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Spread of visited sites of a random walk along the generations of a branching process

P. Andreatti, P. Debs *

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Abstract

In this paper we consider a null recurrent random walk in random environment on a super-critical Galton-Watson tree. We consider the case where the log-Laplace transform ψ of the branching process satisfies $\psi(1) = \psi'(1) = 0$ wherein G. Faraud, Y. Hu and Z. Shi in [9] show that, with probability one, the largest generation visited by the walk, until the instant n , is of the order of $(\log n)^3$. In [3] we prove that the largest generation entirely visited behaves almost surely like $\log n$ up to a constant. Here we study how the walk visits the generations $\ell = (\log n)^{1+\zeta}$, with $0 < \zeta < 2$. We obtain results in probability giving the asymptotic logarithmic behavior of the number of visited sites at a given generation. We prove that there is a phase transition at generation $(\log n)^2$ for the mean of visited sites until n returns to the origin. Also we show that the visited sites spread all over the tree until generation ℓ .

1 Introduction

We start giving an iterative construction of the environment. Let $(A_i, i \geq 1)$ a positive random sequence and N an independent \mathbb{N} -valued random variable following a distribution q , in other words $\mathbb{P}(N = i) = q_i$ for $i \in \mathbb{N}$. Let ϕ the root of the tree and $(A(\phi^i), i \leq N_\phi)$ an independent copy of $(A_i, i \leq N)$. Then, we draw N_ϕ children to ϕ : these individuals are the first generation. Each child ϕ_i is associated with the corresponding $A(\phi^i)$ and so on. At the n -th generation, for each individual x we pick $(A(x^i), i \leq N_x)$ an independent copy of $(A_i, i \leq N)$ where N_x is the number of child of x and $A(x^i)$ is the random variable attached to x^i . The set \mathbb{T} , consisting of the root and its descendants, forms a Galton-Watson tree (GW) of offspring distribution q and where each vertex $x \neq \phi$ is associated with a random variable $A(x)$.

We denote by $|x|$ the generation of x , \overleftarrow{x} the parent of x , and for convenience reasons we add $\overleftarrow{\phi}$, the parent of ϕ . The set of environments denoted \mathbf{E} is the set of all sequences $((A(x^i), i \leq N_x), x \in \mathbb{T})$, with P and E respectively the associated probability measure and expectation.

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We assume that the distribution of $(A_i, i \leq N)$ is non-degenerate and, to obtain a supercritical GW, that $E[N] > 1$. Moreover we add uniform ellipticity conditions

$$P - a.s \exists 0 < \varepsilon_0 < 1, \forall i, \varepsilon_0 \leq A_i \leq 1/\varepsilon_0, \quad (1.1)$$

$$\exists N_0 \in \mathbb{R}, P - a.s N \leq N_0. \quad (1.2)$$

Given $\mathcal{E} \in \mathbf{E}$, we define a \mathbb{T} -valued random walk $(X_n, n \in \mathbb{N})$ starting from ϕ by its transition probabilities,

$$p(x, x^i) = \frac{A(x^i)}{\sum_{j=1}^{N_x} A(x^j) + 1}, \quad p(x, \overleftarrow{x}) = 1 - \sum_{j=1}^{N_x} p(x, x^j), \quad p(\overleftarrow{\phi}, \phi) = 1.$$

Note that our construction implies that $(p(x, \cdot), x \in \mathbb{T})$ is an independent sequence. For all $\mathcal{E} \in \mathbf{E}$, we denote by $\mathbb{P}^{\mathcal{E}}$ the probability measure associated to this walk, the whole system is described under the probability \mathbb{P} , the semi-direct product of P and $\mathbb{P}^{\mathcal{E}}$.

To study asymptotical behaviours associated to $(X_n)_{n \in \mathbb{N}}$, a quantity appears naturally: the potential process V associated to the environment which is actually a branching random walk. It is defined by $V(\phi) := 0$ and

$$V(x) := - \sum_{z \in \llbracket \phi, x \rrbracket} \log A(z), \quad x \in \mathbb{T} \setminus \{\phi\},$$

where $\llbracket \phi, x \rrbracket$ is the set of vertices on the shortest path connecting ϕ to x and $\llbracket \phi, x \rrbracket = \llbracket \phi, x \rrbracket \setminus \{\phi\}$. We also introduce the moment-generating function

$$\psi(t) := \log E \left[\sum_{|x|=1} e^{-tV(x)} \right],$$

characterizing the environment. Note that the hypothesis we discuss above implies that ψ is defined on \mathbb{R} , and $\psi(0) > 0$.

Thanks to the work of M.V. Menshikov and D. Petritis, see [15] and the first part of [8] by G. Faraud, if

$$\psi(1) = \psi'(1) = 0. \quad (1.3)$$

then X is null recurrent. In [9], G. Faraud, Y. Hu, and Z. Shi study the asymptotic behavior of $\max_{0 \leq i \leq n} |X_i| = X_n^*$, i.e. the largest generation visited by the walk. Assuming (1.3), they prove the existence of a constant (known explicitly) $C_0 > 0$ such that \mathbb{P} a.s. on the set of non-extinction of the GW

$$\lim_{n \rightarrow +\infty} \frac{X_n^*}{(\log n)^3} = C_0. \quad (1.4)$$

In a previous paper [3] we were interested in the *largest generation entirely visited by the walk*, that is to say the behavior of $R_n := \sup\{k \geq 1, \forall |z| = k, \mathcal{L}(z, n) \geq 1\}$, with \mathcal{L} the local time of X defined by $\mathcal{L}(z, n) := \sum_{k=1}^n \mathbb{1}_{X_k=z}$. More precisely, if (1.3) is realized, \mathbb{P} a.s. on the set of non-extinction

$$\lim_{n \rightarrow +\infty} \frac{R_n}{\log n} = \frac{1}{\bar{\gamma}}, \quad (1.5)$$

where $\tilde{\gamma} := \sup\{a \in \mathbb{R}, \tilde{J}(a) > 0\}$ with $\tilde{J}(a) := \inf_{t \geq 0} \{\psi(-t) - at\}$.

Although in [3] all recurrent cases are treated, here we focus only on the hypothesis (1.3). According to (1.4) and (1.5), until generation $\log n / \tilde{\gamma}$ all the points are visited but X do not visit generation further than $C_0(\log n)^3$. The aim of this paper is to study the asymptotic of the number of visited sites at a given generation $(\log n)^{1+\zeta}$ with $0 < \zeta < 2$. For this purpose we define the number of visited sites at generation $m \in \mathbb{N}$

$$M_n(m) := \#\{|z| = m, \mathcal{L}(z, n) \geq 1\}.$$

We also introduce the number of visited sites before n returns to the root, more precisely, $K_n(\cdot) := M_{T_\phi^n}(\cdot)$ where $T_x^n = \inf\{k > T_x^{n-1}, X_k = x\}$ for $n \geq 1$ and $T_x^0 = 0$ for $x \in \mathbb{T}$.

Let Z_m the number of descendants at generation $m \in \mathbb{N}$, we have $Z_1 = N$. An obvious consequence of (1.5) is that a.s. for n large enough $M_n(R_n) \sim Z_{\log n / \tilde{\gamma}}$. Moreover $Z_{\log n / \tilde{\gamma}} e^{-\psi(0) \log n / \tilde{\gamma}}$ converge a.s. to a positive random variable, so for all $\varepsilon > 0$ a.s. for any n large enough $M_n(R_n) \geq n^{\psi(0)(1-\varepsilon)/\tilde{\gamma}}$.

Our first results quantify the number of visited points at a given generation $\ell = (\log n)^{1+\zeta}$,

Theorem 1.1 *For all $0 < \zeta < 2$, $\varepsilon > 0$ independent of ζ*

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(\frac{\psi(0)}{\tilde{\gamma}}(1 - \varepsilon) \leq \frac{\log M_n(\ell)}{\log n} \leq 1 - \mathcal{O} \left(\frac{\log \log n}{\log n} \vee \frac{1}{(\log n)^\zeta} \right) \right) = 1. \quad (1.6)$$

Also for all n large enough, there exist two positive constants C_1 and C_2 such that

$$C_1 n \frac{e^{-(\log n)^{1-\zeta}/2\sigma^2}}{(\log n)^{1/2(1+\zeta)+\varepsilon}} \leq \mathbb{E}[K_n(\ell)] \leq C_2 n \frac{e^{-(\log n)^{1-\zeta}/2\sigma^2}}{(\log n)^{1/2(1+\zeta)}}, \quad (1.7)$$

with $\sigma^2 := \psi''(1)$.

(1.6) shows that, at each generation ℓ , the cardinal of visited sites is at least $n^{\psi(0)(1-\varepsilon)/\tilde{\gamma}}$ for any ζ , that is to say like the last generation entirely visited R_n . Also the upper bound of $M_n(\ell)$ is at most of the order of $ne^{-C_3(\log n)^{1-\zeta}}/(\log n)^{C_4}$, with $C_3, C_4 > 0$. This suggests that it may have a phase transition when $\zeta = 1$ i.e at generation $(\log n)^2$. We were not able to show this for $M_n(\ell)$ but we do prove a phase transition for the mean of $K_n(\ell)$ which is the second result of the Theorem.

(1.7) shows that in the neighborhood of generation $(\log n)^2$, the asymptotic behavior of $\mathcal{N}_\zeta := \mathbb{E}[K_n(\ell)]$ changes. We easily check that for all $0 < \zeta < \zeta' < 1$, $\lim_{n \rightarrow +\infty} \mathcal{N}_{\zeta'}/\mathcal{N}_\zeta = +\infty$ and $\lim_{n \rightarrow +\infty} \mathcal{N}_1/\mathcal{N}_{\zeta'} = +\infty$, whereas for all $1 < \zeta < \zeta' < 2$, $\lim_{n \rightarrow +\infty} \mathcal{N}_{\zeta'}/\mathcal{N}_\zeta = 0$ and $\lim_{n \rightarrow +\infty} \mathcal{N}_\zeta/\mathcal{N}_1 = 0$. So the generations of order $(\log n)^2$ are, in mean, the most visited generation until n returns to the origin.

Our second result gives an idea of the way the visited points spread on the tree, for this purpose we introduce clusters: let $z \in \mathbb{T}$ and $m \geq |z|$, we call cluster issued from z at generation m denoted $\mathcal{C}_m(z)$, the set of descendants u of z such that $|u| = m$, in other words

$$\mathcal{C}_m(z) := \{u > z, |u| = m\}.$$

Also recall Neveu's notation for the individuals of a given generation of the tree: if A is a set of individuals we number them from the left to the right calling them $u_1, u_2, \dots, u_{|A|}$

and we denote $\mathbf{inf}\{u, u \in A\} := u_1$, and $\mathbf{sup}\{u, u \in A\} := u_{|A|}$.

At some point we need to quantify the number of individuals between two disjoint clusters with common generations. For given initial and terminal generations, let us denote \mathcal{C} be a set of disjoint clusters. Let $(\mathcal{D}_j, 1 \leq j \leq |\mathcal{C}|)$, with $|\mathcal{C}|$ the cardinal of \mathcal{C} , an ordered (disjoint) sequence of clusters belonging to \mathcal{C} , that is to say for all j , $\mathbf{inf}\{z, z \in \mathcal{D}_j\} \leq \mathbf{inf}\{z, z \in \mathcal{D}_{j+1}\}$ we define the minimal distance between clusters in the following way $\mathbf{D}(\mathcal{C}) := \min_{1 \leq j \leq |\mathcal{C}|-2} (\mathbf{inf}\{z, z \in \mathcal{D}_{j+2}\} - \mathbf{sup}\{z, z \in \mathcal{D}_j\})$. Notice that we do not look at two successive clusters, but two successive clusters separate by one. We now state a second result

Theorem 1.2 *For all $0 < \zeta < 2$, and $\varepsilon > 0$*

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(\max_{|z|=\ell-\psi(0)\log n/\tilde{\gamma}} \min_{y \in \mathcal{C}_\ell(z)} \mathcal{L}(y, n) \geq 1 \right) = 1, \quad (1.8)$$

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(\min_{z \in \mathcal{C}_{\varepsilon\ell^{1/3}(\phi)}} \max_{y > z, |y|=\ell} \mathcal{L}(y, n) \geq 1 \right) = 1. \quad (1.9)$$

Let k_n, h_n and r_n positive sequences of integers such that $k_n r_n + (k_n - 1)h_n = \ell$. For all $1 \leq i \leq k_n$, let us denote \mathcal{C}_i a set of clusters initiated at generation $(i-1)(r_n + h_n)$ and with end points at generation $ir_n + (i-1)h_n$ (see Figure 3), also define the following event for all $m > 0$ and $q > 0$

$$\mathcal{A}_i(m, q) := \bigcup_{\mathcal{C}_i} \left\{ \{|\mathcal{C}_i| \geq q, \mathbf{D}(\mathcal{C}_i) \geq m\} \bigcap_{\mathcal{D} \in \mathcal{C}_i} \{\forall z \in \mathcal{D}, \mathcal{L}(z, n) \geq 1\} \right\}.$$

There exist $0 < \mathbf{k} < 1$, and \mathbf{r} with $0 < \mathbf{k} + \mathbf{r} \leq 1$ such that $k_n = (\log n)^{\mathbf{k}}$, $r_n = (\log n)^{\mathbf{r}}$ and

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(\bigcap_{i=2}^{k_n} \mathcal{A}_i(e^{\psi(0)h_n/2}, e^{\psi(0)r_n(i-1)/2}) \right) = 1. \quad (1.10)$$

(1.8) tells that there exists a cluster of points starting at generation close to ℓ ($\ell - \psi(0)\log n/\tilde{\gamma}$) entirely visited (see also Figure 1). As $|\mathcal{C}_\ell(z)|$ is equal in law to $Z_{\ell-|z|} = Z_{\psi(0)\log n/\tilde{\gamma}}$, we notice that this cluster is large and that in particular (1.8) implies the lower bound in (1.6).

(1.9) tells that we can find visited individuals at generation $\ell = (\log n)^{1+\zeta}$, which have a common ancestor to a generation close to the root, that is to say before generation $\varepsilon\ell^{1/3}$ (see also Figure 4). This implies that (with a probability close to one) at least $e^{\varepsilon(1-\varepsilon)\psi(0)\ell^{1/3}/2}$ individuals of generation ℓ separate by at least $e^{\psi(0)\ell/2}$ individuals of the same generation ℓ , are visited.

Finally (1.10) tells that if we make cuts regularly on the tree we can find many visited clusters (which number increases when descending on the tree) well separated. In particular these visited clusters can not be in a same large visited clusters as they are separated by at least $e^{\psi(0)h_n/2} \sim e^{\psi(0)(\log n)^{1+\zeta-\mathbf{k}}}$ individuals (see also Figure 3). Moreover as \mathbf{k} can be taken smaller than ζ (see Section 4) $e^{\psi(0)(\log n)^{1+\zeta-\mathbf{k}}} \geq e^{\psi(0)(\log n)^{1+\varepsilon}} > n$.

To obtain these results we show that $K_n(\ell)$ can be linked to a random variable depending

only on the random environment and n . For all integer k and real a , let us define the random variable

$$\mathcal{R}_a(k) := \#\{|z| = k, \bar{V}(z) \leq a\}.$$

When $k = \ell$ and $a = \Phi(n)$, such that for small $\varepsilon > 0$ $(1 - \varepsilon) \log n \leq \Phi(n) \leq \log n + o(\log n)$, we call the quantity $\mathcal{R}_{\Phi(n)}(\ell)$ number of *accessible points* of generation ℓ . We obtain the following

Proposition 1.3 *For all $0 < \zeta < 2$ and $\varepsilon > 0$,*

$$\lim_{n \rightarrow +\infty} P \left(\frac{\psi(0)}{\tilde{\gamma}} (1 - \varepsilon) \leq \frac{\log \mathcal{R}_{\Phi(n)}(\ell)}{\Phi(n)} \leq 1 - \mathcal{O} \left(\frac{\log \log n}{\Phi(n)} \vee \frac{\Phi(n)}{(\log n)^{1+\zeta}} \right) \right) = 1, \quad (1.11)$$

$$E[\mathcal{R}_{\Phi(n)}(\ell)] \asymp \Phi(n) \frac{e^{\Phi(n)(1-\Phi(n)/2\sigma^2\ell)}}{\ell^{3/2}}. \quad (1.12)$$

We use the notation $a_n \asymp b_n$ when there exists two positive constants c_1 and c_2 such that $c_1 b_n \leq a_n \leq c_2 b_n$ for all n large enough. The lack of precision for the first result shows no difference between $\mathcal{R}_{\log n}(\ell)$ and $M_n(\ell)$ (see (1.6)), whereas we see differences between the means of $\mathcal{R}_{\log n}(\ell)$ and of $K_n(\ell)$.

The rest of the paper is organized as follow: in Section 2 we study $\mathcal{R}_{\Phi(n)}(\ell)$ and prove Proposition 1.3. In Section 3 we link $\mathcal{R}_{\Phi(n)}(\ell)$ and $M_n(\ell)$, which leads to Theorem 1.1 and (1.8) of Theorem 1.2. In Section 4 we prove the end of Theorem 1.2. Also we add an appendix where we state known results on branching processes and local limit theorems for sums of i.i.d. random variables.

Note that for typographical simplicity, we do not distinguish a real number and its integer part throughout the article.

2 Expectation and bounds of $\mathcal{R}_{\Phi(n)}(\ell)$

In this section we only work with the environment more especially with the number of *accessible points* $\mathcal{R}_{\Phi(n)}(\ell)$.

2.1 Expectation of $\mathcal{R}_{\Phi(n)}(\ell)$ (proof of (1.12))

According to Biggins-Kyprianou identity (see part A of appendix), as $E[\mathcal{R}_{\Phi(n)}(\ell)] = E \left[e^{S_\ell} \mathbf{1}_{\bar{S}_\ell \leq \Phi(n)} \right]$ where S_j is a centered random walk, we only have to prove

Lemma 2.1 *For all $0 < \zeta < 2$*

$$E \left[e^{S_\ell} \mathbf{1}_{\bar{S}_\ell \leq \Phi(n)} \right] \asymp \Phi(n) \frac{e^{\Phi(n)(1-\Phi(n)/2\sigma^2\ell)}}{\ell^{3/2}}$$

Proof.

For every sequence $(u_n)_{n \in \mathbb{N}}$, we denote $\bar{u}_j := \max_{1 \leq i \leq j} u_i$ and $\underline{u}_j := \min_{1 \leq i \leq j} u_i$, also let $\mathcal{S}_j := \{\bar{S}_{j-1} < S_j = \bar{S}_j\}$. First, as $S_0 = 0$

$$\sum_{j=1}^{\ell} E \left[e^{S_\ell} \mathbf{1}_{S_j \leq \Phi(n), \mathcal{S}_j} \right] \leq E \left[e^{S_\ell} \mathbf{1}_{\bar{S}_\ell \leq \Phi(n)} \right] \leq \sum_{j=1}^{\ell} E \left[e^{S_\ell} \mathbf{1}_{S_j \leq \Phi(n), \mathcal{S}_j} \right] + 1.$$

Let $\tilde{S}_i := S_j - S_{j-i}$, with this notation $\{\bar{S}_{j-1} < S_j\} = \{\underline{S}_j > 0\}$ and $\tilde{S}_j = S_j$. Writing S as a sum of i.i.d. random variables, we easily see that $S \stackrel{\mathcal{L}}{=} \tilde{S}$. Then conditioning on $\sigma\{S_k, k \leq j\}$

$$E \left[e^{S_\ell} \mathbb{1}_{S_j \leq \Phi(n), \underline{S}_j} \right] = D_j F_{\ell-j} \quad (2.1)$$

with $F_m := E \left[e^{S_m} \mathbb{1}_{\bar{S}_m \leq 0} \right]$ and $D_j := E \left[e^{S_j} \mathbb{1}_{S_j \leq \Phi(n), \underline{S}_j > 0} \right]$.

According to Remark A.1, there exists $\alpha := |\log \varepsilon_0| > 0$ such that $P(|S_1| \leq \alpha) = 1$. So for $j \leq \sqrt{\Phi(n)}$, $S_j \leq \alpha \sqrt{\Phi(n)}$ a.s and $\sum_{j=1}^{\sqrt{\Phi(n)}} D_j F_{\ell-j} \leq \sqrt{\Phi(n)} e^{\alpha \sqrt{\Phi(n)}}$.

For the rest of the proof we assume that $j > \sqrt{\Phi(n)}$. In [11] p.44, we find that for all $j \leq \ell$

$$F_{\ell-j} \asymp (\ell - j + 1)^{-3/2}, \quad (2.2)$$

then it remains to estimate D_j , first

$$D_j = E \left[e^{S_j} \mathbb{1}_{0 \leq S_j \leq (\varepsilon j) \wedge \Phi(n), \underline{S}_j > 0} \right] + E \left[e^{S_j} \mathbb{1}_{(\varepsilon j) \wedge \Phi(n) < S_j \leq (\alpha j) \wedge \Phi(n), \underline{S}_j > 0} \right] =: G_j + H_j.$$

Thus using Lemma B.4, for all n large enough

$$\begin{aligned} G_j &= \sum_{k=0}^{(\varepsilon j) \wedge \Phi(n)} E \left[e^{S_j} \mathbb{1}_{k \leq S_j < k+1, \underline{S}_j > 0} \right] \asymp \sum_{k=0}^{(\varepsilon j) \wedge \Phi(n)} \frac{k e^k}{j^{3/2}} e^{-k^2/(2j\sigma^2)} \\ &\asymp (\varepsilon j) \wedge \Phi(n) e^{(\varepsilon j) \wedge \Phi(n) \left(1 - \frac{(\varepsilon j) \wedge \Phi(n)}{2\sigma^2 j}\right)} / j^{3/2}. \end{aligned}$$

So, as $\Phi(n)/\ell \leq \varepsilon$, for n large enough

$$\sum_{j=\sqrt{\Phi(n)+1}}^{\ell} G_j F_{\ell-j} \leq C_+ \Phi(n) e^{\Phi(n) \left(1 - \frac{\Phi(n)}{2\ell\sigma^2}\right)} \sum_{j=1}^{\ell} F_{\ell-j} / j^{3/2} \leq C_+ \Phi(n) e^{\Phi(n) \left(1 - \frac{\Phi(n)}{2\ell\sigma^2}\right)} / \ell^{3/2},$$

where C_+ is a positive constant that may grow from line to line. The lower bound is an easy task as $\sum_{j=\sqrt{\Phi(n)+1}}^{\ell} G_j F_{\ell-j} \geq G_\ell F_0$, finally

$$\sum_{j=1}^{\ell} G_j F_{\ell-j} \asymp \Phi(n) e^{\Phi(n) \left(1 - \frac{\Phi(n)}{2\ell\sigma^2}\right)} \ell^{-3/2}. \quad (2.3)$$

To conclude, we show that $\sum H_j F_{\ell-j}$ is negligible compared to $\sum G_j F_{\ell-j}$. We have

$$H_j \leq \sum_{k=(\varepsilon j) \wedge \Phi(n)+1}^{(\alpha j) \wedge \Phi(n)} e^k P(S_j \geq k, \underline{S}_j > 0) =: \tilde{H}_j.$$

For $j \leq \Phi(n)/\alpha$ let $s > 0$, with exponential Markov inequality

$$\tilde{H}_j \leq C_+ e^{j\psi(1-s)} \sum_{k=0}^{\alpha j} e^{k(1-s)} \leq C_+ e^{j(\psi(1-s) + (1-s)\alpha)}.$$

Remark A.2 implies that we can choose $s > 0$ small enough, such that $\psi(1-s) \leq \alpha s/2$. As for $j \leq \Phi(n)/\alpha$, $F_{\ell-j} \leq C_+(\ell-j+1)^{-3/2} \leq 2C_+\ell^{-3/2}$

$$\sum_{j=1}^{\Phi(n)/\alpha} \tilde{H}_j F_{\ell-j} \leq C_+ \frac{e^{(1-s/2)\Phi(n)}}{\ell^{3/2}}. \quad (2.4)$$

For $\Phi(n)/\alpha < j \leq \ell$ we do the same, for all $t > 0$ using again Lemma (B.4)

$$\begin{aligned} \sum_{j=\Phi(n)/\alpha+1}^{\ell} \tilde{H}_j F_{\ell-j} &= \sum_{j=\Phi(n)/\alpha+1}^{\Phi(n)/\varepsilon} \tilde{H}_j F_{\ell-j} \leq C_+ \sum_{j=\Phi(n)/\alpha+1}^{\Phi(n)/\varepsilon} F_{\ell-j} e^{j\psi(1-t)} \sum_{k=(\varepsilon j) \wedge \Phi(n)}^{\Phi(n)} e^{(1-t)k} \\ &\leq C_+ \sum_{j=\Phi(n)/\alpha+1}^{\Phi(n)/\varepsilon} \frac{e^{\Phi(n)(1-t)+j\psi(1-t)}}{(\ell-j+1)^{3/2}} \leq \frac{C_+ e^{\Phi(n)((1-t)+\psi(1-t)/\varepsilon)}}{\ell^{3/2}}. \end{aligned}$$

Using again Remark A.2 with $t = \varepsilon/\sigma^2 \sum_{j=\Phi(n)/\alpha+1}^{\ell} \tilde{H}_j F_{\ell-j} \leq C_+ e^{\Phi(n)(1-\varepsilon/(2\sigma^2))} \ell^{-3/2}$. Collecting the estimate from (2.3), the main contribution in (2.1) is given by (2.3). ■

2.2 Bounds for $\log \mathcal{R}_{\Phi(n)}(\ell)$ (proof of (1.11))

The upper bound is a direct consequence of Markov inequality and (1.12).

For the lower bound, we first need an estimation on the deviation of $\min_{|z|=m} \bar{V}(m)$, this point has been studied in details in [9], we recall their result below

Proposition 2.2 [9] *Let a_n a positive sequence such that $a_n \sim n^{1/3}$, there exists $b_0 > 0$ such that for any $0 < b \leq b_0$*

$$\lim_{n \rightarrow +\infty} \frac{1}{a_n} \log P \left(\min_{|z|=n} \bar{V}(x) \leq ba_n \right) = b - b_0. \quad (2.5)$$

We now give a short proof of the following Lemma, which is a useful corollary of the above Proposition

Lemma 2.3 *Assume that a_n is a positive sequence such that $a_n \sim n^{1/3}$, there exists a constant $\mu > 0$ such that for any n large enough*

$$P \left(\min_{|z|=n} \bar{V}(x) > \mu a_n \right) \leq \lambda_n, \quad (2.6)$$

where $\lambda_n = e^{-c_1 e^{c_1 a_n}}$ if $q_0 + q_1 = 0$, and $\lambda_n = e^{-c_1 a_n}$ otherwise, also $c_1 > 0$ depends only on the distribution P .

Proof.

Clearly for $z_1 < z$, $\bar{V}(z) \leq \bar{V}(z_1) + \tilde{V}(z_1, z)$ where $\tilde{V}(z_1, z) = \max_{z_1 \leq x \leq z} V(x) - V(z_1)$. In the sequel, writing $\tilde{V}(z_1, z)$ implies that $z_1 < z$ implicitly. For $0 < \eta < 1$ and $|z_1| = v_n := \eta b_0 a_n / \alpha$

$$P \left(\min_{|z|=n} \bar{V}(z) > 2\eta b_0 a_n \right) \leq P \left(\min_{|z_1|=v_n} \min_{|z|=n} \tilde{V}(z_1, z) > 2\eta b_0 a_n - \max_{|z_1|=v_n} \bar{V}(z_1) \right).$$

Using that $\max_{|z_1|=v_n} \bar{V}(z_1) \leq \alpha v_n$ by ellipticity and for $\mathcal{A}_n := \{Z_{v_n} \geq e^{\eta\psi(0)v_n}\}$

$$\begin{aligned} P\left(\min_{|z|=n} \bar{V}(z) > 2\eta b_0 a_n\right) &\leq P\left(\min_{|z_1|=v_n} \min_{|z|=n} \tilde{V}(z_1, z) > \eta b_0 a_n\right) \\ &\leq P\left(\min_{|z_1|=v_n} \min_{|z|=n} \tilde{V}(z_1, z) > \eta b_0 a_n, \mathcal{A}_n\right) + P(\bar{\mathcal{A}}_n). \end{aligned}$$

Theorem A.3 tells that if $q_0 + q_1 > 0$, there exists $\nu > 0$ such that $P(\bar{\mathcal{A}}_n) \leq e^{-\nu(1-\eta)\psi(0)v_n}$, otherwise there exists $\beta' > 0$ such that $\log P(\bar{\mathcal{A}}_n) \sim -e^{\beta'(1-\eta)\psi(0)v_n}$. Stationarity gives $\min_{|z|=n} \tilde{V}(z_1, z) \stackrel{\mathcal{L}}{=} \min_{|z|=n-v_n} \bar{V}(z)$ and independence of the sub-branching processes rooted at generation v_n together with (2.5)

$$\begin{aligned} P\left(\min_{|z_1|=v_n} \min_{|z|=n} \tilde{V}(z_1, z) > \eta b_0 a_n, \mathcal{A}_n\right) &\leq P\left(\min_{|z|=n-v_n} \bar{V}(z) > \eta b_0 a_n\right)^{e^{\eta\psi(0)v_n}} \\ &\leq \left(1 - e^{-b_0(1-\eta)a_n}\right)^{e^{\eta\psi(0)v_n}}, \end{aligned}$$

we conclude choosing η sufficiently close to 1 to get $(1-\eta) < \eta^2\psi(0)/\alpha$. ■

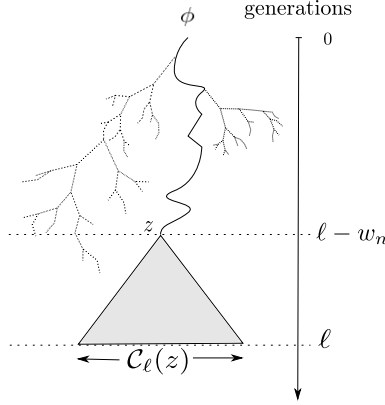


Figure 1: One large cluster

To obtain the lower bound for $\log \mathcal{R}_{\Phi(n)}(\ell)$, we prove the existence of a cluster $\mathcal{C}_\ell(z)$ with $|z| = \ell - w_n$ where $w_n := \Phi(n)(1-\varepsilon)/\tilde{\gamma}$ and such that $\forall z' \in \mathcal{C}_\ell(z)$, $\bar{V}(z') \leq \Phi(n)$. In other words for $|z| < \ell$, Z_ℓ^z the number of descendants of z at generation ℓ , we prove

$$\lim_{n \rightarrow +\infty} P\left(\bigcup_{z=\ell-w_n} \{\#\{z' \in \mathcal{C}_\ell(z), \bar{V}(z') \leq \Phi(n)\} = Z_\ell^z\}\right) = 1,$$

which implies with the help of Theorem A.3 that

$$\lim_{n \rightarrow +\infty} P\left(\mathcal{R}_{\Phi(n)}(\ell) \geq e^{\psi(0)w_n(1-\varepsilon)}\right) = 1.$$

Let $\mathcal{B} := \bigcup_{|z|=\ell-w_n} \{\bar{V}(z) \leq y_n, \mathcal{R}_{\Phi(n)-y_n}^z(\ell) = Z_\ell^z\}$ where $\mathcal{R}_A^z(\ell) := \#\{z' > z, |z'| = \ell,$

$\tilde{V}(z, z') \leq A\}$ and $y_n := \mu\ell^{1/3}$

$$\begin{aligned} P(\mathcal{B}) &\geq P\left(\{\mathcal{R}_{y_n}(\ell - w_n) \geq 1\} \cap \bigcup_{|z|=\ell-w_n, \bar{V}(z) \leq y_n} \{\mathcal{R}_{\Phi(n)-y_n}^z(\ell) = Z_\ell^z\}\right) \\ &= \sum_{k \geq 1} P(\mathcal{R}_{y_n}(\ell - w_n) = k) P\left(\bigcup_{|z|=\ell-w_n, \bar{V}(z) \leq y_n} \{\mathcal{R}_{\Phi(n)-y_n}^z(\ell) = Z_\ell^z\} \middle| \mathcal{R}_{y_n}(\ell - w_n) = k\right). \end{aligned}$$

Let us denote z_1, \dots, z_k, \dots the ordered points at generation $\ell - w_n$ satisfying $\bar{V}(z_i) \leq y_n$. Conditionally on $\{\mathcal{R}_{y_n}(\ell - w_n) = k\}$, z_1 exists and

$$\{\mathcal{R}_{\Phi(n)-y_n}^{z_1}(\ell) = Z_\ell^{z_1}\} \subset \bigcup_{|z|=\ell-w_n, \bar{V}(z) \leq y_n} \{\mathcal{R}_{\Phi(n)-y_n}^z(\ell) = Z_\ell^z\}.$$

Furthermore, by stationarity $\mathcal{R}_{\Phi(n)-y_n}^{z_1}(\ell) \stackrel{L}{=} \mathcal{R}_{\Phi(n)-y_n}(w_n)$, so

$$\begin{aligned} P(\mathcal{B}) &\geq P(\mathcal{R}_{\Phi(n)-y_n}(w_n) = Z_{w_n}) \sum_{k \geq 1} P(\mathcal{R}_{y_n}(\ell - w_n) = k) \\ &\geq P\left(\max_{|z|=w_n} \bar{V}(z) \leq \Phi(n) - y_n\right) P\left(\min_{|z|=\ell-w_n} \bar{V}(z) \leq y_n\right) \end{aligned}$$

The first probability tends to one thanks to a result of Mac-Diarmid [14] (see also [3] Lemma 2.1), the second one behaves in the same way as a consequence of Lemma 2.3. ■

3 Expectation of $K_n(\ell)$, bounds for $\log K_n(\ell)$ and $\log M_n(\ell)$

In this section we prove Theorem 1.1.

3.1 The expectation of $K_n(\ell)$ (proof of 1.7)

We start with general upper and lower bounds for the annealed expectation of $K_n(\ell)$.

Lemma 3.1 *For $n \in \mathbb{N}$:*

$$C_-(nA_n^- + B_n^-) \leq \mathbb{E}[K_n(\ell)] \leq C_+(nA_n^+ + B_n^+)$$

where

$$\begin{aligned} A_n^+ &:= E\left(e^{S_\ell - \bar{S}_\ell} \mathbf{1}_{\sum_{i=1}^\ell e^{S_i} > c_- n / c_+}\right), B_n^+ := E\left(e^{S_\ell} \mathbf{1}_{\bar{S}_\ell \leq \log(n/c_+)}\right), \\ A_n^- &:= E\left(\frac{e^{S_\ell}}{\sum_{i=1}^\ell e^{S_i}} \mathbf{1}_{\bar{S}_\ell > \log(n/c_+)}\right) \text{ and } B_n^- := E\left(e^{S_\ell} \mathbf{1}_{\sum_{i=1}^\ell e^{S_i} \leq c_- n / c_+}\right), \end{aligned}$$

C_- and c_- (respectively c_+) are positive constants that may decrease (respectively increase) from line to line.

Proof.

Let $p_z := \mathbb{P}_\phi^\mathcal{E}(T_z < T_\phi)$, strong Markov property gives $\mathbb{E}^\mathcal{E}[K_n(\ell)] = \sum_{|z|=\ell} (1 - e^{n \log(1-p_z)})$, so for n large enough

$$\frac{np_z}{4} \mathbb{1}_{np_z < 3/4} + C_- \mathbb{1}_{np_z \geq 3/4} \leq 1 - e^{-np_z} \leq 2np_z \mathbb{1}_{np_z < 3/4} + \mathbb{1}_{np_z \geq 3/4}.$$

Using successively the fact that $c_- (\sum_{x \in \llbracket \phi, z \rrbracket} e^{V(x)})^{-1} \leq p_z \leq c_+ e^{-\bar{V}(z)}$ and Biggins-Kyprianou identity (see Appendix A.1)

$$B_n^- \leq E \left[\sum_{|z|=\ell} \mathbb{1}_{np_z \geq 3/4} \right] \leq B_n^+.$$

Similar arguments show $C_- A_n^- \leq E \left[\sum_{|z|=\ell} p_z \mathbb{1}_{np_z < 3/4} \right] \leq C_+ A_n^+$. \blacksquare

We now give upper bounds for B_n^+ and A_n^+ , and a lower bound for A_n^- .

- For B_n^+ , Lemma 2.1 yields $B_n^+ \leq C_+ n \log n e^{-\frac{(\log n)^{1-\zeta}}{2\sigma^2}} \ell^{-3/2}$.
- For A_n^+ , first note that $\{\sum_{i=1}^\ell e^{S_i} > c_- n / c_+\} \subset \{\bar{S}_\ell > d_n\}$, with $d_n = \log(c_- n / c_+ \ell)$. Recalling the arguments given in (2.1), A_n^+ is bounded from above by

$$E \left[e^{S_\ell - \bar{S}_\ell} \mathbb{1}_{\bar{S}_\ell > d_n} \right] = \sum_{j=d_n/\alpha}^\ell P(S_j > d_n, \underline{S}_j > 0) E \left[e^{S_\ell - j} \mathbb{1}_{\bar{S}_{\ell-j} \leq 0} \right] =: \sum_{j=d_n/\alpha}^\ell L_j F_{\ell-j}. \quad (3.1)$$

We split L_j in two cases writing

$$L_j = P(d_n < S_j \leq j / \log \log n, \underline{S}_j > 0) + P(S_j > j / \log \log n, \underline{S}_j > 0) := P_1 + P_2.$$

and following the same lines as the ones to estimate G_j in the proof of Lemma 2.1

$$P_1 \asymp e^{-d_n^2 / (2\sigma^2 j)} j^{-1/2}.$$

Also exponential Markov inequality with $s = (\sigma^2 \log \log n)^{-1}$ yields $P_2 \leq P(s S_j > s j / \log \log n) \leq e^{-j / (2\sigma^2 \log \log n)} j^{-1/2}$. Collecting P_1, P_2 , (2.2) and (3.1)

$$A_n^+ \leq E \left(e^{S_\ell - \bar{S}_\ell} \mathbb{1}_{\bar{S}_\ell > d_n} \right) \asymp \ell^{-1/2} e^{-d_n^2 / (2\sigma^2 \ell)} \sim \ell^{-1/2} e^{-(\log n)^{1-\zeta} / 2\sigma^2}.$$

- For A_n^- , with $\mathcal{B}_\ell := \{\sum_{i=1}^\ell e^{S_i} \leq \ell^\varepsilon e^{\bar{S}_\ell}\}$, and $b_n := \log(n/c_+)$

$$\ell^\varepsilon A_n^- \geq E \left[e^{S_\ell - \bar{S}_\ell} \mathbb{1}_{\bar{S}_\ell > b_n, \mathcal{B}_\ell} \right] = E \left[e^{S_\ell - \bar{S}_\ell} \left(\mathbb{1}_{\bar{S}_\ell > b_n} - \mathbb{1}_{\bar{S}_\ell > b_n, \bar{\mathcal{B}}_\ell} \right) \right] := \Gamma_1 - \Gamma_2. \quad (3.2)$$

Γ_1 can be treated as $E \left[e^{S_\ell - \bar{S}_\ell} \mathbb{1}_{\bar{S}_\ell > d_n} \right]$ so

$$\Gamma_1 \asymp \ell^{-1/2} e^{-(\log n)^{1-\zeta} / 2\sigma^2}. \quad (3.3)$$

Recalling that $\mathcal{S}_j = \{S_j = \bar{S}_\ell, \bar{S}_{j-1} < S_j\}$, $\Gamma_2 = \sum_{j=1}^\ell E \left[e^{S_\ell - S_j} \mathbb{1}_{S_j > b_n, \bar{\mathcal{B}}_\ell, \mathcal{S}_j} \right]$. Note that on \mathcal{S}_j , $\bar{\mathcal{B}}_\ell = \{Y_1(j) + Y_2(j) > \ell^\varepsilon\}$ where $Y_1(j) := \sum_{i=1}^j e^{S_i - S_j}$ and $Y_2(j) := \sum_{i=j+1}^\ell e^{S_i - S_j}$. As for $0 < \delta < 1/2$, $\bar{\mathcal{B}}_\ell \subset \{Y_1(j) > \delta \ell^\varepsilon\} \cup \{Y_2(j) > \delta \ell^\varepsilon\}$

$$\Gamma_2 \leq \sum_{j=b_n/\alpha}^\ell (\Pi_j + \Omega_j) := \sum_{j=b_n/\alpha}^\ell E \left[e^{S_\ell - S_j} \mathbb{1}_{S_j > b_n, \mathcal{S}_j} \left(\mathbb{1}_{Y_1(j) > \delta \ell^\varepsilon} + \mathbb{1}_{Y_2(j) > \delta \ell^\varepsilon} \right) \right] \quad (3.4)$$

For Ω_j , conditioning by $\sigma(S_k, 0 \leq k \leq j)$

$$\Omega_j = P(\mathcal{H}_j) E \left[e^{S_{\ell-j}} \mathbf{1}_{\bar{S}_{\ell-j} \leq 0, Y^+(\ell-j) > \delta \ell^\varepsilon} \right]$$

where $Y^\pm(k) := \sum_{i=1}^k e^{\pm S_i}$ and $\mathcal{H}_m := \{S_m > b_n, \underline{S}_m > 0\}$. Moreover, using (2.2) and the fact that $\{Y^+(\ell-j) > \delta \ell^\varepsilon\} = \emptyset$ for $j \leq \ell - \delta \ell^\varepsilon$

$$E \left[e^{S_{\ell-j}} \mathbf{1}_{\bar{S}_{\ell-j} \leq 0, Y^+(\ell-j) > \delta \ell^\varepsilon} \right] \leq C_+ (\ell - j + 1)^{-3/2} \mathbf{1}_{\ell-j \geq \delta \ell^\varepsilon},$$

and $P(\mathcal{H}_j)$ can be treated like L_j , we obtain for n large enough and $j \geq b_n/\alpha$

$$\Omega_j \leq \frac{C_+}{j^{1/2}} \left(e^{-b_n^2/(2\sigma^2 j)} + e^{-j/(2\sigma^2 \log \log n)} \right) (\ell - j + 1)^{-3/2} \mathbf{1}_{\ell-j \geq \delta \ell^\varepsilon}. \quad (3.5)$$

For Π_j , we have $\Pi_j = P(\mathcal{H}_j, Y^-(j) > \delta \ell^\varepsilon) F_{\ell-j}$. Let $R := \sup\{0 < k \leq j, S_k \leq \gamma \log \ell\}$ with $\gamma > 1$, $\tau_x^+ := \inf\{k > 0, S_k \geq x\}$ for $x \in \mathbb{R}$ and $t_n = (\log \ell)^2$, then

$$\Pi_j = (P(\mathcal{H}_j, Y^-(j) > \delta \ell^\varepsilon, R \leq \tau_{t_n}^+) + P(\mathcal{H}_j, Y^-(j) > \delta \ell^\varepsilon, R > \tau_{t_n}^+)) F_{\ell-j} =: \Gamma_3 + \Gamma_4.$$

– Upper bound for Γ_3 , let $\tau_x^- := \inf\{k > 0, S_k \leq x\}$ for $x \in \mathbb{R}$. Notice that on $\{R \leq \tau_{t_n}^+\}$,

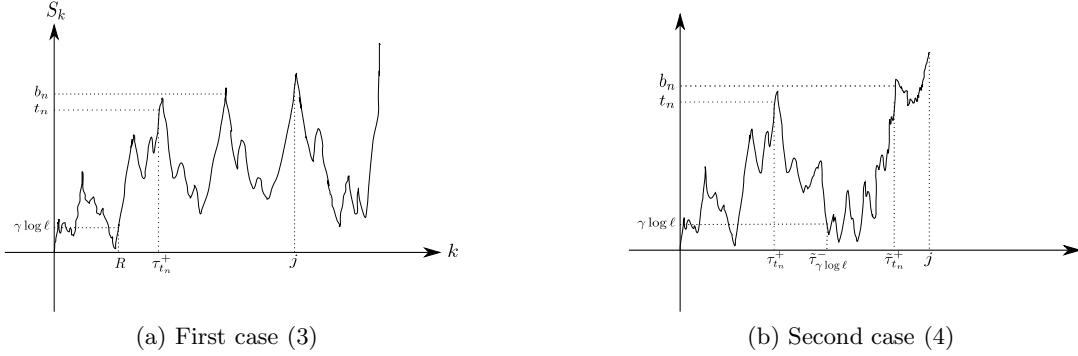


Figure 2: Two cases

$\sum_{k=\tau_{t_n}^++1}^j e^{-S_k} \leq \ell^{1-\gamma}$ implying that $\{Y^-(j) > \delta \ell^\varepsilon\} \subset \{Y^-(\tau_{t_n}^+) > \delta \ell^\varepsilon/2\}$. Thus, using strong Markov property

$$\begin{aligned} P(\mathcal{H}_j, Y^-(j) > \delta \ell^\varepsilon, R \leq \tau_{t_n}^+) &\leq P(\mathcal{H}_j, Y^-(\tau_{t_n}^+) > \delta \ell^\varepsilon/2, R \leq \tau_{t_n}^+) \\ &\leq \sum_{k=\tau_{t_n}^+/\alpha}^j P(\mathcal{H}_j, Y^-(\tau_{t_n}^+) > \delta \ell^\varepsilon/2, \tau_{t_n}^+ = k) \\ &\leq \sum_{k=\tau_{t_n}^+/\alpha}^j \sup_{0 \leq z \leq \alpha} P_{z+t_n}(\mathcal{H}_{j-k}) P(Y^-(\tau_{t_n}^+) > \delta \ell^\varepsilon/2, \tau_0^- > \tau_{t_n}^+ = k). \end{aligned} \quad (3.6)$$

From here we distinguish the cases $\zeta < 1$ and $\zeta \geq 1$.

When $\zeta < 1$, (3.6) is smaller than

$$P(Y^-(\tau_{t_n}^+) > \delta\ell^\varepsilon/2, \tau_0^- > \tau_{t_n}^+) \sup_{t_n/\alpha \leq k \leq j} \sup_{z \in [0, \alpha]} P_{z+t_n}(\mathcal{H}_{j-k}).$$

Lemma B.2 gives an upper bound for the first probability, moreover for all $t_n/\alpha \leq k \leq j$, with the help of Lemma B.3

$$\sup_{z \in [0, \alpha]} P_{z+t_n}(\mathcal{H}_{j-k}) \leq C_+ \frac{t_n}{j^{1/2}} e^{-b_n^2/(2\sigma^2 j)}.$$

So $\Gamma_3 \leq C_+ t_n j^{-1/2} \ell^{-\varepsilon} e^{-b_n^2/(2\sigma^2 j)} (\ell - j + 1)^{-3/2}$.

When $\zeta \geq 1$, (3.6) is now bounded from above by

$$\begin{aligned} & P(Y^-(\tau_{t_n}^+) > \delta\ell^\varepsilon/2, \tau_0^- > \tau_{t_n}^+) \sup_{\frac{t_n}{\alpha} \leq k \leq j/2} \sup_{z \in [0, \alpha]} P_{z+t_n}(\mathcal{H}_{j-k}) + P(\tau_0^- \wedge \tau_{t_n}^+ \geq j/2) \\ & \leq P(Y^-(\tau_{t_n}^+) > \delta\ell^\varepsilon/2, \tau_0^- > \tau_{t_n}^+) P_{t_n+\alpha}(\underline{S}_{j/2} > 0) + 2E[\tau_0^- \wedge \tau_{t_n}^+]/j \leq \frac{C_+ t_n}{\ell^\varepsilon j^{1/2}} + \frac{C_+ t_n}{j}, \end{aligned}$$

using Lemmata B.1 and B.2. So

$$\Gamma_3 \leq \frac{C_+ t_n}{(\ell - j + 1)^{3/2}} \left(\frac{1}{\ell^\varepsilon j^{1/2}} + \frac{1}{j} \right).$$

– *Upper bound for Γ_4* , first note that on $\{R > \tau_{t_n}^+\}$, the following hitting times $\tilde{\tau}_{\gamma \log \ell}^- := \inf\{k \in \llbracket \tau_{t_n}^+, j \rrbracket, S_k \leq \gamma \log \ell\} = \tau_{t_n}^+ + \theta_{\tau_{t_n}^+} \circ \tau_{\gamma \log \ell}^-$, and $\tilde{\tau}_{t_n}^+ := \inf\{k \in \llbracket \tilde{\tau}_{\gamma \log \ell}^-, j \rrbracket, S_k \geq t_n\} = \tilde{\tau}_{\gamma \log \ell}^- + \theta_{\tau_{\gamma \log \ell}^-} \circ \tau_{t_n}^+$ (where θ is the shift operator) exist. With these notations according to Lemma B.1

$$\begin{aligned} & P(\tau_0^- > \tilde{\tau}_{t_n}^+) = P(\tau_{t_n}^+ < \tau_{t_n}^+ + \theta_{\tau_{t_n}^+} \circ \tau_{\gamma \log \ell}^- < \tilde{\tau}_{\gamma \log \ell}^- + \theta_{\tilde{\tau}_{\gamma \log \ell}^-} \circ \tau_{t_n}^+ < \tau_0^-) \\ & \leq P(\tau_{t_n}^+ < \tau_0^-) \sup_{z \in [0, \alpha]} P_{\gamma \log \ell - z}(\tau_{t_n}^+ < \tau_0^-) \leq C_+ \frac{\gamma \log \ell}{(t_n)^2}. \end{aligned}$$

Again at this point we distinguish the cases $\zeta < 1$ or $\zeta \geq 1$.

When $\zeta < 1$ the above inequality yields

$$\begin{aligned} & P(\mathcal{H}_j, R > \tau_{t_n}^+) \leq \sup_{\frac{3t_n}{\alpha} \leq k \leq j} \sup_{z \in [0, \alpha]} P_{z+t_n}(\mathcal{H}_{j-k}) P(\tau_0^- > \tilde{\tau}_{t_n}^+) \\ & \leq C_+ \frac{\gamma \log \ell}{(t_n)^2} \max_{\frac{3t_n}{\alpha} \leq k \leq j} \max_{z \in [0, \alpha]} P_{z+t_n}(\mathcal{H}_{j-k}). \end{aligned}$$

Finally using that $P(\mathcal{H}_j, Y^-(j) > \delta\ell^\varepsilon, R > \tau_{t_n}^+) \leq P(\mathcal{H}_j, R > \tau_{t_n}^+)$, Lemma B.3 and (2.2)

$$\Gamma_4 \leq C_+ \frac{e^{-b_n^2/(2\sigma^2 j)}}{(\log \ell)^3 j^{1/2} (\ell - j + 1)^{3/2}}.$$

When $\zeta \geq 1$, we use similar arguments than for Γ_3 , we have $P(\mathcal{H}_j, Y^-(j) > \delta\ell^\varepsilon, R > \tau_{t_n}^+) \leq P(\mathcal{H}_j, j/2 \geq R > \tau_{t_n}^+) + P(\mathcal{H}_j, R > \tau_{t_n}^+ \vee j/2)$. Moreover $P(\mathcal{H}_j, j/2 \geq R > \tau_{t_n}^+) \leq P(\tau_{t_n}^+ < \tau_0^-)$.

$\sup_{0 \leq x \leq \alpha} P_{\gamma \log \ell + x}(\underline{S}_{j/2} > 0)$ and $P(\mathcal{H}_j, R > \tau_{t_n}^+ \vee j/2) \leq P(\underline{S}_{j/2} > 0) \sup_{0 \leq x \leq \alpha} P_{\gamma \log \ell + x}(\tau_0^- > \tau_{b_n}^+)$. So using Lemma B.1 we obtain

$$\Gamma_4 \leq \frac{C_+}{(\log \ell) j^{1/2} (\ell - j + 1)^{3/2}}.$$

Collecting Γ_3 and Γ_4 and (3.5), we obtain an upper bound for $\Pi_j + \Omega_j$, with (3.4) it leads to

$$\Gamma_2 \leq C_+ \frac{e^{-b_n^2/(2\sigma^2 \ell)}}{(\log \ell)^3 \ell^{1/2}} \sim C_+ \frac{e^{-(\log n)^{1-\zeta}/2\sigma^2}}{(\log \ell)^3 \ell^{1/2}},$$

if $\zeta < 1$ and

$$\Gamma_2 \leq \frac{C_+}{(\log \ell) \ell^{1/2}} \leq 2C_+ \frac{e^{-(\log n)^{1-\zeta}/2\sigma^2}}{(\log \ell) \ell^{1/2}},$$

if $\zeta \geq 1$. Γ_2 is therefore negligible compared to Γ_1 (see (3.3)) and (3.2) implies that $A_n^- \geq C_+ \frac{e^{-(\log n)^{1-\zeta}/2\sigma^2}}{\ell^{\varepsilon+1/2}}$. This finish the proof of (1.7).

3.2 From $\mathcal{R}_{\Phi(n)}(\ell)$ to $K_n(\ell)$ and $M_n(\ell)$ (proof of (1.6))

To obtain a lower bound for $M_n(\ell)$, let $\Phi_1(n) := \log n(1 - 2\varepsilon)$. First we need the following

Lemma 3.1

$$\lim_{n \rightarrow +\infty} \mathbb{P}(K_{n^{1-\varepsilon}}(\ell) \geq \mathcal{R}_{\Phi_1(n)}(\ell)) = 1.$$

Proof.

Applying Corollary C.1 with $\mathcal{A} := \{|z| = \ell, \bar{V}(z) \leq \Phi_1(n)\}$,

$$\mathbb{P}^{\mathcal{E}}(\cup_{z \in \mathcal{A}} \{\mathcal{L}(z, T_\phi^{n^{1-\varepsilon}}) = 0\}) \leq |\mathcal{A}| e^{-c-n^\varepsilon \ell^{-1}} \leq \mathcal{R}_{\Phi_1(n)}(\ell) e^{-c-n^\varepsilon/2}$$

As (1.11) implies that $\lim_{n \rightarrow +\infty} P(\mathcal{R}_{\Phi_1(n)}(\ell) \leq e^{\Phi_1(n)}) = 1$, the proof is achieved. \blacksquare

(1.11) together with the above Lemma give for n large enough

$$\mathbb{P}(K_{n^{1-\varepsilon}}(\ell) \geq e^{\psi(0) \log n(1-2\varepsilon)/\bar{\gamma}}) = 1 - o(1). \quad (3.7)$$

To obtain the lower bound for $M_n(\ell)$ we use the following result that can be deduced from [9] (see [3] Lemma 3.2 and what follows for details)

$$\forall \delta > 0, \quad \lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{L}(\phi, n) \geq n^{1-\delta}) = 1. \quad (3.8)$$

So $\lim_{n \rightarrow +\infty} \mathbb{P}(M_n(\ell) \geq K_{n^{1-\delta}}(\ell)) = 1$, which together with (3.7) implies the lower bound for $M_n(\ell)$.

Denote $u_n := C (\log \log n \vee (\log n)^{1-\zeta})$, where $C > 0$. As $n \leq T_\phi^n$, by Markov inequality and (1.7), $\mathbb{P}(\log M_{\Phi(n)}(\ell) \geq \log n - u_n) \leq \mathbb{P}(K_n(\ell) \geq n e^{-u_n}) \leq C_+ e^{u_n} e^{-(\log n)^{1-\zeta}/2\sigma^2} / \ell^{1/2}$ which gives the upper bound adjusting C properly.

4 Visited points along the GW

In this paragraph we study the way the random walk visits the tree.

4.1 Visits of clusters at deterministic cut (proof of (1.10))

Recall that a cluster initiated at z with end points at generation m is the set $\mathcal{C}_m(z) = \{u > z, |u| = m\}$. Also take $k_n = (\Phi(n))^{\mathbf{k}}$ and $r_n = (\Phi(n))^{\mathbf{r}}$ (more general than in the Theorem for the moment) and let $s_n = (\Phi(n))^{\mathbf{s}}$ with $\mathbf{s} > 0$ and h_n sequences such that

$$h_n = \frac{\ell - k_n r_n}{k_n - 1}, \quad k_n(\alpha r_n + s_n) - s_n \leq \Phi(n)(1 - 2\varepsilon). \quad (4.1)$$

We define recursively clusters at generations $ir_n + (i-1)h_n$ for all $1 \leq i \leq k_n$ in the following way (see also Figure 3): we start the iteration with $\check{z}_0 = \phi$ and

$$\forall z_i \in \mathcal{C}_{ir_n + (i-1)h_n}(\check{z}_{i-1}), \check{z}_i = \mathbf{inf}\{u > z_i, |u| = i(r_n + h_n), \bar{V}(u) \leq i(\alpha r_n + s_n)\}.$$

The individuals of these clusters form a subtree of the GW, moreover for all z of this subtree at generation ℓ , $\bar{V}(z) \leq \Phi(n)(1 - 2\varepsilon)$. For all $1 \leq i \leq k_n$, let \mathcal{C}_i be the set of clusters $\mathcal{C}_{ir_n + (i-1)h_n}(\cdot)$ previously defined. We first give an upper bound for the probability that for all $i \leq k_n$ every clusters in \mathcal{C}_i are fully visited before $T_\phi^{n^{1-\varepsilon}}$

$$\mathbb{P}^{\mathcal{E}} \left(\bigcup_{i=1}^{k_n} \bigcup_{\mathcal{D} \in \mathcal{C}_i} \bigcup_{z \in \mathcal{D}} \{\mathcal{L}(z, T_\phi^{n^{1-\varepsilon}}) = 0\} \right) \leq \sum_{i=1}^{k_n} \sum_{\mathcal{D} \in \mathcal{C}_i} \sum_{z \in \mathcal{D}} \mathbb{P}^{\mathcal{E}} \left(\mathcal{L}(z, T_\phi^{n^{1-\varepsilon}}) = 0 \right).$$

In the previous formula $\bigcup_{i=1}^{k_n} \bigcup_{\mathcal{D} \in \mathcal{C}_i}$ is an abuse of notation as the sets of clusters are defined recursively. We keep this notation in the following. With a similar reasoning as the one for Corollary C.1 and the ellipticity condition for the number of descendants

$$\mathbb{P}^{\mathcal{E}} \left(\bigcup_{i=1}^{k_n} \bigcup_{\mathcal{D} \in \mathcal{C}_i} \bigcup_{z \in \mathcal{D}} \{\mathcal{L}(z, T_\phi^{n^{1-\varepsilon}}) = 0\} \right) \leq \exp(N_0 k_n r_n) \exp(-c_- n^{1-\varepsilon} e^{-\Phi(n)(1-2\varepsilon)} / \ell). \quad (4.2)$$

We now prove the existence of such clusters, this implies new constraints on \mathbf{k} , \mathbf{r} and \mathbf{s} in addition to (4.1). First the event $\mathcal{B} := \{\text{the clusters } \mathcal{C}(\cdot) \text{ exist}\}$ contains

$$\bigcap_{i=1}^{k_n} \bigcap_{z_i \in \mathcal{C}_{ir_n + (i-1)h_n}(\check{z}_{i-1})} \{\bar{V}(z_i) \leq \alpha|z_i|, \exists \check{z}_i > z_i, |\check{z}_i| = i(h_n + r_n), \bar{V}(\check{z}_i) \leq i(\alpha r_n + s_n)\}.$$

Note that the ellipticity condition imply that for any site z , $\{\bar{V}(z) \leq \alpha|z|\}$ a.s. and consequently $\{\bar{V}(z) \leq \alpha|z|, \exists \check{z} > z, |\check{z}| = |z| + a, \bar{V}(\check{z}) \leq \alpha|z| + b\}$ contains a.s. $\{\exists \check{z} > z, |\check{z}| = |z| + a, \tilde{V}(z, \check{z}) \leq b\}$. Then a.s. \mathcal{B} contains

$$\bigcap_{i=1}^{k_n} \left\{ \bigcap_{z_i \in \mathcal{C}_{ir_n + (i-1)h_n}(\check{z}_{i-1})} \{\exists \check{z}_i > z_i, |\check{z}_i| = i(h_n + r_n), \tilde{V}(z_i, \check{z}_i) \leq s_n\} \right\}.$$

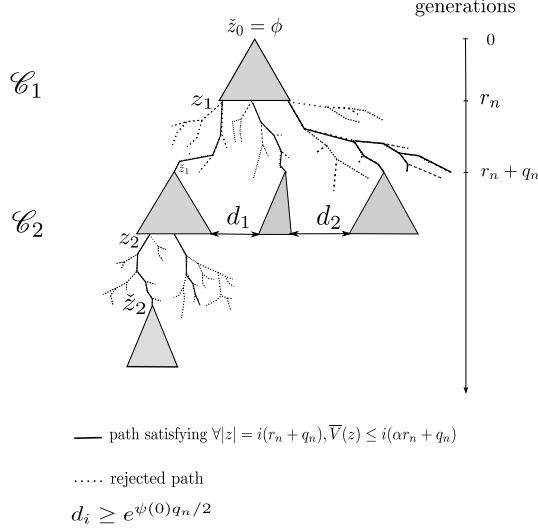


Figure 3: Clusters at regular cuts

The independence of the increments of V and the ellipticity assumptions on the number of descendants ($|\mathcal{C}(\cdot)| \leq e^{N_0 r_n}$) implies

$$P(\mathcal{B}) \geq P(\exists |z| = h_n, \bar{V}(z) \leq s_n) e^{N_0 k_n r_n} = P\left(\min_{|z|=h_n} \bar{V}(z) \leq s_n\right) e^{N_0 k_n r_n}.$$

Assuming that $\mathbf{s} \geq (1 + \zeta - \mathbf{k})/3$, then Lemma 2.3 yields $P(\mathcal{B}) \geq (1 - \lambda_{h_n}) e^{N_0 k_n r_n} \sim \exp(-e^{N_0 k_n r_n} \lambda_{h_n})$. We now choose \mathbf{k} , \mathbf{r} and \mathbf{s} , for that we have to take into account the last constraint in (4.1) $\mathbf{s} \leq 1 - \mathbf{k}$ and that if $q_0 + q_1 \neq 0$ we need that $\mathbf{k} + \mathbf{r} < \mathbf{s}$. We distinguish two cases

- if $0 < \zeta \leq 1$, let $0 < \delta < \zeta/2$, take $\mathbf{s} = (1 + \zeta)/2 - \delta$, $\mathbf{k} = \delta/2$, and $\mathbf{r} = (1 - \zeta)/2 + \delta/2$,
- if $1 < \zeta < 2$, let $0 < \delta < (2 - \zeta)/3$, take $\mathbf{s} = (1 + \zeta)/3$, $\mathbf{k} = \delta$ and $\mathbf{r} = (1 + \zeta - 4\delta)/3$.

Thus in both cases

$$P(\mathcal{B}) \geq 1 - C_+ e^{-c_2 s_n} e^{N_0 k_n r_n} \xrightarrow{n \rightarrow +\infty} 1. \quad (4.3)$$

When $q_0 + q_1 = 0$, the above choices give an even better rate of convergence for $P(\mathcal{B})$. We now move back to (4.2), first note that $k_n r_n < \Phi(n)$, so as $\Phi(n) \leq \log n + o(\log n)$ (4.3) implies

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(\bigcap_{i=1}^{k_n} \bigcap_{\mathcal{D} \in \mathcal{C}_i} \bigcap_{z \in \mathcal{D}} \{ \mathcal{L}(z, T_\phi^{n^{1-\varepsilon}}) \geq 1 \}, \mathcal{B} \right) = 1.$$

According to (3.8), as $\mathbb{P}(n \geq T_\phi^{n^{1-\varepsilon}})$ tends to one we finally obtain

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(\bigcap_{i=1}^{k_n} \bigcap_{\mathcal{D} \in \mathcal{C}_i} \bigcap_{z \in \mathcal{D}} \{ \mathcal{L}(z, n) \geq 1 \}, \mathcal{B} \right) = 1.$$

So with probability one we can find set of clusters at regular cuts on the tree which are fully visited. To finish the proof of (1.10) we first show the existence of a lower bound for the number of visited clusters. Using successively that conditionally on \mathcal{B} , $|\mathcal{C}_i|$ is equal in law to $Z_{(i-1)r_n}$, Theorem A.3 and (4.3)

$$\begin{aligned} P\left(\exists i \in \llbracket 2, k_n \rrbracket, |\mathcal{C}_i| \leq e^{\psi(0)(i-1)r_n/2}\right) &\leq P\left(\exists i \in \llbracket 2, k_n \rrbracket, |\mathcal{C}_i| \leq e^{\psi(0)(i-1)r_n/2}, \mathcal{B}\right) + P(\bar{\mathcal{B}}) \\ &\leq \sum_{i=2}^{k_n} P\left(Z_{(i-1)r_n} \leq e^{\psi(0)(i-1)r_n/2}\right) + P(\bar{\mathcal{B}}) \\ &\leq e^{-\psi(0)\nu r_n/4} + C_+ e^{-c_2 s_n} e^{N_0 k_n r_n}. \end{aligned}$$

Finally we prove that the previously defined visited clusters are very spaced out. Recalling the definition of \mathbf{D} before Theorem 1.2,

$$\begin{aligned} P\left(\exists i \in \llbracket 2, k_n \rrbracket, \mathbf{D}(\mathcal{C}_i) \leq e^{\psi(0)h_n/2}\right) &\leq P(\bar{\mathcal{B}}) + \sum_{i=2}^{k_n} P\left(\mathbf{D}(\mathcal{C}_i) \leq e^{\psi(0)h_n/2}, \mathcal{B}\right) \\ &\leq P(\bar{\mathcal{B}}) + \sum_{i=2}^{k_n} P\left(\bigcup_{\mathcal{D} \in \mathcal{C}_{i-1}} \bigcup_{z \in \mathcal{D}} \{|\mathcal{C}_{i(r_n+h_n)}(z)| \leq e^{\psi(0)h_n/2}\} \cap \mathcal{B}\right). \end{aligned}$$

We now use the facts that conditionally on \mathcal{B} , $|\mathcal{C}_{i(r_n+h_n)}(z)|$ and Z_{h_n} are equal in law, on $\mathcal{D}_i := \{|\mathcal{C}_i| \leq e^{2\psi(0)(i-1)r_n}\}$, $|\mathcal{D}| |\mathcal{C}_i| \leq e^{N_0 r_n} e^{2\psi(0)(i-1)r_n}$ and Theorem A.3

$$\begin{aligned} P\left(\mathbf{D}(\mathcal{C}_i) \leq e^{\psi(0)h_n/2}, \mathcal{B}\right) &\leq P\left(\bigcup_{\mathcal{D} \in \mathcal{C}_{i-1}} \bigcup_{z \in \mathcal{D}} \{|\mathcal{C}_{i(r_n+h_n)}(z)| \leq e^{\psi(0)h_n/2}\} \cap \mathcal{B} \cap \mathcal{D}_i\right) + P(\bar{\mathcal{D}}_i) \\ &\leq e^{N_0 r_n + 2\psi(0)(i-1)r_n} P(Z_{h_n} \leq e^{\psi(0)h_n/2}) + e^{-\psi(0)(i-1)r_n} \\ &\leq e^{N_0 r_n + 2\psi(0)(i-1)r_n - \nu\psi(0)h_n/2} + e^{-\psi(0)(i-1)r_n}. \end{aligned}$$

Consequently

$$P\left(\exists i \in \llbracket 2, k_n \rrbracket, \mathbf{D}(\mathcal{C}_i) \leq e^{\psi(0)h_n/2}\right) \leq e^{3\psi(0)k_n r_n - \nu\psi(0)h_n/2} + e^{-\psi(0)k_n r_n} + C_+ e^{-c_2 s_n} e^{N_0 k_n r_n},$$

moreover as $h_n \sim (\log n)^{1+\zeta-\mathbf{k}}$, $\mathbf{k} + \mathbf{r} < \mathbf{s} < 1$ and $\mathbf{k} < \zeta$ we obtain the result.

4.2 Proof of (1.9)

We assume $1 < \zeta < 2$, (the case $0 < \zeta \leq 1$ can be treated similarly). Let $m = \varepsilon \ell^{1/3}$, $\delta > 0$, define \mathcal{B} the set of points z' such that for all $z, |z| = m$, $z' := \mathbf{inf}\{u > z, |u| = \ell, \bar{V}(u) \leq \Phi(n)(1-\delta)\}$. Corollary C.1 gives

$$\mathbb{P}^{\mathcal{E}}\left(\bigcup_{z' \in \mathcal{B}} \{\mathcal{L}(z', T_\phi^{1-\varepsilon}) = 0\}\right) \leq |\mathcal{B}| e^{-c_- n^{1-\varepsilon}} e^{-\Phi(n)(1-\delta)/\ell}. \quad (4.4)$$

Note that $|\mathcal{B}| \leq Z_m$, also as $P(Z_m \leq e^{2m\psi(0)}) \geq 1 - e^{-m\psi(0)}$ and $\Phi(n) \leq \log n + o(\log n)$, taking $\delta = 2\varepsilon$ the above probability is smaller than $e^{-c_- n^\varepsilon/2}$.

We now prove that $\lim_{n \rightarrow +\infty} P(|\mathcal{B}| = Z_m) = 1$. From [14] (see also [3] Lemma 2.1), $\lim_{n \rightarrow +\infty} P(\max_{|z|=m} \bar{V}(z) \leq 2\tilde{\gamma}m) = 1$, so as for n large enough $\ell^{1/3}/\Phi(n) \leq \delta$ with the same arguments used in the proof of Lemma 2.3

$$\begin{aligned} P(|\mathcal{B}| < Z_m) &\leq P\left(\bigcup_{|z|=m} \{\forall z' > z, |z'| = \ell, \bar{V}(z') > (1 - \delta)\Phi(n)\}\right) \\ &\leq 1 - \left(1 - P\left(\min_{|z|=\ell-m} \bar{V}(z) > (1 - 2\delta)\Phi(n)\right)\right)^{e^{2\psi(0)m}} + o(1) \\ &\leq 2P\left(\min_{|z|=\ell-m} \bar{V}(z) > (1 - 4\varepsilon)\Phi(n)\right) e^{2\psi(0)m} + o(1). \end{aligned}$$

To finish we put ourself in the case $q_0 + q_1 > 0$ (the other case is treated similarly), as $\Phi(n)$ is way larger than $\ell^{1/3}$, using Lemma 2.3

$$P\left(\min_{|z|=\ell-m} \bar{V}(z) > (1 - 4\varepsilon)\Phi(n)\right) \leq e^{-c_1\ell^{1/3}}.$$

We can now choose ε small enough and obtain, $P(|\mathcal{B}| = Z_m) \geq 1 - e^{-c_1\ell^{1/3}/2} - o(1)$. Moving back to (4.4) $\mathbb{P}(\forall z \in \mathcal{B}, \mathcal{L}(z, T_\phi^{n^{1-\varepsilon}}) \geq 1, |\mathcal{B}| = Z_m) \geq 1 - o(1)$. Finally to obtain (1.9) we apply (3.8).

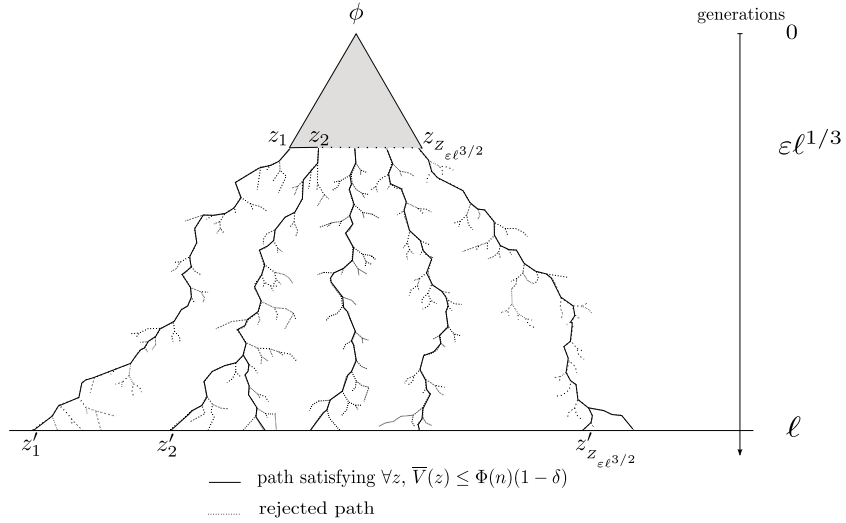


Figure 4: Distant visited sites

A Basic facts for branching processes and Galton-Watson trees

A.1 Biggins-Kypriaou identities and properties of ψ

For any $n \geq 1$ any measurable function $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, +\infty)$, Biggins-Kyprianou identity is given by

$$E \left[\sum_{|x|=n} e^{-V(x)} F(V(x_i), 1 \leq i \leq n) \right] = E[F(S_i, 1 \leq i \leq n)] \quad (\text{A.1})$$

where $(S_i - S_{i-1})_{i \geq 1}$ are i.i.d. random variables, and the law of S_1 is determined by

$$E[f(S_1)] = E \left[\sum_{|x|=1} e^{-V(x)} f(V(x)) \right], \quad (\text{A.2})$$

for any measurable function $f : \mathbb{R} \rightarrow [0, +\infty)$. A proof can be found in [5], see also [17]. We have the following identities

$$\psi(t) = \log E[e^{(1-t)S_1}], \quad \psi'(t) = -\frac{E[S_1 e^{(1-t)S_1}]}{E[e^{(1-t)S_1}]}.$$

In particular, $E[N] = e^{\psi(0)} = E[e^{S_1}]$, $\psi'(0) = -\frac{E[S_1 e^{S_1}]}{E[e^{S_1}]} < 0$, and the hypothesis $\psi'(1) = 0$, is equivalent to $E[S_1] = 0$.

Remark A.1 Let $\alpha := |\log \varepsilon_0|$, we have $\mathbb{P}(|S_1| \leq \alpha) = 1$. Indeed taking $f(x) = \mathbf{1}_{|x| \leq \alpha}$ and using Biggins-Kyprianou, we have $\mathbb{P}(|S_1| < \alpha) = E[f(S_1)] = e^{\psi(1)} = 1$.

Remark A.2 With the hypothesis of ellipticity ψ is smooth, in particular for all s in the neighborhood of 0, $\psi(1-s) = s^2 \psi''(1)/2 + O(s^3)$.

A.2 Left tail of Z_n

Recall that the positive martingale $(W_n)_{n \geq 0} := (Z_n / e^{\psi(0)n})_{n \geq 0}$ converges a.s. to a non degenerate limit W (see for instance [17]). Moreover W has a positive continuous density function denoted w . Bingham [6] shows that for the Schröder case ($q_0 + q_1 > 0$), there exists $0 < \nu < 1$ such that for small x , $w(x) \asymp x^{\nu-1}$ and for the Böttcher case ($q_0 + q_1 = 0$) there exists $\beta \in (0, 1)$ such that when $x \rightarrow 0$, $\log w(x) \asymp -x^{-\frac{\beta}{1-\beta}}$. The results of [4] and then [12] (Theorems 4 and 5) and [13] (Theorem 7) leads to

Theorem A.3 Let $0 < \kappa < 1$ then $P(Z_n \leq e^{\kappa \psi(0)n}) \sim e^{-\nu \psi(0)(1-\kappa)n}$ in the Schröder case, and $\log P(Z_n \leq e^{\kappa \psi(0)n}) \sim \log w(e^{\psi(0)(\kappa-1)n})$ in the Böttcher case.

B Results for sums of i.i.d. random variables

In this section we recall basic facts for sum of i.i.d. random variables applied to $(S_n)_{n \geq 0}$ of Section A. Recall that for all $x \in \mathbb{R}$, $\tau_x^+ = \inf\{n \geq 1, S_n \geq x\}$ and $\tau_x^- = \inf\{n \geq 1, S_n \leq x\}$. The following result can be found in [1] and [10].

Lemma B.1 *For all real $x \in [0, y]$*

$$P_x(\tau_y^+ < \tau_0^-) \asymp \frac{x+1}{y+1}, \quad E(\tau_y^+ \wedge \tau_0^-) \asymp y.$$

Let $0 \leq a < m$ for m large enough $P_a(\tau_0^- > m) \asymp \frac{a}{\sqrt{m}}$.

Lemma B.2 *There exists a constant $C_+ > 1$ such that for all $a \geq 0$ and $M > 0$*

$$P\left(\sum_{i=0}^{\tau_a^+} e^{-S_i} > M, \tau_a^+ < \tau_0^-\right) \leq \frac{C_+}{M}.$$

Proof.

According to [2] p.19, there exists $C_+ > 1$ such that for all $L \geq 1$ and $0 \leq a \leq L$, $\mathbb{E}\left[\sum_{i=0}^{\tau_0^- \wedge \tau_a^+} e^{-S_i}\right] \leq C_+ \frac{a+1}{a}$. To conclude, we just have to note that

$$P\left(\sum_{i=0}^{\tau_a^+} e^{-S_i} > M, \tau_a^+ < \tau_0^-\right) \leq P\left(\sum_{i=0}^{\tau_a^+ \wedge \tau_0^-} e^{-S_i} > M\right) \text{ and use Markov inequality. } \blacksquare$$

The following Lemma may be found in the literature, however as we can prove it easily for our case we present a short proof.

Lemma B.3 *Let $m > 1$, assume that $b = b(m) \geq \sigma^2 \sqrt{m} \log m$, with $\lim_{m \rightarrow +\infty} b/m = 0$, and $a = a(m) > 0$ is such that $\lim_{m \rightarrow +\infty} a/\sqrt{m} = 0$, then for all m large enough*

$$P_a(S_m > b, \underline{S}_m > 0) \leq C_+ \frac{a}{\sqrt{m}} e^{-b^2/2\sigma^2 m}, \quad (\text{B.1})$$

For all $\varepsilon > 0$ and $r > \varepsilon m$

$$P_a(S_m \geq r, \underline{S}_m > 0) \leq C_+ \frac{a}{\sqrt{m}} e^{-sr + m\psi(1-s)}. \quad (\text{B.2})$$

Proof.

For (B.1), let $\omega = 2m^{1/2}/b$

$$\begin{aligned} P_a(S_m > b, \underline{S}_m > 0) &= P_a\left(S_m > b, \underline{S}_m > 0, \tau_{\sqrt{m}}^+ \leq \omega m\right) + P_a\left(S_m > b, \underline{S}_m > 0, \tau_{\sqrt{m}}^+ > \omega m\right) \\ &=: P_3 + P_4. \end{aligned}$$

Strong Markov property and homogeneity give:

$$\begin{aligned} P_a\left(S_m > b, \underline{S}_m > 0, \tau_{\sqrt{m}}^+ = j\right) &\leq P_a\left(\tau_0^- > \tau_{\sqrt{m}}^+ = j\right) \sup_{0 \leq x \leq \alpha} P_{\sqrt{m}+x}\left(S_{m-j} > b, \underline{S}_{m-j} > 0\right) \\ &\leq P_a\left(\tau_0^- > \tau_{\sqrt{m}}^+ = j\right) P\left(S_{m-j} > b - \sqrt{m} - \alpha\right) \end{aligned}$$

implying with Lemma B.1:

$$\begin{aligned}
P_4 &\leq \sum_{j=\omega m+1}^m P_a \left(\tau_0^- > \tau_{\sqrt{m}}^+ = j \right) P \left(S_{m-j} > b - \sqrt{m} - \alpha \right) \\
&\leq P_a \left(\tau_0^- > \tau_{\sqrt{m}}^+ \right) \sup_{\omega m \leq j \leq m} P \left(S_{m-j} > b - \sqrt{m} - \alpha \right) \\
&\leq C_+ \frac{a}{\sqrt{m}} \sup_{\omega m \leq j \leq m} P \left(S_{m-j} > b - \sqrt{m} - \alpha \right),
\end{aligned}$$

A classical result of moderate deviations (see for instance [16], Chapter VIII) implies

$$\begin{aligned}
P \left(S_{m(1-\omega)} > b - \sqrt{m} - \alpha \right) &\leq C_+ e^{-(b-\sqrt{m}-\alpha)^2/(2\sigma^2 m(1-\omega))} \leq C_+ e^{-(b-\sqrt{m}-\alpha)^2(1+\omega)/(2\sigma^2 m)} \\
&\leq C_+ e^{-b^2/2\sigma^2 m}.
\end{aligned}$$

With similar computations this upper bound is still true for $P(S_{m-j} > b - \sqrt{m} - \alpha)$ for $m(1-\varepsilon) \leq j \leq m$. So $P_4 \leq C_+ \frac{a}{\sqrt{m}} e^{-b^2/2\sigma^2 m}$. In the same way

$$\begin{aligned}
P_3 &\leq \sum_{j=\sqrt{m}/\alpha}^{\omega m} P_a \left(\tau_0^- > \tau_{\sqrt{m}}^+ = j \right) \sup_{0 \leq x \leq \alpha} P_{\sqrt{m}+x} \left(S_{m-j} > b, \underline{S}_{m-j} > 0 \right) \\
&\leq \sum_{j=\sqrt{m}/\alpha}^{\omega m} P \left(S_j \geq \sqrt{m} - a \right) P \left(S_{m-j} > b - \sqrt{m} - \alpha \right),
\end{aligned}$$

Let $1 > t > s > 0$, using Markov inequality for both probabilities and the decreasing of ψ in the interval $[0, 1]$, we get

$$P_3 \leq C_+ e^{-t(\sqrt{m}-a)} e^{-s(b-\sqrt{m}-a)} e^{m\psi(1-s)+\omega m(\psi(1-t)-\psi(1-s))} / (e^{\psi(1-t)-\psi(1-s)} - 1)$$

Choosing $s = (b - \sqrt{m} - a)/(\sigma^2(m - \sqrt{m}/\alpha))$, $t = 2b/(\sigma^2 m)$ and applying Remark A.2 we obtain

$$P_3 \leq C_+ \frac{m^2}{b^2} e^{-b^2/2\sigma^2 m} e^{-2b/\sigma^2 \sqrt{m}} \leq C_+ e^{-b^2/2\sigma^2 m} / \sqrt{m},$$

which finish the proof.

(B.2) can be proved in a similar way. ■

The following Lemma states the local behavior of sums of i.i.d. random variables.

Lemma B.4 *Let $\varepsilon > 0$ small and $A > 0$ large. For all m large enough, for all $1 \leq r \leq A\sqrt{m}$, we have*

$$P(S_m \in [r, r+a], \underline{S}_m > 0) = \frac{C_5 r}{m^{3/2}} e^{-r^2/\sigma^2(2m)} + o(m^{-3/2}). \quad (\text{B.3})$$

with $C_5 > 0$. For all $A\sqrt{m} \leq r \leq \varepsilon m$,

$$P(S_m \in [r, r+a], \underline{S}_m > 0) = \frac{C_6 r}{m^{3/2}} e^{-r^2/\sigma^2(2m)} (1 + o(1)), \quad (\text{B.4})$$

with $C_6 > 0$.

Proof.

(B.3) is F. Caravenna [7] result and (B.4) can be obtained with both [7] and similar arguments than in the proof of Lemma B.3. ■

C Probability of hitting time

Lemma C.1 For $x' \in \llbracket \phi, x \rrbracket$:

$$\mathbb{P}_{x'}^{\mathcal{E}}(T_x < T_{x'}) = \frac{e^{V(x')}}{\sum_{z \in \llbracket x', x \rrbracket} e^{V(z)}}, \quad (\text{C.1})$$

$$\mathbb{P}_x^{\mathcal{E}}(T_{x'} < T_x) = \frac{e^{V(x)}}{\sum_{z \in \llbracket x', x \rrbracket} e^{V(z)}}. \quad (\text{C.2})$$

where x'_x is the only children of x' in $\llbracket x', x \rrbracket$.

The proof is classical (see for instance [3]) and an useful direct consequence of this Lemma is the following

Corollary C.1 Let $\mathcal{A} \subset \{z \in \mathbb{T}, |z| = \ell\}$, and $\kappa > 0$, there exists a positive constant c_7 such that:

$$\mathbb{P}_\phi^{\mathcal{E}}(T_\phi^{n^\kappa} < T_z) \leq \exp\left(-c_7 n^\kappa e^{-\bar{V}(z)/\ell}\right), \quad \forall z \in \mathcal{A}, \quad (\text{C.3})$$

$$\mathbb{P}^{\mathcal{E}}\left(\bigcup_{z \in \mathcal{A}} \{\mathcal{L}(z, T_\phi^{n^\kappa}) = 0\}\right) \leq |\mathcal{A}| \exp\left(-c_7 n^\kappa e^{-\max_{z \in \mathcal{A}} \bar{V}(z)/\ell}\right) \quad (\text{C.4})$$

Proof.

Note that the second formula is an obvious consequence of the first one. Thanks to formula (C.1), for $z \in \mathcal{A}$, $\mathbb{P}_\phi^{\mathcal{E}}(T_z < T_\phi) \geq C_- e^{-\bar{V}(z)/\ell}$. Then using the strong Markov property and the recurrence of X ($\lim_{n \rightarrow \infty} \mathbb{P}_\phi^{\mathcal{E}}(T_z < T_\phi) = 0$), for n large enough $\mathbb{P}_\phi^{\mathcal{E}}(T_\phi^{n^\kappa} < T_z) = (1 - \mathbb{P}_\phi^{\mathcal{E}}(T_z < T_\phi))^{n^\kappa} \leq \exp\left(e^{-c_7 n^\kappa \bar{V}(z)/\ell}\right)$. \blacksquare

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