

Let us now precise the assumptions we will need to make on the operator (1). The sequence $\{S_n\}_{n \in \mathbb{Z}}$ is chosen to belong to the following subset:

$$(2) \quad \mathbb{U}(2L)_{\text{inv}} = \left\{ S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{U}(2L), \left| \alpha, \gamma, \delta \in \mathcal{M}_L(\mathbb{C}) \text{ and } \beta \in \text{GL}_L(\mathbb{C}) \right. \right\}.$$

Invertibility of upper right entries of size $L \times L$ of the S_n 's assures that the scattering is effective so that \mathbb{U} does not decouple into a direct sum of two or more parts.

In (2), equivalent to the condition that β is invertible is the condition $\alpha^* \alpha < \mathbb{1}_L$. Furthermore, one has the representation (see [MS])

$$(3) \quad \mathbb{U}(2L)_{\text{inv}} = \{ S(\alpha, U, V) \in \mathbb{U}(2L) \mid \alpha^* \alpha < \mathbb{1}_L \text{ and } U, V \in \mathbb{U}(L) \},$$

where

$$(4) \quad S(\alpha, U, V) = \begin{pmatrix} \alpha & \rho(\alpha)U \\ V\tilde{\rho}(\alpha) & -V\alpha^*U \end{pmatrix},$$

and

$$\rho(\alpha) = (\mathbb{1}_L - \alpha\alpha^*)^{\frac{1}{2}}, \quad \tilde{\rho}(\alpha) = (\mathbb{1}_L - \alpha^*\alpha)^{\frac{1}{2}}.$$

In what follows, we will simply write ρ and $\tilde{\rho}$ where there is no ambiguity. The sequence of $\{\alpha_n\}_{n \in \mathbb{Z}}$ is often called the Verblunsky sequence associated to \mathbb{U} .

Assumption. We will assume that the Verblunsky sequence is constant, equal to some $\alpha \neq \mathbf{0}$.

1.2. Random scattering zippers. We will now introduce the random setting which will allow us to define a random version of the scattering zipper \mathbb{U} . Let $\Omega_0 = \mathbb{U}(L) \times \mathbb{U}(L)$, let \mathcal{B}_0 be the Borelian σ -algebra over $\mathbb{U}(L) \times \mathbb{U}(L)$ for the usual Lie group topology and let $\mathbb{P}_0 = \nu_L \otimes \nu_L$ where ν_L is the Haar measure on $\mathbb{U}(L)$. Then \mathbb{P}_0 is a probability measure on Ω_0 .

We now define the product probability space $(\Omega, \mathcal{B}, \mathbb{P})$:

$$(\Omega, \mathcal{B}, \mathbb{P}) = ((\Omega_0)^{\mathbb{Z}}, \otimes_{n \in \mathbb{Z}} \mathcal{B}_0, \otimes_{n \in \mathbb{Z}} \mathbb{P}_0).$$

Let $\omega \in \Omega$. For every $n \in \mathbb{Z}$, we consider the random unitary matrix $S_n(\omega) \in \mathbb{U}(2L)_{\text{inv}}$ defined by

$$S_n(\omega) = S(\alpha, U_n(\omega), V_n(\omega)),$$

where $((U_n(\omega), V_n(\omega)))_{n \in \mathbb{Z}}$ is a sequence of independent and identically distributed (*i.i.d.* for short) random variables on the probability space $(\Omega_0, \mathcal{B}_0, \mathbb{P}_0)$. Then $(S_n(\omega))_{n \in \mathbb{Z}}$ is a sequence of *i.i.d.* random matrices in $\mathbb{U}(2L)_{\text{inv}}$.

Associated to this sequence $(S_n(\omega))_{n \in \mathbb{Z}}$, one can defined as in (1) the operators \mathbb{V}_ω , \mathbb{W}_ω and $\mathbb{U}_\omega = \mathbb{V}_\omega \mathbb{W}_\omega$. We call **random scattering zipper** the family of random operators $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$.

Notation. Denote the shift $\tau : \Omega \rightarrow \Omega$, $(\tau(\omega))(2n) = \omega(2n + 2)$.

The shift τ is ergodic on $(\Omega, \mathcal{B}, \mathbb{P})$. Moreover, using τ , one show that the family of random operators $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ is $2\mathbb{Z}$ -ergodic. Thus, there exists $\Sigma \subset \mathbb{C}$ such that, for \mathbb{P} -almost every $\omega \in \Omega$, $\Sigma = \sigma(\mathbb{U}_\omega)$. There also exist Σ_{pp} , Σ_{ac} and Σ_{sc} , subsets of \mathbb{C} , such that, for \mathbb{P} -almost every $\omega \in \Omega$, $\Sigma_{\text{pp}} = \sigma_{\text{pp}}(\mathbb{U}_\omega)$, $\Sigma_{\text{ac}} = \sigma_{\text{ac}}(\mathbb{U}_\omega)$ and $\Sigma_{\text{sc}} = \sigma_{\text{sc}}(\mathbb{U}_\omega)$, respectively the pure point, absolutely continuous and singular continuous spectrum of \mathbb{U}_ω .

1.3. Main results. Now that we have precisely defined the model we will study in this paper, let us state the two main results we will prove in the next sections. For the definition of the Lyapunov exponents associated to the random scattering zipper $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$, see Section 2.

Theorem 1. *For every $z \in \mathbb{S}^1$, $\gamma_1(z) > \gamma_2(z) > \dots > \gamma_L(z) > 0$.*

Using Theorem 1, and adapting some results of Kotani's theory, one deduce the absence of absolutely continuous spectrum for $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$.

Theorem 2. *The random scattering zipper $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ has no absolutely continuous spectrum : $\Sigma_{ac} = \emptyset$.*

We will actually show a more general result on the multiplicity of the absolutely continuous spectrum of \mathbb{U}_ω related to the number of vanishing Lyapunov exponents (see Theorem 3), from which we deduce Theorem 2.

Let us finish this introduction by giving the outline of the article. In Section 2 we will present the formalism of transfer matrices for the random scattering zipper and we will define the Lyapunov exponents associated to these transfer matrices. We will also recall some of the first properties of these exponents. In Section 3 we will prove Theorem 1 by reducing it to an algebraic result on the Lie group generated by the transfer matrices of $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ and then by proving this algebraic result. Finally, in Section 4, we will prove Theorem 2 by adapting many ideas of Kotani and Simon which were found in [KS] to the setting of random scattering zippers. In particular we will characterize the multiplicity of the absolutely continuous spectrum in term of the number of vanishing Lyapunov exponents on a subset of full measure of the unit circle.

Results on the positivity of the Lyapunov exponents are already known for some models of unitary band matrices (see [BHJ, HS]). These unitary models deal with scalar coefficients compared to our model which deals with matrix coefficients. The positivity of the Lyapunov exponents for matrix-valued models is also known for matrix-valued Anderson models, both in the discrete and continuous settings (see [G] for the discrete case and [B] for the continuous case). For all these models, the positivity of the Lyapunov exponents implies absence of absolutely continuous spectrum through the results of Kotani's theory (see [BHJ, KS]).

2. TRANSFER MATRICES AND LYAPUNOV EXPONENTS

2.1. Transfer matrices. In this section we will present the formalism of the transfer matrices which allows us to reduce the understanding of the asymptotic behaviour of the solutions of the equation $\mathbb{U}_\omega \phi = z \phi$ (for $z \in \mathbb{S}^1$) to the understanding of the asymptotic behaviour of a product of random matrices in the Lorentz group.

Recall that the Lorentz group $U(L, L)$ of signature (L, L) is defined to be the set of $2L \times 2L$ matrices preserving the form

$$(5) \quad \mathcal{L} = \begin{pmatrix} \mathbb{1}_L & 0 \\ 0 & -\mathbb{1}_L \end{pmatrix}.$$

Let $z \in \mathbb{S}^1$. One define the transfer matrices in the following way (see [MS]) : consider the application $T(z, \cdot) : \Omega \rightarrow \text{U}(L, L)$,

$$\forall \omega \in \Omega, T(z, \omega) = \begin{pmatrix} V_0(\omega) & 0 \\ 0 & (U_0(\omega))^* \end{pmatrix} \hat{T}_0(z) \begin{pmatrix} V_1(\omega) & 0 \\ 0 & (U_1(\omega))^* \end{pmatrix} \hat{T}_1$$

with

$$\hat{T}_0(z) = \begin{pmatrix} z^{-1}\tilde{\rho}^{-1} & \tilde{\rho}^{-1}\alpha^* \\ \alpha\tilde{\rho}^{-1} & z\rho^{-1} \end{pmatrix}, \quad \hat{T}_1 = \begin{pmatrix} \tilde{\rho}^{-1} & \tilde{\rho}^{-1}\alpha^* \\ \alpha\tilde{\rho}^{-1} & \rho^{-1} \end{pmatrix}.$$

Definition 1. Let $z \in \mathbb{S}^1$, $n \in \mathbb{Z}$ and $\omega \in \Omega$. The n -th transfer matrix associated to the operator \mathbb{U}_ω is the matrix $T(z, \tau^n(\omega))$.

The sequence $(T(z, \tau^n(\omega)))_{n \in \mathbb{Z}}$ is an *i.i.d.* sequence of random matrices in $\text{U}(L, L)$ because of the *i.i.d.* character of the sequence $((U_n(\omega), V_n(\omega)))_{n \in \mathbb{Z}}$ in $\text{U}(L)$.

The transfer matrices are obtained directly from the matrices $S_n(\omega)$ through the following bijection between $\text{U}(2L)_{\text{inv}}$ and $\text{U}(L, L)$:

$$\begin{aligned} \text{U}(2L)_{\text{inv}} &\rightarrow \text{U}(L, L) \\ \varphi : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &\mapsto \begin{pmatrix} \gamma - \delta\beta^{-1}\alpha & \delta\beta^{-1} \\ -\beta^{-1}\alpha & \beta^{-1} \end{pmatrix}. \end{aligned}$$

Then we have the relation (see [MS]):

$$\forall \omega \in \Omega, \forall z \in \mathbb{S}^1, \forall k \in \mathbb{Z}, T(z, \tau^{2k}(\omega)) = \varphi(z^{-1}S_{2k}(\omega)) \cdot \varphi(S_{2k+1}(\omega)).$$

To define the Lyapunov exponents in the next section, we need to remark that the transfer matrices $T(z, \cdot)$ generate the cocycle Φ on the ergodic dynamical system $(\Omega, \mathcal{B}, \mathbb{P}, (\tau^n)_{n \in \mathbb{Z}})$ with $\Phi(z, \cdot, \cdot) : \Omega \times \mathbb{Z} \rightarrow \text{U}(L, L)$

$$\Phi(z, \omega, n) = \begin{cases} T(z, \tau^{n-1}(\omega)) \dots T(z, \omega) & \text{if } n > 0 \\ \mathbb{1}_{2L} & \text{if } n = 0 \\ (T(z, \tau^n(\omega)))^{-1} \dots (T(z, \tau^{-1}(\omega)))^{-1} & \text{if } n < 0. \end{cases}$$

Remark. For fixed $z \in \mathbb{S}^1$ and $\omega \in \Omega$, the asymptotic behaviour of the solutions of $\mathbb{U}_\omega \phi = z\phi$ is related to the asymptotic behaviour of the sequence $(\|\Phi(z, \omega, n)\|)_{n \in \mathbb{Z}}$ where $\|\cdot\|$ is any norm on $\text{U}(L, L)$.

2.2. Lyapunov exponents. Using the cocycle Φ , we can now define the Lyapunov exponents associated to the family $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$.

Proposition 1. Let $z \in \mathbb{S}^1$. For \mathbb{P} -almost every $\omega \in \Omega$, the following limits exist :

$$(6) \quad \Psi(z, \omega) := \lim_{n \rightarrow +\infty} ((\Phi(z, \omega, n))^* \Phi(z, \omega, n))^{1/2n} = \lim_{n \rightarrow -\infty} ((\Phi(z, \omega, n))^* \Phi(z, \omega, n))^{1/2|n|}.$$

For every $k \in \{1, \dots, 2L\}$, we denote by $\lambda_k(z, \omega)$ the eigenvalues of $\Psi(z, \omega)$ arranged in decreasing order. Then, there exist real numbers $\lambda_k(z) \geq 0$ such that for \mathbb{P} -almost every $\omega \in \Omega$, $\lambda_k(z, \omega) = \lambda_k(z)$.

Proof: This is a direct consequence of Oseledets theorem applied to the cocycle $\Phi(z, \cdot, \cdot)$. Indeed, according to [Arn, Remark 3.4.10], one can apply [Arn, Theorem 3.4.11] on \mathbb{C}^{2L} instead of \mathbb{R}^{2L} .

□

Notation. We denote by Ω_{Lyap} a subset of Ω such that $\mathbb{P}(\Omega_{\text{Lyap}}) = 1$ and, for every $\omega \in \Omega_{\text{Lyap}}$ the limits (6) exist and, for every $k \in \{1, \dots, 2L\}$, $\lambda_k(z, \omega) = \lambda_k(z)$.

Proposition 1 leads to the definition of the Lyapunov exponents associated to the operator \mathbb{U}_ω .

Definition 2. We define the $2L$ Lyapunov exponents associated to the family $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ by :

$$\forall k \in \{1, \dots, 2L\}, \gamma_k(z) := \log(\lambda_k(z)).$$

Using the fact that the transfer matrices lie in the group $U(L, L)$ and thus for every $n \in \mathbb{Z}$ the matrix $\Phi(z, \omega, n)$ lies in this group too, we get a symmetry relation for the Lyapunov exponents :

$$\forall j \in \{0, \dots, L\}, \gamma_{2L-j+1}(z) = -\gamma_j(z).$$

It implies that the Lyapunov exponents arrange by pairs of opposite real numbers :

$$(7) \quad \gamma_1(z) \geq \gamma_2(z) \geq \dots \geq \gamma_L(z) \geq 0 \geq -\gamma_L(z) \geq \dots \geq -\gamma_1(z).$$

Thus we will only have to consider the L first Lyapunov exponents which are positive numbers :

$$\gamma_1(z) \geq \gamma_2(z) \geq \dots \geq \gamma_L(z) \geq 0.$$

In the next section, we study the Lyapunov exponents associated to the cocycle Φ and we prove that they are all distincts and thus, using (7) $\gamma_1(z), \dots, \gamma_L(z)$ are all positive.

3. POSITIVITY OF THE LYAPUNOV EXPONENTS

3.1. Reduction of Theorem 1 to an algebraic result. This section is devoted to the proof of the positivity of the Lyapunov exponents of the family $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$. We will explain how to reduce the proof of Theorem 1 to an algebraic result on some subgroup of the Lie group $U(L, L)$.

To prove that the Lyapunov exponents are all distinct and strictly positive, we need to consider the Fürstenberg group associated to \mathbb{U}_ω , which is the subgroup of $U(L, L)$ generated by all the transfer matrices :

$$G(z) = \overline{\langle \text{supp} \mu_z \rangle} \subset U(L, L)$$

where μ_z is the common law of all the transfer matrices $T(z, \tau^n(\omega))$, for every $n \in \mathbb{Z}$. The closure is taken for the usual topology on $U(L, L)$ which is the topology induced on $U(L, L)$ by the usual topology on $\mathcal{M}_{2L}(\mathbb{C})$.

Directly from definition and using the fact that $(T(z, \tau^n(\omega)))_{n \in \mathbb{Z}}$ is an *i.i.d.* sequence of random matrices in $U(L, L)$, we get the following internal description of $G(z)$:

$$G(z) = \overline{\left\langle \left\langle \begin{pmatrix} V_0 & 0 \\ 0 & (U_0)^* \end{pmatrix} \hat{T}_0(z) \begin{pmatrix} V_1 & 0 \\ 0 & (U_1)^* \end{pmatrix} \hat{T}_1 \middle| (U_0, V_0, U_1, V_1) \in U(L)^4 \right\rangle \right\rangle} \subset U(L, L).$$

We will use this description of the elements of $G(z)$ to prove in Proposition 2 that actually the subgroup $G(z)$ is the whole group $U(L, L)$. Taking in account the result of Proposition 2, we can prove Theorem 1.

Proof: Assuming that for every $z \in \mathbb{S}^1$, $G(z) = U(L, L)$, we can follow the strategy of [ABJ, Theorem 6.1]. We fix $z \in \mathbb{S}^1$. By Cayley transform, the group $G(z)$ is unitarily equivalent to the complex symplectic group. Indeed, if we denote by $C \in \mathcal{M}_{2L}(\mathbb{C})$ the matrix

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_L & -i\mathbf{1}_L \\ \mathbf{1}_L & i\mathbf{1}_L \end{pmatrix},$$

and by J the matrix

$$J = \begin{pmatrix} 0 & -\mathbf{1}_L \\ \mathbf{1}_L & 0 \end{pmatrix},$$

we have

$$U(L, L) = C\mathrm{Sp}_L(\mathbb{C})C^*$$

where

$$\mathrm{Sp}_L(\mathbb{C}) = \{M \in \mathcal{M}_{2L}(\mathbb{C}) \mid M^*JM = J\}.$$

Since the results of [BL] are stated for the real symplectic group, we introduce, as in [ABJ], the following application which split the real and imaginary parts of the matrices in $\mathcal{M}_{2L}(\mathbb{C})$

$$\begin{aligned} \mathcal{M}_{2L}(\mathbb{C}) &\rightarrow \mathcal{M}_{4L}(\mathbb{R}) \\ \tau : \quad A + iB &\mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \end{aligned}$$

So we have $\tau(C^*U(L, L)C) \subset \mathrm{Sp}_{2L}(\mathbb{R})$. With this setting, we deduce immediately from [ABJ, Lemma 6.3] that $\tau(C^*U(L, L)C)$ is L_{2p} -strongly irreducible for every $p \in \{1, \dots, L\}$ (for a definition of the notion L_p -strong irreducibility, see [BL, Definition A.IV.3.3]). Then, to adapt the proof of [ABJ, Lemma 6.4] to our setting, we only need to perform a permutation of lines and columns of the matrix with $2L$ distinct singular values they define. More precisely, let P denote the permutation matrix which send, for $j \in \{1, \dots, L\}$, the $(L + j)$ -th line of the identity matrix of order $2L$ to the line $2j$ and, for $k \in \{0, \dots, L - 1\}$, the k -th line to the line $2k + 1$. Then, multiplying on left and on right by P the matrix they used in their proof, we obtain an element of $U(L, L)$ with $2L$ distinct singular values. Thus $\tau(C^* \cdot U(L, L) \cdot C)$ is $2p$ -contracting for every $p \in \{1, \dots, L\}$ (for a definition of p -contractivity, see [BL, Definition A.IV.1.1]).

Now we can apply [BL, Proposition A.IV.3.4] to the group $\tau(C^*U(L, L)C)$ to obtain that all Lyapunov exponents are distincts. Since we have the inequalities (7), the L first Lyapunov exponents are strictly positive, which finishes the proof. \square

3.2. Proof of Proposition 2. In this section, we prove the algebraic result on which the proof of Theorem 1 is based on.

Proposition 2. *For any $z \in \mathbb{S}^1$, $G(z) = U(L, L)$.*

Proof: By connexity of $U(L, L)$, it is enough to show that the Lie algebras of $G(z)$ and $U(L, L)$ are equal. The Lie algebra associated to $U(L, L)$ is given by

$$\begin{aligned} \mathrm{Lie}(U(L, L)) &= \mathfrak{u}(L, L) = \{T \in \mathcal{M}_{2L}(\mathbb{C}) \mid T^*\mathcal{L} + \mathcal{L}T = 0\} \\ &= \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathcal{M}_{2L}(\mathbb{C}) \mid A^* = -A, D^* = -D, (A, B, D) \in \mathcal{M}_L(\mathbb{C}) \right\} \end{aligned}$$

We denote $\mathrm{Lie}(G(z)) = \mathfrak{g}(z)$. Now, the proof divides in numerous small Lemma. By taking in account Lemma 2 and Lemma 5, one gets that $\mathfrak{u}(L, L) = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \subset \mathfrak{g}(z)$ and thus $\mathfrak{u}(L, L) = \mathfrak{g}(z)$ which ends the proof. \square

Lemma 1. *For any $(U_0, V_0) \in U(L)^2$, the matrix $\begin{pmatrix} V_0 & 0 \\ 0 & U_0^* \end{pmatrix}$ belongs to $G(z)$.*

Proof: Setting $U_0 = V_0 = U_1 = V_1 = \mathbf{1}_L$, one gets $\hat{T}_0(z)\hat{T}_1 \in G(z)$. As $G(z)$ is a multiplicative group, $(\hat{T}_0(z)\hat{T}_1)^{-1}$ also belongs to $G(z)$. Thus only setting $U_1 = V_1 = \mathbf{1}_L$ and U_0, V_0 free and multiplying by $(\hat{T}_0(z)\hat{T}_1)^{-1}$ yields to the statement. \square

Lemma 2. *The algebra $\mathfrak{g}(z)$ contains $\mathfrak{a}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A^* = -A, D^* = -D, (A, D) \in \mathcal{M}_L(\mathbb{C})^2 \right\}$.*

Proof: By previous Lemma, one has $U(L) \oplus U(L) \subset G(z)$. Therefore this implies $\text{Lie}(U(L)) \oplus \text{Lie}(U(L)) \subset \mathfrak{g}(z)$. As $\text{Lie}(U(L)) = \{A \in \mathcal{M}_L(\mathbb{C}) \mid A^* = -A\}$, this implies the statement. \square

Lemma 3. *The matrix $i \begin{pmatrix} 0 & \tilde{\rho}^{-2}\alpha^* \\ \alpha\tilde{\rho}^{-2} & 0 \end{pmatrix}$ belongs to $\mathfrak{g}(z)$.*

Proof: For $V_0 = U_0 = \mathbf{1}_L$, $\hat{T}_0(z) \begin{pmatrix} V_1 & 0 \\ 0 & U_1^* \end{pmatrix} \hat{T}_1 \in G(z)$.

Therefore, $\hat{T}_1^{-1} \begin{pmatrix} V_1 & 0 \\ 0 & U_1^* \end{pmatrix} \hat{T}_1 = (\hat{T}_0(z)\hat{T}_1)^{-1}\hat{T}_0(z) \begin{pmatrix} V_1 & 0 \\ 0 & U_1^* \end{pmatrix} \hat{T}_1 \in G(z)$.

Now for $j \in \llbracket 1, L \rrbracket$ and any $t \in \mathbb{R}$, picking V_1 to be the matrix $\text{diag}(1, \dots, 1, e^{it}, 1, \dots, 1)$ with e^{it} at the j -th place, and $U_1 = \mathbf{1}_L$ and derivating in $t = 0$ yields to

$$\forall j \in \llbracket 1, L \rrbracket, i\hat{T}_1^{-1} \begin{pmatrix} E_{jj} & 0 \\ 0 & 0 \end{pmatrix} \hat{T}_1 \in \mathfrak{g}(z).$$

In particular summing over j gives

$$i\hat{T}_1^{-1} \begin{pmatrix} \mathbf{1}_L & 0 \\ 0 & 0 \end{pmatrix} \hat{T}_1 \in \mathfrak{g}(z)$$

and the result writing \hat{T}_1 as block matrix. \square

Lemma 4. *There exists an index $(j_0, k_0) \in \llbracket 1, L \rrbracket^2$ and $c \in \mathbb{C}$, with $c \neq 0$ such that*

$$i \begin{pmatrix} 0 & cE_{k_0j_0} \\ \bar{c}E_{j_0k_0} & 0 \end{pmatrix} \in \mathfrak{g}(z).$$

Proof: As we suppose $\alpha \neq 0$ and $\tilde{\rho}^{-2}$ is invertible then $\tilde{\rho}^{-2}\alpha^* \neq 0$. So there exists an index $(j_0, k_0) \in \llbracket 1, L \rrbracket^2$ such that $(\tilde{\rho}^{-2}\alpha^*)_{(j_0, k_0)} = c \neq 0$ and

$$\left[i \begin{pmatrix} 0 & 0 \\ 0 & E_{j_0j_0} \end{pmatrix}, \left[i \begin{pmatrix} E_{k_0k_0} & 0 \\ 0 & 0 \end{pmatrix}, i \begin{pmatrix} 0 & \tilde{\rho}^{-2}\alpha^* \\ \alpha\tilde{\rho}^{-2} & 0 \end{pmatrix} \right] \right] = i \begin{pmatrix} 0 & cE_{k_0j_0} \\ \bar{c}E_{j_0k_0} & 0 \end{pmatrix} \in \mathfrak{g}(z).$$

\square

Lemma 5. $\mathfrak{a}_2 = \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \mid B \in \mathcal{M}_L(\mathbb{C}) \right\} \subset \mathfrak{g}(z)$.

Proof: It is enough to show that the elements $\begin{pmatrix} 0 & yE_{jk} \\ \bar{y}E_{kj} & 0 \end{pmatrix}$ for $(j, k) \in \llbracket 1, L \rrbracket^2$, $y = 1$ and $y = i$ belong to $\mathfrak{g}(z)$, as they form a basis of \mathfrak{a}_2 . From Lemma 2, the matrix $\begin{pmatrix} yE_{jk} - \bar{y}E_{kj} & 0 \\ 0 & 0 \end{pmatrix}$ belongs to $\mathfrak{g}(z)$ as the block $yE_{jk} - \bar{y}E_{kj}$ is anti-hermitian for any $(j, k) \in \llbracket 1, L \rrbracket^2$ and any $y \in \mathbb{C}$. For any $k \in \llbracket 1, L \rrbracket^2$, $k \neq k_0$ and any $y \in \mathbb{C}$ one has,

$$\left[\begin{pmatrix} yE_{kk_0} - \bar{y}E_{k_0k} & 0 \\ 0 & 0 \end{pmatrix}, i \begin{pmatrix} 0 & cE_{k_0j_0} \\ \bar{c}E_{j_0k_0} & 0 \end{pmatrix} \right] = i \begin{pmatrix} 0 & ycE_{kj_0} \\ -\bar{y}\bar{c}E_{j_0k} & 0 \end{pmatrix} \in \mathfrak{g}(z).$$

Then, for any $j \in \llbracket 1, L \rrbracket^2, j \neq j_0$,

$$\left[i \begin{pmatrix} 0 & ycE_{kj_0} \\ -\bar{y}cE_{j_0k} & 0 \end{pmatrix}, i \begin{pmatrix} 0 & 0 \\ E_{jj_0} + E_{j_0j} & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & -ycE_{kj} \\ -\bar{y}cE_{jk} & 0 \end{pmatrix} \in \mathfrak{g}(z).$$

One can choose either $y = c^{-1}$ or $y = ic^{-1}$ to get all the elements of the basis of \mathfrak{a}_2 except the two corresponding to the case $j = j_0, k = k_0$. In this case, one obtains from the same computation that

$$i \begin{pmatrix} 0 & (y - \bar{y})cE_{k_0j_0} \\ (\bar{y} - y)\bar{c}E_{j_0k_0} & 0 \end{pmatrix} \in \mathfrak{g}(z).$$

Since $(y - \bar{y})$ is purely imaginary, real linear combinations with the element given in Lemma 4 complete the basis of the set \mathfrak{a}_2 . \square

Remark 1. *In case $\alpha = \mathbf{0}$, one can show that $\mathfrak{g}(z) = \mathfrak{a}_1 \subsetneq \mathfrak{u}(L, L)$. This justifies our hypothesis $\alpha \neq \mathbf{0}$.*

4. ABSENCE OF ABSOLUTELY CONTINUOUS SPECTRUM

In this section, we will prove that Theorem 1 implies absence of absolutely continuous spectrum for the family $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$. For this purpose, we will prove an analog of the theorem of Ishii and Pastur on the characterization of the absolutely continuous spectrum in term of zeros of the Lyapunov exponents. More precisely, we will obtain a unitary version of the matrix-valued analog of Ishii-Pastur's theorem which was proven in [KS, Theorem 5.4].

We recall that for $\omega \in \Omega_{\text{Lyap}}$, the operator \mathbb{U}_ω has $2L$ Lyapunov exponents which can be regrouped by pairs of opposite real numbers :

$$\gamma_1(z) \geq \dots \geq \gamma_L(z) \geq 0 \geq \gamma_{L+1}(z) = -\gamma_L(z) \geq \dots \geq \gamma_{2L}(z) = -\gamma_1(z).$$

For $j \in \{1, \dots, L\}$, we set

$$Z_j = \{z \in \mathbb{S}^1 \mid \text{there exist } l_1, \dots, l_{2j} \in \{1, \dots, 2L\}, \gamma_{l_1}(z) = \dots = \gamma_{l_{2j}}(z) = 0\}.$$

We also recall that a sequence $\varphi \in (\mathbb{C}^L)^\mathbb{Z}$ is said to be *polynomially bounded* if there exist $C > 0$ and $p \geq 1$ such that

$$\forall n \in \mathbb{Z}, \|\varphi_n\|_{\mathbb{C}^L} \leq C(1 + |n|)^p,$$

where $\|\cdot\|_{\mathbb{C}^L}$ is any norm on \mathbb{C}^L . Now we can state the first result of this section.

Proposition 3. *Let $j \in \{1, \dots, L\}$ and let $z \in Z_j$ be fixed. Let $\omega \in \Omega_{\text{Lyap}}$. Then, every subspace of $\{\varphi \in (\mathbb{C}^L)^\mathbb{Z} \mid \mathbb{U}_\omega \varphi = z\varphi, \varphi \notin \ell^2(\mathbb{Z}) \otimes \mathbb{C}^L \text{ and } \varphi \text{ is polynomially bounded}\}$ has dimension at most $2j$.*

Proof: We set

$$V_{\text{sol}}(z) = \{\varphi \in (\mathbb{C}^L)^\mathbb{Z} \mid \mathbb{U}_\omega \varphi = z\varphi\},$$

$$V_{\text{P}}(z) = \{\varphi \in V_{\text{sol}}(z) \mid \varphi \text{ is polynomially bounded}\},$$

and

$$V_{\ell^2}(z) = \{\varphi \in V_{\text{sol}}(z) \mid \varphi \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^L\}.$$

To prove the proposition, we have to show that

$$\dim V_{\text{P}}(z) \leq 2j + \dim V_{\ell^2}(z).$$

For $\varphi = (\varphi_n)_{n \in \mathbb{Z}}$ and $\psi = (\psi_n)_{n \in \mathbb{Z}}$ in $V_{\text{sol}}(z)$, we set

$$W(\varphi, \psi) = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}^* \mathcal{L} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}.$$

Then, W is an antisymmetric form on $V_{\text{sol}}(z)$. Moreover, since ψ is in $V_{\text{sol}}(z)$, it is uniquely determined by $\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$ and since \mathcal{L} is non-singular, W is non-degenerate. Indeed, let $\psi \in V_{\text{sol}}(z)$ be such that, for every $\varphi \in V_{\text{sol}}(z)$, $W(\varphi, \psi) = 0$. Then, we apply this equality with $2L$ different φ 's, those being such that $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$ are the $2L$ vectors of the canonical basis of \mathbb{C}^{2L} . Writing these $2L$ relations, we obtain a Cramer system with unique solution $\mathcal{L} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = 0$. Since \mathcal{L} is non-singular, we get $\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = 0$ and since ψ is uniquely determined by $\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$, we get $\psi = 0$.

Now, since W is a non-degenerate antisymmetric form on $V_{\text{sol}}(z)$, if V_1 and V_2 are two subspaces of $V_{\text{sol}}(z)$ such that

$$(8) \quad \forall \varphi \in V_1, \forall \psi \in V_2, W(\varphi, \psi) = 0,$$

then

$$(9) \quad \dim V_1 + \dim V_2 \leq 2L.$$

Indeed, V_1 and $\mathcal{L}V_2$ are orthogonal for W .

We set

$$D_{\pm} = \{\varphi \in V_{\text{sol}}(z) \mid \varphi \text{ decays exponentially at } \pm \infty\}.$$

Since $z \in Z_j$, exactly $2j$ among the Lyapunov exponents vanish at z . Thus, by Oseledets theorem,

$$\dim D_{\pm} = L - j.$$

We also have $D_+ \cap D_- \subset V_{\ell^2}(z)$, so

$$(10) \quad \dim (D_+ + D_-) = \dim D_+ + \dim D_- - \dim (D_+ \cap D_-) \geq 2L - 2j - \dim V_{\ell^2}(z)$$

Moreover, if we take $\varphi \in V_{\text{P}}(z)$ and $\psi \in D_+ + D_-$, then, by direct domination, we have

$$(11) \quad \lim_{|n| \rightarrow +\infty} \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}^* \mathcal{L} \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} = 0.$$

But, one can actually show that the sequence $\left(\begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}^* \mathcal{L} \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} \right)_{n \in \mathbb{Z}}$ is constant when we choose φ and ψ in $V_{\text{sol}}(z)$. Indeed, since $|z| = 1$, $T(z, \tau^n(\omega)) \in \text{U}(L, L)$ for every $n \in \mathbb{Z}$. Thus,

$$\begin{aligned} \forall n \in \mathbb{Z}, \begin{pmatrix} \varphi_{n+1} \\ \varphi_{n+2} \end{pmatrix}^* \mathcal{L} \begin{pmatrix} \psi_{n+1} \\ \psi_{n+2} \end{pmatrix} &= (T(z, \tau^{n+1}(\omega)) \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix})^* \mathcal{L} T(z, \tau^{n+1}(\omega)) \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}^* T(z, \tau^{n+1}(\omega))^* \mathcal{L} T(z, \tau^{n+1}(\omega)) \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}^* \mathcal{L} \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix}. \end{aligned}$$

In particular,

$$(12) \quad \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}^* \mathcal{L} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \lim_{|n| \rightarrow +\infty} \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}^* \mathcal{L} \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} = 0.$$

Thus, if $\varphi \in V_{\text{P}}(z)$ and $\psi \in D_+ + D_-$, $W(\varphi, \psi) = 0$. Then, by (9),

$$\dim V_{\text{P}}(z) + \dim (D_+ + D_-) \leq 2L.$$

Combining this inequality with (10), one finally get

$$\dim V_P(z) \leq 2j + \dim V_{\ell^2}(z),$$

which proves the proposition. \square

We introduce the set P_{bdd} of all the complex numbers z of modulus 1 such that the equation $\mathbb{U}_\omega \varphi = z\varphi$ admits a non-trivial polynomially bounded solution. Since our operator \mathbb{U}_ω has band structure, we can use results from [BHJ]. In particular, [BHJ, Lemma 5.4] and [BHJ, Corollary 5.2] applied to our model lead to the following result.

Proposition 4. *For \mathbb{P} -almost every $\omega \in \Omega$,*

$$\sigma(\mathbb{U}_\omega) = \overline{P_{\text{bdd}}} = \overline{\{z \in \mathbb{S}^1 \mid \exists \varphi \in V_P(z) \setminus \{0\}\}}$$

and $E_{\mathbb{S}^1 \setminus P_{\text{bdd}}}(\mathbb{U}_\omega) = 0$, where $E_{\mathbb{S}^1 \setminus P_{\text{bdd}}}(\mathbb{U}_\omega)$ is the spectral projector on $\mathbb{S}^1 \setminus P_{\text{bdd}}$ associated to the unitary operator \mathbb{U}_ω .

With this proposition, we can finally prove the main result of this section, an Ishii-Pastur theorem for our model.

Theorem 3. *For every $\omega \in \Omega_{\text{Lyap}}$, the multiplicity of the absolutely continuous spectrum of \mathbb{U}_ω on Z_j is at most $2j$.*

Proof: Let $\omega \in \Omega_{\text{Lyap}}$. For Δ a Borelian subset of \mathbb{S}^1 , we denote by $E_\Delta(\mathbb{U}_\omega)$ the spectral projection on Δ and by $E_\Delta^{\text{a.c.}}(\mathbb{U}_\omega)$ the spectral projection on the absolutely continuous part of $\sigma(\mathbb{U}_\omega)$ in Δ .

To prove the theorem, we have to prove that

$$\text{rk } E_{Z_j}^{\text{a.c.}}(\mathbb{U}_\omega) \leq 2j.$$

Since by Proposition 4, $E_{\mathbb{S}^1 \setminus P_{\text{bdd}}}(\mathbb{U}_\omega) = 0$, we have

$$E_{Z_j}^{\text{a.c.}}(\mathbb{U}_\omega) = E_{Z_j \cap P_{\text{bdd}}}^{\text{a.c.}}(\mathbb{U}_\omega) = E_{Z_j \cap P_{\text{bdd}} \cap S}^{\text{a.c.}}(\mathbb{U}_\omega) + E_{Z_j \cap P_{\text{bdd}} \cap S^c}^{\text{a.c.}}(\mathbb{U}_\omega),$$

where $S = \{z \in \mathbb{S}^1 \mid \exists \varphi \in V_P(z) \cap V_{\ell^2}(z), \varphi \neq 0\}$. If $z \in S$, then z is an eigenvalue of \mathbb{U}_ω . Since there are only countably many ℓ^2 -eigenvectors for \mathbb{U}_ω , the Haar measure of S is zero. Thus,

$$(13) \quad E_{Z_j \cap P_{\text{bdd}} \cap S}^{\text{a.c.}}(\mathbb{U}_\omega) = 0 \quad \text{and} \quad E_{Z_j}^{\text{a.c.}}(\mathbb{U}_\omega) = E_{Z_j \cap P_{\text{bdd}} \cap S^c}^{\text{a.c.}}(\mathbb{U}_\omega).$$

But, since $z \in Z_j$, we can apply Proposition 3 to get directly that

$$\text{rk } E_{Z_j \cap P_{\text{bdd}} \cap S^c}(\mathbb{U}_\omega) \leq 2j$$

which implies

$$(14) \quad \text{rk } E_{Z_j \cap P_{\text{bdd}} \cap S^c}^{\text{a.c.}}(\mathbb{U}_\omega) \leq 2j.$$

So, combining (13) and (14), we have finally proven that

$$\text{rk } E_{Z_j}^{\text{a.c.}}(\mathbb{U}_\omega) \leq 2j,$$

which achieve the proof of the theorem. \square

We recall that ν_1 denote the Haar measure on \mathbb{S}^1 . We can deduce from Theorem 3 the following corollary.

Corollary 1. *If for ν_1 -almost every $z \in \mathbb{S}^1$, $\gamma_1(z) \geq \dots \geq \gamma_L(z) > 0$, then $\Sigma_{\text{ac}} = \emptyset$.*

Proof: If $z \in \mathbb{S}^1$ is such that $\gamma_L(z) > 0$, then no Lyapunov exponent vanishes at z , which means that $z \in Z_0$. By Theorem 3, $E_{Z_0}^{\text{a.c.}}(\mathbb{U}_\omega) = 0$. Now, if Δ is a Borelian subset of \mathbb{S}^1 , since by hypothesis $\nu_1(Z_0) = 1$,

$$E_\Delta^{\text{a.c.}}(\mathbb{U}_\omega) = E_{\Delta \cap Z_0}^{\text{a.c.}}(\mathbb{U}_\omega) = 0, \text{ for } \mathbb{P}\text{-a.e } \omega \in \Omega.$$

It implies $\Sigma_{\text{ac}} = \emptyset$ by definition of Σ_{ac} . □

Finally, from Corollary 1 and Theorem 1 we deduce immediately Theorem 2.

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