \leq \overline{u}_{\mathcal{S},0}$.

Step 2: Existence of a minimal solution. The set $\mathcal{U}_{\mathcal{S},0}(\Omega)$ is not empty since it contains $\overline{u}_{\mathcal{S},0}$ and we may define

$$\tilde{u}_{\mathcal{S},0} = \sup\{u_{\infty,a} : a \in \mathcal{S}\}, \tag{6.25}$$

and

$$\hat{u}_{\mathcal{S},0} = \inf\{u : u \in \mathcal{U}_{\mathcal{S},0}(\Omega)\}. \tag{6.26}$$

The functions $\tilde{u}_{\mathcal{S},0}$ and $\hat{u}_{\mathcal{S},0}$ are respectively positive sub and supersolutions of (1.6) in Q_∞^Ω. They are bounded from above by $\overline{u}_{\mathcal{S},0}$ and from below by $u_{\infty,a}$ for any $a \in \mathcal{S}$. Since $u_{\infty,a} \leq u$ for any $a \in \mathcal{S}$ and $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$, it follows that $\tilde{u}_{\mathcal{S},0} \leq \hat{u}_{\mathcal{S},0}$. Therefore there exists a solution $\underline{u}_{\mathcal{S},0}$ of (1.6) in Q_∞^Ω which satisfies and

$$\tilde{u}_{\mathcal{S},0} \leq \underline{u}_{\mathcal{S},0} \leq \hat{u}_{\mathcal{S},0}. \tag{6.27}$$

This implies that $\underline{u}_{\mathcal{S},0}$ has initial trace $(\mathcal{S}, 0)$, it vanishes on $\partial\Omega \times (0,\infty)$ and it is therefore the minimal element of $\mathcal{U}_{\mathcal{S},0}(\Omega)$. □

Remark. If $\mathrm{dist}\,(\mathcal{S},\Omega^c)>0$, it is not needed to replace Ω by a larger set Ω_σ in order to construct the maximal solution. The construction of $\overline{u}_{\mathcal{S},0}$ can be done in replacing $u_{n,\sigma,\delta}$ by the solution $u=u_{n,\sigma}$ of

$$\begin{aligned} \partial_t u-\Delta u+t^\alpha u^q&=0 && \text{in } Q^\Omega_\infty\\ u&=0 && \text{in } \partial\Omega\times(0,\infty)\\ u(.,0)&=n\chi_{\mathcal{S}_\delta} && \text{in } \Omega, \end{aligned} \tag{6.28}$$

with $\delta<\delta_0:=\mathrm{dist}\,(\mathcal{S},\Omega^c)$.

The next result is an extension of Proposition 2.2.

Proposition 6.8 *Assume $\alpha>-1$ and $1<q<q_{c,\alpha}$. Let $\{u_n\}$ be a sequence of positive solutions of (1.6) which converges to u locally uniformly in Q^Ω_T, and denote by $(\mathcal{S}_n,\mu_n)$ and $(\mathcal{S},\mu)$ the respective initial trace of u_n and u. If $\mathcal{A}$ is an open subset of $\cap_n\mathcal{R}_n$ and $\mu_n(\mathcal{A})$ remains bounded independently of $n\in\mathbb{N}$ (where $\mathcal{R}_n=\Omega\setminus\mathcal{S}_n$ and $\mathcal{R}=\Omega\setminus\mathcal{S}$), then $\mathcal{A}\subset\mathcal{R}$ and $\chi_{\mathcal{A}}\mu_n\rightharpoonup\chi_{\mathcal{A}}\mu$ in the weak sense of measures. Conversely, if $\mathcal{A}\subset\mathcal{R}$, then for any compact $K\subset\mathcal{A}$, there exist $C_K>0$ and $n_K\in\mathbb{N}$ such that $\mu_n(K)\le C_K$ for any $n\ge n_K$.*

Proof. Clearly (2.14) holds. We keep the notations of the proof of Proposition 2.2 where the first statement has been proved in assuming $\overline{B}_{\tilde r}(z)\subset\mathcal{A}$. Since $\mu_n(\mathcal{A})$ remains bounded, there exists a subsequence $\{n_j\}$ and a positive measure μ' on $\mathcal{A}$ such that $\mu_{n_j}\rightharpoonup\mu'$ in the weak sense of measures in $\mathcal{A}$. Then $u_{\chi_{B_{\tilde r}(z)}\mu_{n_j}}$ converges locally uniformly in $Q^{B_{\tilde r}(z)}_\infty$ to the solution $u_{\chi_{B_{\tilde r}(z)}\mu'}$ of

$$\begin{aligned} \partial_t u-\Delta u+t^\alpha u^q&=0 && \text{in } Q^{B_{\tilde r}(z)}_\infty\\ u&=0 && \text{in } \partial B_{\tilde r}(z)\times(0,\infty)\\ u(.,0)&=\chi_{B_{\tilde r}(z)}\mu' && \text{in } B_{\tilde r}(z). \end{aligned} \tag{6.29}$$

Since $q<q_{c,\alpha}$, the convergence of $u_{\chi_{B_{\tilde r}(z)}\mu_{n_j}}$ and $t^\alpha u^q_{\chi_{B_{\tilde r}(z)}\mu_{n_j}}$ respectively to $u_{\chi_{B_{\tilde r}(z)}\mu_{n_j}}$ and $t^\alpha u^q_{\chi_{B_{\tilde r}(z)}\mu'}$ holds in $L^1(Q^{B_{\tilde r}(z)}_T)$ for any $T>0$. Relation (2.21) reads

$$u_{\chi_{B_{\tilde r}(z)}\mu_n}(x,t)\le u_n(x,t)\le u_{\chi_{B_{\tilde r}(z)}\mu_n}(x,t)+w_{B_{\tilde r}(z)}(x,t)\qquad \text{in } Q^{B_{\tilde r}(z)}_\infty. \tag{6.30}$$

(see Proposition 3.3). Then u_{n_j} and $t^\alpha u^q_{n_j}$ converge to u and $t^\alpha u^q$ respectively, in $L^1(Q^{B_r(z)}_T)$ for any $r<\tilde r$. From (2.17), we derive

$$\iint_{Q^{B_r(z)}_\infty}(-u(\zeta_t+\Delta\zeta)+\zeta t^\alpha u^q)\,dxdt=\int_{B_r(z)}\zeta(x,0)d\mu'(x), \tag{6.31}$$

for any $\zeta\in C^{1,1;1}_c(\overline{Q}^{B_r(z)}_\infty)$ which vanishes for t large enough. This implies that μ' is the initial trace of u in $B_r(z)$, i.e. $\chi_{B_r(z)}\mu'=\chi_{B_r(z)}\mu$ and $\chi_{B_r(z)}\mu_n\rightharpoonup\chi_{B_r(z)}\mu$. Using a partition of unity, we conclude that $\chi_{\mathcal{A}}\mu_n\rightharpoonup\chi_{\mathcal{A}}\mu$.

Conversely, we assume that there exist a compact set $K\subset\mathcal{A}$ and a subsequence μ_{n_j} such that $\mu_{n_j}(K)\to\infty$. Thus, using the diagonal process, there exist $z\in K$ and another subsequence that we still denote μ_{n_j} such that

$$\lim_{n_j\to\infty}\mu_{n_j}(B_\epsilon(z))=\infty\qquad\forall\epsilon>0.$$

Therefore, we can construct a subsequence $\{n_{j_\ell}\} \subset \{n_j\}$ such that

$$\mu_{n_{j_\ell}}(B_{2^{-n_{j_\ell}}}(z)) = m_{n_{j_\ell}} \to \infty$$

when $n_{j_\ell} \to \infty$. Since the solution $u_{\chi_{B_{2^{-n_{j_\ell}}}(z)}\mu_{n_{j_\ell}}}$ of

$$\begin{array}{rll} \partial_t u - \Delta u + t^\alpha u^q = 0 & & \text{in } Q_\infty^{B_{\tilde r}(z)} \\ u = 0 & & \text{in } \partial B_{\tilde r}(z) \times (0,\infty) \\ u(.,0) = \chi_{B_{2^{-n_{j_\ell}}}(z)}\mu_{n_{j_\ell}} & & \text{in } B_{\tilde r}(z). \end{array} \tag{6.32}$$

converges to $u_{\infty\delta_z}^{B_{\tilde r}(z)}$ which is the limit of the solution $u_{k\delta_z}$ of (6.32) with initial data $u(.,0) = k\delta_z$, and is dominated by $u_{n_{j_\ell}}$ in $Q_T^{B_{\tilde r}(z)}$ we conclude that $u \geq u_{\infty\delta_z}^{B_{\tilde r}(z)}$ in $Q_T^{B_{\tilde r}(z)}$, which implies that $z \in \mathcal{S}$, contradiction. □

Proposition 6.9 *Assume u_1 and u_2 are two positive solutions of (1.6) in Q_∞^Ω with initial trace $(\mathcal{S},\mu)$. Then for any $a \in \mathcal{R}$ and $R > 0$ such that $B_R(a) \subset \mathcal{R}$, there holds*

$$|u_1(x,t) - u_2(x,t)| \leq w_{B_R}(x-a,t) \qquad \forall (x,t) \in Q_\infty^{B_R(a)} \tag{6.33}$$

In particular $\lim_{t\to 0} |u_1(x,t) - u_2(x,t)| = 0$ uniformly on any compact subset of $\mathcal{R}$.

Proof. Since u and u' are solution of (1.6) and $B_R(a) \in \mathcal{R}$, for any $i = 1,2$, $R' < R$ and $T > 0$,

$$\iint_{Q_T^{B'_R(a)}} t^\alpha u_i^q(x,t)dxdt + \iint_{Q_T^{B'_R(a)}} u_i(x,t)dxdt < \infty,$$

furthermore

$$\lim_{t\to 0} \int_{B_R(a)} u_i(x,t)\zeta(x)dx = \int_{B_R(a)} \zeta(x)d\mu(x) \qquad \forall \zeta \in C_c(B_R(a)).$$

This implies that u_i has a Sobolev trace on $\partial_\ell Q_T^{B'_R(a)}$ which belongs to L^1 and they are the limit, when $k \to \infty$ of the solutions $u_{i,k}$ of

$$\begin{array}{rll} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u = 0 & & \text{in } Q_\infty^{B'_R(a)} \\ u = \min\{k, u_i\lfloor_{Q_\infty^{B'_R(a)}}\} & & \text{in } \partial_\ell Q_\infty^{B'_R(a)} \\ u(.,0) = \mu & & \text{in } B'_R(a). \end{array} \tag{6.34}$$

Since $u_{2,k} + w_{B'_R}(.-a)$ is a supersolution

$$u_{1,k} \leq u_{2,k} + w_{B'_R}(.-a) \Longrightarrow |u_{1,k} - u_{2,k}| \leq w_{B'_R}(.-a) \text{ in } Q_\infty^{B'_R(a)}.$$

Letting $k \to \infty$, R' to R, we derive (6.33). The second statement is a consequence of the fact that $\lim_{t\to 0} w_{B_R}(.-a) = 0$, uniformly on B'_R by Proposition 3.3. □

Remark. The previous estimate does not use the fact that $\partial\Omega$ is smooth and bounded. If the u_i belong to $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$, estimate (6.33) can be improved since the u_i vanish on $\partial_\ell Q_\infty^{B_R(a)}$, and we obtain,

$$|u_1(x,t) - u_2(x,t)| \leq \min\left\{w_{B_R}(x-a,t), c_\alpha t^{-\frac{1+\alpha}{q-1}}\right\} \qquad \forall (x,t) \in Q_\infty^{B_R(a)}. \tag{6.35}$$

Proposition 6.10 *Assume $\Omega \subseteq \mathbb{R}^N$ is either $\mathbb{R}^N$ or an open domain with a C^2 compact boundary, $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Then for any measure μ in Ω such that $\mu\lfloor_{\Omega_R} \in \mathfrak{M}_+^{b,\rho}(\Omega_R)$ where $\Omega_R = \Omega \cap B_R$, there exists a unique solution u_μ to*

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^\Omega \\ u &= 0 && \text{in } \partial_\ell Q_\infty^\Omega \\ u(.,0) &= \mu && \text{in } \Omega, \end{aligned} \tag{6.36}$$

and the mapping $\mu \mapsto u_\mu$ is increasing. Furthermore, if $\{\mu_n\}$ is a sequence of positive measures such that $\mu_n\lfloor_{\Omega_R} \in \mathfrak{M}_+^{b,\rho}(\Omega_R)$ which converges weakly to $\mu\lfloor_{\Omega_R} \in \mathfrak{M}_+^{b,\rho}(\Omega_R)$, then $\{u_{\mu_n}\} \to u_\mu$ locally uniformly in Q_∞^Ω.

Proof. We recall that u is a solution of (6.36) if $u \in L^1_{loc}(\overline{Q_\infty^\Omega})$, $|u|^q \in L^1_{loc}(\overline{Q_\infty^\Omega}; t^\alpha \rho dxdt)$ satisfies

$$\iint_{Q_\infty^\Omega} \left(-u(\partial_t \zeta + \Delta\zeta) + t^\alpha |u|^{q-1} u\zeta\right) dxdt = \int_\Omega \zeta(.,0) d\mu \tag{6.37}$$

for any test function $\zeta \in C_0^{2,1;1}(\overline{Q_\infty^\Omega})$. When Ω is bounded, existence, uniqueness and stability are proved in [9]. Thus we assume that Ω is unbounded and we assume $R \geq R_0$ such that $\Omega^c \subset B_{R_0}$. There exists a unique solution u_R of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_R} \\ u &= 0 && \text{in } \partial_\ell Q_\infty^{\Omega_R} \\ u(.,0) &= \mu\lfloor_{\Omega_R} && \text{in } \Omega_R. \end{aligned} \tag{6.38}$$

The function u_R is nonnegative, $R \mapsto u_R$ is increasing. For $R > R_1$, u_R admits a Sobolev trace f_{R_1} on $\partial B_{R_1} \times (0,T)$ which is an integrable function, and u_R is the unique solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_{R_1}} \\ u &= 0 && \text{in } \partial\Omega \times (0,\infty) \\ u &= f_{R_1} && \text{in } \partial B_{R_1} \times (0,\infty) \\ u(.,0) &= \mu\lfloor_{\Omega_{R_1}} && \text{in } \Omega_{R_1}. \end{aligned} \tag{6.39}$$

Furthermore, $u_R\lfloor_{Q_\infty^{\Omega_{R_1}}} = \lim_{m\to\infty} v_m$, where v_m is the unique solution of (6.36) where the boundary data on $\partial B_{R_1} \times (0,\infty)$ is replaced by $f_{R_1,m} = f_{R_1} \wedge m$ ($m \in \mathbb{N}^*$). Let v_{R_1} be the unique solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_{R_1}} \\ u &= 0 && \text{in } \partial\Omega \times (0,\infty) \\ u &= 0 && \text{in } \partial B_{R_1} \times (0,\infty) \\ u(.,0) &= \mu\lfloor_{\Omega_{R_1}} && \text{in } \Omega_{R_1}, \end{aligned} \tag{6.40}$$

If $w_{B_{R_1}}$ is the barrier function in $Q_\infty^{\Omega_{R_1}}$ which has been constructed in Proposition 3.3, $v_{R_1} + w_{B_{R_1}}$ is a supersolution for problem (6.39). Since it is larger than v_m in $Q_\infty^{\Omega_{R_1}}$ for any $m > 0$, there holds $u_R \leq v_{R_1} + w_{B_{R_1}}$, for any $R > R_1$. Then $u_R \uparrow u_\mu$ which is a

solution of 1.6 in $Q_\infty^{\Omega_R}$. By Proposition 3.4, $w_{B_{R_1}}$ remains uniformly bounded in $Q_\infty^{\Omega_{R'}}$ for any $R_0 < R' < R_1$. Therefore u_μ shares the same property. If $\zeta \in C_c^{1,1;1}(Q_\infty^{\overline{\Omega}})$ vanishes on $Q_\infty^{\partial\Omega})$ and for $|x| > R' > R_0$, there holds for $R > R' > R_0$ and $T > 0$,

$$\iint_{Q_\infty^\Omega} (-u_R(x,t)(\partial_t\zeta + \Delta\zeta) + \zeta t^\alpha u_R^q)\, dxdt = \int_\Omega \zeta(x,0)d\mu(x) - \int_\Omega \zeta(x,T)u_R(x,T)dx \tag{6.41}$$

If we let $R \to \infty$ we deduce by the monotone convergence theorem that u_μ is a weak solution of (6.36). This proves existence.

For uniqueness, we consider u_μ and u'_μ two solutions of (6.36). By the same argument as in the existence part, for any $R > 0$, u_μ is smaller than the supersolution $u'_\mu + w_{B_{R_1}}$ in $Q_\infty^{\Omega_R}$. Since $\lim_{R\to\infty} w_{B_R} = 0$ by Proposition 3.4 we obtain $u_\mu \le u'_\mu$. Similarly $u'_\mu \le u_\mu$. Uniqueness implies the monotonicity of the mapping $\mu \mapsto u_\mu$.

For proving the stability, assume $\{\mu_n\lfloor_{\Omega_R}\}$ converges to $\mu\lfloor_{\Omega_R}$ in the weak sense of measures in $\mathfrak{M}_+^{b,\rho}(\Omega_R)$ for any $R > R_0$. Then the sequence of solutions $v_{n,R}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_R} \\ u &= 0 && \text{in } \partial\Omega \times (0,\infty) \\ u &= 0 && \text{in } \partial B_R \times (0,\infty) \\ u(.,0) &= \mu_n\lfloor_{\Omega_R} && \text{in } \Omega_R, \end{aligned} \tag{6.42}$$

converges to the solution v_R of (6.40) with $R_1 = R$. In particular $v_{n,R} \to v_R$ and $t^\alpha v_{n,R}^q \to t^\alpha v_R^q$ in $L^1(Q_T^{\Omega_R})$ and by standard regularity result the convergence of $v_{n,R}$ towards v_R holds uniformly on $\overline{\Omega}_R \times [\epsilon, T]$ for any $0 < \epsilon < T$. Furthermore $u_{\mu_n} \le v_{n,R} + W_{B_R}$ in $Q_\infty^{\Omega_R}$. This jointly with standard local regularity results for heat equation, implies that $\{u_{\mu_n}\}$ remains uniformly bounded and hence relatively compact for the topology of uniform convergence on any compact set of $\overline{\Omega} \times [\epsilon, \infty)$. Thus there exist a subsequence $\{u_{\mu_{n_k}}\}$ and a function $u^* \in C^(\overline{Q}_\infty^\Omega)$ such that $u_{\mu_{n_k}} \to u^*$ locally uniformly in $\overline{\Omega} \times (0,\infty)$. Since $t^\alpha u_{\mu_n}^q \le t^\alpha v_{n,R}^q + t^\alpha W_{B_R}^q$, there also holds by the dominated convergence theorem $t^\alpha u_{\mu_{n_k}}^q \to t^\alpha u^{*q}$ in $L^1_{loc}(\overline{\Omega} \times [0,\infty))$. Henceforth letting $n_k \to \infty$ in the expression

$$\begin{aligned} &\iint_{Q_\infty^\Omega} \left(-u_{\mu_{n_k}}(x,t)(\partial_t\zeta + \Delta\zeta) + \zeta t^\alpha u_{\mu_{n_k}}^q\right) dxdt \\ &\qquad = \int_\Omega \zeta(x,0)d\mu_{n_k}(x) - \int_\Omega \zeta(x,T)u_{n_k}(x,T)dx, \end{aligned} \tag{6.43}$$

where $\zeta \in C_c^{1,1;1}(Q_\infty^{\overline{\Omega}})$, we conclude that $u^* = u_\mu$ and that $u_{\mu_n} \to u_\mu$. □

Proposition 6.11 *Assume F is a non-empty relatively closed subset of Ω, $\mathcal{R} = \Omega \setminus F$ and $\mu \in \mathfrak{M}_+(\mathcal{R})$. If we set*

$$\partial_\mu F = \{z \in F : \mu(\mathcal{R} \cap B_r(z)) = \infty, \forall r > 0\}, \tag{6.44}$$

then $\partial_\mu F$ is relatively closed in Ω. If $\mathcal{R}^\mu = \Omega \setminus \partial_\mu F$, it contains $\mathcal{R}$ and if μ^* is the measure defined in $\mathcal{R}^\mu$ by μ on $\mathcal{R}$ and 0 in $\mathcal{R}^\mu \cap \mathcal{R}^c$, then there exist a minimal positive solution

$\underline{u}_{\mu^*}$ and a maximal solution $\overline{u}_{\mu^*}$ of (1.6) vanishing on $\partial\Omega \times (0,\infty)$ satisfying $tr_\Omega(u) = (\partial_\mu F, \mu^*)$. Furthermore $\underline{u}_{\mu^*}$ and $\overline{u}_{\mu^*}$ are respectively the minimal and the maximal element of $\mathcal{U}_{\partial_\mu F,\mu^*}(\Omega)$.

Proof. The set $\partial_\mu F$ is the blow-up set of the measure μ. It is clearly a relatively closed subset of Ω included into $\overline{\mathcal{R}} \setminus \mathcal{R}$.

Step 1: Existence of a minimal solution. For $\delta > 0$, we denote $(\partial_\mu F)_\delta := \{x \in \Omega : \text{dist}\,(x, \partial_\mu F) \le \delta\}$ and $\mathcal{R}^\mu_\delta = \Omega \setminus (\partial_\mu F)_\delta \subset \mathcal{R}^\mu$. We define the Radon measure μ_δ on Ω by

$$\mu_\delta = \begin{cases} \mu & \text{on } \mathcal{R}^\mu_\delta \cap \mathcal{R} \\ 0 & \text{on } F \cup (\partial_\mu F)_\delta \end{cases}$$

Then μ_δ is a positive Radon measure in Ω and by Proposition 6.10 problem (6.36) with initial data μ_δ admits a unique positive solution u_{μ_δ}. Furthermore the mapping $\delta \mapsto u_{\mu_\delta}$ is nonincreasing, and we set $\underline{u}_{\mu^*} = \lim_{\delta\to 0} u_{\mu_\delta}$. Then $\underline{u}_{\mu^*}$ is a positive solution of (1.6) in Q^Ω_∞ which vanishes on $\partial\Omega \times (0,\infty)$ and has initial trace $(\mathcal{S}', \mu')$. If $a \in \mathcal{R}^\mu$, there exists $R > 0$ such that $\overline{B}_R(a) \subset \mathcal{R}^\mu$ and $\delta_a > 0$ such that $\overline{B}_R(a) \subset \mathcal{R}^\mu_\delta$ for $0 < \delta < \delta_a$, that we assume below. By the maximum principle there holds

$$v_{\mu_\delta}(x,t) \le u_\delta(x,t) \le v_{\mu_\delta}(x,t) + w_{B_R}(x-a,t) \qquad \text{in } Q^{B_R(a)}_\infty,$$

where v_{μ_δ} is the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q^{B_R(a)}_\infty \\ u &\ge 0 && \text{in } Q^{B_R(a)}_\infty \\ u(x,t) &= 0 && \text{in } \partial B_R(a) \times (0,\infty) \\ u(.,0) &= \mu_\delta \chi_{B_R(a)} && \text{in } B_R(a). \end{aligned}$$

Letting $\delta \to 0$, then $\mu_\delta \chi_{B_R(a)} \uparrow \mu^* \chi_{B_R(a)}$, which yields to

$$v_{\mu^*\lfloor_{B_R(a)}}(x,t) \le \underline{u}_{\mu^*}(x,t) \le v_{\mu^*\lfloor_{B_R(a)}}(x,t) + w_{B_R}(x-a,t) \qquad \text{in } Q^{B_R(a)}_\infty.$$

By a partition of unity, it implies that for any $\zeta \in C_c(\mathcal{R}^\mu)$, we have

$$\lim_{t\to 0} \int_{\mathcal{R}^\mu} \underline{u}_{\mu^*}(x,t)\zeta(x)dx = \int_{\mathcal{R}^\mu} \zeta d\mu^*(x).$$

Therefore $\mathcal{R}^\mu \subset \mathcal{R}'$ and $\mu'\lfloor_{\mathcal{R}^\mu} = \mu^*$. If $z \in \mathcal{R}' \cap (\mathcal{R}^\mu)^c = \mathcal{R}' \cap \partial_\mu F$, $\mu^*(B_r(z) \cap \mathcal{R}^\mu) = \mu(B_r(z) \cap \mathcal{R}) = \infty$ for any $r > 0$ while there exists $r_0 > 0$ such that $\mu'(B_{r_0}(z)) < \infty$. By Proposition 6.8, for any $r' < r_0$ there exists $C > 0$ such that $\mu_\delta(\overline{B}_r(z)) \le C$. By the monotone convergence theorem, it implies $\mu(\overline{B}_r(z) \cap \mathcal{R}) \le C$, which contradicts the definition of $\partial_\mu F$. Thus $\mathcal{S}' = \partial_\mu F$ and $tr_\Omega(\underline{u}_{\mu^*}) = (\partial_\mu F, \mu^*)$.

Let us assume that Ω is unbounded, R_0 is such that $\Omega^c \subset B_{R_0}$ and $\Omega_R = \Omega \cap B_R$ for $R > R_0$. Let $\delta, \epsilon > 0$, there exists τ_ϵ such that

$$u_{\mu_\delta}(x,\tau) \le \epsilon \qquad \forall (x,\tau) \in \Omega_R \cap (\partial_\mu F)_{\frac{\delta}{2}} \times [0,\tau_\epsilon] \tag{6.45}$$

Let $u \in \mathcal{U}_{\partial_\mu F,\mu^*}(\Omega)$. In order to compare u_{μ_δ} and $v := u + w_{B_R} + \epsilon$ in $\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}} \times (0, \tau_\epsilon]$ we see that

$$u_{\mu_\delta}(.,0) = \chi_{\mathcal{R}^\mu_R \setminus (\partial_\mu F)_\delta} \mu^* \leq u(.,0) = \chi_{\mathcal{R}^\mu_R \setminus (\partial_\mu F)_{\frac{\delta}{2}}} \mu^*$$

and both are bounded Radon measures. Since $u_{\mu_\delta}(t,0) \leq v(x,t)$ in $\partial(\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}}) \times (0, \tau_\epsilon]$ and v is a supersolution, it follows that $u_{\mu_\delta} \leq v$ in $\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}} \times (0, \tau_\epsilon]$. Using (6.45) we conclude that

$$u_{\mu_\delta}(x,\tau) \leq v(x,\tau) \qquad \forall (x,\tau) \in \Omega_R \times [0, \tau_\epsilon]. \tag{6.46}$$

Next, applying the comparison principle in $\Omega_R \times [\tau^*, \infty)$ between the solution u_{μ_δ} and the supersolution $u + \epsilon + w_{B_R}$, we conclude that (6.46) holds in $\Omega_R \times [\tau, \infty)$ and thus in $Q^{B_R}_\infty$. Letting successively $R \to \infty$, $\epsilon \to 0$ and $\delta \to 0$, we conclude that $\underline{u}_{\mu^*} \leq u$, thus u_{μ^*} is the minimal element of $\mathcal{U}_{\partial_\mu F,\mu^*}(\Omega)$.

Step 2: Existence of a maximal solution. Let $\delta > 0$ and $u \in \mathcal{U}_{\partial_\mu F,\mu^*}(\Omega)$. By Proposition 6.9, for any $R > 0$ and $\epsilon > 0$ there exists τ_ϵ such that

$$u(x,t) \leq \underline{u}_{\mu^*}(x,t) + \epsilon \qquad \forall (x,t) \in \Omega_R \setminus (\partial_\mu F)_\delta \times (0, \tau_\epsilon], \tag{6.47}$$

and by (4.1), $u(x,t) \leq c_\alpha t^{-\frac{1+\alpha}{q-1}}$. Let $\tau \in (0, \tau_\epsilon]$ and $w_{\delta,\tau}$ be the solution of (6.36) in $Q^{\Omega_R}_\infty$ with initial data μ replaced by

$$h_{\delta,\tau} = \begin{cases} \underline{u}_{\mu^*}(x,\tau) & \text{if } x \in \Omega_R \setminus (\partial_\mu F)_\delta \\ c_\alpha \tau^{-\frac{1+\alpha}{q-1}} & \text{if } x \in \Omega_R \cap (\partial_\mu F)_\delta \end{cases} \tag{6.48}$$

By (6.47), (6.48) and the maximum principle,

$$u(x,t+\tau) \leq w_{\delta,\tau}(x,t) + \epsilon + w_{B_R}(x,t+\tau) \qquad \forall (x,t) \in Q^{\Omega_R}_\infty. \tag{6.49}$$

Let $\overline{u}_{(\partial_\mu F)_\delta}$ be the maximal element of $\mathcal{U}_{(\partial_\mu F)_\delta,0}(\Omega)$, which exists by Proposition 6.7. Then, by (4.3), for any $\delta' > \delta$, there exists $\tau_{\delta'} \in (0, \tau_\epsilon]$ such that for any $\tau \in (0, \tau_{\delta'}]$

$$\max\{\underline{u}_{\mu^*}(.,.+\tau), \overline{u}_{(\partial_\mu F)_\delta}(.,.+\tau)\} \leq w_{\delta,\tau} \leq \underline{u}_{\mu^*}(.,.+\tau) + \overline{u}_{(\partial_\mu F)_{\delta'}}(.,.+\tau) \qquad \text{in } Q^{\Omega_R}_\infty. \tag{6.50}$$

Up to a sequence $\{\tau_n\}$ converging to 0, $\{w_{\delta,\tau_n}\}$ converges, locally uniformly in Q^Ω_∞ to a solution w_δ of (1.6) in Q^Ω_∞ which satisfies

$$\max\{\underline{u}_{\mu^*}, \overline{u}_{(\partial_\mu F)_\delta}\} \leq w_\delta \leq \underline{u}_{\mu^*} + \overline{u}_{(\partial_\mu F)_{\delta'}} \qquad \text{in } Q^{\Omega_R}_\infty. \tag{6.51}$$

We can replace δ' by δ in this inequality, this proves that w_δ vanishes on $\partial_\ell Q^{\Omega_R}_\infty$ has initial trace $(\mu_\delta, (\partial_\mu F)_\delta)$, therefore (6.49) becomes

$$u(x,t) \leq w_\delta(x,t) + \epsilon + w_{B_R}(x,t) \qquad \forall (x,t) \in Q^{\Omega_R}_\infty. \tag{6.52}$$

Letting successively $R \to \infty$ and $\epsilon \to 0$ we deduce that w_δ is larger that any $u \in \mathcal{U}_{\partial_\mu F,\mu^*}(\Omega)$ in $Q^{\Omega_R}_\infty$. Since $h_{\delta,\tau}$ decreases with δ, w_δ shares this property and the limit, denoted by $\overline{u}_{\mu^*}$ is a solution of (1.6) in Q^Ω_∞ which vanishes on $\partial_\ell Q^{\Omega_R}_\infty$ which is large than u, thus it is the maximal element of $\mathcal{U}_{\partial_\mu F,\mu^*}(\Omega)$. □

Proposition 6.12 *Under the assumptions of Proposition* 6.11*, we set* $F_\delta := \{x \in \Omega : \mathrm{dist}\,(x, F) \le \delta\}$ *and* $\mathcal{R}_\delta := \Omega \setminus F_\delta \subset \mathcal{R}$*. If we define the measure* $\tilde{\mu}_\delta$ *in* Ω *by*

$$\tilde{\mu}_\delta = \begin{cases} \mu & \text{on } \mathcal{R}_\delta \\ 0 & \text{on } F_\delta, \end{cases}$$

then $u_{\tilde{\mu}_\delta} \uparrow \underline{u}^*$ *when* $\delta \downarrow 0$*.*

Proof. There holds $\tilde{\mu}_\delta \le \mu_\delta$ which implies $u_{\tilde{\mu}_\delta} \le u_{\mu_\delta}$. When $\delta \to 0$, $u_{\tilde{\mu}_\delta} \uparrow \underline{u}^* \le \underline{u}_{\mu^*}$, thus $\underline{u}^*$ is a positive solution of (1.6) in Q^Ω_∞ which vanishes on $\partial_\ell Q^\Omega_\infty$. Then $tr_\Omega(\underline{u}^*) = (\mathcal{S}'', \mu'')$ and $\mathcal{S}'' \subset \partial_\mu F$ and $\mu'' \le \mu^*$ on $\mathcal{R}^\mu$. Furthermore $\mu'' = \mu = \mu^*$ on $\mathcal{R}$, as in the proof of Proposition 6.11. Since $\mu^* = 0$ on $\mathcal{R}^\mu \setminus \mathcal{R}$ it follows that $\mu'' = \mu^*$ on $\mathcal{R}^\mu$. Let $a \in \mathcal{R}'' \cap \partial_\mu F$ and $R > 0$ such that $\overline{B}_R(a) \subset \mathcal{R}''$. Then $\mu''(\overline{B}_R(a)) < \infty$. Therefore

$$\mu''(\overline{B}_R(a) \cap \mathcal{R}) = \mu(\overline{B}_R(a) \cap \mathcal{R}) < \infty,$$

contradiction. Thus $\mathcal{S}'' = \partial_\mu F$ and $tr_\Omega(\underline{u}^*) = (\partial_\mu F, \mu^*)$. Since $\underline{u}^* \le \underline{u}_{\mu^*}$ and $\underline{u}_{\mu^*}$ is minimal, it follows that $\underline{u}^* = \underline{u}_{\mu^*}$. □

Proposition 6.13 *Assume* $v \in C(\overline{\Omega} \times (0, \infty))$ *is a positive sub-solution (resp. supersolution) of (1.6) in* Q^Ω_∞ *which vanishes on* $\partial\ell Q^\Omega_\infty$*. Then there exists a minimal solution* $\pi_+(v)$ *larger than* v *(resp. a maximal solution* $\pi_-(v)$ *smaller than* v *and vanishing on* $\partial\ell Q^\Omega_\infty$*).*

Proof. Assume v is a subsolution. Let $\tau > 0$ and let u_τ be the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (\tau, \infty) \\ u \ge 0, u &= 0 && \text{in } \partial\Omega \times (\tau, \infty) \\ u(., \tau) &= v(., \tau) && \text{in } \Omega. \end{aligned} \tag{6.53}$$

Existence and uniqueness follows from Proposition 6.10. Furthermore $u_\tau \ge v$ in $\Omega \times (\tau, \infty)$. This implies that for $0 < \tau < \tau'$, $u_\tau \ge u_{\tau'}$. Since $u_\tau(x, t) \le c_\alpha (t - \tau)^{-\frac{1+\alpha}{q-1}}$, there exists $\pi_+(v) = \lim_{\tau \to 0} u_\tau$, and $\pi_+(v)$ is a positive solution of (1.6) in Q^Ω_∞ and is larger than v. If u is any positive solution of (1.6) in Q^Ω_∞, vanishing on $\partial\ell Q^\Omega_\infty$ and larger than v, for any $\tau > 0$ it is larger than $u(., \tau)$, thus it is larger than u_τ on $\Omega \times (\tau, \infty)$. Therefore $u \ge \pi_+(v)$.

Assume now that u is a supersolution. We define u_τ by (6.53). Then $u_\tau \le v$ and $u_\tau \le u_{\tau'}$ for $0 < \tau < \tau'$. Then $\pi_-(v) = \lim_{\tau \to 0} u_\tau$, and $\pi_-(v)$ is a positive solution of (1.6) in Q^Ω_∞ and is smaller than v, and thus vanishing on $\partial\ell Q^\Omega_\infty$. Similarly as above $\pi_-(v)$ is larger than any positive solution smaller than v. □

Theorem 6.14 *Assume* $\Omega \subseteq \mathbb{R}^N$ *is either* $\mathbb{R}^N$ *or an open domain with a* C^2 *compact boundary,* $\alpha > -1$ *and* $1 < q < q_{c,\alpha}$*. Then for any* $\nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega)$ *there exist a maximal positive solution* $\overline{u}_{\mathcal{S},\mu}$ *and a minimal positive solution* $\underline{u}_{\mathcal{S},\mu}$ *of (1.6) in* Q^Ω_∞ *vanishing on* $\partial\Omega \times (0, \infty)$ *with initial trace* ν*. Furthermore, if* $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$*, then*

$$\overline{u}_{\mathcal{S},\mu} - \underline{u}_{\mathcal{S},\mu} \le \overline{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0}. \tag{6.54}$$

Proof. Step 1: Construction of the maximal and minimal solutions. The functions $\underline{u}_{\mu^*}$, $\underline{u}_{\mathcal{S},0}$, $\overline{u}_{\mu^*}$ and $\overline{u}_{\mathcal{S},0}$ have been defined in Proposition 6.11 and Proposition 6.7. Since $\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\}$ is a subsolution of (1.6) which is smaller than the supersolution $\overline{u}_{\mu^*} + \overline{u}_{\mathcal{S},0}$ we set

$$(i) \quad \underline{u}_{\mathcal{S},\mu} = \pi_+(\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\}) \quad \text{and} \quad (ii) \quad \overline{u}_{\mathcal{S},\mu} = \pi_-(\overline{u}_{\mu^*} + \overline{u}_{\mathcal{S},0}). \tag{6.55}$$

Then $\underline{u}_{\mathcal{S},\mu}$ and $\overline{u}_{\mathcal{S},\mu}$ are solutions which satisfy

$$\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \le \underline{u}_{\mathcal{S},\mu} \le \overline{u}_{\mathcal{S},\mu} \le \overline{u}_{\mu^*} + \overline{u}_{\mathcal{S},0}. \tag{6.56}$$

Therefore $\underline{u}_{\mathcal{S},\mu}$ and $\overline{u}_{\mathcal{S},\mu}$ vanish on $\partial_\ell Q^\Omega_\infty$, they have initial trace μ on $\mathcal{R}$ and are larger than any $u_{\infty,a}$ for $a \in \mathcal{S}$ (notice that $\partial_\mu \mathcal{S} \subset \mathcal{S}$). This implies that $\underline{u}_{\mathcal{S},\mu}$ and $\overline{u}_{\mathcal{S},\mu}$ belong to $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$.

Let $u \in \mathcal{U}_{\mathcal{S},\mu}(\Omega)$. Then for $\sigma > \delta > 0$ and $\epsilon, R > 0$, there exists $\tau_1 > 0$ such that

$$u(x,t) \le u_{\sigma,\delta}(x,t) \text{ for } (x,t) \in (\mathcal{S}_{\frac{\delta}{2}} \cap \Omega_R) \times (0, \tau_1].$$

There exists $\tau_2 \in (0, \tau_1]$ such that

$$u(x,t) \le \overline{u}_{\mu^*}(x,t) + \epsilon + w_{B_R}(x,t) \text{ for } (x,t) \in (\Omega_R \setminus \mathcal{S}_{\frac{\delta}{2}}) \times (0, \tau_2].$$

Therefore

$$u(x,t) \le u_{\sigma,\delta}(x,t) + \overline{u}_{\mu^*}(x,t) + \epsilon + w_{B_R}(x,t) \text{ for } (x,t) \in \Omega_R \times (0, \tau_2].$$

Therefore

$$u \le u_{\sigma,\delta} + u_{\mu^*} + \epsilon + w_{B_R} \quad \text{in } Q^{\Omega_R}_\infty \tag{6.57}$$

Letting successively $R \to \infty$, $\epsilon \to 0$, $\delta \to 0$ and $\sigma \to 0$ we obtain $u \le \overline{u}_{\mathcal{S},0} + \overline{u}_{\mu^*}$, and therefore $u \le \pi_+(\overline{u}_{\mathcal{S},0} + \overline{u}_{\mu^*}) = \overline{u}_{\mathcal{S},\mu}$. Next, we also have

$$u \ge \tilde{u}_{\mathcal{S},0} := \sup\{u_{\infty,a} : a \in \mathcal{S}\} \Longrightarrow u \ge \underline{u}_{\mathcal{S},0} = \pi_+(\tilde{u}_{\mathcal{S},0}),$$

by (6.25). With the notations of Proposition 6.12 with $F = \mathcal{S}$, for any $R > 0$, $\delta > 0$ and $\epsilon > 0$, there exists τ_ϵ such that

$$u_{\tilde{\mu}_\delta}(x,t) \le u(x,t) + w_{B_R}(x,t) + \epsilon \qquad \text{in } \Omega_R \times (0, \tau_\epsilon],$$

because the support of $\tilde{\mu}_\delta$ is included in $\Omega \setminus \mathcal{S}_\delta$. Therefore this last inequality holds in $Q^{\Omega_R}_\infty$ and consequently

$$\sup\{u_{\tilde{\mu}_\delta}, \underline{u}_{\mathcal{S},0}\} \le u + \epsilon + w_{B_R} \qquad \text{in } Q^{\Omega_R}_\infty,$$

and we can let $R \to \infty$ and $\epsilon \to 0$ to obtain $\sup\{u_{\tilde{\mu}_\delta}, \underline{u}_{\mathcal{S},0}\} \le u$ in Q^Ω_∞. Letting $\delta \to 0$ and using Proposition 6.12 we get $\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \le \underline{u}_{\mathcal{S},\mu} \le u$.

Step 2: Alternative construction. For $0 < \delta < \sigma$ and $n \in \mathbb{N}_*$ we denote by $\overline{u}_{n,\sigma,\delta,\mu}$ the solution of

$$\begin{array}{rl} \partial_t u - \Delta u + t^\alpha u^q = 0 & \text{in } Q^{\Omega_\sigma}_\infty \\ u \ge 0 & \text{in } Q^{\Omega_\sigma}_\infty \\ u(x,t) = 0 & \text{in } \partial_\ell Q^{\Omega_\sigma}_\infty \\ u(.,0) = \mu_\delta + n\chi_{\mathcal{S}_\delta} & \text{in } \Omega_\sigma. \end{array} \tag{6.58}$$

We denote here $\Omega_\sigma = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x,\Omega) < \sigma\}$, $\mathcal{S}_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x,\mathcal{S}) < \delta\}$ and $\mathcal{R}_\delta = \Omega \cap \mathcal{S}_\delta^c$ and $\mu_\delta = \chi_{\mathcal{R}_\delta}\mu$. The same arguments of monotonicity as in Proposition 6.7 and Proposition 6.11 show that

$$\lim_{\sigma\to 0}\lim_{\delta\to 0}\lim_{n\to\infty} = \overline{u}_{n,\sigma,\delta,\mu} = \overline{u}_{\mathcal{S},\mu}. \tag{6.59}$$

If $\tau > 0$ we denote by $\underline{u}_{\tau,\delta,\mu}$ the solution of

$$\begin{array}{rl} \partial_t u - \Delta u + t^\alpha u^q = 0 & \text{in } Q_\infty^\Omega \\ u \geq 0 & \text{in } Q_\infty^\Omega \\ u(x,t) = 0 & \text{in } \partial_\ell Q_\infty^\Omega \\ u(.,0) = \mu_\delta + \chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau) & \text{in } \Omega, \end{array} \tag{6.60}$$

Using estimate (6.3) and Proposition 6.9 is is easy to prove that $\underline{u}_{\tau,\delta,\mu} \leq u$ for any $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$. Furthermore

$$\max\{u_{\mu_\delta}, u_{\chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)}\} \leq \underline{u}_{\tau,\delta,\mu} \leq u_{\mu_\delta} + \underline{u}_{\mathcal{S},0}(.,.+\tau),$$

since have

$$u_{\chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)} \leq \underline{u}_{\mathcal{S},0}(.,\tau) \leq u_{\chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)} + C\left(\frac{\delta^2}{\tau}\right)^{\frac{1+\alpha}{q-1}-\frac{1}{2}} e^{-\frac{\delta^2}{4\tau}}$$

by (5.6) with $N = 1$. Set $c(\delta,\tau) = C\left(\frac{\delta^2}{\tau}\right)^{\frac{1+\alpha}{q-1}-\frac{1}{2}} e^{-\frac{\delta^2}{4\tau}}$, then

$$\max\{u_{\mu_\delta}, u_{\chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)}\} \geq \max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}(.,\tau) - c(\delta,\tau)\} \geq \max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}(.,\tau)\} - c(\delta,\tau).$$

Therefore, if $\underline{u}_{\tau_n,\delta,\mu} \to \underline{u}_{0,\delta,\mu}$ locally uniformly in Q_∞^Ω, then $\underline{u}_{0,\delta,\mu}$ is a solution of (1.6) in Q_∞^Ω which satisfies

$$\max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{\delta,0,\mu} \leq u_{\mu_\delta} + \underline{u}_{\mathcal{S},0}, \tag{6.61}$$

and is smaller than any $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$. There exists $\delta_n \to 0$ such that $\underline{u}_{0,\delta_n,\mu} \to \underline{u}_{0,0,\mu}$. Then

$$\max\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{0,0,\mu} \leq \underline{u}_{\mu^*} + \underline{u}_{\mathcal{S},0}, \tag{6.62}$$

and $\underline{u}_{0,0,\mu}$ is an element of $\mathcal{U}_{\mathcal{S},0}(\Omega)$ smaller than any $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$. Thus $\underline{u}_{0,0,\mu} = \underline{u}_{\mathcal{S},\mu}$ and

$$\lim_{\delta\to 0}\lim_{\tau\to 0}\underline{u}_{\tau,\delta,\mu} = \underline{u}_{\mathcal{S},\mu}. \tag{6.63}$$

Step 3: Proof of (6.54). We assume $\inf\{|z-z'| : z \in \mathcal{S}, z' \in \Omega^c\} = \delta_0 > 0$, so that we can take $\sigma = 0$ in the construction of $\overline{u}_{\mathcal{S},\mu}$. Put $\tau = (\frac{c_\alpha}{n})^{\frac{q-1}{1+\alpha}}$ and

$$\overline{Z}_{n,\delta,\mu} = \overline{u}_{n,\delta,\mu} - \overline{u}_{n,\delta,0}\,,\ \underline{Z}_{\tau,\delta,\mu} = \underline{u}_{\tau,\delta,\mu} - \underline{u}_{\tau,\delta,0} \text{ and } W_{n,\delta,\mu} = \overline{Z}_{n,\delta,\mu} - \underline{Z}_{\tau,\delta,\mu}. \tag{6.64}$$

Then $w = W_{n,\delta,\mu}$ satisfies

$$\partial_t w - \Delta w + t^\alpha\left(\overline{u}_{n,\delta,\mu}^q - \overline{u}_{n,\delta,0}^q - \underline{u}_{\tau,\delta,\mu}^q + \underline{u}_{\tau,\delta,0}^q\right) \tag{6.65}$$

in Q^Ω_∞ and we can write

$$\begin{aligned}\overline{u}^q_{n,\delta,\mu} - \overline{u}^q_{n,\delta,0} - \underline{u}^q_{\tau,\delta,\mu} + \underline{u}^q_{\tau,\delta,0} &= (\overline{u}^q_{n,\delta,\mu} - \underline{u}^q_{\tau,\delta,\mu}) - (\overline{u}^q_{n,\delta,0} - \underline{u}^q_{\tau,\delta,0}) \\ &= d_\mu(\overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}) - d_0(\overline{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0})\end{aligned} \tag{6.66}$$

where

$$d_\mu(x,t) = \begin{cases} \dfrac{\overline{u}^q_{n,\delta,\mu} - \underline{u}^q_{\tau,\delta,\mu}}{\overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}} & \text{if } \overline{u}_{n,\delta,\mu} \neq \underline{u}_{\tau,\delta,\mu} \\ 0 & \text{if } \overline{u}_{n,\delta,\mu} = \underline{u}_{\tau,\delta,\mu} \end{cases} \tag{6.67}$$

and d_0 is defined accordingly. Since

$$\overline{u}_{n,\delta,\mu} \geq \max\{\underline{u}_{\tau,\delta,\mu}, \overline{u}_{n,\delta,0}\} \text{ and } \underline{u}_{\tau,\delta,0} \leq \min\{\underline{u}_{\tau,\delta,\mu}, \overline{u}_{n,\delta,0}\},$$

there holds $d_\mu \geq d_0 \geq$ by convexity. Using the fact that $\overline{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0} \geq 0$ is infers that

$$d_\mu(\overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}) - d_0(\overline{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}) \geq d_\mu(\overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu} - \overline{u}_{n,\delta,0} + \underline{u}_{\tau,\delta,0})$$

Finally (6.65) becomes

$$\partial_t w - \Delta w + t^\alpha d_\mu w \leq 0 \qquad \text{in } Q^\Omega_\infty. \tag{6.68}$$

Furthermore, in the sense of measures,

$$w(.,0) = \mu_\delta + n\chi_{\mathcal{S}_\delta} - n\chi_{\mathcal{S}_\delta} - (\mu_\delta + \chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)) + \chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau) = 0.$$

Because $w = 0$ in $\partial_\ell Q^\Omega_\infty$ it follows $w \leq 0$ by the maximum principe. Therefore

$$\overline{Z}_{n,\delta,\mu} \leq \underline{Z}_{\tau,\delta,\mu} \Longrightarrow \overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu} \leq \overline{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}. \tag{6.69}$$

If we let successively $n \to \infty$ (and therefore $\tau \to 0$) and $\delta \to 0$, we obtain (6.54). □

Remark. We do not know if (6.54) holds if we do not assume $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} = \delta_0 > 0$. However, if for $\theta > 0$ we set $\mathcal{S}_\theta = \mathcal{S} \cap \{x \in \Omega : \text{dist}\,(x, \Omega^c) \geq \theta\}$, then we have

$$\overline{u}_{\mathcal{S}_\theta,\mu} - \underline{u}_{\mathcal{S}_\theta,\mu} \leq \overline{u}_{\mathcal{S}_\theta,0} - \underline{u}_{\mathcal{S}_\theta,0}. \tag{6.70}$$

Furthermore all the four above functions increases when θ decreases to 0. If we set

$$\begin{array}{ll} (i) & \lim_{\theta\to 0} \overline{u}_{\mathcal{S}_\theta,\mu} = \overline{u}_{\underline{\mathcal{S}},\mu} \\ (ii) & \lim_{\theta\to 0} \overline{u}_{\mathcal{S}_\theta,0} = \overline{u}_{\underline{\mathcal{S}},0} \\ (iii) & \lim_{\theta\to 0} \underline{u}_{\mathcal{S}_\theta,\mu} = \underline{u}_{\underline{\mathcal{S}},\mu} \\ (iv) & \lim_{\theta\to 0} \underline{u}_{\mathcal{S}_\theta,0} = \underline{u}_{\underline{\mathcal{S}},0}, \end{array} \tag{6.71}$$

then we infer that

$$\overline{u}_{\underline{\mathcal{S}},\mu} - \underline{u}_{\underline{\mathcal{S}},\mu} \leq \overline{u}_{\underline{\mathcal{S}},0} - \underline{u}_{\underline{\mathcal{S}},0}. \tag{6.72}$$

□

Our final result is the following existence and uniqueness theorem the proof is close to the one of [7, Th 3.5], therefore we present only the main ideas and the needed changes.

Theorem 6.15 *Assume $\Omega \subseteq \mathbb{R}^N$ is either $\mathbb{R}^N$ or an open domain with a C^2 compact boundary, $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Then for any $\nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega)$ such that $\rho\mu$ is bounded in any neighborhood of $\partial\Omega$ and $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$, the set $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$ contains one and only one element.*

Proof. We assume that $\mathcal{S} \neq \{\emptyset\}$ otherwhile uniqueness is already known. Thanks to (6.54) it is enough to prove that $\overline{u}_{\mathcal{S},0} = \underline{u}_{\mathcal{S},0}$.

Case 1: $\Omega = \mathbb{R}^N$. For $\sigma > 0$ and $a \in \mathcal{S}$ set $\mathcal{P}_\sigma(a) = \{(x,t) \in Q_\infty : |x - a| \leq \sigma\sqrt{t}\}$ and $\mathcal{P}_\sigma = \cup_a \in \mathcal{P}_\sigma(a)$. Because of (4.1), (5.6), (6.3) and (5.12), for any $\sigma > 0$, there exists $C_\sigma > 1$ such that

$$\overline{u}_{\mathcal{S},0}(x,t) \leq C_\sigma \underline{u}_{\mathcal{S},0}(x,t) \qquad \forall (x,t) \in \mathcal{P}_\sigma. \tag{6.73}$$

Fix $\sigma > 0$. If $y \in \mathbb{R}^N \setminus \mathcal{S}$, we set $r(y) = \mathrm{dist}\,(y, \mathcal{S})$. Using Proposition 3.2 and estimate (3.12), we obtain that

$$\overline{u}_{\mathcal{S},0}(x,t) \leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\frac{r(y) - |x-y|}{\sqrt{t}}\right) \qquad \forall (x,t) \in Q_\infty^{B_{r(y)}(y)}. \tag{6.74}$$

If we take $x = y$ in the above estimate, we derive from (3.3),

$$\overline{u}_{\mathcal{S},0}(y,t) \leq C_1 \left(\frac{r(y)}{\sqrt{t}}\right)^{2\frac{1+\alpha}{q-1}} e^{-\frac{r^2(y)}{4t}}(1 + o(1)). \tag{6.75}$$

Since there exists $z \in \mathcal{S} \cap \partial B_{r(y)}(y)$, we have also from (6.3) and (5.6)

$$\underline{u}_{\mathcal{S},0}(y,t) \geq C_2 \left(\frac{r(y)}{\sqrt{t}}\right)^{2\frac{1+\alpha}{q-1} - N} e^{-\frac{r^2(y)}{4t}}(1 + o(1)). \tag{6.76}$$

Thus there exists $C_3 > 1$ such that

$$\overline{u}_{\mathcal{S},0}(y,t) \leq \underline{u}_{\mathcal{S},0}(y, C_3 t). \tag{6.77}$$

For $\tau > 0$, we denote by $u_{1,\tau}$ and $u_{2,\tau}$ the solutions of (1.6) in $Q_{\tau,\infty} := \mathbb{R}^N \times (\tau, \infty)$ with respective initial data

$$\begin{aligned} u_1(x,\tau) &= C_\sigma \underline{u}_{\mathcal{S},0}(x,\tau)\chi_{\mathcal{P}_\sigma}(x,\tau) \\ u_2(x,\tau) &= \underline{u}_{\mathcal{S},0}(x, C_3\tau)(1 - \chi_{\mathcal{P}_\sigma}(x,\tau)). \end{aligned} \tag{6.78}$$

Then $u_1 + u_2$ is a supersolution of (1.6) in $Q_{\tau,\infty}$. Therefore

$$(u_1 + u_2)(x,\tau) \geq \overline{u}_{\mathcal{S},0}(x,\tau)\ \forall x \in \mathbb{R}^N \Longrightarrow (u_1 + u_2)(x,t) \geq \overline{u}_{\mathcal{S},0}(x,t)\ \forall (x,t) \in Q_{\tau,\infty}. \tag{6.79}$$

Since $C_\sigma \underline{u}_{\mathcal{S},0}$ is a supersolution of (1.6) which is larger than u_1 at $t = \tau$, then $u_1 \leq C_\sigma \underline{u}_{\mathcal{S},0}$ in $Q_{\tau,\infty}$. Next the function $(x,t) \mapsto w(x,t) := \underline{u}_{\mathcal{S},0}(x, t + (C_3 - 1)\tau)$ satisfies

$$\begin{aligned} \partial_t w - \Delta w + h(t,\tau)t^\alpha w^q &= 0 && \text{in } Q_{\tau,\infty} \\ w(.,\tau) &= \underline{u}_{\mathcal{S},0}(., C_3\tau) && \text{in } \mathbb{R}^N \end{aligned}$$

where

$$h(t,\tau) = \left(\frac{t+(C_3-1)\tau}{t}\right)^{\alpha}.$$

Then

$$h(t,\tau) \le C_4 := \max\{1, C_3^{\alpha}\} \qquad \forall t \ge \tau.$$

This implies that $C_4^{1/(q-1)}w$ is a supersolution of (1.6) in $Q_{\tau,\infty}$ which is larger than u_2 for $t=\tau$. Thus it dominates u_2 in $Q_{\tau,\infty}$. Combining the above estimates on u_1 and u_2 with (6.79), we derive that for any $\tau > 0$, there holds

$$\overline{u}_{\mathcal{S},0}(x,t) \le C_5\left(\underline{u}_{\mathcal{S},0}(x,t+(C_3-1)\tau) + \underline{u}_{\mathcal{S},0}(x,t)\right) \qquad \forall (x,t) \in Q_{\tau,\infty}. \tag{6.80}$$

where $C_5 = \max\left\{C_4^{1/(q-1)}, C_\sigma\right\}$. Letting $\tau \to 0$ we finally obtain the key estimate with $C = 2C_5$,

$$\overline{u}_{\mathcal{S},0} \le C\underline{u}_{\mathcal{S},0} \quad \text{ in } Q_\infty. \tag{6.81}$$

Then end of the proof is the same as in [7, Th 3.5], but we recall it for the sake of completeness: If $\overline{u}_{\mathcal{S},0} \ne \underline{u}_{\mathcal{S},0}$, then strict inequality holds. For $0 < \beta < C^{-1}$ the function $\tilde{u} = \underline{u}_{\mathcal{S},0} - \alpha(\overline{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0})$ is a supersolution of (1.6) in Q_∞ (this due to the convexity of $r \mapsto r^q$) which satisfies $\beta\underline{u}_{\mathcal{S},0} \le \tilde{u} < \underline{u}_{\mathcal{S},0}$. For $0 < \gamma < \beta$, the function $\gamma\underline{u}_{\mathcal{S},0}$ is a subsolution smaller than $\tilde{u}$. Then there exists a solution u' of (1.6) in Q_∞ which satisfies

$$\gamma\underline{u}_{\mathcal{S},0} \le u' \le \tilde{u} < \underline{u}_{\mathcal{S},0}. \tag{6.82}$$

This implies that $tr_{\mathbb{R}^N}(u') = (\mathcal{S},0)$, which contradicts the minimality of $\underline{u}_{\mathcal{S},0}$.

Case 2: $\partial\Omega$ is nonempty and compact. Again the proof is similar to the one of [7, Th 3.5]. We denote by $\underline{u}^{\Omega}_{\mathcal{S},0}$ and $\overline{u}^{\Omega}_{\mathcal{S},0}$ the minimal and the maximal solutions of (1.6) in Q^{Ω}_{∞} with initial trace $(\mathcal{S},0)$ and we assume that $\mathrm{dist}\,(\mathcal{S},\Omega^c) = \delta > 0$. We also set $\underline{u}^{\mathbb{R}^N}_{\mathcal{S},0} = \underline{u}_{\mathcal{S},0}$ and $\overline{u}^{\mathbb{R}^N}_{\mathcal{S},0} = \overline{u}_{\mathcal{S},0}$. Clearly $\overline{u}^{\Omega}_{\mathcal{S},0} \le \overline{u}_{\mathcal{S},0}$ in Q^{Ω}_{∞}. Furthermore, if we denote by $k(t)$ the maximum of $\overline{u}_{\mathcal{S},0}(x,t)$ for $x \in \partial\Omega$, then $\lim_{t\to 0} k(t) = 0$ (this is due to the fact that $\mathrm{dist}\,(\mathcal{S},\Omega^c) > 0$). Clearly, the construction of the minimal solutions shows that $\underline{u}_{\mathcal{S},0}(x,t) \le \underline{u}^{\Omega}_{\mathcal{S},0}(x,t) + k(t)$ in Q^{Ω}_{∞}. Therefore

$$\overline{u}^{\Omega}_{\mathcal{S},0}(x,t) \le \underline{u}^{\Omega}_{\mathcal{S},0}(x,t) + k(t) \quad \forall (x,t) \in Q^{\Omega}_{\infty}. \tag{6.83}$$

If we fix $\tau > 0$, $t \mapsto \underline{u}^{\Omega}_{\mathcal{S},0}(x,t) + k(t)$ is a supersolution of (1.6) in Q^{Ω}_{∞} which is larger that $\overline{u}^{\Omega}_{\mathcal{S},0}(x,t)$ at $t=\tau$. This implies

$$\overline{u}^{\Omega}_{\mathcal{S},0}(x,t) \le \underline{u}^{\Omega}_{\mathcal{S},0}(x,t) + k(\tau) \quad \forall (x,t) \in Q^{\Omega}_{\tau,\infty}. \tag{6.84}$$

If we let $\tau \to 0$, we deduce that $\overline{u}^{\Omega}_{\mathcal{S},0} = \underline{u}^{\Omega}_{\mathcal{S},0}$. □

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