



**HAL**  
open science

# Initial trace of solutions of general semilinear heat equation

Waad Al Sayed, Laurent Veron

► **To cite this version:**

Waad Al Sayed, Laurent Veron. Initial trace of solutions of general semilinear heat equation. 2013.  
hal-00800654v1

**HAL Id: hal-00800654**

**<https://hal.science/hal-00800654v1>**

Preprint submitted on 14 Mar 2013 (v1), last revised 7 Apr 2013 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Initial trace of solutions of general semilinear heat equation

Waad Al Sayed

College of Sciences and Humanities, Fahed Bin Sultan University, Tabuk, Saudi arabia

Laurent Véron

Department of Mathematics, Université François Rabelais, Tours, France

**Abstract** We study the initial trace problem for positive solutions of semilinear heat equations with strong absorption. We emphasize the particular case where  $u$  solves  $\partial_t u - \Delta u + t^\alpha u^q = 0$ , with  $q > 1$  and  $\alpha > -1$ .

## 1 Introduction

In this paper we study the initial trace problem for positive solutions of

$$\partial_t u - \Delta u + g(x, t, u) = 0 \text{ dans } Q_T^\Omega := \Omega \times (0, T) \quad (1.1)$$

where  $\Omega$  is a smooth domain in  $\mathbb{R}^N$  containing  $O$  and  $g \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ ,  $g(x, t, \cdot)$  is nondecreasing  $\forall (x, t) \in \Omega \times \mathbb{R}$  and  $g(x, t, r) \geq 0$  if  $r \geq 0$ . We give general conditions for the existence of an initial trace which belongs to the class of positive Borel measures.

**Theorem A** *Assume  $g$  satisfies the above conditions and that equation (1.1) possesses a barrier at any  $z \in \Omega$ . Then if  $u \in C^1(Q_T^\Omega) \cap C(\Omega \times [0, T] \setminus \{(O, 0)\})$  is a positive solution of (1.1) which vanishes on  $\Omega \setminus \{O\}$  at  $t = 0$  it admits an initial trace which is a outer regular positive Borel measure.*

The initial trace of positive solutions of (1.1) exists in the following sense: *there exists a relatively closed set  $\mathcal{S} \subset \Omega$  and a positive Radon measure  $\mu$  on  $\mathcal{R} := \Omega \setminus \mathcal{S}$  with the following properties:*

(i) *for any  $x_0 \in \mathcal{S}$  and any  $\epsilon > 0$*

$$\lim_{t \rightarrow 0} \int_{B_\epsilon(x_0) \cap \Omega} u(x, t) dx = \infty, \quad (1.2)$$

(ii) *for any  $\zeta \in C_c(\mathcal{R})$*

$$\lim_{t \rightarrow 0} \int_{\Omega} u(x, t) \zeta(x) dx = \int_{\Omega} \zeta d\mu. \quad (1.3)$$

The couple  $(\mathcal{S}, \mu)$  is unique and characterizes a unique positive outer regular Borel measure  $\nu$  on  $\Omega$ . The barrier property is always verified if  $g$  satisfies the Keller-Osserman condition, that is  $g(x, t, r) \geq h(r) \geq 0$  where  $h$  is nondecreasing and

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} \quad \text{where } H(s) = \int_0^s h(t)dt. \quad (1.4)$$

It is also satisfied if  $g(r) = t^\alpha |u|^{q-1} u$ , with  $q > 1$  and  $\alpha > -1$ . This notion has been introduced by Marcus and Véron [10] in the study of the boundary trace of positive solutions of

$$-\Delta u + g(x, u) = 0 \text{ dans } \Omega. \quad (1.5)$$

In the second part we concentrate on the particular case of equation

$$\partial_t u - \Delta u + t^\alpha u^q = 0 \text{ dans } Q_T^\Omega \quad (1.6)$$

where  $T > 0$ ,  $\alpha > -1$  and  $q > 1$ . Among the main tools are the solutions with an isolated singularity at  $(0, O)$ , which are solutions  $u$  of (1.6) in  $Q_T^\Omega$ , which belong to  $C^{2,1}(Q_T^\Omega) \cap C(\Omega \times [0, T] \setminus \{(0, 0)\})$  and satisfy

$$u(x, 0) = 0 \text{ in } \Omega \setminus \{O\}. \quad (1.7)$$

When  $\alpha = 0$ , Brezis and Friedman prove in [2] that if  $B_{2R} \subset \Omega$ , the following estimate holds:

$$u(x, t) \leq \frac{C(N, q, R)}{(|x|^2 + t)^{\frac{1}{q-1}}} \quad \forall (x, t) \in B_R \setminus \{0\} \times [0, T]. \quad (1.8)$$

They proved that if  $1 < q < q_N := 1 + \frac{2}{N}$  and  $k > 0$  there exist singular solutions with initial data  $u(\cdot, 0) = k\delta_O$ , unique if  $u$  vanishes on  $\partial\Omega \times [0, T]$ . In this range of exponents, Brezis, Peletier, Terman obtain in [1] the existence and uniqueness of a *very singular solution* of (1.6), always with  $\alpha = 0$ , that is a positive solution in  $Q_\infty$  under the form

$$v_0(x, t) = t^{-1/(q-1)} V_0\left(\frac{x}{\sqrt{t}}\right),$$

where  $V_0 > 0$  is  $C^2$  and satisfies

$$\begin{aligned} -\Delta V_0 - \frac{1}{2}\eta \cdot \nabla V_0 - \frac{1}{q-1} V_0 + V_0^q &= 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|\eta| \rightarrow \infty} |\eta|^{\frac{2}{q-1}} V_0(\eta) &= 0. \end{aligned} \quad (1.9)$$

Actually, Kamin and Peletier proved that  $v_0$  is the limit of the solutions  $u_k$  of (1.6) in  $Q_\infty$  which satisfy  $u(\cdot, 0) = k\delta_O$ . The very singular singular solution play a fundamental role in Marcus and Véron's description [7] of the initial trace of positive solutions of (1.6) with  $\alpha = 0$ . In [9], Marcus et Véron study this equation when  $\alpha \geq 0$  and  $1 < q < q_{\alpha, N} = 1 + \frac{2(1+\alpha)}{N}$ . They prove the existence of a self-similar solution of (1.6) under the form

$$v_\alpha(x, t) = t^{-\frac{1+\alpha}{q-1}} V_\alpha\left(\frac{x}{\sqrt{t}}\right),$$

which satisfies

$$\lim_{t \rightarrow 0} v_\alpha(x, t) = 0 \quad \forall x \neq 0$$

and

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} v_\alpha(x, t) dx = \infty \quad \forall \epsilon > 0.$$

The function  $V_\alpha$  verifies

$$-\Delta V_\alpha - \frac{1}{2}\eta \cdot \nabla V_\alpha - \frac{1+\alpha}{q-1}V_\alpha + V_\alpha^q = 0 \text{ in } \mathbb{R}^N. \quad (1.10)$$

Furthermore

$$V_\alpha(\eta) = C|\eta|^{\frac{2(1+\alpha)}{q-1}-N} e^{-\frac{|\eta|^2}{4}} (1 + o(1)) \text{ as } |\eta| \rightarrow \infty.$$

If  $1 < q < q_{\alpha, N}$ , they show that for every  $k > 0$  there exists a unique solution  $u_k$  of (1.6) with initial data  $k\delta_O$ . Furthermore  $\lim_{k \rightarrow \infty} u_k = v_\alpha$ . Actually the limitation  $\alpha \geq 0$  can be relaxed to  $\alpha > -1$  as we will see it later on.

In this article we extend Brezis-Friedman removability result to equation (1.6) and prove that if  $q \geq q_{\alpha, N}$ , any solution in  $Q_T$  which satisfies (1.7) is identically 0. The starting point of our study is the following extension of estimate (1.8) valid for any  $\alpha > -1$  and  $q > 1$ .

$$u(x, t) \leq \frac{C(N, q, \alpha, R)}{\left(|x|^2 + t\right)^{\frac{1+\alpha}{q-1}}} \quad \forall (x, t) \in B_R \setminus \{0\} \times [0, T]. \quad (1.11)$$

We prove that the following classification results holds.

**Theorem B** *Assume  $1 < q < q_{\alpha, N}$  and  $u \in C^1(Q_T^O) \cap C(\Omega \times [0, T] \setminus \{(O, 0)\})$  is a positive solution of (1.6) which vanishes on  $\Omega \setminus \{O\}$  at  $t = 0$ . Then*

(i) *either there exists  $k \geq 0$  such that  $u(\cdot, 0) = k\delta_O$  and*

$$u(x, t) \sim kE(x, t) \quad \text{as } (x, t) \rightarrow (O, 0), \quad (1.12)$$

where  $E(x, t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$ ,

(ii) or

$$u(x, t) \sim v_\alpha(x, t) \quad \text{as } (x, t) \rightarrow (O, 0). \quad (1.13)$$

In the supercritical case we obtain a removability statement.

**Theorem C** *Assume  $q \geq q_{\alpha, N}$  and  $u$  is as in Theorem A, then  $u$  can be extended by continuity as a function in  $C(\Omega \times [0, T])$ .*

We prove that equation (1.6) admits a barrier at any  $z \in \Omega$ . More precisely we construct a positive solution  $w_{B_R}$  of (1.6) in  $Q_\infty^{B_R}$  which tends to 0 locally uniformly in  $B_R$  when  $t \rightarrow 0$  and which blows-up uniformly on  $\partial B_R \times [\tau, \infty)$ , for any  $\tau > 0$ . Using sharp asymptotics of the function  $V_\alpha$ , we have the following result which extends Theorem A.

**Theorem D** *Assume  $1 < q < q_{\alpha, N}$  and  $\Omega \subset \mathbb{R}^N$  is open with a  $C^2$  compact boundary, eventually empty. Then for any couple  $(\mathcal{S}, \mu)$  where  $\mathcal{S} \subset \Omega$  is relatively closed and  $\mu \in \mathfrak{M}_+(\Omega \setminus \mathcal{S})$ , there exists a maximal and a minimal  $u \in C^1(Q_T^O) \cap C^1(\overline{\Omega} \times (0, T])$  which both satisfy (i) and (ii) and vanish on  $\partial\Omega \times (0, T]$ . If  $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$  and  $\mu$  is bounded in a neighborhood of  $\partial\Omega$ , then  $u$  is unique.*

## 2 Initial trace

In this section  $\Omega \subset \mathbb{R}^N$  is an open set,  $Q_T^\Omega = \Omega \times (0, T)$ ,  $\partial_\ell Q_T^\Omega = \Omega \times \{0\} \cup \partial\Omega \times [0, T)$  and  $g \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ . If  $u$  is defined in  $\Omega \times \mathbb{R}_+$ , we denote by  $g \circ u$  the function  $(x, t) \mapsto g(x, t, u(x, t))$ . We say that  $g$  belongs to  $\mathcal{H}$  (resp  $\mathcal{H}_0$ ) if

$$\begin{aligned} g(x, t, r) &\geq 0 \quad \forall (x, t, r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \\ &\text{(resp. } g \in \mathcal{H} \text{ and } r \mapsto g(x, t, r) \text{ is nondecreasing)}. \end{aligned} \quad (2.1)$$

We denote by  $\mathfrak{M}(\Omega)$  the set of Radon measures in  $\Omega$ , and by  $\mathfrak{M}^b(\Omega)$  (resp.  $\mathfrak{M}^{b,\rho}(\Omega)$ ) the subset of Radon measures such that

$$\int_\Omega d|\mu| < \infty \quad \left( \text{resp. } \int_\Omega \rho d|\mu| < \infty \right),$$

where  $\rho(x) := \text{dist}(x, \partial\Omega)$ . Their positive cones are respectively  $\mathfrak{M}_+(\Omega)$ ,  $\mathfrak{M}_+^b(\Omega)$  and  $\mathfrak{M}_+^{b,\rho}(\Omega)$ .

**Definition 2.1** *Let  $\mathcal{S}$  be a relatively closed subset of  $\Omega$  and  $\mu$  a Radon measure on  $\mathcal{R} := \Omega \setminus \mathcal{S}$ . We say that a nonnegative function  $u \in C(Q_T^\Omega)$  admits the couple  $(\mathcal{S}, \mu)$  for initial trace if*

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}} u(x, t) \zeta(x) dx = \int_{\mathcal{R}} \zeta d\mu \quad \forall \zeta \in C_c(\mathcal{R}),$$

and

$$\lim_{t \rightarrow 0} \int_U u(x, t) dx = \infty \quad \forall U \subset \Omega, U \text{ open}, U \cap \mathcal{S} \neq \emptyset$$

The set  $\mathcal{S}$  is the set of singular initial points of  $u$  and its complement  $\mathcal{R}$  the set of regular initial points. We write  $\text{tr}_\Omega(u) = (\mathcal{S}, \mu)$ .

Let  $\tilde{\mu}$  be the extension of  $\mu$  as a locally bounded Borel measure. To the couple  $(\mathcal{S}, \tilde{\mu})$  we can associate a unique Borel measure  $\nu$  defined by

$$\nu(E) = \begin{cases} \infty & \forall E \subset \Omega : E \text{ Borel}, E \cap \mathcal{S} \neq \emptyset \\ \tilde{\mu}(E) & \forall E \subset \Omega : E \text{ Borel}, E \subset \mathcal{R}. \end{cases} \quad (2.2)$$

**Lemma 2.1** *Assume  $\Omega$  is a bounded open set with a  $C^2$  boundary,  $T > 0$ ,  $g \in \mathcal{H}$ , and let  $u \in C(\overline{\Omega} \times (0, T])$  be a positive solution of*

$$\partial_t u - \Delta u + g \circ u = 0 \quad \text{in } Q_T^\Omega. \quad (2.3)$$

If  $g \circ u \in L^1_\rho(Q_T^\Omega)$ , then  $u \in L^\infty(0, T, L^1_\rho(\Omega))$  and there exists  $\mu \in \mathfrak{M}_+^\rho(\Omega)$  such that

$$\lim_{t \rightarrow 0} \int_\Omega u(x, t) \zeta(x) dx = \int_\Omega \zeta d\mu \quad \forall \zeta \in C_c(\Omega). \quad (2.4)$$

*Proof.* If  $\phi_1 > 0$  is the first eigenfunction of  $-\Delta$  in  $W_0^{1,2}(\Omega)$  and  $\lambda_1$  is the corresponding eigenvalue, we have

$$\frac{d}{dt} \int_{\Omega} u \phi_1 dx + \lambda_1 \int_{\Omega} u \phi_1 dx + \int_{\Omega} g \circ u \phi_1 dx + \int_{\partial\Omega} u \frac{\partial \phi_1}{\partial \nu} dS = 0,$$

where  $\nu$  is the normal vector. Set  $X = \int_{\Omega} u \phi_1 dx$ , then by Hopf Lemma,

$$X' + \lambda_1 X + \int_{\Omega} g \circ u \phi_1 dx \geq 0$$

which yields to

$$\frac{d}{dt} (e^{\lambda_1 t} X) + e^{\lambda_1 t} \int_{\Omega} g \circ u \phi_1 dx \geq 0.$$

For  $s \in (t, T)$

$$\frac{d}{dt} \left( e^{\lambda_1 t} X - \int_t^T e^{\lambda_1 s} \int_{\Omega} g \circ u \phi_1 dx ds \right) \geq 0,$$

which means that the mapping

$$t \rightarrow e^{\lambda_1 t} X - \int_t^T e^{\lambda_1 s} \int_{\Omega} g \circ u \phi_1 dx ds$$

is nondecreasing. Therefore

$$e^{\lambda_1 t} X - \int_t^T e^{\lambda_1 s} \int_{\Omega} g \circ u \phi_1 dx ds \leq e^{\lambda_1 T} X.$$

and finally

$$X \leq e^{\lambda_1(T-t)} X + e^{-\lambda_1 t} \int_t^T e^{\lambda_1 s} \int_{\Omega} g \circ u \phi_1 dx ds$$

Since  $\rho^{-1}\phi_1$  is positively bounded from above and from below,  $u \in L^\infty(0, T, L_\rho^1(\Omega))$  and there exists a sequence  $\{t_n\}$  decreasing to 0 and a measure  $\mu \in \mathfrak{M}_+^p(\Omega)$  such that

$$\lim_{t_n \rightarrow 0} \int_{\Omega} u(x, t_n) \zeta dx = \int_{\Omega} \zeta d\mu \quad \forall \zeta \in C_c(\Omega).$$

If  $\zeta \in C_c^2(\Omega)$  there holds

$$\int_{\Omega} u(x, t_n) \zeta dx = \int_{t_n}^T \int_{\Omega} (g \circ u \zeta - u \Delta \zeta) dx dt + \int_{\Omega} u(x, T) \zeta dx,$$

thus

$$\int_{\Omega} \zeta d\mu = \iint_{Q_T^{\Omega}} (g \circ u \zeta - u \Delta \zeta) dx dt + \int_{\Omega} u(x, T) \zeta dx.$$

This implies that  $\mu$  is uniquely determined and  $u(\cdot, t)$  converges to  $\mu$  in the weak sense of measures.  $\square$

**Corollary 2.1** Assume  $\Omega$  is an open domain,  $g \in \mathcal{H}$  and  $u \in C^2(Q_T^\Omega)$  is a positive solution of (2.4). Suppose that for any  $z \in \Omega$  there exists an open neighborhood  $U \subset \Omega$  such that

$$\int_0^T \int_U g \circ u dx dt < \infty.$$

Then  $u(x, t) \in L^\infty(0, T, L_{loc}^1(\Omega))$  and there exists a positive Radon measure  $\mu$  on  $U$  such that

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}} u(x, t) f(x) dx = \int_{\mathcal{R}} d\mu \quad \forall f \in C_c(\mathcal{R})$$

*Proof.* We apply the previous lemma in replacing  $U$  by a ball  $B_\epsilon(z)$  and conclude by a partition of unity.  $\square$

The following class of nonlinearity has been introduced by Marcus and Véron [10] in order to study the boundary trace of solutions of elliptic equations

**Definition 2.2** A function  $g \in \mathcal{H}$  is a coercive nonlinearity in  $Q_T^\Omega$  if, for every subdomain  $\Omega'$  of  $\Omega$  and every  $\epsilon \in (0, T)$ , the set of positive solutions of (1.1) in  $Q_{\epsilon, T}^{\Omega'} := \Omega' \times (\epsilon, T)$  is uniformly bounded in compact subsets of  $Q_{\epsilon, T}^{\Omega'}$ .

**Definition 2.3** Let  $z \in \Omega$ . We say that equation (1.1) possesses a strong barrier at  $z$  if there exists a number  $r_0 \in (0, \rho(z))$  such that, for every  $r \in (0, r_0)$ , there exists a positive super solution  $w = w_{r, z}$  of (1.1) in  $B_r(z) \times (0, T)$  such that

$$w \in C(B_r(z) \times [0, T]), \quad \lim_{|x-z| \rightarrow r} w(x, t) = \infty \text{ locally uniformly if } t \in (0, T). \quad (2.5)$$

**Lemma 2.1** Assume  $g \in \mathcal{H}$  is a coercive nonlinearity in  $Q_T^\Omega$ , then the set of solutions of (1.1) in  $Q_T^\Omega$  is uniformly bounded from above in every compact subset of  $Q_T^\Omega$ . Furthermore, if  $g \in \mathcal{H}_0$ ,  $A \subset \Omega$  is open and (1.1) possesses a strong barrier at every point of  $z \in A$ , then the set of solutions  $u$  of (1.1) such that  $u \in C(A \times [0, T])$  and  $u(x, 0) = 0$  on  $A$  is uniformly bounded from above in every compact subset of  $A \times [0, T]$ .

*Proof.* Let  $K$  be a compact subset of  $Q_T^\Omega$  and let  $\Omega'$  be a smooth, bounded domain of  $\Omega$  and  $\epsilon > 0$  such that  $K \subset Q_{\epsilon, T}^{\Omega'}$ . Let  $U = U_{Q_{\infty}^{\Omega'}}$  be the minimal large solution of (1.1) in  $Q_T^{\Omega'}$ , i.e. the limit, when  $k \rightarrow \infty$ , of solutions with Cauchy-Dirichlet data  $k$  on  $\partial_\ell Q_{\epsilon, T}^{\Omega'} := \Omega' \times \{\epsilon\} \cup \partial\Omega' \times [\epsilon, T]$ . By the maximum principle, if  $u \in C(Q_T^\Omega)$  is a solution of (1.1), then  $u \leq U$  in  $\Omega'$ .

For the second statement, let  $K$  be a compact subset of  $A$ . For any  $z \in K$  there exists  $r_z > 0$  such that for any  $r \in (0, r_z)$  there exists a positive super solution of (1.1) in  $Q_T^{B_r(z)}$  which satisfies (2.5). Since  $K$  is compact, there exist  $z_1, \dots, z_p$  such that  $K \subset \cup_{j=1}^p B_{r_{z_j}/2}(z_j)$ .

For any  $j \in \{1, \dots, p\}$  we denote by  $w_j$  the super solution in  $Q_T^{B_{2r_{z_j}/3}(z_j)}$ . By comparison principle, there holds

$$u(x, t) \leq \sup\{w_j(x, t) : (x, t) \in B_{r_{z_j}/2}(z_j) \times (0, T)\} := M_j, \quad (2.6)$$

for  $(x, t) \in Q_T^{B_{r_{z_j}/2}(z_j)}$ . Therefore  $u \leq M = \max_{j=1, \dots, p} M_j$  in  $K \times (0, T)$ .  $\square$

**Lemma 2.2** Let  $g \in \mathcal{H}$  and  $u \in C^2(Q_T^\Omega)$  be a positive solution of (1.1) and suppose  $z \in \Omega$  is such that

$$\int_0^T \int_{B_\epsilon(z) \cap \Omega} g \circ u dx dt = \infty \quad \forall \epsilon > 0. \quad (2.7)$$

Suppose that at least one of the following sets of conditions holds:

(i) There exists an open neighborhood  $U' \subset \Omega$  of  $z$  such that  $u \in L^1(U' \times (0, T))$ .

(ii) The following hold:

1-  $g \in \mathcal{H}_0$ ,

2- (1.1) possesses a strong barrier at  $z$ .

Then,

$$\lim_{t \rightarrow 0} \int_{B_\epsilon(z) \cap \Omega} u(x, t) dx = \infty \quad \forall \epsilon > 0. \quad (2.8)$$

*Proof.* Assume that  $\Omega$  is bounded. First consider the case when condition (i) holds. Let  $\phi \in C^{2,1}(U' \times [0, T])$  with compact support in  $U' \times [0, T]$  and such that  $\phi(x, 0) = 1$  in a neighborhood of  $z$ . Then

$$\int_t^T \int_{U'} (u(-\phi_t - \Delta\phi) + g \circ u \phi) dx dt = \int_{U'} u(t) \phi dx - \int_{U'} u(T) \phi dx. \quad (2.9)$$

By assumption  $\int_t^T \int_{U'} u(\phi_t + \Delta\phi) dx dt$  is bounded. We let  $t$  tend to 0, the result follows from (2.7).

Next we assume that condition (ii) holds,  $u \notin L^1(U' \times (0, T))$  for any neighborhood  $U'$  of  $z$  and that the conclusion is not valid. Thus there exist  $r^* > 0$ , such that  $\overline{B}_{r^*}(z) \subset U_z$  and a sequence  $\{t_n\}$  decreasing to 0 such that

$$\int_{B_{r^*}(z)} u(x, t_n) dx \leq M$$

for some  $M > 0$ . Furthermore  $g$  is coercive in  $B_{r^*}(z) \times (0, T)$ . Let  $\{h_{n,k}\} \subset C^\infty(Q_T^\Omega)$  an increasing sequence with respect to  $k$  and  $n$  of nonnegative functions such that  $h_{n,k} = 0$  on  $B_{r^*}(z) \times \{0\}$ ,  $0 \leq h_{n,k} \leq k$  and  $h_{n,k} = k$  on  $(t_n, T) \times \partial B_{r^*}(z)$ . Let  $w_{h_{n,k}}$  be the solution of (1.1) in  $B_{r^*}(z) \times (0, T)$  such that  $w_{h_{n,k}} = h_{n,k}$  on  $\partial_\ell Q_T^{B_{r^*}(z)}$ . By the maximum principle and condition (ii)-1, the sequence  $\{w_{h_{n,k}}\}$  is monotone increasing with respect to  $k$  and  $n$ . Condition (ii)-3 implies that, for every  $r < r^*$  and  $\beta < T$ , the sequence is bounded in  $\overline{B}_r(z) \times [0, \beta]$ , and since  $u$  is locally bounded in  $Q_T^\Omega$  there exists  $k = k(n)$  such that  $k \geq u$  on  $(t_n, T) \times \partial B_r(z)$  and  $k(n) \rightarrow \infty$  when  $n \rightarrow \infty$ . Then  $w = \lim_{n \rightarrow \infty} w_{h_{n,k}}$  is a solution of (1.1) which blows up on  $\partial B_{r^*}(z) \times (0, T)$  and vanishes on  $B_{r^*}(z) \times \{0\}$ . Let  $v_n$  be the solution of the heat equation in  $B_{r^*}(z) \times (t_n, T)$  such that  $v_n(\cdot, t_n) = u(\cdot, t_n)$  in  $B_{r^*}(z)$  and  $v_n = 0$  on  $\partial B_{r^*}(z) \times (t_n, T)$ . Then  $w_{h_{n,k(n)}} + v_n$  is a super solution of (1.1) in  $B_{r^*}(z) \times (t_n, T)$  which dominates  $u$  on  $\partial_\ell Q_{t_n, T}^{B_{r^*}(z)}$ . By the maximum principle,

$$u \leq w_{h_{n,k(n)}} + v_n \quad \text{in } Q_{t_n, T}^{B_{r^*}(z)}.$$



And we have in particular

$$\int_{B_r(z)} u(x, t) dx \leq \int_{B_r(z)} (w_{h_n, k(n)} + v_n)(x, t) dx \leq M + \int_{B_r(z)} w(x, t) dx \quad \forall t \in (t_n, T).$$

Since it holds for any  $n$ , it implies  $u \in L^1(Q_T^{B_r(z)})$ , which leads to a contradiction.  $\square$

*Example 1.* If  $g(x, t, r) = h(r)$  where  $h$  is continuous, nondecreasing and  $h(0) \geq 0$ , then

$$u_t - \Delta u + h(u) = 0 \quad (2.10)$$

possesses a strong barrier at any  $z \in \Omega$  if and only if  $h$  satisfies the Keller-Osserman condition, that is there exists some  $a \geq 0$  such that

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \text{ where } H(s) = \int_0^s h(\tau) d\tau. \quad (2.11)$$

The super solution can be chosen to be the maximal solution  $\phi_r$  of the elliptic equation

$$-\Delta \phi + h(\phi) = 0 \quad \text{in } B_r(z). \quad (2.12)$$

If we assume moreover that  $h$  is super-additive, i.e.  $h(a+b) \geq h(a) + h(b)$  for all  $a, b \geq 0$ , then there holds

$$\int_a^\infty \frac{ds}{h(s)} < \infty, \quad (2.13)$$

and any solution  $u$  of (2.10) is dominated in  $Q_T^{B_r(z)}$  by  $\phi_r(x) + \psi(t)$  where  $\psi$  is defined by inversion from

$$\int_{\psi(t)}^\infty \frac{ds}{h(s)} = t \quad \forall t > 0.$$

*Example 2.* If  $g(x, t, r) = a(x)b(t)h(r)$  where  $a \in C(\Omega)$ ,  $b \in C((0, T))$ ,  $a, b > 0$ , then  $g$  is a coercive nonlinearity if  $h$  is super-additive and satisfies the Keller-Osserman condition. This is not sufficient for the existence of a barrier as it is shown in [9] with  $h(r) = r^q$  ( $q > 1$ )  $a \equiv 1$  and  $b(t) = e^{-\frac{1}{t}}$ .

**Proposition 2.2** *Let  $g \in \mathcal{H}_0$  such that at any  $z \in \Omega$  there exists a strong barrier. We assume also*

$$g(x, t, a) + g(x, t, b) \leq g(x, t, a + b) \quad \forall (x, t, a, b) \in Q_T^\Omega \times \mathbb{R}_+ \times \mathbb{R}_+. \quad (2.14)$$

*Let  $\{u_n\}$  be a sequence of positive solutions of (1.1) which converges to  $u$  locally uniformly in  $Q_T^\Omega$ . Denote by  $tr_\Omega(u_n) = (\mathcal{S}_n, \mu_n)$  and  $tr_\Omega(u) = (\mathcal{S}, \mu)$  their respective initial trace. If  $\mathcal{A} \subset \cap_n \mathcal{R}_n$  is open and if  $\mu_n(\mathcal{A}_n)$  remains bounded independently of  $n \in \mathbb{N}$ , then  $\mathcal{A} \subset \mathcal{R} := \Omega \setminus \mathcal{S}$ .*

*Proof.* Let  $z \in A$  and  $\tilde{r} \in (0, \rho(z))$  such that for any  $r \in (0, \tilde{r}]$  there exists a positive super solution  $w_{r,z}$  satisfying (2.5) and  $\overline{B_{\tilde{r}}(z)} \subset \mathcal{A}$ . For any  $n \in \mathbb{N}$  and  $\tau \in (0, T)$ , we denote by  $u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n}$  the solution of

$$\begin{aligned} \partial_t u - \Delta u + g \circ u &= 0 && \text{in } B_{\tilde{r}}(z) \times (\tau, T) \\ u(\cdot, \tau) &= \chi_{B_{\tilde{r}}(z)} u_n(\cdot, \tau) && \text{in } B_{\tilde{r}}(z) \\ u &= 0 && \text{in } \partial B_{\tilde{r}}(z) \times (\tau, T) \end{aligned} \quad (2.15)$$

By the maximum principle  $u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} \leq u_n$  in  $B_{\tilde{r}}(z) \times (\tau, T)$ , and  $g \circ u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} \leq g \circ u_n$ . Furthermore, if  $\zeta \in C^{1,1;1}(Q_T^{\overline{B_{\tilde{r}}(z)}})$  vanishes on  $\partial B_{\tilde{r}}(z) \times [0, T)$  and for  $t = T$ , there holds

$$\iint_{B_{\tilde{r}}(z) \times (\tau, T)} \left( -u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} (\partial_t \zeta + \Delta \zeta) + \zeta g \circ u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} \right) dx dt = \int_{B_{\tilde{r}}(z)} u_n(x, \tau) \zeta(x, \tau) dx \quad (2.16)$$

Since  $u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n}$  and  $g \circ u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n}$  are bounded independently of  $\tau$ , standard regularity theory for parabolic equations implies that they converge a.e. in  $B_{\tilde{r}}(z) \times (0, T)$  when  $\tau \rightarrow 0$  to  $u_{\chi_{B_{\tilde{r}}(z)} \mu_n}$  and  $g \circ u_{\chi_{B_{\tilde{r}}(z)} \mu_n}$ . Furthermore

$$\lim_{\tau \rightarrow 0} \int_{B_{\tilde{r}}(z)} u_n(x, \tau) \zeta(x, \tau) dx = \int_{B_{\tilde{r}}(z)} \zeta(x, 0) d\mu_n(x).$$

Using the dominated convergence theorem, it follows from (2.15) that

$$\iint_{Q_T^{B_{\tilde{r}}(z)}} \left( -u_{\chi_{B_{\tilde{r}}(z)} \mu_n} (\partial_t \zeta + \Delta \zeta) + \zeta g \circ u_{\chi_{B_{\tilde{r}}(z)} \mu_n} \right) dx dt = \int_{B_{\tilde{r}}(z)} \zeta(x, 0) d\mu_n(x), \quad (2.17)$$

and  $u_{\chi_{B_{\tilde{r}}(z)} \mu_n}$  is the (unique) solution of

$$\begin{aligned} \partial_t u - \Delta u + g \circ u &= 0 && \text{in } Q_T^{B_{\tilde{r}}(z)} \\ u(\cdot, 0) &= \chi_{B_{\tilde{r}}(z)} \mu_n && \text{in } B_{\tilde{r}}(z) \\ u &= 0 && \text{in } \partial B_{\tilde{r}}(z) \times (0, T). \end{aligned} \quad (2.18)$$

Furthermore, if  $\eta$  is the solution of the backward problem

$$\begin{aligned} \partial_t \eta + \Delta \eta &= -1 && \text{in } Q_T^{B_{\tilde{r}}(z)} \\ \eta(\cdot, T) &= 0 && \text{in } B_{\tilde{r}}(z) \\ \eta &= 0 && \text{in } \partial B_{\tilde{r}}(z) \times (0, T), \end{aligned} \quad (2.19)$$

there holds

$$\iint_{Q_T^{B_{\tilde{r}}(z)}} \left( u_{\chi_{B_{\tilde{r}}(z)} \mu_n} + \eta g \circ u_{\chi_{B_{\tilde{r}}(z)} \mu_n} \right) dx dt = \int_{B_{\tilde{r}}(z)} \eta(x, 0) d\mu_n(x) \leq M, \quad (2.20)$$

for some  $M > 0$  independent of  $n$ . Next we set  $Z_{\tau, n} := u_{\tau, \chi_{B_{\tilde{r}}(z)} \mu_n} + w_{\tilde{r}, z}$ . It is a super solution of (2.10) in  $(\tau, T) \times B_{\tilde{r}}(z)$  which is infinite on  $\partial B_{\tilde{r}}(z) \times [\tau, T)$  and dominates  $u_n$  in  $B_{\tilde{r}}(z)$  at  $t = \tau$ . Thus  $Z_{\tau, n} \geq u_n$  in  $(\tau, T) \times B_{\tilde{r}}(z)$ . Letting  $\tau \rightarrow 0$  we finally obtain

$$u_{\chi_{B_{\tilde{r}}(z)} \mu_n}(x, t) \leq u_n(x, t) \leq u_{\chi_{B_{\tilde{r}}(z)} \mu_n}(x, t) + w_{\tilde{r}, z}(x, t) \quad \forall (x, t) \in Q_T^{B_{\tilde{r}}(z)}. \quad (2.21)$$

For any  $r < \tilde{r}$  and  $T' < T$ , there exists  $\delta, \sigma > 0$  such that  $\eta(x, t) \geq \delta$  and  $w_{\tilde{r}, z}(x, t) \leq \sigma$  for all  $(x, t) \in Q_{T'}^{B_r(z)}$ . It follows from (2.20), (2.21) and Fatou's lemma that  $u$  and  $g \circ u$  are integrable in  $L^1(Q_{T'}^{B_r(z)})$ . By Lemma 2.1,  $B_{r'}(z) \subset \mathcal{R}$ . Since it holds for any  $z \in \mathcal{A}$ , the result is proved.  $\square$

### 3 Construction of a barrier

In the next results we construct the barrier function

**Lemma 3.1** *Assume  $\alpha > -1$  and  $q > 1$ , then there exists a unique positive function  $W_\alpha \in C^2([0, \infty))$  satisfying*

$$\begin{aligned} W'' + \frac{r}{2}W' + \frac{1+\alpha}{q-1}W - W^q &= 0 \quad \text{in } (0, \infty) \\ \lim_{r \rightarrow 0} W(r) &= \infty \\ \lim_{r \rightarrow \infty} r^{\frac{2}{q-1}}W(r) &= 0. \end{aligned} \tag{3.1}$$

Furthermore  $W_\alpha$  is decreasing and

$$W_\alpha(r) = Cr^{\frac{2(1+\alpha)}{q-1}-1}e^{-\frac{r^2}{4}}(1 + o(1)) \quad \text{as } r \rightarrow \infty. \tag{3.2}$$

*Proof.* Consider the functional

$$J(\phi) := \frac{1}{2} \int_0^\infty \left( \phi'^2 - \frac{1+\alpha}{q-1} \phi^2 + \frac{2}{q+1} |\phi|^{q+1} \right) e^{\frac{r^2}{4}} dr, \tag{3.3}$$

defined over the convex set

$$H_k := \{ \phi \in W_2^1(0, \infty; e^{\frac{r^2}{4}} dr) \cap L^{q+1}(0, \infty; e^{\frac{r^2}{4}} dr) : \phi(0) = k \}.$$

Note that if  $\phi \in H_k$ ,

$$\begin{aligned} e^{\frac{r^2}{4}} \phi^2(r) &= \int_r^\infty (e^{\frac{s^2}{4}} \phi^2(s))' ds \\ &= 2 \int_r^\infty e^{\frac{s^2}{4}} \phi \phi'(s) ds + \frac{1}{2} \int_r^\infty s e^{\frac{s^2}{4}} \phi^2(s) ds \end{aligned}$$

In this set  $J$  admits a positive minimizer  $w_k$  which is the unique solution of

$$\begin{aligned} w'' + \frac{r}{2}w + \frac{1+\alpha}{q-1}w' - w^q &= 0 \quad \text{in } (0, \infty) \\ w(0) &= k. \end{aligned} \tag{3.4}$$

Furthermore,  $w_k = \lim_{n \rightarrow \infty} w_{k,n}$  where  $w_{k,n}$  is the unique positive solution of

$$\begin{aligned} w'' + \frac{r}{2}w' + \frac{1+\alpha}{q-1}w - w^q &= 0 \quad \text{in } (0, n) \\ w(0) &= k \\ w(n) &= 0. \end{aligned} \tag{3.5}$$

and, by the maximum principle,  $(k, n) \mapsto w_{k,n}$  is increasing. If we consider the linear equation

$$z'' + \frac{r}{2}z' + \frac{1+\alpha}{q-1}z = 0 \quad \text{in } (0, \infty) \tag{3.6}$$

it admits two linearly independent positive solutions  $z_1$  and  $z_2$  with the following asymptotic behaviour as  $r \rightarrow \infty$

$$z_1(r) = r^{-\frac{2(1+\alpha)}{q-1}}(1 + o(1)) \quad (3.7)$$

and

$$z_2(r) = r^{\frac{2(1+\alpha)}{q-1}-1} e^{-\frac{r^2}{4}}(1 + o(1)) \quad (3.8)$$

(see [8, Appendix]. Since any solution of 3.4, and 3.5 as well, satisfies an a priori estimate of Keller-Osserman type (see [15])

$$w(r) \leq Cr^{-\frac{2}{q-1}} \quad \text{for } 0 < r < 1, \quad (3.9)$$

there holds

$$w_{k,n} \leq Cz_2(r) \quad \text{for } k \geq r \geq 1.$$

Letting  $n$  and  $k$  go to infinity successively, it follows that  $W_\alpha = \lim_{k,n \rightarrow \infty} w_{k,n}$  exists. It is a positive solution of problem (3.1) and it satisfies

$$W_\alpha(r) \leq C \left( r^{-\frac{2}{q-1}} + z_2(r) \right) \quad \text{for } r > 0.$$

The singular behaviour at  $r = 0$  is standard (see e.g. [15]) and yields to

$$W_\alpha(r) = \frac{2(q+1)}{(q-1)^2} r^{-\frac{2}{q-1}}(1 + o(1)) \quad \text{as } r \rightarrow 0. \quad (3.10)$$

Thus uniqueness follows by the maximum principle and estimates (3.3) and (3.12) is obtained via standard linearization, using the upper estimate at infinity.  $\square$

In the sequel we set

$$w_\alpha(s, t) = t^{-\frac{1+\alpha}{q-1}} W_\alpha \left( \frac{s}{\sqrt{t}} \right) \quad \forall s > 0, t > 0. \quad (3.11)$$

**Proposition 3.2** *Assume  $\alpha > -1$  and  $q > 1$ . Then for any  $R > 0$ , there exists  $C = C(q, \alpha, R) > 0$  such that any solution  $u$  of (1.6) in  $Q_\infty^{BR}$  which vanishes on  $B_R \times \{0\}$  satisfies*

$$u(x, t) \leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left( \frac{R-|x|}{\sqrt{t}} \right) \quad \forall (x, t) \in Q_\infty^{BR}. \quad (3.12)$$

*Proof.* For  $m > 0$ , set  $S_m = \{x = (x_1, \dots, x_N) : |x_j| < m\}$ . For  $R' < R$

$$\tilde{w}_{R'}(x, t) = t^{-\frac{1+\alpha}{q-1}} \sum_{j=1}^N \left( W_\alpha \left( \frac{R'-x_j}{\sqrt{t}} \right) + W_\alpha \left( \frac{R'+x_j}{\sqrt{t}} \right) \right) \quad \forall (x, t) \in Q_\infty^{S_{R'}} \quad (3.13)$$

Then  $\tilde{w}_{R'}$  is a super solution of (1.6) in  $Q_\infty^{S_{R'}}$  which is infinite on  $\partial S_{R'} \times (0, \infty)$ , thus  $u \leq \tilde{w}_{R'}$ . Letting  $R' \rightarrow R$  yields to  $u \leq \tilde{w}_R$  in  $Q_\infty^{S_R}$ . Since the equation is invariant by rotation, for any  $x \in B_R$ , there is a rotation  $\mathcal{R}$  such that  $\mathcal{R}(x)$  has only a positive  $x_1$ -coordinate. Thus

$$\begin{aligned} u(x, t) &\leq \tilde{w}_R(|x|, t) \\ &\leq t^{-\frac{1+\alpha}{q-1}} \left( W_\alpha \left( \frac{R-|x|}{\sqrt{t}} \right) + (2N-1)W_\alpha \left( \frac{R}{\sqrt{t}} \right) \right) \\ &\leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left( \frac{R-|x|}{\sqrt{t}} \right), \end{aligned} \quad (3.14)$$

which is (3.12).  $\square$

**Proposition 3.3** *Assume  $\alpha > -1$ ,  $q > 1$  and  $R > 0$ . Then there exists a unique positive solution  $w_{B_R}$  of (1.6) in  $Q_\infty^{B_R}$ , continuous in  $B_R \times [0, \infty)$ , which vanishes on  $B_R \times \{0\}$  and satisfies  $\lim_{|x| \rightarrow R} w_{B_R}(x, t) = \infty$ , locally uniformly in  $(0, \infty)$ . In particular*

$$t^{-\frac{1+\alpha}{q-1}} W_\alpha \left( \frac{R-|x|}{\sqrt{t}} \right) \leq w_{B_R}(x, t) \leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left( \frac{R-|x|}{\sqrt{t}} \right) \quad \forall (x, t) \in Q_\infty^{B_R}. \quad (3.15)$$

*Proof.* For  $k > 0$ , let  $w_{B_R}^k$  be the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{B_R} \\ u &= k && \text{in } \partial_\ell Q_\infty^{B_R} \\ u(\cdot, 0) &= 0 && \text{in } B_R. \end{aligned} \quad (3.16)$$

By (3.12),  $w_{B_R}^k(x, t) \leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left( \frac{R-|x|}{\sqrt{t}} \right)$ . There there exists  $w_{B_R} = \lim_{k \rightarrow \infty} w_{B_R}^k$  and  $w_{B_R}$  is a solution of (1.6) in  $Q_\infty^{B_R}$  which vanishes on  $B_R \times \{0\}$  and is infinite on  $\partial B_R \times (0, \infty)$ . Consider the similarity transformation  $T_m$  which leaves equation (1.6) invariant

$$T_m[u](x, t) = m^{\frac{1+\alpha}{q-1}} u(\sqrt{m}x, mt) \quad \forall m > 0,$$

then  $T_m[w_{B_R}^k] = w_{B_{\frac{R}{\sqrt{m}}}}^{m^{\frac{1+\alpha}{q-1}} k}$  which implies

$$T_m[w_{B_R}] = w_{B_{\frac{R}{\sqrt{m}}}} \quad \forall m > 0.$$

If  $u \in C(B_R \times [0, \infty))$  is any positive solution of problem

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{B_R} \\ \lim_{|x| \rightarrow R} u(x, t) &= \infty && \text{locally uniformly on } (0, \infty) \\ u(\cdot, 0) &= 0 && \text{in } B_R, \end{aligned} \quad (3.17)$$

then for any  $m > 1$  and  $\epsilon > 0$ , there exists  $\tau_\epsilon > 0$  such that  $u(x, t) \leq \epsilon$  in  $B_{\frac{R}{\sqrt{m}}}$  for  $0 \leq t \leq \tau_\epsilon$ . Therefore, for any  $T > 0$ ,

$$u \leq \epsilon + w_{B_{\frac{R}{\sqrt{m}}}} \quad \text{on } \partial_\ell Q_T^{B_{\frac{R}{\sqrt{m}}}} \cup \overline{B_{\frac{R}{\sqrt{m}}}} \times \{0\}$$

Since  $\epsilon + w_{B_{\frac{R}{\sqrt{m}}}}$  is a super solution  $u \leq \epsilon + w_{B_{\frac{R}{\sqrt{m}}}}$  in  $Q_T^{B_{\frac{R}{\sqrt{m}}}}$ . Letting  $\epsilon \rightarrow 0$ ,  $m \rightarrow 1$  and  $T \rightarrow \infty$  yields to  $u \leq w_{B_R}$ . In the same way, with  $0 < m < 1$ , we obtain  $u \geq w_{B_R}$ .  $\square$

The next result is an immediate consequence of Proposition 3.2.

**Proposition 3.4** *Assume  $\alpha > -1$ ,  $q > 1$  and  $K \subset \Omega$  is compact. Let  $u$  be any solution  $u$  of (1.6) in  $Q_\infty^\Omega$  which vanishes on  $\Omega \setminus \{K\} \times \{0\}$  and on  $\partial\Omega \times [0, \infty)$ , then*

$$u(x, t) \leq 2Nt^{-\frac{1+\alpha}{q-1}} W_\alpha \left( \frac{\text{dist}(x, K)}{\sqrt{t}} \right). \quad (3.18)$$

## 4 Upper estimates

We start with the following upper estimate already obtained by Shishkov and Véron [12] in the case  $\alpha \geq 0$ .

**Proposition 4.1** *Let  $q > 1$  and  $\alpha > -1$ . If  $u$  is a solution of (1.6) vanishing on  $\partial\Omega \times [0, T)$ , there holds*

$$u(x, t) \leq c_\alpha t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x, t) \in Q_T^\Omega, \quad (4.1)$$

with  $c_\alpha = \left(\frac{\alpha+1}{q-1}\right)^{\frac{1}{q-1}}$ .

*Proof.* Let  $\phi(t) = c_\alpha t^{-\frac{\alpha+1}{q-1}}$  be the maximal solution of

$$\begin{aligned} \phi' + t^\alpha \phi^q &= 0 \\ \phi(0) &= \infty. \end{aligned}$$

with  $c_\alpha = \left(\frac{\alpha+1}{q-1}\right)^{\frac{1}{q-1}}$ .

*Case  $\alpha \geq 0$ .* For  $\tau > 0$ , we denote by  $\Phi_{1,\tau}$  the solution of

$$\begin{aligned} -\Delta \Phi_{1,\tau} + \tau^\alpha \Phi_{1,\tau}^q &= 0 \quad \text{in } B_1 \\ \lim_{|x| \rightarrow 1} \Phi_{1,\tau}(x) &= \infty, \end{aligned} \quad (4.2)$$

and for  $R > 0$

$$\Phi_{R,\tau}(x) = R^{\frac{-2}{q-1}} \Phi_{1,\tau}\left(\frac{x}{R}\right).$$

Note that  $\Phi_{R,\tau}(x)$  is the solution of the problem (4.2) in the ball  $B_R$ . The function  $\Phi_{R,\tau}$  tends to 0 uniformly on every compact set of  $\mathbb{R}^N$  when  $R \rightarrow \infty$ . Set

$$\tilde{v}(x, t) = \phi(t - \tau) + \Phi_{R,\tau}(x),$$

then  $\tilde{v}$  is a supersolution of (1.6) in  $B_R \times [\tau, T)$  which is infinite on  $\partial B_R \times [\tau, T) \cup B_R \times \{0\}$ . Then  $u(x, t) \leq \tilde{v}(x, t)$ . Letting  $R \rightarrow \infty$  and  $\tau \rightarrow 0$ , we obtain

$$u(x, t) \leq c_\alpha t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x, t) \in Q_T.$$

*Case  $-1 < \alpha < 0$ .* Let  $\tau > 0$  and  $\phi_\tau(t) = c_\alpha (t^{\alpha+1} - \tau^{\alpha+1})^{-\frac{1}{q-1}}$  be the solution of

$$\begin{aligned} \phi_\tau' + t^\alpha \phi_\tau^q &= 0 \quad \text{on } (\tau, \infty) \\ \phi_\tau(\tau) &= \infty \end{aligned}$$

If  $\Phi_{1,T}$  is the solution of (4.2) with  $\tau = T$ , we set

$$\Phi_{R,T}(x) = R^{\frac{-2}{q-1}} \Phi_{1,T}\left(\frac{x}{R}\right).$$

Clearly  $\Phi_{R,T}$  tends to 0 uniformly on every compact of  $\mathbb{R}^N$  when  $R \rightarrow \infty$ . Set

$$\hat{v}(x, t) = \phi_\tau(t) + \Phi_{R,T}(x),$$

$\hat{v}$  is a supersolution of (1.6) in  $B_R \times (\tau, T)$ , thus  $u(x, t) \leq \hat{v}(x, t)$ , as in the first case. Letting  $R \rightarrow \infty$  and  $\tau \rightarrow 0$ , we obtain the desired estimate.  $\square$

Combining Proposition 3.3 and Proposition 4.1 we obtain,

**Corollary 4.2** *Assume  $q > 1$ ,  $\alpha > -1$  and  $K \subset \Omega$  is compact. If  $u \in C^2(Q_T^\Omega) \cap C(\overline{Q_T^\Omega} \setminus K \times \{0\})$  is a solution of (1.6) which vanishes on  $\partial\Omega \times [0, T) \cup \{(\Omega \setminus K) \times \{0\}\}$ , there holds*

$$u(x, t) \leq \min \left\{ 2NW_\alpha \left( \frac{\text{dist}(x, K)}{\sqrt{t}} \right), c_\alpha \right\} t^{-\frac{\alpha+1}{q-1}} \quad \text{for all } (x, t) \in Q_T^\Omega. \quad (4.3)$$

In the particular case where  $K = \{O\}$ , (4.3) yields to

$$u(x, t) \leq \min \left\{ 2NW_\alpha \left( \frac{|x|}{\sqrt{t}} \right), c_\alpha \right\} t^{-\frac{\alpha+1}{q-1}} \leq \frac{c_1}{(|x|^2 + t)^{\frac{1+\alpha}{q-1}}} \quad \text{for all } (x, t) \in Q_T^\Omega, \quad (4.4)$$

for some  $c_1 = c_1(\alpha, q) > 0$ .

*Remark.* If  $\Omega$  is replaced by  $\mathbb{R}^N$ , the previous estimates (4.1), (4.3) and (4.4) remains valid. Furthermore,  $K$  needs only to be closed.

## 5 Isolated singularities

In this section we present the results of classifications of isolated singularities of positive solutions of (1.6), always in the range  $q > 1$  and  $\alpha > -1$ . Since the proofs are somewhat similar to the ones of classification of boundary singularities of positive solutions of

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T^\Omega \quad (5.1)$$

obtained by Marcus and Véron in [8], we only indicate their main ideas.

If we look for solution of (1.6) in  $Q_T := Q_T^{\mathbb{R}^N}$  under the form

$$u(x, t) = t^\gamma V\left(\frac{x}{\sqrt{t}}\right)$$

it is immediate that  $\gamma = -\frac{1+\alpha}{q-1}$  and  $V$  is a solution of

$$-\Delta V - \frac{\eta}{2} \cdot \nabla V - \frac{1+\alpha}{q-1} V + V^q = 0 \quad \text{in } \mathbb{R}^N. \quad (5.2)$$

It is proved by Escobedo and Kavian [4] that if  $\frac{1+\alpha}{q-1} > \frac{N}{2}$ , or equivalently if

$$1 < q < q_{c,\alpha} := 1 + \frac{2(1+\alpha)}{q-1}, \quad (5.3)$$

there exists a positive solution of (5.2) which is the minimizer of the functional

$$\omega \mapsto J(\omega) := \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla \omega|^2 - \frac{1+\alpha}{q-1} \omega^2 + \frac{2}{q+1} |\omega|^{q+1} \right) e^{\frac{|\eta|^2}{4}} d\eta. \quad (5.4)$$

This minimizing solution  $V_\alpha$  is unique and radial. Furthermore, by adapting the results of [8, Th 2.1], it satisfies

$$V_\alpha(\eta) = C |\eta|^{2\frac{1+\alpha}{q-1}-N} e^{-\frac{|\eta|^2}{4}} (1 + o(1)) \quad \text{as } |\eta| \rightarrow \infty. \quad (5.5)$$

The function

$$v_\alpha(x, t) = t^{-\frac{1+\alpha}{q-1}} V_\alpha \left( \frac{x}{\sqrt{t}} \right) \quad (5.6)$$

is a positive solution of (5.2) in  $Q_\infty$ , continuous in  $\overline{Q}_T \setminus \{(O, 0)\}$ , vanishes on  $\mathbb{R}^N \times \{0\} \setminus \{(O, 0)\}$  and satisfies

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} v_\alpha(x, t) dx = \infty \quad \forall \epsilon > 0. \quad (5.7)$$

It is called the *very singular solution* of (1.6).

When  $\mathbb{R}^N$  is replaced by a proper open domain  $\Omega$  with a compact  $C^2$  boundary there exists no self-similar solution to (1.6). For any  $k > 0$  and  $a \in \Omega$  there exists a unique solution  $u := u_{k\delta_a}$  to the initial value problem

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^\Omega \\ u &= 0 && \text{on } \partial\Omega \times [0, \infty) \\ u(\cdot, 0) &= k\delta_a && \text{in } \Omega. \end{aligned} \quad (5.8)$$

(see e.g. [9]). The function  $u$  belongs to  $L^q(Q_T^\Omega; t^\alpha dx dt) \cap L^1(Q_T^\Omega)$ ,  $T > 0$  arbitrary, and satisfies

$$\iint_{Q_T^\Omega} (-u(\partial_t \zeta + \Delta \zeta) + t^\alpha u^q \zeta) dx dt = k\zeta(a) \quad (5.9)$$

for all  $\zeta \in C^{1,1;1}(\overline{Q}_T^\Omega)$  which vanishes on  $\partial\Omega \times [0, T]$  and on  $\Omega \times \{T\}$ . It is unique in the class of weak solutions, i.e. the functions belonging to  $L^q(Q_T^\Omega; t^\alpha dx dt) \cap L^1(Q_T^\Omega)$  and satisfying the above identity. When  $k \rightarrow \infty$ ,  $u_{k\delta_a} \uparrow u_{\infty\delta_a}$ , where  $u_{\infty\delta_a} := u_{\infty,a}$  is a solution of (1.6) in  $Q_\infty^\Omega$  which vanishes on  $\partial\Omega \times [0, \infty)$  and on  $\overline{\Omega} \times \{0\} \setminus \{(a, 0)\}$  and satisfies (5.7). Finally

$$u_{k\delta_a} \sim kE(x - a, t) \quad \text{when } (x, t) \rightarrow (a, 0) \quad (5.10)$$

and

$$u_{\infty,a} \sim v_\alpha(x - a, t) \quad \text{when } (x, t) \rightarrow (a, 0). \quad (5.11)$$

The following classification of isolated singularities holds

**Theorem 5.1** *Assume  $\alpha > -1$ ,  $1 < q < q_{c,\alpha}$  and  $a \in \Omega$ . If  $u \in \overline{Q}_T^\Omega \setminus \{(a, 0)\}$  is a positive solution of (1.6) which vanishes on  $\Omega \times \{0\} \setminus \{(a, 0)\}$ , then*

(i) *either there exists  $k \geq 0$  such that*

$$u(x, t) \sim kE(x - a, t) \quad \text{when } (x, t) \rightarrow (a, 0), \quad (5.12)$$

*and  $u$  is a solution of*

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_T^\Omega \\ u(\cdot, 0) &= k\delta_a && \text{in } \Omega, \end{aligned} \quad (5.13)$$

(ii) *or*

$$u(x, t) \sim v_\alpha(x - a, t) \quad \text{when } (x, t) \rightarrow (a, 0), \quad (5.14)$$

*and  $u$  satisfies (5.7).*



If  $q \geq q_{c,\alpha}$  there holds the following result which extends Brezis and Freidman's classical one.

**Theorem 5.2** *Assume  $\alpha > -1$ ,  $q \geq q_{c,\alpha}$  and  $a \in \Omega$ . If  $u \in \overline{Q_T^\Omega} \setminus \{(a,0)\}$  is a positive solution of (1.6) which vanishes on  $\Omega \times \{0\} \setminus \{(a,0)\}$ , then it can be extended as a continuous function  $\tilde{u}$  which vanishes on  $\Omega \times \{0\}$ .*

*Proof.* Up to modifying a few parameters the proof is similar to Brezis-Friedman's construction. The first step is to prove that  $u \in L^q(Q_T^{BR})$  for some  $R > 0$ . This is done by using (4.4) and appropriate test functions. Then the same choice of test functions yields to the fact that  $u$  satisfies

$$\lim_{t \rightarrow 0} \int_{\Omega} u(x,t) \zeta(x) dx = 0 \quad \forall \zeta \in C_0^\infty(\Omega).$$

Finally the extension of  $u$  by zero at  $t = 0$  satisfies the equation in  $\Omega \times [0, T)$ .  $\square$

*Remark.* We recall that  $E(x,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$ . Then if  $1 \leq r < q_{c,\alpha}$ , there holds

$$\iint_{Q_T} E^r(x,t) t^\alpha dx dt < \infty, \quad (5.15)$$

while if  $r \geq q_{c,\alpha}$

$$\iint_{Q_T} E^r(x,t) t^\alpha dx dt = \infty. \quad (5.16)$$

## 6 The trace theorem

In all this section we assume that  $\Omega \subset \mathbb{R}^N$  is an open domain with a compact  $C^2$  boundary,  $\alpha > -1$  and  $1 < q < q_{c,\alpha}$ , and let  $u \in C(\overline{\Omega} \times (0, T])$  be a positive solution of (1.6) in  $Q_T^\Omega$  which vanishes on  $\partial\Omega \times (0, T]$ . By Section 2  $u$  possesses an initial trace  $tr_\Omega(u) = (\mathcal{S}, \mu)$  where  $\mathcal{S}$  is a relatively closed subset of  $\Omega$  and  $\mu$  is a Radon measure on  $\mathcal{R} := \Omega \setminus \mathcal{S}$ . To this couple we can associate a unique outer regular Borel measure  $\nu$  defined by

$$\nu(E) = \begin{cases} \mu(E) & \text{if } E \subset \mathcal{R} \\ \infty & \text{if } E \cap \mathcal{S} \neq \emptyset \end{cases} \quad (6.1)$$

for any Borel subset  $E$  of  $\Omega$ . Conversely, to any outer Borel measure  $\nu$  on  $\Omega$  we can associate the regular set  $\mathcal{R} \subset \Omega$  which is the set of points  $y \in \Omega$  which possess an open neighborhood  $\mathcal{O}_y$  such that  $\nu(\mathcal{O}_y) < \infty$ . Clearly  $\mathcal{R}$  is open and the restriction of  $\nu$  to  $\mathcal{R}$  is a positive Radon measure. The set  $\mathcal{S} = \Omega \setminus \mathcal{R}$  is relatively closed and it is the singular part of  $\nu$ . It has the property that  $\nu(E) = \infty$  for any Borel set  $E$  such that  $E \cap \mathcal{S} \neq \emptyset$ . We shall denote by  $\mathfrak{B}^{reg}(\Omega)$  the set of outer regular Borel measures in  $\Omega$  and by  $\mathfrak{B}_c^{reg}(\Omega)$  the subset of  $\mathfrak{B}^{reg}(\Omega)$  for which  $\mathcal{S}$  is a compact subset of  $\Omega$ . Thus  $u$  is a solution of the following problem,

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_T^\Omega \\ u \geq 0, u &= 0 && \text{on } \partial\Omega \times [0, T) \\ tr_\Omega(u) &= \nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega). \end{aligned} \quad (6.2)$$

**Definition 6.1** *We denote by  $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$  the set of solutions of problem (6.2).*

The first step in the characterization of the singular set is the following lower estimate.

**Theorem 6.2** *Let  $u \in \mathcal{U}_{\mathcal{S}, \mu}(\Omega)$  and  $a \in \mathcal{S}$ , then*

$$u(x, t) \geq u_{\infty, a}(x, t). \quad (6.3)$$

Furthermore, if  $\mathcal{S}$  has a nonempty interior  $\mathcal{A}$ , there holds

$$\lim_{t \rightarrow 0} t^{\frac{1+\alpha}{q-1}} u(x, t) = c_\alpha, \quad (6.4)$$

uniformly on compact subsets of  $\mathcal{A}$ .

We first give a proof of (6.53) in the case where either  $-1 < \alpha \leq 0$ , or  $\alpha > 0$  and  $1 < q < q_{c,0}$ .

**Proposition 6.3** *Assume either  $-1 < \alpha \leq 0$  and  $1 < q < q_{c,\alpha}$ , or  $\alpha > 0$  and  $1 < q < q_{c,0}$ , then inequality (6.53) holds.*

*Proof.* For any  $\epsilon > 0$ , there holds

$$\lim_{t \rightarrow 0} \int_{B_\epsilon(a)} u(x, t) dx = \infty. \quad (6.5)$$

If  $k > 0$  is fixed, and  $\{t_n\}$  is a sequence decreasing to 0. There exists  $t_{n_1}$  such that

$$\int_{B_{2^{-1}}(a)} u(x, t_{n_1}) dx > k,$$

and there exists  $m = m_1(k)$  such that

$$\int_{B_{2^{-1}}(a)} m_1(k) \wedge u(x, t_{n_1}) dx = k,$$

where  $A \wedge B = \inf\{A, B\}$ . Assume we have constructed  $t_{n_j} < t_{n_{j-1}}$  and  $m_j(k) > 0$  such that

$$\int_{B_{2^{-j}}(a)} m_j(k) \wedge u(x, t_{n_j}) dx = k.$$

Since (6.5) holds with  $\epsilon = 2^{-j-1}$ , there exists  $t_{n_{j+1}} < t_{n_j}$  such that

$$\int_{B_{2^{-j-1}}(a)} u(x, t_{n_{j+1}}) dx > k.$$

and thus  $m_{j+1}(k) > 0$  such that

$$\int_{B_{2^{-j-1}}(a)} m_{j+1}(k) \wedge u(x, t_{n_{j+1}}) dx = k.$$

Next we denote by  $u_j$  the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (t_{n_j}, T) \\ u &= 0 && \text{on } \partial\Omega \times [t_{n_j}, T) \\ u(\cdot, t_{n_j}) &= m_{j+1}(k) \wedge u(\cdot, t_{n_{j+1}}) && \text{in } \Omega. \end{aligned} \quad (6.6)$$

By the maximum principle  $u_j \leq u$  in  $\Omega \times [t_{n_j}, T)$ , or equivalently  $u_j(x, t + t_{n_j}) \leq u(x, t + t_{n_j})$  in  $Q_{T-t_{n_j}}^\Omega$ . Clearly

$$u_{j,0} := u(\cdot, t_{n_j}) \rightharpoonup k\delta_a \quad \text{as } j \rightarrow \infty$$

in the weak sense of measures. In order to prove that  $u_j$  converges to  $u_{k\delta_a}$  we notice that  $u_j(x, t + t_{n_j}) \leq E * u_{j,0}(x, t)$  in  $Q_\infty^\Omega$ . If  $-1 < \alpha \leq 0$  and  $r \in (1, q_{c,\alpha})$

$$\begin{aligned} \iint_{Q_T^\Omega} u_j^r(x, t + t_{n_j})(t + t_{n_j})^\alpha dx dt &\leq \iint_{Q_T} (E * u_{j,0})^r(x, t)(t + t_{n_j})^\alpha dx dt \\ &\leq \|u_{j,0}\|_{L^1}^r \iint_{Q_T} E^r(x, t)t^\alpha dx dt, \end{aligned}$$

using Young's inequality and (5.15). If  $\alpha \geq 0$  and  $r \in (1, q_{c,0})$

$$\begin{aligned} \iint_{Q_T^\Omega} u_j^r(x, t + t_{n_j})(t + t_{n_j})^\alpha dx dt &\leq 2^\alpha T^\alpha \iint_{Q_T} (E * u_{j,0})^r(x, t) dx dt \\ &\leq 2^\alpha T^\alpha \|u_{j,0}\|_{L^1}^r \iint_{Q_T} E^r(x, t) dx dt. \end{aligned}$$

Furthermore, for  $s \in (1, q_{c,0})$

$$\begin{aligned} \iint_{Q_T^\Omega} u_j^s(x, t + t_{n_j}) dx dt &\leq \iint_{Q_T} (E * u_{j,0})^s(x, t) dx dt \\ &\leq \|u_{j,0}\|_{L^1}^s \iint_{Q_T} E^s(x, t) dx dt. \end{aligned}$$

Thus the set of functions  $\{u_j^q(\cdot, \cdot + t_{n_j})(\cdot + t_{n_j})^\alpha\}$  and  $\{u_j(\cdot, \cdot + t_{n_j})\}$  are uniformly integrable in  $L^1(Q_T^\Omega)$ . Since  $u_j$  satisfies

$$\iint_{Q_T^\Omega} (-u_j(x, t + t_{n_j}) (\partial_t \zeta + \Delta \zeta) + (t + t_{n_j})^\alpha u_j^q(x, t + t_{n_j}) \zeta) dx dt = \int_\Omega u_{j,0} \zeta dx \quad (6.7)$$

for any test function  $\zeta \in C^{1,1;1}(\overline{Q_T^\Omega})$  vanishing on  $\partial\Omega \times [0, T] \cup \Omega \times \{T\}$  and converges a.e. in  $Q_T^\Omega$  to some  $u^*$ , it follows by Vitali's theorem that

$$\iint_{Q_T^\Omega} (-v(\partial_t \zeta + \Delta \zeta) + t^\alpha v^q \zeta) dx dt = k\zeta(a) \quad (6.8)$$

thus  $u^* = u_{k\delta_a}$  by uniqueness, which implies the claim since  $u \geq u^*$ .  $\square$

When  $\alpha > 0$  and  $q_{c,0} \leq q < q_{c,\alpha}$ , this argument cannot work since the sequence  $u_{j,0}$  could concentrate too quickly with respect to  $t$  to a Dirac mass and isolated singularities are removable for solutions of

$$\partial_t u - \Delta u + cu^q = 0.$$

We develop below a proof valid for any  $\alpha > -1$  and  $1 < q < q_{c,\alpha}$ , which is based upon the parabolic Harnack inequality and shows that such a fast concentration never occurs.

**Lemma 6.4** Assume  $\alpha > -1$  and  $1 < q < q_{c,\alpha}$ . Let  $\{(x_n, t_n)\} \subset Q_T^\Omega$  be a sequence converging to  $(a, 0)$  and  $\ell > 0$ . Put  $V_n = B_{\ell\sqrt{t_n}}(x_n)$  and suppose that there exist nonnegative functions  $h_n \in L^\infty(\mathbb{R}^N)$  with support in  $V_n$  such that  $0 \leq h_n \leq c_1 t_n^{-\frac{N}{2}}$  and

$$h_n \rightharpoonup k\delta_0. \quad (6.9)$$

Then the solutions  $u_n$  of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (t_n, T) \\ u &= 0 && \text{on } \partial\Omega \times [t_n, T) \\ u(\cdot, t_n) &= h_n && \text{in } \Omega, \end{aligned} \quad (6.10)$$

satisfy  $u_n \rightarrow u_{k\delta_0}$  when  $n \rightarrow \infty$ .

*Proof.* The estimate  $h_n \leq c t_n^{-\frac{N}{2}} \chi_{V_n}$  can be written under the form

$$h_n(x) \leq c_2 E(x - x_n, t_n) \chi_{V_n}$$

where  $c_2 = (4\pi)^{\frac{N}{2}} e^{\frac{\ell}{4}} c_1$ . By the maximum principle

$$u_n(x, t) \leq c_2 E(x - x_n, t) \quad \text{in } \Omega \times (t_n, \infty).$$

By (5.15), (5.16), the sequences  $\{E^q(\cdot - x_n, \cdot) t^\alpha\}$  and  $\{E(\cdot - x_n, \cdot)\}$  are uniformly integrable in  $Q_T^\Omega$ , therefore, if we extend  $u_n$  by zero in  $Q_{t_n}^\Omega$  and denote by  $\tilde{u}_n$  the extended function defined in  $Q_T^\Omega$ , we infer that the sequences  $\{t^\alpha \tilde{u}_n^q\}$  and  $\{\tilde{u}_n\}$  are uniformly integrable in  $Q_T^\Omega$ . Using standard regularity estimates there exists a function  $u^*$  defined in  $Q_T^\Omega$  and a subsequence  $u_{n_j}$  such that  $\tilde{u}_{n_j} \rightarrow u^*$  locally uniformly in  $Q_T^\Omega$ . It follows by uniform integrability and Vitali's convergence theorem that

$$\tilde{u}_{n_j} \rightarrow u^* \quad \text{in } L^q(Q_T^\Omega; t^\alpha dx dt) \cap L^1(Q_T^\Omega).$$

Let  $\zeta \in C^{1,1;1}(\overline{Q_T^\Omega})$  vanishing on  $\partial\Omega \times [0, T] \cup \Omega \times \{T\}$ , then

$$\iint_{Q_T^\Omega} \left( -\tilde{u}_{n_j} (\partial_t \zeta + \Delta \zeta) + t^\alpha \tilde{u}_{n_j}^q \zeta \right) dx dt = \int_\Omega \zeta(\cdot, t_{n_j}) h_n dx.$$

Using the previous convergence results and the assumption (6.9), we derive

$$\iint_{Q_T^\Omega} \left( -u^* (\partial_t \zeta + \Delta \zeta) + t^\alpha u^{*q} \zeta \right) dx dt = k\zeta(O).$$

Thus  $u^* = u_{k\delta_0}$  and  $\tilde{u}_n \rightarrow u_{k\delta_0}$  locally uniformly in  $Q_T^\Omega$ .  $\square$

**Lemma 6.5** Let  $u$  be a positive solution of (1.6) in  $Q_T^\Omega$  vanishing on  $\partial\Omega \times [0, T)$ . Then for any  $\Omega' \subset \overline{\Omega} \subset \Omega$  there exists a constant  $C = C(N, q, \alpha, \Omega') > 0$  such that

$$u(y, s) \leq u(x, t) e^{C \left( \frac{|x-y|^2}{t-s} + \frac{t}{s} + 1 \right)} \quad \forall (x, t), (y, s) \in Q_T^{\Omega'}, s < t. \quad (6.11)$$

*Proof.* By (4.1),  $V(x, t) := t^\alpha u^{q-1} \leq c_\alpha^{q-1} t^{-1}$ . If we write (1.6) under the form

$$\partial_t u - \Delta u + V(x, t)u = 0 \quad \text{in } Q_T^\Omega \quad (6.12)$$

it follows (6.11) from parabolic Harnack inequality (see e.g. [5, Lemma 3.16] although the result is much older).  $\square$

If  $G \subset \mathbb{R}^N$  is a bounded open subset, we denote by  $\phi_G$  is the first eigenfunction of  $-\Delta$  in  $W_0^{1,2}(G)$  normalized by  $\sup \phi_G = 1$  with corresponding eigenvalue  $\lambda_G$ .

**Lemma 6.6** *Let  $G \subset \mathbb{R}^N$  be a bounded open subset with a smooth boundary  $\omega \in C^1(Q_T^G)$ ,  $\omega \geq 0$ , such that*

$$\int_0^T \|\omega(\cdot, t)\|_{L^\infty(G)} dt < \infty. \quad (6.13)$$

*If  $v \in C^{2;1}(\overline{G} \times (0, T])$  is a positive solution of*

$$\partial_t v - \Delta v + \omega v = 0 \quad \text{in } Q_T^G, \quad (6.14)$$

*then  $v\phi_G \in L^1(Q_T^G)$ ,  $\omega v\phi_G^3 \in L^1(Q_T^G)$  and there exists  $\mu \in \mathfrak{M}_+(G)$  such that*

$$\lim_{t \rightarrow 0} \int_G v(x, t)\zeta(x)dx = \int_G \zeta d\mu \quad \forall \zeta \in C_c(\overline{G}). \quad (6.15)$$

*Proof.* Set  $\gamma(t) = \|\omega(\cdot, t)\|_{L^\infty(G)}$ . Denote  $X(t) = \int_G v(x, t)\phi_G(x)dx$ . Then, from (6.14),

$$X' + \lambda_G X + \gamma(t)X \geq 0$$

This implies that the function

$$t \mapsto e^{t\lambda_G + \int_0^t \gamma(s)ds} X(t)$$

(which exists thanks to (6.13)) is nondecreasing. Therefore there exists  $X(0) = \lim_{t \rightarrow 0} X(t)$  and  $v\phi_G \in L^1(Q_T^G)$ . Furthermore, if we set  $Y(t) = \int_G v(x, t)\phi_G^3(x)dx$

$$\frac{d}{dt}(e^{3t\lambda_G} X(t)) + e^{3t\lambda_1} \int_\Omega (\omega\phi_G^3 - 6\phi_G |\nabla\phi_G|^2)v dx = 0.$$

Since  $\nabla\phi_G$  is bounded and  $v\phi_G \in L^1(Q_T^G)$ , it implies

$$e^{3t\lambda_G} Y(T) + \int_0^T e^{3t\lambda_G} \int_\Omega \omega(x, t)v(x, t)\phi_G^2 dx dt = Y(0) + 6 \int_0^T e^{3t\lambda_G} \int_\Omega \phi_G |\nabla\phi_G|^2 v dx dt,$$

which implies that  $\omega v\phi_G^3 \in L^1(Q_T^G)$ . The argument given in the proof of Lemma 2.1 shows that  $v$  admits an initial trace which belongs to  $\mathfrak{M}_+(G)$ .  $\square$

*Proof of Theorem 6.2.* We define the parabolic distance in  $\mathbb{R}^N \times \mathbb{R}$  by

$$d_P((x, t), (y, s)) = \sqrt{|x - y|^2 + |t - s|}$$

*Step 1.* We first prove that if  $u$  satisfies

$$\limsup_{d_P((x,t),(a,0)) \rightarrow 0} t^{\frac{N}{2}} u(x,t) < \infty, \quad (6.16)$$

then  $a \in \mathcal{R}(u)$ . If (6.16) holds there exists  $\epsilon, c > 0$  such that

$$u(x,t) \leq ct^{-\frac{N}{2}} \quad \forall (x,t) \text{ s.t. } \sqrt{|x-a|^2 + t} \leq \epsilon.$$

If we set  $\omega(x,t) = t^\alpha u^{q-1}(x,t)$ , then

$$\omega(x,t) \leq c^{q-1} t^{-(\frac{N(q-1)}{2} - \alpha)} \quad \forall (x,t) \in B_{\frac{\epsilon}{\sqrt{2}}}(a) \times (0, \frac{\epsilon^2}{2}].$$

Since  $q < q_{c,\alpha}$ ,  $\frac{N(q-1)}{2} - \alpha < 1$ , thus the assumptions of Lemma 6.6 are fulfilled and there exists a positive Radon measure  $\mu$  in  $B_{\frac{\epsilon}{\sqrt{2}}}(a)$  such that

$$\int_{\Omega} u(x,t)\zeta(x)dx \rightarrow \int_{\Omega} \zeta d\mu \quad \text{when } t \rightarrow 0, \quad \forall \zeta \in C_c^\infty(B_{\frac{\epsilon}{\sqrt{2}}}(a)).$$

Furthermore  $t^\alpha u^q \in L^1(B_{\frac{\epsilon}{\sqrt{3}}}(a) \times (0, T))$ , which is the claim.

*Step 2.* Since  $a \in \mathcal{S}(u)$ , there holds

$$\limsup_{d_P((x,t),(a,0)) \rightarrow 0} t^{\frac{N}{2}} u(x,t) = \infty. \quad (6.17)$$

Then there exists a sequence  $\{(x_n, s_n)\}$  converging to  $(a, 0)$  such that

$$u(x_n, t_n) \geq nt_n^{-\frac{N}{2}} \quad (6.18)$$

We apply Lemma 6.5 with  $s = s_n$ ,  $t = 2s_n := t_n$ ,  $y = x_n$   $|x - x_n| \leq \sqrt{s_n}$ . Then

$$u(x, t_n) \geq Cnt_n^{-\frac{N}{2}} \quad \forall x \in V_n := B_{\frac{\sqrt{t_n}}{2}}(x_n)$$

This implies

$$\int_{V_n} u(x, t_n) dx \geq C_N n.$$

For  $k < n$  fixed, we denote by  $v := v_{n,k}$  the solution of

$$\begin{aligned} \partial_t v - \Delta v + t^\alpha v^q &= 0 && \text{in } \Omega \times (t_n, \infty) \\ v &= 0 && \text{in } \partial\Omega \times (t_n, \infty) \\ v(\cdot, t_n) &= Ckt_n^{-\frac{N}{2}} \chi_{V_n} && \text{in } \Omega. \end{aligned} \quad (6.19)$$

By the maximum principle  $u \geq v_{n,k}$  in  $\Omega \times (t_n, \infty)$ . Furthermore

$$\int_{V_n} v(\cdot, t_n) dx = C_N k.$$

Thus  $v(\cdot, t_n) \rightharpoonup C_N k \delta_a$  in the weak sense of measures. It follows from Lemma 6.4 that  $v_{n,k} \rightarrow u_{C_N k \delta_a}$  locally uniformly in  $Q_T^\Omega$ . Therefore  $u \geq u_{C_N k \delta_a}$  in  $Q_T^\Omega$ . Since  $k$  is arbitrary, we obtain (6.53).

*Step 3.* Formula (6.4) holds. Denote by  $S_m(a) = \{x \in \mathbb{R}^N : |x_j| < m\}$ . If  $\bar{S}_R(a) \subset \mathcal{S}$ , the function

$$(x, t) \mapsto c_\alpha (t - \tau)^{-\frac{1+\alpha}{q-1}} + w_{B_R}(x - a, t - \tau)$$

is a super solution of (1.6) in  $S_R(a) \times (\tau, \infty)$  which is infinite on  $S_R(a) \times \{\tau\} \cup \partial S_R(a) \times [\tau, \infty)$  by Proposition 3.3, while  $u$  is finite, thus it dominates  $u$  in  $S_R(a) \times (\tau, \infty)$ . Letting  $\tau \rightarrow 0$  yields to

$$u(x, t) \leq c_\alpha t^{-\frac{1+\alpha}{q-1}} + w_{B_R}(x - a, t) \quad \forall (x, t) \in Q_T^{S_R(a)}. \quad (6.20)$$

Conversely, the function

$$(x, t) \mapsto u(x, t - \tau) + w_{B_R}(x - a, t - \tau)$$

is a super solution in  $S_R(a) \times (\tau, \infty)$  which dominates  $c_\alpha t^{-\frac{1+\alpha}{q-1}}$  on  $S_R(a) \times \{\tau\} \cup \partial S_R(a) \times [\tau, \infty)$ , thus as above, we obtain (6.20). Since

$$\lim_{t \rightarrow 0} w_{B_R}(x - a, t) = 0$$

uniformly on  $B_{R'}(a)$  for any  $R' < R$ , we derive (6.4).  $\square$

**Proposition 6.7** *For any relatively closed  $\mathcal{S} \in \Omega$ , the set  $\mathcal{U}_{\mathcal{S},0}(\Omega)$  is not empty and it admits a minimal element  $\underline{u}_{\mathcal{S},0}$  and a maximal element  $\bar{u}_{\mathcal{S},0}$ .*

*Proof. Step 1: Existence of a maximal solution.* The maximal solution is constructed by thickening  $\Omega$  and  $\mathcal{S}$  in defining for  $0 < \sigma$

$$\Omega_\sigma = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \sigma\}, \quad \mathcal{S}_\sigma = \{x \in \mathbb{R}^N : \text{dist}(x, \bar{\mathcal{S}}) \leq \sigma\}$$

If  $z \in \partial\Omega$ , we denote by  $\mathbf{n}_z$  the outward unit normal vector to  $\Omega$  at  $z$ . Since  $\partial\Omega$  is compact and  $C^2$ , there exists  $\sigma_0 > 0$  such that for any  $(z, \sigma) \in \partial\Omega \times [0, \sigma_0]$ , the mapping  $\Pi : (z, \sigma) \mapsto z + \sigma \mathbf{n}_z$  is a  $C^2$  diffeomorphism from  $\partial\Omega \times [0, \sigma_0]$  to  $\Theta'_\sigma := \bar{\Omega}_{\sigma_0} \setminus \Omega$ . The mapping  $\Pi$  defines the flow coordinates near  $\partial\Omega$ .

If  $0 < \delta < \sigma$ , there exists a unique solution  $u = u_{n,\sigma,\delta}$  of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{\Omega_\sigma} \\ u &= 0 && \text{in } \partial\Omega_\sigma \times (0, \infty) \\ u(\cdot, 0) &= n \chi_{\mathcal{S}_\delta} && \text{in } \Omega_\sigma. \end{aligned} \quad (6.21)$$

Notice that  $\mathcal{S}_\delta$  is closed in  $\Omega_\sigma$  and  $\inf\{|z - z'| : z \in \mathcal{S}_\delta, z' \in \Omega_\delta^c\} = \delta - \sigma$ . Existence is standard as well as uniqueness in the case where  $\Omega$  is bounded. If  $\Omega^c$  is bounded the proof goes as in the uniqueness proof in Proposition 6.10. When  $n \rightarrow \infty$ ,  $\{u_{n,\sigma,\delta}\} \uparrow u_{\sigma,\delta}$  which is a solution of (1.6) in  $Q_\infty^{\Omega_\sigma}$ . Since  $u_{n,\sigma,\delta}$  satisfies (4.1), for any  $r, \tau > 0$  and any  $a \in \partial\Omega_\sigma$ ,  $u_{n,\sigma,\delta}$  remains uniformly continuous with respect to  $n$  in  $\bar{Q}_\infty^{\Omega_\sigma} \cap (\bar{B}_r(a) \times [2^{-2}\tau, 2\tau])$ . Consequently  $u_{\sigma,\delta}(x, t) = 0$  on  $\partial\Omega_\sigma \times (0, \infty) \cap (\bar{B}_r(a) \times [2^{-2}\tau, 2\tau])$ . Therefore  $u_{\sigma,\delta}$  vanishes

on  $\partial\Omega_\sigma \times (0, \infty)$ . If  $a \in \Omega_\sigma$  and  $\text{dist}(a, \mathcal{S}_\delta) = r > 0$ ,  $u_{n,\sigma,\delta}(x, t) \leq w_{B_r}(x - a, t)$  in  $Q_{\overline{B_r}(a)}^\infty$ . This implies that  $u_{n,\sigma,\delta}$  remains uniformly continuous with respect to  $n$  in  $\overline{B_{r'}}(a) \times [0, T]$  for any  $0 < r' < r$  and  $T > 0$ . Since  $u_{n,\sigma,\delta}(x, t) \rightarrow 0$  in  $\overline{B_{r'}}(a)$ ,  $u_{\sigma,\delta}$  inherits the same property. Consequently  $u_{\sigma,\delta}$  has initial trace  $(\mathcal{S}_\delta, 0)$  in  $\Omega_\sigma$ . By the maximum principle the mapping  $(n, \delta) \mapsto u_{n,\sigma,\delta}$  is increasing with respect to  $n$  and decreasing with respect to  $\delta$ . Furthermore, if  $0 < \delta' < \sigma' < \sigma$  and  $0 < \delta < \sigma$ , there holds  $u_{n,\sigma',\delta'} < u_{n,\sigma,\delta}$  in  $Q_{\infty}^{\sigma'}$ , which implies  $u_{\sigma',\delta'} < u_{\sigma,\delta}$ . We define

$$\overline{u}_{\mathcal{S},0} = \lim_{\sigma \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} u_{n,\sigma,\delta}. \quad (6.22)$$

Then  $\overline{u}_{\mathcal{S},0}$  is a solution of (1.6) in  $Q_\infty^\Omega$ . Since  $\lim_{t \rightarrow 0} u_{\sigma,\delta}(x, t) = 0$  uniformly on any compact subset  $K \subset \mathcal{R} = \Omega \setminus \mathcal{S}$ ,  $\overline{u}_{\mathcal{S},0}$  has initial trace 0 on  $\mathcal{R}$ . If  $a \in \mathcal{S}$ , we denote by  $\tilde{u}_{\infty,a}$  the function defined in  $Q_\infty^{\Omega_\sigma}$  by

$$\tilde{u}_{\infty,a}(x, t) = \begin{cases} u_{\infty,a}(x, t) & \text{if } (x, t) \in Q_\infty^\Omega \\ 0 & \text{if } (x, t) \in Q_\infty^{\Omega_\sigma} \setminus Q_\infty^\Omega \end{cases}$$

Then  $\tilde{u}_{\infty,a}$  is a subsolution of (1.6) in  $Q_\infty^{\Omega_\sigma}$  which is smaller than  $u_{\infty,a,\sigma}$  which is the limit, when  $k \rightarrow \infty$  of the solution  $u_{k\delta_a,\sigma}$  of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 & \text{in } Q_\infty^{\Omega_\sigma} \\ u &= 0 & \text{in } \partial\Omega_\sigma \times (0, \infty) \\ u(\cdot, 0) &= k\delta_a & \text{in } \Omega_\sigma. \end{aligned} \quad (6.23)$$

There holds, by Theorem 6.2,

$$u_{\sigma,\delta}(x, t) \geq u_{\infty,a,\sigma} \geq \tilde{u}_{\infty,a}(x, t) \quad \forall (x, t) \in Q_\infty^{\Omega_\sigma}.$$

Letting successively  $\delta \rightarrow 0$  and  $\sigma \rightarrow 0$  yields to  $\overline{u}_{\mathcal{S},0} \geq \tilde{u}_{\infty,a} = u_{\infty,a}$  in  $Q_\infty^{\Omega_\sigma}$ . Therefore any  $a \in \mathcal{S}$  is a singular initial point of  $\overline{u}_{\mathcal{S},0}$ . Since  $\mathcal{S} \cup \mathcal{R} = \Omega$ , it follows that  $\text{tr}_\Omega(u) = (\mathcal{S}, 0)$ . Since  $u_{\sigma,\delta}$  satisfies (4.1) and  $\partial\Omega_\sigma$  has bounded curvature, independent of  $\sigma$ , there holds classically

$$|\nabla u_{\sigma,\delta}(x, t)| \leq ct^{-\frac{q+\alpha}{q-1}} \quad \forall (x, t) \in \partial\Omega_\sigma \times (0, \infty). \quad (6.24)$$

If  $z \in \partial\Omega$  and then by the mean value theorem there exists  $\theta \in (0, 1)$  such that,

$$0 \leq u_{\sigma,\delta}(z, t) = u_{\sigma,\delta}(z + \sigma \mathbf{n}_z, t) - \sigma \nabla u_{\sigma,\delta}(z + \theta \sigma \mathbf{n}_z, t) \cdot \mathbf{n}_z \leq c\sigma t^{-\frac{q+\alpha}{q-1}}.$$

This implies that  $\overline{u}_{\mathcal{S},0}$  vanishes on  $\partial\Omega_\sigma \times (0, \infty)$ .

Let  $u$  be any positive solution of (1.6) in  $Q_\infty^\Omega$ , vanishing on  $\partial\Omega \times (0, \infty)$ , with initial trace  $(\mathcal{S}, 0)$ . For  $0 < \delta < \sigma$  fixed and for  $R, \epsilon > 0$ , there exists  $\tau_\epsilon > 0$  such that, for any  $\tau \in (0, \tau_\epsilon]$ ,

$$u(x, \tau) \leq \epsilon \quad \forall x \in \overline{B_R} \cap \Omega \setminus \mathcal{S}_\sigma.$$

This is due to the fact that  $u(x, \tau) \rightarrow 0$  when  $\tau \rightarrow 0$ , uniformly on compact subset of  $\overline{B_R} \cap \mathcal{R}$ . Assume that  $\Omega$  is unbounded (the case where  $\Omega$  is bounded is simpler since it does not require to introduce the barrier  $w_{B_R}$ ) and let  $R > 0$  large enough so  $\Omega^c \subset B_R$ . By (4.1)-(4.3) there exists  $0 < \tau_1 \leq \tau_0$  such that for any  $\tau \in (0, \tau_1]$ ,

$$u(x, \tau) \leq w_{B_R}(x, \tau) + u_{\sigma,\delta}(x, \tau) \quad \forall x \in \overline{B_R} \cap \Omega \cap \mathcal{S}_\sigma.$$



Furthermore  $u(x, t) < w_{B_R}(x, t)$  for all  $t > 0$  and  $x \in \partial B_R \cap \Omega$ . Since  $\epsilon + w_{B_R} + u_{\sigma, \delta}$  is a supersolution for (1.6) in  $B_R \cap \Omega \times (0, \infty)$ , it follows that

$$u(x, t) \leq \epsilon + w_{B_R}(x, t) + u_{\sigma, \delta}(x, t) \quad \forall (x, t) \in B_R \cap \Omega \times (0, \infty).$$

Letting successively  $\delta \rightarrow 0$ ,  $\sigma \rightarrow 0$ ,  $R \rightarrow \infty$  (here we use the fact that  $w_{B_R}(x, t) \rightarrow 0$  when  $R \rightarrow \infty$  by Proposition 3.4) and  $\epsilon \rightarrow 0$  yields to  $u \leq \bar{u}_{\mathcal{S}, 0}$ .

*Step 2: Existence of a minimal solution.* The set  $\mathcal{U}_{\mathcal{S}, 0}(\Omega)$  is not empty since it contains  $\bar{u}_{\mathcal{S}, 0}$  and we may define

$$\tilde{u}_{\mathcal{S}, 0} = \sup\{u_{\infty, a} : a \in \mathcal{S}\}, \quad (6.25)$$

and

$$\hat{u}_{\mathcal{S}, 0} = \inf\{u : u \in \mathcal{U}_{\mathcal{S}, 0}(\Omega)\}. \quad (6.26)$$

The functions  $\tilde{u}_{\mathcal{S}, 0}$  and  $\hat{u}_{\mathcal{S}, 0}$  are respectively positive sub and super solutions of (1.6) in  $Q_{\infty}^{\Omega}$ . They are bounded from above by  $\bar{u}_{\mathcal{S}, 0}$  and from below by  $u_{\infty, a}$  for any  $a \in \mathcal{S}$ . Since  $u_{\infty, a} \leq u$  for any  $a \in \mathcal{S}$  and  $u \in \mathcal{U}_{\mathcal{S}, 0}(\Omega)$ , it follows that  $\tilde{u}_{\mathcal{S}, 0} \leq \hat{u}_{\mathcal{S}, 0}$ . Therefore there exists a solution  $\underline{u}_{\mathcal{S}, 0}$  of (1.6) in  $Q_{\infty}^{\Omega}$  which satisfies and

$$\tilde{u}_{\mathcal{S}, 0} \leq \underline{u}_{\mathcal{S}, 0} \leq \hat{u}_{\mathcal{S}, 0}. \quad (6.27)$$

This implies that  $\underline{u}_{\mathcal{S}, 0}$  has initial trace  $(\mathcal{S}, 0)$ , it vanishes on  $\partial\Omega \times (0, \infty)$  and it is therefore the minimal element of  $\mathcal{U}_{\mathcal{S}, 0}(\Omega)$ .  $\square$

*Remark.* If  $\text{dist}(\mathcal{S}, \Omega^c) > 0$ , it is not needed to replace  $\Omega$  by a larger set  $\Omega_{\sigma}$  in order to construct the maximal solution. The construction of  $\bar{u}_{\mathcal{S}, 0}$  can be done in replacing  $u_{n, \sigma, \delta}$  by the solution  $u = u_{n, \sigma}$  of

$$\begin{aligned} \partial_t u - \Delta u + t^{\alpha} u^q &= 0 && \text{in } Q_{\infty}^{\Omega} \\ u &= 0 && \text{in } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) &= n\chi_{\mathcal{S}_{\delta}} && \text{in } \Omega, \end{aligned} \quad (6.28)$$

with  $\delta < \delta_0 := \text{dist}(\mathcal{S}, \Omega^c)$ .

The next result is an extension of Proposition 2.2.

**Proposition 6.8** *Assume  $\alpha > -1$  and  $1 < q < q_{c, \alpha}$ . Let  $\{u_n\}$  be a sequence of positive solutions of (1.6) which converges to  $u$  locally uniformly in  $Q_T^{\Omega}$ , and denote by  $(\mathcal{S}_n, \mu_n)$  and  $(\mathcal{S}, \mu)$  the respective initial trace of  $u_n$  and  $u$ . If  $\mathcal{A}$  is an open subset of  $\cap_n \mathcal{R}_n$  and  $\mu_n(\mathcal{A})$  remains bounded independently of  $n \in \mathbb{N}$  (where  $\mathcal{R}_n = \Omega \setminus \mathcal{S}_n$  and  $\mathcal{R} = \Omega \setminus \mathcal{S}$ ), then  $\mathcal{A} \subset \mathcal{R}$  and  $\chi_{\mathcal{A}} \mu_n \rightharpoonup \chi_{\mathcal{A}} \mu$  in the weak sense of measures. Conversely, if  $\mathcal{A} \subset \mathcal{R}$ , then for any compact  $K \subset \mathcal{A}$ , there exist  $C_K > 0$  and  $n_K \in \mathbb{N}$  such that  $\mu_n(K) \leq C_K$  for any  $n \geq n_K$ .*

*Proof.* Clearly (2.14) holds. We keep the notations of the proof of Proposition 2.2 where the first statement has been proved in assuming  $\bar{B}_{\bar{r}}(z) \subset \mathcal{A}$ . Since  $\mu_n(\mathcal{A})$  remains bounded, there exists a subsequence  $\{n_j\}$  and a positive measure  $\mu'$  on  $\mathcal{A}$  such that  $\mu_{n_j} \rightharpoonup \mu'$  in the weak sense of measures in  $\mathcal{A}$ . Then  $u_{\chi_{B_{\bar{r}}(z)} \mu_{n_j}}$  converges locally uniformly in  $Q_{\infty}^{B_{\bar{r}}(z)}$  to the solution  $u_{\chi_{B_{\bar{r}}(z)} \mu'}$  of

$$\begin{aligned} \partial_t u - \Delta u + t^{\alpha} u^q &= 0 && \text{in } Q_{\infty}^{B_{\bar{r}}(z)} \\ u &= 0 && \text{in } \partial B_{\bar{r}}(z) \times (0, \infty) \\ u(\cdot, 0) &= \chi_{B_{\bar{r}}(z)} \mu' && \text{in } B_{\bar{r}}(z). \end{aligned} \quad (6.29)$$

Since  $q < q_{c,\alpha}$ , the convergence of  $u_{\chi_{B_{\tilde{r}}(z)}\mu_{n_j}}$  and  $t^\alpha u_{\chi_{B_{\tilde{r}}(z)}\mu_{n_j}}^q$  respectively to  $u_{\chi_{B_{\tilde{r}}(z)}\mu_{n_j}}$  and  $t^\alpha u_{\chi_{B_{\tilde{r}}(z)}\mu'}^q$  holds in  $L^1(Q_T^{B_{\tilde{r}}(z)})$  for any  $T > 0$ . Relation (2.21) reads

$$u_{\chi_{B_{\tilde{r}}(z)}\mu_n}(x, t) \leq u_n(x, t) \leq u_{\chi_{B_{\tilde{r}}(z)}\mu_n}(x, t) + w_{B_{\tilde{r}}(z)}(x, t) \quad \text{in } Q_\infty^{B_{\tilde{r}}(z)}. \quad (6.30)$$

(see Proposition 3.3). Then  $u_{n_j}$  and  $t^\alpha u_{n_j}^q$  converge to  $u$  and  $t^\alpha u^q$  respectively, in  $L^1(Q_T^{B_r(z)})$  for any  $r < \tilde{r}$ . From (2.17), we derive

$$\iint_{Q_\infty^{B_r(z)}} (-u(\zeta_t + \Delta\zeta) + \zeta t^\alpha u^q) dxdt = \int_{B_r(z)} \zeta(x, 0) d\mu'(x), \quad (6.31)$$

for any  $\zeta \in C_c^{1,1}(\overline{Q_\infty^{B_r(z)}})$  which vanishes for  $t$  large enough. This implies that  $\mu'$  is the initial trace of  $u$  in  $B_r(z)$ , i.e.  $\chi_{B_r(z)}\mu' = \chi_{B_r(z)}\mu$  and  $\chi_{B_r(z)}\mu_n \rightarrow \chi_{B_r(z)}\mu$ . Using a partition of unity, we conclude that  $\chi_{\mathcal{A}}\mu_n \rightarrow \chi_{\mathcal{A}}\mu$ .

Conversely, we assume that there exist a compact set  $K \subset \mathcal{A}$  and a subsequence  $\mu_{n_j}$  such that  $\mu_{n_j}(K) \rightarrow \infty$ . Thus, using the diagonal process, there exist  $z \in K$  and another subsequence that we still denote  $\mu_{n_j}$  such that

$$\lim_{n_j \rightarrow \infty} \mu_{n_j}(B_\epsilon(z)) = \infty \quad \forall \epsilon > 0.$$

Therefore, we can construct a subsequence  $\{n_{j_\ell}\} \subset \{n_j\}$  such that

$$\mu_{n_{j_\ell}}(B_{2^{-n_{j_\ell}}}(z)) = m_{n_{j_\ell}} \rightarrow \infty$$

when  $n_{j_\ell} \rightarrow \infty$ . Since the solution  $u_{\chi_{B_{2^{-n_{j_\ell}}}(z)}\mu_{n_{j_\ell}}}$  of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{B_{\tilde{r}}(z)} \\ u &= 0 && \text{in } \partial B_{\tilde{r}}(z) \times (0, \infty) \\ u(\cdot, 0) &= \chi_{B_{2^{-n_{j_\ell}}}(z)}\mu_{n_{j_\ell}} && \text{in } B_{\tilde{r}}(z). \end{aligned} \quad (6.32)$$

converges to  $u_{\infty\delta_z}^{B_{\tilde{r}}(z)}$  which is the limit of the solution  $u_{k\delta_z}$  of (6.32) with initial data  $u(\cdot, 0) = k\delta_z$ , and is dominated by  $u_{n_{j_\ell}}$  in  $Q_T^{B_{\tilde{r}}(z)}$  we conclude that  $u \geq u_{\infty\delta_z}^{B_{\tilde{r}}(z)}$  in  $Q_T^{B_{\tilde{r}}(z)}$ , which implies that  $z \in \mathcal{S}$ , contradiction.  $\square$

**Proposition 6.9** *Assume  $u_1$  and  $u_2$  are two positive solutions of (1.6) in  $Q_\infty^\Omega$  with initial trace  $(\mathcal{S}, \mu)$ . Then for any  $a \in \mathcal{R}$  and  $R > 0$  such that  $B_R(a) \subset \mathcal{R}$ , there holds*

$$|u_1(x, t) - u_2(x, t)| \leq w_{B_R}(x - a, t) \quad \forall (x, t) \in Q_\infty^{B_R(a)} \quad (6.33)$$

*In particular  $\lim_{t \rightarrow 0} |u_1(x, t) - u_2(x, t)| = 0$  uniformly on any compact subset of  $\mathcal{R}$ .*

*Proof.* Since  $u$  and  $u'$  are solution of (1.6) and  $B_R(a) \in \mathcal{R}$ , for any  $i = 1, 2$ ,  $R' < R$  and  $T > 0$ ,

$$\iint_{Q_T^{B_{R'}(a)}} t^\alpha u_i^q(x, t) dxdt + \iint_{Q_T^{B_{R'}(a)}} u_i(x, t) dxdt < \infty,$$

furthermore

$$\lim_{t \rightarrow 0} \int_{B_R(a)} u_i(x, t) \zeta(x) dx = \int_{B_R(a)} \zeta(x) d\mu(x) \quad \forall \zeta \in C_c(B_R(a)).$$

This implies that  $u_i$  has a Sobolev trace on  $\partial_\ell Q_T^{B'_R(a)}$  which belongs to  $L^1$  and they are the limit, when  $k \rightarrow \infty$  of the solutions  $u_{i,k}$  of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{B'_R(a)} \\ u &= \min\{k, u_i|_{Q_\infty^{B'_R(a)}}\} && \text{in } \partial_\ell Q_\infty^{B'_R(a)} \\ u(\cdot, 0) &= \mu && \text{in } B'_R(a). \end{aligned} \quad (6.34)$$

Since  $u_{2,k} + w_{B'_R}(\cdot - a)$  is a supersolution

$$u_{1,k} \leq u_{2,k} + w_{B'_R}(\cdot - a) \implies |u_{1,k} - u_{2,k}| \leq w_{B'_R}(\cdot - a) \text{ in } Q_\infty^{B'_R(a)}.$$

Letting  $k \rightarrow \infty$ ,  $R'$  to  $R$ , we derive (6.33). The second statement is a consequence of the fact that  $\lim_{t \rightarrow 0} w_{B_R}(\cdot - a) = 0$ , uniformly on  $B'_R$  by Proposition 3.3.  $\square$

*Remark.* The previous estimate does not use the fact that  $\partial\Omega$  is smooth and bounded. If the  $u_i$  belong to  $\mathcal{U}_{S,\mu}(\Omega)$ , estimate (6.33) can be improved since the  $u_i$  vanish on  $\partial_\ell Q_\infty^{B'_R(a)}$ , and we obtain,

$$|u_1(x, t) - u_2(x, t)| \leq \min \left\{ w_{B_R}(x - a, t), c_\alpha t^{-\frac{1+\alpha}{q-1}} \right\} \quad \forall (x, t) \in Q_\infty^{B'_R(a)}. \quad (6.35)$$

**Proposition 6.10** *Assume  $\Omega \subseteq \mathbb{R}^N$  is either  $\mathbb{R}^N$  or an open domain with a  $C^2$  compact boundary,  $\alpha > -1$  and  $1 < q < q_{c,\alpha}$ . Then for any measure  $\mu$  in  $\Omega$  such that  $\mu|_{\Omega_R} \in \mathfrak{M}_+^{b,\rho}(\Omega_R)$  where  $\Omega_R = \Omega \cap B_R$ , there exists a unique solution  $u_\mu$  to*

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^\Omega \\ u &= 0 && \text{in } \partial_\ell Q_\infty^\Omega \\ u(\cdot, 0) &= \mu && \text{in } \Omega, \end{aligned} \quad (6.36)$$

and the mapping  $\mu \mapsto u_\mu$  is increasing. Furthermore, if  $\{\mu_n\}$  is a sequence of positive measures such that  $\mu_n|_{\Omega_R} \in \mathfrak{M}_+^{b,\rho}(\Omega_R)$  which converges weakly to  $\mu|_{\Omega_R} \in \mathfrak{M}_+^{b,\rho}(\Omega_R)$ , then  $\{u_{\mu_n}\} \rightarrow u_\mu$  locally uniformly in  $Q_\infty^\Omega$ .

*Proof.* We recall that  $u$  is a solution of (6.36) if  $u \in L_{loc}^1(\overline{Q_\infty^\Omega})$ ,  $|u|^q \in L_{loc}^1(\overline{Q_\infty^\Omega}; t^\alpha \rho dx dt)$  satisfies

$$\iint_{Q_\infty^\Omega} \left( -u(\partial_t \zeta + \Delta \zeta) + t^\alpha |u|^{q-1} u \zeta \right) dx dt = \int \zeta(\cdot, 0) d\mu \quad (6.37)$$

for any test function  $\zeta \in C_0^{2,1;1}(\overline{Q_\infty^\Omega})$ . When  $\Omega$  is bounded, existence, uniqueness and stability are proved in [9]. Thus we assume that  $\Omega$  is unbounded and we assume  $R \geq R_0$  such that  $\Omega^c \subset B_{R_0}$ . There exists a unique solution  $u_R$  of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_R} \\ u &= 0 && \text{in } \partial_\ell Q_\infty^{\Omega_R} \\ u(\cdot, 0) &= \mu|_{\Omega_R} && \text{in } \Omega_R. \end{aligned} \quad (6.38)$$

The function  $u_R$  is nonnegative,  $R \mapsto u_R$  is increasing. For  $R > R_1$ ,  $u_R$  admits a Sobolev trace  $f_{R_1}$  on  $\partial B_{R_1} \times (0, T)$  which is an integrable function, and  $u_R$  is the unique solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_{R_1}} \\ u &= 0 && \text{in } \partial\Omega \times (0, \infty) \\ u &= f_{R_1} && \text{in } \partial B_{R_1} \times (0, \infty) \\ u(\cdot, 0) &= \mu|_{\Omega_{R_1}} && \text{in } \Omega_{R_1}. \end{aligned} \quad (6.39)$$

Furthermore,  $u_R|_{Q_\infty^{\Omega_{R_1}}} = \lim_{m \rightarrow \infty} v_m$ , where  $v_m$  is the unique solution of (6.36) where the boundary data on  $\partial B_{R_1} \times (0, \infty)$  is replaced by  $f_{R_1, m} = f_{R_1} \wedge m$  ( $m \in \mathbb{N}^*$ ). Let  $v_{R_1}$  be the unique solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_{R_1}} \\ u &= 0 && \text{in } \partial\Omega \times (0, \infty) \\ u &= 0 && \text{in } \partial B_{R_1} \times (0, \infty) \\ u(\cdot, 0) &= \mu|_{\Omega_{R_1}} && \text{in } \Omega_{R_1}, \end{aligned} \quad (6.40)$$

If  $w_{B_{R_1}}$  is the barrier function in  $Q_\infty^{\Omega_{R_1}}$  which has been constructed in Proposition 3.3,  $v_{R_1} + w_{B_{R_1}}$  is a supersolution for problem (6.39). Since it is larger than  $v_m$  in  $Q_\infty^{\Omega_{R_1}}$  for any  $m > 0$ , there holds  $u_R \leq v_{R_1} + w_{B_{R_1}}$ , for any  $R > R_1$ . Then  $u_R \uparrow u_\mu$  which is a solution of 1.6 in  $Q_\infty^{\Omega_{R'}}$ . By Proposition 3.4,  $w_{B_{R_1}}$  remains uniformly bounded in  $Q_\infty^{\Omega_{R'}}$  for any  $R_0 < R' < R_1$ . Therefore  $u_\mu$  shares the same property. If  $\zeta \in C_c^{1,1;1}(Q_\infty^{\overline{\Omega}})$  vanishes on  $Q_\infty^{\partial\Omega}$  and for  $|x| > R' > R_0$ , there holds for  $R > R' > R_0$  and  $T > 0$ ,

$$\iint_{Q_\infty^{\overline{\Omega}}} (-u_R(x, t)(\partial_t \zeta + \Delta \zeta) + \zeta t^\alpha u_R^q) dx dt = \int_\Omega \zeta(x, 0) d\mu(x) - \int_\Omega \zeta(x, T) u_R(x, T) dx \quad (6.41)$$

If we let  $R \rightarrow \infty$  we deduce by the monotone convergence theorem that  $u_\mu$  is a weak solution of (6.36). This proves existence.

For uniqueness, we consider  $u_\mu$  and  $u'_\mu$  two solutions of (6.36). By the same argument as in the existence part, for any  $R > 0$ ,  $u_\mu$  is smaller than the supersolution  $u'_\mu + w_{B_{R_1}}$  in  $Q_\infty^{\Omega_{R'}}$ . Since  $\lim_{R \rightarrow \infty} w_{B_R} = 0$  by Proposition 3.4 we obtain  $u_\mu \leq u'_\mu$ . Similarly  $u'_\mu \leq u_\mu$ . Uniqueness implies the monotonicity of the mapping  $\mu \mapsto u_\mu$ .

For proving the stability, assume  $\{\mu_n|_{\Omega_R}\}$  converges to  $\mu|_{\Omega_R}$  in the weak sense of measures in  $\mathfrak{M}_+^{b,\rho}(\Omega_R)$  for any  $R > R_0$ . Then the sequence of solutions  $v_{n,R}$  of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha |u|^{q-1} u &= 0 && \text{in } Q_\infty^{\Omega_R} \\ u &= 0 && \text{in } \partial\Omega \times (0, \infty) \\ u &= 0 && \text{in } \partial B_R \times (0, \infty) \\ u(\cdot, 0) &= \mu_n|_{\Omega_R} && \text{in } \Omega_R, \end{aligned} \quad (6.42)$$

converges to the solution  $v_R$  of (6.40) with  $R_1 = R$ . In particular  $v_{n,R} \rightarrow v_R$  and  $t^\alpha v_{n,R}^q \rightarrow t^\alpha v_R^q$  in  $L^1(Q_T^{\Omega_R})$  and by standard regularity result the convergence of  $v_{n,R}$  towards  $v_R$  holds uniformly on  $\overline{\Omega_R} \times [\epsilon, T]$  for any  $0 < \epsilon < T$ . Furthermore  $u_{\mu_n} \leq v_{n,R} + W_{B_R}$  in  $Q_\infty^{\Omega_R}$ . This

jointly with standard local regularity results for heat equation, implies that  $\{u_{\mu_n}\}$  remains uniformly bounded and hence relatively compact for the topology of uniform convergence on any compact set of  $\overline{\Omega} \times [\epsilon, \infty)$ . Thus there exist a subsequence  $\{u_{\mu_{n_k}}\}$  and a function  $u^* \in C(\overline{Q_\infty^\Omega})$  such that  $u_{\mu_{n_k}} \rightarrow u^*$  locally uniformly in  $\overline{\Omega} \times (0, \infty)$ . Since  $t^\alpha u_{\mu_n}^q \leq t^\alpha v_{n,R}^q + t^\alpha W_{B_R}^q$ , there also holds by the dominated convergence theorem  $t^\alpha u_{\mu_{n_k}}^q \rightarrow t^\alpha u^{*q}$  in  $L_{loc}^1(\overline{\Omega} \times [0, \infty))$ . Henceforth letting  $n_k \rightarrow \infty$  in the expression

$$\begin{aligned} & \iint_{Q_\infty^\Omega} \left( -u_{\mu_{n_k}}(x, t)(\partial_t \zeta + \Delta \zeta) + \zeta t^\alpha u_{\mu_{n_k}}^q \right) dx dt \\ &= \int_\Omega \zeta(x, 0) d\mu_{n_k}(x) - \int_\Omega \zeta(x, T) u_{n_k}(x, T) dx, \end{aligned} \quad (6.43)$$

where  $\zeta \in C_c^{1,1;1}(Q_\infty^\Omega)$ , we conclude that  $u^* = u_\mu$  and that  $u_{\mu_n} \rightarrow u_\mu$ .  $\square$

**Proposition 6.11** *Assume  $F$  is a non-empty relatively closed subset of  $\Omega$ ,  $\mathcal{R} = \Omega \setminus F$  and  $\mu \in \mathfrak{M}_+(\mathcal{R})$ . If we set*

$$\partial_\mu F = \{z \in F : \mu(\mathcal{R} \cap B_r(z)) = \infty, \forall r > 0\}, \quad (6.44)$$

then  $\partial_\mu F$  is relatively closed in  $\Omega$ . If  $\mathcal{R}^\mu = \Omega \setminus \partial_\mu F$ , it contains  $\mathcal{R}$  and if  $\mu^*$  is the measure defined in  $\mathcal{R}^\mu$  by  $\mu$  on  $\mathcal{R}$  and 0 in  $\mathcal{R}^\mu \cap \mathcal{R}^c$ , then there exist a minimal positive solution  $\underline{u}_{\mu^*}$  and a maximal solution  $\overline{u}_{\mu^*}$  of (1.6) vanishing on  $\partial\Omega \times (0, \infty)$  satisfying  $\text{tr}_\Omega(u) = (\partial_\mu F, \mu^*)$ . Furthermore  $\underline{u}_{\mu^*}$  and  $\overline{u}_{\mu^*}$  are respectively the minimal and the maximal element of  $\mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$ .

*Proof.* The set  $\partial_\mu F$  is the blow-up set of the measure  $\mu$ . It is clearly a relatively closed subset of  $\Omega$  included into  $\overline{\mathcal{R}} \setminus \mathcal{R}$ .

*Step 1: Existence of a minimal solution.* For  $\delta > 0$ , we denote  $(\partial_\mu F)_\delta := \{x \in \Omega : \text{dist}(x, \partial_\mu F) \leq \delta\}$  and  $\mathcal{R}_\delta^\mu = \Omega \setminus (\partial_\mu F)_\delta \subset \mathcal{R}^\mu$ . We define the Radon measure  $\mu_\delta$  on  $\Omega$  by

$$\mu_\delta = \begin{cases} \mu & \text{on } \mathcal{R}_\delta^\mu \cap \mathcal{R} \\ 0 & \text{on } F \cup (\partial_\mu F)_\delta \end{cases}$$

Then  $\mu_\delta$  is a positive Radon measure in  $\Omega$  and by Proposition 6.10 problem (6.36) with initial data  $\mu_\delta$  admits a unique positive solution  $u_{\mu_\delta}$ . Furthermore the mapping  $\delta \mapsto u_{\mu_\delta}$  is nonincreasing, and we set  $\underline{u}_{\mu^*} = \lim_{\delta \rightarrow 0} u_{\mu_\delta}$ . Then  $\underline{u}_{\mu^*}$  is a positive solution of (1.6) in  $Q_\infty^\Omega$  which vanishes on  $\partial\Omega \times (0, \infty)$  and has initial trace  $(\mathcal{S}', \mu')$ . If  $a \in \mathcal{R}^\mu$ , there exists  $R > 0$  such that  $\overline{B}_R(a) \subset \mathcal{R}^\mu$  and  $\delta_a > 0$  such that  $\overline{B}_R(a) \subset \mathcal{R}_\delta^\mu$  for  $0 < \delta < \delta_a$ , that we assume below. By the maximum principle there holds

$$v_{\mu_\delta}(x, t) \leq u_\delta(x, t) \leq v_{\mu_\delta}(x, t) + w_{B_R}(x - a, t) \quad \text{in } Q_\infty^{B_R(a)},$$

where  $v_{\mu_\delta}$  is the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{B_R(a)} \\ u &\geq 0 && \text{in } Q_\infty^{B_R(a)} \\ u(x, t) &= 0 && \text{in } \partial B_R(a) \times (0, \infty) \\ u(\cdot, 0) &= \mu_\delta \chi_{B_R(a)} && \text{in } B_R(a). \end{aligned}$$

Letting  $\delta \rightarrow 0$ , then  $\mu_\delta \chi_{B_R(a)} \uparrow \mu^* \chi_{B_R(a)}$ , which yields to

$$v_{\mu^* \lfloor_{B_R(a)}}(x, t) \leq \underline{u}_{\mu^*}(x, t) \leq v_{\mu^* \lfloor_{B_R(a)}}(x, t) + w_{B_R}(x - a, t) \quad \text{in } Q_\infty^{B_R(a)}.$$

By a partition of unity, it implies that for any  $\zeta \in C_c(\mathcal{R}^\mu)$ , we have

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}^\mu} \underline{u}_{\mu^*}(x, t) \zeta(x) dx = \int_{\mathcal{R}^\mu} \zeta d\mu^*(x).$$

Therefore  $\mathcal{R}^\mu \subset \mathcal{R}'$  and  $\mu' \lfloor_{\mathcal{R}^\mu} = \mu^*$ . If  $z \in \mathcal{R}' \cap (\mathcal{R}^\mu)^c = \mathcal{R}' \cap \partial_\mu F$ ,  $\mu^*(B_r(z) \cap \mathcal{R}^\mu) = \mu(B_r(z) \cap \mathcal{R}) = \infty$  for any  $r > 0$  while there exists  $r_0 > 0$  such that  $\mu'(B_{r_0}(z)) < \infty$ . By Proposition 6.8, for any  $r' < r_0$  there exists  $C > 0$  such that  $\mu_\delta(\overline{B}_{r'}(z)) \leq C$ . By the monotone convergence theorem, it implies  $\mu(\overline{B}_{r'}(z) \cap \mathcal{R}) \leq C$ , which contradicts the definition of  $\partial_\mu F$ . Thus  $\mathcal{S}' = \partial_\mu F$  and  $tr_\Omega(\underline{u}_{\mu^*}) = (\partial_\mu F, \mu^*)$ .

Let us assume that  $\Omega$  is unbounded,  $R_0$  is such that  $\Omega^c \subset B_{R_0}$  and  $\Omega_R = \Omega \cap B_R$  for  $R > R_0$ . Let  $\delta, \epsilon > 0$ , there exists  $\tau_\epsilon$  such that

$$u_{\mu_\delta}(x, \tau) \leq \epsilon \quad \forall (x, \tau) \in \Omega_R \cap (\partial_\mu F)_{\frac{\delta}{2}} \times [0, \tau_\epsilon] \quad (6.45)$$

Let  $u \in \mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$ . In order to compare  $u_{\mu_\delta}$  and  $v := u + w_{B_R} + \epsilon$  in  $\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}} \times (0, \tau_\epsilon]$  we see that

$$u_{\mu_\delta}(\cdot, 0) = \chi_{\mathcal{R}_R^\mu \setminus (\partial_\mu F)_\delta} \mu^* \leq u(\cdot, 0) = \chi_{\mathcal{R}_R^\mu \setminus (\partial_\mu F)_{\frac{\delta}{2}}} \mu^*$$

and both are bounded Radon measures. Since  $u_{\mu_\delta}(t, 0) \leq v(x, t)$  in  $\partial(\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}}) \times (0, \tau_\epsilon]$  and  $v$  is a supersolution, it follows that  $u_{\mu_\delta} \leq v$  in  $\Omega_R \setminus (\partial_\mu F)_{\frac{\delta}{2}} \times (0, \tau_\epsilon]$ . Using (6.45) we conclude that

$$u_{\mu_\delta}(x, \tau) \leq v(x, \tau) \quad \forall (x, \tau) \in \Omega_R \times [0, \tau_\epsilon]. \quad (6.46)$$

Next, applying the comparison principle in  $\Omega_R \times [\tau^*, \infty)$  between the solution  $u_{\mu_\delta}$  and the supersolution  $u + \epsilon + w_{B_R}$ , we conclude that (6.46) holds in  $\Omega_R \times [\tau, \infty)$  and thus in  $Q_\infty^{B_R}$ . Letting successively  $R \rightarrow \infty$ ,  $\epsilon \rightarrow 0$  and  $\delta \rightarrow 0$ , we conclude that  $\underline{u}_{\mu^*} \leq u$ , thus  $u_{\mu^*}$  is the minimal element of  $\mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$ .

*Step 2: Existence of a maximal solution.* Let  $\delta > 0$  and  $u \in \mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$ . By Proposition 6.9, for any  $R > 0$  and  $\epsilon > 0$  there exists  $\tau_\epsilon$  such that

$$u(x, t) \leq \underline{u}_{\mu^*}(x, t) + \epsilon \quad \forall (x, t) \in \Omega_R \setminus (\partial_\mu F)_\delta \times (0, \tau_\epsilon], \quad (6.47)$$

and by (4.1),  $u(x, t) \leq c_\alpha t^{-\frac{1+\alpha}{q-1}}$ . Let  $\tau \in (0, \tau_\epsilon]$  and  $w_{\delta, \tau}$  be the solution of (6.36) in  $Q_\infty^{\Omega_R}$  with initial data  $\mu$  replaced by

$$h_{\delta, \tau} = \begin{cases} \underline{u}_{\mu^*}(x, \tau) & \text{if } x \in \Omega_R \setminus (\partial_\mu F)_\delta \\ c_\alpha \tau^{-\frac{1+\alpha}{q-1}} & \text{if } x \in \Omega_R \cap (\partial_\mu F)_\delta \end{cases} \quad (6.48)$$

By (6.47), (6.48) and the maximum principle,

$$u(x, t + \tau) \leq w_{\delta, \tau}(x, t) + \epsilon + w_{B_R}(x, t + \tau) \quad \forall (x, t) \in Q_\infty^{\Omega_R}. \quad (6.49)$$

Let  $\bar{u}_{(\partial_\mu F)_\delta}$  be the maximal element of  $\mathcal{U}_{(\partial_\mu F)_\delta, 0}(\Omega)$ , which exists by Proposition 6.7. Then, by (4.3), for any  $\delta' > \delta$ , there exists  $\tau_{\delta'} \in (0, \tau_\epsilon]$  such that for any  $\tau \in (0, \tau_{\delta'}]$

$$\max\{\underline{u}_{\mu^*}(\cdot, \cdot + \tau), \bar{u}_{(\partial_\mu F)_\delta}(\cdot, \cdot + \tau)\} \leq w_{\delta, \tau} \leq \underline{u}_{\mu^*}(\cdot, \cdot + \tau) + \bar{u}_{(\partial_\mu F)_{\delta'}}(\cdot, \cdot + \tau) \quad \text{in } Q_\infty^{\Omega_R}. \quad (6.50)$$

Up to a sequence  $\{\tau_n\}$  converging to 0,  $\{w_{\delta, \tau_n}\}$  converges, locally uniformly in  $Q_\infty^\Omega$  to a solution  $w_\delta$  of (1.6) in  $Q_\infty^\Omega$  which satisfies

$$\max\{\underline{u}_{\mu^*}, \bar{u}_{(\partial_\mu F)_\delta}\} \leq w_\delta \leq \underline{u}_{\mu^*} + \bar{u}_{(\partial_\mu F)_{\delta'}} \quad \text{in } Q_\infty^{\Omega_R}. \quad (6.51)$$

We can replace  $\delta'$  by  $\delta$  in this inequality, this proves that  $w_\delta$  vanishes on  $\partial_\ell Q_\infty^{\Omega_R}$  has initial trace  $(\mu_\delta, (\partial_\mu F)_\delta)$ , therefore (6.49) becomes

$$u(x, t) \leq w_\delta(x, t) + \epsilon + w_{B_R}(x, t) \quad \forall (x, t) \in Q_\infty^{\Omega_R}. \quad (6.52)$$

Letting successively  $R \rightarrow \infty$  and  $\epsilon \rightarrow 0$  we deduce that  $w_\delta$  is larger than any  $u \in \mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$  in  $Q_\infty^{\Omega_R}$ . Since  $h_{\delta, \tau}$  decreases with  $\delta$ ,  $w_\delta$  shares this property and the limit, denoted by  $\bar{u}_{\mu^*}$  is a solution of (1.6) in  $Q_\infty^\Omega$  which vanishes on  $\partial_\ell Q_\infty^{\Omega_R}$  which is large than  $u$ , thus it is the maximal element of  $\mathcal{U}_{\partial_\mu F, \mu^*}(\Omega)$ .  $\square$

**Proposition 6.12** *Under the assumptions of Proposition 6.11, we set  $F_\delta := \{x \in \Omega : \text{dist}(x, F) \leq \delta\}$  and  $\mathcal{R}_\delta := \Omega \setminus F_\delta \subset \mathcal{R}$ . If we define the measure  $\tilde{\mu}_\delta$  in  $\Omega$  by*

$$\tilde{\mu}_\delta = \begin{cases} \mu & \text{on } \mathcal{R}_\delta \\ 0 & \text{on } F_\delta, \end{cases}$$

then  $u_{\tilde{\mu}_\delta} \uparrow \underline{u}^*$  when  $\delta \downarrow 0$ .

*Proof.* There holds  $\tilde{\mu}_\delta \leq \mu_\delta$  which implies  $u_{\tilde{\mu}_\delta} \leq u_{\mu_\delta}$ . When  $\delta \rightarrow 0$ ,  $u_{\tilde{\mu}_\delta} \uparrow \underline{u}^* \leq \underline{u}_{\mu^*}$ , thus  $\underline{u}^*$  is a positive solution of (1.6) in  $Q_\infty^\Omega$  which vanishes on  $\partial_\ell Q_\infty^\Omega$ . Then  $\text{tr}_\Omega(\underline{u}^*) = (\mathcal{S}'', \mu'')$  and  $\mathcal{S}'' \subset \partial_\mu F$  and  $\mu'' \leq \mu^*$  on  $\mathcal{R}^\mu$ . Furthermore  $\mu'' = \mu = \mu^*$  on  $\mathcal{R}$ , as in the proof of Proposition 6.11. Since  $\mu^* = 0$  on  $\mathcal{R}^\mu \setminus \mathcal{R}$  it follows that  $\mu'' = \mu^*$  on  $\mathcal{R}^\mu$ . Let  $a \in \mathcal{R}'' \cap \partial_\mu F$  and  $R > 0$  such that  $\bar{B}_R(a) \subset \mathcal{R}''$ . Then  $\mu''(\bar{B}_R(a)) < \infty$ . Therefore

$$\mu''(\bar{B}_R(a) \cap \mathcal{R}) = \mu(\bar{B}_R(a) \cap \mathcal{R}) < \infty,$$

contradiction. Thus  $\mathcal{S}'' = \partial_\mu F$  and  $\text{tr}_\Omega(\underline{u}^*) = (\partial_\mu F, \mu^*)$ . Since  $\underline{u}^* \leq \underline{u}_{\mu^*}$  and  $\underline{u}_{\mu^*}$  is minimal, it follows that  $\underline{u}^* = \underline{u}_{\mu^*}$ .  $\square$

**Proposition 6.13** *Assume  $v \in C(\bar{\Omega} \times (0, \infty))$  is a positive sub-solution (resp. supersolution) of (1.6) in  $Q_\infty^\Omega$  which vanishes on  $\partial_\ell Q_\infty^\Omega$ . Then there exists a minimal solution  $\pi_+(v)$  larger than  $v$  (resp. a maximal solution  $\pi_-(v)$  smaller than  $v$  and vanishing on  $\partial_\ell Q_\infty^\Omega$ ).*

*Proof.* Assume  $v$  is a subsolution. Let  $\tau > 0$  and let  $u_\tau$  be the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } \Omega \times (\tau, \infty) \\ u \geq 0, u &= 0 && \text{in } \partial\Omega \times (\tau, \infty) \\ u(\cdot, \tau) &= v(\cdot, \tau) && \text{in } \Omega. \end{aligned} \quad (6.53)$$

Existence and uniqueness follows from Proposition 6.10. Furthermore  $u_\tau \geq v$  in  $\Omega \times (\tau, \infty)$ . This implies that for  $0 < \tau < \tau'$ ,  $u_\tau \geq u_{\tau'}$ . Since  $u_\tau(x, t) \leq c_\alpha(t - \tau)^{-\frac{1+\alpha}{q-1}}$ , there exists  $\pi_+(v) = \lim_{\tau \rightarrow 0} u_\tau$ , and  $\pi_+(v)$  is a positive solution of (1.6) in  $Q_\infty^\Omega$  and is larger than  $v$ . If  $u$  is any positive solution of (1.6) in  $Q_\infty^\Omega$ , vanishing on  $\partial\ell Q_\infty^\Omega$  and larger than  $v$ , for any  $\tau > 0$  it is larger than  $u(\cdot, \tau)$ , thus it is larger than  $u_\tau$  on  $\Omega \times (\tau, \infty)$ . Therefore  $u \geq \pi_+(v)$ .

Assume now that  $u$  is a supersolution. We define  $u_\tau$  by (6.53). Then  $u_\tau \leq v$  and  $u_\tau \leq u_{\tau'}$  for  $0 < \tau < \tau'$ . Then  $\pi_-(v) = \lim_{\tau \rightarrow 0} u_\tau$ , and  $\pi_-(v)$  is a positive solution of (1.6) in  $Q_\infty^\Omega$  and is smaller than  $v$ , and thus vanishing on  $\partial\ell Q_\infty^\Omega$ . Similarly as above  $\pi_-(v)$  is larger than any positive solution smaller than  $v$ .  $\square$

**Theorem 6.14** *Assume  $\Omega \subseteq \mathbb{R}^N$  is either  $\mathbb{R}^N$  or an open domain with a  $C^2$  compact boundary,  $\alpha > -1$  and  $1 < q < q_{c,\alpha}$ . Then for any  $\nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega)$  there exist a maximal positive solution  $\bar{u}_{\mathcal{S},\mu}$  and a minimal positive solution  $\underline{u}_{\mathcal{S},\mu}$  of (1.6) in  $Q_\infty^\Omega$  vanishing on  $\partial\Omega \times (0, \infty)$  with initial trace  $\nu$ . Furthermore, if  $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$ , then*

$$\bar{u}_{\mathcal{S},\mu} - \underline{u}_{\mathcal{S},\mu} \leq \bar{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0}. \quad (6.54)$$

*Proof. Step 1: Construction of the maximal and minimal solutions.* The functions  $\underline{u}_{\mu^*}$ ,  $\underline{u}_{\mathcal{S},0}$ ,  $\bar{u}_{\mu^*}$  and  $\bar{u}_{\mathcal{S},0}$  have been defined in Proposition 6.11 and Proposition 6.7. Since  $\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\}$  is a subsolution of (1.6) which is smaller than the supersolution  $\bar{u}_{\mu^*} + \bar{u}_{\mathcal{S},0}$  we set

$$(i) \quad \underline{u}_{\mathcal{S},\mu} = \pi_+(\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\}) \quad \text{and} \quad (ii) \quad \bar{u}_{\mathcal{S},\mu} = \pi_-(\bar{u}_{\mu^*} + \bar{u}_{\mathcal{S},0}). \quad (6.55)$$

Then  $\underline{u}_{\mathcal{S},\mu}$  and  $\bar{u}_{\mathcal{S},\mu}$  are solutions which satisfy

$$\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{\mathcal{S},\mu} \leq \bar{u}_{\mathcal{S},\mu} \leq \bar{u}_{\mu^*} + \bar{u}_{\mathcal{S},0}. \quad (6.56)$$

Therefore  $\underline{u}_{\mathcal{S},\mu}$  and  $\bar{u}_{\mathcal{S},\mu}$  vanish on  $\partial\ell Q_\infty^\Omega$ , they have initial trace  $\mu$  on  $\mathcal{R}$  and are larger than any  $u_{\infty,a}$  for  $a \in \mathcal{S}$  (notice that  $\partial_\mu \mathcal{S} \subset \mathcal{S}$ ). This implies that  $\underline{u}_{\mathcal{S},\mu}$  and  $\bar{u}_{\mathcal{S},\mu}$  belong to  $\mathcal{U}_{\mathcal{S},\mu}(\Omega)$ .

Let  $u \in \mathcal{U}_{\mathcal{S},\mu}(\Omega)$ . Then for  $\sigma > \delta > 0$  and  $\epsilon, R > 0$ , there exists  $\tau_1 > 0$  such that

$$u(x, t) \leq u_{\sigma,\delta}(x, t) \text{ for } (x, t) \in (\mathcal{S}_{\frac{\delta}{2}} \cap \Omega_R) \times (0, \tau_1].$$

There exists  $\tau_2 \in (0, \tau_1]$  such that

$$u(x, t) \leq \bar{u}_{\mu^*}(x, t) + \epsilon + w_{B_R}(x, t) \text{ for } (x, t) \in (\Omega_R \setminus \mathcal{S}_{\frac{\delta}{2}}) \times (0, \tau_2].$$

Therefore

$$u(x, t) \leq u_{\sigma,\delta}(x, t) + \bar{u}_{\mu^*}(x, t) + \epsilon + w_{B_R}(x, t) \text{ for } (x, t) \in \Omega_R \times (0, \tau_2].$$

Therefore

$$u \leq u_{\sigma,\delta} + u_{\mu^*} + \epsilon + w_{B_R} \quad \text{in } Q_\infty^{\Omega_R} \quad (6.57)$$

Letting successively  $R \rightarrow \infty$ ,  $\epsilon \rightarrow 0$ ,  $\delta \rightarrow 0$  and  $\sigma \rightarrow 0$  we obtain  $u \leq \bar{u}_{\mathcal{S},0} + \bar{u}_{\mu^*}$ , and therefore  $u \leq \pi_+(\bar{u}_{\mathcal{S},0} + \bar{u}_{\mu^*}) = \bar{u}_{\mathcal{S},\mu}$ . Next, we also have

$$u \geq \tilde{u}_{\mathcal{S},0} := \sup\{u_{\infty,a} : a \in \mathcal{S}\} \implies u \geq \underline{u}_{\mathcal{S},0} = \pi_+(\tilde{u}_{\mathcal{S},0}),$$



by (6.25). With the notations of Proposition 6.12 with  $F = \mathcal{S}$ , for any  $R > 0$ ,  $\delta > 0$  and  $\epsilon > 0$ , there exists  $\tau_\epsilon$  such that

$$u_{\tilde{\mu}_\delta}(x, t) \leq u(x, t) + w_{B_R}(x, t) + \epsilon \quad \text{in } \Omega_R \times (0, \tau_\epsilon],$$

because the support of  $\tilde{\mu}_\delta$  is included in  $\Omega \setminus \mathcal{S}_\delta$ . Therefore this last inequality holds in  $Q_\infty^{\Omega_R}$  and consequently

$$\sup\{u_{\tilde{\mu}_\delta}, \underline{u}_{\mathcal{S},0}\} \leq u + \epsilon + w_{B_R} \quad \text{in } Q_\infty^{\Omega_R},$$

and we can let  $R \rightarrow \infty$  and  $\epsilon \rightarrow 0$  to obtain  $\sup\{u_{\tilde{\mu}_\delta}, \underline{u}_{\mathcal{S},0}\} \leq u$  in  $Q_\infty^\Omega$ . Letting  $\delta \rightarrow 0$  and using Proposition 6.12 we get  $\sup\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{\mathcal{S},\mu} \leq u$ .

*Step 2: Alternative construction.* For  $0 < \delta < \sigma$  and  $n \in \mathbb{N}_*$  we denote by  $\bar{u}_{n,\sigma,\delta,\mu}$  the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^{\Omega_\sigma} \\ u &\geq 0 && \text{in } Q_\infty^{\Omega_\sigma} \\ u(x, t) &= 0 && \text{in } \partial_\ell Q_\infty^{\Omega_\sigma} \\ u(\cdot, 0) &= \mu_\delta + n\chi_{\mathcal{S}_\delta} && \text{in } \Omega_\sigma. \end{aligned} \tag{6.58}$$

We denote here  $\Omega_\sigma = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \sigma\}$ ,  $\mathcal{S}_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{S}) < \delta\}$  and  $\mathcal{R}_\delta = \Omega \cap \mathcal{S}_\delta^c$  and  $\mu_\delta = \chi_{\mathcal{R}_\delta} \mu$ . The same arguments of monotonicity as in Proposition 6.7 and Proposition 6.11 show that

$$\lim_{\sigma \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \bar{u}_{n,\sigma,\delta,\mu} = \bar{u}_{\mathcal{S},\mu}. \tag{6.59}$$

If  $\tau > 0$  we denote by  $\underline{u}_{\tau,\delta,\mu}$  the solution of

$$\begin{aligned} \partial_t u - \Delta u + t^\alpha u^q &= 0 && \text{in } Q_\infty^\Omega \\ u &\geq 0 && \text{in } Q_\infty^\Omega \\ u(x, t) &= 0 && \text{in } \partial_\ell Q_\infty^\Omega \\ u(\cdot, 0) &= \mu_\delta + \chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau) && \text{in } \Omega, \end{aligned} \tag{6.60}$$

Using estimate (6.53) and Proposition 6.9 is is easy to prove that  $\underline{u}_{\tau,\delta,\mu} \leq u$  for any  $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$ . Furthermore

$$\max\{u_{\mu_\delta}, u_{\chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)}\} \leq \underline{u}_{\tau,\delta,\mu} \leq u_{\mu_\delta} + \underline{u}_{\mathcal{S},0}(\cdot, \cdot + \tau),$$

since have

$$u_{\chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)} \leq \underline{u}_{\mathcal{S},0}(\cdot, \tau) \leq u_{\chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)} + C \left( \frac{\delta^2}{\tau} \right)^{\frac{1+\alpha}{q-1} - \frac{1}{2}} e^{-\frac{\delta^2}{4\tau}}$$

by (5.5) with  $N = 1$ . Set  $c(\delta, \tau) = C \left( \frac{\delta^2}{\tau} \right)^{\frac{1+\alpha}{q-1} - \frac{1}{2}} e^{-\frac{\delta^2}{4\tau}}$ , then

$$\max\{u_{\mu_\delta}, u_{\chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)}\} \geq \max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}(\cdot, \tau) - c(\delta, \tau)\} \geq \max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}(\cdot, \tau)\} - c(\delta, \tau).$$

Therefore, if  $\underline{u}_{\tau_n,\delta,\mu} \rightarrow \underline{u}_{0,\delta,\mu}$  locally uniformly in  $Q_\infty^\Omega$ , then  $\underline{u}_{0,\delta,\mu}$  is a solution of (1.6) in  $Q_\infty^\Omega$  which satisfies

$$\max\{u_{\mu_\delta}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{0,\delta,\mu} \leq u_{\mu_\delta} + \underline{u}_{\mathcal{S},0}, \tag{6.61}$$

and is smaller than any  $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$ . There exists  $\delta_n \rightarrow 0$  such that  $\underline{u}_{0,\delta_n,\mu} \rightarrow \underline{u}_{0,0,\mu}$ . Then

$$\max\{\underline{u}_{\mu^*}, \underline{u}_{\mathcal{S},0}\} \leq \underline{u}_{0,0,\mu} \leq \underline{u}_{\mu^*} + \underline{u}_{\mathcal{S},0}, \quad (6.62)$$

and  $\underline{u}_{0,0,\mu}$  is an element of  $\mathcal{U}_{\mathcal{S},0}(\Omega)$  smaller than any  $u \in \mathcal{U}_{\mathcal{S},0}(\Omega)$ . Thus  $\underline{u}_{0,0,\mu} = \underline{u}_{\mathcal{S},\mu}$  and

$$\lim_{\delta \rightarrow 0} \lim_{\tau \rightarrow 0} \underline{u}_{\tau,\delta,\mu} = \underline{u}_{\mathcal{S},\mu}. \quad (6.63)$$

*Step 3: Proof of (6.54).* We assume  $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} = \delta_0 > 0$ , so that we can take  $\sigma = 0$  in the construction of  $\bar{u}_{\mathcal{S},\mu}$ . Put  $\tau = (\frac{c\alpha}{n})^{\frac{q-1}{1+\alpha}}$  and

$$\bar{Z}_{n,\delta,\mu} = \bar{u}_{n,\delta,\mu} - \bar{u}_{n,\delta,0}, \quad \underline{Z}_{\tau,\delta,\mu} = \underline{u}_{\tau,\delta,\mu} - \underline{u}_{\tau,\delta,0} \quad \text{and} \quad W_{n,\delta,\mu} = \bar{Z}_{n,\delta,\mu} - \underline{Z}_{\tau,\delta,\mu}. \quad (6.64)$$

Then  $w = W_{n,\delta,\mu}$  satisfies

$$\partial_t w - \Delta w + t^\alpha \left( \bar{u}_{n,\delta,\mu}^q - \bar{u}_{n,\delta,0}^q - \underline{u}_{\tau,\delta,\mu}^q + \underline{u}_{\tau,\delta,0}^q \right) \quad (6.65)$$

in  $Q_\infty^\Omega$  and we can write

$$\begin{aligned} \bar{u}_{n,\delta,\mu}^q - \bar{u}_{n,\delta,0}^q - \underline{u}_{\tau,\delta,\mu}^q + \underline{u}_{\tau,\delta,0}^q &= (\bar{u}_{n,\delta,\mu}^q - \underline{u}_{\tau,\delta,\mu}^q) - (\bar{u}_{n,\delta,0}^q - \underline{u}_{\tau,\delta,0}^q) \\ &= d_\mu(\bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}) - d_0(\bar{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}) \end{aligned} \quad (6.66)$$

where

$$d_\mu(x, t) = \begin{cases} \frac{\bar{u}_{n,\delta,\mu}^q - \underline{u}_{\tau,\delta,\mu}^q}{\bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}} & \text{if } \bar{u}_{n,\delta,\mu} \neq \underline{u}_{\tau,\delta,\mu} \\ 0 & \text{if } \bar{u}_{n,\delta,\mu} = \underline{u}_{\tau,\delta,\mu} \end{cases} \quad (6.67)$$

and  $d_0$  is defined accordingly. Since

$$\bar{u}_{n,\delta,\mu} \geq \max\{\underline{u}_{\tau,\delta,\mu}, \bar{u}_{n,\delta,0}\} \quad \text{and} \quad \underline{u}_{\tau,\delta,0} \leq \min\{\underline{u}_{\tau,\delta,\mu}, \bar{u}_{n,\delta,0}\},$$

there holds  $d_\mu \geq d_0 \geq$  by convexity. Using the fact that  $\bar{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0} \geq 0$  is infers that

$$d_\mu(\bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu}) - d_0(\bar{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}) \geq d_\mu(\bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu} - \bar{u}_{n,\delta,0} + \underline{u}_{\tau,\delta,0})$$

Finally (6.65) becomes

$$\partial_t w - \Delta w + t^\alpha d_\mu w \leq 0 \quad \text{in } Q_\infty^\Omega. \quad (6.68)$$

Furthermore, in the sense of measures,

$$w(\cdot, 0) = \mu_\delta + n\chi_{\mathcal{S}_\delta} - n\chi_{\mathcal{S}_\delta} - (\mu_\delta + \chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau)) + \chi_{\mathcal{S}_\delta} \underline{u}_{\mathcal{S},0}(\cdot, \tau) = 0.$$

Because  $w = 0$  in  $\partial_\ell Q_\infty^\Omega$  it follows  $w \leq 0$  by the maximum principle. Therefore

$$\bar{Z}_{n,\delta,\mu} \leq \underline{Z}_{\tau,\delta,\mu} \implies \bar{u}_{n,\delta,\mu} - \underline{u}_{\tau,\delta,\mu} \leq \bar{u}_{n,\delta,0} - \underline{u}_{\tau,\delta,0}. \quad (6.69)$$

If we let successively  $n \rightarrow \infty$  (and therefore  $\tau \rightarrow 0$ ) and  $\delta \rightarrow 0$ , we obtain (6.54).  $\square$

*Remark.* We do not know if (6.54) holds if we do not assume  $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} = \delta_0 > 0$ . However, if for  $\theta > 0$  we set  $\mathcal{S}_\theta = \mathcal{S} \cap \{x \in \Omega : \text{dist}(x, \Omega^c) \geq \theta\}$ , then we have

$$\bar{u}_{\mathcal{S}_\theta, \mu} - \underline{u}_{\mathcal{S}_\theta, \mu} \leq \bar{u}_{\mathcal{S}_\theta, 0} - \underline{u}_{\mathcal{S}_\theta, 0}. \quad (6.70)$$

Furthermore all the four above functions increases when  $\theta$  decreases to 0. If we set

$$\begin{aligned} (i) & \quad \lim_{\theta \rightarrow 0} \bar{u}_{\mathcal{S}_\theta, \mu} = \bar{u}_{\mathcal{S}, \mu} \\ (ii) & \quad \lim_{\theta \rightarrow 0} \bar{u}_{\mathcal{S}_\theta, 0} = \bar{u}_{\mathcal{S}, 0} \\ (iii) & \quad \lim_{\theta \rightarrow 0} \underline{u}_{\mathcal{S}_\theta, \mu} = \underline{u}_{\mathcal{S}, \mu} \\ (iv) & \quad \lim_{\theta \rightarrow 0} \underline{u}_{\mathcal{S}_\theta, 0} = \underline{u}_{\mathcal{S}, 0} \end{aligned} \quad (6.71)$$

then we infer that

$$\bar{u}_{\mathcal{S}, \mu} - \underline{u}_{\mathcal{S}, \mu} \leq \bar{u}_{\mathcal{S}, 0} - \underline{u}_{\mathcal{S}, 0}. \quad (6.72)$$

Our final result is the following existence and uniqueness theorem the proof is similar to the one of [7, Th 3.5].

**Theorem 6.15** *Assume  $\Omega \subseteq \mathbb{R}^N$  is either  $\mathbb{R}^N$  or an open domain with a  $C^2$  compact boundary,  $\alpha > -1$  and  $1 < q < q_{c, \alpha}$ . Then for any  $\nu \approx (\mathcal{S}, \mu) \in \mathfrak{B}^{reg}(\Omega)$  such that  $\rho_\mu$  is bounded in any neighborhood of  $\partial\Omega$  and  $\inf\{|z - z'| : z \in \mathcal{S}, z' \in \Omega^c\} > 0$ , the set  $\mathcal{U}_{\mathcal{S}, \mu}(\Omega)$  contains one and only one element.*

## References

- [1] H. Brezis, L. A. Peletier, and D. Terman. A very singular solution of the heat equation with absorption. *Arch. Rational Mech. Anal.*, 95(3):185–209, 1986.
- [2] H. Brezis and A. Friedman. Nonlinear parabolic equations involving measures as initial conditions. *J. Math. Pures Appl. (9)*, 62(1):73–97, 1983.
- [3] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. II.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1989. Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.
- [4] M. Escobedo and O. Kavian. Variational problems related to self-similar solutions of the heat equation. *Nonlinear Anal.*, 11(10):1103–1133, 1987.
- [5] K. Gkikas and L. Véron. Initial value problems for diffusion equations with singular potential, *Contemporary Math.*, to appear.
- [6] S. Kamin and L. A. Peletier. Large time behaviour of solutions of the heat equation with absorption. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):393–408, 1985.
- [7] M. Marcus and L. Véron. Initial trace of positive solutions of some nonlinear parabolic equations. *Comm. Partial Differential Equations*, 24(7-8):1445–1499, 1999.
- [8] M. Marcus and L. Véron. Semilinear parabolic equations with measure boundary data and isolated singularities. *J. Anal. Math.*, 85:245–290, 2001.

- [9] M. Marcus and L. Véron. Initial trace of positive solutions to semilinear parabolic inequalities. *Advanced Nonlinear Studies*, 2:395–436, 2002.
- [10] M. Marcus and L. Véron. The boundary trace and generalized boundary value problem for semilinear elliptic equations with coercive absorption. *Comm. Pure Appl. Math.*, LVI:689-731, 2003.
- [11] I. Moutoussamy and L. Véron. Isolated singularities and asymptotic behaviour of the solutions of a semi-linear heat equation. *Asymptotic Anal.*, 9(3):259–289, 1994.
- [12] A. Shishkov and L. Véron. The balance between diffusion and absorption in semilinear parabolic equations *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei*,(9) Mat. Appl. 15: 138 (2006).
- [13] L. Véron. Geometric invariance of singular solutions of some nonlinear partial differential equations. *Indiana Univ. Math. J.*, 38(1):75–100, 1989.
- [14] L. Véron. Semilinear elliptic equations with uniform blow-up on the boundary. *J. Anal. Math.*, 59:231–250, 1992.
- [15] L. Véron. Singularities of some quasilinear equations. In *Nonlinear diffusion equations and their equilibrium states, II (Berkeley, CA, 1986)*, volume 13 of *Math. Sci. Res. Inst. Publ.*, pages 333–365. Springer, New York, 1988.