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Initial trace of solutions of general semilinear heat equation

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Abstract We study the initial trace problem for positive solutions of semilinear heat equations with strong absorption. We emphasize the particular case where u solves $\partial_t u - \Delta u + t^\alpha u^q = 0$, with $q > 1$ and $\alpha > -1$.

1 Introduction

In this paper we study the initial trace problem for positive solutions of

$$\partial_t u - \Delta u + g(x, t, u) = 0 \text{ dans } Q_T^\Omega := \Omega \times (0, T) \quad (1.1)$$

where Ω is a smooth domain in \mathbb{R}^N containing O and $g \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R})$, $g(x, t, \cdot)$ is nondecreasing $\forall (x, t) \in \Omega \times \mathbb{R}$ and $g(x, t, r) \geq 0$ if $r \geq 0$. We give general conditions for the existence of an initial trace which belongs to the class of positive Borel measures.

Theorem A *Assume g satisfies the above conditions and that equation (1.1) possesses a barrier at any $z \in \Omega$. Then if $u \in C^1(Q_T^\Omega) \cap C(\Omega \times [0, T] \setminus \{(O, 0)\})$ is a positive solution of (1.1) which vanishes on $\Omega \setminus \{O\}$ at $t = 0$ it admits an initial trace which is a outer regular positive Borel measure.*

The initial trace of positive solutions of (1.1) exists in the following sense: *there exists a relatively closed set $\mathcal{S} \subset \Omega$ and a positive Radon measure μ on $\mathcal{R} := \Omega \setminus \mathcal{S}$ with the following properties:*

(i) *for any $x_0 \in \mathcal{S}$ and any $\epsilon > 0$*

$$\lim_{t \rightarrow 0} \int_{B_\epsilon(x_0) \cap \Omega} u(x, t) dx = \infty, \quad (1.2)$$

(ii) *for any $\zeta \in C_c(\mathcal{R})$*

$$\lim_{t \rightarrow 0} \int_{\Omega} u(x, t) \zeta(x) dx = \int_{\Omega} \zeta d\mu. \quad (1.3)$$

The couple (\mathcal{S}, μ) is unique and characterizes a unique positive outer regular Borel measure ν on Ω . The barrier property is always verified if g satisfies the Keller-Osserman condition, that is $g(x, t, r) \geq h(r) \geq 0$ where h is nondecreasing and

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} \quad \text{where } H(s) = \int_0^s h(t)dt. \quad (1.4)$$

It is also satisfied if $g(r) = t^\alpha |u|^{q-1} u$, with $q > 1$ and $\alpha > -1$. This notion has been introduced by Marcus and Véron [10] in the study of the boundary trace of positive solutions of

$$-\Delta u + g(x, u) = 0 \text{ dans } \Omega. \quad (1.5)$$

In the second part we concentrate on the particular case of equation

$$\partial_t u - \Delta u + t^\alpha u^q = 0 \text{ dans } Q_T^\Omega \quad (1.6)$$

where $T > 0$, $\alpha > -1$ and $q > 1$. Among the main tools are the solutions with an isolated singularity at $(0, O)$, which are solutions u of (1.6) in Q_T^Ω , which belong to $C^{2,1}(Q_T^\Omega) \cap C(\Omega \times [0, T] \setminus \{(0, 0)\})$ and satisfy

$$u(x, 0) = 0 \text{ in } \Omega \setminus \{O\}. \quad (1.7)$$

When $\alpha = 0$, Brezis and Friedman prove in [2] that if $B_{2R} \subset \Omega$, the following estimate holds:

$$u(x, t) \leq \frac{C(N, q, R)}{(|x|^2 + t)^{\frac{1}{q-1}}} \quad \forall (x, t) \in B_R \setminus \{0\} \times [0, T]. \quad (1.8)$$

They proved that if $1 < q < q_N := 1 + \frac{2}{N}$ and $k > 0$ there exist singular solutions with initial data $u(\cdot, 0) = k\delta_O$, unique if u vanishes on $\partial\Omega \times [0, T]$. In this range of exponents, Brezis, Peletier, Terman obtain in [1] the existence and uniqueness of a *very singular solution* of (1.6), always with $\alpha = 0$, that is a positive solution in Q_∞ under the form

$$v_0(x, t) = t^{-1/(q-1)} V_0\left(\frac{x}{\sqrt{t}}\right),$$

where $V_0 > 0$ is C^2 and satisfies

$$\begin{aligned} -\Delta V_0 - \frac{1}{2}\eta \cdot \nabla V_0 - \frac{1}{q-1} V_0 + V_0^q &= 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|\eta| \rightarrow \infty} |\eta|^{\frac{2}{q-1}} V_0(\eta) &= 0. \end{aligned} \quad (1.9)$$

Actually, Kamin and Peletier proved that v_0 is the limit of the solutions u_k of (1.6) in Q_∞ which satisfy $u(\cdot, 0) = k\delta_O$. The very singular singular solution play a fundamental role in Marcus and Véron's description [7] of the initial trace of positive solutions of (1.6) with $\alpha = 0$. In [9], Marcus et Véron study this equation when $\alpha \geq 0$ and $1 < q < q_{\alpha, N} = 1 + \frac{2(1+\alpha)}{N}$. They prove the existence of a self-similar solution of (1.6) under the form

$$v_\alpha(x, t) = t^{-\frac{1+\alpha}{q-1}} V_\alpha\left(\frac{x}{\sqrt{t}}\right),$$

which satisfies

$$\lim_{t \rightarrow 0} v_\alpha(x, t) = 0 \quad \forall x \neq 0$$

and

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} v_\alpha(x, t) dx = \infty \quad \forall \epsilon > 0.$$

The function V_α verifies

$$-\Delta V_\alpha - \frac{1}{2}\eta \cdot \nabla V_\alpha - \frac{1+\alpha}{q-1}V_\alpha + V_\alpha^q = 0 \text{ in } \mathbb{R}^N. \quad (1.10)$$

Furthermore

$$V_\alpha(\eta) = C|\eta|^{\frac{2(1+\alpha)}{q-1}-N} e^{-\frac{|\eta|^2}{4}} (1 + o(1)) \text{ as } |\eta| \rightarrow \infty.$$

If $1 < q < q_{\alpha, N}$, they show that for every $k > 0$ there exists a unique solution u_k of (1.6) with initial data $k\delta_O$. Furthermore $\lim_{k \rightarrow \infty} u_k = v_\alpha$. Actually the limitation $\alpha \geq 0$ can be relaxed to $\alpha > -1$ as we will see it later on.

In this article we extend Brezis-Friedman removability result to equation (1.6) and prove that if $q \geq q_{\alpha, N}$, any solution in Q_T which satisfies (1.7) is identically 0. The starting point of our study is the following extension of estimate (1.8) valid for any $\alpha > -1$ and $q > 1$.

$$u(x, t) \leq \frac{C(N, q, \alpha, R)}{\left(|x|^2 + t\right)^{\frac{1+\alpha}{q-1}}} \quad \forall (x, t) \in B_R \setminus \{0\} \times [0, T]. \quad (1.11)$$

We prove that the following classification results holds.

Theorem B *Assume $1 < q < q_{\alpha, N}$ and $u \in C^1(Q_T^O) \cap C(\Omega \times [0, T] \setminus \{(O, 0)\})$ is a positive solution of (1.6) which vanishes on $\Omega \setminus \{O\}$ at $t = 0$. Then*

(i) *either there exists $k \geq 0$ such that $u(\cdot, 0) = k\delta_O$ and*

$$u(x, t) \sim kE(x, t) \quad \text{as } (x, t) \rightarrow (O, 0), \quad (1.12)$$

where $E(x, t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$,

(ii) *or*

$$u(x, t) \sim v_\alpha(x, t) \quad \text{as } (x, t) \rightarrow (O, 0). \quad (1.13)$$

In the supercritical case we obtain a removability statement.

Theorem C *Assume $q \geq q_{\alpha, N}$ and u is as in Theorem A, then u can be extended by continuity as a function in $C(\Omega \times [0, T])$.*

We prove that equation (1.6) admits a barrier at any $z \in \Omega$. More precisely we construct a positive solution w_{B_R} of (1.6) in $Q_\infty^{B_R}$ which tends to 0 locally uniformly in B_R when $t \rightarrow 0$ and which blows-up uniformly on $\partial B_R \times [\tau, \infty)$, for any $\tau > 0$. Using sharp asymptotics of the function V_α , we have the following result which extends Theorem A.

Theorem D *Assume $1 < q < q_{\alpha, N}$ and $\Omega \subset \mathbb{R}^N$ is open with a C^2 compact boundary, eventually empty. Then for any couple (\mathcal{S}, μ) where $\mathcal{S} \subset \Omega$ is relatively closed and $\mu \in \mathfrak{M}_+(\Omega \setminus \mathcal{S})$, there exists a maximal and a minimal $u \in C^1(Q_T^O) \cap C^1(\overline{\Omega} \times (0, T])$ which both satisfy (i) and (ii) and vanish on $\partial\Omega \times (0, T]$. If $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$ and μ is bounded in a neighborhood of $\partial\Omega$, then u is unique.*

2 Initial trace

In this section $\Omega \subset \mathbb{R}^N$ is an open set, $Q_T^\Omega = \Omega \times (0, T)$, $\partial_\ell Q_T^\Omega = \Omega \times \{0\} \cup \partial\Omega \times [0, T)$ and $g \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R})$. If u is defined in $\Omega \times \mathbb{R}_+$, we denote by $g \circ u$ the function $(x, t) \mapsto g(x, t, u(x, t))$. We say that g belongs to \mathcal{H} (resp \mathcal{H}_0) if

$$\begin{aligned} g(x, t, r) &\geq 0 \quad \forall (x, t, r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \\ &\text{(resp. } g \in \mathcal{H} \text{ and } r \mapsto g(x, t, r) \text{ is nondecreasing)}. \end{aligned} \quad (2.1)$$

We denote by $\mathfrak{M}(\Omega)$ the set of Radon measures in Ω , and by $\mathfrak{M}^b(\Omega)$ (resp. $\mathfrak{M}^{b,\rho}(\Omega)$) the subset of Radon measures such that

$$\int_\Omega d|\mu| < \infty \quad \left(\text{resp. } \int_\Omega \rho d|\mu| < \infty \right),$$

where $\rho(x) := \text{dist}(x, \partial\Omega)$. Their positive cones are respectively $\mathfrak{M}_+(\Omega)$, $\mathfrak{M}_+^b(\Omega)$ and $\mathfrak{M}_+^{b,\rho}(\Omega)$.

Definition 2.1 *Let \mathcal{S} be a relatively closed subset of Ω and μ a Radon measure on $\mathcal{R} := \Omega \setminus \mathcal{S}$. We say that a nonnegative function $u \in C(Q_T^\Omega)$ admits the couple (\mathcal{S}, μ) for initial trace if*

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}} u(x, t) \zeta(x) dx = \int_{\mathcal{R}} \zeta d\mu \quad \forall \zeta \in C_c(\mathcal{R}),$$

and

$$\lim_{t \rightarrow 0} \int_U u(x, t) dx = \infty \quad \forall U \subset \Omega, U \text{ open}, U \cap \mathcal{S} \neq \emptyset$$

The set \mathcal{S} is the set of singular initial points of u and its complement \mathcal{R} the set of regular initial points. We write $\text{tr}_\Omega(u) = (\mathcal{S}, \mu)$.

Let $\tilde{\mu}$ be the extension of μ as a locally bounded Borel measure. To the couple $(\mathcal{S}, \tilde{\mu})$ we can associate a unique Borel measure ν defined by

$$\nu(E) = \begin{cases} \infty & \forall E \subset \Omega : E \text{ Borel}, E \cap \mathcal{S} \neq \emptyset \\ \tilde{\mu}(E) & \forall E \subset \Omega : E \text{ Borel}, E \subset \mathcal{R}. \end{cases} \quad (2.2)$$

Lemma 2.1 *Assume Ω is a bounded open set with a C^2 boundary, $T > 0$, $g \in \mathcal{H}$, and let $u \in C(\overline{\Omega} \times (0, T])$ be a positive solution of*

$$\partial_t u - \Delta u + g \circ u = 0 \quad \text{in } Q_T^\Omega. \quad (2.3)$$

If $g \circ u \in L^1_\rho(Q_T^\Omega)$, then $u \in L^\infty(0, T, L^1_\rho(\Omega))$ and there exists $\mu \in \mathfrak{M}_+^\rho(\Omega)$ such that

$$\lim_{t \rightarrow 0} \int_\Omega u(x, t) \zeta(x) dx = \int_\Omega \zeta d\mu \quad \forall \zeta \in C_c(\Omega). \quad (2.4)$$

Proof. If $\phi_1 > 0$ is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ and λ_1 is the corresponding eigenvalue, we have

$$\frac{d}{dt} \int_{\Omega} u \phi_1 dx + \lambda_1 \int_{\Omega} u \phi_1 dx + \int_{\Omega} g \circ u \phi_1 dx + \int_{\partial\Omega} u \frac{\partial \phi_1}{\partial \nu} dS = 0,$$

where ν is the normal vector. Set $X = \int_{\Omega} u \phi_1 dx$, then by Hopf Lemma,

$$X' + \lambda_1 X + \int_{\Omega} g \circ u \phi_1 dx \geq 0$$

which yields to

$$\frac{d}{dt} (e^{\lambda_1 t} X) + e^{\lambda_1 t} \int_{\Omega} g \circ u \phi_1 dx \geq 0.$$

For $s \in (t, T)$

$$\frac{d}{dt} \left(e^{\lambda_1 t} X - \int_t^T e^{\lambda_1 s} \int_{\Omega} g \circ u \phi_1 dx ds \right) \geq 0,$$

which means that the mapping

$$t \rightarrow e^{\lambda_1 t} X - \int_t^T e^{\lambda_1 s} \int_{\Omega} g \circ u \phi_1 dx ds$$

is nondecreasing. Therefore

$$e^{\lambda_1 t} X - \int_t^T e^{\lambda_1 s} \int_{\Omega} g \circ u \phi_1 dx ds \leq e^{\lambda_1 T} X.$$

and finally

$$X \leq e^{\lambda_1(T-t)} X + e^{-\lambda_1 t} \int_t^T e^{\lambda_1 s} \int_{\Omega} g \circ u \phi_1 dx ds$$

Since $\rho^{-1}\phi_1$ is positively bounded from above and from below, $u \in L^\infty(0, T, L_\rho^1(\Omega))$ and there exists a sequence $\{t_n\}$ decreasing to 0 and a measure $\mu \in \mathfrak{M}_+^p(\Omega)$ such that

$$\lim_{t_n \rightarrow 0} \int_{\Omega} u(x, t_n) \zeta dx = \int_{\Omega} \zeta d\mu \quad \forall \zeta \in C_c(\Omega).$$

If $\zeta \in C_c^2(\Omega)$ there holds

$$\int_{\Omega} u(x, t_n) \zeta dx = \int_{t_n}^T \int_{\Omega} (g \circ u \zeta - u \Delta \zeta) dx dt + \int_{\Omega} u(x, T) \zeta dx,$$

thus

$$\int_{\Omega} \zeta d\mu = \iint_{Q_T^{\Omega}} (g \circ u \zeta - u \Delta \zeta) dx dt + \int_{\Omega} u(x, T) \zeta dx.$$

This implies that μ is uniquely determined and $u(\cdot, t)$ converges to μ in the weak sense of measures. \square

Corollary 2.1 Assume Ω is an open domain, $g \in \mathcal{H}$ and $u \in C^2(Q_T^\Omega)$ is a positive solution of (2.4). Suppose that for any $z \in \Omega$ there exists an open neighborhood $U \subset \Omega$ such that

$$\int_0^T \int_U g \circ u dx dt < \infty.$$

Then $u(x, t) \in L^\infty(0, T, L_{loc}^1(\Omega))$ and there exists a positive Radon measure μ on U such that

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}} u(x, t) f(x) dx = \int_{\mathcal{R}} d\mu \quad \forall f \in C_c(\mathcal{R})$$

Proof. We apply the previous lemma in replacing U by a ball $B_\epsilon(z)$ and conclude by a partition of unity. \square

The following class of nonlinearity has been introduced by Marcus and Véron [10] in order to study the boundary trace of solutions of elliptic equations

Definition 2.2 A function $g \in \mathcal{H}$ is a coercive nonlinearity in Q_T^Ω if, for every subdomain Ω' of Ω and every $\epsilon \in (0, T)$, the set of positive solutions of (1.1) in $Q_{\epsilon, T}^{\Omega'} := \Omega' \times (\epsilon, T)$ is uniformly bounded in compact subsets of $Q_{\epsilon, T}^{\Omega'}$.

Definition 2.3 Let $z \in \Omega$. We say that equation (1.1) possesses a strong barrier at z if there exists a number $r_0 \in (0, \rho(z))$ such that, for every $r \in (0, r_0)$, there exists a positive super solution $w = w_{r, z}$ of (1.1) in $B_r(z) \times (0, T)$ such that

$$w \in C(B_r(z) \times [0, T]), \quad \lim_{|x-z| \rightarrow r} w(x, t) = \infty \text{ locally uniformly if } t \in (0, T). \quad (2.5)$$

Lemma 2.1 Assume $g \in \mathcal{H}$ is a coercive nonlinearity in Q_T^Ω , then the set of solutions of (1.1) in Q_T^Ω is uniformly bounded from above in every compact subset of Q_T^Ω . Furthermore, if $g \in \mathcal{H}_0$, $A \subset \Omega$ is open and (1.1) possesses a strong barrier at every point of $z \in A$, then the set of solutions u of (1.1) such that $u \in C(A \times [0, T])$ and $u(x, 0) = 0$ on A is uniformly bounded from above in every compact subset of $A \times [0, T]$.

Proof. Let K be a compact subset of Q_T^Ω and let Ω' be a smooth, bounded domain of Ω and $\epsilon > 0$ such that $K \subset Q_{\epsilon, T}^{\Omega'}$. Let $U = U_{Q_{\infty}^{\Omega'}}$ be the minimal large solution of (1.1) in $Q_T^{\Omega'}$, i.e. the limit, when $k \rightarrow \infty$, of solutions with Cauchy-Dirichlet data k on $\partial_\ell Q_{\epsilon, T}^{\Omega'} := \Omega' \times \{\epsilon\} \cup \partial\Omega' \times [\epsilon, T]$. By the maximum principle, if $u \in C(Q_T^\Omega)$ is a solution of (1.1), then $u \leq U$ in Ω' .

For the second statement, let K be a compact subset of A . For any $z \in K$ there exists $r_z > 0$ such that for any $r \in (0, r_z)$ there exists a positive super solution of (1.1) in $Q_T^{B_r(z)}$ which satisfies (2.5). Since K is compact, there exist z_1, \dots, z_p such that $K \subset \cup_{j=1}^p B_{r_{z_j}/2}(z_j)$.

For any $j \in \{1, \dots, p\}$ we denote by w_j the super solution in $Q_T^{B_{2r_{z_j}/3}(z_j)}$. By comparison principle, there holds

$$u(x, t) \leq \sup\{w_j(x, t) : (x, t) \in B_{r_{z_j}/2}(z_j) \times (0, T)\} := M_j, \quad (2.6)$$

for $(x, t) \in Q_T^{B_{r_{z_j}/2}(z_j)}$. Therefore $u \leq M = \max_{j=1, \dots, p} M_j$ in $K \times (0, T)$. \square

Lemma 2.2 Let $g \in \mathcal{H}$ and $u \in C^2(Q_T^\Omega)$ be a positive solution of (1.1) and suppose $z \in \Omega$ is such that

$$\int_0^T \int_{B_\epsilon(z) \cap \Omega} g \circ u dx dt = \infty \quad \forall \epsilon > 0. \quad (2.7)$$

Suppose that at least one of the following sets of conditions holds:

(i) There exists an open neighborhood $U' \subset \Omega$ of z such that $u \in L^1(U' \times (0, T))$.

(ii) The following hold:

1- $g \in \mathcal{H}_0$,

2- (1.1) possesses a strong barrier at z .

Then,

$$\lim_{t \rightarrow 0} \int_{B_\epsilon(z) \cap \Omega} u(x, t) dx = \infty \quad \forall \epsilon > 0. \quad (2.8)$$

Proof. Assume that Ω is bounded. First consider the case when condition (i) holds. Let $\phi \in C^{2,1}(U' \times [0, T])$ with compact support in $U' \times [0, T]$ and such that $\phi(x, 0) = 1$ in a neighborhood of z . Then

$$\int_t^T \int_{U'} (u(-\phi_t - \Delta\phi) + g \circ u \phi) dx dt = \int_{U'} u(t) \phi dx - \int_{U'} u(T) \phi dx. \quad (2.9)$$

By assumption $\int_t^T \int_{U'} u(\phi_t + \Delta\phi) dx dt$ is bounded. We let t tend to 0, the result follows from (2.7).

Next we assume that condition (ii) holds, $u \notin L^1(U' \times (0, T))$ for any neighborhood U' of z and that the conclusion is not valid. Thus there exist $r^* > 0$, such that $\overline{B}_{r^*}(z) \subset U_z$ and a sequence $\{t_n\}$ decreasing to 0 such that

$$\int_{B_{r^*}(z)} u(x, t_n) dx \leq M$$

for some $M > 0$. Furthermore g is coercive in $B_{r^*}(z) \times (0, T)$. Let $\{h_{n,k}\} \subset C^\infty(Q_T^\Omega)$ an increasing sequence with respect to k and n of nonnegative functions such that $h_{n,k} = 0$ on $B_{r^*}(z) \times \{0\}$, $0 \leq h_{n,k} \leq k$ and $h_{n,k} = k$ on $(t_n, T) \times \partial B_{r^*}(z)$. Let $w_{h_{n,k}}$ be the solution of (1.1) in $B_{r^*}(z) \times (0, T)$ such that $w_{h_{n,k}} = h_{n,k}$ on $\partial_\ell Q_T^{B_{r^*}(z)}$. By the maximum principle and condition (ii)-1, the sequence $\{w_{h_{n,k}}\}$ is monotone increasing with respect to k and n . Condition (ii)-3 implies that, for every $r < r^*$ and $\beta < T$, the sequence is bounded in $\overline{B}_r(z) \times [0, \beta]$, and since u is locally bounded in Q_T^Ω there exists $k = k(n)$ such that $k \geq u$ on $(t_n, T) \times \partial B_r(z)$ and $k(n) \rightarrow \infty$ when $n \rightarrow \infty$. Then $w = \lim_{n \rightarrow \infty} w_{h_{n,k}}$ is a solution of (1.1) which blows up on $\partial B_{r^*}(z) \times (0, T)$ and vanishes on $B_{r^*}(z) \times \{0\}$. Let v_n be the solution of the heat equation in $B_{r^*}(z) \times (t_n, T)$ such that $v_n(\cdot, t_n) = u(\cdot, t_n)$ in $B_{r^*}(z)$ and $v_n = 0$ on $\partial B_{r^*}(z) \times (t_n, T)$. Then $w_{h_{n,k(n)}} + v_n$ is a super solution of (1.1) in $B_{r^*}(z) \times (t_n, T)$ which dominates u on $\partial_\ell Q_{t_n, T}^{B_{r^*}(z)}$. By the maximum principle,

$$u \leq w_{h_{n,k(n)}} + v_n \quad \text{in } Q_{t_n, T}^{B_{r^*}(z)}.$$

And we have in particular

$$\int_{B_r(z)} u(x, t) dx \leq \int_{B_r(z)} (w_{h_n, k(n)} + v_n)(x, t) dx \leq M + \int_{B_r(z)} w(x, t) dx \quad \forall t \in (t_n, T).$$

Since it holds for any n , it implies $u \in L^1(Q_T^{B_r(z)})$, which leads to a contradiction. \square

Example 1. If $g(x, t, r) = h(r)$ where h is continuous, nondecreasing and $h(0) \geq 0$, then

$$u_t - \Delta u + h(u) = 0 \quad (2.10)$$

possesses a strong barrier at any $z \in \Omega$ if and only if h satisfies the Keller-Osserman condition, that is there exists some $a \geq 0$ such that

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \text{ where } H(s) = \int_0^s h(\tau) d\tau. \quad (2.11)$$

The super solution can be chosen to be the maximal solution ϕ_r of the elliptic equation

$$-\Delta \phi + h(\phi) = 0 \quad \text{in } B_r(z). \quad (2.12)$$

If we assume moreover that h is super-additive, i.e. $h(a+b) \geq h(a) + h(b)$ for all $a, b \geq 0$, then there holds

$$\int_a^\infty \frac{ds}{h(s)} < \infty, \quad (2.13)$$

and any solution u of (2.10) is dominated in $Q_T^{B_r(z)}$ by $\phi_r(x) + \psi(t)$ where ψ is defined by inversion from

$$\int_{\psi(t)}^\infty \frac{ds}{h(s)} = t \quad \forall t > 0.$$

Example 2. If $g(x, t, r) = a(x)b(t)h(r)$ where $a \in C(\Omega)$, $b \in C((0, T))$, $a, b > 0$, then g is a coercive nonlinearity if h is super-additive and satisfies the Keller-Osserman condition. This is not sufficient for the existence of a barrier as it is shown in [9] with $h(r) = r^q$ ($q > 1$) $a \equiv 1$ and $b(t) = e^{-\frac{1}{t}}$.

Proposition 2.2 *Let $g \in \mathcal{H}_0$ such that at any $z \in \Omega$ there exists a strong barrier. We assume also*

$$g(x, t, a) + g(x, t, b) \leq g(x, t, a + b) \quad \forall (x, t, a, b) \in Q_T^\Omega \times \mathbb{R}_+ \times \mathbb{R}_+. \quad (2.14)$$

Let $\{u_n\}$ be a sequence of positive solutions of (1.1) which converges to u locally uniformly in Q_T^Ω . Denote by $tr_\Omega(u_n) = (\mathcal{S}_n, \mu_n)$ and $tr_\Omega(u) = (\mathcal{S}, \mu)$ their respective initial trace. If $\mathcal{A} \subset \cap_n \mathcal{R}_n$ is open and if $\mu_n(\mathcal{A}_n)$ remains bounded independently of $n \in \mathbb{N}$, then $\mathcal{A} \subset \mathcal{R} := \Omega \setminus \mathcal{S}$.

Proof. Let $z \in A$ and $\tilde{r} \in (0, \rho(z))$ such that for any $r \in (0, \tilde{r}]$ there exists a positive super solution $w_{r,z}$ satisfying (2.5) and $\overline{B_{\tilde{r}}(z)} \subset \mathcal{A}$. For any $n \in \mathbb{N}$ and $\tau \in (0, T)$, we denote by $u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n}$ the solution of

$$\begin{aligned} \partial_t u - \Delta u + g \circ u &= 0 && \text{in } B_{\tilde{r}}(z) \times (\tau, T) \\ u(\cdot, \tau) &= \chi_{B_{\tilde{r}}(z)} u_n(\cdot, \tau) && \text{in } B_{\tilde{r}}(z) \\ u &= 0 && \text{in } \partial B_{\tilde{r}}(z) \times (\tau, T) \end{aligned} \quad (2.15)$$

By the maximum principle $u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} \leq u_n$ in $B_{\tilde{r}}(z) \times (\tau, T)$, and $g \circ u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} \leq g \circ u_n$. Furthermore, if $\zeta \in C^{1,1;1}(Q_T^{\overline{B_{\tilde{r}}(z)}})$ vanishes on $\partial B_{\tilde{r}}(z) \times [0, T]$ and for $t = T$, there holds

$$\iint_{B_{\tilde{r}}(z) \times (\tau, T)} \left(-u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} (\partial_t \zeta + \Delta \zeta) + \zeta g \circ u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} \right) dx dt = \int_{B_{\tilde{r}}(z)} u_n(x, \tau) \zeta(x, \tau) dx \quad (2.16)$$

Since $u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n}$ and $g \circ u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n}$ are bounded independently of τ , standard regularity theory for parabolic equations implies that they converge a.e. in $B_{\tilde{r}}(z) \times (0, T)$ when $\tau \rightarrow 0$ to $u_{\chi_{B_{\tilde{r}}(z)} \mu_n}$ and $g \circ u_{\chi_{B_{\tilde{r}}(z)} \mu_n}$. Furthermore

$$\lim_{\tau \rightarrow 0} \int_{B_{\tilde{r}}(z)} u_n(x, \tau) \zeta(x, \tau) dx = \int_{B_{\tilde{r}}(z)} \zeta(x, 0) d\mu_n(x).$$

Using the dominated convergence theorem, it follows from (2.15) that

$$\iint_{Q_T^{B_{\tilde{r}}(z)}} \left(-u_{\chi_{B_{\tilde{r}}(z)} \mu_n} (\partial_t \zeta + \Delta \zeta) + \zeta g \circ u_{\chi_{B_{\tilde{r}}(z)} \mu_n} \right) dx dt = \int_{B_{\tilde{r}}(z)} \zeta(x, 0) d\mu_n(x), \quad (2.17)$$

and $u_{\chi_{B_{\tilde{r}}(z)} \mu_n}$ is the (unique) solution of

$$\begin{aligned} \partial_t u - \Delta u + g \circ u &= 0 && \text{in } Q_T^{B_{\tilde{r}}(z)} \\ u(\cdot, 0) &= \chi_{B_{\tilde{r}}(z)} \mu_n && \text{in } B_{\tilde{r}}(z) \\ u &= 0 && \text{in } \partial B_{\tilde{r}}(z) \times (0, T). \end{aligned} \quad (2.18)$$

Furthermore, if η is the solution of the backward problem

$$\begin{aligned} \partial_t \eta + \Delta \eta &= -1 && \text{in } Q_T^{B_{\tilde{r}}(z)} \\ \eta(\cdot, T) &= 0 && \text{in } B_{\tilde{r}}(z) \\ \eta &= 0 && \text{in } \partial B_{\tilde{r}}(z) \times (0, T), \end{aligned} \quad (2.19)$$

there holds

$$\iint_{Q_T^{B_{\tilde{r}}(z)}} \left(u_{\chi_{B_{\tilde{r}}(z)} \mu_n} + \eta g \circ u_{\chi_{B_{\tilde{r}}(z)} \mu_n} \right) dx dt = \int_{B_{\tilde{r}}(z)} \eta(x, 0) d\mu_n(x) \leq M, \quad (2.20)$$

for some $M > 0$ independent of n . Next we set $Z_{\tau, n} := u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} + w_{\tilde{r}, z}$. It is a super solution of (2.10) in $(\tau, T) \times B_{\tilde{r}}(z)$ which is infinite on $\partial B_{\tilde{r}}(z) \times [\tau, T]$ and dominates u_n in $B_{\tilde{r}}(z)$ at $t = \tau$. Thus $Z_{\tau, n} \geq u_n$ in $(\tau, T) \times B_{\tilde{r}}(z)$. Letting $\tau \rightarrow 0$ we finally obtain

$$u_{\chi_{B_{\tilde{r}}(z)} \mu_n}(x, t) \leq u_n(x, t) \leq u_{\chi_{B_{\tilde{r}}(z)} \mu_n}(x, t) + w_{\tilde{r}, z}(x, t) \quad \forall (x, t) \in Q_T^{B_{\tilde{r}}(z)}. \quad (2.21)$$

For any $r < \tilde{r}$ and $T' < T$, there exists $\delta, \sigma > 0$ such that $\eta(x, t) \geq \delta$ and $w_{\tilde{r}, z}(x, t) \leq \sigma$ for all $(x, t) \in Q_{T'}^{B_r(z)}$. It follows from (2.20), (2.21) and Fatou's lemma that u and $g \circ u$ are integrable in $L^1(Q_{T'}^{B_r(z)})$. By Lemma 2.1, $B_{r'}(z) \subset \mathcal{R}$. Since it holds for any $z \in \mathcal{A}$, the result is proved. \square

3 Construction of a barrier

In the next results we construct the barrier function

Lemma 3.1 *Assume $\alpha > -1$ and $q > 1$, then there exists a unique positive function $W_\alpha \in C^2([0, \infty))$ satisfying*

$$\begin{aligned} W'' + \frac{r}{2}W' + \frac{1+\alpha}{q-1}W - W^q &= 0 \quad \text{in } (0, \infty) \\ \lim_{r \rightarrow 0} W(r) &= \infty \\ \lim_{r \rightarrow \infty} r^{\frac{2}{q-1}}W(r) &= 0. \end{aligned} \tag{3.1}$$

Furthermore W_α is decreasing and

$$W_\alpha(r) = Cr^{\frac{2(1+\alpha)}{q-1}-1}e^{-\frac{r^2}{4}}(1 + o(1)) \quad \text{as } r \rightarrow \infty. \tag{3.2}$$

Proof. Consider the functional

$$J(\phi) := \frac{1}{2} \int_0^\infty \left(\phi'^2 - \frac{1+\alpha}{q-1} \phi^2 + \frac{2}{q+1} |\phi|^{q+1} \right) e^{\frac{r^2}{4}} dr, \tag{3.3}$$

defined over the convex set

$$H_k := \{ \phi \in W_2^1(0, \infty; e^{\frac{r^2}{4}} dr) \cap L^{q+1}(0, \infty; e^{\frac{r^2}{4}} dr) : \phi(0) = k \}.$$

Note that if $\phi \in H_k$,

$$\begin{aligned} e^{\frac{r^2}{4}} \phi^2(r) &= \int_r^\infty (e^{\frac{s^2}{4}} \phi^2(s))' ds \\ &= 2 \int_r^\infty e^{\frac{s^2}{4}} \phi \phi'(s) ds + \frac{1}{2} \int_r^\infty s e^{\frac{s^2}{4}} \phi^2(s) ds \end{aligned}$$

In this set J admits a positive minimizer w_k which is the unique solution of

$$\begin{aligned} w'' + \frac{r}{2}w + \frac{1+\alpha}{q-1}w' - w^q &= 0 \quad \text{in } (0, \infty) \\ w(0) &= k. \end{aligned} \tag{3.4}$$

Furthermore, $w_k = \lim_{n \rightarrow \infty} w_{k,n}$ where $w_{k,n}$ is the unique positive solution of

$$\begin{aligned} w'' + \frac{r}{2}w' + \frac{1+\alpha}{q-1}w - w^q &= 0 \quad \text{in } (0, n) \\ w(0) &= k \\ w(n) &= 0. \end{aligned} \tag{3.5}$$

and, by the maximum principle, $(k, n) \mapsto w_{k,n}$ is increasing. If we consider the linear equation

$$z'' + \frac{r}{2}z' + \frac{1+\alpha}{q-1}z = 0 \quad \text{in } (0, \infty) \tag{3.6}$$

it admits two linearly independent positive solutions z_1 and z_2 with the following asymptotic behaviour as $r \rightarrow \infty$

$$z_1(r) = r^{-\frac{2(1+\alpha)}{q-1}}(1 + o(1)) \quad (3.7)$$

and

$$z_2(r) = r^{\frac{2(1+\alpha)}{q-1}-1} e^{-\frac{r^2}{4}}(1 + o(1)) \quad (3.8)$$

(see [8, Appendix]. Since any solution of 3.4, and 3.5 as well, satisfies an a priori estimate of Keller-Osserman type (see [15])

$$w(r) \leq Cr^{-\frac{2}{q-1}} \quad \text{for } 0 < r < 1, \quad (3.9)$$

there holds

$$w_{k,n} \leq Cz_2(r) \quad \text{for } k \geq r \geq 1.$$

Letting n and k go to infinity successively, it follows that $W_\alpha = \lim_{k,n \rightarrow \infty} w_{k,n}$ exists. It is a positive solution of problem (3.1) and it satisfies

$$W_\alpha(r) \leq C \left(r^{-\frac{2}{q-1}} + z_2(r) \right) \quad \text{for } r > 0.$$

The singular behaviour at $r = 0$ is standard (see e.g. [15]) and yields to

$$W_\alpha(r) = \frac{2(q+1)}{(q-1)^2} r^{-\frac{2}{q-1}}(1 + o(1)) \quad \text{as } r \rightarrow 0. \quad (3.10)$$

Thus uniqueness follows by the maximum principle and estimates (3.3) and (3.12) is obtained via standard linearization, using the upper estimate at infinity. \square

In the sequel we set

$$w_\alpha(s, t) = t^{-\frac{1+\alpha}{q-1}} W_\alpha \left(\frac{s}{\sqrt{t}} \right) \quad \forall s > 0, t > 0. \quad (3.11)$$

Proposition 3.2 *Assume $\alpha > -1$ and $q > 1$. Then for any $R > 0$, there exists $C = C(q, \alpha, R) > 0$ such that any solution u of (1.6) in Q_∞^{BR} which vanishes on $B_R \times \{0\}$ satisfies*

$$u(x, t) \leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left(\frac{R-|x|}{\sqrt{t}} \right) \quad \forall (x, t) \in Q_\infty^{BR}. \quad (3.12)$$

Proof. For $m > 0$, set $S_m = \{x = (x_1, \dots, x_N) : |x_j| < m\}$. For $R' < R$

$$\tilde{w}_{R'}(x, t) = t^{-\frac{1+\alpha}{q-1}} \sum_{j=1}^N \left(W_\alpha \left(\frac{R'-x_j}{\sqrt{t}} \right) + W_\alpha \left(\frac{R'+x_j}{\sqrt{t}} \right) \right) \quad \forall (x, t) \in Q_\infty^{S_{R'}} \quad (3.13)$$

Then $\tilde{w}_{R'}$ is a super solution of (1.6) in $Q_\infty^{S_{R'}}$ which is infinite on $\partial S_{R'} \times (0, \infty)$, thus $u \leq \tilde{w}_{R'}$. Letting $R' \rightarrow R$ yields to $u \leq \tilde{w}_R$ in $Q_\infty^{S_R}$. Since the equation is invariant by rotation, for any $x \in B_R$, there is a rotation \mathcal{R} such that $\mathcal{R}(x)$ has only a positive x_1 -coordinate. Thus

$$\begin{aligned} u(x, t) &\leq \tilde{w}_R(|x|, t) \\ &\leq t^{-\frac{1+\alpha}{q-1}} \left(W_\alpha \left(\frac{R-|x|}{\sqrt{t}} \right) + (2N-1)W_\alpha \left(\frac{R}{\sqrt{t}} \right) \right) \\ &\leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left(\frac{R-|x|}{\sqrt{t}} \right), \end{aligned} \quad (3.14)$$

which is (3.12). \square

Proposition 3.3 *Assume $\alpha > -1$, $q > 1$ and $R > 0$. Then there exists a unique positive solution w_{B_R} of (1.6) in $Q_\infty^{B_R}$, continuous in $B_R \times [0, \infty)$, which vanishes on $B_R \times \{0\}$ and satisfies $\lim_{|x| \rightarrow R} w_{B_R}(x, t) = \infty$, locally uniformly in $(0, \infty)$. In particular*

$$t^{-\frac{1+\alpha}{q-1}} W_\alpha \left(\frac{R-|x|}{\sqrt{t}} \right) \leq w_{B_R}(x, t) \leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left(\frac{R-|x|}{\sqrt{t}} \right) \quad \forall (x, t) \in Q_\infty^{B_R}. \quad (3.15)$$

Proof. For $k > 0$, let $w_{B_R}^k$ be the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{B_R} \\ u &= k && \text{in } \partial_\ell Q_\infty^{B_R} \\ u(\cdot, 0) &= 0 && \text{in } B_R. \end{aligned} \quad (3.16)$$

By (3.12), $w_{B_R}^k(x, t) \leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left(\frac{R-|x|}{\sqrt{t}} \right)$. There there exists $w_{B_R} = \lim_{k \rightarrow \infty} w_{B_R}^k$ and w_{B_R} is a solution of (1.6) in $Q_\infty^{B_R}$ which vanishes on $B_R \times \{0\}$ and is infinite on $\partial B_R \times (0, \infty)$. Consider the similarity transformation T_m which leaves equation (1.6) invariant

$$T_m[u](x, t) = m^{\frac{1+\alpha}{q-1}} u(\sqrt{m}x, mt) \quad \forall m > 0,$$

then $T_m[w_{B_R}^k] = w_{B_{\frac{R}{\sqrt{m}}}}^{m^{\frac{1+\alpha}{q-1}} k}$ which implies

$$T_m[w_{B_R}] = w_{B_{\frac{R}{\sqrt{m}}}} \quad \forall m > 0.$$

If $u \in C(B_R \times [0, \infty))$ is any positive solution of problem

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{B_R} \\ \lim_{|x| \rightarrow R} u(x, t) &= \infty && \text{locally uniformly on } (0, \infty) \\ u(\cdot, 0) &= 0 && \text{in } B_R, \end{aligned} \quad (3.17)$$

then for any $m > 1$ and $\epsilon > 0$, there exists $\tau_\epsilon > 0$ such that $u(x, t) \leq \epsilon$ in $B_{\frac{R}{\sqrt{m}}}$ for $0 \leq t \leq \tau_\epsilon$. Therefore, for any $T > 0$,

$$u \leq \epsilon + w_{B_{\frac{R}{\sqrt{m}}}} \quad \text{on } \partial_\ell Q_T^{B_{\frac{R}{\sqrt{m}}}} \cup \overline{B_{\frac{R}{\sqrt{m}}}} \times \{0\}$$

Since $\epsilon + w_{B_{\frac{R}{\sqrt{m}}}}$ is a super solution $u \leq \epsilon + w_{B_{\frac{R}{\sqrt{m}}}}$ in $Q_T^{B_{\frac{R}{\sqrt{m}}}}$. Letting $\epsilon \rightarrow 0$, $m \rightarrow 1$ and $T \rightarrow \infty$ yields to $u \leq w_{B_R}$. In the same way, with $0 < m < 1$, we obtain $u \geq w_{B_R}$. \square

The next result is an immediate consequence of Proposition 3.2.

Proposition 3.4 *Assume $\alpha > -1$, $q > 1$ and $K \subset \Omega$ is compact. Let u be any solution u of (1.6) in Q_∞^Ω which vanishes on $\Omega \setminus \{K\} \times \{0\}$ and on $\partial\Omega \times [0, \infty)$, then*

$$u(x, t) \leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left(\frac{\text{dist}(x, K)}{\sqrt{t}} \right). \quad (3.18)$$

4 Upper estimates

We start with the following upper estimate already obtained by Shishkov and Véron [12] in the case $\alpha \geq 0$.

Proposition 4.1 *Let $q > 1$ and $\alpha > -1$. If u is a solution of (1.6) vanishing on $\partial\Omega \times [0, T)$, there holds*

$$u(x, t) \leq c_\alpha t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x, t) \in Q_T^\Omega, \quad (4.1)$$

with $c_\alpha = \left(\frac{\alpha+1}{q-1}\right)^{\frac{1}{q-1}}$.

Proof. Let $\phi(t) = c_\alpha t^{-\frac{\alpha+1}{q-1}}$ be the maximal solution of

$$\begin{aligned} \phi' + t^\alpha \phi^q &= 0 \\ \phi(0) &= \infty. \end{aligned}$$

with $c_\alpha = \left(\frac{\alpha+1}{q-1}\right)^{\frac{1}{q-1}}$.

Case $\alpha \geq 0$. For $\tau > 0$, we denote by $\Phi_{1,\tau}$ the solution of

$$\begin{aligned} -\Delta \Phi_{1,\tau} + \tau^\alpha \Phi_{1,\tau}^q &= 0 \quad \text{in } B_1 \\ \lim_{|x| \rightarrow 1} \Phi_{1,\tau}(x) &= \infty, \end{aligned} \quad (4.2)$$

and for $R > 0$

$$\Phi_{R,\tau}(x) = R^{\frac{-2}{q-1}} \Phi_{1,\tau}\left(\frac{x}{R}\right).$$

Note that $\Phi_{R,\tau}(x)$ is the solution of the problem (4.2) in the ball B_R . The function $\Phi_{R,\tau}$ tends to 0 uniformly on every compact set of \mathbb{R}^N when $R \rightarrow \infty$. Set

$$\tilde{v}(x, t) = \phi(t - \tau) + \Phi_{R,\tau}(x),$$

then \tilde{v} is a supersolution of (1.6) in $B_R \times [\tau, T)$ which is infinite on $\partial B_R \times [\tau, T) \cup B_R \times \{0\}$. Then $u(x, t) \leq \tilde{v}(x, t)$. Letting $R \rightarrow \infty$ and $\tau \rightarrow 0$, we obtain

$$u(x, t) \leq c_\alpha t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x, t) \in Q_T.$$

Case $-1 < \alpha < 0$. Let $\tau > 0$ and $\phi_\tau(t) = c_\alpha (t^{\alpha+1} - \tau^{\alpha+1})^{-\frac{1}{q-1}}$ be the solution of

$$\begin{aligned} \phi_\tau' + t^\alpha \phi_\tau^q &= 0 \quad \text{on } (\tau, \infty) \\ \phi_\tau(\tau) &= \infty \end{aligned}$$

If $\Phi_{1,T}$ is the solution of (4.2) with $\tau = T$, we set

$$\Phi_{R,T}(x) = R^{\frac{-2}{q-1}} \Phi_{1,T}\left(\frac{x}{R}\right).$$

Clearly $\Phi_{R,T}$ tends to 0 uniformly on every compact of \mathbb{R}^N when $R \rightarrow \infty$. Set

$$\hat{v}(x, t) = \phi_\tau(t) + \Phi_{R,T}(x),$$

\hat{v} is a supersolution of (1.6) in $B_R \times (\tau, T)$, thus $u(x, t) \leq \hat{v}(x, t)$, as in the first case. Letting $R \rightarrow \infty$ and $\tau \rightarrow 0$, we obtain the desired estimate. \square

Combining Proposition 3.3 and Proposition 4.1 we obtain,

Corollary 4.2 *Assume $q > 1$, $\alpha > -1$ and $K \subset \Omega$ is compact. If $u \in C^2(Q_T^\Omega) \cap C(\overline{Q_T^\Omega} \setminus K \times \{0\})$ is a solution of (1.6) which vanishes on $\partial\Omega \times [0, T) \cup \{(\Omega \setminus K) \times \{0\}\}$, there holds*

$$u(x, t) \leq \min \left\{ 2NW_\alpha \left(\frac{\text{dist}(x, K)}{\sqrt{t}} \right), c_\alpha \right\} t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x, t) \in Q_T^\Omega. \quad (4.3)$$

In the particular case where $K = \{O\}$, (4.3) yields to

$$u(x, t) \leq \min \left\{ 2NW_\alpha \left(\frac{|x|}{\sqrt{t}} \right), c_\alpha \right\} t^{-\frac{\alpha+1}{q-1}} \leq \frac{c_1}{(|x|^2 + t)^{\frac{1+\alpha}{q-1}}} \quad \text{for all } (x, t) \in Q_T^\Omega, \quad (4.4)$$

for some $c_1 = c_1(\alpha, q) > 0$.

Remark. If Ω is replaced by \mathbb{R}^N , the previous estimates (4.1), (4.3) and (4.4) remains valid. Furthermore, K needs only to be closed.

5 Isolated singularities

In this section we present the results of classifications of isolated singularities of positive solutions of (1.6), always in the range $q > 1$ and $\alpha > -1$. Since the proofs are somewhat similar to the ones of classification of boundary singularities of positive solutions of

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T^\Omega \quad (5.1)$$

obtained by Marcus and Véron in [8], we only indicate their main ideas.

If we look for solution of (1.6) in $Q_T := Q_T^{\mathbb{R}^N}$ under the form

$$u(x, t) = t^\gamma V\left(\frac{x}{\sqrt{t}}\right)$$

it is immediate that $\gamma = -\frac{1+\alpha}{q-1}$ and V is a solution of

$$-\Delta V - \frac{\eta}{2} \cdot \nabla V - \frac{1+\alpha}{q-1} V + V^q = 0 \quad \text{in } \mathbb{R}^N. \quad (5.2)$$

It is proved by Escobedo and Kavian [4] that if $\frac{1+\alpha}{q-1} > \frac{N}{2}$, or equivalently if

$$1 < q < q_{c,\alpha} := 1 + \frac{2(1+\alpha)}{q-1}, \quad (5.3)$$

there exists a positive solution of (5.2) which is the minimizer of the functional

$$\omega \mapsto J(\omega) := \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla \omega|^2 - \frac{1+\alpha}{q-1} \omega^2 + \frac{2}{q+1} |\omega|^{q+1} \right) e^{\frac{|\eta|^2}{4}} d\eta. \quad (5.4)$$

This minimizing solution V_α is unique and radial. Furthermore, by adapting the results of [8, Th 2.1], it satisfies

$$V_\alpha(\eta) = C |\eta|^{2\frac{1+\alpha}{q-1}-N} e^{-\frac{|\eta|^2}{4}} (1 + o(1)) \quad \text{as } |\eta| \rightarrow \infty. \quad (5.5)$$

The function

$$v_\alpha(x, t) = t^{-\frac{1+\alpha}{q-1}} V_\alpha \left(\frac{x}{\sqrt{t}} \right) \quad (5.6)$$

is a positive solution of (5.2) in Q_∞ , continuous in $\overline{Q}_T \setminus \{(O, 0)\}$, vanishes on $\mathbb{R}^N \times \{0\} \setminus \{(O, 0)\}$ and satisfies

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} v_\alpha(x, t) dx = \infty \quad \forall \epsilon > 0. \quad (5.7)$$

It is called the *very singular solution* of (1.6).

When \mathbb{R}^N is replaced by a proper open domain Ω with a compact C^2 boundary there exists no self-similar solution to (1.6). For any $k > 0$ and $a \in \Omega$ there exists a unique solution $u := u_{k\delta_a}$ to the initial value problem

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^\Omega \\ u &= 0 && \text{on } \partial\Omega \times [0, \infty) \\ u(\cdot, 0) &= k\delta_a && \text{in } \Omega. \end{aligned} \quad (5.8)$$

(see e.g. [9]). The function u belongs to $L^q(Q_T^\Omega; t^\alpha dx dt) \cap L^1(Q_T^\Omega)$, $T > 0$ arbitrary, and satisfies

$$\iint_{Q_T^\Omega} (-u(\partial_t \zeta + \Delta \zeta) + t^\alpha u^q \zeta) dx dt = k\zeta(a) \quad (5.9)$$

for all $\zeta \in C^{1,1;1}(\overline{Q}_T^\Omega)$ which vanishes on $\partial\Omega \times [0, T]$ and on $\Omega \times \{T\}$. It is unique in the class of weak solutions, i.e. the functions belonging to $L^q(Q_T^\Omega; t^\alpha dx dt) \cap L^1(Q_T^\Omega)$ and satisfying the above identity. When $k \rightarrow \infty$, $u_{k\delta_a} \uparrow u_{\infty\delta_a}$, where $u_{\infty\delta_a} := u_{\infty,a}$ is a solution of (1.6) in Q_∞^Ω which vanishes on $\partial\Omega \times [0, \infty)$ and on $\overline{\Omega} \times \{0\} \setminus \{(a, 0)\}$ and satisfies (5.7). Finally

$$u_{k\delta_a} \sim kE(x - a, t) \quad \text{when } (x, t) \rightarrow (a, 0) \quad (5.10)$$

and

$$u_{\infty,a} \sim v_\alpha(x - a, t) \quad \text{when } (x, t) \rightarrow (a, 0). \quad (5.11)$$

The following classification of isolated singularities holds

Theorem 5.1 *Assume $\alpha > -1$, $1 < q < q_{c,\alpha}$ and $a \in \Omega$. If $u \in \overline{Q}_T^\Omega \setminus \{(a, 0)\}$ is a positive solution of (1.6) which vanishes on $\Omega \times \{0\} \setminus \{(a, 0)\}$, then*

(i) *either there exists $k \geq 0$ such that*

$$u(x, t) \sim kE(x - a, t) \quad \text{when } (x, t) \rightarrow (a, 0), \quad (5.12)$$

and u is a solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_T^\Omega \\ u(\cdot, 0) &= k\delta_a && \text{in } \Omega, \end{aligned} \quad (5.13)$$

(ii) *or*

$$u(x, t) \sim v_\alpha(x - a, t) \quad \text{when } (x, t) \rightarrow (a, 0), \quad (5.14)$$

and u satisfies (5.7).

If $q \geq q_{c,\alpha}$ there holds the following result which extends Brezis and Freidman's classical one.

Theorem 5.2 *Assume $\alpha > -1$, $q \geq q_{c,\alpha}$ and $a \in \Omega$. If $u \in \overline{Q_T^\Omega} \setminus \{(a,0)\}$ is a positive solution of (1.6) which vanishes on $\Omega \times \{0\} \setminus \{(a,0)\}$, then it can be extended as a continuous function \tilde{u} which vanishes on $\Omega \times \{0\}$.*

Proof. Up to modifying a few parameters the proof is similar to Brezis-Friedman's construction. The first step is to prove that $u \in L^q(Q_T^{BR})$ for some $R > 0$. This is done by using (4.4) and appropriate test functions. Then the same choice of test functions yields to the fact that u satisfies

$$\lim_{t \rightarrow 0} \int_{\Omega} u(x,t) \zeta(x) dx = 0 \quad \forall \zeta \in C_0^\infty(\Omega).$$

Finally the extension of u by zero at $t = 0$ satisfies the equation in $\Omega \times [0, T)$. \square

Remark. We recall that $E(x,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$. Then if $1 \leq r < q_{c,\alpha}$, there holds

$$\iint_{Q_T} E^r(x,t) t^\alpha dx dt < \infty, \quad (5.15)$$

while if $r \geq q_{c,\alpha}$

$$\iint_{Q_T} E^r(x,t) t^\alpha dx dt = \infty. \quad (5.16)$$

6 The trace theorem

In all this section we assume that $\Omega \subset \mathbb{R}^N$ is an open domain with a compact C^2 boundary, $\alpha > -1$ and $1 < q < q_{c,\alpha}$, and let $u \in C(\overline{\Omega} \times (0, T])$ be a positive solution of (1.6) in Q_T^Ω which vanishes on $\partial\Omega \times (0, T]$. By Section 2 u possesses an initial trace $tr_\Omega(u) = (\mathcal{S}, \mu)$ where \mathcal{S} is a relatively closed subset of Ω and μ is a Radon measure on $\mathcal{R} := \Omega \setminus \mathcal{S}$. To this couple we can associate a unique outer regular Borel measure ν defined by

$$\nu(E) = \begin{cases} \mu(E) & \text{if } E \subset \mathcal{R} \\ \infty & \text{if } E \cap \mathcal{S} \neq \emptyset \end{cases} \quad (6.1)$$

for any Borel subset E of Ω . Conversely, to any outer Borel measure ν on Ω we can associate the regular set $\mathcal{R} \subset \Omega$ which is the set of points $y \in \Omega$ which possess an open neighborhood \mathcal{O}_y such that $\nu(\mathcal{O}_y) < \infty$. Clearly \mathcal{R} is open and the restriction of ν to \mathcal{R} is a positive Radon measure. The set $\mathcal{S} = \Omega \setminus \mathcal{R}$ is relatively closed and it is the singular part of ν . It has the property that $\nu(E) = \infty$ for any Borel set E such that $E \cap \mathcal{S} \neq \emptyset$. We shall denote by $\mathfrak{B}^{reg}(\Omega)$ the set of outer regular Borel measures in Ω and by $\mathfrak{B}_c^{reg}(\Omega)$ the subset of $\mathfrak{B}^{reg}(\Omega)$ for which \mathcal{S} is a compact subset of Ω . Thus u is a solution of the following problem,

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_T^\Omega \\ u \geq 0, u &= 0 && \text{on } \partial\Omega \times [0, T) \\ tr_\Omega(u) &= \nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega). \end{aligned} \quad (6.2)$$

Definition 6.1 *We denote by $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$ the set of solutions of problem (6.2).*

The first step in the characterization of the singular set is the following lower estimate.

Theorem 6.2 *Let $u \in \mathcal{U}_{\mathcal{S}, \mu}(\Omega)$ and $a \in \mathcal{S}$, then*

$$u(x, t) \geq u_{\infty, a}(x, t). \quad (6.3)$$

Furthermore, if \mathcal{S} has a nonempty interior \mathcal{A} , there holds

$$\lim_{t \rightarrow 0} t^{\frac{1+\alpha}{q-1}} u(x, t) = c_\alpha, \quad (6.4)$$

uniformly on compact subsets of \mathcal{A} .

We first give a proof of (6.53) in the case where either $-1 < \alpha \leq 0$, or $\alpha > 0$ and $1 < q < q_{c,0}$.

Proposition 6.3 *Assume either $-1 < \alpha \leq 0$ and $1 < q < q_{c,\alpha}$, or $\alpha > 0$ and $1 < q < q_{c,0}$, then inequality (6.53) holds.*

Proof. For any $\epsilon > 0$, there holds

$$\lim_{t \rightarrow 0} \int_{B_\epsilon(a)} u(x, t) dx = \infty. \quad (6.5)$$

If $k > 0$ is fixed, and $\{t_n\}$ is a sequence decreasing to 0. There exists t_{n_1} such that

$$\int_{B_{2^{-1}}(a)} u(x, t_{n_1}) dx > k,$$

and there exists $m = m_1(k)$ such that

$$\int_{B_{2^{-1}}(a)} m_1(k) \wedge u(x, t_{n_1}) dx = k,$$

where $A \wedge B = \inf\{A, B\}$. Assume we have constructed $t_{n_j} < t_{n_{j-1}}$ and $m_j(k) > 0$ such that

$$\int_{B_{2^{-j}}(a)} m_j(k) \wedge u(x, t_{n_j}) dx = k.$$

Since (6.5) holds with $\epsilon = 2^{-j-1}$, there exists $t_{n_{j+1}} < t_{n_j}$ such that

$$\int_{B_{2^{-j-1}}(a)} u(x, t_{n_{j+1}}) dx > k.$$

and thus $m_{j+1}(k) > 0$ such that

$$\int_{B_{2^{-j-1}}(a)} m_{j+1}(k) \wedge u(x, t_{n_{j+1}}) dx = k.$$

Next we denote by u_j the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (t_{n_j}, T) \\ u &= 0 && \text{on } \partial\Omega \times [t_{n_j}, T) \\ u(\cdot, t_{n_j}) &= m_{j+1}(k) \wedge u(\cdot, t_{n_{j+1}}) && \text{in } \Omega. \end{aligned} \quad (6.6)$$

By the maximum principle $u_j \leq u$ in $\Omega \times [t_{n_j}, T)$, or equivalently $u_j(x, t + t_{n_j}) \leq u(x, t + t_{n_j})$ in $Q_{T-t_{n_j}}^\Omega$. Clearly

$$u_{j,0} := u(\cdot, t_{n_j}) \rightharpoonup k\delta_a \quad \text{as } j \rightarrow \infty$$

in the weak sense of measures. In order to prove that u_j converges to $u_{k\delta_a}$ we notice that $u_j(x, t + t_{n_j}) \leq E * u_{j,0}(x, t)$ in Q_∞^Ω . If $-1 < \alpha \leq 0$ and $r \in (1, q_{c,\alpha})$

$$\begin{aligned} \iint_{Q_T^\Omega} u_j^r(x, t + t_{n_j})(t + t_{n_j})^\alpha dxdt &\leq \iint_{Q_T} (E * u_{j,0})^r(x, t)(t + t_{n_j})^\alpha dxdt \\ &\leq \|u_{j,0}\|_{L^1}^r \iint_{Q_T} E^r(x, t)t^\alpha dxdt, \end{aligned}$$

using Young's inequality and (5.15). If $\alpha \geq 0$ and $r \in (1, q_{c,0})$

$$\begin{aligned} \iint_{Q_T^\Omega} u_j^r(x, t + t_{n_j})(t + t_{n_j})^\alpha dxdt &\leq 2^\alpha T^\alpha \iint_{Q_T} (E * u_{j,0})^r(x, t) dxdt \\ &\leq 2^\alpha T^\alpha \|u_{j,0}\|_{L^1}^r \iint_{Q_T} E^r(x, t) dxdt. \end{aligned}$$

Furthermore, for $s \in (1, q_{c,0})$

$$\begin{aligned} \iint_{Q_T^\Omega} u_j^s(x, t + t_{n_j}) dxdt &\leq \iint_{Q_T} (E * u_{j,0})^s(x, t) dxdt \\ &\leq \|u_{j,0}\|_{L^1}^s \iint_{Q_T} E^s(x, t) dxdt. \end{aligned}$$

Thus the set of functions $\{u_j^q(\cdot, \cdot + t_{n_j})(\cdot + t_{n_j})^\alpha\}$ and $\{u_j(\cdot, \cdot + t_{n_j})\}$ are uniformly integrable in $L^1(Q_T^\Omega)$. Since u_j satisfies

$$\iint_{Q_T^\Omega} (-u_j(x, t + t_{n_j}) (\partial_t \zeta + \Delta \zeta) + (t + t_{n_j})^\alpha u_j^q(x, t + t_{n_j}) \zeta) dxdt = \int_\Omega u_{j,0} \zeta dx \quad (6.7)$$

for any test function $\zeta \in C^{1,1;1}(\overline{Q_T^\Omega})$ vanishing on $\partial\Omega \times [0, T] \cup \Omega \times \{T\}$ and converges a.e. in Q_T^Ω to some u^* , it follows by Vitali's theorem that

$$\iint_{Q_T^\Omega} (-v(\partial_t \zeta + \Delta \zeta) + t^\alpha v^q \zeta) dxdt = k\zeta(a) \quad (6.8)$$

thus $u^* = u_{k\delta_a}$ by uniqueness, which implies the claim since $u \geq u^*$. \square

When $\alpha > 0$ and $q_{c,0} \leq q < q_{c,\alpha}$, this argument cannot work since the sequence $u_{j,0}$ could concentrate too quickly with respect to t to a Dirac mass and isolated singularities are removable for solutions of

$$\partial_t u - \Delta u + cu^q = 0.$$

We develop below a proof valid for any $\alpha > -1$ and $1 < q < q_{c,\alpha}$, which is based upon the parabolic Harnack inequality and shows that such a fast concentration never occurs.

Lemma 6.4 Assume $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Let $\{(x_n, t_n)\} \subset Q_T^\Omega$ be a sequence converging to $(a, 0)$ and $\ell > 0$. Put $V_n = B_{\ell\sqrt{t_n}}(x_n)$ and suppose that there exist nonnegative functions $h_n \in L^\infty(\mathbb{R}^N)$ with support in V_n such that $0 \leq h_n \leq c_1 t_n^{-\frac{N}{2}}$ and

$$h_n \rightharpoonup k\delta_0. \quad (6.9)$$

Then the solutions u_n of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (t_n, T) \\ u &= 0 && \text{on } \partial\Omega \times [t_n, T) \\ u(\cdot, t_n) &= h_n && \text{in } \Omega, \end{aligned} \quad (6.10)$$

satisfy $u_n \rightarrow u_{k\delta_0}$ when $n \rightarrow \infty$.

Proof. The estimate $h_n \leq c t_n^{-\frac{N}{2}} \chi_{V_n}$ can be written under the form

$$h_n(x) \leq c_2 E(x - x_n, t_n) \chi_{V_n}$$

where $c_2 = (4\pi)^{\frac{N}{2}} e^{\frac{\ell}{4}} c_1$. By the maximum principle

$$u_n(x, t) \leq c_2 E(x - x_n, t) \quad \text{in } \Omega \times (t_n, \infty).$$

By (5.15), (5.16), the sequences $\{E^q(\cdot - x_n, \cdot) t^\alpha\}$ and $\{E(\cdot - x_n, \cdot)\}$ are uniformly integrable in Q_T^Ω , therefore, if we extend u_n by zero in $Q_{t_n}^\Omega$ and denote by \tilde{u}_n the extended function defined in Q_T^Ω , we infer that the sequences $\{t^\alpha \tilde{u}_n^q\}$ and $\{\tilde{u}_n\}$ are uniformly integrable in Q_T^Ω . Using standard regularity estimates there exists a function u^* defined in Q_T^Ω and a subsequence u_{n_j} such that $\tilde{u}_{n_j} \rightarrow u^*$ locally uniformly in Q_T^Ω . It follows by uniform integrability and Vitali's convergence theorem that

$$\tilde{u}_{n_j} \rightarrow u^* \quad \text{in } L^q(Q_T^\Omega; t^\alpha dx dt) \cap L^1(Q_T^\Omega).$$

Let $\zeta \in C^{1,1;1}(\overline{Q_T^\Omega})$ vanishing on $\partial\Omega \times [0, T] \cup \Omega \times \{T\}$, then

$$\iint_{Q_T^\Omega} \left(-\tilde{u}_{n_j} (\partial_t \zeta + \Delta \zeta) + t^\alpha \tilde{u}_{n_j}^q \zeta \right) dx dt = \int_\Omega \zeta(\cdot, t_{n_j}) h_n dx.$$

Using the previous convergence results and the assumption (6.9), we derive

$$\iint_{Q_T^\Omega} (-u^* (\partial_t \zeta + \Delta \zeta) + t^\alpha u^{*q} \zeta) dx dt = k\zeta(O).$$

Thus $u^* = u_{k\delta_0}$ and $\tilde{u}_n \rightarrow u_{k\delta_0}$ locally uniformly in Q_T^Ω . \square

Lemma 6.5 Let u be a positive solution of (1.6) in Q_T^Ω vanishing on $\partial\Omega \times [0, T)$. Then for any $\Omega' \subset \overline{\Omega} \subset \Omega$ there exists a constant $C = C(N, q, \alpha, \Omega') > 0$ such that

$$u(y, s) \leq u(x, t) e^{C \left(\frac{|x-y|^2}{t-s} + \frac{t}{s} + 1 \right)} \quad \forall (x, t), (y, s) \in Q_T^{\Omega'}, s < t. \quad (6.11)$$

Proof. By (4.1), $V(x, t) := t^\alpha u^{q-1} \leq c_\alpha^{q-1} t^{-1}$. If we write (1.6) under the form

$$\partial_t u - \Delta u + V(x, t)u = 0 \quad \text{in } Q_T^\Omega \quad (6.12)$$

it follows (6.11) from parabolic Harnack inequality (see e.g. [5, Lemma 3.16] although the result is much older). \square

If $G \subset \mathbb{R}^N$ is a bounded open subset, we denote by ϕ_G is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ normalized by $\sup \phi_G = 1$ with corresponding eigenvalue λ_G .

Lemma 6.6 *Let $G \subset \mathbb{R}^N$ be a bounded open subset with a smooth boundary $\omega \in C^1(Q_T^G)$, $\omega \geq 0$, such that*

$$\int_0^T \|\omega(\cdot, t)\|_{L^\infty(G)} dt < \infty. \quad (6.13)$$

If $v \in C^{2;1}(\overline{G} \times (0, T])$ is a positive solution of

$$\partial_t v - \Delta v + \omega v = 0 \quad \text{in } Q_T^G, \quad (6.14)$$

then $v\phi_G \in L^1(Q_T^G)$, $\omega v\phi_G^3 \in L^1(Q_T^G)$ and there exists $\mu \in \mathfrak{M}_+(G)$ such that

$$\lim_{t \rightarrow 0} \int_G v(x, t)\zeta(x)dx = \int_G \zeta d\mu \quad \forall \zeta \in C_c(\overline{G}). \quad (6.15)$$

Proof. Set $\gamma(t) = \|\omega(\cdot, t)\|_{L^\infty(G)}$. Denote $X(t) = \int_G v(x, t)\phi_G(x)dx$. Then, from (6.14),

$$X' + \lambda_G X + \gamma(t)X \geq 0$$

This implies that the function

$$t \mapsto e^{t\lambda_G + \int_0^t \gamma(s)ds} X(t)$$

(which exists thanks to (6.13)) is nondecreasing. Therefore there exists $X(0) = \lim_{t \rightarrow 0} X(t)$ and $v\phi_G \in L^1(Q_T^G)$. Furthermore, if we set $Y(t) = \int_G v(x, t)\phi_G^3(x)dx$

$$\frac{d}{dt}(e^{3t\lambda_G} X(t)) + e^{3t\lambda_G} \int_\Omega (\omega\phi_G^3 - 6\phi_G |\nabla\phi_G|^2)v dx = 0.$$

Since $\nabla\phi_G$ is bounded and $v\phi_G \in L^1(Q_T^G)$, it implies

$$e^{3t\lambda_G} Y(T) + \int_0^T e^{3t\lambda_G} \int_\Omega \omega(x, t)v(x, t)\phi_G^2 dx dt = Y(0) + 6 \int_0^T e^{3t\lambda_G} \int_\Omega \phi_G |\nabla\phi_G|^2 v dx dt,$$

which implies that $\omega v\phi_G^3 \in L^1(Q_T^G)$. The argument given in the proof of Lemma 2.1 shows that v admits an initial trace which belongs to $\mathfrak{M}_+(G)$. \square

Proof of Theorem 6.2. We define the parabolic distance in $\mathbb{R}^N \times \mathbb{R}$ by

$$d_P((x, t), (y, s)) = \sqrt{|x - y|^2 + |t - s|}$$

Step 1. We first prove that if u satisfies

$$\limsup_{d_P((x,t),(a,0)) \rightarrow 0} t^{\frac{N}{2}} u(x,t) < \infty, \quad (6.16)$$

then $a \in \mathcal{R}(u)$. If (6.16) holds there exists $\epsilon, c > 0$ such that

$$u(x,t) \leq ct^{-\frac{N}{2}} \quad \forall (x,t) \text{ s.t. } \sqrt{|x-a|^2 + t} \leq \epsilon.$$

If we set $\omega(x,t) = t^\alpha u^{q-1}(x,t)$, then

$$\omega(x,t) \leq c^{q-1} t^{-(\frac{N(q-1)}{2} - \alpha)} \quad \forall (x,t) \in B_{\frac{\epsilon}{\sqrt{2}}}(a) \times (0, \frac{\epsilon^2}{2}].$$

Since $q < q_{c,\alpha}$, $\frac{N(q-1)}{2} - \alpha < 1$, thus the assumptions of Lemma 6.6 are fulfilled and there exists a positive Radon measure μ in $B_{\frac{\epsilon}{\sqrt{2}}}(a)$ such that

$$\int_{\Omega} u(x,t)\zeta(x)dx \rightarrow \int_{\Omega} \zeta d\mu \quad \text{when } t \rightarrow 0, \quad \forall \zeta \in C_c^\infty(B_{\frac{\epsilon}{\sqrt{2}}}(a)).$$

Furthermore $t^\alpha u^q \in L^1(B_{\frac{\epsilon}{\sqrt{3}}}(a) \times (0, T))$, which is the claim.

Step 2. Since $a \in \mathcal{S}(u)$, there holds

$$\limsup_{d_P((x,t),(a,0)) \rightarrow 0} t^{\frac{N}{2}} u(x,t) = \infty. \quad (6.17)$$

Then there exists a sequence $\{(x_n, s_n)\}$ converging to $(a, 0)$ such that

$$u(x_n, t_n) \geq nt_n^{-\frac{N}{2}} \quad (6.18)$$

We apply Lemma 6.5 with $s = s_n$, $t = 2s_n := t_n$, $y = x_n$ $|x - x_n| \leq \sqrt{s_n}$. Then

$$u(x, t_n) \geq Cnt_n^{-\frac{N}{2}} \quad \forall x \in V_n := B_{\frac{\sqrt{t_n}}{2}}(x_n)$$

This implies

$$\int_{V_n} u(x, t_n) dx \geq C_N n.$$

For $k < n$ fixed, we denote by $v := v_{n,k}$ the solution of

$$\begin{aligned} \partial_t v - \Delta v + t^\alpha v^q &= 0 && \text{in } \Omega \times (t_n, \infty) \\ v &= 0 && \text{in } \partial\Omega \times (t_n, \infty) \\ v(\cdot, t_n) &= Ckt_n^{-\frac{N}{2}} \chi_{V_n} && \text{in } \Omega. \end{aligned} \quad (6.19)$$

By the maximum principle $u \geq v_{n,k}$ in $\Omega \times (t_n, \infty)$. Furthermore

$$\int_{V_n} v(\cdot, t_n) dx = C_N k.$$

Thus $v(\cdot, t_n) \rightharpoonup C_N k \delta_a$ in the weak sense of measures. It follows from Lemma 6.4 that $v_{n,k} \rightarrow u_{C_N k \delta_a}$ locally uniformly in Q_T^Ω . Therefore $u \geq u_{C_N k \delta_a}$ in Q_T^Ω . Since k is arbitrary, we obtain (6.53).

Step 3. Formula (6.4) holds. Denote by $S_m(a) = \{x \in \mathbb{R}^N : |x_j| < m\}$. If $\bar{S}_R(a) \subset \mathcal{S}$, the function

$$(x, t) \mapsto c_\alpha (t - \tau)^{-\frac{1+\alpha}{q-1}} + w_{B_R}(x - a, t - \tau)$$

is a super solution of (1.6) in $S_R(a) \times (\tau, \infty)$ which is infinite on $S_R(a) \times \{\tau\} \cup \partial S_R(a) \times [\tau, \infty)$ by Proposition 3.3, while u is finite, thus it dominates u in $S_R(a) \times (\tau, \infty)$. Letting $\tau \rightarrow 0$ yields to

$$u(x, t) \leq c_\alpha t^{-\frac{1+\alpha}{q-1}} + w_{B_R}(x - a, t) \quad \forall (x, t) \in Q_T^{S_R(a)}. \quad (6.20)$$

Conversely, the function

$$(x, t) \mapsto u(x, t - \tau) + w_{B_R}(x - a, t - \tau)$$

is a super solution in $S_R(a) \times (\tau, \infty)$ which dominates $c_\alpha t^{-\frac{1+\alpha}{q-1}}$ on $S_R(a) \times \{\tau\} \cup \partial S_R(a) \times [\tau, \infty)$, thus as above, we obtain (6.20). Since

$$\lim_{t \rightarrow 0} w_{B_R}(x - a, t) = 0$$

uniformly on $B_{R'}(a)$ for any $R' < R$, we derive (6.4). \square

Proposition 6.7 *For any relatively closed $\mathcal{S} \in \Omega$, the set $\mathcal{U}_{\mathcal{S},0}(\Omega)$ is not empty and it admits a minimal element $\underline{u}_{\mathcal{S},0}$ and a maximal element $\bar{u}_{\mathcal{S},0}$.*

Proof. Step 1: Existence of a maximal solution. The maximal solution is constructed by thickening Ω and \mathcal{S} in defining for $0 < \sigma$

$$\Omega_\sigma = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \sigma\}, \quad \mathcal{S}_\sigma = \{x \in \mathbb{R}^N : \text{dist}(x, \bar{\mathcal{S}}) \leq \sigma\}$$

If $z \in \partial\Omega$, we denote by \mathbf{n}_z the outward unit normal vector to Ω at z . Since $\partial\Omega$ is compact and C^2 , there exists $\sigma_0 > 0$ such that for any $(z, \sigma) \in \partial\Omega \times [0, \sigma_0]$, the mapping $\Pi : (z, \sigma) \mapsto z + \sigma \mathbf{n}_z$ is a C^2 diffeomorphism from $\partial\Omega \times [0, \sigma_0]$ to $\Theta'_\sigma := \bar{\Omega}_{\sigma_0} \setminus \Omega$. The mapping Π defines the flow coordinates near $\partial\Omega$.

If $0 < \delta < \sigma$, there exists a unique solution $u = u_{n,\sigma,\delta}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{\Omega_\sigma} \\ u &= 0 && \text{in } \partial\Omega_\sigma \times (0, \infty) \\ u(\cdot, 0) &= n \chi_{\mathcal{S}_\delta} && \text{in } \Omega_\sigma. \end{aligned} \quad (6.21)$$

Notice that \mathcal{S}_δ is closed in Ω_σ and $\inf\{|z - z'| : z \in \mathcal{S}_\delta, z' \in \Omega_\delta^c\} = \delta - \sigma$. Existence is standard as well as uniqueness in the case where Ω is bounded. If Ω^c is bounded the proof goes as in the uniqueness proof in Proposition 6.10. When $n \rightarrow \infty$, $\{u_{n,\sigma,\delta}\} \uparrow u_{\sigma,\delta}$ which is a solution of (1.6) in $Q_\infty^{\Omega_\sigma}$. Since $u_{n,\sigma,\delta}$ satisfies (4.1), for any $r, \tau > 0$ and any $a \in \partial\Omega_\sigma$, $u_{n,\sigma,\delta}$ remains uniformly continuous with respect to n in $\bar{Q}_\infty^{\Omega_\sigma} \cap (\bar{B}_r(a) \times [2^{-2}\tau, 2\tau])$. Consequently $u_{\sigma,\delta}(x, t) = 0$ on $\partial\Omega_\sigma \times (0, \infty) \cap (\bar{B}_r(a) \times [2^{-2}\tau, 2\tau])$. Therefore $u_{\sigma,\delta}$ vanishes

on $\partial\Omega_\sigma \times (0, \infty)$. If $a \in \Omega_\sigma$ and $\text{dist}(a, \mathcal{S}_\delta) = r > 0$, $u_{n,\sigma,\delta}(x, t) \leq w_{B_r}(x - a, t)$ in $Q_{\overline{B_r}(a)}^\infty$. This implies that $u_{n,\sigma,\delta}$ remains uniformly continuous with respect to n in $\overline{B_{r'}}(a) \times [0, T]$ for any $0 < r' < r$ and $T > 0$. Since $u_{n,\sigma,\delta}(x, t) \rightarrow 0$ in $\overline{B_{r'}}(a)$, $u_{\sigma,\delta}$ inherits the same property. Consequently $u_{\sigma,\delta}$ has initial trace $(\mathcal{S}_\delta, 0)$ in Ω_σ . By the maximum principle the mapping $(n, \delta) \mapsto u_{n,\sigma,\delta}$ is increasing with respect to n and decreasing with respect to δ . Furthermore, if $0 < \delta' < \sigma' < \sigma$ and $0 < \delta < \sigma$, there holds $u_{n,\sigma',\delta'} < u_{n,\sigma,\delta}$ in $Q_{\infty}^{\sigma'}$, which implies $u_{\sigma',\delta'} < u_{\sigma,\delta}$. We define

$$\overline{u}_{\mathcal{S},0} = \lim_{\sigma \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} u_{n,\sigma,\delta}. \quad (6.22)$$

Then $\overline{u}_{\mathcal{S},0}$ is a solution of (1.6) in Q_∞^Ω . Since $\lim_{t \rightarrow 0} u_{\sigma,\delta}(x, t) = 0$ uniformly on any compact subset $K \subset \mathcal{R} = \Omega \setminus \mathcal{S}$, $\overline{u}_{\mathcal{S},0}$ has initial trace 0 on \mathcal{R} . If $a \in \mathcal{S}$, we denote by $\tilde{u}_{\infty,a}$ the function defined in $Q_\infty^{\Omega_\sigma}$ by

$$\tilde{u}_{\infty,a}(x, t) = \begin{cases} u_{\infty,a}(x, t) & \text{if } (x, t) \in Q_\infty^\Omega \\ 0 & \text{if } (x, t) \in Q_\infty^{\Omega_\sigma} \setminus Q_\infty^\Omega \end{cases}$$

Then $\tilde{u}_{\infty,a}$ is a subsolution of (1.6) in $Q_\infty^{\Omega_\sigma}$ which is smaller than $u_{\infty,a,\sigma}$ which is the limit, when $k \rightarrow \infty$ of the solution $u_{k\delta_a,\sigma}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 & \text{in } Q_\infty^{\Omega_\sigma} \\ u &= 0 & \text{in } \partial\Omega_\sigma \times (0, \infty) \\ u(\cdot, 0) &= k\delta_a & \text{in } \Omega_\sigma. \end{aligned} \quad (6.23)$$

There holds, by Theorem 6.2,

$$u_{\sigma,\delta}(x, t) \geq u_{\infty,a,\sigma} \geq \tilde{u}_{\infty,a}(x, t) \quad \forall (x, t) \in Q_\infty^{\Omega_\sigma}.$$

Letting successively $\delta \rightarrow 0$ and $\sigma \rightarrow 0$ yields to $\overline{u}_{\mathcal{S},0} \geq \tilde{u}_{\infty,a} = u_{\infty,a}$ in $Q_\infty^{\Omega_\sigma}$. Therefore any $a \in \mathcal{S}$ is a singular initial point of $\overline{u}_{\mathcal{S},0}$. Since $\mathcal{S} \cup \mathcal{R} = \Omega$, it follows that $\text{tr}_\Omega(u) = (\mathcal{S}, 0)$. Since $u_{\sigma,\delta}$ satisfies (4.1) and $\partial\Omega_\sigma$ has bounded curvature, independent of σ , there holds classically

$$|\nabla u_{\sigma,\delta}(x, t)| \leq ct^{-\frac{q+\alpha}{q-1}} \quad \forall (x, t) \in \partial\Omega_\sigma \times (0, \infty). \quad (6.24)$$

If $z \in \partial\Omega$ and then by the mean value theorem there exists $\theta \in (0, 1)$ such that,

$$0 \leq u_{\sigma,\delta}(z, t) = u_{\sigma,\delta}(z + \sigma \mathbf{n}_z, t) - \sigma \nabla u_{\sigma,\delta}(z + \theta \sigma \mathbf{n}_z, t) \cdot \mathbf{n}_z \leq c\sigma t^{-\frac{q+\alpha}{q-1}}.$$

This implies that $\overline{u}_{\mathcal{S},0}$ vanishes on $\partial\Omega_\sigma \times (0, \infty)$.

Let u be any positive solution of (1.6) in Q_∞^Ω , vanishing on $\partial\Omega \times (0, \infty)$, with initial trace $(\mathcal{S}, 0)$. For $0 < \delta < \sigma$ fixed and for $R, \epsilon > 0$, there exists $\tau_\epsilon > 0$ such that, for any $\tau \in (0, \tau_\epsilon]$,

$$u(x, \tau) \leq \epsilon \quad \forall x \in \overline{B_R} \cap \Omega \setminus \mathcal{S}_\sigma.$$

This is due to the fact that $u(x, \tau) \rightarrow 0$ when $\tau \rightarrow 0$, uniformly on compact subset of $\overline{B_R} \cap \mathcal{R}$. Assume that Ω is unbounded (the case where Ω is bounded is simpler since it does not require to introduce the barrier w_{B_R}) and let $R > 0$ large enough so $\Omega^c \subset B_R$. By (4.1)-(4.3) there exists $0 < \tau_1 \leq \tau_0$ such that for any $\tau \in (0, \tau_1]$,

$$u(x, \tau) \leq w_{B_R}(x, \tau) + u_{\sigma,\delta}(x, \tau) \quad \forall x \in \overline{B_R} \cap \Omega \cap \mathcal{S}_\sigma.$$

Furthermore $u(x, t) < w_{B_R}(x, t)$ for all $t > 0$ and $x \in \partial B_R \cap \Omega$. Since $\epsilon + w_{B_R} + u_{\sigma, \delta}$ is a supersolution for (1.6) in $B_R \cap \Omega \times (0, \infty)$, it follows that

$$u(x, t) \leq \epsilon + w_{B_R}(x, t) + u_{\sigma, \delta}(x, t) \quad \forall (x, t) \in B_R \cap \Omega \times (0, \infty).$$

Letting successively $\delta \rightarrow 0$, $\sigma \rightarrow 0$, $R \rightarrow \infty$ (here we use the fact that $w_{B_R}(x, t) \rightarrow 0$ when $R \rightarrow \infty$ by Proposition 3.4) and $\epsilon \rightarrow 0$ yields to $u \leq \bar{u}_{\mathcal{S}, 0}$.

Step 2: Existence of a minimal solution. The set $\mathcal{U}_{\mathcal{S}, 0}(\Omega)$ is not empty since it contains $\bar{u}_{\mathcal{S}, 0}$ and we may define

$$\tilde{u}_{\mathcal{S}, 0} = \sup\{u_{\infty, a} : a \in \mathcal{S}\}, \quad (6.25)$$

and

$$\hat{u}_{\mathcal{S}, 0} = \inf\{u : u \in \mathcal{U}_{\mathcal{S}, 0}(\Omega)\}. \quad (6.26)$$

The functions $\tilde{u}_{\mathcal{S}, 0}$ and $\hat{u}_{\mathcal{S}, 0}$ are respectively positive sub and super solutions of (1.6) in Q_{∞}^{Ω} . They are bounded from above by $\bar{u}_{\mathcal{S}, 0}$ and from below by $u_{\infty, a}$ for any $a \in \mathcal{S}$. Since $u_{\infty, a} \leq u$ for any $a \in \mathcal{S}$ and $u \in \mathcal{U}_{\mathcal{S}, 0}(\Omega)$, it follows that $\tilde{u}_{\mathcal{S}, 0} \leq \hat{u}_{\mathcal{S}, 0}$. Therefore there exists a solution $\underline{u}_{\mathcal{S}, 0}$ of (1.6) in Q_{∞}^{Ω} which satisfies and

$$\tilde{u}_{\mathcal{S}, 0} \leq \underline{u}_{\mathcal{S}, 0} \leq \hat{u}_{\mathcal{S}, 0}. \quad (6.27)$$

This implies that $\underline{u}_{\mathcal{S}, 0}$ has initial trace $(\mathcal{S}, 0)$, it vanishes on $\partial\Omega \times (0, \infty)$ and it is therefore the minimal element of $\mathcal{U}_{\mathcal{S}, 0}(\Omega)$. \square

Remark. If $\text{dist}(\mathcal{S}, \Omega^c) > 0$, it is not needed to replace Ω by a larger set Ω_{σ} in order to construct the maximal solution. The construction of $\bar{u}_{\mathcal{S}, 0}$ can be done in replacing $u_{n, \sigma, \delta}$ by the solution $u = u_{n, \sigma}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^{\alpha} u^q &= 0 && \text{in } Q_{\infty}^{\Omega} \\ u &= 0 && \text{in } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) &= n\chi_{\mathcal{S}_{\delta}} && \text{in } \Omega, \end{aligned} \quad (6.28)$$

with $\delta < \delta_0 := \text{dist}(\mathcal{S}, \Omega^c)$.

The next result is an extension of Proposition 2.2.

Proposition 6.8 *Assume $\alpha > -1$ and $1 < q < q_{c, \alpha}$. Let $\{u_n\}$ be a sequence of positive solutions of (1.6) which converges to u locally uniformly in Q_T^{Ω} , and denote by (\mathcal{S}_n, μ_n) and (\mathcal{S}, μ) the respective initial trace of u_n and u . If \mathcal{A} is an open subset of $\cap_n \mathcal{R}_n$ and $\mu_n(\mathcal{A})$ remains bounded independently of $n \in \mathbb{N}$ (where $\mathcal{R}_n = \Omega \setminus \mathcal{S}_n$ and $\mathcal{R} = \Omega \setminus \mathcal{S}$), then $\mathcal{A} \subset \mathcal{R}$ and $\chi_{\mathcal{A}} \mu_n \rightharpoonup \chi_{\mathcal{A}} \mu$ in the weak sense of measures. Conversely, if $\mathcal{A} \subset \mathcal{R}$, then for any compact $K \subset \mathcal{A}$, there exist $C_K > 0$ and $n_K \in \mathbb{N}$ such that $\mu_n(K) \leq C_K$ for any $n \geq n_K$.*

Proof. Clearly (2.14) holds. We keep the notations of the proof of Proposition 2.2 where the first statement has been proved in assuming $\bar{B}_{\bar{r}}(z) \subset \mathcal{A}$. Since $\mu_n(\mathcal{A})$ remains bounded, there exists a subsequence $\{n_j\}$ and a positive measure μ' on \mathcal{A} such that $\mu_{n_j} \rightharpoonup \mu'$ in the weak sense of measures in \mathcal{A} . Then $u_{\chi_{B_{\bar{r}}(z)} \mu_{n_j}}$ converges locally uniformly in $Q_{\infty}^{B_{\bar{r}}(z)}$ to the solution $u_{\chi_{B_{\bar{r}}(z)} \mu'}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^{\alpha} u^q &= 0 && \text{in } Q_{\infty}^{B_{\bar{r}}(z)} \\ u &= 0 && \text{in } \partial B_{\bar{r}}(z) \times (0, \infty) \\ u(\cdot, 0) &= \chi_{B_{\bar{r}}(z)} \mu' && \text{in } B_{\bar{r}}(z). \end{aligned} \quad (6.29)$$

Since $q < q_{c,\alpha}$, the convergence of $u_{\chi_{B_{\tilde{r}}(z)}\mu_{n_j}}$ and $t^\alpha u_{\chi_{B_{\tilde{r}}(z)}\mu_{n_j}}^q$ respectively to $u_{\chi_{B_{\tilde{r}}(z)}\mu_{n_j}}$ and $t^\alpha u_{\chi_{B_{\tilde{r}}(z)}\mu'}^q$ holds in $L^1(Q_T^{B_{\tilde{r}}(z)})$ for any $T > 0$. Relation (2.21) reads

$$u_{\chi_{B_{\tilde{r}}(z)}\mu_n}(x, t) \leq u_n(x, t) \leq u_{\chi_{B_{\tilde{r}}(z)}\mu_n}(x, t) + w_{B_{\tilde{r}}(z)}(x, t) \quad \text{in } Q_\infty^{B_{\tilde{r}}(z)}. \quad (6.30)$$

(see Proposition 3.3). Then u_{n_j} and $t^\alpha u_{n_j}^q$ converge to u and $t^\alpha u^q$ respectively, in $L^1(Q_T^{B_r(z)})$ for any $r < \tilde{r}$. From (2.17), we derive

$$\iint_{Q_\infty^{B_r(z)}} (-u(\zeta_t + \Delta\zeta) + \zeta t^\alpha u^q) dxdt = \int_{B_r(z)} \zeta(x, 0) d\mu'(x), \quad (6.31)$$

for any $\zeta \in C_c^{1,1}(\overline{Q_\infty^{B_r(z)}})$ which vanishes for t large enough. This implies that μ' is the initial trace of u in $B_r(z)$, i.e. $\chi_{B_r(z)}\mu' = \chi_{B_r(z)}\mu$ and $\chi_{B_r(z)}\mu_n \rightarrow \chi_{B_r(z)}\mu$. Using a partition of unity, we conclude that $\chi_{\mathcal{A}}\mu_n \rightarrow \chi_{\mathcal{A}}\mu$.

Conversely, we assume that there exist a compact set $K \subset \mathcal{A}$ and a subsequence μ_{n_j} such that $\mu_{n_j}(K) \rightarrow \infty$. Thus, using the diagonal process, there exist $z \in K$ and another subsequence that we still denote μ_{n_j} such that

$$\lim_{n_j \rightarrow \infty} \mu_{n_j}(B_\epsilon(z)) = \infty \quad \forall \epsilon > 0.$$

Therefore, we can construct a subsequence $\{n_{j_\ell}\} \subset \{n_j\}$ such that

$$\mu_{n_{j_\ell}}(B_{2^{-n_{j_\ell}}}(z)) = m_{n_{j_\ell}} \rightarrow \infty$$

when $n_{j_\ell} \rightarrow \infty$. Since the solution $u_{\chi_{B_{2^{-n_{j_\ell}}}(z)}\mu_{n_{j_\ell}}}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{B_{\tilde{r}}(z)} \\ u &= 0 && \text{in } \partial B_{\tilde{r}}(z) \times (0, \infty) \\ u(\cdot, 0) &= \chi_{B_{2^{-n_{j_\ell}}}(z)}\mu_{n_{j_\ell}} && \text{in } B_{\tilde{r}}(z). \end{aligned} \quad (6.32)$$

converges to $u_{\infty\delta_z}^{B_{\tilde{r}}(z)}$ which is the limit of the solution $u_{k\delta_z}$ of (6.32) with initial data $u(\cdot, 0) = k\delta_z$, and is dominated by $u_{n_{j_\ell}}$ in $Q_T^{B_{\tilde{r}}(z)}$ we conclude that $u \geq u_{\infty\delta_z}^{B_{\tilde{r}}(z)}$ in $Q_T^{B_{\tilde{r}}(z)}$, which implies that $z \in \mathcal{S}$, contradiction. \square

Proposition 6.9 *Assume u_1 and u_2 are two positive solutions of (1.6) in Q_∞^Ω with initial trace (\mathcal{S}, μ) . Then for any $a \in \mathcal{R}$ and $R > 0$ such that $B_R(a) \subset \mathcal{R}$, there holds*

$$|u_1(x, t) - u_2(x, t)| \leq w_{B_R}(x - a, t) \quad \forall (x, t) \in Q_\infty^{B_R(a)} \quad (6.33)$$

In particular $\lim_{t \rightarrow 0} |u_1(x, t) - u_2(x, t)| = 0$ uniformly on any compact subset of \mathcal{R} .

Proof. Since u and u' are solution of (1.6) and $B_R(a) \in \mathcal{R}$, for any $i = 1, 2$, $R' < R$ and $T > 0$,

$$\iint_{Q_T^{B_{R'}(a)}} t^\alpha u_i^q(x, t) dxdt + \iint_{Q_T^{B_{R'}(a)}} u_i(x, t) dxdt < \infty,$$

furthermore

$$\lim_{t \rightarrow 0} \int_{B_R(a)} u_i(x, t) \zeta(x) dx = \int_{B_R(a)} \zeta(x) d\mu(x) \quad \forall \zeta \in C_c(B_R(a)).$$

This implies that u_i has a Sobolev trace on $\partial_\ell Q_T^{B'_R(a)}$ which belongs to L^1 and they are the limit, when $k \rightarrow \infty$ of the solutions $u_{i,k}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{B'_R(a)} \\ u &= \min\{k, u_i|_{Q_\infty^{B'_R(a)}}\} && \text{in } \partial_\ell Q_\infty^{B'_R(a)} \\ u(\cdot, 0) &= \mu && \text{in } B'_R(a). \end{aligned} \quad (6.34)$$

Since $u_{2,k} + w_{B'_R}(\cdot - a)$ is a supersolution

$$u_{1,k} \leq u_{2,k} + w_{B'_R}(\cdot - a) \implies |u_{1,k} - u_{2,k}| \leq w_{B'_R}(\cdot - a) \text{ in } Q_\infty^{B'_R(a)}.$$

Letting $k \rightarrow \infty$, R' to R , we derive (6.33). The second statement is a consequence of the fact that $\lim_{t \rightarrow 0} w_{B_R}(\cdot - a) = 0$, uniformly on B'_R by Proposition 3.3. \square

Remark. The previous estimate does not use the fact that $\partial\Omega$ is smooth and bounded. If the u_i belong to $\mathcal{U}_{S,\mu}(\Omega)$, estimate (6.33) can be improved since the u_i vanish on $\partial_\ell Q_\infty^{B'_R(a)}$, and we obtain,

$$|u_1(x, t) - u_2(x, t)| \leq \min \left\{ w_{B_R}(x - a, t), c_\alpha t^{-\frac{1+\alpha}{q-1}} \right\} \quad \forall (x, t) \in Q_\infty^{B'_R(a)}. \quad (6.35)$$

Proposition 6.10 *Assume $\Omega \subseteq \mathbb{R}^N$ is either \mathbb{R}^N or an open domain with a C^2 compact boundary, $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Then for any measure μ in Ω such that $\mu|_{\Omega_R} \in \mathfrak{M}_+^{b,\rho}(\Omega_R)$ where $\Omega_R = \Omega \cap B_R$, there exists a unique solution u_μ to*

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^\Omega \\ u &= 0 && \text{in } \partial_\ell Q_\infty^\Omega \\ u(\cdot, 0) &= \mu && \text{in } \Omega, \end{aligned} \quad (6.36)$$

and the mapping $\mu \mapsto u_\mu$ is increasing. Furthermore, if $\{\mu_n\}$ is a sequence of positive measures such that $\mu_n|_{\Omega_R} \in \mathfrak{M}_+^{b,\rho}(\Omega_R)$ which converges weakly to $\mu|_{\Omega_R} \in \mathfrak{M}_+^{b,\rho}(\Omega_R)$, then $\{u_{\mu_n}\} \rightarrow u_\mu$ locally uniformly in Q_∞^Ω .

Proof. We recall that u is a solution of (6.36) if $u \in L_{loc}^1(\overline{Q_\infty^\Omega})$, $|u|^q \in L_{loc}^1(\overline{Q_\infty^\Omega}; t^\alpha \rho dx dt)$ satisfies

$$\iint_{Q_\infty^\Omega} \left(-u(\partial_t \zeta + \Delta \zeta) + t^\alpha |u|^{q-1} u \zeta \right) dx dt = \int \zeta(\cdot, 0) d\mu \quad (6.37)$$

for any test function $\zeta \in C_0^{2,1;1}(\overline{Q_\infty^\Omega})$. When Ω is bounded, existence, uniqueness and stability are proved in [9]. Thus we assume that Ω is unbounded and we assume $R \geq R_0$ such that $\Omega^c \subset B_{R_0}$. There exists a unique solution u_R of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_R} \\ u &= 0 && \text{in } \partial_\ell Q_\infty^{\Omega_R} \\ u(\cdot, 0) &= \mu|_{\Omega_R} && \text{in } \Omega_R. \end{aligned} \quad (6.38)$$

The function u_R is nonnegative, $R \mapsto u_R$ is increasing. For $R > R_1$, u_R admits a Sobolev trace f_{R_1} on $\partial B_{R_1} \times (0, T)$ which is an integrable function, and u_R is the unique solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_{R_1}} \\ u &= 0 && \text{in } \partial\Omega \times (0, \infty) \\ u &= f_{R_1} && \text{in } \partial B_{R_1} \times (0, \infty) \\ u(\cdot, 0) &= \mu|_{\Omega_{R_1}} && \text{in } \Omega_{R_1}. \end{aligned} \quad (6.39)$$

Furthermore, $u_R|_{Q_\infty^{\Omega_{R_1}}} = \lim_{m \rightarrow \infty} v_m$, where v_m is the unique solution of (6.36) where the boundary data on $\partial B_{R_1} \times (0, \infty)$ is replaced by $f_{R_1, m} = f_{R_1} \wedge m$ ($m \in \mathbb{N}^*$). Let v_{R_1} be the unique solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_{R_1}} \\ u &= 0 && \text{in } \partial\Omega \times (0, \infty) \\ u &= 0 && \text{in } \partial B_{R_1} \times (0, \infty) \\ u(\cdot, 0) &= \mu|_{\Omega_{R_1}} && \text{in } \Omega_{R_1}, \end{aligned} \quad (6.40)$$

If $w_{B_{R_1}}$ is the barrier function in $Q_\infty^{\Omega_{R_1}}$ which has been constructed in Proposition 3.3, $v_{R_1} + w_{B_{R_1}}$ is a supersolution for problem (6.39). Since it is larger than v_m in $Q_\infty^{\Omega_{R_1}}$ for any $m > 0$, there holds $u_R \leq v_{R_1} + w_{B_{R_1}}$, for any $R > R_1$. Then $u_R \uparrow u_\mu$ which is a solution of 1.6 in $Q_\infty^{\Omega_{R'}}$. By Proposition 3.4, $w_{B_{R_1}}$ remains uniformly bounded in $Q_\infty^{\Omega_{R'}}$ for any $R_0 < R' < R_1$. Therefore u_μ shares the same property. If $\zeta \in C_c^{1,1;1}(Q_\infty^{\overline{\Omega}})$ vanishes on $Q_\infty^{\partial\Omega}$ and for $|x| > R' > R_0$, there holds for $R > R' > R_0$ and $T > 0$,

$$\iint_{Q_\infty^{\overline{\Omega}}} (-u_R(x, t)(\partial_t \zeta + \Delta \zeta) + \zeta t^\alpha u_R^q) dx dt = \int_\Omega \zeta(x, 0) d\mu(x) - \int_\Omega \zeta(x, T) u_R(x, T) dx \quad (6.41)$$

If we let $R \rightarrow \infty$ we deduce by the monotone convergence theorem that u_μ is a weak solution of (6.36). This proves existence.

For uniqueness, we consider u_μ and u'_μ two solutions of (6.36). By the same argument as in the existence part, for any $R > 0$, u_μ is smaller than the supersolution $u'_\mu + w_{B_{R_1}}$ in $Q_\infty^{\Omega_{R'}}$. Since $\lim_{R \rightarrow \infty} w_{B_R} = 0$ by Proposition 3.4 we obtain $u_\mu \leq u'_\mu$. Similarly $u'_\mu \leq u_\mu$. Uniqueness implies the monotonicity of the mapping $\mu \mapsto u_\mu$.

For proving the stability, assume $\{\mu_n|_{\Omega_R}\}$ converges to $\mu|_{\Omega_R}$ in the weak sense of measures in $\mathfrak{M}_+^{b,\rho}(\Omega_R)$ for any $R > R_0$. Then the sequence of solutions $v_{n,R}$ of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_R} \\ u &= 0 && \text{in } \partial\Omega \times (0, \infty) \\ u &= 0 && \text{in } \partial B_R \times (0, \infty) \\ u(\cdot, 0) &= \mu_n|_{\Omega_R} && \text{in } \Omega_R, \end{aligned} \quad (6.42)$$

converges to the solution v_R of (6.40) with $R_1 = R$. In particular $v_{n,R} \rightarrow v_R$ and $t^\alpha v_{n,R}^q \rightarrow t^\alpha v_R^q$ in $L^1(Q_T^{\Omega_R})$ and by standard regularity result the convergence of $v_{n,R}$ towards v_R holds uniformly on $\overline{\Omega_R} \times [\epsilon, T]$ for any $0 < \epsilon < T$. Furthermore $u_{\mu_n} \leq v_{n,R} + W_{B_R}$ in $Q_\infty^{\Omega_R}$. This

jointly with standard local regularity results for heat equation, implies that $\{u_{\mu_n}\}$ remains uniformly bounded and hence relatively compact for the topology of uniform convergence on any compact set of $\overline{\Omega} \times [\epsilon, \infty)$. Thus there exist a subsequence $\{u_{\mu_{n_k}}\}$ and a function $u^* \in C(\overline{Q_\infty^\Omega})$ such that $u_{\mu_{n_k}} \rightarrow u^*$ locally uniformly in $\overline{\Omega} \times (0, \infty)$. Since $t^\alpha u_{\mu_n}^q \leq t^\alpha v_{n,R}^q + t^\alpha W_{B_R}^q$, there also holds by the dominated convergence theorem $t^\alpha u_{\mu_{n_k}}^q \rightarrow t^\alpha u^{*q}$ in $L_{loc}^1(\overline{\Omega} \times [0, \infty))$. Henceforth letting $n_k \rightarrow \infty$ in the expression

$$\begin{aligned} & \iint_{Q_\infty^\Omega} \left(-u_{\mu_{n_k}}(x, t)(\partial_t \zeta + \Delta \zeta) + \zeta t^\alpha u_{\mu_{n_k}}^q \right) dx dt \\ &= \int_\Omega \zeta(x, 0) d\mu_{n_k}(x) - \int_\Omega \zeta(x, T) u_{n_k}(x, T) dx, \end{aligned} \quad (6.43)$$

where $\zeta \in C_c^{1,1}(\overline{Q_\infty^\Omega})$, we conclude that $u^* = u_\mu$ and that $u_{\mu_n} \rightarrow u_\mu$. \square

Proposition 6.11 *Assume F is a non-empty relatively closed subset of Ω , $\mathcal{R} = \Omega \setminus F$ and $\mu \in \mathfrak{M}_+(\mathcal{R})$. If we set*

$$\partial_\mu F = \{z \in F : \mu(\mathcal{R} \cap B_r(z)) = \infty, \forall r > 0\}, \quad (6.44)$$

then $\partial_\mu F$ is relatively closed in Ω . If $\mathcal{R}^\mu = \Omega \setminus \partial_\mu F$, it contains \mathcal{R} and if μ^* is the measure defined in \mathcal{R}^μ by μ on \mathcal{R} and 0 in $\mathcal{R}^\mu \cap \mathcal{R}^c$, then there exist a minimal positive solution \underline{u}_{μ^*} and a maximal solution \overline{u}_{μ^*} of (1.6) vanishing on $\partial\Omega \times (0, \infty)$ satisfying $\text{tr}_\Omega(u) = (\partial_\mu F, \mu^*)$. Furthermore \underline{u}_{μ^*} and \overline{u}_{μ^*} are respectively the minimal and the maximal element of $\mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$.

Proof. The set $\partial_\mu F$ is the blow-up set of the measure μ . It is clearly a relatively closed subset of Ω included into $\overline{\mathcal{R}} \setminus \mathcal{R}$.

Step 1: Existence of a minimal solution. For $\delta > 0$, we denote $(\partial_\mu F)_\delta := \{x \in \Omega : \text{dist}(x, \partial_\mu F) \leq \delta\}$ and $\mathcal{R}_\delta^\mu = \Omega \setminus (\partial_\mu F)_\delta \subset \mathcal{R}^\mu$. We define the Radon measure μ_δ on Ω by

$$\mu_\delta = \begin{cases} \mu & \text{on } \mathcal{R}_\delta^\mu \cap \mathcal{R} \\ 0 & \text{on } F \cup (\partial_\mu F)_\delta \end{cases}$$

Then μ_δ is a positive Radon measure in Ω and by Proposition 6.10 problem (6.36) with initial data μ_δ admits a unique positive solution u_{μ_δ} . Furthermore the mapping $\delta \mapsto u_{\mu_\delta}$ is nonincreasing, and we set $\underline{u}_{\mu^*} = \lim_{\delta \rightarrow 0} u_{\mu_\delta}$. Then \underline{u}_{μ^*} is a positive solution of (1.6) in Q_∞^Ω which vanishes on $\partial\Omega \times (0, \infty)$ and has initial trace (\mathcal{S}', μ') . If $a \in \mathcal{R}^\mu$, there exists $R > 0$ such that $\overline{B}_R(a) \subset \mathcal{R}^\mu$ and $\delta_a > 0$ such that $\overline{B}_R(a) \subset \mathcal{R}_\delta^\mu$ for $0 < \delta < \delta_a$, that we assume below. By the maximum principle there holds

$$v_{\mu_\delta}(x, t) \leq u_\delta(x, t) \leq v_{\mu_\delta}(x, t) + w_{B_R}(x - a, t) \quad \text{in } Q_\infty^{B_R(a)},$$

where v_{μ_δ} is the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{B_R(a)} \\ u &\geq 0 && \text{in } Q_\infty^{B_R(a)} \\ u(x, t) &= 0 && \text{in } \partial B_R(a) \times (0, \infty) \\ u(\cdot, 0) &= \mu_\delta \chi_{B_R(a)} && \text{in } B_R(a). \end{aligned}$$

Letting $\delta \rightarrow 0$, then $\mu_\delta \chi_{B_R(a)} \uparrow \mu^* \chi_{B_R(a)}$, which yields to

$$v_{\mu^* \lfloor_{B_R(a)}}(x, t) \leq \underline{u}_{\mu^*}(x, t) \leq v_{\mu^* \lfloor_{B_R(a)}}(x, t) + w_{B_R}(x - a, t) \quad \text{in } Q_\infty^{B_R(a)}.$$

By a partition of unity, it implies that for any $\zeta \in C_c(\mathcal{R}^\mu)$, we have

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}^\mu} \underline{u}_{\mu^*}(x, t) \zeta(x) dx = \int_{\mathcal{R}^\mu} \zeta d\mu^*(x).$$

Therefore $\mathcal{R}^\mu \subset \mathcal{R}'$ and $\mu' \lfloor_{\mathcal{R}^\mu} = \mu^*$. If $z \in \mathcal{R}' \cap (\mathcal{R}^\mu)^c = \mathcal{R}' \cap \partial_\mu F$, $\mu^*(B_r(z) \cap \mathcal{R}^\mu) = \mu(B_r(z) \cap \mathcal{R}) = \infty$ for any $r > 0$ while there exists $r_0 > 0$ such that $\mu'(B_{r_0}(z)) < \infty$. By Proposition 6.8, for any $r' < r_0$ there exists $C > 0$ such that $\mu_\delta(\overline{B}_{r'}(z)) \leq C$. By the monotone convergence theorem, it implies $\mu(\overline{B}_{r'}(z) \cap \mathcal{R}) \leq C$, which contradicts the definition of $\partial_\mu F$. Thus $\mathcal{S}' = \partial_\mu F$ and $tr_\Omega(\underline{u}_{\mu^*}) = (\partial_\mu F, \mu^*)$.

Let us assume that Ω is unbounded, R_0 is such that $\Omega^c \subset B_{R_0}$ and $\Omega_R = \Omega \cap B_R$ for $R > R_0$. Let $\delta, \epsilon > 0$, there exists τ_ϵ such that

$$u_{\mu_\delta}(x, \tau) \leq \epsilon \quad \forall (x, \tau) \in \Omega_R \cap (\partial_\mu F)_{\frac{\delta}{2}} \times [0, \tau_\epsilon] \quad (6.45)$$

Let $u \in \mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$. In order to compare u_{μ_δ} and $v := u + w_{B_R} + \epsilon$ in $\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}} \times (0, \tau_\epsilon]$ we see that

$$u_{\mu_\delta}(\cdot, 0) = \chi_{\mathcal{R}_R^\mu \setminus (\partial_\mu F)_\delta} \mu^* \leq u(\cdot, 0) = \chi_{\mathcal{R}_R^\mu \setminus (\partial_\mu F)_{\frac{\delta}{2}}} \mu^*$$

and both are bounded Radon measures. Since $u_{\mu_\delta}(t, 0) \leq v(x, t)$ in $\partial(\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}}) \times (0, \tau_\epsilon]$ and v is a supersolution, it follows that $u_{\mu_\delta} \leq v$ in $\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}} \times (0, \tau_\epsilon]$. Using (6.45) we conclude that

$$u_{\mu_\delta}(x, \tau) \leq v(x, \tau) \quad \forall (x, \tau) \in \Omega_R \times [0, \tau_\epsilon]. \quad (6.46)$$

Next, applying the comparison principle in $\Omega_R \times [\tau^*, \infty)$ between the solution u_{μ_δ} and the supersolution $u + \epsilon + w_{B_R}$, we conclude that (6.46) holds in $\Omega_R \times [\tau, \infty)$ and thus in $Q_\infty^{B_R}$. Letting successively $R \rightarrow \infty$, $\epsilon \rightarrow 0$ and $\delta \rightarrow 0$, we conclude that $\underline{u}_{\mu^*} \leq u$, thus u_{μ^*} is the minimal element of $\mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$.

Step 2: Existence of a maximal solution. Let $\delta > 0$ and $u \in \mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$. By Proposition 6.9, for any $R > 0$ and $\epsilon > 0$ there exists τ_ϵ such that

$$u(x, t) \leq \underline{u}_{\mu^*}(x, t) + \epsilon \quad \forall (x, t) \in \Omega_R \setminus (\partial_\mu F)_\delta \times (0, \tau_\epsilon], \quad (6.47)$$

and by (4.1), $u(x, t) \leq c_\alpha t^{-\frac{1+\alpha}{q-1}}$. Let $\tau \in (0, \tau_\epsilon]$ and $w_{\delta, \tau}$ be the solution of (6.36) in $Q_\infty^{\Omega_R}$ with initial data μ replaced by

$$h_{\delta, \tau} = \begin{cases} \underline{u}_{\mu^*}(x, \tau) & \text{if } x \in \Omega_R \setminus (\partial_\mu F)_\delta \\ c_\alpha \tau^{-\frac{1+\alpha}{q-1}} & \text{if } x \in \Omega_R \cap (\partial_\mu F)_\delta \end{cases} \quad (6.48)$$

By (6.47), (6.48) and the maximum principle,

$$u(x, t + \tau) \leq w_{\delta, \tau}(x, t) + \epsilon + w_{B_R}(x, t + \tau) \quad \forall (x, t) \in Q_\infty^{\Omega_R}. \quad (6.49)$$

Let $\bar{u}_{(\partial_\mu F)_\delta}$ be the maximal element of $\mathcal{U}_{(\partial_\mu F)_\delta, 0}(\Omega)$, which exists by Proposition 6.7. Then, by (4.3), for any $\delta' > \delta$, there exists $\tau_{\delta'} \in (0, \tau_\epsilon]$ such that for any $\tau \in (0, \tau_{\delta'}]$

$$\max\{\underline{u}_{\mu^*}(\cdot, \cdot + \tau), \bar{u}_{(\partial_\mu F)_\delta}(\cdot, \cdot + \tau)\} \leq w_{\delta, \tau} \leq \underline{u}_{\mu^*}(\cdot, \cdot + \tau) + \bar{u}_{(\partial_\mu F)_{\delta'}}(\cdot, \cdot + \tau) \quad \text{in } Q_\infty^{\Omega_R}. \quad (6.50)$$

Up to a sequence $\{\tau_n\}$ converging to 0, $\{w_{\delta, \tau_n}\}$ converges, locally uniformly in Q_∞^Ω to a solution w_δ of (1.6) in Q_∞^Ω which satisfies

$$\max\{\underline{u}_{\mu^*}, \bar{u}_{(\partial_\mu F)_\delta}\} \leq w_\delta \leq \underline{u}_{\mu^*} + \bar{u}_{(\partial_\mu F)_{\delta'}} \quad \text{in } Q_\infty^{\Omega_R}. \quad (6.51)$$

We can replace δ' by δ in this inequality, this proves that w_δ vanishes on $\partial_\ell Q_\infty^{\Omega_R}$ has initial trace $(\mu_\delta, (\partial_\mu F)_\delta)$, therefore (6.49) becomes

$$u(x, t) \leq w_\delta(x, t) + \epsilon + w_{B_R}(x, t) \quad \forall (x, t) \in Q_\infty^{\Omega_R}. \quad (6.52)$$

Letting successively $R \rightarrow \infty$ and $\epsilon \rightarrow 0$ we deduce that w_δ is larger than any $u \in \mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$ in $Q_\infty^{\Omega_R}$. Since $h_{\delta, \tau}$ decreases with δ , w_δ shares this property and the limit, denoted by \bar{u}_{μ^*} is a solution of (1.6) in Q_∞^Ω which vanishes on $\partial_\ell Q_\infty^{\Omega_R}$ which is large than u , thus it is the maximal element of $\mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$. \square

Proposition 6.12 *Under the assumptions of Proposition 6.11, we set $F_\delta := \{x \in \Omega : \text{dist}(x, F) \leq \delta\}$ and $\mathcal{R}_\delta := \Omega \setminus F_\delta \subset \mathcal{R}$. If we define the measure $\tilde{\mu}_\delta$ in Ω by*

$$\tilde{\mu}_\delta = \begin{cases} \mu & \text{on } \mathcal{R}_\delta \\ 0 & \text{on } F_\delta, \end{cases}$$

then $u_{\tilde{\mu}_\delta} \uparrow \underline{u}^*$ when $\delta \downarrow 0$.

Proof. There holds $\tilde{\mu}_\delta \leq \mu_\delta$ which implies $u_{\tilde{\mu}_\delta} \leq u_{\mu_\delta}$. When $\delta \rightarrow 0$, $u_{\tilde{\mu}_\delta} \uparrow \underline{u}^* \leq \underline{u}_{\mu^*}$, thus \underline{u}^* is a positive solution of (1.6) in Q_∞^Ω which vanishes on $\partial_\ell Q_\infty^\Omega$. Then $\text{tr}_\Omega(\underline{u}^*) = (\mathcal{S}'', \mu'')$ and $\mathcal{S}'' \subset \partial_\mu F$ and $\mu'' \leq \mu^*$ on \mathcal{R}^μ . Furthermore $\mu'' = \mu = \mu^*$ on \mathcal{R} , as in the proof of Proposition 6.11. Since $\mu^* = 0$ on $\mathcal{R}^\mu \setminus \mathcal{R}$ it follows that $\mu'' = \mu^*$ on \mathcal{R}^μ . Let $a \in \mathcal{R}'' \cap \partial_\mu F$ and $R > 0$ such that $\bar{B}_R(a) \subset \mathcal{R}''$. Then $\mu''(\bar{B}_R(a)) < \infty$. Therefore

$$\mu''(\bar{B}_R(a) \cap \mathcal{R}) = \mu(\bar{B}_R(a) \cap \mathcal{R}) < \infty,$$

contradiction. Thus $\mathcal{S}'' = \partial_\mu F$ and $\text{tr}_\Omega(\underline{u}^*) = (\partial_\mu F, \mu^*)$. Since $\underline{u}^* \leq \underline{u}_{\mu^*}$ and \underline{u}_{μ^*} is minimal, it follows that $\underline{u}^* = \underline{u}_{\mu^*}$. \square

Proposition 6.13 *Assume $v \in C(\bar{\Omega} \times (0, \infty))$ is a positive sub-solution (resp. supersolution) of (1.6) in Q_∞^Ω which vanishes on $\partial_\ell Q_\infty^\Omega$. Then there exists a minimal solution $\pi_+(v)$ larger than v (resp. a maximal solution $\pi_-(v)$ smaller than v and vanishing on $\partial_\ell Q_\infty^\Omega$).*

Proof. Assume v is a subsolution. Let $\tau > 0$ and let u_τ be the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (\tau, \infty) \\ u \geq 0, u &= 0 && \text{in } \partial\Omega \times (\tau, \infty) \\ u(\cdot, \tau) &= v(\cdot, \tau) && \text{in } \Omega. \end{aligned} \quad (6.53)$$

Existence and uniqueness follows from Proposition 6.10. Furthermore $u_\tau \geq v$ in $\Omega \times (\tau, \infty)$. This implies that for $0 < \tau < \tau'$, $u_\tau \geq u_{\tau'}$. Since $u_\tau(x, t) \leq c_\alpha(t - \tau)^{-\frac{1+\alpha}{q-1}}$, there exists $\pi_+(v) = \lim_{\tau \rightarrow 0} u_\tau$, and $\pi_+(v)$ is a positive solution of (1.6) in Q_∞^Ω and is larger than v . If u is any positive solution of (1.6) in Q_∞^Ω , vanishing on $\partial\ell Q_\infty^\Omega$ and larger than v , for any $\tau > 0$ it is larger than $u(\cdot, \tau)$, thus it is larger than u_τ on $\Omega \times (\tau, \infty)$. Therefore $u \geq \pi_+(v)$.

Assume now that u is a supersolution. We define u_τ by (6.53). Then $u_\tau \leq v$ and $u_\tau \leq u_{\tau'}$ for $0 < \tau < \tau'$. Then $\pi_-(v) = \lim_{\tau \rightarrow 0} u_\tau$, and $\pi_-(v)$ is a positive solution of (1.6) in Q_∞^Ω and is smaller than v , and thus vanishing on $\partial\ell Q_\infty^\Omega$. Similarly as above $\pi_-(v)$ is larger than any positive solution smaller than v . \square

Theorem 6.14 *Assume $\Omega \subseteq \mathbb{R}^N$ is either \mathbb{R}^N or an open domain with a C^2 compact boundary, $\alpha > -1$ and $1 < q < q_{c,\alpha}$. Then for any $\nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega)$ there exist a maximal positive solution $\bar{u}_{\mathcal{S},\mu}$ and a minimal positive solution $\underline{u}_{\mathcal{S},\mu}$ of (1.6) in Q_∞^Ω vanishing on $\partial\Omega \times (0, \infty)$ with initial trace ν . Furthermore, if $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$, then*

$$\bar{u}_{\mathcal{S},\mu} - \underline{u}_{\mathcal{S},\mu} \leq \bar{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0}. \quad (6.54)$$

Proof. Step 1: Construction of the maximal and minimal solutions. The functions \underline{u}_{μ^*} , $\underline{u}_{\mathcal{S},0}$, \bar{u}_{μ^*} and $\bar{u}_{\mathcal{S},0}$ have been defined in Proposition 6.11 and Proposition 6.7. Since $\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\}$ is a subsolution of (1.6) which is smaller than the supersolution $\bar{u}_{\mu^*} + \bar{u}_{\mathcal{S},0}$ we set

$$(i) \quad \underline{u}_{\mathcal{S},\mu} = \pi_+(\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\}) \quad \text{and} \quad (ii) \quad \bar{u}_{\mathcal{S},\mu} = \pi_-(\bar{u}_{\mu^*} + \bar{u}_{\mathcal{S},0}). \quad (6.55)$$

Then $\underline{u}_{\mathcal{S},\mu}$ and $\bar{u}_{\mathcal{S},\mu}$ are solutions which satisfy

$$\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{\mathcal{S},\mu} \leq \bar{u}_{\mathcal{S},\mu} \leq \bar{u}_{\mu^*} + \bar{u}_{\mathcal{S},0}. \quad (6.56)$$

Therefore $\underline{u}_{\mathcal{S},\mu}$ and $\bar{u}_{\mathcal{S},\mu}$ vanish on $\partial\ell Q_\infty^\Omega$, they have initial trace μ on \mathcal{R} and are larger than any $u_{\infty,a}$ for $a \in \mathcal{S}$ (notice that $\partial_\mu \mathcal{S} \subset \mathcal{S}$). This implies that $\underline{u}_{\mathcal{S},\mu}$ and $\bar{u}_{\mathcal{S},\mu}$ belong to $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$.

Let $u \in \mathcal{U}_{\mathcal{S},\mu}(\Omega)$. Then for $\sigma > \delta > 0$ and $\epsilon, R > 0$, there exists $\tau_1 > 0$ such that

$$u(x, t) \leq u_{\sigma,\delta}(x, t) \text{ for } (x, t) \in (\mathcal{S}_{\frac{\delta}{2}} \cap \Omega_R) \times (0, \tau_1].$$

There exists $\tau_2 \in (0, \tau_1]$ such that

$$u(x, t) \leq \bar{u}_{\mu^*}(x, t) + \epsilon + w_{B_R}(x, t) \text{ for } (x, t) \in (\Omega_R \setminus \mathcal{S}_{\frac{\delta}{2}}) \times (0, \tau_2].$$

Therefore

$$u(x, t) \leq u_{\sigma,\delta}(x, t) + \bar{u}_{\mu^*}(x, t) + \epsilon + w_{B_R}(x, t) \text{ for } (x, t) \in \Omega_R \times (0, \tau_2].$$

Therefore

$$u \leq u_{\sigma,\delta} + u_{\mu^*} + \epsilon + w_{B_R} \quad \text{in } Q_\infty^{\Omega_R} \quad (6.57)$$

Letting successively $R \rightarrow \infty$, $\epsilon \rightarrow 0$, $\delta \rightarrow 0$ and $\sigma \rightarrow 0$ we obtain $u \leq \bar{u}_{\mathcal{S},0} + \bar{u}_{\mu^*}$, and therefore $u \leq \pi_+(\bar{u}_{\mathcal{S},0} + \bar{u}_{\mu^*}) = \bar{u}_{\mathcal{S},\mu}$. Next, we also have

$$u \geq \tilde{u}_{\mathcal{S},0} := \sup\{u_{\infty,a} : a \in \mathcal{S}\} \implies u \geq \underline{u}_{\mathcal{S},0} = \pi_+(\tilde{u}_{\mathcal{S},0}),$$

by (6.25). With the notations of Proposition 6.12 with $F = \mathcal{S}$, for any $R > 0$, $\delta > 0$ and $\epsilon > 0$, there exists τ_ϵ such that

$$u_{\tilde{\mu}_\delta}(x, t) \leq u(x, t) + w_{B_R}(x, t) + \epsilon \quad \text{in } \Omega_R \times (0, \tau_\epsilon],$$

because the support of $\tilde{\mu}_\delta$ is included in $\Omega \setminus \mathcal{S}_\delta$. Therefore this last inequality holds in $Q_\infty^{\Omega_R}$ and consequently

$$\sup\{u_{\tilde{\mu}_\delta}, \underline{u}_{\mathcal{S},0}\} \leq u + \epsilon + w_{B_R} \quad \text{in } Q_\infty^{\Omega_R},$$

and we can let $R \rightarrow \infty$ and $\epsilon \rightarrow 0$ to obtain $\sup\{u_{\tilde{\mu}_\delta}, \underline{u}_{\mathcal{S},0}\} \leq u$ in Q_∞^Ω . Letting $\delta \rightarrow 0$ and using Proposition 6.12 we get $\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{\mathcal{S},\mu} \leq u$.

Step 2: Alternative construction. For $0 < \delta < \sigma$ and $n \in \mathbb{N}_*$ we denote by $\bar{u}_{n,\sigma,\delta,\mu}$ the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{\Omega_\sigma} \\ u &\geq 0 && \text{in } Q_\infty^{\Omega_\sigma} \\ u(x, t) &= 0 && \text{in } \partial_\ell Q_\infty^{\Omega_\sigma} \\ u(\cdot, 0) &= \mu_\delta + n\chi_{\mathcal{S}_\delta} && \text{in } \Omega_\sigma. \end{aligned} \tag{6.58}$$

We denote here $\Omega_\sigma = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \sigma\}$, $\mathcal{S}_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{S}) < \delta\}$ and $\mathcal{R}_\delta = \Omega \cap \mathcal{S}_\delta^c$ and $\mu_\delta = \chi_{\mathcal{R}_\delta} \mu$. The same arguments of monotonicity as in Proposition 6.7 and Proposition 6.11 show that

$$\lim_{\sigma \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \bar{u}_{n,\sigma,\delta,\mu} = \bar{u}_{\mathcal{S},\mu}. \tag{6.59}$$

If $\tau > 0$ we denote by $\underline{u}_{\tau,\delta,\mu}$ the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^\Omega \\ u &\geq 0 && \text{in } Q_\infty^\Omega \\ u(x, t) &= 0 && \text{in } \partial_\ell Q_\infty^\Omega \\ u(\cdot, 0) &= \mu_\delta + \chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau) && \text{in } \Omega, \end{aligned} \tag{6.60}$$

Using estimate (6.53) and Proposition 6.9 is easy to prove that $\underline{u}_{\tau,\delta,\mu} \leq u$ for any $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$. Furthermore

$$\max\{u_{\mu_\delta}, u_{\chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)}\} \leq \underline{u}_{\tau,\delta,\mu} \leq u_{\mu_\delta} + \underline{u}_{\mathcal{S},0}(\cdot, \cdot + \tau),$$

since have

$$u_{\chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)} \leq \underline{u}_{\mathcal{S},0}(\cdot, \tau) \leq u_{\chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)} + C \left(\frac{\delta^2}{\tau} \right)^{\frac{1+\alpha}{q-1} - \frac{1}{2}} e^{-\frac{\delta^2}{4\tau}}$$

by (5.5) with $N = 1$. Set $c(\delta, \tau) = C \left(\frac{\delta^2}{\tau} \right)^{\frac{1+\alpha}{q-1} - \frac{1}{2}} e^{-\frac{\delta^2}{4\tau}}$, then

$$\max\{u_{\mu_\delta}, u_{\chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)}\} \geq \max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}(\cdot, \tau) - c(\delta, \tau)\} \geq \max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}(\cdot, \tau)\} - c(\delta, \tau).$$

Therefore, if $\underline{u}_{\tau_n,\delta,\mu} \rightarrow \underline{u}_{0,\delta,\mu}$ locally uniformly in Q_∞^Ω , then $\underline{u}_{0,\delta,\mu}$ is a solution of (1.6) in Q_∞^Ω which satisfies

$$\max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{0,\delta,\mu} \leq u_{\mu_\delta} + \underline{u}_{\mathcal{S},0}, \tag{6.61}$$

and is smaller than any $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$. There exists $\delta_n \rightarrow 0$ such that $\underline{u}_{0,\delta_n,\mu} \rightarrow \underline{u}_{0,0,\mu}$. Then

$$\max\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{0,0,\mu} \leq \underline{u}_{\mu^*} + \underline{u}_{\mathcal{S},0}, \quad (6.62)$$

and $\underline{u}_{0,0,\mu}$ is an element of $\mathcal{U}_{\mathcal{S},0}(\Omega)$ smaller than any $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$. Thus $\underline{u}_{0,0,\mu} = \underline{u}_{\mathcal{S},\mu}$ and

$$\lim_{\delta \rightarrow 0} \lim_{\tau \rightarrow 0} \underline{u}_{\tau,\delta,\mu} = \underline{u}_{\mathcal{S},\mu}. \quad (6.63)$$

Step 3: Proof of (6.54). We assume $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} = \delta_0 > 0$, so that we can take $\sigma = 0$ in the construction of $\bar{u}_{\mathcal{S},\mu}$. Put $\tau = (\frac{c\alpha}{n})^{\frac{q-1}{1+\alpha}}$ and

$$\bar{Z}_{n,\delta,\mu} = \bar{u}_{n,\delta,\mu} - \bar{u}_{n,\delta,0}, \quad \underline{Z}_{\tau,\delta,\mu} = \underline{u}_{\tau,\delta,\mu} - \underline{u}_{\tau,\delta,0} \quad \text{and} \quad W_{n,\delta,\mu} = \bar{Z}_{n,\delta,\mu} - \underline{Z}_{\tau,\delta,\mu}. \quad (6.64)$$

Then $w = W_{n,\delta,\mu}$ satisfies

$$\partial_t w - \Delta w + t^\alpha \left(\bar{u}_{n,\delta,\mu}^q - \bar{u}_{n,\delta,0}^q - \underline{u}_{\tau,\delta,\mu}^q + \underline{u}_{\tau,\delta,0}^q \right) \quad (6.65)$$

in Q_∞^Ω and we can write

$$\begin{aligned} \bar{u}_{n,\delta,\mu}^q - \bar{u}_{n,\delta,0}^q - \underline{u}_{\tau,\delta,\mu}^q + \underline{u}_{\tau,\delta,0}^q &= (\bar{u}_{n,\delta,\mu}^q - \underline{u}_{\tau,\delta,\mu}^q) - (\bar{u}_{n,\delta,0}^q - \underline{u}_{\tau,\delta,0}^q) \\ &= d_\mu(\bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}) - d_0(\bar{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}) \end{aligned} \quad (6.66)$$

where

$$d_\mu(x, t) = \begin{cases} \frac{\bar{u}_{n,\delta,\mu}^q - \underline{u}_{\tau,\delta,\mu}^q}{\bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}} & \text{if } \bar{u}_{n,\delta,\mu} \neq \underline{u}_{\tau,\delta,\mu} \\ 0 & \text{if } \bar{u}_{n,\delta,\mu} = \underline{u}_{\tau,\delta,\mu} \end{cases} \quad (6.67)$$

and d_0 is defined accordingly. Since

$$\bar{u}_{n,\delta,\mu} \geq \max\{\underline{u}_{\tau,\delta,\mu}, \bar{u}_{n,\delta,0}\} \quad \text{and} \quad \underline{u}_{\tau,\delta,0} \leq \min\{\underline{u}_{\tau,\delta,\mu}, \bar{u}_{n,\delta,0}\},$$

there holds $d_\mu \geq d_0 \geq$ by convexity. Using the fact that $\bar{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0} \geq 0$ is infers that

$$d_\mu(\bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}) - d_0(\bar{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}) \geq d_\mu(\bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu} - \bar{u}_{n,\delta,0} + \underline{u}_{\tau,\delta,0})$$

Finally (6.65) becomes

$$\partial_t w - \Delta w + t^\alpha d_\mu w \leq 0 \quad \text{in } Q_\infty^\Omega. \quad (6.68)$$

Furthermore, in the sense of measures,

$$w(\cdot, 0) = \mu_\delta + n\chi_{\mathcal{S}_\delta} - n\chi_{\mathcal{S}_\delta} - (\mu_\delta + \chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)) + \chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau) = 0.$$

Because $w = 0$ in $\partial_\ell Q_\infty^\Omega$ it follows $w \leq 0$ by the maximum principle. Therefore

$$\bar{Z}_{n,\delta,\mu} \leq \underline{Z}_{\tau,\delta,\mu} \implies \bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu} \leq \bar{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}. \quad (6.69)$$

If we let successively $n \rightarrow \infty$ (and therefore $\tau \rightarrow 0$) and $\delta \rightarrow 0$, we obtain (6.54). \square

Remark. We do not know if (6.54) holds if we do not assume $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} = \delta_0 > 0$. However, if for $\theta > 0$ we set $\mathcal{S}_\theta = \mathcal{S} \cap \{x \in \Omega : \text{dist}(x, \Omega^c) \geq \theta\}$, then we have

$$\bar{u}_{\mathcal{S}_\theta, \mu} - \underline{u}_{\mathcal{S}_\theta, \mu} \leq \bar{u}_{\mathcal{S}_\theta, 0} - \underline{u}_{\mathcal{S}_\theta, 0}. \quad (6.70)$$

Furthermore all the four above functions increases when θ decreases to 0. If we set

$$\begin{aligned} (i) \quad & \lim_{\theta \rightarrow 0} \bar{u}_{\mathcal{S}_\theta, \mu} = \bar{u}_{\mathcal{S}, \mu} \\ (ii) \quad & \lim_{\theta \rightarrow 0} \bar{u}_{\mathcal{S}_\theta, 0} = \bar{u}_{\mathcal{S}, 0} \\ (iii) \quad & \lim_{\theta \rightarrow 0} \underline{u}_{\mathcal{S}_\theta, \mu} = \underline{u}_{\mathcal{S}, \mu} \\ (iv) \quad & \lim_{\theta \rightarrow 0} \underline{u}_{\mathcal{S}_\theta, 0} = \underline{u}_{\mathcal{S}, 0} \end{aligned} \quad (6.71)$$

then we infer that

$$\bar{u}_{\mathcal{S}, \mu} - \underline{u}_{\mathcal{S}, \mu} \leq \bar{u}_{\mathcal{S}, 0} - \underline{u}_{\mathcal{S}, 0}. \quad (6.72)$$

Our final result is the following existence and uniqueness theorem the proof is similar to the one of [7, Th 3.5].

Theorem 6.15 *Assume $\Omega \subseteq \mathbb{R}^N$ is either \mathbb{R}^N or an open domain with a C^2 compact boundary, $\alpha > -1$ and $1 < q < q_{c, \alpha}$. Then for any $\nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega)$ such that ρ_μ is bounded in any neighborhood of $\partial\Omega$ and $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$, the set $\mathcal{U}_{\mathcal{S}, \mu}(\Omega)$ contains one and only one element.*

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