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Abstract We study the initial trace problem for positive solutions of semilinear heat equations with strong absorption. We amphasize the particular case where u satisfies $\partial u - \Delta u + t^\alpha u^q = 0$, with $q > 1$ and $\alpha > -1$.

1 Introduction

In this paper we study the initial trace problem for positive solutions of

$$\partial_t u - \Delta u + g(x,t,u) = 0 \text{ dans } \ Q_T^\Omega := \Omega \times (0,T) \tag{1.1}$$

where Ω is a smooth domain in $\mathbb{R}^N$ containing O and $g \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R})$, $g(x,t,.)$ is nondecreasing $\forall (x,t) \in \Omega \times \mathbb{R}$ and satisfies $rg(x,t,r) \geq 0$ for $r \in \mathbb{R}$. Our first result states the existence of an ini.

Theorem A *Assume g satisfies the above conditions and that equation* (1.1) *possesses a barrier at any $z \in \Omega$. If $u \in C^1(Q_T^\Omega)$ is a positive solution of (1.1), it admits an initial trace which belongs to the class of outer regular positive Borel measure in Ω.*

The barrier assumption will be made precise later on in full generality. It is fulfilled if $g(x,t,r) = h(x)t^\alpha |r|^{q-1} r$ with $\alpha > -1$, $q > 1$ and $h \in L^\infty_{loc}(\Omega)$ satisfies $\inf \operatorname{ess} h > 0$ for any compact subset $K \subset \Omega$, or if g satisfies the Keller-Osserman condition, that is $g(x,t,r) \geq h(r) \geq 0$ where h is nondecreasing and

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} \qquad \text{where } H(s) = \int_0^s h(t)dt. \tag{1.2}$$

The initial trace of positive solutions of (1.1) exists in the following sense: *there exists a relatively closed set $\mathcal{S} \subset \Omega$ and a positive Radon measure μ on $\mathcal{R} := \Omega \setminus \mathcal{S}$ with the following properties:*
(i) for any $x_0 \in \mathcal{S}$ and any $\epsilon > 0$

$$\lim_{t \to 0} \int_{B_\epsilon(x_0) \cap \Omega} u(x,t)dx = \infty, \tag{1.3}$$

(ii) for any $\zeta \in C_c(\mathcal{R})$

$$\lim_{t\to 0}\int_\Omega u(x,t)\zeta(x)dx = \int_\Omega \zeta d\mu. \tag{1.4}$$

The couple $(\mathcal{S},\mu)$ is unique and characterizes a unique positive outer regular Borel measure ν on Ω.

A similar notion of boundary trace has been introduced by Marcus and Véron [10] in the study of positive solutions of

$$-\Delta u + g(x,u) = 0 \text{ in } \Omega. \tag{1.5}$$

This notion in itself has turned out to be a very usefull tool for classifying the positive solutions of (1.5).

In the second part we concentrate on the particular case of equation

$$\partial_t u - \Delta u + t^\alpha |u|^{q-1} u = 0 \text{ dans } Q_T^\Omega \tag{1.6}$$

where $T>0$, $\alpha > -1$ and $q>1$. Among the main useful tools, are the positive solutions with an isolated singularity at $(0,O)$: they are solutions u of (1.6) in Q_T^Ω, which belong to $C^{2,1}\left(Q_T^\Omega\right) \cap C\left(\Omega\times[0,T)\backslash\{(O,0)\}\right)$ and satisfy

$$u(x,0) = 0 \text{ in } \Omega\backslash\{O\}. \tag{1.7}$$

When $\alpha = 0$, Brezis and Friedman prove in [2] that if $B_{2R}\subset\Omega$, then any such solution satisfies

$$u(x,t) \le \frac{C(N,q,R)}{\left(|x|^2+t\right)^{\frac{1}{q-1}}} \quad \forall (x,t)\in B_R\backslash\{0\}\times[0,T]. \tag{1.8}$$

They also prove that if $1<q<q_N := 1+\frac{2}{N}$ and $k>0$ there exist singular solutions with initial data $u(.,0) = k\delta_O$, unique if u vanishes on $\partial\Omega\times[0,T]$. In this range of exponents, Brezis, Peletier, Terman obtain in [1] the existence and uniqueness of a *very singular solution* of (1.6), always with $\alpha = 0$: it is a positive solution in Q_∞ under the form

$$v_0(x,t) = t^{-1/(q-1)} V_0\left(\frac{x}{\sqrt{t}}\right),$$

where $V_0>0$ is C^2 and satisfies

$$\begin{array}{c} -\Delta V_0 - \frac{1}{2}\eta.\nabla V_0 - \frac{1}{q-1}V_0 + V_0^q = 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|\eta|\to\infty} |\eta|^{\frac{2}{q-1}} V_0(\eta) = 0. \end{array} \tag{1.9}$$

Actually, Kamin and Peletier show that v_0 is the limit of the solutions u_k of (1.6) in Q_∞ which satisfy $u(.,0) = k\delta_O$. The very singular singular solution plays a fundamental role in Marcus and Véron's description [7] of the initial trace of positive solutions of (1.6) with $\alpha = 0$. In [9], Marcus et Véron study this equation when $\alpha \ge 0$ and $1<q<q_{\alpha,N} = 1+\frac{2(1+\alpha)}{N}$. They obtain the existence of a self-similar solution of (1.6) under the form

$$v_\alpha(x,t) = t^{-\frac{1+\alpha}{q-1}} V_\alpha\left(\frac{x}{\sqrt{t}}\right),$$

which satisfies

$$\lim_{t\to 0} v_\alpha(x,t) = 0 \quad \forall x \neq 0$$

and

$$\lim_{t\to 0} \int_{B_\epsilon} v_\alpha(x,t)dx = \infty \quad \forall \epsilon > 0.$$

The function V_α is nonnegative and verifies

$$-\Delta V_\alpha - \frac{1}{2}\eta.\nabla V_\alpha - \frac{1+\alpha}{q-1}V_\alpha + V_\alpha^q = 0 \text{ in } \mathbb{R}^N. \tag{1.10}$$

Furthermore

$$V_\alpha(\eta) = C|\eta|^{\frac{2(1+\alpha)}{q-1}-N} e^{\frac{-|\eta|^2}{4}} (1+o(1)) \text{ as } |\eta| \to \infty. \tag{1.11}$$

If $1 < q < q_{\alpha,N}$, they show that for every $k > 0$ there exists a unique solution u_k of (1.6) with initial data $k\delta_O$. Furthermore $\lim_{k\to\infty} u_k = v_\alpha$. Actually the limitation $\alpha \geq 0$ can be relaxed to $\alpha > -1$ has we will see it later on.

In this article we extend Brezis-Friedman removability result to equation (1.6) and prove that if $q \geq q_{\alpha,N}$, any solution in Q_T which satisfies (1.7) is identically 0. The starting point of our study is the following extension of estimate (1.8) valid for any $\alpha > -1$ and $q > 1$.

$$u(x,t) \leq \frac{C\left(N,q,\alpha,R\right)}{\left(|x|^2+t\right)^{\frac{1+\alpha}{q-1}}} \quad \forall (x,t) \in B_R\backslash\{0\} \times [0,T]. \tag{1.12}$$

The obstacle for obtaining such an estimate arises when $\alpha > 0$ and the absorption term $t^\alpha u^q$ is degenerate near $t = 0$. We overcome this difficulty by a delicate construction of 1-dim self-similar supersolutions. Thanks to this estimate, we obtain that the following classification results.

Theorem B *Assume $1 < q < q_{\alpha,N}$ and $u \in C^1(Q_T^\Omega) \cap C(\Omega \times [0,T]\backslash\{(O,0))\})$ is a solution of (1.6) which vanishes on $\Omega \setminus \{O\}$ at $t = 0$. Then*

(i) either there exists $k \geq 0$ such that $u(.,0) = k\delta_O$ and

$$u(x,t) \sim kE(x,t) \qquad \text{as } (x,t) \to (O,0), \tag{1.13}$$

where $E(x,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$,

(ii) or

$$u(x,t) \sim v_\alpha(x,t) \qquad \text{as } (x,t) \to (O,0). \tag{1.14}$$

In the supercritical case the following removability statement holds.

Theorem C *Assume $q \geq q_{\alpha,N}$ and $u \in C^1(Q_T^\Omega) \cap C(\Omega \times [0,T]\backslash\{(O,0))\})$ is a positive solution of (1.6) which vanishes on $\Omega \setminus \{O\}$ at $t = 0$. Then u can be extended by continuity as a function in $C(\Omega \times [0,T])$.*

We prove that equation (1.6) admits a barrier at any $z \in \Omega$. More precisely we construct a positive solution w_{B_R} of (1.6) in $Q_\infty^{B_R}$ which tends to 0 locally uniformly in B_R when

$t \to 0$ and which blows-up uniformly on $\partial B_R \times [\tau, \infty)$, for any $\tau > 0$. Applying Theorem A, we infer that any positive solution admits an initial trace which is an outer regular Borel measure $\nu \approx (\mathcal{S}, \mu)$. Using sharp parabolic Harnack inequality and a concentration principle, we prove the following result which is the key-stone for analyzing the behaviour of u on the set $\mathcal{S}$.

Theorem D *Assume $1 < q < q_{\alpha,N}$ and $u \in C^2(Q_T^\Omega)$ is a positive solution of (1.6) with initial trace $(\mathcal{S}, \mu)$. Then for any $a \in \mathcal{S}$ there holds*

$$u(x,t) \geq u_{\infty,a}(x,t) \qquad \forall (x,t) \in Q_T^\Omega \tag{1.15}$$

where $u_{\infty,a} = \lim_{k\to\infty} u_{k\delta_a}$ where $u_{k\delta_a}$ is the solution of (1.6) in Q_T^Ω with initial trace $k\delta_a$ and which vanishes on $\partial\Omega \times [0,T]$.

It is important to notice that the behaviour of $u_{\infty,a}$ near $(a,0)$ is given by (1.14) and (1.11). Using sharp asymptotics of the function V_α, we have the following result which extends Theorem A.

Theorem E *Assume $1 < q < q_{\alpha,N}$ and $\Omega \subset \mathbb{R}^N$ is open with a C^2 compact boundary, eventually empty. Then for any couple $(\mathcal{S}, \mu)$ where $\mathcal{S} \subset \Omega$ is relatively closed and $\mu \in \mathfrak{M}_+(\Omega \setminus \mathcal{S})$, there exists a maximal and a minimal $u \in C^1(Q_T^\Omega) \cap C^1(\overline{\Omega} \times (0,T])$ which both satisfy (i) and (ii) and vanish on $\partial\Omega \times (0,T]$. If $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$ and μ is bounded in a neighborhood of $\partial\Omega$, then u is unique.*

2 Initial trace

In this section $\Omega \subset \mathbb{R}^N$ is an open set, $Q_T^\Omega = \Omega \times (0,T)$, $\partial_\ell Q_T^\Omega = \Omega \times \{0\} \cup \partial\Omega \times [0,T)$ and $g \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R})$. If u is defined in $\Omega \times \mathbb{R}_+$, we denote by $g \circ u$ the function $(x,t) \mapsto g(x,t,u(x,t))$. We say that g belongs to $\mathcal{H}$ (resp $\mathcal{H}_0$) if

$$\begin{aligned} &g(x,t,r) \geq 0 \quad \forall (x,t,r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \\ &(\text{resp. } g \in \mathcal{H} \text{ and } r \mapsto g(x,t,r) \text{ is nondecreasing}). \end{aligned} \tag{2.1}$$

We denote by $\mathfrak{M}(\Omega)$ the set of Radon measures in Ω, and by $\mathfrak{M}^b(\Omega)$ (resp. $\mathfrak{M}^{b,\rho}(\Omega)$) the subset of Radon measures such that

$$\int_\Omega d\,|\mu| < \infty \quad \left(\text{resp. } \int_\Omega \rho d\,|\mu| < \infty\right),$$

where $\rho(x) := \operatorname{dist}(x, \partial\Omega)$. Their positive cones are respectively $\mathfrak{M}_+(\Omega)$, $\mathfrak{M}_+^b(\Omega)$ and $\mathfrak{M}_+^{b,\rho}(\Omega)$.

Definition 2.1 *Let $\mathcal{S}$ be a relatively closed subset of Ω and μ a Radon measure on $\mathcal{R} := \Omega \setminus \mathcal{S}$. We say that a nonnegative function $u \in C(Q_T^\Omega)$ admits the couple $(\mathcal{S}, \mu)$ for initial trace if*

$$\lim_{t\to 0} \int_{\mathcal{R}} u(x,t)\zeta(x)dx = \int_{\mathcal{R}} \zeta d\mu \qquad \forall \zeta \in C_c(\mathcal{R}),$$

and

$$\lim_{t\to 0}\int_U u(x,t)dx=\infty \qquad \forall U\subset\Omega, U \text{ open}, U\cap\mathcal{S}\neq\emptyset$$

The set $\mathcal{S}$ is the set of singular initial points of u and its complement $\mathcal{R}$ the set of regular initial points. We write $tr_\Omega(u)=(\mathcal{S},\mu)$.

Let $\tilde{\mu}$ be the extension of μ as a locally bounded Borel measure. To the couple $(\mathcal{S},\tilde{\mu})$ we can associate a unique Borel measure ν defined by

$$\nu(E)=\begin{cases}\infty & \forall E\subset\Omega: E \text{ Borel}, E\cap\mathcal{S}\neq\emptyset\\ \tilde{\mu}(E) & \forall E\subset\Omega: E \text{ Borel}, E\subset\mathcal{R}.\end{cases} \tag{2.2}$$

Lemma 2.1 *Assume Ω is a bounded open set with a C^2 boundary, $T>0$, $g\in\mathcal{H}$, and let $u\in C(\overline{\Omega}\times(0,T])$ be a positive solution of*

$$\partial_t u-\Delta u+g\circ u=0 \qquad \text{in } Q_T^\Omega. \tag{2.3}$$

If $g\circ u\in L^1_\rho(Q_T^\Omega)$, then $u\in L^\infty\left(0,T,L^1_\rho(\Omega)\right)$ and there exists $\mu\in\mathfrak{M}^\rho_+(\Omega)$ such that

$$\lim_{t\to 0}\int_\Omega u(x,t)\zeta(x)dx=\int_\Omega\zeta d\mu \qquad \forall\zeta\in C_c(\Omega). \tag{2.4}$$

Proof. If $\phi_1>0$ is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ and λ_1 is the corresponding eigenvalue, we have

$$\frac{d}{dt}\int_\Omega u\phi_1dx+\lambda_1\int_\Omega u\phi_1dx+\int_\Omega g\circ u\,\phi_1dx+\int_{\partial\Omega}u\frac{\partial\phi_1}{\partial\nu}dS=0,$$

where ν is the normal vector. Set $X=\displaystyle\int_\Omega u\phi_1dx$, then by Hopf Lemma,

$$X'+\lambda_1X+\int_\Omega g\circ u\,\phi_1dx\geq 0$$

which yields to

$$\frac{d}{dt}\left(e^{\lambda_1 t}X\right)+e^{\lambda_1 t}\int_\Omega g\circ u\,\phi_1dx\geq 0.$$

For $s\in(t,T)$

$$\frac{d}{dt}\left(e^{\lambda_1 t}X-\int_t^T e^{\lambda_1 s}\int_\Omega g\circ u\,\phi_1dxds\right)\geq 0,$$

which means that the mapping

$$t\to e^{\lambda_1 t}X-\int_t^T e^{\lambda_1 s}\int_\Omega g\circ u\,\phi_1dxds$$

is nondecreasing. Therefore

$$e^{\lambda_1 t}X-\int_t^T e^{\lambda_1 s}\int_\Omega g\circ u\,\phi_1dxds\leq e^{\lambda_1 T}X.$$

and finally

$$X \leq e^{\lambda_1(T-t)}X + e^{-\lambda_1 t}\int_t^T e^{\lambda_1 s}\int_\Omega g\circ u\,\phi_1 dxds$$

Since $\rho^{-1}\phi_1$ is positively bounded from above and from below, $u \in L^\infty\left(0,T,L^1_\rho(\Omega)\right)$ and there exists a sequence $\{t_n\}$ decreasing to 0 and a measure $\mu \in \mathfrak{M}^\rho_+(\Omega)$ such that

$$\lim_{t_n\to 0}\int_\Omega u(x,t_n)\zeta dx = \int_\Omega \zeta d\mu \qquad \forall \zeta \in C_c(\Omega).$$

If $\zeta \in C^2_c(\Omega)$ there holds

$$\int_\Omega u(x,t_n)\zeta dx = \int_{t_n}^T\int_\Omega \left(g\circ u\,\zeta - u\Delta\zeta\right)dxdt + \int_\Omega u(x,T)\zeta dx,$$

thus

$$\int_\Omega \zeta d\mu = \iint_{Q^\Omega_T}\left(g\circ u\,\zeta - u\Delta\zeta\right)dxdt + \int_\Omega u(x,T)\zeta dx.$$

This implies that μ is uniquely determined and $u(.,t)$ converges to μ in the weak sense of measures. □

Corollary 2.1 *Assume Ω is an open domain, $g \in \mathcal{H}$ and $u \in C^2(Q^\Omega_T)$ is a positive solution of (2.4). Suppose that for any $z \in \Omega$ there exists an open neighborhood $U \subset \Omega$ such that*

$$\int_0^T\int_U g\circ u dxdt < \infty.$$

Then $u(x,t) \in L^\infty\left(0,T,L^1_{loc}(\Omega)\right)$ and there exists a positive Radon measure μ on U such that

$$\lim_{t\to 0}\int_{\mathcal{R}} u(x,t)f(x)dx = \int_{\mathcal{R}} f d\mu \quad \forall f \in C_c(\mathcal{R}).$$

Proof. We apply the previous lemma in replacing U by a ball $B_\epsilon(z)$ and conclude by a partition of unity. □

The following class of nonlinearity has been introduced by Marcus and Véron [10] in order to study the boundary trace of solutions of elliptic equations

Definition 2.2 *A function $g \in \mathcal{H}$ is a coercive nonlinearity in Q^Ω_T if, for every subdomain Ω' of Ω and every $\epsilon \in (0,T)$, the set of positive solutions of (1.1) in $Q^{\Omega'}_{\epsilon,T} := \Omega'\times(\epsilon,T)$ is uniformly bounded in compact subsets of $Q^{\Omega'}_{\epsilon,T}$.*

Definition 2.3 *Let $z \in \Omega$. We say that equation (1.1) possesses a strong barrier at z if there exists a number $r_0 \in (0,\rho(z))$ such that, for every $r \in (0,r_0)$, there exists a positive super solution $w = w_{r,z}$ of (1.1) in $B_r(z)\times(0,T)$ such that*

$$w \in C\left(B_r(z)\times[0,T)\right), \lim_{|x-z|\to r} w(x,t) = \infty \text{ locally uniformly if } t \in (0,T). \tag{2.5}$$

Lemma 2.1 *Assume $g \in \mathcal{H}$ is a coercive nonlinearity in Q_T^Ω, then the set of solutions of (1.1) in Q_T^Ω is uniformly bounded from above in every compact subset of Q_T^Ω. Furthermore, if $g \in \mathcal{H}_0$, $A \subset \Omega$ is open and (1.1) possesses a strong barrier at every point of $z \in A$, then the set of solutions u of (1.1) such that $u \in C(A \times [0,T))$ and $u(x,0) = 0$ on A is uniformly bounded from above in every compact subset of $A \times [0,T)$.*

Proof. Let K be a compact subset of Q_T^Ω and let Ω' be a smooth, bounded domain of Ω and $\epsilon > 0$ such that $K \subset Q_{\epsilon,T}^{\Omega'}$ Let $U = U_{Q_\infty^{\Omega'}}$ be the minimal large solution of (1.1) in $Q_T^{\Omega'}$, i.e. the limit, when $k \to \infty$, of solutions with Cauchy-Dirichlet data k on $\partial_\ell Q_{\epsilon,T}^{\Omega'} := \Omega' \times \{\epsilon\} \cup \partial\Omega' \times [\epsilon, T)$. By the maximum principle, if $u \in C(Q_T^\Omega)$ is a solution of (1.1), then $u \le U$ in Ω'.
For the second statement, let K be a compact subset of A. For any $z \in K$ there exists $r_z > 0$ such that for any $r \in (0, r_z)$ there exists a positive super solution of (1.1) in $Q_T^{B_r(z)}$ which satisfies (2.5). Since K is compact, there exist $z_1, ..., z_p$ such that $K \subset \cup_{j=1}^p B_{r_{z_j}/2}(z_j)$. For any $j \in \{1, ..., p\}$ we denote by w_j the super solution in $Q_T^{B_{2r_{z_j}/3}(z_j)}$. By comparison principle, there holds

$$u(x,t) \le \sup\{w_j(x,t) : (x,t) \in B_{r_{z_j}/2}(z_j) \times (0,T)\} := M_j, \tag{2.6}$$

for $(x,t) \in Q_T^{B_{r_{z_j}/2}(z_j)}$. Therefore $u \le M = \max_{j=1,...,p} M_j$ in $K \times (0,T)$. □

Lemma 2.2 *Let $g \in \mathcal{H}$ and $u \in C^2(Q_T^\Omega)$ be a positive solution of (1.1) and suppose $z \in \Omega$ is such that*

$$\int_0^T \int_{B_\epsilon(z) \cap \Omega} g \circ u dx dt = \infty \qquad \forall \epsilon > 0. \tag{2.7}$$

Suppose that at least one of the following sets of conditions holds:
(i) There exists an open neighborhood $U' \subset \Omega$ of z such that $u \in L^1(U' \times (0,T))$.
(ii) The following hold:
1- $g \in \mathcal{H}_0$,
2- (1.1) possesses a strong barrier at z.
Then,

$$\lim_{t \to 0} \int_{B_\epsilon(z) \cap \Omega} u(x,t) dx = \infty \qquad \forall \epsilon > 0. \tag{2.8}$$

Proof. Assume that Ω is bounded. First consider the case when condition (i). holds. Let $\phi \in C^{2,1}(U' \times [0,T))$ with compact support in $U' \times [0,T)$ and such that $\phi(x,0) = 1$ in a neighborhood of z. Then

$$\int_t^T \int_{U'} (u(-\phi_t - \Delta\phi) + g \circ u\, \phi)\, dx dt = \int_{U'} u(t)\phi dx - \int_{U'} u(T)\phi dx. \tag{2.9}$$

By assumption $\int_t^T \int_{U'} u(\phi_t + \Delta\phi) dx dt$ is bounded. We let t tend to 0, the result follows from (2.7).

Next we assume that condition (ii) holds, $u \notin L^1(U' \times (0,T))$ for any neighborhood U' of z and that the conclusion is not valid. Thus there exist $r^* > 0$, such that $\overline{B}_{r^*}(z) \subset U_z$ and a sequence $\{t_n\}$ decreasing to 0 such that

$$\int_{B_{r^*}(z)} u(x,t_n)dx \leq M$$

for some $M > 0$. Furthermore g is coercive in $B_{r^*}(z) \times (0,T)$. Let $\{h_{n,k}\} \subset C^\infty(Q_T^\Omega)$ an increasing sequence with respect to k and n of nonnegative functions such that $h_{n,k} = 0$ on $B_{r^*}(z) \times \{0\}$, $0 \leq h_{n,k} \leq k$ and $h_{n,k} = k$ on $(t_n, T) \times \partial B_{r^*}(z)$. Let $w_{h_{n,k}}$ be the solution of (1.1) in $B_{r^*}(z) \times (0,T)$ such that $w_{h_{n,k}} = h_{n,k}$ on $\partial_\ell Q_T^{B_{r^*}(z)}$. By the maximum principle and condition (ii)-1, the sequence $\{w_{h_{n,k}}\}$ is monotone increasing with respect to k and n. Condition (ii)-2 implies that, for every $r < r^*$ and $\beta < T$, the sequence is bounded in $\overline{B}_r(z) \times [0,\beta]$, and since u is locally bounded in Q_T^Ω there exists $k = k(n)$ such that $k \geq u$ on $(t_n, T) \times \partial B_r(z)$ and $k(n) \to \infty$ when $n \to \infty$. Then $w = \lim_{n\to\infty} w_{h_{n,k}}$ is a solution of (1.1) which blows up on $\partial B_{r^*}(z) \times (0,T)$ and vanishes on $B_{r^*}(z) \times \{0\}$. Let v_n be the solution of the heat equation in $B_{r^*}(z) \times (t_n, T)$ such that $v_n(.,t_n) = u(.,t_n)$ in $B_{r^*}(z)$ and $v_n = 0$ on $\partial B_{r^*}(z) \times (t_n, T)$. Then $w_{h_{n,k(n)}} + v_n$ is a super solution of (1.1) in $B_{r^*}(z) \times (t_n, T)$ which dominates u on $\partial_\ell Q_{t_n,T}^{B_{r^*}(z)}$. By the maximum principle,

$$u \leq w_{h_{n,k(n)}} + v_n \qquad \text{in } Q_{t_n,T}^{B_{r^*}(z)}.$$

And we have in particular

$$\int_{B_r(z)} u(x,t)dx \leq \int_{B_r(z)} \left(w_{h_{n,k(n)}} + v_n\right)(x,t)dx \leq M + \int_{B_r(z)} w(x,t)dx \qquad \forall t \in (t_n, T).$$

Since it holds for any n, it implies $u \in L^1(Q_T^{B_r(z)})$, which leads to a contradiction. □

Example 1. If $g(x,t,r) = h(r)$ where h is continuous, nondecreasing and $h(0) \geq 0$, then

$$u_t - \Delta u + h(u) = 0 \tag{2.10}$$

possesses a strong barrier at any $z \in \Omega$ if and only if h satisfies the Keller-Osserman condition, that is there exists some $a \geq 0$ such that

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \text{ where } H(s) = \int_0^s h(\tau)d\tau. \tag{2.11}$$

The super solution can be chosen to be the maximal solution ϕ_r of the elliptic equation

$$-\Delta\phi + h(\phi) = 0 \qquad \text{in } B_r(z). \tag{2.12}$$

If we assume moreover that h is super-additive, i.e. $h(a+b) \geq h(a) + h(b)$ for all $a, b \geq 0$, then there holds

$$\int_a^\infty \frac{ds}{h(s)} < \infty, \tag{2.13}$$

and any solution u of (2.10) is dominated in $Q_T^{B_r(z)}$ by $\phi_r(x)+\psi(t)$ where ψ is defined by inversion from

$$\int_{\psi(t)}^{\infty}\frac{ds}{h(s)}=t\quad\forall t>0.$$

Example 2. If $g(x,t,r)=a(x)b(t)h(r)$ where $a\in C(\Omega)$, $b\in C((0,T))$, $a,b>0$, then g is a coercive nonlinearity if h is super-additive and satisfies the Keller-Osserman condition. This is not sufficient for the existence of a barrier as it is shown in [9] with $h(r)=r^q$ $(q>1)$ $a\equiv 1$ and $b(t)=e^{-\frac{1}{t}}$.

Proposition 2.2 *Let $g\in\mathcal{H}_0$ such that at any $z\in\Omega$ there exists a strong barrier. We assume also*

$$g(x,t,a)+g(x,t,b)\le g(x,t,a+b)\qquad\forall(x,t,a,b)\in Q_T^{\Omega}\times\mathbb{R}_+\times\mathbb{R}_+. \tag{2.14}$$

Let $\{u_n\}$ be a sequence of positive solutions of (1.1) which converges to u locally uniformly in Q_T^{Ω}. Denote by $tr_\Omega(u_n)=(\mathcal{S}_n,\mu_n)$ and $tr_\Omega(u)=(\mathcal{S},\mu)$ their respective initial trace. If $\mathcal{A}\subset\cap_n\mathcal{R}_n$ is open and if $\mu_n(\mathcal{A}_n)$ remains bounded independently of $n\in\mathbb{N}$, then $\mathcal{A}\subset\mathcal{R}:=\Omega\setminus\mathcal{S}$.

Proof. Let $z\in\mathcal{A}$ and $\tilde r\in(0,\rho(z))$ such that for any $r\in(0,\tilde r]$ there exists a positive super solution $w_{r,z}$ satisfying (2.5) and $\overline{B}_{\tilde r}(z)\subset\mathcal{A}$. For any $n\in\mathbb{N}$ and $\tau\in(0,T)$, we denote by $u_{\tau,\chi_{B_{\tilde r}(z)}\mu_n}$ the solution of

$$\begin{array}{rll}\partial_t u-\Delta u+g\circ u=0 & & \text{in } B_{\tilde r}(z)\times(\tau,T)\\ u(.,\tau)=\chi_{B_{\tilde r}(z)}u_n(.,\tau) & & \text{in } B_{\tilde r}(z)\\ u=0 & & \text{in } \partial B_{\tilde r}(z)\times(\tau,T)\end{array} \tag{2.15}$$

By the maximum principle $u_{\tau,\chi_{B_{\tilde r}(z)}\mu_n}\le u_n$ in $B_{\tilde r}(z)\times(\tau,T)$, and $g\circ u_{\tau,\chi_{B_{\tilde r}(z)}\mu_n}\le g\circ u_n$ Furthermore, if $\zeta\in C^{1,1;1}(Q_T^{\overline{B}_{\tilde r}(z)})$ vanishes on $\partial B_{\tilde r}(z)\times[0,T)$ and for $t=T$, there holds

$$\iint_{B_{\tilde r}(z)\times(\tau,T)}\left(-u_{\tau,\chi_{B_{\tilde r}(z)}\mu_n}(\partial_t\zeta+\Delta\zeta)+\zeta g\circ u_{\tau,\chi_{B_{\tilde r}(z)}\mu_n}\right)dxdt=\int_{B_{\tilde r}(z)}u_n(x,\tau)\zeta(x,\tau)dx \tag{2.16}$$

Since $u_{\tau,\chi_{B_{\tilde r}(z)}\mu_n}$ and $g\circ u_{\tau,\chi_{B_{\tilde r}(z)}\mu_n}$ are bounded independently of τ, standard regularity theory for parabolic equations implies that they converge a.e. in $B_{\tilde r}(z)\times(0,T)$ when $\tau\to 0$ to $u_{\chi_{B_{\tilde r}(z)}\mu_n}$ and $g\circ u_{\chi_{B_{\tilde r}(z)}\mu_n}$. Furthermore

$$\lim_{\tau\to 0}\int_{B_{\tilde r}(z)}u_n(x,\tau)\zeta(x,\tau)dx=\int_{B_{\tilde r}(z)}\zeta(x,0)d\mu_n(x).$$

Using the dominated convergence theorem, it follows from (2.15) that

$$\iint_{Q_T^{B_{\tilde r}(z)}}\left(-u_{\chi_{B_{\tilde r}(z)}\mu_n}(\partial_t\zeta+\Delta\zeta)+\zeta g\circ u_{\chi_{B_{\tilde r}(z)}\mu_n}\right)dxdt=\int_{B_{\tilde r}(z)}\zeta(x,0)d\mu_n(x), \tag{2.17}$$

and $u_{\chi_{B_{\tilde{r}}(z)}\mu_n}$ is the (unique) solution of

$$\begin{array}{rl} \partial_t u - \Delta u + g \circ u = 0 & \text{in } Q_T^{B_{\tilde{r}}(z)} \\ u(.,0) = \chi_{B_{\tilde{r}}(z)}\mu_n & \text{in } B_{\tilde{r}}(z) \\ u = 0 & \text{in } \partial B_{\tilde{r}}(z) \times (0,T). \end{array} \tag{2.18}$$

Furthermore, if η is the solution of the backward problem

$$\begin{array}{rl} \partial_t \eta + \Delta \eta = -1 & \text{in } Q_T^{B_{\tilde{r}}(z)} \\ \eta(.,T) = 0 & \text{in } B_{\tilde{r}}(z) \\ \eta = 0 & \text{in } \partial B_{\tilde{r}}(z) \times (0,T), \end{array} \tag{2.19}$$

there holds

$$\iint_{Q_T^{B_{\tilde{r}}(z)}} \left(u_{\chi_{B_{\tilde{r}}(z)}\mu_n} + \eta g \circ u_{\chi_{B_{\tilde{r}}(z)}\mu_n} \right) dxdt = \int_{B_{\tilde{r}}(z)} \eta(x,0) d\mu_n(x) \leq M, \tag{2.20}$$

for some $M > 0$ independent of n. Next we set $Z_{\tau,n} := u_{\tau \chi_{B_{\tilde{r}}(z)}\mu_n} + w_{\tilde{r},z}$. It is a super solution of (2.10) in $(\tau,T) \times B_{\tilde{r}}(z)$ which is infinite on $\partial B_{\tilde{r}}(z) \times [\tau,T)$ and dominates u_n in $B_{\tilde{r}}(z)$ at $t = \tau$. Thus $Z_{\tau,n} \geq u_n$ in $(\tau,T) \times B_{\tilde{r}}(z)$. Letting $\tau \to 0$ we finally obtain

$$u_{\chi_{B_{\tilde{r}}(z)}\mu_n}(x,t) \leq u_n(x,t) \leq u_{\chi_{B_{\tilde{r}}(z)}\mu_n}(x,t) + w_{\tilde{r},z}(x,t) \qquad \forall (x,t) \in Q_T^{B_{\tilde{r}}(z)}. \tag{2.21}$$

For any $r < \tilde{r}$ and $T' < T$, there exists $\delta, \sigma > 0$ such that $\eta(x,t) \geq \delta$ and $w_{\tilde{r},z}(x,t) \leq \sigma$ for all $(x,t) \in Q_{T'}^{B_r(z)}$. It follows from (2.20), (2.21) and Fatou's lemma that u and $g \circ u$ are integrable in $L^1(Q_{T'}^{B_r(z)})$. By Lemma 2.1, $B_{r'}(z) \subset \mathcal{R}$. Since it holds for any $z \in \mathcal{A}$, the result is proved. □

3 Construction of a barrier

In the next results we construct the barrier function

Lemma 3.1 *Assume $\alpha > -1$ and $q > 1$, then there exists a unique positive function $W_\alpha \in C^2([0,\infty))$ satisfying*

$$\begin{array}{c} W'' + \dfrac{r}{2} W' + \dfrac{1+\alpha}{q-1} W - W^q = 0 \quad \text{in } (0,\infty) \\ \lim_{r \to 0} W(r) = \infty \\ \lim\limits_{r \to \infty} r^{\frac{2}{q-1}} W(r) = 0. \end{array} \tag{3.1}$$

Furthermore W_α is decreasing and

$$W_\alpha(r) = C r^{\frac{2(1+\alpha)}{q-1} - 1} e^{-\frac{r^2}{4}} (1 + \circ(1)) \quad \text{as } r \to \infty. \tag{3.2}$$

Proof. Consider the functional

$$J(\phi) := \frac{1}{2}\int_0^\infty \left(\phi'^2 - \frac{1+\alpha}{q-1}\phi^2 + \frac{2}{q+1}\,|\phi|^{q+1}\right) e^{\frac{r^2}{4}} dr, \tag{3.3}$$

defined over the convex set

$$H_k := \{\phi \in W_2^1(0,\infty; e^{\frac{r^2}{4}} dr) \cap L^{q+1}(0,\infty; e^{\frac{r^2}{4}} dr) : \phi(0) = k\}.$$

Note that if $\phi \in H_k$,

$$\begin{aligned} e^{\frac{r^2}{4}}\phi^2(r) &= \int_r^\infty (e^{\frac{s^2}{4}}\phi^2(s))' ds \\ &= 2\int_r^\infty e^{\frac{s^2}{4}}\phi\phi'(s) ds + \tfrac{1}{2}\int_r^\infty s e^{\frac{s^2}{4}}\phi^2(s) ds \end{aligned}$$

In this set J admits a positive minimizer w_k which is the unique solution of

$$\begin{aligned} w'' + \frac{r}{2}w + \frac{1+\alpha}{q-1}w' - w^q = 0 \quad \text{in } (0,\infty) \\ w(0) = k. \end{aligned} \tag{3.4}$$

Furthermore, $w_k = \lim_{n\to\infty} w_{k,n}$ where $w_{k,n}$ is the unique positive solution of

$$\begin{aligned} w'' + \frac{r}{2}w' + \frac{1+\alpha}{q-1}w - w^q = 0 \quad \text{in } (0,n) \\ w(0) = k \\ w(n) = 0. \end{aligned} \tag{3.5}$$

and, by the maximum principle, $(k,n) \mapsto w_{k,n}$ is increasing. If we consider the linear equation

$$z'' + \frac{r}{2}z' + \frac{1+\alpha}{q-1}z = 0 \quad \text{in } (0,\infty) \tag{3.6}$$

it admits two linearly independent positive solutions z_1 and z_2 with the following asymptotic behaviour as $r \to \infty$

$$z_1(r) = r^{-\frac{2(1+\alpha)}{q-1}}(1 + \circ(1)) \tag{3.7}$$

and

$$z_2(r) = r^{\frac{2(1+\alpha)}{q-1}-1} e^{-\frac{r^2}{4}}(1 + \circ(1)) \tag{3.8}$$

(see [8, Appendix]. Since any solution of 3.4, and 3.5 as well, satisfies an a priori estimate of Keller-Osserman type (see [15])

$$w(r) \le C r^{-\frac{2}{q-1}} \qquad \text{for } 0 < r < 1, \tag{3.9}$$

there holds

$$w_{k,n} \le C z_2(r) \qquad \text{for } k \ge r \ge 1.$$

Letting n and k go to infinity successively, it follows that $W_\alpha = \lim_{k,n\to\infty} w_{k,n}$ exists. It is a positive solution of problem (3.1) and it satisfies

$$W_\alpha(r) \le C\left(r^{-\frac{2}{q-1}} + z_2(r)\right) \qquad \text{for } r > 0.$$

The singular behaviour at $r = 0$ is standard (see e.g. [15]) and yields to

$$W_\alpha(r) = \frac{2(q+1)}{(q-1)^2} r^{-\frac{2}{q-1}}(1 + \circ(1)) \qquad \text{as } r \to 0. \tag{3.10}$$

Thus uniqueness follows by the maximum principle and estimates (3.3) is obtained via standard linearization, using the upper estimate at infinity. □

In the sequel we set

$$w_\alpha(s,t) = t^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\frac{s}{\sqrt{t}}\right) \qquad \forall s > 0,\, t > 0. \tag{3.11}$$

Proposition 3.2 *Assume $\alpha > -1$ and $q > 1$. Then for any $R > 0$, there exists $C = C(q,\alpha,R) > 0$ such that any solution u of (1.6) in $Q_\infty^{B_R}$ which vanishes on $B_R \times \{0\}$ satisfies*

$$u(x,t) \le 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\tfrac{R-|x|}{\sqrt{t}}\right) \quad \forall (x,t) \in Q_\infty^{B_R}. \tag{3.12}$$

Proof. For $m > 0$, set $S_m = \{x = (x_1, ..., x_N) : |x_j| < m\}$. For $R' < R$

$$\tilde{w}_{R'}(x,t) = t^{-\frac{1+\alpha}{q-1}} \sum_{j=1}^{N} \left(W_\alpha\left(\tfrac{R'-x_j}{\sqrt{t}}\right) + W_\alpha\left(\tfrac{R'+x_j}{\sqrt{t}}\right)\right) \quad \forall (x,t) \in Q_\infty^{S_{R'}} \tag{3.13}$$

Then $\tilde{w}_{R'}$ is a super solution of (1.6) in $Q_\infty^{S_{R'}}$ which is infinite on $\partial S_{R'} \times (0,\infty)$, thus $u \le \tilde{w}_{R'}$. Letting $R' \to R$ yields to $u \le \tilde{w}_R$ in $Q_\infty^{S_R}$. Since the equation is invariant by rotation, for any $x \in B_R$, there is a rotation $\mathcal{R}$ such that $\mathcal{R}(x)$ has only a positive x_1- coordinate. Thus

$$\begin{aligned} u(x,t) &\le \tilde{w}_R(|x|,t) \\ &\le t^{-\frac{1+\alpha}{q-1}} \left(W_\alpha\left(\tfrac{R-|x|}{\sqrt{t}}\right) + (2N-1) W_\alpha\left(\tfrac{R}{\sqrt{t}}\right)\right) \\ &\le 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\tfrac{R-|x|}{\sqrt{t}}\right), \end{aligned} \tag{3.14}$$

which is (3.12). □

Proposition 3.3 *Assume $\alpha > -1$, $q > 1$ and $R > 0$. Then there exists a unique positive solution w_{B_R} of (1.6) in $Q_\infty^{B_R}$, continuous in $B_R \times [0,\infty)$, which vanishes on $B_R \times \{0\}$ and satisfies $\lim_{|x|\to R} w_{B_R}(x,t) = \infty$, locally uniformly in $(0,\infty)$. In particular*

$$t^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\tfrac{R-|x|}{\sqrt{t}}\right) \le w_{B_R}(x,t) \le 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\tfrac{R-|x|}{\sqrt{t}}\right) \quad \forall (x,t) \in Q_\infty^{B_R}. \tag{3.15}$$

Proof. For $k > 0$, let $w^k_{B_R}$ be the solution of

$$\begin{array}{rll} \partial_t u - \Delta u + t^\alpha u^q = 0 & & \text{in } Q_\infty^{B_R} \\ u = k & & \text{in } \partial_\ell Q_\infty^{B_R} \\ u(.,0) = 0 & & \text{in } B_R. \end{array} \tag{3.16}$$

By (3.12), $w^k_{B_R}(x,t) \le 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\frac{R-|x|}{\sqrt{t}}\right)$. There there exists $w_{B_R} = \lim_{k\to\infty} w^k_{B_R}$ and w_{B_R} is a solution of (1.6) in $Q_\infty^{B_R}$ which vanishes on $B_R \times \{0\}$ and is infinite on $\partial B_R \times (0,\infty)$. Consider the similarity transformation T_m which leaves equation (1.6) invariant

$$T_m[u](x,t) = m^{\frac{1+\alpha}{q-1}} u(\sqrt{m}x, mt) \qquad \forall m > 0,$$

then $T_m[w^k_{B_R}] = w^{m^{\frac{1+\alpha}{q-1}}k}_{B_{\frac{R}{\sqrt{m}}}}$ which implies

$$T_m[w_{B_R}] = w_{B_{\frac{R}{\sqrt{m}}}} \qquad \forall m > 0.$$

If $u \in C(B_R \times [0,\infty))$ is any positive solution of problem

$$\begin{array}{rll} \partial_t u - \Delta u + t^\alpha u^q = 0 & & \text{in } Q_\infty^{B_R} \\ \lim\limits_{|x|\to R} u(x,t) = \infty & & \text{locally uniformly on } (0,\infty) \\ u(.,0) = 0 & & \text{in } B_R, \end{array} \tag{3.17}$$

then for any $m > 1$ and $\epsilon > 0$, there exists $\tau_\epsilon > 0$ such that $u(x,t) \le \epsilon$ in $B_{\frac{R}{\sqrt{m}}}$ for $0 \le t \le \tau_\epsilon$. Therefore, for any $T > 0$,

$$u \le \epsilon + w_{B_{\frac{R}{\sqrt{m}}}} \qquad \text{on } \partial_\ell Q_T^{B_{\frac{R}{\sqrt{m}}}} \cup \overline{B_{\frac{R}{\sqrt{m}}}} \times \{0\}$$

Since $\epsilon + w_{B_{\frac{R}{\sqrt{m}}}}$ is a super solution $u \le \epsilon + w_{B_{\frac{R}{\sqrt{m}}}}$ in $Q_T^{B_{\frac{R}{\sqrt{m}}}}$. Letting $\epsilon \to 0$, $m \to 1$ and $T \to \infty$ yields to $u \le w_{B_R}$. In the same way, with $0 < m < 1$, we obtain $u \ge w_{B_R}$. □

The next result is an immediate consequence of Proposition 3.2.

Proposition 3.4 *Assume $\alpha > -1$, $q > 1$ and $K \subset \Omega$ is compact. Let u be any solution u of (1.6) in Q_∞^Ω which vanishes on $\Omega \setminus \{K\} \times \{0\}$ and on $\partial\Omega \times [0,\infty)$, then*

$$u(x,t) \le 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha\left(\frac{\text{dist}\,(x,K)}{\sqrt{t}}\right). \tag{3.18}$$

4 Upper estimates

We start with the following upper estimate already obtained by Shishkov and Véron [12] in the case $\alpha \ge 0$.

Proposition 4.1 *Let $q > 1$ and $\alpha > -1$. If u is a solution of (1.6) vanishing on $\partial\Omega \times [0,T)$, there holds*

$$u(x,t) \le c_\alpha t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x,t) \in Q_T^\Omega, \tag{4.1}$$

with $c_\alpha = \left(\frac{\alpha+1}{q-1}\right)^{\frac{1}{q-1}}$.

Proof. Let $\phi(t) = c_\alpha t^{-\frac{\alpha+1}{q-1}}$ be the maximal solution of

$$\begin{aligned} \phi' + t^\alpha \phi^q &= 0 \\ \phi(0) &= \infty. \end{aligned}$$

with $c_\alpha = \left(\frac{\alpha+1}{q-1}\right)^{\frac{1}{q-1}}$.

Case $\alpha \ge 0$. For $\tau > 0$, we denote by $\Phi_{1,\tau}$ the solution of

$$\begin{aligned} -\Delta \Phi_{1,\tau} + \tau^\alpha \Phi_{1,\tau}^q &= 0 \quad \text{in } B_1 \\ \lim_{|x|\to 1} \Phi_{1,\tau}(x) &= \infty, \end{aligned} \tag{4.2}$$

and for $R > 0$

$$\Phi_{R,\tau}(x) = R^{\frac{-2}{q-1}} \Phi_{1,\tau}\left(\frac{x}{R}\right).$$

Note that $\Phi_{R,\tau}(x)$ is the solution of the problem (4.2) in the ball B_R. The function $\Phi_{R,\tau}$ tends to 0 uniformly on every compact set of $\mathbb{R}^N$ when $R \to \infty$. Set

$$\tilde{v}(x,t) = \phi(t-\tau) + \Phi_{R,\tau}(x),$$

then $\tilde{v}$ is a supersolution of (1.6) in $B_R \times [\tau, T)$ which is infinite on $\partial B_R \times [\tau, T) \cup B_R \times \{0\}$. Then $u(x,t) \le \tilde{v}(x,t)$. Letting $R \to \infty$ and $\tau \to 0$, we obtain

$$u(x,t) \le c_\alpha t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x,t) \in Q_T.$$

Case $-1 < \alpha < 0$. Let $\tau > 0$ and $\phi_\tau(t) = c_\alpha \left(t^{\alpha+1} - \tau^{\alpha+1}\right)^{-\frac{1}{q-1}}$ be the solution of

$$\begin{aligned} \phi_\tau' + t^\alpha \phi_\tau^q &= 0 \quad \text{on } (\tau, \infty) \\ \phi_\tau(\tau) &= \infty \end{aligned}$$

If $\Phi_{1,T}$ is the solution of (4.2) with $\tau = T$, we set

$$\Phi_{R,T}(x) = R^{\frac{-2}{q-1}} \Phi_{1,T}\left(\frac{x}{R}\right).$$

Clearly $\Phi_{R,T}$ tends to 0 uniformly on every compact of $\mathbb{R}^N$ when $R \to \infty$. Set

$$\hat{v}(x,t) = \phi_\tau(t) + \Phi_{R,T}(x),$$

$\hat{v}$ is a supersolution of (1.6) in $B_R \times (\tau, T)$, thus $u(x,t) \le \hat{v}(x,t)$, as in the first case. Letting $R \to \infty$ and $\tau \to 0$, we obtain the desired estimate. □

Combining Proposition 3.2 and Proposition 4.1 we obtain,

Corollary 4.2 *Assume $q > 1$, $\alpha > -1$ and $K \subset \Omega$ is compact. If $u \in C^2(Q_T^\Omega) \cap C(\overline{Q}_T^\Omega \setminus K \times \{0\})$ is a solution of (1.6) which vanishes on $\partial\Omega \times [0,T) \cup \{(\Omega \setminus K) \times \{0\}\}$, there holds*

$$u(x,t) \le \min\left\{2NW_\alpha\left(\frac{\operatorname{dist}(x,K)}{\sqrt{t}}\right), c_\alpha\right\} t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x,t) \in Q_T^\Omega. \tag{4.3}$$

In the particular case where $K = \{O\}$, (4.3) yields to

$$u(x,t) \le \min\left\{2NW_\alpha\left(\frac{|x|}{\sqrt{t}}\right), c_\alpha\right\} t^{-\frac{\alpha+1}{q-1}} \le \frac{c_1}{\left(|x|^2+t\right)^{\frac{1+\alpha}{q-1}}} \quad \text{for all } (x,t) \in Q_T^\Omega, \tag{4.4}$$

for some $c_1 = c_1(\alpha, q) > 0$.

Remark. If Ω is replaced by $\mathbb{R}^N$, the previous estimates (4.1), (4.3) and (4.4) remain valid. Furthermore, K needs only to be closed.

5 Isolated singularities

In this section we present the results of classifications of isolated singularities of positive solutions of (1.6), always in the range $q > 1$ and $\alpha > -1$. Since the proofs are somewhat similar to the ones of classification of boundary singularities of positive solutions of

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T^\Omega \tag{5.1}$$

obtained by Marcus and Véron in [8], we only indicate their main ideas.

If we look for solution of (1.6) in $Q_T := Q_T^{\mathbb{R}^N}$ under the form

$$u(x,t) = t^\gamma V(\frac{x}{\sqrt{t}})$$

it is immediate that $\gamma = -\frac{1+\alpha}{q-1}$ and V is a solution of

$$-\Delta V - \frac{\eta}{2}.\nabla V - \frac{1+\alpha}{q-1}V + V^q = 0 \quad \text{in } \mathbb{R}^N. \tag{5.2}$$

It is proved by Escobedo and Kavian [4] that if $\frac{1+\alpha}{q-1} > \frac{N}{2}$, or equivalently if

$$1 < q < q_{c,\alpha} := 1 + \frac{2(1+\alpha)}{N}, \tag{5.3}$$

there exists a positive solution of (5.2) which is the minimizer of the functional

$$\omega \mapsto J(\omega) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla\omega|^2 - \frac{1+\alpha}{q-1}\omega^2 + \frac{2}{q+1}|\omega|^{q+1}\right)e^{\frac{|\eta|^2}{4}}d\eta. \tag{5.4}$$

This minimizing solution V_α is unique and radial. Furthermore, by adapting the results of [8, Th 2.1], it satisfies

$$V_\alpha(\eta) = C|\eta|^{2\frac{1+\alpha}{q-1}-N}e^{-\frac{|\eta|^2}{4}}(1+\circ(1)) \quad \text{as } |\eta| \to \infty. \tag{5.5}$$

The function

$$v_\alpha(x,t) = t^{-\frac{1+\alpha}{q-1}} V_\alpha\left(\frac{x}{\sqrt{t}}\right) \tag{5.6}$$

is a positive solution of (5.2) in Q_∞, continuous in $\overline{Q}_T \setminus \{(O,0)\}$, vanishes on $\mathbb{R}^N \times \{0\} \setminus \{(O,0)\}$ and satisfies

$$\lim_{t\to 0} \int_{B_\epsilon} v_\alpha(x,t)dx = \infty \qquad \forall \epsilon > 0. \tag{5.7}$$

It is called the *very singular solution of (1.6).*

When $\mathbb{R}^N$ is replaced by a a proper open domain Ω with a compact C^2 boundary there exists no self-similar solution to (1.6). For any $k > 0$ and $a \in \Omega$ there exists a unique solution $u := u_{k\delta_a}$ to the initial value problem

$$\begin{array}{rl} \partial_t u - \Delta u + t^\alpha u^q = 0 & \text{in } Q_\infty^\Omega \\ u = 0 & \text{on } \partial\Omega \times [0,\infty) \\ u(.,0) = k\delta_a & \text{in } \Omega. \end{array} \tag{5.8}$$

(see e.g. [9]). The function u belongs to $L^q(Q_T^\Omega; t^\alpha dxdt) \cap L^1(Q_T^\Omega)$, $T > 0$ arbitrary, and satisfies

$$\iint_{Q_T^\Omega} \left(-u\left(\partial_t\zeta + \Delta\zeta\right) + t^\alpha u^q \zeta\right) dxdt = k\zeta(a) \tag{5.9}$$

for all $\zeta \in C^{1,1;1}(\overline{Q}_T^\Omega)$ which vanishes on $\partial\Omega \times [0,T]$ and on $\Omega \times \{T\}$. It is unique in the class of of weak solutions, i.e. the functions belonging to $L^q(Q_T^\Omega; t^\alpha dxdt) \cap L^1(Q_T^\Omega)$ and satisfying the above identity. When $k \to \infty$, $u_{k\delta_a} \uparrow u_{\infty\delta_a}$, where $u_{\infty\delta_a} := u_{\infty,a}$ is a solution of (1.6) in Q_∞^Ω which vanishes on $\partial\Omega \times [0,\infty)$ and on $\overline{\Omega} \times \{0\} \setminus \{(a,0)\}$ and satisfies (5.7). Finally, if $E(x,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$ denotes the heat kernel in $\mathbb{R}^N$,

$$u_{k\delta_a} \sim kE(x-a,t) \qquad \text{when } (x,t) \to (a,0) \tag{5.10}$$

and

$$u_{\infty,a} \sim v_\alpha(x-a,t) \qquad \text{when } (x,t) \to (a,0). \tag{5.11}$$

The following classification of isolated singularities holds

Theorem 5.1 *Assume $\alpha > -1$, $1 < q < q_{c,\alpha}$ and $a \in \Omega$. If $u \in \overline{Q}_T^\Omega \setminus \{(a,0)\}$ is a positive solution of (1.6) which vanishes on $\Omega \times \{0\} \setminus \{(a,0)\}$, then*
(i) either there exists $k \geq 0$ such that

$$u(x,t) \sim kE(x-a,t) \qquad \textit{when } (x,t) \to (a,0), \tag{5.12}$$

and u is a solution of

$$\begin{array}{rl} \partial_t u - \Delta u + t^\alpha u^q = 0 & \textit{in } Q_T^\Omega \\ u(.,0) = k\delta_a & \textit{in } \Omega, \end{array} \tag{5.13}$$

(ii) or

$$u(x,t) \sim v_\alpha(x-a,t) \qquad \textit{when } (x,t) \to (a,0), \tag{5.14}$$

and u satisfies (5.7).

If $q \geq q_{c,\alpha}$ there holds the following result which extends Brezis and Freidman's classical one.

Theorem 5.2 *Assume $\alpha > -1$, $q \geq q_{c,\alpha}$ and $a \in \Omega$. If $u \in \overline{Q}_T^{\Omega} \setminus \{(a,0)\}$ is a positive solution of (1.6) which vanishes on $\Omega \times \{0\} \setminus \{(a,0)\}$, then it can be extended as a continuous function $\tilde{u}$ which vanishes on $\Omega \times \{0\}$.*

Proof. Up to modifying a few parameters the proof is similar to Brezis-Friedman's construction. The first step is to prove that $u \in L^q(Q_T^{B_R})$ for some $R > 0$. This is done by using (4.4) and appropriate test functions. Then the same choice of test functions yields to the fact that u satisfies

$$\lim_{t \to 0} \int_{\Omega} u(x,t)\zeta(x)dx = 0 \qquad \forall \zeta \in C_0^{\infty}(\Omega).$$

Finally the extension of u by zero at $t = 0$ satisfies the equation in $\Omega \times [0,T)$. □

Remark. We recall that $E(x,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$. Then if $1 \leq r < q_{c,\alpha}$, there holds

$$\iint_{Q_T} E^r(x,t)t^{\alpha}dxdt < \infty, \tag{5.15}$$

while if $r \geq q_{c,\alpha}$

$$\iint_{Q_T} E^r(x,t)t^{\alpha}dxdt = \infty. \tag{5.16}$$

6 The trace theorem

In all this section we assume that $\Omega \subset \mathbb{R}^N$ is an open domain with a compact C^2 boundary, $\alpha > -1$ and $1 < q < q_{c,\alpha}$, and let $u \in C(\overline{\Omega} \times (0,T])$ be a positive solution of (1.6) in Q_T^{Ω} which vanishes on $\partial\Omega \times (0,T]$. By Section 2 u possesses an initial trace $tr_{\Omega}(u) = (\mathcal{S}, \mu)$ where $\mathcal{S}$ is a relatively closed subset of Ω and μ is a Radon measure on $\mathcal{R} := \Omega \setminus \mathcal{S}$. To this couple we can associate a unique outer regular Borel measure ν defined by

$$\nu(E) = \begin{cases} \mu(E) & \text{if } E \subset \mathcal{R} \\ \infty & \text{if } E \cap \mathcal{S} \neq \emptyset \end{cases} \tag{6.1}$$

for any Borel subset E of Ω. Conversely, to any outer Borel measure ν on Ω we can associate the regular set $\mathcal{R} \subset \Omega$ which is the set of points $y \in \Omega$ which possess an open neighborhood $\mathcal{O}_y$ such that $\nu(\mathcal{O}_y) < \infty$. Clearly $\mathcal{R}$ is open and the restriction of ν to $\mathcal{R}$ is a positive Radon measure. The set $\mathcal{S} = \Omega \setminus \mathcal{R}$ is relatively closed and it is the singular part of ν. It has the property that $\nu(E) = \infty$ for any Borel set E such that $E \cap \mathcal{S} \neq \emptyset$. We shall denote by $\mathfrak{B}^{reg}(\Omega)$ the set of outer regular Borel measures in Ω and by $\mathfrak{B}_c^{reg}(\Omega)$ the subset of $\mathfrak{B}^{reg}(\Omega)$ for which $\mathcal{S}$ is a compact subset of Ω. Thus u is a solution of the following problem,

$$\begin{array}{rll} \partial_t u - \Delta u + t^{\alpha}u^q = 0 & & \text{in } Q_T^{\Omega} \\ u \geq 0,\ u = 0 & & \text{on } \partial\Omega \times [0,T) \\ tr_{\Omega}(u) = \nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega). & & \end{array} \tag{6.2}$$

Definition 6.1 *We denote by $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$ the set of solutions of problem (6.2).*

The first step in the characterization of the singular set is the following lower estimate.

Theorem 6.2 *Let $u \in \mathcal{U}_{\mathcal{S},\mu}(\Omega)$ and $a \in \mathcal{S}$, then*

$$u(x,t) \geq u_{\infty,a}(x,t). \tag{6.3}$$

Furthermore, if $\mathcal{S}$ has a nonempty interior $\mathcal{A}$, there holds

$$\lim_{t\to 0} t^{\frac{1+\alpha}{q-1}} u(x,t) = c_\alpha, \tag{6.4}$$

uniformly on compact subsets of $\mathcal{A}$.

We first give a proof of (6.4) in the case where either $-1 < \alpha \leq 0$, or $\alpha > 0$ and $1 < q < q_{c,0}$.

Proposition 6.3 *Assume either $-1 < \alpha \leq 0$ and $1 < q < q_{c,\alpha}$, or $\alpha > 0$ and $1 < q < q_{c,0}$, then inequality (6.3) holds.*

Proof. For any $\epsilon > 0$, there holds

$$\lim_{t\to 0} \int_{B_\epsilon(a)} u(x,t)dx = \infty. \tag{6.5}$$

If $k > 0$ is fixed, and $\{t_n\}$ is a sequence decreasing to 0. There exists t_{n_1} such that

$$\int_{B_{2^{-1}}(a)} u(x,t_{n_1})dx > k,$$

and there exists $m = m_1(k)$ such that

$$\int_{B_{2^{-1}}(a)} m_1(k) \wedge u(x,t_{n_1})dx = k,$$

where $A \wedge B = \inf\{A,B\}$. Assume we have constructed $t_{n_j} < t_{n_{j-1}}$ and $m_j(k) > 0$ such that

$$\int_{B_{2^{-j}}(a)} m_j(k) \wedge u(x,t_{n_j})dx = k.$$

Since (6.5) holds with $\epsilon = 2^{-j-1}$, there exists $t_{n_{j+1}} < t_{n_j}$ such that

$$\int_{B_{2^{-j-1}}(a)} u(x,t_{n_j})dx > k.$$

and thus $m_{j+1}(k) > 0$ such that

$$\int_{B_{2^{-j-1}}(a)} m_{j+1}(k) \wedge u(x,t_{n_{j+1}})dx = k.$$

Next we denote by u_j the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (t_{n_j}, T) \\ u &= 0 && \text{on } \partial\Omega \times [t_{n_j}, T) \\ u(.,t_{n_j}) &= m_{j+1}(k) \wedge u(.,t_{n_{j+1}}) && \text{in } \Omega. \end{aligned} \tag{6.6}$$

By the maximum principle $u_j \le u$ in $\Omega\times[t_{n_j},T)$, or equivalently $u_j(x,t+t_{n_j}) \le u(x,t+t_{n_j})$ in $Q^\Omega_{T-t_{n_j}}$. Clearly

$$u_{j,0} := u(.,t_{n_j}) \rightharpoonup k\delta_a \quad \text{as } j\to\infty$$

in the weak sense of measures. In order to prove that u_j converges to $u_{k\delta_a}$ we notice that $u_j(x,t+t_{n_j}) \le E * u_{j,0}(x,t)$ in Q^Ω_∞. If $-1<\alpha\le 0$ and $r\in(1,q_{c,\alpha})$

$$\begin{aligned}\iint_{Q^\Omega_T} u_j^r(x,t+t_{n_j})(t+t_{n_j})^\alpha dxdt &\le \iint_{Q_T} (E*u_{j,0})^r(x,t)(t+t_{n_j})^\alpha dxdt\\ &\le \|u_{j,0}\|_{L^1}^r \iint_{Q_T} E^r(x,t)t^\alpha dxdt,\end{aligned}$$

using Young's inequality and (5.15). If $\alpha\ge 0$ and $r\in(1,q_{c,0})$

$$\begin{aligned}\iint_{Q^\Omega_T} u_j^r(x,t+t_{n_j})(t+t_{n_j})^\alpha dxdt &\le 2^\alpha T^\alpha \iint_{Q_T} (E*u_{j,0})^r(x,t)dxdt\\ &\le 2^\alpha T^\alpha \|u_{j,0}\|_{L^1}^r \iint_{Q_T} E^r(x,t)dxdt.\end{aligned}$$

Furthermore, for $s\in(1,q_{c,0})$

$$\begin{aligned}\iint_{Q^\Omega_T} u_j^s(x,t+t_{n_j})dxdt &\le \iint_{Q_T} (E*u_{j,0})^s(x,t)dxdt\\ &\le \|v_{j,0}\|_{L^1}^s \iint_{Q_T} E^s(x,t)dxdt.\end{aligned}$$

Thus the set of functions $\{u_j^q(.,.+t_{n_j})(.+t_{n_j})^\alpha\}$ and $\{u_j(.,.+t_{n_j})\}$ are uniformly integrable in $L^1(Q^\Omega_T)$. Since u_j satisfies

$$\iint_{Q^\Omega_T} \left(-u_j(x,t+t_{n_j})\left(\partial_t\zeta+\Delta\zeta\right)+(t+t_{n_j})^\alpha u_j^q(x,t+t_{n_j})\zeta\right)dxdt = \int_\Omega u_{j,0}\zeta dx \qquad (6.7)$$

for any test function $\zeta\in C^{1,1;1}(\overline{Q}^\Omega_T)$ vanishing on $\partial\Omega\times[0,T]\cup\Omega\times\{T\}$ and converges a.e. in Q^Ω_T to some u^*, it follows by Vitali's theorem that

$$\iint_{Q^\Omega_T} \left(-v\left(\partial_t\zeta+\Delta\zeta\right)+t^\alpha v^q\zeta\right)dxdt = k\zeta(a) \qquad (6.8)$$

thus $u^*=u_{k\delta_a}$ by uniqueness, which implies the claim since $u\ge u^*$. □

When $\alpha>0$ and $q_{c,0}\le q<q_{c,\alpha}$, this argument cannot work since the sequence $u_{j,0}$ could concentrate too quickly with respect to t to a Dirac mass and isolated singularities are removable for solutions of

$$\partial_t u-\Delta u+cu^q=0.$$

We develop below a proof valid for any $\alpha>-1$ and $1<q<q_{c,\alpha}$, which is based upon the parabolic Harnack inequality and shows that such a fast concentration never occurs.

Lemma 6.4 *Assume $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Let $\{(x_n, t_n)\} \subset Q_T^\Omega$ be a sequence converging to $(a, 0)$ and $\ell > 0$. Put $V_n = B_{\ell\sqrt{t_n}}(x_n)$ and suppose that there exist nonnegative functions $h_n \in L^\infty(\mathbb{R}^N)$ with support in V_n such that $0 \le h_n \le c_1 t_n^{-\frac{N}{2}}$ and*

$$h_n \rightharpoonup k\delta_0. \tag{6.9}$$

Then the solutions u_n of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (t_n, T) \\ u &= 0 && \text{on } \partial\Omega \times [t_n, T) \\ u(., t_n) &= h_n && \text{in } \Omega, \end{aligned} \tag{6.10}$$

satisfy $u_n \to u_{k\delta_a}$ when $n \to \infty$.

Proof. The estimate $h_n \le c t_n^{-\frac{N}{2}} \chi_{V_n}$ can be written under the form

$$h_n(x) \le c_2 E(x - x_n, t_n) \chi_{V_n}$$

where $c_2 = (4\pi)^{\frac{N}{2}} e^{\frac{\ell}{4}} c_1$. By the maximum principle

$$u_n(x,t) \le c_2 E(x - x_n, t) \quad \text{in } \Omega \times (t_n, \infty).$$

By (5.15), (5.16), the sequences $\{E^q(.-x_n, .)t^\alpha\}$ and $\{E(.-x_n, .)\}$ are uniformly integrable in Q_T^Ω, therefore, if we extend u_n by zero in $Q_{t_n}^\Omega$ and denote by $\tilde{u}_n$ the extended function defined in Q_T^Ω, we infer that the sequences $\{t^\alpha \tilde{u}_n^q\}$ and $\{\tilde{u}_n\}$ are uniformly integrable in Q_T^Ω. Using standard regularity estimates there exists a function u^* defined in Q_T^Ω and a subsequence u_{n_j} such that $\tilde{u}_{n_j} \to u^*$ locally uniformly in Q_T^Ω. It follows by uniform integrability and Vitali's convergence theorem that

$$\tilde{u}_{n_j} \to u^* \qquad \text{in } L^q(Q_T^\Omega; t^\alpha dxdt) \cap L^1(Q_T^\Omega).$$

Let $\zeta \in C^{1,1;1}(\overline{Q}_T^\Omega)$ vanishing on $\partial\Omega \times [0,T] \cup \Omega \times \{T\}$, then

$$\iint_{Q_T^\Omega} \left(-\tilde{u}_{n_j}(\partial_t \zeta + \Delta\zeta) + t^\alpha \tilde{u}_{n_j}^q \zeta \right) dxdt = \int_\Omega \zeta(., t_{n_j}) h_n dx.$$

Using the previous convergence results and the assumption (6.9), we derive

$$\iint_{Q_T^\Omega} (-u^*(\partial_t \zeta + \Delta\zeta) + t^\alpha u^{*q} \zeta)\, dxdt = k\zeta(a).$$

Thus $u^* = u_{k\delta_a}$ and $\tilde{u}_n \to u_{k\delta_a}$ locally uniformly in Q_T^Ω. □

Lemma 6.5 *Let u be a positive solution of (1.6) in Q_T^Ω vanishing on $\partial\Omega \times [0,T)$. Then for any $\Omega' \subset \overline{\Omega} \subset \Omega$ there exists a constant $C = C(N, q, \alpha, \Omega') > 0$ such that*

$$u(y,s) \le u(x,t) e^{C\left(\frac{|x-y|^2}{t-s} + \frac{t}{s} + 1\right)} \qquad \forall (x,t), (y,s) \in Q_T^{\Omega'},\ s < t. \tag{6.11}$$

Proof. By (4.1), $V(x,t) := t^\alpha u^{q-1} \le c_\alpha^{q-1} t^{-1}$. If we write (1.6) under the form

$$\partial_t u - \Delta u + V(x,t)u = 0 \quad \text{in } Q_T^\Omega \tag{6.12}$$

it follows (6.11) from parabolic Harnack inequality (see e.g. [5, Lemma 3.16] although the result is much older). □

If $G \subset \mathbb{R}^N$ is a bounded open subset, we denote by ϕ_G is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ normalized by $\sup \phi_G = 1$ with corresponding eigenvalue λ_G.

Lemma 6.6 *Let $G \subset \mathbb{R}^N$ be a bounded open subset with a smooth boundary $\omega \in C^1(Q_T^G)$, $\omega \ge 0$, such that*

$$\int_0^T \|\omega(.,t)\|_{L^\infty(G)}\, dt < \infty. \tag{6.13}$$

If $v \in C^{2;1}(\overline{G} \times (0,T])$ is a positive solution of

$$\partial_t v - \Delta v + \omega v = 0 \quad \text{in } Q_T^G, \tag{6.14}$$

then $v\phi_G \in L^1(Q_T^G)$, $\omega v \phi_G^3 \in L^1(Q_T^G)$ and there exists $\mu \in \mathfrak{M}_+(G)$ such that

$$\lim_{t\to 0} \int_G v(x,t)\zeta(x)dx = \int_G \zeta d\mu \qquad \forall \zeta \in C_c(\overline{G}). \tag{6.15}$$

Proof. Set $\gamma(t) = \|\omega(.,t)\|_{L^\infty(G)}$. Denote $X(t) = \int_G v(x,t)\phi_G(x)dx$. Then, from (6.14),

$$X' + \lambda_G X + \gamma(t)X \ge 0$$

This implies that the function

$$t \mapsto e^{t\lambda_G + \int_0^t \gamma(s)ds} X(t)$$

(which exists thanks to (6.13)) is nondecreasing. Therefore there exists $X(0) = \lim_{t\to 0} X(t)$ and $v\phi_G \in L^1(Q_T^G)$. Furthermore, if we set $Y(t) = \int_G v(x,t)\phi_G^3(x)dx$

$$\frac{d}{dt}(e^{3t\lambda_G} X(t)) + e^{3t\lambda_1} \int_\Omega (\omega\phi_G^3 - 6\phi_G |\nabla\phi_G|^2)v dx = 0.$$

Since $\nabla\phi_G$ is bounded and $v\phi_G \in L^1(Q_T^G)$, it implies

$$e^{3t\lambda_G} Y(T) + \int_0^T e^{3t\lambda_G} \int_\Omega \omega(x,t)v(x,t)\phi_G^2 dxdt = Y(0) + 6\int_0^T e^{3t\lambda_G} \int_\Omega \phi_G |\nabla\phi_G|^2)v dxdt,$$

which implies that $\omega v\phi_G^3 \in L^1(Q_T^G)$. The argument given in the proof of Lemma 2.1 shows that v admits an initial trace which belongs to $\mathfrak{M}_+(G)$. □

Proof of Theorem 6.2. We define the parabolic distance in $\mathbb{R}^N \times \mathbb{R}$ by

$$d_P((x,t),(y,s)) = \sqrt{|x-y|^2 + |t-s|}$$

Step 1. We first prove that if u satisfies

$$\limsup_{d_P((x,t))(a,0))\to 0} t^{\frac{N}{2}} u(x,t) < \infty, \tag{6.16}$$

then $a \in \mathcal{R}(u)$. If (6.16) holds there exists $\epsilon, c > 0$ such that

$$u(x,t) \le c t^{-\frac{N}{2}} \qquad \forall (x,t) \text{ s.t. } \sqrt{|x-a|^2 + t} \le \epsilon.$$

If we set $\omega(x,t) = t^\alpha u^{q-1}(x,t)$, then

$$\omega(x,t) \le c^{q-1} t^{-(\frac{N(q-1)}{2} - \alpha)} \qquad \forall (x,t) \in B_{\frac{\epsilon}{\sqrt{2}}}(a) \times (0, \frac{\epsilon^2}{2}].$$

Since $q < q_{c,\alpha}$, then $\frac{N(q-1)}{2}) - \alpha < 1$; thus the assumptions of Lemma 6.6 are fulfilled and there exists a positive Radon measure μ in $B_{\frac{\epsilon}{\sqrt{2}}}(a)$ such that

$$\int_\Omega u(x,t)\zeta(x)dx \to \int_\Omega \zeta d\mu \text{ when } t \to 0, \quad \forall \zeta \in C_c^\infty(B_{\frac{\epsilon}{\sqrt{2}}}(a)).$$

Furthermore $t^\alpha u^q \in L^1(B_{\frac{\epsilon}{\sqrt{3}}}(a) \times (0,T))$, which is the claim.

Step 2. Since $a \in \mathcal{S}(u)$, there holds

$$\limsup_{d_P((x,t))(a,0))\to 0} t^{\frac{N}{2}} u(x,t) = \infty. \tag{6.17}$$

Then there exists a sequence $\{(x_n, s_n)\}$ converging to $(a,0)$ such that

$$u(x_n, t_n) \ge n t_n^{-\frac{N}{2}} \tag{6.18}$$

We apply Lemma 6.5 with $s = s_n$, $t = 2s_n := t_n$, $y = x_n$ $|x - x_n| \le \sqrt{s_n}$. Then

$$u(x,t_n) \ge C n t_n^{-\frac{N}{2}} \qquad \forall x \in V_n := B_{\frac{\sqrt{t_n}}{2}}(x_n)$$

This implies

$$\int_{V_n} u(x,t_n)dx \ge C_N n.$$

For $k < n$ fixed, we denote by $v := v_{n,k}$ the solution of

$$\begin{aligned} \partial_t v - \Delta v + t^\alpha v^q &= 0 && \text{in } \Omega \times (t_n, \infty) \\ v &= 0 && \text{in } \partial\Omega \times (t_n, \infty) \\ v(.,t_n) &= C k t_n^{-\frac{N}{2}} \chi_{V_n} && \text{in } \Omega. \end{aligned} \tag{6.19}$$

By the maximum principle $u \ge v_{n,k}$ in $\Omega \times (t_n, \infty)$. Furthermore

$$\int_{V_n} v(.,t_n)dx = C_N k.$$

Thus $v(.,t_n) \rightharpoonup C_N k\delta_a$ in the weak sense of measures. It follows from Lemma 6.4 that $v_{n,k} \to u_{C_N k\delta_a}$ locally uniformly in Q_T^Ω. Therefore $u \geq u_{C_N k\delta_a}$ in Q_T^Ω. Since k is arbitrary, we obtain (6.3).

Step 3. Formula (6.4) holds. Denote by $S_m(a) = \{x \in \mathbb{R}^N : |x_j| < m\}$. If $\overline{S}_R(a) \subset \mathcal{S}$, the function

$$(x,t) \mapsto c_\alpha (t-\tau)^{-\frac{1+\alpha}{q-1}} + w_{B_R}(x-a, t-\tau)$$

is a super solution of (1.6) in $S_R(a) \times (\tau, \infty)$ which is infinite on $S_R(a) \times \{\tau\} \cup \partial S_R(a) \times [\tau, \infty)$ by Proposition 3.3, while u is finite, thus it dominates u in $S_R(a) \times (\tau, \infty)$. Letting $\tau \to 0$ yields to

$$u(x,t) \leq c_\alpha t^{-\frac{1+\alpha}{q-1}} + w_{B_R}(x-a,t) \qquad \forall (x,t) \in Q_T^{S_R(a)}. \tag{6.20}$$

Conversely, the function

$$(x,t) \mapsto u(x, t-\tau) + w_{B_R}(x-a, t-\tau)$$

is a super solution in $S_R(a) \times (\tau, \infty)$ which dominates $c_\alpha t^{-\frac{1+\alpha}{q-1}}$ on $S_R(a) \times \{\tau\} \cup \partial S_R(a) \times [\tau, \infty)$, thus as above, we obtain (6.20). Since

$$\lim_{t \to 0} w_{B_R}(x-a,t) = 0$$

uniformly on $B_{R'}(a)$ for any $R' < R$, we derive (6.4). □

Proposition 6.7 *For any relatively closed $\mathcal{S} \in \Omega$, the set $\mathcal{U}_{\mathcal{S},0}(\Omega)$ is not empty and it admits a minimal element $\underline{u}_{\mathcal{S},0}$ and a maximal element $\overline{u}_{\mathcal{S},0}$.*

Proof. Step 1: Existence of a maximal solution. The maximal solution is constructed by thickening Ω and $\mathcal{S}$ in defining for $0 < \sigma$

$$\Omega_\sigma = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, \Omega) < \sigma\}, \qquad \mathcal{S}_\sigma = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, \overline{\mathcal{S}}) \leq \sigma\}$$

If $z \in \partial\Omega$, we denote by $\mathbf{n}_z$ the outward unit normal vector to Ω at z. Since $\partial\Omega$ is compact and C^2, there exists $\sigma_0 > 0$ such that for any $(z,\sigma) \in \partial\Omega \times [0\sigma_0]$, the mapping $\Pi : (z,\sigma) \mapsto z + \sigma \mathbf{n}_z$ is a C^2 diffeomorphism from $\partial\Omega \times [0, \sigma_0]$ to $\Theta'_{\sigma_0} := \overline{\Omega}_{\sigma_0} \setminus \Omega$. The mapping Π defines the flow coordinates near $\partial\Omega$.

If $0 < \delta < \sigma$, there exists a unique solution $u = u_{n,\sigma,\delta}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{\Omega_\sigma} \\ u &= 0 && \text{in } \partial\Omega_\sigma \times (0,\infty) \\ u(.,0) &= n\chi_{\mathcal{S}_\delta} && \text{in } \Omega_\sigma. \end{aligned} \tag{6.21}$$

Notice that $\mathcal{S}_\delta$ is closed in Ω_σ and $\inf\{|z - z'| : z \in \mathcal{S}_\delta, z' \in \Omega_\delta^c\} = \delta - \sigma$. Existence is standard as well as uniqueness in the case where Ω is bounded. If Ω^c is bounded the proof goes as in the uniqueness proof in Proposition 6.10. When $n \to \infty$, $\{u_{n,\sigma,\delta}\} \uparrow u_{\sigma,\delta}$ which is a solution of (1.6) in $Q_\infty^{\Omega_\sigma}$. Since $u_{n,\sigma,\delta}$ satisfies (4.1), for any $r, \tau > 0$ and any $a \in \partial\Omega_\sigma$, $u_{n,\sigma,\delta}$ remains uniformly continuous with respect to n in $\overline{Q}_\infty^{\Omega_\sigma} \cap \big(\overline{B}_r(a) \times [2^{-2}\tau, 2\tau]\big)$. Consequently $u_{\sigma,\delta}(x,t) = 0$ on $\partial\Omega_\sigma \times (0,\infty) \cap \big(\overline{B}_r(a) \times [2^{-2}\tau, 2\tau]\big)$. Therefore $u_{\sigma,\delta}$ vanishes

on $\partial\Omega_\sigma \times (0,\infty)$. If $a \in \Omega_\sigma$ and $\mathrm{dist}\,(a,\mathcal{S}_\delta) = r > 0$, $u_{n,\sigma,\delta}(x,t) \le w_{B_r}(x-a,t)$ in $Q^\infty_{B_r(a)}$. This implies that $u_{n,\sigma,\delta}$ remains uniformly continuous with respect to n in $\overline{B}_{r'}(a) \times [0,T)$ for any $0 < r' < r$ and $T > 0$. Since $u_{n,\sigma,\delta}(x,t) \to 0$ in $\overline{B}_{r'}(a)$, $u_{\sigma,\delta}$ inherits the same property. Consequently $u_{\sigma,\delta}$ has initial trace $(\mathcal{S}_\delta, 0)$ in Ω_σ. By the maximum principle the mapping $(n,\delta) \mapsto u_{n,\sigma,\delta}$ is increasing with respect to n and decreasing with respect to δ. Furthermore, if $0 < \delta' < \sigma' < \sigma$ and $0 < \delta < \sigma$, there holds $u_{n,\sigma',\delta'} < u_{n,\sigma,\delta}$ in $Q^{\Omega_{\sigma'}}_\infty$, which implies $u_{\sigma',\delta'} < u_{\sigma,\delta}$. We define

$$\overline{u}_{\mathcal{S},0} = \lim_{\sigma\to 0}\lim_{\delta\to 0}\lim_{n\to\infty} u_{n,\sigma,\delta}. \tag{6.22}$$

Then $\overline{u}_{\mathcal{S},0}$ is a solution of (1.6) in Q^Ω_∞. Since $\lim_{t\to 0} u_{\sigma,\delta}(x,t) = 0$ uniformly on any compact subset $K \subset \mathcal{R} = \Omega \setminus \mathcal{S}$, $\overline{u}_{\mathcal{S},0}$ has initial trace 0 on $\mathcal{R}$. If $a \in \mathcal{S}$, we denote by $\tilde{u}_{\infty,a}$ the function defined in $Q^{\Omega_\sigma}_\infty$ by

$$\tilde{u}_{\infty,a}(x,t) = \begin{cases} u_{\infty,a}(x,t) & \text{if } (x,t) \in Q^\Omega_\infty \\ 0 & \text{if } (x,t) \in Q^{\Omega_\sigma}_\infty \setminus Q^\Omega_\infty \end{cases}$$

Then $\tilde{u}_{\infty,a}$ is a subsolution of (1.6) in $Q^{\Omega_\sigma}_\infty$ which is smaller than $u_{\infty,a,\sigma}$ which is the limit, when $k \to \infty$ of the solution $u_{k\delta_a,\sigma}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q = 0 \qquad & \text{in } Q^{\Omega_\sigma}_\infty \\ u = 0 \qquad & \text{in } \partial\Omega_\sigma \times (0,\infty) \\ u(.,0) = k\delta_a \qquad & \text{in } \Omega_\sigma. \end{aligned} \tag{6.23}$$

There holds, by Theorem 6.2,

$$u_{\sigma,\delta}(x,t) \ge u_{\infty,a,\sigma} \ge \tilde{u}_{\infty,a}(x,t) \qquad \forall (x,t) \in Q^{\Omega_\sigma}_\infty.$$

Letting successively $\delta \to 0$ and $\sigma \to 0$ yields to $\overline{u}_{\mathcal{S},0} \ge \tilde{u}_{\infty,a} = u_{\infty,a}$ in $Q^{\Omega_\sigma}_\infty$. Therefore any $a \in \mathcal{S}$ is a singular initial point of $\overline{u}_{\mathcal{S},0}$. Since $\mathcal{S} \cup \mathcal{R} = \Omega$, it follows that $tr_\Omega(u) = (\mathcal{S}, 0)$. Since $u_{\sigma,\delta}$ satisfies (4.1) and $\partial\Omega_\sigma$ has bounded curvature, independent of σ, there holds classicaly

$$|\nabla u_{\sigma,\delta}(x,t)| \le c t^{-\frac{q+\alpha}{q-1}} \qquad \forall (x,t) \in \partial\Omega_\sigma \times (0,\infty). \tag{6.24}$$

If $z \in \partial\Omega$ and then by the mean value theorem there exists $\theta \in (0,1)$ such that,

$$0 \le u_{\sigma,\delta}(z,t) = u_{\sigma,\delta}(z + \sigma\mathbf{n}_z, t) - \sigma\nabla u_{\sigma,\delta}(z + \theta\sigma\mathbf{n}_z, t).\mathbf{n}_z \le c\sigma t^{-\frac{q+\alpha}{q-1}}.$$

This implies that $\overline{u}_{\mathcal{S},0}$ vanishes on $\partial\Omega_\sigma \times (0,\infty)$.

Let u be any positive solution of (1.6) in Q^Ω_∞, vanishing on $\partial\Omega \times (0,\infty)$, with initial trace $(\mathcal{S}, 0)$. For $0 < \delta < \sigma$ fixed and for $R, \epsilon > 0$, there exists $\tau_\epsilon > 0$ such that, for any $\tau \in (0, \tau_\epsilon]$,

$$u(x,\tau) \le \epsilon \qquad \forall x \in \overline{B}_R \cap \Omega \setminus \mathcal{S}_\sigma.$$

This is due to the fact that $u(x,\tau) \to 0$ when $\tau \to 0$, uniformly on compact subset of $\overline{B}_R \cap \mathcal{R}$. Assume that Ω is unbounded (the case where Ω is bounded is simpler since it does not require to introduce the barrier w_{B_R}) and let $R > 0$ large enough so $\Omega^c \subset B_R$. By (4.1)-(4.3) there exists $0 < \tau_1 \le \tau_0$ such that for any $\tau \in (0, \tau_1]$,

$$u(x,\tau) \le w_{B_R}(x,\tau) + u_{\sigma,\delta}(x,\tau) \qquad \forall x \in \overline{B}_R \cap \Omega \cap \mathcal{S}_\sigma.$$

Furthermore $u(x,t) < w_{B_R}(x,t)$ for all $t > 0$ and $x \in \partial B_R \cap \Omega$. Since $\epsilon + w_{B_R} + u_{\sigma,\delta}$ is a supersolution for (1.6) in $B_R \cap \Omega \times (0,\infty)$, it follows that

$$u(x,t) \le \epsilon + w_{B_R}(x,t) + u_{\sigma,\delta}(x,t) \qquad \forall (x,t) \in B_R \cap \Omega \times (0,\infty).$$

Letting successively $\delta \to 0$, $\sigma \to 0$, $R \to \infty$ (here we use the fact that $w_{B_R}(x,t) \to 0$ when $R \to \infty$ by Proposition 3.4) and $\epsilon \to 0$ yields to $u \le \overline{u}_{\mathcal{S},0}$.

Step 2: Existence of a minimal solution. The set $\mathcal{U}_{\mathcal{S},0}(\Omega)$ is not empty since it contains $\overline{u}_{\mathcal{S},0}$ and we may define

$$\tilde{u}_{\mathcal{S},0} = \sup\{u_{\infty,a} : a \in \mathcal{S}\}, \tag{6.25}$$

and

$$\hat{u}_{\mathcal{S},0} = \inf\{u : u \in \mathcal{U}_{\mathcal{S},0}(\Omega)\}. \tag{6.26}$$

The functions $\tilde{u}_{\mathcal{S},0}$ and $\hat{u}_{\mathcal{S},0}$ are respectively positive sub and super solutions of (1.6) in Q^Ω_∞. They are bounded from above by $\overline{u}_{\mathcal{S},0}$ and from below by $u_{\infty,a}$ for any $a \in \mathcal{S}$. Since $u_{\infty,a} \le u$ for any $a \in \mathcal{S}$ and $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$, it follows that $\tilde{u}_{\mathcal{S},0} \le \hat{u}_{\mathcal{S},0}$. Therefore there exists a solution $\underline{u}_{\mathcal{S},0}$ of (1.6) in Q^Ω_∞ which satisfies and

$$\tilde{u}_{\mathcal{S},0} \le \underline{u}_{\mathcal{S},0} \le \hat{u}_{\mathcal{S},0}. \tag{6.27}$$

This implies that $\underline{u}_{\mathcal{S},0}$ has initial trace $(\mathcal{S},0)$, it vanishes on $\partial\Omega \times (0,\infty)$ and it is therefore the minimal element of $\mathcal{U}_{\mathcal{S},0}(\Omega)$. □

Remark. If $\mathrm{dist}\,(\mathcal{S},\Omega^c) > 0$, it is not needed to replace Ω by a larger set Ω_σ in order to construct the maximal solution. The construction of $\overline{u}_{\mathcal{S},0}$ can be done in replacing $u_{n,\sigma,\delta}$ by the solution $u = u_{n,\sigma}$ of

$$\begin{array}{rl} \partial_t u - \Delta u + t^\alpha u^q = 0 & \text{in } Q^\Omega_\infty \\ u = 0 & \text{in } \partial\Omega \times (0,\infty) \\ u(.,0) = n\chi_{\mathcal{S}_\delta} & \text{in } \Omega, \end{array} \tag{6.28}$$

with $\delta < \delta_0 := \mathrm{dist}\,(\mathcal{S},\Omega^c)$.

The next result is an extension of Proposition 2.2.

Proposition 6.8 *Assume $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Let $\{u_n\}$ be a sequence of positive solutions of (1.6) which converges to u locally uniformly in Q^Ω_T, and denote by $(\mathcal{S}_n,\mu_n)$ and $(\mathcal{S},\mu)$ the respective initial trace of u_n and u. If $\mathcal{A}$ is an open subset of $\cap_n \mathcal{R}_n$ and $\mu_n(\mathcal{A})$ remains bounded independently of $n \in \mathbb{N}$ (where $\mathcal{R}_n = \Omega \setminus \mathcal{S}_n$ and $\mathcal{R} = \Omega \setminus \mathcal{S}$), then $\mathcal{A} \subset \mathcal{R}$ and $\chi_{\mathcal{A}}\mu_n \rightharpoonup \chi_{\mathcal{A}}\mu$ in the weak sense of measures. Conversely, if $\mathcal{A} \subset \mathcal{R}$, then for any compact $K \subset \mathcal{A}$, there exist $C_K > 0$ and $n_K \in \mathbb{N}$ such that $\mu_n(K) \le C_K$ for any $n \ge n_K$.*

Proof. Clearly (2.14) holds. We keep the notations of the proof of Proposition 2.2 where the first statement has been proved in assuming $\overline{B}_{\tilde{r}}(z) \subset \mathcal{A}$. Since $\mu_n(\mathcal{A})$ remains bounded, there exists a subsequence $\{n_j\}$ and a positive measure μ' on $\mathcal{A}$ such that $\mu_{n_j} \rightharpoonup \mu'$ in the weak sense of measures in $\mathcal{A}$. Then $u_{\chi_{B_{\tilde{r}}(z)}\mu_{n_j}}$ converges locally uniformly in $Q^{B_{\tilde{r}}(z)}_\infty$ to the solution $u_{\chi_{B_{\tilde{r}}(z)}\mu'}$ of

$$\begin{array}{rl} \partial_t u - \Delta u + t^\alpha u^q = 0 & \text{in } Q^{B_{\tilde{r}}(z)}_\infty \\ u = 0 & \text{in } \partial B_{\tilde{r}}(z) \times (0,\infty) \\ u(.,0) = \chi_{B_{\tilde{r}}(z)}\mu' & \text{in } B_{\tilde{r}}(z). \end{array} \tag{6.29}$$

Since $q < q_{c,\alpha}$, the convergence of $u_{\chi_{B_{\tilde r}(z)}\mu_{n_j}}$ and $t^\alpha u^q_{\chi_{B_{\tilde r}(z)}\mu_{n_j}}$ respectively to $u_{\chi_{B_{\tilde r}(z)}\mu_{n_j}}$ and $t^\alpha u^q_{\chi_{B_{\tilde r}(z)}\mu'}$ holds in $L^1(Q_T^{B_{\tilde r}(z)})$ for any $T > 0$. Relation (2.21) reads

$$u_{\chi_{B_{\tilde r}(z)}\mu_n}(x,t) \le u_n(x,t) \le u_{\chi_{B_{\tilde r}(z)}\mu_n}(x,t) + w_{B_{\tilde r}(z)}(x,t) \qquad \text{in } Q_\infty^{B_{\tilde r}(z)}. \tag{6.30}$$

(see Proposition 3.3). Then u_{n_j} and $t^\alpha u^q_{n_j}$ converge to u and $t^\alpha u^q$ respectively, in $L^1(Q_T^{B_r(z)})$ for any $r < \tilde r$. From (2.17), we derive

$$\iint_{Q_\infty^{B_r(z)}} (-u(\zeta_t + \Delta\zeta) + \zeta t^\alpha u^q)\, dxdt = \int_{B_r(z)} \zeta(x,0) d\mu'(x), \tag{6.31}$$

for any $\zeta \in C_c^{1,1;1}(\overline{Q}_\infty^{B_r(z)})$ which vanishes for t large enough. This implies that μ' is the initial trace of u in $B_r(z)$, i.e. $\chi_{B_r(z)}\mu' = \chi_{B_r(z)}\mu$ and $\chi_{B_r(z)}\mu_n \rightharpoonup \chi_{B_r(z)}\mu$. Using a partition of unity, we conclude that $\chi_{\mathcal{A}}\mu_n \rightharpoonup \chi_{\mathcal{A}}\mu$.

Conversely, we assume that there exist a compact set $K \subset \mathcal{A}$ and a subsequence μ_{n_j} such that $\mu_{n_j}(K) \to \infty$. Thus, using the diagonal process, there exist $z \in K$ and another subsequence that we still denote μ_{n_j} such that

$$\lim_{n_j \to \infty} \mu_{n_j}(B_\epsilon(z)) = \infty \qquad \forall \epsilon > 0.$$

Therefore, we can construct a subsequence $\{n_{j_\ell}\} \subset \{n_j\}$ such that

$$\mu_{n_{j_\ell}}(B_{2^{-n_{j_\ell}}}(z)) = m_{n_{j_\ell}} \to \infty$$

when $n_{j_\ell} \to \infty$. Since the solution $u_{\chi_{B_{2^{-n_{j_\ell}}}(z)}\mu_{n_{j_\ell}}}$ of

$$\begin{array}{rl} \partial_t u - \Delta u + t^\alpha u^q = 0 & \text{in } Q_\infty^{B_{\tilde r}(z)} \\ u = 0 & \text{in } \partial B_{\tilde r}(z) \times (0,\infty) \\ u(.,0) = \chi_{B_{2^{-n_{j_\ell}}}(z)}\mu_{n_{j_\ell}} & \text{in } B_{\tilde r}(z). \end{array} \tag{6.32}$$

converges to $u_{\infty\delta_z}^{B_{\tilde r}(z)}$ which is the limit of the solution $u_{k\delta_z}$ of (6.32) with initial data $u(.,0) = k\delta_z$, and is dominated by $u_{n_{j_\ell}}$ in $Q_T^{B_{\tilde r}(z)}$ we conclude that $u \ge u_{\infty\delta_z}^{B_{\tilde r}(z)}$ in $Q_T^{B_{\tilde r}(z)}$, which implies that $z \in \mathcal{S}$, contradiction. □

Proposition 6.9 *Assume u_1 and u_2 are two positive solutions of (1.6) in Q_∞^Ω with initial trace $(\mathcal{S}, \mu)$. Then for any $a \in \mathcal{R}$ and $R > 0$ such that $B_R(a) \subset \mathcal{R}$, there holds*

$$|u_1(x,t) - u_2(x,t)| \le w_{B_R}(x-a,t) \qquad \forall (x,t) \in Q_\infty^{B_R(a)} \tag{6.33}$$

In particular $\lim_{t\to 0} |u_1(x,t) - u_2(x,t)| = 0$ uniformly on any compact subset of $\mathcal{R}$.

Proof. Since u and u' are solution of (1.6) and $B_R(a) \in \mathcal{R}$, for any $i = 1, 2$, $R' < R$ and $T > 0$,

$$\iint_{Q_T^{B'_R(a)}} t^\alpha u_i^q(x,t) dxdt + \iint_{Q_T^{B'_R(a)}} u_i(x,t) dxdt < \infty,$$

furthermore

$$\lim_{t\to 0}\int_{B_R(a)} u_i(x,t)\zeta(x)dx = \int_{B_R(a)}\zeta(x)d\mu(x) \qquad \forall \zeta\in C_c(B_R(a)).$$

This implies that u_i has a Sobolev trace on $\partial_\ell Q_T^{B'_R(a)}$ which belongs to L^1 and they are the limit, when $k\to\infty$ of the solutions $u_{i,k}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1}u = 0 \qquad & \text{in } Q_\infty^{B'_R(a)} \\ u = \min\{k, u_i\lfloor_{Q_\infty^{B'_R(a)}}\} \qquad & \text{in } \partial_\ell Q_\infty^{B'_R(a)} \\ u(.,0)=\mu \qquad & \text{in } B'_R(a). \end{aligned} \tag{6.34}$$

Since $u_{2,k}+w_{B'_R}(.-a)$ is a supersolution

$$u_{1,k}\le u_{2,k}+w_{B'_R}(.-a) \Longrightarrow |u_{1,k}-u_{2,k}|\le w_{B'_R}(.-a) \text{ in } Q_\infty^{B'_R(a)}.$$

Letting $k\to\infty$, R' to R, we derive (6.33). The second statement is a consequence of the fact that $\lim_{t\to 0} w_{B_R}(.-a)=0$, uniformly on B'_R by Proposition 3.3. □

Remark. The previous estimate does not use the fact that $\partial\Omega$ is smooth and bounded. If the u_i belong to $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$, estimate (6.33) can be improved since the u_i vanish on $\partial_\ell Q_\infty^{B_R(a)}$, and we obtain,

$$|u_1(x,t)-u_2(x,t)|\le \min\left\{w_{B_R}(x-a,t), c_\alpha t^{-\frac{1+\alpha}{q-1}}\right\} \qquad \forall (x,t)\in Q_\infty^{B_R(a)}. \tag{6.35}$$

Proposition 6.10 *Assume $\Omega\subseteq\mathbb{R}^N$ is either $\mathbb{R}^N$ or an open domain with a C^2 compact boundary, $\alpha>-1$ and $1<q<q_{c,\alpha}$. Then for any measure μ in Ω such that $\mu\lfloor_{\Omega_R}\in \mathfrak{M}_+^{b,\rho}(\Omega_R)$ where $\Omega_R=\Omega\cap B_R$, there exists a unique solution u_μ to*

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1}u = 0 \qquad & \text{in } Q_\infty^\Omega \\ u = 0 \qquad & \text{in } \partial_\ell Q_\infty^\Omega \\ u(.,0)=\mu \qquad & \text{in } \Omega, \end{aligned} \tag{6.36}$$

and the mapping $\mu\mapsto u_\mu$ is increasing. Furthermore, if $\{\mu_n\}$ is a sequence of positive measures such that $\mu_n\lfloor_{\Omega_R}\in\mathfrak{M}_+^{b,\rho}(\Omega_R)$ which converges weakly to $\mu\lfloor_{\Omega_R}\in\mathfrak{M}_+^{b,\rho}(\Omega_R)$, then $\{u_{\mu_n}\}\to u_\mu$ locally uniformly in Q_∞^Ω.

Proof. We recall that u is a solution of (6.36) if $u\in L^1_{loc}(\overline{Q_\infty^\Omega})$, $|u|^q\in L^1_{loc}(\overline{Q_\infty^\Omega};t^\alpha\rho dxdt)$ satisfies

$$\iint_{Q_\infty^\Omega}\left(-u(\partial_t\zeta+\Delta\zeta)+t^\alpha|u|^{q-1}u\zeta\right)dxdt=\int_\Omega\zeta(.,0)d\mu \tag{6.37}$$

for any test function $\zeta\in C_0^{2,1;1}(\overline{Q_\infty^\Omega})$. When Ω is bounded, existence, uniqueness and stability are proved in [9]. Thus we assume that Ω is unbounded and we assume $R\ge R_0$ such that $\Omega^c\subset B_{R_0}$. There exists a unique solution u_R of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1}u = 0 \qquad & \text{in } Q_\infty^{\Omega_R} \\ u = 0 \qquad & \text{in } \partial_\ell Q_\infty^{\Omega_R} \\ u(.,0)=\mu\lfloor_{\Omega_R} \qquad & \text{in } \Omega_R. \end{aligned} \tag{6.38}$$

The function u_R is nonnegative, $R \mapsto u_R$ is increasing. For $R > R_1$, u_R admits a Sobolev trace f_{R_1} on $\partial B_{R_1} \times (0,T)$ which is an integrable function, and u_R is the unique solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_{R_1}} \\ u &= 0 && \text{in } \partial\Omega \times (0,\infty) \\ u &= f_{R_1} && \text{in } \partial B_{R_1} \times (0,\infty) \\ u(.,0) &= \mu\lfloor_{\Omega_{R_1}} && \text{in } \Omega_{R_1}. \end{aligned} \tag{6.39}$$

Furthermore, $u_R\lfloor_{Q_\infty^{\Omega_{R_1}}} = \lim_{m\to\infty} v_m$, where v_m is the unique solution of (6.36) where the boundary data on $\partial B_{R_1} \times (0,\infty)$ is replaced by $f_{R_1,m} = f_{R_1} \wedge m$ ($m \in \mathbb{N}^*$). Let v_{R_1} be the unique solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_{R_1}} \\ u &= 0 && \text{in } \partial\Omega \times (0,\infty) \\ u &= 0 && \text{in } \partial B_{R_1} \times (0,\infty) \\ u(.,0) &= \mu\lfloor_{\Omega_{R_1}} && \text{in } \Omega_{R_1}, \end{aligned} \tag{6.40}$$

If $w_{B_{R_1}}$ is the barrier function in $Q_\infty^{\Omega_{R_1}}$ which has been constructed in Proposition 3.3, $v_{R_1} + w_{B_{R_1}}$ is a supersolution for problem (6.39). Since it is larger than v_m in $Q_\infty^{\Omega_{R_1}}$ for any $m > 0$, there holds $u_R \le v_{R_1} + w_{B_{R_1}}$, for any $R > R_1$. Then $u_R \uparrow u_\mu$ which is a solution of 1.6 in $Q_\infty^{\Omega_R}$. By Proposition 3.4, $w_{B_{R_1}}$ remains uniformly bounded in $Q_\infty^{\Omega_{R'}}$ for any $R_0 < R' < R_1$. Therefore u_μ shares the same property. If $\zeta \in C_c^{1,1;1}(Q_\infty^{\overline{\Omega}})$ vanishes on $Q_\infty^{\partial\Omega}$) and for $|x| > R' > R_0$, there holds for $R > R' > R_0$ and $T > 0$,

$$\iint_{Q_\infty^\Omega} (-u_R(x,t)(\partial_t\zeta + \Delta\zeta) + \zeta t^\alpha u_R^q)\, dxdt = \int_\Omega \zeta(x,0) d\mu(x) - \int_\Omega \zeta(x,T) u_R(x,T) dx \tag{6.41}$$

If we let $R \to \infty$ we deduce by the monotone convergence theorem that u_μ is a weak solution of (6.36). This proves existence.

For uniqueness, we consider u_μ and u'_μ two solutions of (6.36). By the same argument as in the existence part, for any $R > 0$, u_μ is smaller than the supersolution $u'_\mu + w_{B_{R_1}}$ in $Q_\infty^{\Omega_R}$. Since $\lim_{R\to\infty} w_{B_R} = 0$ by Proposition 3.4 we obtain $u_\mu \le u'_\mu$. Similarly $u'_\mu \le u_\mu$. Uniqueness implies the monotonicity of the mapping $\mu \mapsto u_\mu$.

For proving the stability, assume $\{\mu_n\lfloor_{\Omega_R}\}$ converges to $\mu\lfloor_{\Omega_R}$ in the weak sense of measures in $\mathfrak{M}_+^{b,\rho}(\Omega_R)$ for any $R > R_0$. Then the sequence of solutions $v_{n,R}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_R} \\ u &= 0 && \text{in } \partial\Omega \times (0,\infty) \\ u &= 0 && \text{in } \partial B_R \times (0,\infty) \\ u(.,0) &= \mu_n\lfloor_{\Omega_R} && \text{in } \Omega_R, \end{aligned} \tag{6.42}$$

converges to the solution v_R of (6.40) with $R_1 = R$. In particular $v_{n,R} \to v_R$ and $t^\alpha v_{n,R}^q \to t^\alpha v_R^q$ in $L^1(Q_T^{\Omega_R})$ and by standard regularity result the convergence of $v_{n,R}$ towards v_R holds uniformly on $\overline{\Omega}_R \times [\epsilon, T]$ for any $0 < \epsilon < T$. Furthermore $u_{\mu_n} \le v_{n,R} + W_{B_R}$ in $Q_\infty^{\Omega_R}$. This

jointly with standard local regularity results for heat equation, implies that $\{u_{\mu_n}\}$ remains uniformly bounded and hence relatively compact for the topology of uniform convergence on any compact set of $\overline{\Omega}\times[\epsilon,\infty)$. Thus there exist a subsequence $\{u_{\mu_{n_k}}\}$ and a function $u^*\in C(\overline{Q}^\Omega_\infty)$ such that $u_{\mu_{n_k}}\to u^*$ locally uniformly in $\overline{\Omega}\times(0,\infty)$. Since $t^\alpha u^q_{\mu_n}\le t^\alpha v^q_{n,R}+t^\alpha W^q_{B_R}$, there also holds by the dominated convergence theorem $t^\alpha u^q_{\mu_{n_k}}\to t^\alpha u^{*q}$ in $L^1_{loc}(\overline{\Omega}\times[0,\infty))$. Henceforth letting $n_k\to\infty$ in the expression

$$\begin{aligned}\iint_{Q^\Omega_\infty}\left(-u_{\mu_{n_k}}(x,t)(\partial_t\zeta+\Delta\zeta)+\zeta t^\alpha u^q_{\mu_{n_k}}\right)dxdt\\ =\int_\Omega\zeta(x,0)d\mu_{n_k}(x)-\int_\Omega\zeta(x,T)u_{n_k}(x,T)dx,\end{aligned}\tag{6.43}$$

where $\zeta\in C^{1,1;1}_c(Q^{\overline{\Omega}}_\infty)$, we conclude that $u^*=u_\mu$ and that $u_{\mu_n}\to u_\mu$. □

Proposition 6.11 *Assume F is a non-empty relatively closed subset of Ω, $\mathcal{R}=\Omega\setminus F$ and $\mu\in\mathfrak{M}_+(\mathcal{R})$. If we set*

$$\partial_\mu F=\{z\in F:\mu(\mathcal{R}\cap B_r(z))=\infty,\ \forall r>0\},\tag{6.44}$$

then $\partial_\mu F$ is relatively closed in Ω. If $\mathcal{R}^\mu=\Omega\setminus\partial_\mu F$, it contains $\mathcal{R}$ and if μ^* is the measure defined in $\mathcal{R}^\mu$ by μ on $\mathcal{R}$ and 0 in $\mathcal{R}^\mu\cap\mathcal{R}^c$, then there exist a minimal positive solution $\underline{u}_{\mu^*}$ and a maximal solution $\overline{u}_{\mu^*}$ of (1.6) vanishing on $\partial\Omega\times(0,\infty)$ satisfying $tr_\Omega(u)=(\partial_\mu F,\mu^*)$. Furthermore $\underline{u}_{\mu^*}$ and $\overline{u}_{\mu^*}$ are respectively the minimal and the maximal element of $\mathcal{U}_{\partial_\mu F,\mu^*}(\Omega)$.

Proof. The set $\partial_\mu F$ is the blow-up set of the measure μ. It is clearly a relatively closed subset of Ω included into $\overline{\mathcal{R}}\setminus\mathcal{R}$.

Step 1: Existence of a minimal solution. For $\delta>0$, we denote $(\partial_\mu F)_\delta:=\{x\in\Omega:\text{dist}\,(x,\partial_\mu F)\le\delta\}$ and $\mathcal{R}^\mu_\delta=\Omega\setminus(\partial_\mu F)_\delta\subset\mathcal{R}^\mu$. We define the Radon measure μ_δ on Ω by

$$\mu_\delta=\begin{cases}\mu & \text{on } \mathcal{R}^\mu_\delta\cap\mathcal{R}\\ 0 & \text{on } F\cup(\partial_\mu F)_\delta\end{cases}$$

Then μ_δ is a positive Radon measure in Ω and by Proposition 6.10 problem (6.36) with initial data μ_δ admits a unique positive solution u_{μ_δ}. Furthermore the mapping $\delta\mapsto u_{\mu_\delta}$ is nonincreasing, and we set $\underline{u}_{\mu^*}=\lim_{\delta\to0}u_{\mu_\delta}$. Then $\underline{u}_{\mu^*}$ is a positive solution of (1.6) in Q^Ω_∞ which vanishes on $\partial\Omega\times(0,\infty)$ and has initial trace $(\mathcal{S}',\mu')$. If $a\in\mathcal{R}^\mu$, there exists $R>0$ such that $\overline{B}_R(a)\subset\mathcal{R}^\mu$ and $\delta_a>0$ such that $\overline{B}_R(a)\subset\mathcal{R}^\mu_\delta$ for $0<\delta<\delta_a$, that we assume below. By the maximum principle there holds

$$v_{\mu_\delta}(x,t)\le u_\delta(x,t)\le v_{\mu_\delta}(x,t)+w_{B_R}(x-a,t)\qquad\text{in } Q^{B_R(a)}_\infty,$$

where v_{μ_δ} is the solution of

$$\begin{aligned}\partial_t u-\Delta u+t^\alpha u^q&=0 && \text{in } Q^{B_R(a)}_\infty\\ u&\ge0 && \text{in } Q^{B_R(a)}_\infty\\ u(x,t)&=0 && \text{in } \partial B_R(a)\times(0,\infty)\\ u(.,0)&=\mu_\delta\chi_{B_R(a)} && \text{in } B_R(a).\end{aligned}$$

Letting $\delta \to 0$, then $\mu_\delta \chi_{B_R(a)} \uparrow \mu^* \chi_{B_R(a)}$, which yields to

$$v_{\mu^* \lfloor_{B_R(a)}}(x,t) \le \underline{u}_{\mu^*}(x,t) \le v_{\mu^* \lfloor_{B_R(a)}}(x,t) + w_{B_R}(x-a,t) \qquad \text{in } Q_\infty^{B_R(a)}.$$

By a partition of unity, it implies that for any $\zeta \in C_c(\mathcal{R}^\mu)$, we have

$$\lim_{t\to 0} \int_{\mathcal{R}^\mu} \underline{u}_{\mu^*}(x,t)\zeta(x)dx = \int_{\mathcal{R}^\mu} \zeta d\mu^*(x).$$

Therefore $\mathcal{R}^\mu \subset \mathcal{R}'$ and $\mu' \lfloor_{\mathcal{R}^\mu} = \mu^*$. If $z \in \mathcal{R}' \cap (\mathcal{R}^\mu)^c = \mathcal{R}' \cap \partial_\mu F$, $\mu^*(B_r(z) \cap \mathcal{R}^\mu) = \mu(B_r(z) \cap \mathcal{R}) = \infty$ for any $r > 0$ while there exists $r_0 > 0$ such that $\underline{\mu}'(B_{r_0}(z)) < \infty$. By Proposition 6.8, for any $r' < r_0$ there exists $C > 0$ such that $\mu_\delta(\overline{B}_r(z)) \le C$. By the monotone convergence theorem, it implies $\mu(\overline{B}_r(z) \cap \mathcal{R}) \le C$, which contradicts the definition of $\partial_\mu F$. Thus $\mathcal{S}' = \partial_\mu F$ and $tr_\Omega(\underline{u}_{\mu^*}) = (\partial_\mu F, \mu^*)$.

Let us assume that Ω is unbounded, R_0 is such that $\Omega^c \subset B_{R_0}$ and $\Omega_R = \Omega \cap B_R$ for $R > R_0$. Let $\delta, \epsilon > 0$, there exists τ_ϵ such that

$$u_{\mu_\delta}(x,\tau) \le \epsilon \qquad \forall (x,\tau) \in \Omega_R \cap (\partial_\mu F)_{\frac{\delta}{2}} \times [0,\tau_\epsilon] \tag{6.45}$$

Let $u \in \mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$. In order to compare u_{μ_δ} and $v := u + w_{B_R} + \epsilon$ in $\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}} \times (0,\tau_\epsilon]$ we see that

$$u_{\mu_\delta}(.,0) = \chi_{\mathcal{R}^\mu_R \setminus (\partial_\mu F)_\delta} \mu^* \le u(.,0) = \chi_{\mathcal{R}^\mu_R \setminus (\partial_\mu F)_{\frac{\delta}{2}}} \mu^*$$

and both are bounded Radon measures. Since $u_{\mu_\delta}(t,0) \le v(x,t)$ in $\partial(\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}}) \times (0,\tau_\epsilon]$ and v is a supersolution, it follows that $u_{\mu_\delta} \le v$ in $\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}} \times (0,\tau_\epsilon]$. Using (6.45) we conclude that

$$u_{\mu_\delta}(x,\tau) \le v(x,\tau) \qquad \forall (x,\tau) \in \Omega_R \times [0,\tau_\epsilon]. \tag{6.46}$$

Next, applying the comparison principle in $\Omega_R \times [\tau^*, \infty)$ between the solution u_{μ_δ} and the supersolution $u + \epsilon + w_{B_R}$, we conclude that (6.46) holds in $\Omega_R \times [\tau, \infty)$ and thus in $Q_\infty^{B_R}$. Letting successively $R \to \infty$, $\epsilon \to 0$ and $\delta \to 0$, we conclude that $\underline{u}_{\mu^*} \le u$, thus u_{μ^*} is the minimal element of $\mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$.

Step 2: Existence of a maximal solution. Let $\delta > 0$ and $u \in \mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$. By Proposition 6.9, for any $R > 0$ and $\epsilon > 0$ there exists τ_ϵ such that

$$u(x,t) \le \underline{u}_{\mu^*}(x,t) + \epsilon \qquad \forall (x,t) \in \Omega_R \setminus (\partial_\mu F)_\delta \times (0,\tau_\epsilon], \tag{6.47}$$

and by (4.1), $u(x,t) \le c_\alpha t^{-\frac{1+\alpha}{q-1}}$. Let $\tau \in (0,\tau_\epsilon]$ and $w_{\delta,\tau}$ be the solution of (6.36) in $Q_\infty^{\Omega_R}$ with initial data μ replaced by

$$h_{\delta,\tau} = \begin{cases} \underline{u}_{\mu^*}(x,\tau) & \text{if } x \in \Omega_R \setminus (\partial_\mu F)_\delta \\ c_\alpha \tau^{-\frac{1+\alpha}{q-1}} & \text{if } x \in \Omega_R \cap (\partial_\mu F)_\delta \end{cases} \tag{6.48}$$

By (6.47), (6.48) and the maximum principle,

$$u(x,t+\tau) \le w_{\delta,\tau}(x,t) + \epsilon + w_{B_R}(x,t+\tau) \qquad \forall (x,t) \in Q_\infty^{\Omega_R}. \tag{6.49}$$

Let $\overline{u}_{(\partial_\mu F)_\delta}$ be the maximal element of $\mathcal{U}_{(\partial_\mu F)_\delta,0}(\Omega)$, which exists by Proposition 6.7. Then, by (4.3), for any $\delta' > \delta$, there exists $\tau_{\delta'} \in (0, \tau_\epsilon]$ such that for any $\tau \in (0, \tau_{\delta'}]$

$$\max\{\underline{u}_{\mu^*}(.,.+\tau), \overline{u}_{(\partial_\mu F)_\delta}(.,.+\tau)\} \leq w_{\delta,\tau} \leq \underline{u}_{\mu^*}(.,.+\tau)+\overline{u}_{(\partial_\mu F)_{\delta'}}(.,.+\tau) \qquad \text{in } Q_\infty^{\Omega_R}. \tag{6.50}$$

Up to a sequence $\{\tau_n\}$ converging to 0, $\{w_{\delta,\tau_n}\}$ converges, locally uniformly in Q_∞^Ω to a solution w_δ of (1.6) in Q_∞^Ω which satisfies

$$\max\{\underline{u}_{\mu^*}, \overline{u}_{(\partial_\mu F)_\delta}\} \leq w_\delta \leq \underline{u}_{\mu^*} + \overline{u}_{(\partial_\mu F)_{\delta'}} \qquad \text{in } Q_\infty^{\Omega_R}. \tag{6.51}$$

We can replace δ' by δ in this inequality, this proves that w_δ vanishes on $\partial_\ell Q_\infty^{\Omega_R}$ has initial trace $(\mu_\delta, (\partial_\mu F)_\delta)$, therefore (6.49) becomes

$$u(x,t) \leq w_\delta(x,t) + \epsilon + w_{B_R}(x,t) \qquad \forall (x,t) \in Q_\infty^{\Omega_R}. \tag{6.52}$$

Letting successively $R \to \infty$ and $\epsilon \to 0$ we deduce that w_δ is larger that any $u \in \mathcal{U}_{\partial_\mu F,\mu^*}(\Omega)$ in $Q_\infty^{\Omega_R}$. Since $h_{\delta,\tau}$ decreases with δ, w_δ shares this property and the limit, denoted by $\overline{u}_{\mu^*}$ is a solution of (1.6) in Q_∞^Ω which vanishes on $\partial_\ell Q_\infty^{\Omega_R}$ which is large than u, thus it is the maximal element of $\mathcal{U}_{\partial_\mu F,\mu^*}(\Omega)$. □

Proposition 6.12 *Under the assumptions of Proposition* 6.11*, we set* $F_\delta := \{x \in \Omega : \text{dist}\,(x,F) \leq \delta\}$ *and* $\mathcal{R}_\delta := \Omega \setminus F_\delta \subset \mathcal{R}$*. If we define the measure* $\tilde{\mu}_\delta$ *in* Ω *by*

$$\tilde{\mu}_\delta = \begin{cases} \mu & \text{on } \mathcal{R}_\delta \\ 0 & \text{on } F_\delta, \end{cases}$$

then $u_{\tilde{\mu}_\delta} \uparrow \underline{u}^*$ *when* $\delta \downarrow 0$.

Proof. There holds $\tilde{\mu}_\delta \leq \mu_\delta$ which implies $u_{\tilde{\mu}_\delta} \leq u_{\mu_\delta}$. When $\delta \to 0$, $u_{\tilde{\mu}_\delta} \uparrow \underline{u}^* \leq \underline{u}_{\mu^*}$, thus $\underline{u}^*$ is a positive solution of (1.6) in Q_∞^Ω which vanishes on $\partial_\ell Q_\infty^\Omega$. Then $tr_\Omega(\underline{u}^*) = (\mathcal{S}'', \mu'')$ and $\mathcal{S}'' \subset \partial_\mu F$ and $\mu'' \leq \mu^*$ on $\mathcal{R}^\mu$. Furthermore $\mu'' = \mu = \mu^*$ on $\mathcal{R}$, as in the proof of Proposition 6.11. Since $\mu^* = 0$ on $\mathcal{R}^\mu \setminus \mathcal{R}$ it follows that $\mu'' = \mu^*$ on $\mathcal{R}^\mu$. Let $a \in \mathcal{R}'' \cap \partial_\mu F$ and $R > 0$ such that $\overline{B}_R(a) \subset \mathcal{R}''$. Then $\mu''(\overline{B}_R(a)) < \infty$. Therefore

$$\mu''(\overline{B}_R(a) \cap \mathcal{R}) = \mu(\overline{B}_R(a) \cap \mathcal{R}) < \infty,$$

contradiction. Thus $\mathcal{S}'' = \partial_\mu F$ and $tr_\Omega(\underline{u}^*) = (\partial_\mu F, \mu^*)$. Since $\underline{u}^* \leq \underline{u}_{\mu^*}$ and $\underline{u}_{\mu^*}$ is minimal, it follows that $\underline{u}^* = \underline{u}_{\mu^*}$. □

Proposition 6.13 *Assume* $v \in C(\overline{\Omega} \times (0,\infty))$ *is a positive sub-solution (resp. supersolution) of (1.6) in* Q_∞^Ω *which vanishes on* $\partial\ell Q_\infty^\Omega$*. Then there exists a minimal solution* $\pi_+(v)$ *larger than* v *(resp. a maximal solution* $\pi_-(v)$ *smaller than* v *and vanishing on* $\partial\ell Q_\infty^\Omega$*).*

Proof. Assume v is a subsolution. Let $\tau > 0$ and let u_τ be the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q = 0 \qquad & \text{in } \Omega \times (\tau, \infty) \\ u \geq 0, u = 0 \qquad & \text{in } \partial\Omega \times (\tau, \infty) \\ u(.,\tau) = v(.,\tau) \qquad & \text{in } \Omega. \end{aligned} \tag{6.53}$$

Existence and uniqueness follows from Proposition 6.10. Furthermore $u_\tau \geq v$ in $\Omega \times (\tau, \infty)$. This implies that for $0 < \tau < \tau'$, $u_\tau \geq u_{\tau'}$. Since $u_\tau(x,t) \leq c_\alpha (t-\tau)^{-\frac{1+\alpha}{q-1}}$, there exists $\pi_+(v) = \lim_{\tau \to 0} u_\tau$, and $\pi_+(v)$ is a positive solution of (1.6) in Q^Ω_∞ and is larger than v. If u is any positive solution of (1.6) in Q^Ω_∞, vanishing on $\partial\ell Q^\Omega_\infty$ and larger than v, for any $\tau > 0$ it is larger than $u(.,\tau)$, thus it is larger than u_τ on $\Omega \times (\tau, \infty)$. Therefore $u \geq \pi_+(v)$.

Assume now that u is a supersolution. We define u_τ by (6.53). Then $u_\tau \leq v$ and $u_\tau \leq u_{\tau'}$ for $0 < \tau < \tau'$. Then $\pi_-(v) = \lim_{\tau \to 0} u_\tau$, and $\pi_-(v)$ is a positive solution of (1.6) in Q^Ω_∞ and is smaller than v, and thus vanishing on $\partial\ell Q^\Omega_\infty$. Similarly as above $\pi_-(v)$ is larger than any positive solution smaller than v. □

Theorem 6.14 *Assume $\Omega \subseteq \mathbb{R}^N$ is either $\mathbb{R}^N$ or an open domain with a C^2 compact boundary, $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Then for any $\nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega)$ there exist a maximal positive solution $\overline{u}_{\mathcal{S},\mu}$ and a minimal positive solution $\underline{u}_{\mathcal{S},\mu}$ of (1.6) in Q^Ω_∞ vanishing on $\partial\Omega \times (0,\infty)$ with initial trace ν. Furthermore, if $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$, then*

$$\overline{u}_{\mathcal{S},\mu} - \underline{u}_{\mathcal{S},\mu} \leq \overline{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0}. \tag{6.54}$$

Proof. Step 1: Construction of the maximal and minimal solutions. The functions $\underline{u}_{\mu^*}$, $\underline{u}_{\mathcal{S},0}$, $\overline{u}_{\mu^*}$ and $\overline{u}_{\mathcal{S},0}$ have been defined in Proposition 6.11 and Proposition 6.7. Since $\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\}$ is a subsolution of (1.6) which is smaller than the supersolution $\overline{u}_{\mu^*} + \overline{u}_{\mathcal{S},0}$ we set

$$(i)\quad \underline{u}_{\mathcal{S},\mu} = \pi_+(\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\}) \quad \text{and} \quad (ii)\quad \overline{u}_{\mathcal{S},\mu} = \pi_-(\overline{u}_{\mu^*} + \overline{u}_{\mathcal{S},0}). \tag{6.55}$$

Then $\underline{u}_{\mathcal{S},\mu}$ and $\overline{u}_{\mathcal{S},\mu}$ are solutions which satisfy

$$\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{\mathcal{S},\mu} \leq \overline{u}_{\mathcal{S},\mu} \leq \overline{u}_{\mu^*} + \overline{u}_{\mathcal{S},0}. \tag{6.56}$$

Therefore $\underline{u}_{\mathcal{S},\mu}$ and $\overline{u}_{\mathcal{S},\mu}$ vanish on $\partial_\ell Q^\Omega_\infty$, they have initial trace μ on $\mathcal{R}$ and are larger than any $u_{\infty,a}$ for $a \in \mathcal{S}$ (notice that $\partial_\mu \mathcal{S} \subset \mathcal{S}$). This implies that $\underline{u}_{\mathcal{S},\mu}$ and $\overline{u}_{\mathcal{S},\mu}$ belong to $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$.

Let $u \in \mathcal{U}_{\mathcal{S},\mu}(\Omega)$. Then for $\sigma > \delta > 0$ and $\epsilon, R > 0$, there exists $\tau_1 > 0$ such that

$$u(x,t) \leq u_{\sigma,\delta}(x,t) \text{ for } (x,t) \in (\mathcal{S}_{\frac{\delta}{2}} \cap \Omega_R) \times (0, \tau_1].$$

There exists $\tau_2 \in (0, \tau_1]$ such that

$$u(x,t) \leq \overline{u}_{\mu^*}(x,t) + \epsilon + w_{B_R}(x,t) \text{ for } (x,t) \in (\Omega_R \setminus \mathcal{S}_{\frac{\delta}{2}}) \times (0, \tau_2].$$

Therefore

$$u(x,t) \leq u_{\sigma,\delta}(x,t) + \overline{u}_{\mu^*}(x,t) + \epsilon + w_{B_R}(x,t) \text{ for } (x,t) \in \Omega_R \times (0, \tau_2].$$

Therefore

$$u \leq u_{\sigma,\delta} + u_{\mu^*} + \epsilon + w_{B_R} \quad \text{ in } Q^{\Omega_R}_\infty \tag{6.57}$$

Letting successively $R \to \infty$, $\epsilon \to 0$, $\delta \to 0$ and $\sigma \to 0$ we obtain $u \leq \overline{u}_{\mathcal{S},0} + \overline{u}_{\mu^*}$, and therefore $u \leq \pi_+(\overline{u}_{\mathcal{S},0} + \overline{u}_{\mu^*}) = \overline{u}_{\mathcal{S},\mu}$. Next, we also have

$$u \geq \tilde{u}_{\mathcal{S},0} := \sup\{u_{\infty,a} : a \in \mathcal{S}\} \Longrightarrow u \geq \underline{u}_{\mathcal{S},0} = \pi_+(\tilde{u}_{\mathcal{S},0}),$$

by (6.25). With the notations of Proposition 6.12 with $F = \mathcal{S}$, for any $R > 0$, $\delta > 0$ and $\epsilon > 0$, there exists τ_ϵ such that

$$u_{\tilde{\mu}_\delta}(x,t) \le u(x,t) + w_{B_R}(x,t) + \epsilon \qquad \text{in } \Omega_R \times (0,\tau_\epsilon],$$

because the support of $\tilde{\mu}_\delta$ is included in $\Omega \setminus \mathcal{S}_\delta$. Therefore this last inequality holds in $Q_\infty^{\Omega_R}$ and consequently

$$\sup\{u_{\tilde{\mu}_\delta}, \underline{u}_{\mathcal{S},0}\} \le u + \epsilon + w_{B_R} \qquad \text{in } Q_\infty^{\Omega_R},$$

and we can let $R \to \infty$ and $\epsilon \to 0$ to obtain $\sup\{u_{\tilde{\mu}_\delta}, \underline{u}_{\mathcal{S},0}\} \le u$ in Q_∞^Ω. Letting $\delta \to 0$ and using Proposition 6.12 we get $\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \le \underline{u}_{\mathcal{S},\mu} \le u$.

Step 2: Alternative construction. For $0 < \delta < \sigma$ and $n \in \mathbb{N}_*$ we denote by $\overline{u}_{n,\sigma,\delta,\mu}$ the solution of

$$\begin{array}{rl} \partial_t u - \Delta u + t^\alpha u^q = 0 & \text{in } Q_\infty^{\Omega_\sigma} \\ u \ge 0 & \text{in } Q_\infty^{\Omega_\sigma} \\ u(x,t) = 0 & \text{in } \partial_\ell Q_\infty^{\Omega_\sigma} \\ u(.,0) = \mu_\delta + n\chi_{\mathcal{S}_\delta} & \text{in } \Omega_\sigma. \end{array} \tag{6.58}$$

We denote here $\Omega_\sigma = \{x \in \mathbb{R}^N : \text{dist}\,(x,\Omega) < \sigma\}$, $\mathcal{S}_\delta = \{x \in \mathbb{R}^N : \text{dist}\,(x,\mathcal{S}) < \delta\}$ and $\mathcal{R}_\delta = \Omega \cap \mathcal{S}_\delta^c$ and $\mu_\delta = \chi_{\mathcal{R}_\delta}\mu$. The same arguments of monotonicity as in Proposition 6.7 and Proposition 6.11 show that

$$\lim_{\sigma\to 0}\lim_{\delta\to 0}\lim_{n\to\infty} = \overline{u}_{n,\sigma,\delta,\mu} = \overline{u}_{\mathcal{S},\mu}. \tag{6.59}$$

If $\tau > 0$ we denote by $\underline{u}_{\tau,\delta,\mu}$ the solution of

$$\begin{array}{rl} \partial_t u - \Delta u + t^\alpha u^q = 0 & \text{in } Q_\infty^{\Omega} \\ u \ge 0 & \text{in } Q_\infty^{\Omega} \\ u(x,t) = 0 & \text{in } \partial_\ell Q_\infty^{\Omega} \\ u(.,0) = \mu_\delta + \chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau) & \text{in } \Omega, \end{array} \tag{6.60}$$

Using estimate (6.3) and Proposition 6.9 is is easy to prove that $\underline{u}_{\tau,\delta,\mu} \le u$ for any $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$. Furthermore

$$\max\{u_{\mu_\delta}, u_{\chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)}\} \le \underline{u}_{\tau,\delta,\mu} \le u_{\mu_\delta} + \underline{u}_{\mathcal{S},0}(.,.+\tau),$$

since have

$$u_{\chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)} \le \underline{u}_{\mathcal{S},0}(.,\tau) \le u_{\chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)} + C\left(\frac{\delta^2}{\tau}\right)^{\frac{1+\alpha}{q-1}-\frac{1}{2}} e^{-\frac{\delta^2}{4\tau}}$$

by (5.5) with $N = 1$. Set $c(\delta,\tau) = C\left(\frac{\delta^2}{\tau}\right)^{\frac{1+\alpha}{q-1}-\frac{1}{2}} e^{-\frac{\delta^2}{4\tau}}$, then

$$\max\{u_{\mu_\delta}, u_{\chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)}\} \ge \max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}(.,\tau) - c(\delta,\tau)\} \ge \max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}(.,\tau)\} - c(\delta,\tau).$$

Therefore, if $\underline{u}_{\tau_n,\delta,\mu} \to \underline{u}_{0,\delta,\mu}$ locally uniformly in Q_∞^Ω, then $\underline{u}_{0,\delta,\mu}$ is a solution of (1.6) in Q_∞^Ω which satisfies

$$\max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}\} \le \underline{u}_{\delta,0,\mu} \le u_{\mu_\delta} + \underline{u}_{\mathcal{S},0}, \tag{6.61}$$

and is smaller than any $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$. There exists $\delta_n \to 0$ such that $\underline{u}_{0,\delta_n,\mu} \to \underline{u}_{0,0,\mu}$. Then

$$\max\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \le \underline{u}_{0,0,\mu} \le \underline{u}_{\mu^*} + \underline{u}_{\mathcal{S},0}, \tag{6.62}$$

and $\underline{u}_{0,0,\mu}$ is an element of $\mathcal{U}_{\mathcal{S},0}(\Omega)$ smaller than any $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$. Thus $\underline{u}_{0,0,\mu} = \underline{u}_{\mathcal{S},\mu}$ and

$$\lim_{\delta\to 0}\lim_{\tau\to 0} \underline{u}_{\tau,\delta,\mu} = \underline{u}_{\mathcal{S},\mu}. \tag{6.63}$$

Step 3: Proof of (6.54). We assume $\inf\{|z-z'| : z \in \mathcal{S}, z' \in \Omega^c\} = \delta_0 > 0$, so that we can take $\sigma = 0$ in the construction of $\overline{u}_{\mathcal{S},\mu}$. Put $\tau = (\frac{c_\alpha}{n})^{\frac{q-1}{1+\alpha}}$ and

$$\overline{Z}_{n,\delta,\mu} = \overline{u}_{n,\delta,\mu} - \overline{u}_{n,\delta,0}\,,\ \underline{Z}_{\tau,\delta,\mu} = \underline{u}_{\tau,\delta,\mu} - \underline{u}_{\tau,\delta,0} \text{ and } W_{n,\delta,\mu} = \overline{Z}_{n,\delta,\mu} - \underline{Z}_{\tau,\delta,\mu}. \tag{6.64}$$

Then $w = W_{n,\delta,\mu}$ satisfies

$$\partial_t w - \Delta w + t^\alpha \left(\overline{u}^q_{n,\delta,\mu} - \overline{u}^q_{n,\delta,0} - \underline{u}^q_{\tau,\delta,\mu} + \underline{u}^q_{\tau,\delta,0}\right) \tag{6.65}$$

in Q^Ω_∞ and we can write

$$\begin{aligned}\overline{u}^q_{n,\delta,\mu} - \overline{u}^q_{n,\delta,0} - \underline{u}^q_{\tau,\delta,\mu} + \underline{u}^q_{\tau,\delta,0} &= (\overline{u}^q_{n,\delta,\mu} - \underline{u}^q_{\tau,\delta,\mu}) - (\overline{u}^q_{n,\delta,0} - \underline{u}^q_{\tau,\delta,0}) \\ &= d_\mu(\overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}) - d_0(\overline{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0})\end{aligned} \tag{6.66}$$

where

$$d_\mu(x,t) = \begin{cases} \dfrac{\overline{u}^q_{n,\delta,\mu} - \underline{u}^q_{\tau,\delta,\mu}}{\overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}} & \text{if } \overline{u}_{n,\delta,\mu} \neq \underline{u}_{\tau,\delta,\mu} \\ 0 & \text{if } \overline{u}_{n,\delta,\mu} = \underline{u}_{\tau,\delta,\mu} \end{cases} \tag{6.67}$$

and d_0 is defined accordingly. Since

$$\overline{u}_{n,\delta,\mu} \ge \max\{\underline{u}_{\tau,\delta,\mu}, \overline{u}_{n,\delta,0}\} \text{ and } \underline{u}_{\tau,\delta,0} \le \min\{\underline{u}_{\tau,\delta,\mu}, \overline{u}_{n,\delta,0}\},$$

there holds $d_\mu \ge d_0 \ge$ by convexity. Using the fact that $\overline{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0} \ge 0$ is infers that

$$d_\mu(\overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}) - d_0(\overline{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}) \ge d_\mu(\overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu} - \overline{u}_{n,\delta,0} + \underline{u}_{\tau,\delta,0})$$

Finally (6.65) becomes

$$\partial_t w - \Delta w + t^\alpha d_\mu w \le 0 \qquad \text{in } Q^\Omega_\infty. \tag{6.68}$$

Furthermore, in the sense of measures,

$$w(.,0) = \mu_\delta + n\chi_{\mathcal{S}_\delta} - n\chi_{\mathcal{S}_\delta} - (\mu_\delta + \chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau)) + \chi_{\mathcal{S}_\delta}\underline{u}_{\mathcal{S},0}(.,\tau) = 0.$$

Because $w = 0$ in $\partial_\ell Q^\Omega_\infty$ it follows $w \le 0$ by the maximum principe. Therefore

$$\overline{Z}_{n,\delta,\mu} \le \underline{Z}_{\tau,\delta,\mu} \Longrightarrow \overline{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu} \le \overline{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}. \tag{6.69}$$

If we let successively $n \to \infty$ (and therefore $\tau \to 0$) and $\delta \to 0$, we obtain (6.54). □

Remark. We do not know if (6.54 holds if we do not assume $\inf\{|z-z'| : z \in \mathcal{S}, z' \in \Omega^c\} = \delta_0 > 0$. However, if for $\theta > 0$ we set $\mathcal{S}_\theta = \mathcal{S} \cap \{x \in \Omega : \mathrm{dist}\,(x, \Omega^c) \geq \theta\}$, then we have

$$\overline{u}_{\mathcal{S}_\theta,\mu} - \underline{u}_{\mathcal{S}_\theta,\mu} \leq \overline{u}_{\mathcal{S}_\theta,0} - \underline{u}_{\mathcal{S}_\theta,0}. \tag{6.70}$$

Furthermore all the four above functions increases when θ decreases to 0. If we set

$$\begin{array}{ll} (i) & \lim_{\theta\to 0} \overline{u}_{\mathcal{S}_\theta,\mu} = \overline{u}_{\underline{\mathcal{S}},\mu} \\ (ii) & \lim_{\theta\to 0} \overline{u}_{\mathcal{S}_\theta,0} = \overline{u}_{\underline{\mathcal{S}},0} \\ (iii) & \lim_{\theta\to 0} \underline{u}_{\mathcal{S}_\theta,\mu} = \underline{u}_{\underline{\mathcal{S}},\mu} \\ (iv) & \lim_{\theta\to 0} \underline{u}_{\mathcal{S}_\theta,0} = \underline{u}_{\underline{\mathcal{S}},0} \end{array} \tag{6.71}$$

then we infer that

$$\overline{u}_{\underline{\mathcal{S}},\mu} - \underline{u}_{\underline{\mathcal{S}},\mu} \leq \overline{u}_{\underline{\mathcal{S}},0} - \underline{u}_{\underline{\mathcal{S}},0}. \tag{6.72}$$

Our final result is the following existence and uniqueness theorem the proof is similar to the one of [7, Th 3.5].

Theorem 6.15 *Assume $\Omega \subseteq \mathbb{R}^N$ is either $\mathbb{R}^N$ or an open domain with a C^2 compact boundary, $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Then for any $\nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega)$ such that $\rho\mu$ is bounded in any neighborhood of $\partial\Omega$ and $\inf\{|z-z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$, the set $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$ contains one and only one element.*

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