

# THE CARTAN-HADAMARD CONJECTURE AND THE LITTLE PRINCE

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ABSTRACT. The generalized Cartan-Hadamard conjecture says that if  $\Omega$  is a domain with fixed volume in a complete, simply connected Riemannian  $n$ -manifold  $M$  with sectional curvature  $K \leq \kappa \leq 0$ , then  $\partial\Omega$  has the least possible boundary volume when  $\Omega$  is a round  $n$ -ball with constant curvature  $K = \kappa$ . The case  $n = 2$  and  $\kappa = 0$  is an old result of Weil. We give a unified proof of this conjecture in dimensions  $n = 2$  and  $n = 4$  when  $\kappa = 0$ , and a special case of the conjecture for  $\kappa < 0$  and a version for  $\kappa > 0$ . Our argument uses a new interpretation, based on optimal transport, of Croke's proof for  $n = 4$  and  $\kappa = 0$ . The generalization to  $n = 4$  and  $\kappa \neq 0$  is a new result. As Croke implicitly did, we relax the curvature condition  $K \leq \kappa$  to a weaker candle condition  $\text{Candle}(\kappa)$  or  $\text{LCD}(\kappa)$ .

We also find a counterexample to a naïve version of the Cartan-Hadamard conjecture: We establish that for every  $A, V > 0$ , there is a 3-ball with curvature  $K \leq -1$ , volume  $V$ , and surface area  $A$ .

We begin with a pointwise isoperimetric problem called “the problem of the Little Prince.” Its proof becomes part of the more general method.

## 1. INTRODUCTION

In this article, we will prove new, sharp isoperimetric inequalities for a manifold with boundary  $\Omega$ , or a domain in a manifold. Before turning to motivation and context, we state a special case of one of our main results (Theorem 1.4).

**Theorem 1.1.** *Let  $\Omega$  be a compact Riemannian manifold with boundary, of dimension  $n = 2$  or  $n = 4$ . Suppose that  $\Omega$  has unique geodesics, has sectional curvature bounded above by  $+1$ , and that the volume of  $\Omega$  is at most half the volume of the sphere  $X_{n,1}$  of constant curvature 1. Then the volume of  $\partial\Omega$  is at least the volume of the boundary of a spherical cap in  $X_{n,1}$  with the same volume as  $\Omega$ .*

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**1.1. The generalized Cartan-Hadamard conjecture.** An isoperimetric inequality has the form

$$|\partial\Omega| \geq I(|\Omega|) \tag{1}$$

where  $I$  is some function. (We use  $|\cdot|$  to denote volume and  $|\partial\cdot|$  to denote boundary volume or perimeter; see Section 2.1.) The largest function  $I = I_M$  such that (1) holds for all domains of a Riemannian  $n$ -manifold  $M$  is called the *isoperimetric profile* of  $M$ .

Besides the intrinsic appeal of the isoperimetric profile and isoperimetric inequalities generally, they imply other important comparisons. For example, they yield estimates on the first eigenvalue  $\lambda_1(\Omega)$  of the Laplace operator by the Faber-Krahn argument [Cha84]. As a second example, the first author has shown [Klo14] that they imply a lower bound on a certain isometric defect of a continuous map  $\phi : M \rightarrow N$  between Riemannian manifolds. Both of these applications also yield sharp inequalities when the isoperimetric optimum is a metric ball, which will be the case for the main results in this article.

The isoperimetric profile is unknown for most manifolds. Even for symmetric spaces such as complex hyperbolic spaces, the isoperimetric profile is only conjectured. The simply connected  $n$ -manifolds  $X_{n,\kappa}$  of constant curvature  $\kappa$  (*i.e.* round spheres when  $\kappa > 0$ , Euclidean space when  $\kappa = 0$ , real hyperbolic spaces when  $\kappa < 0$ ) are the major exceptions: We know that in such a space, a metric ball  $B_{n,\kappa}(r)$  minimizes perimeter among domains of given volume. In each dimension  $n$ , let  $I_{n,\kappa}$  be the isoperimetric profile of  $X_{n,\kappa}$ . In this case, the volume  $|B_{n,\kappa}(r)|$  and its boundary volume  $|\partial B_{n,\kappa}(r)|$  are easily computable. Thus the profile is explicit, given by

$$I_{n,\kappa}(|B_{n,\kappa}(r)|) = |\partial B_{n,\kappa}(r)|.$$

Instead of calculating the isoperimetric profile of a given manifold, we can look for a sharp isoperimetric inequality in a whole class of manifolds. Since  $I_{n,\kappa}(V)$  decreases as a function of  $\kappa$  for each fixed  $V$ , a natural class to consider are manifolds with sectional curvature bounded above by some  $\kappa$ . This motivates the following well-known conjecture.

**Conjecture 1.2** (Generalized Cartan-Hadamard Conjecture). *If  $M$  is a complete, simply connected  $n$ -manifold with sectional curvature  $K$  bounded above by some  $\kappa \leq 0$ , then every domain  $\Omega \subseteq M$  satisfies*

$$|\partial\Omega| \geq I_{n,\kappa}(|\Omega|). \tag{2}$$

(If  $M$  is not simply connected, then there are elementary counterexamples, such as a closed manifold, with or without a small ball removed.)

The history of Conjecture 1.2 is as follows [Oss78, Dru10, Ber03]. In 1926, Weil [Wei26] established Conjecture 1.2 when  $n = 2$  and  $\kappa = 0$  for

Riemannian disks  $\Omega$ , without assuming an ambient manifold  $M$ , thus answering a question of Paul Lévy. Weil's result was established independently by Beckenbach and Radó [BR33], who are sometimes credited with the result. When  $n = 2$ , the case of disks implies the result for other topologies of  $\Omega$  in the presence of  $M$ . It was first established by Bol [Bol41] when  $n = 2$  and  $\kappa \neq 0$ . Rather later, Conjecture 1.2 was published independently by Aubin [Aub76] and Burago-Zalgaller for  $\kappa \leq 0$  [BZ88], and by Gromov [Gro81, Gro99]. The case  $\kappa = 0$  is called the Cartan-Hadamard conjecture, because a complete, simply connected manifold with  $K \leq 0$  is called a Cartan-Hadamard manifold.

Soon afterward, Croke proved Conjecture 1.2 in dimension  $n = 4$  with  $\kappa = 0$  [Cro84]. Kleiner [Kle92] proved Conjecture 1.2 in dimension  $n = 3$ , for all  $\kappa \leq 0$ , by a completely different method. (See also Ritoré [Rit05].) Morgan and Johnson [MJ00] established Conjecture 1.2 for *small* domains (see also Druet [Dru02] where the curvature hypothesis is on *scalar* curvature); however their argument does not yield any explicit size condition.

Actually, Croke does not assume an ambient Cartan-Hadamard manifold  $M$ , only the more general hypothesis that  $\Omega$  has unique geodesics. We believe that the hypotheses of Conjecture 1.2 are negotiable, and it has some generalization to  $\kappa > 0$ . But the conjecture is not as flexible as one might think; in particular, Conjecture 1.2 is false for Riemannian 3-balls. (See Theorem 1.9 below and Section 4.) With this in mind, we propose the following.

**Conjecture 1.3.** *If  $\Omega$  is a manifold with boundary with unique geodesics, if its sectional curvature is bounded above by some  $\kappa > 0$ , and if  $|\Omega| \leq |X_{n,\kappa}|/2$ , then  $|\partial\Omega| \geq I_{n,\kappa}(|\Omega|)$ .*

The volume restriction in Conjecture 1.3 is justified for two reasons. First, the comparison ball in  $X_{n,\kappa}$  only has unique geodesics when  $|\Omega| < |X_{n,\kappa}|/2$ . Second, Croke [Cro80] proved a curvature-free inequality, using only the condition of unique geodesics, that implies a sharp extension of Conjecture 1.3 when  $|\Omega| \geq |X_{n,\kappa}|/2$  (Theorem 1.14).

Of course, one can extend Conjecture 1.3 to negative curvature bounds. The resulting statement is strictly stronger than Conjecture 1.2, since every domain in a Cartan-Hadamard manifold has unique geodesics, but there are unique-geodesic manifolds that cannot embed in a Cartan-Hadamard manifold of the same dimension (Figure 3).

Another type of generalization of Conjecture 1.2 is one that assumes a bound on some other type of curvature. For example, Gromov [Gro81] suggests that Conjecture 1.2 still holds when  $K \leq \kappa$  is replaced by

$$K \leq 0, \quad \text{Ric} \leq (n-1)^2 \kappa g. \quad (3)$$

Meanwhile Croke [Cro84] only uses a non-local condition that we call  $\text{Candle}(0)$  rather than the curvature condition  $K \leq 0$ ; we state this as Theorem 1.12.

Our previous work [KK12] subsumes both of these two generalizations. More precisely, most of our results will be stated in terms of two volume comparison conditions,  $\text{Candle}(\kappa)$  and  $\text{LCD}(\kappa)$  (defined in Section 2.2), which follow from  $K \leq \kappa$  by Günther's inequality [Gün60, BC64]. We proved in [KK12] that they also are implied by a weaker curvature bound, on what we call the *root-Ricci curvature*  $\sqrt{\text{Ric}}$ . In turn, the mixed bound (3) implies our root-Ricci curvature condition.

**1.2. Main results.** For simplicity, we will consider isoperimetric inequalities only for compact, smooth Riemannian manifolds  $\Omega$  with smooth boundary  $\partial\Omega$ ; or for compact, smooth domains  $\Omega$  in Riemannian manifolds  $M$ . Our constructions will directly establish inequalities for all such  $\Omega$ . We therefore don't have to assume a minimizer or prove that one exists. Our results automatically extend to any limit of smooth objects in a topology in which volume and boundary volume vary continuously, *e.g.*, to domains with piecewise smooth boundary. Note that our uniqueness result, Theorem 1.7, does not automatically generalize to a limit of smooth objects; but its proof might well generalize to some limits of this type.

Our two strongest results are in the next two subsections. They both include Croke's theorem in dimension  $n = 4$  as a special case. Each theorem has a volume restriction that we can take to be vacuous when  $\kappa = 0$ .

### 1.2.1. The positive case.

**Theorem 1.4.** *Let  $\Omega$  be a compact Riemannian manifold with boundary, of dimension  $n \in \{2, 4\}$ . Suppose that  $\Omega$  has unique geodesics and is  $\text{Candle}(\kappa)$  with  $\kappa \geq 0$  (*e.g.*,  $K \leq \kappa$ ), and that  $|\Omega| \leq |X_{n,\kappa}|/2$ . Then  $|\partial\Omega| \geq I_{n,\kappa}(|\Omega|)$ .*

This is our fully general version of Theorem 1.1. As mentioned, Theorem 1.14 provides an optimal extension of Theorem 1.4 to the case  $|\Omega| \geq |X_{n,\kappa}|/2$ .

**1.2.2. The negative case.** When  $\kappa$  is negative and  $n = 4$ , we only get a partial result. (But see Section 9.) To state it, we let  $r_{n,\kappa}(V)$  be the radius of a ball of volume  $V$  in  $X_{n,\kappa}$ . We define  $\text{chord}(\Omega)$  to be the length of the longest geodesic in  $\Omega$ ; we have the elementary inequality

$$\text{chord}(\Omega) \leq \text{diam}(\Omega).$$

**Theorem 1.5.** *Let  $M$  be a Cartan-Hadamard manifold of dimension  $n \in \{2, 4\}$  which is  $\text{LCD}(\kappa)$  with  $\kappa \leq 0$  (*e.g.*,  $K \leq \kappa$ ). Let  $\Omega$  be a domain in  $M$ ,*

and if  $n = 4$ , suppose that

$$\tanh(\text{chord}(\Omega)\sqrt{-\kappa}) \tanh(r_{n,\kappa}(|\Omega|)\sqrt{-\kappa}) \leq \frac{1}{2}. \quad (4)$$

Then  $|\partial\Omega| \geq I_{n,\kappa}(|\Omega|)$ .

Actually, Theorem 1.5 only needs  $M$  to be convex with unique geodesics rather than Cartan-Hadamard. However, we do not know whether that is a more general hypothesis for  $\Omega$ . (See Section 4.)

The smallness condition (4) means that Theorem 1.5 is only a partial solution to Conjecture 1.2 when  $n = 4$ . Note that since  $\tanh(x) < 1$  for all  $x$ , it suffices that *either* the chord length *or* the volume of  $\Omega$  is small. *I.e.*, it suffices that

$$\sqrt{-\kappa} \min(\text{chord}(\Omega), r_{n,\kappa}(|\Omega|)) \leq \text{arctanh}\left(\frac{1}{2}\right) = \frac{\log(3)}{2}.$$

If we think of Conjecture 1.2 as parametrized by dimension, volume, and the curvature bound  $\kappa$ , then Theorem 1.5 is a complete solution for a range of values of the parameters.

1.2.3. *Pointwise illumination.* We prove a pointwise inequality which, in dimension 2, generalizes Weil's isoperimetric inequality [Wei26]. We state it in terms of *illumination* of the boundary of a domain  $\Omega$  by light sources lying in  $\Omega$ , defined rigorously in Section 3.

**Theorem 1.6.** *Let  $\Omega$  be a compact Riemannian  $n$ -manifold with boundary, with unique geodesics, and which is Candle(0); and let  $p \in \partial\Omega$ . If we fix the volume  $|\Omega|$ , then the illumination of  $p$  by a uniform light source in  $\Omega$  is maximized when  $\Omega$  is flat and is the polar plot*

$$r = k \cos(\theta)^{1/(n-1)} \quad (5)$$

for some constant  $k$ , with  $p$  at the origin. In particular, in dimension  $n = 2$ , the optimum  $\Omega$  is a round disk.

Theorem 1.6 generalizes the elementary Proposition 3.1, the problem of the Little Prince, which was part of the inspiration for the present work.

When  $n = 2$ , Theorem 1.6 shows that a flat round disk maximizes illumination simultaneously at all points of its boundary, and therefore maximizes the *average* illumination over the boundary. But, as a consequence of the divergence theorem, the *total* illumination over the boundary is proportional to  $|\Omega|$ . A flat round disk must therefore minimize  $|\partial\Omega|$ , which is precisely Weil's theorem.

1.2.4. *Equality cases.* We also characterize the equality cases in Theorems 1.4 and 1.5, with a moderate weakening when  $\kappa = 0$ .

**Theorem 1.7.** *Suppose that  $\Omega$  is optimal in Theorem 1.4 or 1.5. When  $\kappa = 0$ , suppose further that  $\Omega$  is  $\sqrt{\text{Ric}}$  class 0. Then  $\Omega$  is isometric to a metric ball in  $X_{n,\kappa}$ .*

Again, see Section 2.2 for the definition of root-Ricci curvature  $\sqrt{\text{Ric}}$ . In particular,  $\sqrt{\text{Ric}}$  class 0 implies  $\text{Candle}(0)$ , but it does not implies  $K \leq 0$  when  $n > 2$ .

We will prove Theorem 1.7 in Section 8.1; see also Section 9.

1.2.5. *Relative inequalities and multiple images.* Choe [Cho03, Cho06] generalizes Weil's and Croke's theorems in dimensions 2 and 4 to domains  $\Omega \subseteq M$  that share part of their boundary with a convex domain  $C$ , where now the surface volume to minimize is  $|\partial\Omega \setminus \partial C|$ . The optimum in both cases is half of a Euclidean ball.

Choe's method in dimension 4 is to consider reflecting geodesics that reflect from  $\partial C$  like light rays. (This dynamic is also called *billiards*, but we use optics as our principal metaphor.) Such an  $\Omega$  cannot have unique reflecting geodesics; rather two points in  $\Omega$  are connected by at most two geodesics. We generalize Choe's result by bounding the number of connecting geodesics by any positive integer.

**Theorem 1.8.** *Let  $\Omega$  be a compact  $n$ -manifold with boundary with  $n = 2$  or 4, let  $\kappa \geq 0$ , and let  $W \subset \partial\Omega$  be a (possibly empty)  $(n - 1)$ -dimensional submanifold. Suppose that  $\Omega$  is  $\text{Candle}(\kappa)$  for geodesics that reflect from  $W$  as a mirror, and suppose that every pair of points in  $\Omega$  can be linked by at most  $m$  (possibly reflecting) geodesics. Suppose also that*

$$|\Omega| \leq \frac{|X_{n,\kappa}|}{2m}.$$

Then

$$|\partial\Omega \setminus \partial W| \geq \frac{I_{n,\kappa}(m|\Omega|)}{m}. \quad (6)$$

Note that Günther's inequality generalizes to this case (Proposition 5.8): If  $\Omega$  satisfies  $K \leq \kappa$ , and if the mirror region  $W$  is locally concave, then  $\Omega$  is  $\text{LCD}(\kappa)$  and therefore  $\text{Candle}(\kappa)$  for reflecting geodesics.

Theorem 1.8 is sharp, as can be seen from various examples. Let  $G$  be a finite group that acts on the ball  $B_{n,\kappa}(r)$  by isometries. Then the orbifold quotient  $\Omega = B_{n,\kappa}(r)/G$  matches the bound of Theorem 1.8, if we take the reflection walls to be mirrors and if we take  $m = |G|$ . Although  $\Omega$  has lower-dimensional strata where it fails to be a smooth manifold, we can remove

thin neighborhoods of them and smooth all ridges to make a manifold with nearly the same volume and boundary volume.

We could state a version of Theorem 1.8 for  $\kappa < 0$  using the  $\text{LCD}(\kappa)$  condition, but it would be much more restricted because we would require an ambient  $M$  in which every two points are connected by *exactly*  $m$  geodesics. We do not know any interesting example of such an  $M$ . (E.g., if the boundary of  $M$  is totally geodesic, then this case is equivalent to simply doubling  $M$  and  $\Omega$  across  $\partial M$ .)

1.2.6. *Counterexamples in dimension 3.* We find counterexamples to justify the hypotheses of an ambient Cartan-Hadamard manifold and unique geodesics in Conjectures 1.2 and 1.3. One might like to replace these geometric hypotheses by purely topological ones, but we show that even the strongest topological assumption does not imply any isoperimetric inequality.

**Theorem 1.9.** *For every two numbers  $A, V > 0$ , there is a Riemannian 3-ball  $\Omega$  with sectional curvature  $K \leq -1$  such that  $|\Omega| = V$  and  $|\partial\Omega| = A$ .*

1.3. **The linear programming model.** Our method to prove Theorems 1.4 and 1.5 (and indirectly Theorem 1.6) is a reinterpretation and generalization of Croke's argument, based on optical transport, optimal transport, and linear programming.

We simplify our manifold  $\Omega$  to a measure  $\mu_\Omega$  on the set of triples  $(\ell, \alpha, \beta)$ , where  $\ell$  is length of a complete geodesic  $\gamma \subseteq \Omega$  and  $\alpha$  and  $\beta$  are its boundary angles. Thus  $\mu_\Omega$  is always a measure on the set  $\mathbb{R}_{\geq 0} \times [0, \pi/2)^2$ , regardless of the geometry or even the dimension of  $\Omega$ . We then establish a set of linear constraints on  $\mu_\Omega$ , by combining Theorem 5.3 (more precisely equations (21) and (22)) with Lemmas 5.4, 5.5, and 5.6. The result is the basic LP Problem 6.1 and an extension 7.2. The constraints of the model depend on the volume  $V = |\Omega|$  and the boundary volume  $A = |\partial\Omega|$ , among other parameters.

Given such a linear programming model, we can ask for which pairs  $(V, A)$  the model is *feasible*; i.e., does there exist a measure  $\mu$  that satisfies the constraints? On the one hand, this is a vastly simpler problem than the original Conjecture 1.2, an optimization over all possible domains  $\Omega$ . On the other hand, the isoperimetric problem, minimizing  $A$  for any fixed  $V$ , becomes an interesting question in its own right in the linear model.

Regarding the first point, finite linear programming is entirely algorithmic: It can be solved in practice, and provably in polynomial time in general. Our linear programming models are infinite, which is more complicated and should technically be called convex programming. Nonetheless,



each model has the special structure of optimal transport problems, with finitely many extra parameters. Optimal transport is even nicer than general linear programming. All of our models are algorithmic in principle. In fact, our proofs of optimality in the two most difficult cases are computer-assisted using Sage [Sage].

Regarding the second point, our model is successful in two different ways: First, even though it is a relaxation, it sometimes yields a sharp isoperimetric inequality, *i.e.*, Theorems 1.4, 1.5, and 1.8. Second, our models subsume several previously published isoperimetric inequalities. We mention six significant ones. Note that the first four, Theorems 1.10-1.13, are special cases of Theorems 1.4, 1.5, and 1.8 as mentioned in Section 1.2. The other two results are separate, but they also hold in our linear programming models.

**Theorem 1.10** (Variation of Weil [Wei26] and Bol [Bol41]). *Let  $\Omega$  be a compact Riemannian surface with curvature  $K \leq \kappa \geq 0$  with unique geodesics, and suppose that  $\sqrt{\kappa}|\Omega| \leq 2\pi$ . Then for fixed area  $|\Omega|$ , the perimeter  $|\partial\Omega|$  is minimized when  $|\Omega|$  has constant curvature  $K = \kappa$  and is round.*

**Theorem 1.11** (Variation of Bol [Bol41]). *Suppose that  $\Omega \subseteq M$  is a domain in a Cartan-Hadamard surface  $M$  that satisfies  $K \leq \kappa \leq 0$ . Then for fixed area  $|\Omega|$ , the perimeter  $|\partial\Omega|$  is minimized when  $|\Omega|$  has constant curvature  $K = \kappa$  and is round.*

**Theorem 1.12** (Croke [Cro84]). *If  $\Omega$  is a compact 4-manifold with boundary, with unique geodesics, and which is  $\text{Candle}(0)$ , then for each fixed volume  $|\Omega|$ , the boundary volume  $|\partial\Omega|$  is minimized when  $\Omega$  is flat and round.*

**Theorem 1.13** (Choe [Cho03, Cho06]). *Let  $M$  be a Cartan-Hadamard manifold of dimension  $n \in \{2, 4\}$ , and let  $\Omega \subseteq M$  be a domain whose interior is disjoint from a convex domain  $C \subseteq M$ . Then*

$$|\partial\Omega \setminus \partial C| \geq \frac{I_0(2|\Omega|)}{2}.$$

**Theorem 1.14** (Croke [Cro80]). *If  $\Omega$  is an  $n$ -manifold with boundary with unique geodesics, then  $|\partial\Omega| \geq |\partial Y_{n,\rho}|$  where  $Y_{n,\rho}$  is a hemisphere with constant curvature  $\rho$  and  $\rho$  is chosen so that  $|\Omega| = |Y_{n,\rho}|$ .*

Note that when  $|\Omega| \geq |X_{n,\kappa}|/2$ , we obtain  $\rho \leq \kappa$ , so that Croke's inequality extends Theorem 1.4, as promised. See the end of Section 8.5.2 for further remarks about this result.

**Theorem 1.15** (Yau [Yau75]). *Let  $M$  be a Cartan-Hadamard  $n$ -manifold which is  $\text{LCD}(\kappa)$  with  $\kappa < 0$ . Then every domain  $\Omega \subseteq M$  satisfies*

$$|\partial\Omega| \geq (n-1)\sqrt{-\kappa}|\Omega|.$$



Finally, we state the result that our models subsume all of these bounds.

**Theorem 1.16.** *Let  $\mu$  be a measure that satisfies the LP Problem 6.1, with formal dimension  $n$ , formal curvature bound  $\kappa$ , formal volume  $V(\mu)$ , and formal boundary volume  $A(\mu)$ . Then  $\mu$  satisfies Theorem 1.4 and therefore 1.10. If  $\mu$  satisfies LP Problem 7.2, then it satisfies Theorems 1.15 and 1.5, and therefore 1.11. If  $\mu$  satisfies LP Problem 8.3, then it satisfies 1.14. If  $\mu$  satisfies the LP model 8.1, then it satisfies Theorem 1.8 and therefore 1.13.*

We will prove some cases of Theorem 1.16 in the course of proving our other results; the remaining cases will be done in Section 8.5.

Our linear programming models are similar to the important Delsarte linear programming method in the theory of error-correcting codes and sphere packings [Del72, CS93, CE03]. Delsarte’s original result was that many previously known bounds for error-correcting codes are subsumed by a linear programming model. But his model also implies new bounds, including sharp bounds. For example, consider the kissing number problem for a sphere in  $n$  Euclidean dimensions. The geometric maximum is of course an integer, but in a linear programming model this may no longer be true. Nonetheless, Odlyzko and Sloane established a sharp geometric bound in the Delsarte model, which happens to be an integer and the correct one, in dimensions 2, 8, and 24. (The bounds are, respectively, 6, 240, and 196,560 kissing spheres.) The basic Delsarte bound for the sphere kissing problem is quite strong in other dimensions, but it is not usually an integer and not usually sharp even if rounded down to an integer.

Another interesting common feature of the Delsarte method and ours is that they are both sets of linear constraints satisfied by a “two-point correlation function”, *i.e.*, a measure derived from taking pairs of points in the geometry.

#### 1.4. Other results.

1.4.1. *Croke in all dimensions.* There is a natural version of Croke’s theorem in all dimensions. This is a generalized, sharp isoperimetric inequality in which the volume of  $\Omega$  is replaced by some other functional when the dimension  $n \neq 4$ . This result might not really be new; we state it here to further illustrate of our linear programming model.

If  $\Omega$  is a manifold with boundary and unique geodesics, then the space  $G$  of geodesic chords in  $\Omega$  carries a natural measure  $\mu_G$ , called Liouville measure or étendue (Section 5).

**Theorem 1.17.** *Let  $\Omega$  be a compact manifold with boundary of dimension  $n \geq 4$ , with unique geodesics, and with non-positive sectional curvature.*

Let

$$L(\Omega) = \int_G \ell(\gamma)^{n-3} d\mu_G(\gamma)$$

If  $B_{n,0}(r)$  is the round, Euclidean ball such that

$$L(\Omega) = L(B_{n,0}(r)),$$

then

$$|\partial\Omega| \geq |\partial B_{n,0}(r)|.$$

By Theorem 5.3 (Santaló's equality),

$$\omega_{n-1}|\Omega| = \int_G \ell(\gamma) d\mu_G(\gamma).$$

(Here  $\omega_n = |X_{n,1}|$  is the  $n$ -sphere volume; see Section 2.1.) Thus Theorem 1.17 is Croke's Theorem if  $n = 4$ . The theorem is plainly a sharp isoperimetric bound for the boundary volume  $|\partial\Omega|$  in all cases given the value of  $L(\Omega)$ , which happens to be proportional to the volume  $|\Omega|$  only when  $n = 4$ .

Similar results are possible with a curvature bound  $K < \kappa$ , only with more complicated integrands  $F(\ell)$  over the space  $G$ .

1.4.2. *Non-sharp bounds and future work.* We mention three cases in which the methods of this paper yield improved non-sharp results.

First, when  $n = 3$  and  $\kappa = 0$ , Problem 6.1 yields a non-sharp version of Kleiner's theorem under the weaker hypotheses of Candle(0) and unique geodesics. Croke [Cro84] established the isoperimetric inequality in this case up to a factor of  $\sqrt[3]{36/32} = 1.040\dots$ . Meanwhile Theorem 1.6 implies the same isoperimetric inequality up to a factor of  $\sqrt[3]{27/25} = 1.026\dots$ . The wrinkle is that Croke's proof uses only (34), while Theorem 1.6 uses only (35). These two inequalities compensate for each other's inefficiency, and the combined linear programming problem should produce a superior if still non-sharp bound.

Second, it is a well-known conjecture that a metric ball is the unique optimum to the isoperimetric problem for domains in the complex hyperbolic plane  $\mathbb{C}H^2$ . (The same conjecture is proposed for any non-positively curved symmetric space of rank 1.) Suppose that we normalize the metric on  $\mathbb{C}H^2$  so that the curvature is pinched between  $-4$  and  $-1$ . Then it is easy to check that  $\mathbb{C}H^2$  is LCD( $-16/9$ ), and then the Theorem 1.5 is, to our knowledge, better than what was previously established for moderately small volumes. Even so, this is a crude bound because what we would really want to do is make a version of Problem 7.2 using the specific candle function of  $\mathbb{C}H^2$ .

Third, even for domains in Cartan-Hadamard manifolds with  $K \leq -1$  (or more generally LCD( $-1$ )), we can relax the smallness condition (4) in

Theorem 1.5 simply by increasing the curvature bound  $\kappa$  from  $\kappa = -1$ . This is still a good bound for a range of volumes until it is eventually surpassed by Theorem 1.15. This too is a crude bound that can surely be improved, given that both Theorem 1.5 and Theorem 1.15 hold in the same linear programming model, Problem 7.2.

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## 2. CONVENTIONS

**2.1. Basic conventions.** If  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is an integrable function, we let

$$f^{(-1)}(x) \stackrel{\text{def}}{=} \int_0^x f(t) dt$$

be its antiderivative that vanishes at 0, and then by induction its  $n$ th antiderivative  $f^{(-n)}$ . This is in keeping with the standard notation that  $f^{(n)}$  is the  $n$ th derivative of  $f$  for  $n > 0$ .

If  $M$  is a Riemannian manifold, we let  $\nu_M$  denote the Riemannian measure on  $M$ . As usual,  $TM$  is the tangent bundle of  $M$ , while we use  $UM$  to denote the unit tangent bundle. Also, if  $\Omega$  is a manifold with boundary  $\partial\Omega$ , then we let

$$U^+ \partial\Omega \stackrel{\text{def}}{=} \{u = (p, \nu) \mid p \in \partial\Omega, \nu \in U_p\Omega \text{ inward pointing}\}.$$

We let  $|M|$  be the volume of  $M$ :

$$|M| \stackrel{\text{def}}{=} \int_M d\nu_M.$$

We let

$$\omega_n = |X_{n,1}| = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$$

be the volume of the unit  $n$ -dimensional sphere  $X_{n,1} = S^n \subseteq \mathbb{R}^{n+1}$ .

**2.2. Candles.** Our main results are stated in terms of conditions  $\text{Candle}(\kappa)$  and  $\text{LCD}(\kappa)$  that follow from the sectional curvature condition  $K \leq \kappa$  by Günther's comparison theorem [Gün60, BC64]. These conditions are non-local, but in previous work [KK12], we showed that they follow from another local condition, more general than  $K \leq \kappa$  that we called  $\sqrt{\text{Ric}}$  class  $(\rho, \kappa)$ . The original motivation is that Croke's theorem only needs that the manifold  $D$  is  $\text{Candle}(0)$ , and even then only for pairs of boundary points. Informally, a Riemannian manifold  $M$  is  $\text{Candle}(\kappa)$  if a candle at any given

distance  $r$  from an observer is dimmer than it would be at distance  $r$  in a geometry of constant curvature  $\kappa$ .

More rigorously, let  $M$  be a Riemannian manifold and let  $\gamma = \gamma_u$  be a geodesic in  $M$  that begins at  $p = \gamma(0)$  with initial velocity  $u \in U_p M$ . Then the *candle function*  $j_M(\gamma, r)$  of  $M$  is by definition the normalized Jacobian of the exponential map

$$u \mapsto \gamma_u(r) = \exp_p(ru),$$

given by the equation

$$dv_M(\gamma_u(r)) = j_M(\gamma_u, r) dv_{U_p M}(u) dr$$

for  $r > 0$ , where  $v_M$  is the Riemannian volume on  $M$  and  $v_{U_p M}$  is the Riemannian measure on the round unit sphere  $U_p M$ . More generally, if  $a < b$ , we define

$$j_M(\gamma, a, b) = j_M(\gamma_a, b - a),$$

where  $\gamma_a$  is the same geodesic as  $\gamma$  but with parameter shifted by  $a$ . We also define

$$j_M(\gamma, b, a) = j_M(\bar{\gamma}_b, b - a),$$

where  $\bar{\gamma}_b$  is the same geodesic as  $\gamma$ , but reversed and based at  $\gamma(b)$ . (But see Corollary 5.2.)

The candle function of the constant-curvature geometry  $X_{n, \kappa}$  is independent of the geodesic. We denote it by  $s_{n, \kappa}(r)$ ; it is given by the following explicit formulas:

$$s_{n, \kappa}(r) = \begin{cases} \left( \frac{\sin(r\sqrt{\kappa})}{\sqrt{\kappa}} \right)^{n-1} & \text{if } \kappa > 0, r \leq \frac{\pi}{\sqrt{\kappa}} \\ r^{n-1} & \text{if } \kappa = 0 \\ \left( \frac{\sinh(r\sqrt{-\kappa})}{\sqrt{-\kappa}} \right)^{n-1} & \text{if } \kappa < 0. \end{cases} \quad (7)$$

We will also need the extension  $s_{n, \kappa}(r) = 0$  when  $\kappa > 0$  and  $r \geq \pi/\sqrt{\kappa}$ .

**Definition.** An  $n$ -manifold  $M$  is  $\text{Candle}(\kappa)$  if

$$j_M(\gamma, r) \geq s_{n, \kappa}(r)$$

for all  $\gamma$  and  $r$ . It is  $\text{LCD}(\kappa)$ , for *logarithmic candle derivative*, if

$$\log(j_M(\gamma, r))' \geq \log(s_{n, \kappa}(r))'$$

for all  $\gamma$  and  $r$ . (Here the derivative is with respect to  $r$ .) The  $\text{LCD}(\kappa)$  condition implies the  $\text{Candle}(\kappa)$  condition by integration. If  $\kappa > 0$ , then these conditions are only required up to the focal distance  $\pi/\sqrt{\kappa}$  in the comparison geometry.

To illustrate how  $\text{Candle}(\kappa)$  is more general than  $K \leq \kappa$ , we mention *root-Ricci curvature* [KK12]. Suppose that  $M$  is a manifold such that  $K \leq 0$  and let  $\kappa < 0$ . For any unit tangent vector  $u \in U_p M$  with  $p \in M$ , we define

$$\sqrt{\text{Ric}}(u) \stackrel{\text{def}}{=} \text{Tr}(\sqrt{-R(\cdot, u, \cdot, u)}).$$

Here  $R(u, v, w, x)$  is the Riemann curvature tensor expressed as a tetralinear form, and the square root is the positive square root of a positive semidefinite matrix or operator. We say that  $M$  is of  $\sqrt{\text{Ric}}$  class  $\kappa$  if  $K \leq 0$  and

$$\sqrt{\text{Ric}}(u) \geq (n-1)\sqrt{-\kappa}.$$

Then

$$\begin{aligned} K \leq \kappa &\implies \sqrt{\text{Ric}} \text{ class } \kappa \implies \text{LCD}(\kappa) \\ &\implies \text{Candle}(\kappa) \implies \text{Ric} \leq (n-1)\kappa g. \end{aligned}$$

The second implication, from  $\sqrt{\text{Ric}}$  to LCD, is the main result of [KK12]. (We also established a version of the result that applies for any  $\kappa \in \mathbb{R}$ . This version uses a generalized  $\sqrt{\text{Ric}}$  class  $(\rho, \kappa)$  condition that also requires  $K \leq \rho$  for a constant  $\rho > \max(\kappa, 0)$ .) All implications are strict when  $n > 2$ . By contrast in dimension 2, the last condition trivially equals the first one, so all of the conditions are equivalent.

We conclude with two examples of 4-manifolds of  $\sqrt{\text{Ric}}$  class  $-1$ , and which are therefore LCD( $-1$ ), but that do not satisfy  $K \leq -1$ :

- The complex hyperbolic plane, normalized to have sectional curvature between  $-\frac{9}{4}$  and  $-\frac{9}{16}$ .
- The product of two simply connected surfaces that each satisfy  $K < -9$ .

Actually, the most important regime where  $\text{Candle}(\kappa)$  is weaker than  $K \leq \kappa$  is at short distances. Since  $j'(\gamma, 0)$ , the derivative at 0 of the candle function, is only sensitive to the Ricci curvature of  $M$ , we can write informally that

$$\text{Candle}(\kappa) \stackrel{\approx}{\iff} \text{Ric} \leq (n-1)\kappa g$$

as  $\text{diam}(M) \rightarrow 0$ .

### 3. THE LITTLE PRINCE AND OTHER STORIES

**3.1. The problem of the Little Prince.** As Saint-Exupéry related to inhabitants of our planet, the Little Prince lives on his own planet, also known as asteroid B-612 (Figure 1). Since this planet is not very big, its gravitational pull is small and its habitation is precarious. The question arises as to what shape it should be to maximize the normal component of gravity for the Little Prince, assuming that the planet has a fixed mass, and a uniform, fixed mass density. Let  $\Omega$  be the shape of the planet. The divergence theorem

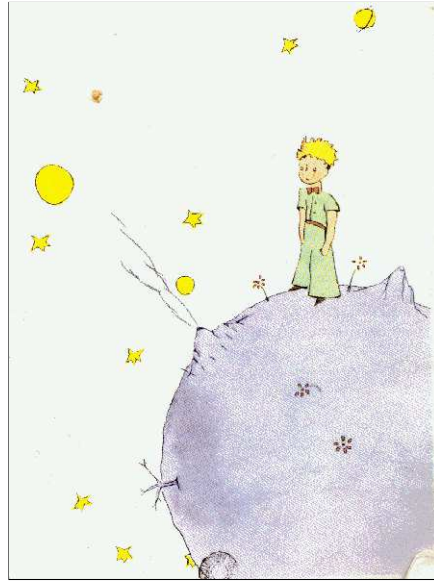


FIGURE 1. The Little Prince on his not-very-big planet, actually an asteroid.

tells us that the average normal gravity is proportional to  $|\Omega|/|\partial\Omega|$ , so maximizing the average would be exactly the isoperimetric problem. Suppose instead that the Little Prince has a favorite location, and does not mind less gravity elsewhere. (After all, in the illustrations he usually stands on top of the planet.)

We cannot be sure of the dimension of the Little Prince or his planet. The illustrations are 2-dimensional, but the Prince visits the Sahara Desert which suggests that he is 3-dimensional. In any case higher-dimensional universes, which are a fashionable topic in physics these days, would each presumably have their own Little Prince. So we assume that the Little Prince is  $n$ -dimensional for some  $n \geq 2$ . We first assume Newtonian gravity and therefore a Euclidean planet; recall that in  $n$  dimensions, a divergenceless central gravitational force is proportional to  $r^{1-n}$ .

**Proposition 3.1** (Little Prince Problem). *Let  $\Omega$  be the shape of a planet in  $n$  Euclidean dimensions with a pointwise gravitational force proportional to  $r^{1-n}$ . Suppose that the planet has a fixed volume  $|\Omega|$  and a uniform, fixed mass density, and let  $p \in \partial\Omega$ . Then the total normal gravitational force  $F(\Omega, p)$  at  $p$  is maximized when  $\Omega$  is bounded by the surface  $r = k \cos(\theta)^{1/(n-1)}$  for some constant  $k$ , in spherical coordinates centered at  $p$ .*

The problem of the Little Prince in 3 dimensions is sometimes used as an undergraduate physics exercise [McD03]. It has also been previously used to prove the isoperimetric inequality in 2 dimensions [HHM99]. However, our further goal is inequalities for curved spaces such as Theorem 1.6.

*Proof.* For convenience, we assume that the gravitational constant and the mass density of the planet are both 1. Given  $x \in \Omega$ , let  $r = r(x)$  and  $\theta = \theta(x)$  be the radius and first angle in spherical coordinates with the point  $p$  at the origin, and such that the normal component of gravity is in the direction  $\theta = 0$ . Then the total gravitational effect of a volume element  $dx$  at  $x$  is  $\cos(\theta)r^{1-n} dx$ , so the total gravitational force is

$$F(\Omega, p) = \int_{x \in \Omega} \cos(\theta)r^{1-n} dx.$$

In general, if  $f(x)$  is a continuous function and we want to choose a region  $\Omega$  with fixed volume to maximize

$$\int_{\Omega} f(x) dx,$$

then by the “bathtub principle”,  $\Omega$  should be bounded by a level curve of  $f$ , *i.e.*,

$$\Omega = f^{-1}([k, \infty))$$

for some constant  $k$ . Our  $f$  is not continuous at the origin, but the principle still applies. Thus  $\Omega$  is bounded by a surface of the form

$$r = k \cos(\theta)^{1/(n-1)}. \quad \square$$

As explained above in words, the integral over  $\partial\Omega$  of the normal component of gravity is proportional to  $|\Omega|$  by the divergence theorem. More rigorously: We switch to a vector expression for gravitational force and we do not assume that  $p = 0$ . Then

$$F(\Omega, p) = \int_{\Omega} -(x-p)|x-p|^{-n} dx.$$

Since  $-(x-p)|x-p|^{-n}$  is divergenceless except at its singularity, for each fixed  $x \in \text{Int}(\Omega)$ ,

$$\int_{\partial\Omega} \langle -w(p), x-p \rangle |x-p|^{-n} dp = \omega_{n-1},$$

where  $w(p)$  is the outward unit normal vector at  $p$ . Thus

$$\int_{\partial\Omega} \langle -w(p), F(\Omega, p) \rangle dp = \omega_{n-1} |\Omega|$$

by switching integrals. Then

$$\omega_{n-1} |\Omega| \leq |\partial\Omega| F_{\max}, \quad (8)$$



where  $F_{\max}$  is the upper bound established by Proposition 3.1.

In particular, when  $n = 2$ , the optimum  $\Omega$  is the polar plot of  $r = k \cos(\theta)$ , which is a round circle. In this case

$$\langle -z(p), F(\Omega, p) \rangle = F_{\max}$$

at all points simultaneously. Thus when  $n = 2$ , equation (8) is exactly the sharp isoperimetric inequality (2).

**3.2. Illumination and Theorem 1.6.** Proposition 3.1 is close to a special case of Theorem 1.6. To make it an actual special case, we slightly change its mathematics and its interpretation, but we will retain the sharp isoperimetric corollary using the divergence theorem. Instead of the shape of a planet, we suppose that  $\Omega$  is the shape of a uniformly lit room, and we let  $I(\Omega, p)$  be the total intensity of light at a point on the wall  $p \in \partial\Omega$ . More rigorously, if  $\text{Vis}(\Omega, p)$  is the subset of  $\Omega$  which is visible from  $p$  (assuming that the walls are opaque), then

$$I(\Omega, p) = \int_{\text{Vis}(\Omega, p)} \langle z(p), x - p \rangle |x - p|^{-n} dx.$$

We still have

$$\int_{\text{Vis}(\partial\Omega, x)} \langle z(p), x - p \rangle |x - p|^{-n} dp = \omega_{n-1}$$

and we can still exchange integrals. Moreover,

$$I(\Omega, p) = F(\Omega, p)$$

when  $\Omega$  is convex. Thus, this variation of Proposition 3.1 is also true and also implies (2).

We now consider the case when  $\Omega$  is a curved Riemannian manifold, that is, Theorem 1.6. The proof is a simplified version of the proof of Theorems 1.4 and 1.5. Before giving the proof, we give a rigorous definition of illumination in the curved setting. (The definition agrees with the natural geometric assumption that light rays travel along geodesics.)

Let  $\Omega$  be a compact Riemannian  $n$ -manifold with boundary and unique geodesics. For any  $x \in \Omega$ , we define a tangent vector field  $v_x$  on  $\Omega$  which is uniquely determined by the following conditions:

1.  $v_x$  is radial; *i.e.*, if  $y$  is visible from  $x$ , then  $v_x(y)$  is tangent to the geodesic from  $x$  to  $y$ .
2.  $\text{div } v_x = \omega_{n-1} \delta_x$ , where  $\delta_x$  is a Dirac delta measure at  $x$ .
3.  $v_x(y) = 0$  if  $y$  is not visible from  $x$ .

Concerning property 2, note that  $v_x$  has a well-defined divergence and can be used for the divergence theorem, even though it is discontinuous when  $\Omega$  is non-convex.

We fix a point  $p \in \partial\Omega$  and again let  $w(p)$  be the outward unit normal vector to  $\partial\Omega$  at  $p$ . Then the illumination at  $p$  is defined by

$$I(\Omega, p) = \int_{\text{Vis}(\Omega, p)} \langle w(p), v_x(p) \rangle d\nu_\Omega(x).$$

*Proof of Theorem 1.6.* First, we express  $I(\Omega, p)$  as an integral over  $U = U_p^+ \partial\Omega$ , the unit inward tangent vectors at  $p$ . Given  $u \in U$ , let  $\ell(u)$  be the length of the maximal geodesic segment defined by  $u$  and let  $\alpha(u)$  be the angle of  $u$  with the inward normal  $-w(p)$ . Then, in polar coordinates we get

$$\begin{aligned} I(\Omega, p) &= \int_U \int_0^{\ell(u)} \cos(\alpha(u)) dt d\nu_U(u) \\ &= \int_U \ell(u) \cos(\alpha(u)) d\nu_U(u). \end{aligned}$$

The first equality expresses the fact that the norm  $\|v_x(p)\|$  is exactly reciprocal to the Jacobian of the exponential map from  $p$ . In other words, it is based on an optical symmetry principle (Corollary 5.2): If two identical candles are at  $x$  and  $p$ , then each one looks exactly as bright from the position of the other one.

Second, the Candle(0) hypothesis tells us that

$$\begin{aligned} |\Omega| &\geq |\text{Vis}(\Omega, p)| = \int_{\text{Vis}(\Omega, p)} d\nu_\Omega(x) \\ &\geq \int_U \int_0^{\ell(u)} t^{n-1} dt d\nu_U(u), \end{aligned}$$

so that

$$|\Omega| \geq \int_U \frac{\ell(u)^n}{n} d\nu_U(u). \quad (9)$$

Third, we apply the linear programming philosophy that will be important in the rest of the paper.

All of our integrands depend only on  $\ell$  and  $\alpha$ . Thus we can summarize all available information by projecting the measure  $\nu_U$  to a measure

$$\sigma_\Omega = (\ell, \alpha)_*(d\nu_U)$$

on the space of pairs

$$(\ell, \alpha) \in \mathbb{R}_{\geq 0} \times [0, \frac{\pi}{2}).$$

Then we want to maximize

$$I = \int_{\ell, \alpha} \ell \cos(\alpha) d\sigma_\Omega \quad (10)$$

subject to the constraint

$$\int_{\ell, \alpha} \frac{\ell^n}{n} d\sigma_{\Omega} \leq V. \quad (11)$$

We have one other linear piece of information: If we project volume on the hemisphere  $U$  into the angle coordinate  $\alpha \in [0, \frac{\pi}{2})$ , then the result is

$$\alpha_*(d\sigma_{\Omega}) = \alpha_*(d\nu_U) = \omega_{n-2} \sin(\alpha)^{n-2} d\alpha, \quad (12)$$

since the latitude on  $U$  at angle  $\alpha$  is an  $(n-2)$ -sphere with radius  $\sin(\alpha)$ .

We temporarily ignore geometry and maximize (10) for an abstract positive measure  $\sigma = \sigma_{\Omega}$  that satisfies (11) and (12). To do this, choose  $a > 0$ , and let

$$f(\alpha) = \sup_{\ell > 0} \left( \ell \cos(\alpha) - \frac{a\ell^n}{n} \right). \quad (13)$$

We obtain

$$0 \leq \int_{\ell, \alpha} \left( f(\alpha) + \frac{a\ell^n}{n} - \ell \cos(\alpha) \right) d\sigma(\ell, \alpha) \quad (14)$$

$$\leq \int_0^{\pi/2} f(\alpha) \omega_{n-2} \sin(\alpha)^{n-2} d\alpha + aV - I. \quad (15)$$

The integral on the right side of (15) is function of  $a$  only. Finally (15) is an upper bound on  $I$ , one that achieves equality if (11) is an equality and  $\sigma = \sigma_{\Omega}$  is supported on the locus

$$\cos(\alpha) = a\ell^{n-1},$$

because that is the maximand of (13). The first condition tells us that  $\Omega$  is flat and visible from  $p$ . The second gives us the polar plot (5) if we take  $k = a^{-1/(n-1)}$ .  $\square$

*Remark.* It is illuminating to give an alternate Croke-style end to the proof of Theorem 1.6. Namely, Hölder's inequality says that

$$\begin{aligned} I &= \int_{\ell, \alpha} \ell \cos(\alpha) d\sigma_{\Omega} \\ &\leq \left( \int_{\ell, \alpha} \ell^n d\sigma_{\Omega} \right)^{\frac{1}{n}} \left( \int_{\ell, \alpha} \cos(\alpha)^{\frac{n}{n-1}} d\sigma_{\Omega} \right)^{\frac{n-1}{n}} \\ &\leq (nV)^{\frac{1}{n}} \left( \int_{\ell, \alpha} \cos(\alpha)^{\frac{n}{n-1}} \omega_{n-2} \sin(\alpha)^{n-2} d\alpha \right)^{\frac{n-1}{n}}. \end{aligned}$$

The last expression depends only on  $V$  and  $n$ , while the inequality is an equality if (11) is an equality, and if

$$\ell^n \propto \cos(\alpha)^{\frac{n}{n-1}}.$$

The first condition again tells us that  $\Omega$  is flat and visible from  $p$ ; the second one gives us the same promised shape (5).

The Croke-style argument looks simpler than our proof of Theorem 1.6, but what was elegance becomes misleading for our purposes. For one reason, our use of the auxiliary  $f(\alpha)$  amounts to a proof of this special case of Hölder’s inequality. Thus our argument is not really different; it is just another way to describe the linear optimization. For another, we will see more complicated linear programming problems in the full generality of Theorems 1.4 and 1.5 that do not reduce to Hölder’s inequality.

#### 4. TOPOLOGY AND GEODESICS

In this section we will analyze the effect of topology and geodesics on isoperimetric inequalities. The end of the section is a proof of Theorem 1.9.

Weil and Bol established the sharp isoperimetric inequality (2) for Riemannian disks  $\Omega$  with curvature  $K \leq \kappa$ , without any ambient manifold  $M$  and for any  $\kappa \in \mathbb{R}$ . The cases  $\kappa \leq 0$  of the Weil and Bol theorems is equivalent to the  $n = 2$  case of Conjecture 1.2 [Dru10].

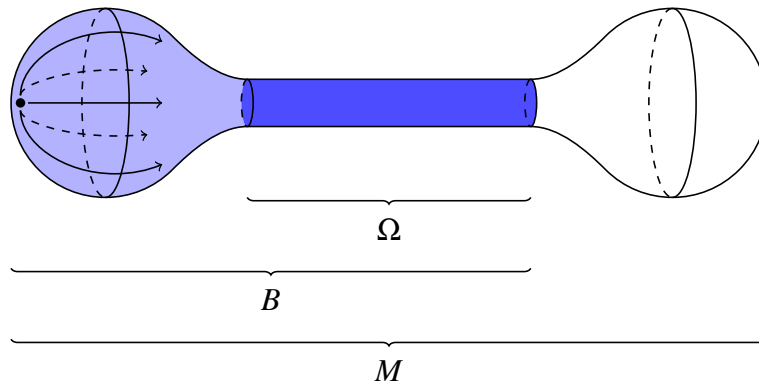


FIGURE 2. A counterexample  $\Omega \subseteq B \subseteq M$  to Aubin’s conjecture with  $\kappa > 0$ , in which both  $|\Omega|$  and  $|\partial\Omega|$  are unrestricted.

The case  $\kappa > 0$  is more delicate, even in 2 dimensions. Aubin [Aub76] assumed that  $B$  is a Riemannian ball with  $K \leq \kappa$  and then that  $\Omega \subseteq B$ ; but this formulation does not work. Even if  $B$  is a metric ball with an injective exponential map, and even if in addition  $B \subset M$  and  $M$  is complete and simply connected with the same  $K \leq \kappa$ , there may be no control over the size of  $\partial\Omega$ . We can let  $M$  be a “barbell” consisting of two large, nearly round 2-spheres connected by a rod (Figure 2). Then  $\Omega$  can be just the rod, while  $B$  is  $\Omega$  union one end of the barbell.  $B$  is also a metric ball with

an injective exponential map. Then  $\Omega$  is an annulus  $S^1 \times I$  in which both the meridian  $S^1$  and the longitude  $I$  can have any length. Thus both  $|\Omega|$  and  $|\partial\Omega|$  can have any value. This was observed by Morgan and Johnson [MJ00] to justify their hypothesis small-volume hypothesis; of course, their upper bound on the volume depends on the ambient manifold.

Theorem 1.9 says that Weil’s theorem fails completely for negatively curved Riemannian 3-balls. Our proof is similar to Hass’s construction [Has94] of a negatively curved 3-ball with concave boundary.

If  $\Omega$  is a smooth domain in a Cartan-Hadamard manifold as in Conjecture 1.2, then it has unique geodesics, but unique geodesics is a strictly weaker hypothesis even in 2 dimensions. For example, if  $\Omega$  is a thin, flat annulus with an angle deficit (Figure 3), then it has unique geodesics, but its inner circle cannot be filled without positive curvature. Theorem 1.9 tells us that we need some geometric condition on a manifold  $\Omega$  to obtain an isoperimetric inequality, because even the strictest topological condition — that  $\Omega$  be diffeomorphic to a ball — is not enough. One natural condition is that  $\Omega$  has unique geodesics. (But see Section 5.5 for a generalization.) Actually, Joel Hass has pointed out to us that if negatively curved balls do not satisfy any isoperimetric inequality, it is also not at all clear that an isoperimetric minimizer in a Cartan-Hadamard manifold must be a ball.

**Question 4.1.** *If  $M$  is a Cartan-Hadamard manifold and  $\Omega$  minimizes  $|\partial\Omega|$  for some fixed value of  $|\Omega|$ , then is it convex? Is it a topological ball?*

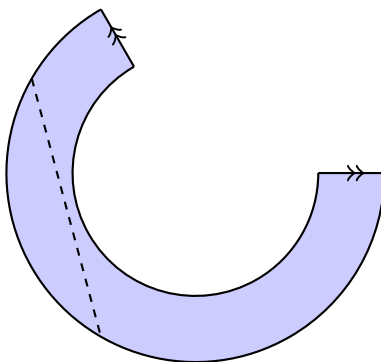


FIGURE 3. A conical, Euclidean annulus that has unique geodesics but does not embed in a Cartan-Hadamard surface.

In two dimensions, if  $\Omega$  is a non-positively curved disk, then it has unique geodesics. (Proof: If a disk does not have unique geodesics, then it contains a geodesic “digon”. By the Gauss-Bonnet theorem, a geodesic digon cannot have non-positive curvature.) Thus the  $\kappa = 0$ ,  $n = 2$  case of Theorem 1.4

implies Weil's theorem; but, as explained in the previous paragraph, it is more general.

In higher dimensions, there are non-positively curved smooth balls with closed geodesics. Hass's construction has closed geodesics, and so does the construction in Theorem 1.9.

*Proof of Theorem 1.9.* We will construct a metric on a 3-ball  $\Omega$  with  $K \leq -1/2$  such that

$$|\Omega| > V \quad |\partial\Omega| < A. \quad (16)$$

Then, on the one hand, we can homothetically shrink  $\Omega$  by any factor  $r < 2^{-1/2}$ , which then induces the condition  $K \leq -1$ . It also reduces both  $|\Omega|$  and  $|\partial\Omega|$  while keeping constant the ratio  $|\partial\Omega|/|\Omega|^{2/3}$ . On the other hand, we can add a thin finger to  $\Omega$  that increases  $|\partial\Omega|$  by any amount while changing  $|\Omega|$  arbitrarily little. Given that  $A$  and  $V$  are arbitrary, these modifications allow us to convert the inequalities in (16) to the equalities in Theorem 1.9.

Let  $L \subseteq S^3$  be a hyperbolic knot, and give  $S^3 \setminus L$  its complete hyperbolic structure with curvature  $K = -1$ . We can choose  $L$  so that  $|S^3 \setminus L|$  is arbitrarily high by a theorem of Adams [Ada05]. A collar around  $L$  becomes a parabolic cusp, which can be truncated to obtain a manifold  $M$  with a horospheric torus boundary  $\partial M$ . All choices for the horospheric truncation yield homothetic, Euclidean metrics on  $\partial M$ . As the truncation moves to the end of the cusp, the metric on  $\partial M$  converges to zero and  $|M| \rightarrow |S^3 \setminus L|$ . Let  $\gamma \subset \partial M$  be a geodesic meridian circle. If we attach a 2-handle  $D^2 \times I$  to  $M$  along  $\gamma$ , then the result is diffeomorphic to a ball. To finish the proof, it suffices to construct a metric on  $D^2 \times [-1, 1]$  that has bounded surface area, curvature  $K \leq -1/2$ , and such that a neighborhood of  $\partial D^2 \times \{0\}$  is isometric to a neighborhood of  $\gamma$  in the cusp (in  $S^3 \setminus L \setminus M$ ). More precisely, we want a metric on  $D^2 \times [-1, 1]$  that shrinks with  $\partial M$  and  $\gamma$ , so that the addition of this handle only changes the surface area by a bounded factor.

We construct the handle as the union of a warped, cusp-like neck  $N$  and a hyperbolic cork  $C$ .  $N$  is a union of a family of horospheric annuli, while  $C$  is a solid pseudo-cylinder, *i.e.*, a 3-ball which is a coordinate cylinder in the upper half-space model of hyperbolic space. We use the coordinates  $(x, y, z)$  in the upper half-space model

$$\mathbb{H}^3 = \{(x, y, z) \mid z > 0\}$$

of hyperbolic 3-space  $\mathbb{H}^3 = X_{3,-1}$ , with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

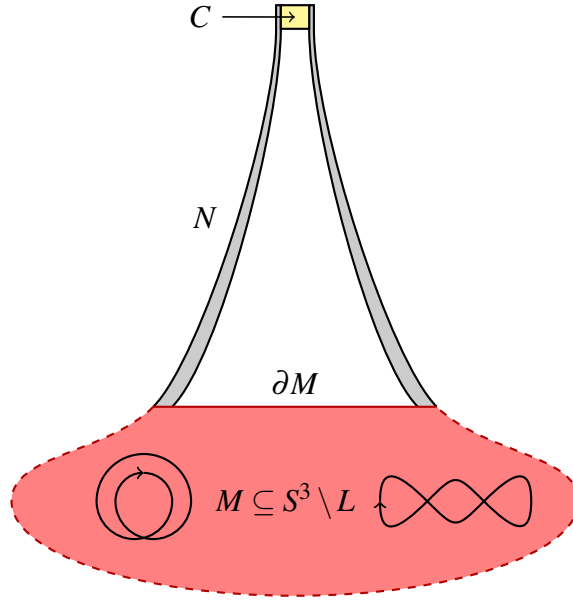


FIGURE 4. A schematic diagram of  $M \cup N \cup C$ , which consists of a hyperbolic knot complement  $M$  with horospheric torus boundary  $\partial M$ , and a 2-handle  $N \cup C$  that makes  $M \cup N \cup C$  into a 3-ball.  $M$  is complicated and has many closed geodesics. The horospheric annuli in the neck  $N$  transition from cylindrical to planar. The cork  $C$  fills in the planar annuli.

A collar  $A$  around  $\gamma$  in the torus  $\partial M$  is a *cylindrical*, Euclidean annulus with a horospheric realization in hyperbolic geometry. To model it, we consider the coordinates

$$(r, \theta, z) \in [1 - \varepsilon, 1 + \varepsilon] \times \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}_{\geq 0}$$

for some small  $\varepsilon > 0$ , with the metric

$$ds^2 = \frac{dr^2 + d\theta^2 + dz^2}{z^2}. \quad (17)$$

This is of course the same as  $\mathbb{H}^3$ , except with  $x = r$  and periodic  $y = \theta$ . If  $\gamma$  has length  $\ell$ , then we can place  $A$  at height

$$z = h = \sqrt{\frac{2\pi}{\ell}},$$



To match the cork, we want *planar*, horospheric annuli instead. We model this with the same coordinates, but with the metric

$$ds^2 = \frac{dr^2 + r^2 d\theta^2 + dz^2}{z^2}. \quad (18)$$

This is also the same as  $\mathbb{H}^3$ , but in cylindrical coordinates

$$(x, y, z) = (r \cos(\theta), r \sin(\theta), z).$$

We will need an interpolation between the metrics (17) and (18) given by

$$ds^2 = \frac{dr^2 + (1 + f(z)(r-1))^2 d\theta^2 + dz^2}{z^2} \quad (19)$$

with  $0 \leq f(z) \leq 1$ . If  $f(z)$  is a constant function, this is still a family of horospheric annuli with a locally hyperbolic total space. Each annulus is conical, or planar with an angle deficit as in Figure 3. To see this more explicitly when  $0 < f(z) < 1$ , we make the change of variables

$$(r, \theta) = \left( \rho - \frac{1-f(z)}{f(z)}, \frac{\psi}{f(z)} \right).$$

Then

$$ds^2 = \frac{d\rho^2 + \rho^2 d\psi^2 + dz^2}{z^2},$$

which is the polar metric with

$$(\rho, \psi) \in \left[ \frac{1}{f(z)} - \varepsilon, \frac{1}{f(z)} + \varepsilon \right] \times \mathbb{R}/(2\pi f(z)\mathbb{Z}).$$

Since the total angle is  $2\pi f(z) < 2\pi$ , the metric is conical.

To make the neck  $N$ , we let  $f(z)$  be a smooth, monotonic transition function from  $f(z) = 0$  when  $z \leq h$  to  $f(z) = 1$  when  $z \geq k \gg h$ . In other words,  $N$  has coordinates

$$(r, \theta, z) \in [1 - \varepsilon, 1 + \varepsilon] \times \mathbb{R}/(2\pi\mathbb{Z}) \times [h - \varepsilon, k + \varepsilon]$$

and metric (19). We assume that  $f(z)$  changes very slowly relative to geometric length  $\log(z)$ , more precisely that

$$f'(z) \ll \frac{1}{z}, \quad f''(z) \ll \frac{1}{z^2}.$$

Then a derivative estimate shows that the Riemannian curvature of  $N$  is close to that of hyperbolic geometry; in particular, we can ensure that  $K \leq -1/2$ . Finally, the cork  $C$  has coordinates

$$(r, \theta, z) \in [0, 1 - \varepsilon] \times \mathbb{R}/(2\pi\mathbb{Z}) \times [k, k + \varepsilon]$$

and standard hyperbolic metric (18).

The union  $M \cup N \cup C$  is shown schematically in Figure 4. It is a topological 3-ball with all of the desired properties, except that it is a manifold with corners. We can smooth the boundary to make a 3-ball  $\Omega$  with  $K \leq -1/2$  and that satisfies (16).  $\square$

## 5. GEODESIC INTEGRALS

In this section we will study Santaló's integral formula [San04, §19.4] in the formalism of geodesic flow and symplectic geometry. See McDuff and Salamon [MS98, §5.4] for properties of symplectic quotients. The formulas we derive are those of Croke [Cro84]; see also Teufel [Teu93].

**5.1. Symplectic geometry.** Let  $W$  be an open symplectic  $2n$ -manifold with a symplectic form  $\omega_W$ . Then  $W$  also has a canonical volume form  $\mu_W = \omega_W^{\wedge n}$  which is called the Liouville measure on  $W$ . Let  $h : W \rightarrow \mathbb{R}$  be a Hamiltonian, by definition any smooth function on  $W$ , suppose that 0 is a regular value of  $h$ , and let  $H = h^{-1}(0)$  then be the corresponding smooth level surface. Then  $\omega_W$  converts the 1-form  $dh$  to a vector field  $\xi$  which is tangent to  $H$ . Suppose that every orbit  $\gamma$  of  $\xi$  only exists for a finite time interval. Let  $G$  be the set of orbits of  $\xi$  on  $H$ ; it is a type of *symplectic quotient* of  $W$ .  $G$  is a smooth open manifold except that it might not be Hausdorff.

The manifold  $G$  is also symplectic with a canonical 2-form  $\omega_G$  and its own Liouville measure  $\mu_G$ .  $H$  cannot be symplectic since it is odd-dimensional, but it does have a Liouville measure  $\mu_H$ . (In fact  $G$  and  $\omega_G$  only depend on  $H$ , and not otherwise on  $h$ , while  $\mu_H$  depends on the specific choice of  $h$ .) Let  $(a(\gamma), b(\gamma))$  be the time interval of existence of  $\gamma \in G$ ; here only the difference

$$\ell(\gamma) = b(\gamma) - a(\gamma)$$

is well-defined by the geometry. In this general setting, if  $f : H \rightarrow \mathbb{R}$  is a suitably integrable function, then

$$\int_H f(x) d\mu_H(x) = \int_{\gamma \in G} \int_{a(\gamma)}^{b(\gamma)} f(\gamma(t)) dt d\mu_G(\gamma). \quad (20)$$

Or, if  $\sigma$  is a measure on  $H$ , we can consider the push-forward  $(\pi_G)_*(\sigma)$  of  $\sigma$  under the projection  $\pi_G : H \rightarrow G$ . Taking the special case that  $f$  is constant on orbits of  $\xi$ , the relation (20) says that

$$(\pi_G)_*(\mu_H) = \ell \mu_G.$$

**5.2. The space of geodesics and étendue.** If  $M$  is a smooth  $n$ -manifold, then  $W = T^*M$  is canonically a symplectic manifold. If  $M$  has a Riemannian

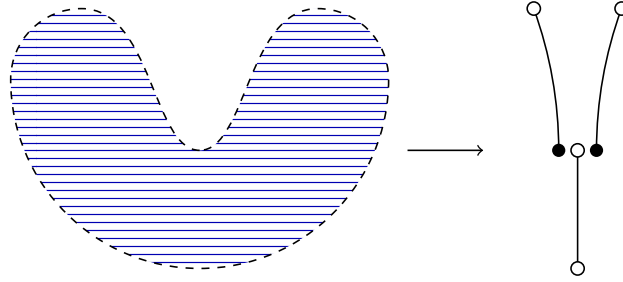


FIGURE 5. A manifold  $M$  in which the space of geodesics is not Hausdorff. The horizontal chords make a “zipper” 1-manifold.

metric  $g$ , then  $g$  gives us a canonical identification  $TM \cong T^*M$ . It also gives us a Hamiltonian  $h : TM \rightarrow \mathbb{R}$  defined as

$$h(v) = (g(v, v) - 1)/2.$$

The level surface  $h^{-1}(0)$  is evidently the unit tangent bundle  $UM$ . It is less evident, but still routine, that the Hamiltonian flow  $\xi$  of  $h$  is the geodesic flow on  $UM$ . Suppose further that  $M$  only has bounded-time geodesics. Then the corresponding symplectic quotient  $G$  is the space of oriented geodesics on  $M$ . The structure on  $G$  that particularly interests us is its Liouville measure  $\mu_G$ . The Liouville measure on  $H = UM$  is also important, and happens to equal the Riemannian measure  $\nu_{UM}$ . Even in this special case,  $G$  might not be Hausdorff if the geodesics of  $M$  merge or split, as in Figure 5.

The Liouville measure  $\mu_G$  is important in geometric optics [Smi07], where among other names it is called *étendue*<sup>1</sup>. Lagrange established that *étendue* is conserved. Mathematically, this says exactly that the  $(2n - 2)$ -form  $\mu_G$ , which is definable on  $H$ , descends to  $G$ . More explicitly, suppose (in the full generality of Section 5.1) that  $K_1, K_2 \subseteq H$  are two transverse open disks that are identified by the holonomy map

$$\phi : K_1 \xrightarrow{\cong} K_2$$

induced by the set of orbits. Then the Liouville measures  $K_1$  and  $K_2$  match, *i.e.*,  $\phi_*(\mu_G) = \mu_G$ . (As in the proof of Liouville’s theorem,  $\phi$  is even a symplectomorphism.)

Now suppose that  $\Omega$  is a compact Riemannian manifold with boundary and with finite geodesics, and let  $M$  be the interior of  $\Omega$ . Then  $G$ , the space of oriented geodesics of  $\Omega$  or  $M$ , is canonically identified in two ways to  $U^+\partial\Omega$ . We can let  $\gamma = \gamma_u$  be the geodesic parallel to  $u$ , or we can let  $\gamma = \bar{\gamma}_u$  be the geodesic anti-parallel to  $u$ . These are both examples of identifying

<sup>1</sup>In English, not just in French.

part of  $G$ , in this case all of  $G$ , with transverse submanifolds as in the previous paragraph. Let

$$\sigma_+ : U^+ \partial \Omega \xrightarrow{\cong} G, \quad \sigma_- : U^+ \partial \Omega \xrightarrow{\cong} G$$

be the two corresponding identifications.

The maps  $\sigma_{\pm}$  are smooth bijections; when  $\partial \Omega$  is convex, they are diffeomorphisms. In general, the inverses  $\sigma_{\pm}^{-1}$  are smooth away from the non-Hausdorff points of  $G$ . These correspond to geodesics tangent to  $\partial \Omega$ , and they are a set of measure 0 in  $G$ . Thus, the composition

$$\phi = \sigma_-^{-1} \circ \sigma_+ : U^+ \partial \Omega \xrightarrow{\cong} U^+ \partial \Omega$$

is an involution of  $U^+ \partial \Omega$  that preserves the measure  $\mu_G$ . (It is even almost everywhere a local symplectomorphism with respect to  $\omega_G$ .) We call the map  $\phi$  the *optical transport* of  $\Omega$ .

We define several types of coordinates on  $G$ ,  $U\Omega$  and  $U^+ \partial \Omega$ . Let  $u = (x, v)$  be the position and vector components of a tangent vector  $u \in U\Omega$ , and let  $u = (p, v)$  be the same for  $u \in U^+ \partial \Omega$ . On  $G$  itself, we already have the length function  $\ell(\gamma)$ . In addition, if  $\gamma = \gamma_u$  for

$$u = (p, v) \in U^+ \partial \Omega,$$

let  $\alpha(\gamma)$  be the angle between  $v$  and the inward normal vector  $w(p)$ . If  $\gamma = \overline{\gamma}_u$ , then let  $\beta(\gamma)$  be that angle instead.

The map  $\sigma_+$  relates the Liouville measure  $\mu_G$  with Riemannian measure  $\nu_{U^+ \partial \Omega}$ . More loosely, the projection  $\pi$  from Section 5.1 relates  $\mu_G$  with  $\nu_{U\Omega}$ . Then by slight abuse of notation,

$$d\mu_G = \cos(\alpha) d\nu_{U^+ \partial \Omega} = \frac{d\nu_{U\Omega}}{\ell}. \quad (21)$$

In words,  $\mu_G$  is close to  $\nu_{U^+ \partial \Omega}$  but not the same: If a beam of light is incident to a surface at an angle of  $\alpha$ , then its illumination has a factor of  $\cos(\alpha)$ . The measure  $(\pi_G)_*(\nu_{U\Omega})$  is also close but not the same, because the étendue of a family of geodesics does not grow with the length of the geodesics.

Another important comparison of measures relates geodesics to pairs of points.

**Lemma 5.1.** *Suppose that  $p, q \in \Omega$  lie on a geodesic  $\gamma$  and that  $p \neq q$ . Let  $p = \gamma(a)$  and  $q = \gamma(b)$ . Then*

$$d\nu_{\Omega \times \Omega}(p, q) = j_{\Omega}(\gamma, a, b) d\mu_G(\gamma) da db.$$

*Proof.* On the one hand, a localized version of formula (21) is

$$d\mu_G(\gamma) da = d\nu_{U\Omega}(u) = d\nu_{U_p \Omega}(v) d\nu_{\Omega}(p)$$

when  $u = (p, v)$  and  $\gamma$  is the geodesic such that  $\gamma(a) = p$  and  $\gamma'(a) = v$ . On the other hand, by the definition of the candle function, we have for all fixed  $p$ :

$$dv_{\Omega}(q) = j_{\Omega}(\gamma, a, b) dv_{U_p\Omega}(v) db$$

when  $\gamma(a) = p$ ,  $\gamma(b) = q$  and  $v = \gamma'(a)$ . Thus, we obtain a pair of equalities of measures:

$$d\mu_G(\gamma) da db = dv_{\Omega}(p) dv_{U_p\Omega}(v) db = \frac{dv_{\Omega}(p) dv_{\Omega}(q)}{j_{\Omega}(\gamma, a, b)}. \quad \square$$

Lemma 5.1 has the important corollary that the candle function is symmetric. To generalize from an  $\Omega$  with finite geodesics to an arbitrary  $M$ , we can let  $\Omega$  be a neighborhood of the geodesic  $\gamma$ , immersed in  $M$ .

**Corollary 5.2** (Folklore [Yau75, Lem. 5]). *In any Riemannian manifold  $M$ ,*

$$j_M(\gamma, a, b) = j_M(\gamma, b, a).$$

Combining (20) with (21) yields Santaló's equality.

**Theorem 5.3** (Santaló [San04, §19.4]). *If  $\Omega$  is as above, and if  $f : U\Omega \rightarrow \mathbb{R}$  is a continuous function, then*

$$\int_{U\Omega} f(u) dv_{U\Omega}(u) = \int_{U^+\partial\Omega} \int_0^{\ell(\gamma_u)} f(\gamma_u(t)) \cos(\alpha(u)) dt dv_{U^+\partial\Omega}(u).$$

Finally, we will consider another reduction of the space  $G$ , the projection

$$\pi_{\text{lab}} : G \rightarrow \mathbb{R}_{\geq 0} \times [0, \pi/2)^2, \quad \pi_{\text{lab}}(\gamma) = (\ell(\gamma), \alpha(\gamma), \beta(\gamma)).$$

Let

$$\mu_{\Omega} = (\pi_{\text{lab}})_*(\mu_G)$$

be the push-forward of Liouville measure. Then  $\mu_{\Omega}$  is a measure-theoretic reduction of the optical transport map  $\phi$ , and is close to a transportation measure in the sense of Monge-Kantorovich. More precisely, equation (21) yields a formula for the  $\alpha$  and  $\beta$  marginals of  $\mu_{\Omega}$ , so we can view  $\mu_{\Omega}$ , or rather its projection to  $[0, \pi/2)^2$ , as a transportation measure from one marginal to the other. The projection onto the  $\alpha$  coordinate is

$$\alpha_*(\mu_{\Omega}) \stackrel{\text{def}}{=} \int_{\ell, \beta} d\mu_{\Omega} = |\partial\Omega| \omega_{n-2} \sin(\alpha)^{n-2} \cos(\alpha) d\alpha,$$

where as in (12) we use the volume of a latitude sphere on  $U_p^+\partial\Omega$ . Using the abbreviation

$$z(\theta) = \frac{\omega_{n-2} \sin(\theta)^{n-1}}{n-1},$$

we can give a simplified formula for both marginals:

$$\alpha_*(\mu_{\Omega}) = |\partial\Omega| dz(\alpha), \quad \beta_*(\mu_{\Omega}) = |\partial\Omega| dz(\beta). \quad (22)$$

A final important property of  $\mu_\Omega$  that follows from its construction is that it is symmetric in  $\alpha$  and  $\beta$ .

**5.3. The core inequalities.** In this section, we establish three geometric comparisons that convert our curvature hypotheses to linear inequalities that can then be used for linear programming. (Section 5.2 does not use either unique geodesics or a curvature hypothesis. Thus, the results there are not strong enough to establish an isoperimetric inequality.)

**Lemma 5.4.** *If  $\Omega$  is Candle( $\kappa$ ) and has unique geodesics, then:*

$$\int_{\ell, \alpha, \beta} \frac{s_{n, \kappa}(\ell)}{\cos(\alpha) \cos(\beta)} d\mu_\Omega \leq |\partial\Omega|^2 \quad (\text{Croke}) \quad (23)$$

$$\int_{\ell, \alpha, \beta} \frac{s_{n, \kappa}^{(-1)}(\ell)}{\cos(\alpha)} d\mu_\Omega \leq |\partial\Omega| |\Omega| \quad (\text{Little Prince}) \quad (24)$$

$$\int_{\ell, \alpha, \beta} s_{n, \kappa}^{(-2)}(\ell) d\mu_\Omega \leq |\Omega|^2 \quad (\text{Teufel}) \quad (25)$$

The first case of Lemma 5.4, equation (23), is due to Croke [Cro84]. Equation (24) generalizes the integral over  $p \in \partial\Omega$  of equation (9) in Theorem 1.6. Finally equation (25) generalizes an isoperimetric inequality of Teufel [Teu91]. Nonetheless all three inequalities can be proven in a similar way.

*Proof.* We define a *partial* map

$$\tau : \Omega \times \Omega \rightarrow G$$

by letting  $\tau(p, q)$  be the unique geodesic  $\gamma \in G$  that passes through  $p$  and  $q$ , if it exists. We define  $\tau(p, q)$  only when  $p \neq q$  and only when  $\gamma$  is available. Also, if  $\gamma$  exists, we parametrize it by length starting at the initial endpoint at 0.

By construction,

$$\begin{aligned} \|\tau_*(v_{\partial\Omega \times \partial\Omega})\| &\leq |\partial\Omega|^2 \\ \|\tau_*(v_{\partial\Omega \times \Omega})\| &\leq |\partial\Omega| |\Omega| \\ \|\tau_*(v_{\Omega \times \Omega})\| &\leq |\Omega|^2. \end{aligned}$$

Using Lemma 5.1, we can write integrals for each of the left sides

$$\begin{aligned} \|\tau_*(\mathbf{v}_{\partial\Omega \times \partial\Omega})\| &= \int_G \frac{j(\gamma, \ell)}{\cos(\alpha) \cos(\beta)} d\mu_G(\gamma) \\ \|\tau_*(\mathbf{v}_{\partial\Omega \times \Omega})\| &= \int_G \int_0^\ell \frac{j(\gamma, r)}{\cos(\alpha)} dr d\mu_G(\gamma) \\ \|\tau_*(\mathbf{v}_{\Omega \times \Omega})\| &= \int_G \int_0^\ell \int_0^t j(\gamma, r, t) dr dt d\mu_G(\gamma). \end{aligned}$$

Because  $\Omega$  is Candle( $\kappa$ ),

$$\begin{aligned} j(\gamma, \ell) &\geq s_{n, \kappa}(\ell), \\ \int_0^\ell j(\gamma, t) dt &\geq s_{n, \kappa}^{(-1)}(\ell), \\ \int_0^\ell \int_0^t j(\gamma, r, t) dr dt &\geq s_{n, \kappa}^{(-2)}(\ell). \end{aligned}$$

We thus obtain

$$\begin{aligned} \int_G \frac{s_{n, \kappa}(\ell)}{\cos(\alpha) \cos(\beta)} d\mu_G(\gamma) &\leq |\partial\Omega|^2 \\ \int_G \frac{s_{n, \kappa}^{(-1)}(\ell)}{\cos(\alpha)} d\mu_G(\gamma) &\leq |\partial\Omega| |\Omega| \\ \int_G s_{n, \kappa}^{(-2)}(\ell) d\mu_G(\gamma) &\leq |\Omega|^2. \end{aligned}$$

Because these integrands only depend on  $\ell$ ,  $\alpha$ , and  $\beta$ , we can now descend from  $\mu_G$  to  $\mu_\Omega$ .  $\square$

**5.4. Extended inequalities.** Lemma 5.4 will yield a linear programming model that is strong enough to prove Theorem 1.4, but not Theorem 1.5 nor many of the other cases of Theorem 1.16. In this section, we will establish several variations of Lemma 5.4 using alternate hypotheses.

The following lemma is the refinement needed for Theorem 1.5 and Theorem 1.15.

**Lemma 5.5.** *Suppose that  $\Omega$  is a compact domain in an LCD(-1) Cartan-Hadamard  $n$ -manifold  $M$  and let*

$$\text{chord}(\Omega) \leq L \in (0, \infty].$$



Then

$$\int_{\ell, \alpha, \beta} \left( \frac{s_{n,-1}^{(-1)}(\ell)}{\cos(\alpha)} - \frac{(n-1)s_{n,-1}^{(-2)}(\ell)}{\tanh(L)} \right) d\mu_{\Omega} \leq |\partial\Omega||\Omega| - \frac{(n-1)|\Omega|^2}{\tanh(L)} \quad (26)$$

$$\int_{\ell, \alpha, \beta} \left( \frac{s_{n,-1}(\ell)}{\cos(\alpha)\cos(\beta)} - \frac{(n-1)s_{n,-1}^{(-1)}(\ell)}{\tanh(L)\cos(\alpha)} \right) d\mu_{\Omega} \leq |\partial\Omega|^2 - \frac{(n-1)|\partial\Omega||\Omega|}{\tanh(L)}. \quad (27)$$

*Proof of (26).* We abbreviate

$$s(\ell) \stackrel{\text{def}}{=} s_{n,-1}(\ell),$$

and we switch  $\alpha$  and  $\beta$  in the integral.

Let  $G$  be the space of geodesics of  $\Omega$  and recall the partial map

$$\tau : \Omega \times \Omega \rightarrow G$$

used in the proof of Lemma 5.4 and the measures  $\nu_{\partial\Omega \times \Omega}$  and  $\nu_{\Omega \times \Omega}$ . We consider the signed measure

$$\sigma_{\Omega \times \Omega} \stackrel{\text{def}}{=} \nu_{\Omega \times \partial\Omega} - (n-1)\nu_{\Omega \times \Omega}.$$

To be precise, if  $(p, q) \in \Omega \times \partial\Omega$ , then  $\gamma = \tau(p, q)$  is the geodesic that passes through  $p$  and ends at  $q$ . We claim two things about the pushforward  $\tau_*(\sigma_{\Omega \times \Omega})$ :

1. That the net measure omitted by  $\tau$  is non-negative:

$$\|\tau_*(\sigma_{\Omega \times \Omega})\| \leq |\partial\Omega||\Omega| - (n-1)|\Omega|^2.$$

2. That the measure that is pushed forward is underestimated by the comparison candle function:

$$\int_G \left( \frac{s^{(-1)}(\ell)}{\cos(\beta)} - \frac{(n-1)s^{(-2)}(\ell)}{\tanh(L)} \right) d\mu_G(\gamma) \leq \|\tau_*(\sigma_{\Omega \times \Omega})\|.$$

Just as in the proof of Lemma 5.4, equation (26) follows from these two claims.

To prove the second claim, let  $\gamma \in G$  be a maximal geodesic of  $\Omega$  with unit speed and domain  $[0, \ell]$ . We abbreviate the candle function along  $\gamma$ :

$$j(t) \stackrel{\text{def}}{=} j(\gamma, t), \quad j(r, t) \stackrel{\text{def}}{=} j(\gamma, r, t).$$

Since  $M$  and therefore  $\Omega$  is LCD(-1), we have the inequality

$$\frac{j'(t)}{j(t)} \geq \frac{s'(t)}{s(t)}.$$

We can rephrase this as saying that

$$\frac{\partial j}{\partial t}(r,t) - \frac{s'(t-r)}{s(t-r)}j(r,t) = \frac{\partial j}{\partial t}(r,t) - \frac{n-1}{\tanh(t-r)}j(r,t)$$

is minimized (with a value of 0) in the  $K = -1$  case. Now

$$\tanh(t-r) \leq \tanh(L),$$

while  $\text{LCD}(-1)$  implies  $\text{Candle}(-1)$ , *i.e.*,

$$j(r,t) \geq s(t-r).$$

It follows that

$$\frac{\partial j}{\partial t}(r,t) - \frac{(n-1)j(r,t)}{\tanh(L)} \geq s'(t-r) - \frac{(n-1)s(t-r)}{\tanh(L)}. \quad (28)$$

We can integrate with respect to  $r$  and  $t$  to obtain:

$$\begin{aligned} \int_0^\ell \int_r^\ell \left[ \frac{\partial j}{\partial t}(r,t) - \frac{(n-1)j(r,t)}{\tanh(L)} \right] dt dr \\ = \int_0^\ell j(r,\ell) dr - \frac{n-1}{\tanh(L)} \int_0^\ell \int_r^\ell j(r,t) dr dt \\ \geq s^{(-1)}(\ell) - \frac{(n-1)s^{(-2)}(\ell)}{\tanh(L)}. \end{aligned}$$

Then, if the terminating angle of  $\gamma$  is  $\beta$ , we can again use the  $\text{Candle}(-1)$  condition to obtain

$$\int_0^\ell \frac{j(r,\ell)}{\cos(\beta)} dr - \frac{n-1}{\tanh(L)} \int_0^\ell \int_0^t j(r,t) dr dt \geq \frac{s^{(-1)}(\ell)}{\cos(\beta)} - \frac{(n-1)s^{(-2)}(\ell)}{\tanh(L)}.$$

Since the left side is the fiber integral of  $\tau_*(\sigma_{\Omega \times \Omega})$ , as in the proof of Lemma 5.4, this establishes the second claim.

To establish the first claim, for each  $p \in \Omega$ , we consider the set  $\Omega \setminus \text{Vis}(\Omega, p)$  consisting of points  $q \in \Omega$  that are *not* visible from  $p$ . The union of all of these is exactly the pairs  $(p, q)$  where  $\tau$  is not defined. If  $\gamma$  is a geodesic in  $M$  emanating from  $p$ , we can restrict further to its intersection

$$\gamma \cap (\Omega \setminus \text{Vis}(\Omega, p))$$

We claim that the integral of  $\sigma_{\Omega \times \Omega}$  on each of these intersections, with the appropriate Jacobian factor, is non-negative.

To verify this claim, we suppose that the intersection is non-empty, and we parametrize  $\gamma$  at unit speed so that  $\gamma(0) = p$ . Let  $I$  be the set of times  $t$  such that

$$\gamma(t) \in \Omega \setminus \text{Vis}(\Omega, p),$$

let  $\{t_k\}$  be the set of right endpoints of  $I$  where  $\gamma$  leaves  $\Omega$ , and for each  $k$ , let  $\beta_k \in [0, \pi/2]$  be the angle that  $\gamma$  exits  $\Omega$  at  $\gamma(t_k)$ . Let  $\ell$  be the rightmost point of  $I$ . Then the infinitesimal portion of  $\sigma_{\Omega \times \Omega}$  on  $\gamma(I)$  is

$$\sum_k \frac{j(t_k)}{\cos(\beta_k)} - \frac{n-1}{\tanh(L)} \int_I j(t) dt \geq j(\ell) - \frac{n-1}{\tanh(L)} \int_0^\ell j(t) dt.$$

(In other words, we geometrically simplify to the worst case:  $I = [0, \ell]$  and  $\beta = 0$ .) The derivative of the right side is now

$$j'(\ell) - \frac{n-1}{\tanh(L)} j(\ell) \geq 0. \quad (29)$$

The inequality holds because it is the same as (28), except with the right side simplified to 0. This establishes the first claim and thus (26).  $\square$

*Proof of (27).* The proof has exactly the same ideas as the proof of (26), only with some changes to the formulas. We keep the same abbreviations. This time we define

$$\sigma_{\partial\Omega \times \Omega} \stackrel{\text{def}}{=} \nu_{\partial\Omega \times \partial\Omega} - (n-1)\nu_{\partial\Omega \times \Omega},$$

we consider  $\tau_*(\sigma_{\partial\Omega \times \Omega})$ , and we claim:

1. That the net measure omitted by  $\tau$  is non-negative:

$$\|\tau_*(\sigma_{\partial\Omega \times \Omega})\| \leq |\partial\Omega|^2 - (n-1)|\partial\Omega||\Omega|^2.$$

2. That the integral underestimates the pushforward:

$$\int_G \left( \frac{s(\ell)}{\cos(\alpha)\cos(\beta)} - \frac{(n-1)s^{(-1)}(\ell)}{\cos(\alpha)\tanh(L)} \right) d\mu_G(\gamma) \leq \|\tau_*(\sigma_{\partial\Omega \times \Omega})\|.$$

To prove the second claim, we define  $\gamma$  and  $j$  as before and we again obtain (28). In this case, we integrate only with respect to  $t \in [0, \ell]$  to obtain

$$j(0, \ell) - \frac{n-1}{\tanh(L)} \int_0^\ell j(r, t) \geq s(\ell) - \frac{(n-1)s^{(-1)}}{\tanh(L)}.$$

Now divide through by  $\cos(\alpha)$ , and we use the Candle(-1) property to divide the first term  $\cos(\beta)$ , to obtain

$$\begin{aligned} \frac{j(0, \ell)}{\cos(\alpha)\cos(\beta)} - \frac{n-1}{\cos(\alpha)\tanh(L)} \int_0^\ell j(r, t) \\ \geq \frac{s(\ell)}{\cos(\alpha)\cos(\beta)} - \frac{(n-1)s^{(-1)}}{\cos(\alpha)\tanh(L)}. \end{aligned}$$

The left side is the fiber integral of  $\tau_*(\sigma_{\partial\Omega \times \Omega})$ , so this establishes the second claim.

The proof of the first claim is identical to the case of (26), except that  $p \in \partial\Omega$ , and we divide through by  $\cos(\alpha)$ .  $\square$

Meanwhile Theorem 1.14 requires the following striking inequality that depends only on the condition of unique geodesics rather than any bound on curvature. We omit the proof as the lemma is equivalent to Lemma 9 of Croke [Cro80].

**Lemma 5.6** (Croke-Berger-Kazdan). *If  $\Omega$  is a compact Riemannian manifold with boundary and with unique geodesics, then*

$$\int_{\ell, \alpha, \beta} s_{n, (\pi/\ell)^2}^{(-2)}(\ell) \, d\mu_{\Omega} \leq |\Omega|^2.$$

**5.5. Mirrors and multiple images.** In this section, we establish the geometric inequalities needed for Theorem 1.8. Let  $M$  be a Riemannian manifold with boundary  $\partial M$  (although  $M$  might not be compact), and consider geodesics that reflect from  $\partial M$  with equal angle of incidence and angle of reflection. Let  $\tilde{G}$  be the space of these geodesics, for simplicity considering only those geodesics that are never tangent to  $\partial M$ . Then the results of Section 5.2 still apply, with only slight modifications. In particular  $M$  might have a compactification  $\Omega$  with  $\partial M = W \subseteq \partial\Omega$ . Then (21) applies if we replace  $\partial\Omega$  by  $\partial\Omega \setminus W$ ; Lemma 5.1 holds; etc.

If  $\Omega$  has a mirror  $W$  as part of its boundary, then some pairs of points have at least two connecting, reflecting geodesics. We can suppose in general that every two points in  $(\Omega, W)$  are connected by at most  $m$  geodesics (which is also interesting even if  $W$  is empty), and we can suppose that  $(\Omega, W)$  is Candle( $\kappa$ ) in the sense of reflecting geodesics. In this case it is straightforward to generalize Lemma 5.4. The generalization will yield the linear programming model for Theorem 1.8.

**Lemma 5.7.** *If  $(\Omega, W)$  is Candle( $\kappa$ ) and has at most  $m$  reflecting geodesics between any pair of points, then:*

$$\int_{\alpha, \beta, \ell} \frac{s_{n, \kappa}(\ell)}{\cos(\alpha) \cos(\beta)} \, d\mu_{\Omega} \leq m |\partial\Omega|^2 \quad (30)$$

$$\int_{\alpha, \beta, \ell} \frac{s_{n, \kappa}^{(-1)}(\ell)}{\cos(\alpha)} \, d\mu_{\Omega} \leq m |\partial\Omega| |\Omega| \quad (31)$$

$$\int_{\alpha, \beta, \ell} s_{n, \kappa}^{(-2)}(\ell) \, d\mu_{\Omega} \leq m |\Omega|^2. \quad (32)$$

*Proof.* The proof is nearly identical to that of Lemma 5.4. In this case

$$\tau : \Omega \times \Omega \rightarrow G$$

is not a partial map, but rather a multivalued correspondence which is at most 1 to  $m$  everywhere. We can define a pushforward measure such as  $\tau_*(\nu_{\Omega \times \Omega})$  by counting multiplicities.

By construction:

$$\begin{aligned} \|\tau_*(\mathbf{v}_{\partial\Omega\times\partial\Omega})\| &\leq m|\partial\Omega|^2 \\ \|\tau_*(\mathbf{v}_{\partial\Omega\times\Omega})\| &\leq m|\partial\Omega||\Omega| \\ \|\tau_*(\mathbf{v}_{\Omega\times\Omega})\| &\leq m|\Omega|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\tau_*(\mathbf{v}_{\partial\Omega\times\partial\Omega})\| &= \int_G \frac{j(\gamma, \ell)}{\cos(\alpha)\cos(\beta)} d\mu_G(\gamma) \\ \|\tau_*(\mathbf{v}_{\partial\Omega\times\Omega})\| &= \int_G \int_0^\ell \frac{j(\gamma, r)}{\cos(\alpha)} dr d\mu_G(\gamma) \\ \|\tau_*(\mathbf{v}_{\Omega\times\Omega})\| &= \int_G \int_0^\ell \int_0^t j(\gamma, r, t) dr dt d\mu_G(\gamma). \end{aligned}$$

Using the Candle( $\kappa$ ) hypothesis, we obtain the desired inequalities.  $\square$

Finally, the following generalization of Günther's inequality [Gün60, BC64] shows that the Candle( $\kappa$ ) condition is actually useful for reflecting geodesics.

**Proposition 5.8.** *Let  $M$  be a Riemannian manifold with  $K \leq \kappa$  for some  $\kappa \in \mathbb{R}$ , and suppose that  $\partial M$  is concave relative to the interior. If  $\kappa > 0$ , suppose also that  $\text{chord}(M) < \pi/\sqrt{\kappa}$ . Then  $M$  is LCD( $\kappa$ ) with respect to geodesics that reflect from  $\partial M$ .*

Proposition 5.8 generalizes Lemma 3.2 of Choe [Cho06], which claims Candle(0) using (in the proof) the same hypotheses when  $\kappa = 0$ . However, the argument given there omits many details about reflection from a convex surface. (Which is thus concave from the other side as we describe it.) We will prove Proposition 5.8 in Section 8.2.

## 6. LINEAR PROGRAMMING AND OPTIMAL TRANSPORT

**6.1. Linear programming.** In this section we abstract the results of Section 5.2 and 5.3 into a linear programming model. Let

$$\mu = \mu_\Omega, \quad V = |\Omega|, \quad A = |\partial\Omega|.$$

(In fact, both  $V$  and  $A$  are linear functions of  $\mu$  by (21) and (22).) Then equations (21), (22) (23), (24), and (25), and symmetrization in  $\alpha$  and  $\beta$ , can be summarized as a problem in linear programming:

**LP Problem 6.1.** *Fix  $n$ ,  $\kappa$ ,  $A$ , and  $V$ , and let*

$$z(\theta) = \frac{\omega_{n-2} \sin(\theta)^{n-1}}{n-1}.$$

Is there a positive measure  $\mu(\ell, \alpha, \beta)$  on  $\mathbb{R}_{\geq 0} \times [0, \pi/2)^2$ , which is symmetric in  $\alpha$  and  $\beta$ , and such that

$$\alpha_*(\mu) = \int_{\ell, \beta} d\mu = A dz(\alpha) \quad (33)$$

$$\int_{\ell, \alpha, \beta} s_{n, \kappa}(\ell) \sec(\alpha) \sec(\beta) d\mu \leq A^2 \quad (34)$$

$$\int_{\ell, \alpha, \beta} s_{n, \kappa}^{(-1)}(\ell) (\sec(\alpha) + \sec(\beta)) d\mu \leq 2AV \quad (35)$$

$$\int_{\ell, \alpha, \beta} s_{n, \kappa}^{(-2)}(\ell) d\mu \leq V^2 \quad (36)$$

$$\int_{\ell, \alpha, \beta} \ell d\mu = \omega_{n-1} V? \quad (37)$$

(We could have written Problem 6.1 without symmetrization in  $\alpha$  and  $\beta$ . It would have been equivalent, but more complicated.)

We will analyze this linear programming problem using a tool which is variously known as Farkas' lemma, the Farkas-Minkowski theorem, and linear programming duality. The original version of the result is formulated in the finite case.

**Theorem 6.2** (Farkas-Minkowski). *Consider a finite system of linear inequalities  $\sum_i M_{j,i} x_i \leq b_j$ , or  $Mx \leq b$ , for real-valued variables  $x = \{x_i\}$ . Then*

1. *The relations  $Mx \leq b$  are infeasible if and only if some non-negative linear combination of them is the falsehood  $0 \leq -1$ .*
2. *A linear bound  $\sum_i c_i x_i \leq c_0$  holds if and only if it is a linear combination of the rows of  $Mx \leq b$ .*
3. *If  $Mx \leq b$  is feasible and  $\sum_i c_i x_i$  is bounded, then it has a maximum  $c_0$ , which is also the minimum of  $\sum_j b_j y_j$  subject to  $y_j \geq 0$  and  $\sum_j M_{j,i} y_j = c_i$ , or  $M^T y = c$ . This is the dual problem of the minimum bound derived from the rows of  $Mx \leq b$ .*

We cannot directly apply Theorem 6.2 to Problem 6.1 because it is an infinite-dimensional problem. The theorem still holds in infinite dimensions, or in finite dimensions with infinitely many inequalities, with an extra hypothesis such as compactness. We do not know a simple way to make Problem 6.1 compact, but in practice it behaves as if it were. Fortunately, we only really need the “if” direction of Theorem 6.2, which is trivial in every setting, finite or infinite. Thus we formulate the dual problem as follows.

**LP Problem 6.3** (Dual to Problem 6.1). *Given  $n, \kappa, A$ , and  $V$ , are there numbers  $a, b, c \geq 0$  and  $d \in \mathbb{R}$  and a continuous function  $f : [0, \pi/2) \rightarrow \mathbb{R}$*

such that

$$as_{n,\kappa}(\ell) \sec(\alpha) \sec(\beta) + bs_{n,\kappa}^{(-1)}(\ell) (\sec(\alpha) + \sec(\beta)) + cs_{n,\kappa}^{(-2)}(\ell) - d\ell + f(\alpha) + f(\beta) \geq 0 \quad (38)$$

$$aA^2 + 2bAV + cV^2 - d\omega_{n-1}V + 2A \int_0^{\pi/2} f(\alpha) dz(\alpha) < 0? \quad (39)$$

Here are three remarks about Problem 6.3:

1. Duality tells us that if Problem 6.3 is feasible, then Problem 6.1 is infeasible.
2. Linear programming duality in general might allow  $f(\alpha)$  to be merely integrable, or even to replace  $f(\alpha) dz(\alpha)$  by a Borel measure. However, Proposition 6.4 from optimal transport theory tells us that an optimal  $f(\alpha)$  is continuous.
3. The constant  $d$  can have either sign. We subtract it that it will be positive in practice.

In addition to viewing Problem 6.1 as a linear feasibility problem, we will also view it as a minimization problem in  $A$ . But Problem 6.1 is non-linear if (34) is used, equivalently if  $a > 0$  in Problem 6.3. For any fixed set of values of  $a, b, c, d, f$  in Problem 6.3, equation (39) becomes a relation  $P(A) < 0$  for a function  $P(A)$ . Suppose now that there is a minimum feasible value  $A_0$  of  $A$ . Then the cleanest possibility is that  $P(A) < 0$  for all  $A \in [0, A_0)$ . Since  $P(A)$  is a convex quadratic polynomial when  $a > 0$ , this amounts to the condition that  $P(A_0) = 0$  and that

$$P(0) = cV^2 - d\omega_{n-1}V < 0,$$

equivalently,

$$cV < d\omega_{n-1}. \quad (40)$$

On the other hand, if a fixed set of values of  $a, b, c, d, f$  is optimal in any sense, then  $P(A) < 0$  on some interval  $(A_1, A_0)$ . To ensure that we are in this case, we require that  $P(A_0) = 0$  and

$$P'(A_0) = 2aA + 2bV + 2A \int_0^{\pi/2} f(\alpha) dz(\alpha) > 0,$$

equivalently that

$$A_0P'(A_0) - P(A_0) = aA_0^2 + d\omega_3V - cV^2 > 0. \quad (41)$$

We can then look for other values of the dual variables to eliminate  $A \in [0, A_1]$  to justify the linear programming model, even though it will not be needed to prove our main results.



**6.2. Optimal transport.** In this section, we interpret Problem 6.1 as an optimal transport problem. See Villani [Vil09, Ch.3-5] for background material on optimal transport. Following Villani, we assume that  $A dz(\alpha)$  is a distribution of boulangeries and  $A dz(\beta)$  is a distribution of cafés. Moreover, for each boulangerie  $\alpha$  and café  $\beta$ , there are a range of possible roads parametrized by  $\ell$ . By (33),  $\mu$  is a transport of baguettes<sup>2</sup> from the boulangeries to the cafés. Problem 6.1 then asks whether the transport is feasible given the constraints that we must pay separate road tolls in Polish zlotys (34), Czech korunas (35), and Hungarian forints (36); and given an exact labor requirement (37) (neither more nor less). As an optimization problem, the objective is to minimize the amount of bread, which is proportional to  $A$ , given a fixed labor expenditure  $V$ . (It could be a multinational, subsidized employment program in a region with only moderate demand for baguettes.)

As a moral corollary of Theorem 6.2, we can reduce all four resource limits to one in a linear combination to make a cost function

$$E(\ell, \alpha, \beta) = as_{n,\kappa}(\ell) \sec(\alpha) \sec(\beta) + bs_{n,\kappa}^{(-1)}(\ell) (\sec(\alpha) + \sec(\beta)) + cs_{n,\kappa}^{(-2)}(\ell) - d\ell. \quad (42)$$

In the economics interpretation, the coefficients are wage rates and currency conversions. The last term is naturally subtracted if employment is the goal of the program and thus a negative cost. It may be non-trivial to choose the coefficients  $a, b, c, d$  optimally or even well. Once they are chosen, Problem 6.1 is then almost a standard optimal transport problem. The two differences are:

1. We have a choice of “roads” parametrized by  $\ell$ . Given a scalar cost, we can convert it to a standard optimal transport problem if we choose the most efficient road for each pair  $(\alpha, \beta)$  and let the cost be  $\min_{\ell} E$ .
2. The transport  $\mu$  does not usually have to be symmetric in  $\alpha$  and  $\beta$ . We can live without this constraint because Problem 6.1 is itself symmetric in  $\alpha$  and  $\beta$ , if we add the relation  $\beta_*(\mu) = A dz(\beta)$ , which is the other half of (22). We can symmetrize any solution using

$$\tilde{\mu}(\ell, \alpha, \beta) \stackrel{\text{def}}{=} \frac{\mu(\ell, \alpha, \beta) + \mu(\ell, \beta, \alpha)}{2}.$$

Having fixed  $a, b, c, d$ , the remaining dual variable in Problem 6.3 is  $f(\alpha)$ . Its sole constraint is

$$F(\ell, \alpha, \beta) \stackrel{\text{def}}{=} E(\ell, \alpha, \beta) + f(\alpha) + f(\beta) \geq 0. \quad (43)$$

---

<sup>2</sup>Even though in Section 5.2, we transported photons.

In optimal transport terminology,  $f(\alpha)$  is known as a *Kantorovich potential*. We will call the left side,  $F(\ell, \alpha, \beta)$ , the *adjusted cost function*. In standard optimal transport, we would have two potentials  $f(\alpha)$  and  $g(\beta)$  satisfying the equation

$$E(\ell, \alpha, \beta) + f(\alpha) + g(\beta) \geq 0. \quad (44)$$

But, just as symmetry is optional in Problem 6.1, it is also optional in Problem 6.3; we can symmetrize a solution to make  $f = g$ .

**Proposition 6.4.** *An optimal potential  $f(\alpha)$  in Problem 6.3 is a convex function of  $\sec(\alpha)$  and therefore continuous.*

Proposition 6.4 is a standard type of result in optimal transport theory. A potential that satisfies an equation such as (45) below is called *cost convex*.

*Proof.* We assume two potentials  $f(\alpha)$  and  $g(\beta)$ . In the asymmetric variation of Problem 6.3, they are chosen to minimize

$$\int_0^{\pi/2} f(\alpha) \, dz(\alpha) + \int_0^{\pi/2} g(\beta) \, dz(\beta).$$

For each fixed  $g(\beta)$ , we can minimize this integral subject to the constraint (44) by choosing

$$f(\alpha) = \sup_{\ell, \beta} [-E(\ell, \alpha, \beta) - g(\beta)]. \quad (45)$$

For each fixed value of  $\ell$  and  $\beta$ , the supremized function on the right side is linear in  $\sec(\alpha)$  by (42). It follows that  $f(\alpha)$  is convex in  $\sec(\alpha)$  and thus continuous; the same is true of  $g(\beta)$ . If this asymmetric optimization yields  $f \neq g$ , then their average  $(f + g)/2$  has all of the desired properties.  $\square$

**6.3. Sharpness.** If a dual pair of feasible solutions  $\{x_i\}$  and  $\{y_j\}$  in the last part of Theorem 6.2 satisfies

$$\sum_i c_i x_i = c_0 = \sum_j b_j y_j,$$

then it is called an *optimal pair*. An optimal pair is a simultaneous proof that  $c_0$  is the maximum of the left side and the minimum of the right side. Theorem 6.2 says that if both the primal and dual problems are feasible, then an optimal pair exists. In addition, a feasible pair is an optimal pair if and only if  $\sum_i M_{j,i} x_i = b_j$  for every  $j$  such that  $y_j > 0$ .

We can morally expect an optimal pair of solutions to Problems 6.1 and 6.3, at least with the optimality condition (41). We fix  $n$ ,  $\kappa$ , and  $V$ . Suppose that we find a solution  $a, b, c, d, f$  to Problem 6.3 that satisfies (40) and proves that  $A \geq A_0$ . Then we have proven the isoperimetric inequality  $|\partial\Omega| \geq A_0$ . Suppose that we also find a measure  $\mu$  that satisfies Problem 6.1 with  $A_0 = A(\mu)$ . Then we have also proven that Problem 6.1 or Lemma 5.4

cannot prove any better isoperimetric inequality. We also do not need to calculate  $A_0$ ; we can instead confirm optimality by checking that  $\mu$  is supported on the equality locus in (43).

In this case,  $A_0$  is not necessarily a sharp isoperimetric inequality; it is merely sharp for the linear programming relaxation. If finally  $\mu = \mu_\Omega$  for a domain  $\Omega$  that satisfies the hypotheses, then we have proven a sharp isoperimetric inequality.

Equation (43) makes it look as if an optimal  $\mu$  is supported on the diagonal  $\alpha = \beta$ , but we do not know that this is always true. If an optimal  $\mu$  is supported on the diagonal, then this is an especially good circumstance. We summarize what happens in this case with a proposition.

**Proposition 6.5.** *Fix  $n$ ,  $\kappa$ , and  $V$ . Suppose that  $a, b, c > 0$  and  $d \in \mathbb{R}$  are candidate values that satisfy (40), and that  $\ell(\alpha)$  is a candidate dependence of  $\ell$  on  $\alpha$ . Let  $\mu$  be the unique measure that satisfies (33) and that is supported on  $\alpha = \beta$  and  $\ell = \ell(\alpha)$ , and let*

$$f(\alpha) = -\frac{E(\ell(\alpha), \alpha, \alpha)}{2}. \quad (46)$$

*If  $\mu$  satisfies Problem 6.1, and if  $f$  satisfies (43), then they are both optimal. If in addition  $\mu = \mu_\Omega$  for an admissible domain  $\Omega$ , then  $A = |\partial\Omega|$  is a sharp isoperimetric value.*

We can derive a consistency check from Proposition 6.5, to see when we have any hope to establish a sharp isoperimetric inequality for a round ball  $\Omega$  in a constant-curvature geometry  $X_{n,\kappa}$ . The consistency check will be much weaker than equation (43), which will be the real work to verify Proposition 6.5 and thus prove Theorems 1.4 and 1.5. Of course Lemma 5.4 shows that  $\mu_\Omega$  must satisfy Problem 6.1, so this part of Proposition 6.5 is guaranteed.

The consistency check is, first, that every geodesic chord in  $\Omega$  should meet the boundary  $\partial\Omega$  at the same angle  $\alpha = \beta$  at both ends; and its length  $\ell = \ell(\alpha)$  should be a function of  $\alpha$ . Happily, round balls have this property. Second, if the potential  $f(\alpha)$  that then results from (46) satisfies (43), then

$$f(\alpha) = -\frac{\min_\ell E(\ell, \alpha, \alpha)}{2}.$$

We can use the derivative test to see whether there exist  $a, b, c, d$  such that  $E(\ell, \alpha, \alpha)$  has a critical point at  $\ell(\alpha)$ . Of course, a critical point need not be a local minimum, much less a global minimum, if there even is a global minimum. Even so, this limited consistency check is an overdetermined equation that has no solutions for  $a, b, c, d$  in most dimensions  $n$ . We will see a minor miracle in dimension  $n = 2$  and a greater miracle in dimension  $n = 4$ , for all values of  $\kappa$ .

Up to rescaling, we can assume that  $\kappa \in \{-1, 0, 1\}$ . If  $\Omega = B_{n,\kappa}(r)$ , then the length of a geodesic chord that makes an angle of  $\alpha$  from the normal to  $\partial\Omega$  is given by the relation

$$\cos(\alpha) = T_{\kappa,r}(\ell) \stackrel{\text{def}}{=} \begin{cases} \frac{\tan(\ell/2)}{\tan(r)} & \text{if } \kappa = 1 \\ \frac{\ell}{2r} & \text{if } \kappa = 0. \\ \frac{\tanh(\ell/2)}{\tanh(r)} & \text{if } \kappa = -1 \end{cases} \quad (47)$$

The remaining consistency check is to solve for  $a, b, c, d$  using the derivative of (42):

$$\frac{\partial E}{\partial \ell}(\ell, \alpha, \alpha) = a \frac{s'_{n,\kappa}(\ell)}{\cos(\alpha)^2} + 2b \frac{s_{n,\kappa}(\ell)}{\cos(\alpha)} + cs_{n,\kappa}^{(-1)}(\ell) - d = 0.$$

Combining with (47), we obtain

$$a \frac{s'_{n,\kappa}(\ell)}{T_{\kappa,r}(\ell)^2} + 2b \frac{s_{n,\kappa}(\ell)}{T_{\kappa,r}(\ell)} + cs_{n,\kappa}^{(-1)}(\ell) - d = 0. \quad (48)$$

We take this as an equation for the coefficients  $a, b, c, d$  that should hold for all  $0 \leq \ell \leq 2r$ . Note that the factor of  $\tan(r)$ ,  $r$ , or  $\tanh(r)$  that appears in  $T_{\kappa,r}$  factors out of the question of whether there is a solution, since this factor can be absorbed into the constants  $a$  and  $b$ .

**6.4. Extended models.** In this section, we remark that Problem 6.1 turns out to be only strong enough for Theorems 1.4, 1.10, and 1.12. In later sections we will make other, similar linear programming models: Problems 7.2, 8.1, 8.2, and 8.3. The arguments of Sections 6.1, 6.2, and 6.3 will all still apply.

## 7. PROOF OF THE MAIN RESULTS

In this section we will prove Theorems 1.4 and 1.5. Again, we assume that  $\kappa \in \{-1, 0, 1\}$  and we abbreviate  $s = s_{n,\kappa}$ . We warn the reader that we made pervasive use of symbolic algebra software in Sections 7.2 and 7.3. Also, each case when we solve for dual coefficients  $a, b, c, d$ , the solution is unique up to rescaling by a positive real number.

Here are two general remarks about dimension  $n = 2$ . First, for every value of  $\kappa$ , there is a separation (50) in this dimension. This means that we could have proved the results with a simpler measure  $\mu(\ell, \alpha)$  that depends on only one angle. Second,  $a = 0$  in all of these cases, so we can accept  $A$  as a linear variable and ignore the optimality conditions (40) and (41).

**7.1. Weil's and Croke's theorems.** As a warm-up to the more difficult cases with  $\kappa \neq 0$ , we include certain steps that are convenient for later derivations. In particular, we introduce the change of variables

$$(x, y) \stackrel{\text{def}}{=} \left( \frac{\sec(\alpha)}{r}, \frac{\sec(\beta)}{r} \right) \quad (49)$$

in place of  $\alpha$  and  $\beta$ . We will give them the range  $x, y \in \mathbb{R}_{\geq 0}$ . By abuse of notation, we can change variables without changing the names of functions; for example, we can write

$$E(\ell, \alpha, \beta) = E(\ell, \alpha(x), \beta(y)) = E(\ell, x, y).$$

If  $\kappa = 0$ , then

$$\begin{aligned} s(\ell) &= \ell^{n-1}, & s'(\ell) &= (n-1)\ell^{n-2}, \\ s^{(-1)}(\ell) &= \frac{\ell^n}{n}, & s^{(-2)}(\ell) &= \frac{\ell^{n+1}}{n(n+1)}. \end{aligned}$$

Equation (48) becomes

$$4(n-1)r^2 a \ell^{n-4} + 4rb \ell^{n-2} + \frac{c \ell^n}{n} - d = 0.$$

Obviously this has solutions if  $n \in \{2, 4\}$  and not otherwise; this point was known to Croke (personal communication).

When  $n = 2$ , the solution is

$$a = 0, \quad b = \frac{1}{r}, \quad c = 0, \quad d = 4.$$

From (42), we thus obtain

$$E(\ell, \alpha, \beta) = \frac{\ell^2(\sec(\alpha) + \sec(\beta))}{2r} - 4\ell$$

Then (46) and (47) give us the potential

$$f(\alpha) = -\frac{E(2r \cos(\alpha), \alpha, \alpha)}{2} = 2r \cos(\alpha).$$

Then the adjusted cost (43) separates as

$$F(\ell, \alpha, \beta) = G(\ell, \alpha) + G(\ell, \beta) \quad (50)$$

with

$$\begin{aligned} G(\ell, \alpha) &= \frac{\ell^2 \sec(\alpha)}{2r} - 2\ell + 2r \cos(\alpha) \\ &= \frac{(2 \cos(\alpha) - \ell)^2}{2r \cos(\alpha)} \geq 0. \end{aligned}$$

This establishes Weil's theorem.

When  $n = 4$ , the solution to (48) is

$$a = \frac{1}{r^2}, \quad b = 0, \quad c = 0, \quad d = 4.$$

These coefficients plainly satisfy condition (40). The cost function is

$$E(\ell, \alpha, \beta) = \frac{\ell^3 \sec(\alpha) \sec(\beta)}{r^2} - 12\ell,$$

the potential is

$$f(\alpha) = -\frac{E(2r \cos(\alpha), \alpha, \alpha)}{2} = 8r \cos(\alpha),$$

and their sum is

$$F(\ell, \alpha, \beta) = \frac{\ell^3 \sec(\alpha) \sec(\beta)}{r^2} - 12\ell + 8r(\cos(\alpha) + \cos(\beta)).$$

Using the change of variables (49),

$$F(\ell, x, y) = \ell^3 xy - 12\ell + \frac{8}{x} + \frac{8}{y}.$$

We want to show that  $F \geq 0$ . For each fixed value of  $xy$ ,  $F$  is minimized when  $x = y$ . We can then calculate

$$F(\ell, x, x) = \ell^3 x^2 - 12\ell + \frac{16}{x} = \frac{(\ell x + 4)(\ell x - 2)^2}{x} \geq 0.$$

This establishes Croke's theorem.

Following the comments after the proof of Theorem 1.6 in Section 3.2, our proof of Croke's theorem is only superficially different from Croke's proof. The extra point here is that Croke's theorem (and Weil's theorem along with it) hold in Model 6.1, which establishes part of Theorem 1.16.

**7.2. The positive case.** In this section we will establish Theorem 1.4. We will let  $\kappa = 1$ , before we do that, we note that  $\kappa = 0$  is a limiting case of  $\kappa > 0$ . Section 7.1 established that a sharp result in the case  $\kappa = 0$  is only possible when  $n \in \{2, 4\}$ , this justifies the same restriction in Theorem 1.4.

We will use the change of variables

$$(x, y) \stackrel{\text{def}}{=} \left( \frac{\sec(\alpha)}{\tan(r)}, \frac{\sec(\beta)}{\tan(r)} \right) \quad (51)$$

with the range  $x, y \in \mathbb{R}_{\geq 0}$ . Note that equation (47) simplifies to

$$\tan\left(\frac{\ell}{2}\right) = \frac{1}{x}. \quad (52)$$

7.2.1. *Dimension 2.* In dimension  $n = 2$ ,

$$\begin{aligned} s(\ell) &= \sin(\ell), & s'(\ell) &= \cos(\ell), \\ s^{(-1)}(\ell) &= 1 - \cos(\ell), & s^{(-2)}(\ell) &= \ell - \sin(\ell) \end{aligned}$$

when  $\ell < \pi$ , and

$$s(\ell) = 0, \quad s^{(-1)}(\ell) = 2, \quad s^{(-2)} = 2\ell - \pi$$

for  $\ell \geq \pi$ . Equation (48), with (47), becomes

$$\frac{a \tan(r)^2 \cos(\ell)}{\tan(\ell/2)^2} + \frac{2b \tan(r) \sin(\ell)}{\tan(\ell/2)} + c(1 - \cos(\ell)) - d = 0.$$

The solution is

$$a = 0, \quad b = \frac{1}{\tan(r)}, \quad c = 2, \quad d = 4.$$

In the variables (51), the cost function (42) is

$$E(\ell, x, y) = (1 - \cos(\ell))(x + y) - 2 \sin(\ell) - 2\ell,$$

for  $\ell \leq \pi$ , and is constant for larger values of  $\ell$ :

$$E(\ell, x, y) = E(\pi, x, y) \quad \forall \ell \geq \pi. \quad (53)$$

The potential (46) is

$$f(x) = 2 \arctan\left(\frac{1}{x}\right).$$

The adjusted cost (43) again separates according to (50), where this time

$$G(\ell, x) = (1 - \cos(\ell))x - \sin(\ell) - \ell + 2 \arctan\left(\frac{1}{x}\right).$$

We can minimize  $G$  with the derivative test either in  $\ell$  or in  $x$ . The latter is slightly simpler and gives us

$$\frac{\partial G}{\partial x}(\ell, x) = \frac{x^2(1 - \cos(\ell)) - (\cos(\ell) + 1)}{x^2 + 1}.$$

We learn that  $\partial G/\partial x$  crosses 0 exactly once, when  $x$  and  $\ell$  satisfy (52); this is therefore the minimum of  $G$  for each fixed  $\ell$ . Since the relation (52) is used to define the potential  $f(x)$ , it is automatic that this minimum value is 0; the substitution  $x = 1/\tan(\ell/2)$  also establishes it. Thus  $G(\ell, x) \geq 0$ , which confirms (43) and establishes the  $n = 2$  case of Theorem 1.4.

7.2.2. *Dimension 4.* In dimension  $n = 4$ ,

$$\begin{aligned} s(\ell) &= \sin(\ell)^3, \\ s'(\ell) &= 3 \sin(\ell)^2 \cos(\ell), \\ s^{(-1)}(\ell) &= \frac{\cos(\ell)^3 - 3 \cos(\ell) + 2}{3}, \\ s^{(-2)}(\ell) &= \frac{6\ell - \sin(\ell)^3 - 6 \sin(\ell)}{9} \end{aligned}$$

when  $\ell < \pi$ , and

$$s(\ell) = 0, \quad s^{(-1)}(\ell) = \frac{4}{3}, \quad s^{(-2)}(\ell) = \frac{4\ell - 2\pi}{3}$$

when  $\ell \geq \pi$ . Equation (48) becomes

$$\begin{aligned} \frac{3a \tan(r)^2 \cos(\ell) \sin(\ell)^2}{\tan(\ell/2)^2} + \frac{2b \tan(r) \sin(\ell)^3}{\tan(\ell/2)} \\ + \frac{c(\cos(\ell)^3 - 3 \cos(\ell) + 2)}{3} - d = 0. \end{aligned}$$

The solution is

$$a = \frac{1}{\tan(r)^2}, \quad b = \frac{3}{\tan(r)}, \quad c = 9, \quad d = 12.$$

The clean optimality condition (40) becomes

$$9V < 12\omega_3 = 24\pi^2.$$

Since  $V$  is at most the volume of a hemisphere, we have

$$9V < \frac{9\omega_4}{2} = 12\pi^2.$$

Thus (40) holds.

The cost function (42) is

$$\begin{aligned} E(\ell, x, y) &= \sin(\ell)^3 xy + (\cos(\ell)^3 - 3 \cos(\ell) + 2)(x + y) \\ &\quad - \sin(\ell)^3 - 6 \sin(\ell) - 6\ell. \end{aligned}$$

for  $\ell \leq \pi$ , while once again  $E$  is constant  $\ell$  for  $\ell \geq \pi$ , as in (53). The potential from (46) and (52) is

$$f(x) = 6 \arctan\left(\frac{1}{x}\right) + \frac{2x}{x^2 + 1}.$$

We will include the values  $x = 0$  and  $y = 0$  in our calculations, so it is helpful to recall that

$$\arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - \arctan(x).$$



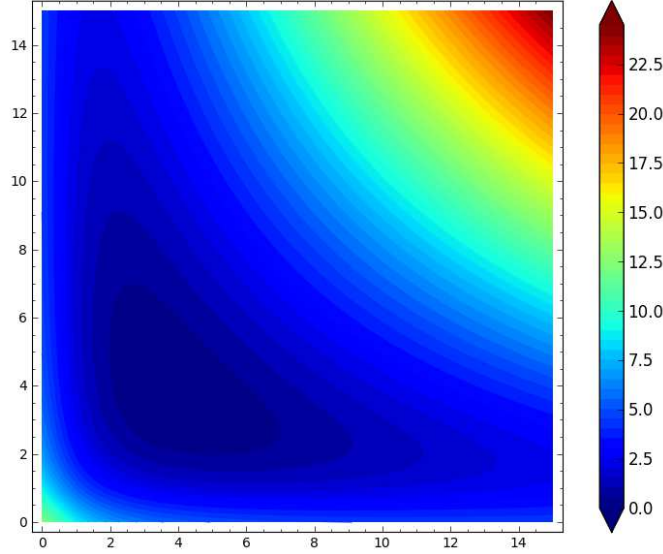


FIGURE 6. The slice  $F(\pi/6, x, y)$  in the case  $\kappa = 1$ .

The adjusted cost (43) is

$$\begin{aligned}
 F(\ell, x, y) = & \sin(\ell)^3 xy + (\cos(\ell)^3 - 3 \cos(\ell) + 2)(x + y) \\
 & - \sin(\ell)^3 - 6 \sin(\ell) - 6\ell + 6\pi - 6 \arctan(x) \\
 & + \frac{2x}{x^2 + 1} - 6 \arctan(y) + \frac{2y}{y^2 + 1}. \quad (54)
 \end{aligned}$$

The remainder of the proof of Theorem 1.4 is given by the following lemma. Although the lemma is evident from contour plots (e.g., Figure 6), the authors found it surprisingly tricky to prove rigorously.

**Lemma 7.1.** *The function  $F(\ell, x, y)$  on  $[0, \pi] \times \mathbb{R}_{\geq 0}^2$  given by (54) is non-negative, and vanishes only when*

$$x = y = \frac{1}{\tan(\ell/2)}.$$

*Proof.* We will use these immediate properties of the potential  $f(x)$ :

$$f(0) = 3\pi, \quad f(x) > 0.$$

We first check the non-compact direction of the domain of  $F$ . There exists a constant  $k > 0$  such that

$$s^{(-1)}(\ell) \geq k\ell^4.$$

(Because  $\ell^4/s^{(-1)}(\ell)$  is continuous on  $[0, \pi]$  and therefore bounded. In fact  $k = s^{(-1)}(\pi)/\pi^4$  works.) Thus

$$\begin{aligned} F(\ell, x, y) &= s(\ell)xy + 3s^{(-1)}(\ell)(x+y) + 9s^{(-2)}(\ell) - 12\ell + f(x) + f(y) \\ &\geq 3k(x+y)\ell^4 - 12\ell \end{aligned}$$

by discarding positive terms and simplifying  $s^{(-1)}(\ell)$ . Thus

$$\begin{aligned} \liminf_{x+y \rightarrow \infty} \left( \min_{\ell} F(\ell, x, y) \right) &\geq \liminf_{x+y \rightarrow \infty} \left( \min_{\ell \geq 0} (3k(x+y)\ell^4 - 12\ell) \right) \\ &= \liminf_{x+y \rightarrow \infty} \frac{-9}{\sqrt[3]{k(x+y)}} = 0. \end{aligned}$$

The inequality comes from discarding positive terms, while the equality follows just from the properties of  $s^{(-1)}(\ell)$  that it is continuous, and that it is positive for  $\ell > 0$ .

Having confined the negative values of  $F$  to some compact region, we will calculate derivatives and boundary values to show that there are no negative values. First, taking  $\ell = 0$ , we get

$$F(0, x, y) = f(x) + f(y) > 0.$$

Second, taking  $\ell = \pi$ , we get

$$F(\pi, x, y) = 4(x+y) - 6\pi + f(x) + f(y).$$

Here we check that

$$\begin{aligned} \frac{\partial F}{\partial x}(\pi, x, y) &= \frac{4x^4}{(x^2+1)^2} \geq 0 \\ \frac{\partial F}{\partial y}(\pi, x, y) &= \frac{4y^4}{(y^2+1)^2} \geq 0 \\ F(\pi, 0, 0) &= 0. \end{aligned}$$

Fourth, taking  $x = y = 0$ , we obtain

$$F(\ell, 0, 0) = 9s^{(-2)}(\ell) - 12\ell + 6\pi.$$

We check in this case that

$$F(\pi, 0, 0), \quad \frac{\partial F}{\partial \ell}(\ell, 0, 0) = 9s^{(-1)}(\ell) - 12 \leq 0.$$

The fifth case is the case  $y = 0$  with  $x$  and  $\ell$  interior, which by symmetry is equivalent to the case  $x = 0$  with  $y$  and  $\ell$  interior. The sixth and final case is the interior for all three coordinates. We will handle the fifth and sixth cases together. Using the final change of variables

$$t \stackrel{\text{def}}{=} \tan\left(\frac{\ell}{2}\right),$$

and with the help of Sage, we learn that

$$\begin{aligned}\frac{\partial F}{\partial \ell}(t, x, y) &= -12 \frac{(t^4 - t^2)xy - 2t^3(x + y) + 3t^2 + 1}{(t^2 + 1)^3}, \\ \frac{\partial F}{\partial x}(t, x, y) &= 4 \frac{(2t^3y - 3t^2 - 1)(x^2 + 1)^2 + x^4(t^2 + 1)^3}{(t^2 + 1)^3(x^2 + 1)^2}.\end{aligned}$$

The conceptual point is that the partial derivatives are rational functions in  $x$ ,  $y$ , and  $t$ . We can rigorously determine the common zeroes of their numerators by finding their associated prime ideals in the ring  $\mathbb{Q}[x, y, t]$  using the “associated\_primes” function in Sage<sup>3</sup> (In other words, we use the Lasker-Noether factorization theorem converted to an algorithm by the Gröebner basis method.) The solution set is characterized by five prime ideals. Four of the ideals are not consistent with the conditions  $x > 0$  and  $y, \ell \geq 0$ : Two contain  $t^2 + 1$ , one contains  $x$ , and one contains

$$2x^2y + 3x^2t + xyt + x + y > 0.$$

The fifth ideal yields the desired locus  $x = y = 1/t$ .

A careful examination of the equality cases shows that  $x = y = 1/t$  is the only possibility for the minimum value  $F = 0$ .  $\square$

**7.3. The negative case.** In this section we will establish Theorem 1.5. As in Section 7.2, we let  $\kappa = -1$  and we must take  $n \in \{2, 4\}$ . We cannot use Problem 6.1, because in both dimensions, one of the dual coefficients turns out to be negative. Instead we use the following model, which is provided by Lemma 5.5.

**LP Problem 7.2.** *Given  $n, A, V$ , and  $L$ , let*

$$q = \frac{n - 1}{\tanh(L)}.$$

---

<sup>3</sup>See the attached Sage files in the source file of the arXiv version of this paper.

Is there a symmetric, positive measure  $\mu(\ell, \alpha, \beta)$  such that

$$\begin{aligned}\alpha_*(\mu) &= \int_{\ell, \beta} d\mu = A dz(\alpha) \\ \int_{\ell, \alpha, \beta} \left( s(\ell) \sec(\beta) - qs^{(-1)}(\ell) \right) \sec(\alpha) d\mu_\Omega &\leq A^2 - qAV. \\ \int_{\ell, \alpha, \beta} \left( s^{(-1)}(\ell) \sec(\alpha) - qs^{(-2)}(\ell) \right) d\mu_\Omega &\leq AV - qV^2 \\ \int_{\ell, \alpha, \beta} s^{(-2)}(\ell) d\mu &\leq V^2 \\ \int_{\ell, \alpha, \beta} \ell d\mu &= \omega_{n-1}V?\end{aligned}$$

We will need the dual problem, which we can state without changing variables.

**LP Problem 7.3** (Dual to Problem 7.2). *Given  $n, A, V,$  and  $L,$  let*

$$q = \frac{n-1}{\tanh(L)}.$$

Are there numbers  $a, b, c, d \in \mathbb{R}$  and a continuous function  $f : [0, \pi/2) \rightarrow \mathbb{R}$  such that

$$\begin{aligned}a \geq 0 \quad 2b + qa \geq 0 \quad c + q(2b + qa) \geq 0 & \quad (55) \\ as_{n,-1}(\ell) \sec(\alpha) \sec(\beta) + bs_{n,-1}^{(-1)}(\ell) (\sec(\alpha) + \sec(\beta)) \\ + cs_{n,-1}^{(-2)}(\ell) - d\ell + f(\alpha) + f(\beta) \geq 0 \\ aA^2 + 2bAV + cV^2 - d\omega_{n-1}V + 2A \int_0^{\pi/2} f(\alpha) dz(\alpha) < 0?\end{aligned}$$

We will use the change of variables

$$(x, y) \stackrel{\text{def}}{=} \left( \frac{\sec(\alpha)}{\tanh(r)}, \frac{\sec(\beta)}{\tanh(r)} \right) \quad (56)$$

with the range  $x, y \in (1, \infty)$ . Equation (47) simplifies to

$$\tanh\left(\frac{\ell}{2}\right) = \frac{1}{x}. \quad (57)$$

7.3.1. *Dimension 2.* In dimension  $n = 2,$

$$\begin{aligned}s(\ell) &= \sinh(\ell), & s'(\ell) &= \cosh(\ell), \\ s^{(-1)}(\ell) &= \cosh(\ell) - 1, & s^{(-2)}(\ell) &= \sinh(\ell) - \ell.\end{aligned}$$

Equation (48), with (47), becomes

$$\frac{a \tanh(r)^2 \cosh(\ell)}{\tanh(\ell/2)^2} + \frac{2b \tanh(r) \sinh(\ell)}{\tanh(\ell/2)} + c(\cosh(\ell) - 1) - d = 0.$$

The solution is

$$a = 0, \quad b = \frac{1}{\tanh(r)}, \quad c = -2, \quad d = 4.$$

We need to check the third case of condition (55), which reduces to

$$c + 2qb = -2 + \frac{2}{\tanh(r) \tanh(L)} \geq 0.$$

Since the tanh function is bounded above by 1, this is immediate.

In the variables (56), the cost function (42) is

$$E(\ell, x, y) = (\cosh(\ell) - 1)(x + y) - \sinh(\ell) - 2\ell.$$

The potential (46) is

$$f(x) = 2 \operatorname{arctanh}\left(\frac{1}{x}\right).$$

The adjusted cost (43) separates according to (50) with

$$G(\ell, x) = (\cosh(\ell) - 1)x - \sinh(\ell) - \ell + 2 \operatorname{arctanh}\left(\frac{1}{x}\right).$$

We minimize  $G$  using the derivative test in  $x$  to obtain

$$\frac{\partial G}{\partial x}(\ell, x) = \frac{x^2(\cosh(\ell) - 1) - (\cosh(\ell) + 1)}{x^2 - 1}.$$

We learn that the minimum of  $G$  in  $x$  for each fixed  $\ell$  occurs when  $x$  and  $\ell$  satisfy (57) and it is easy to confirm that the value is 0. Thus  $G(\ell, x) \geq 0$ , which confirms (43) and establishes the  $n = 2$  case of Theorem 1.5.

7.3.2. *Dimension 4.* In dimension  $n = 4$ ,

$$\begin{aligned} s(\ell) &= \sinh(\ell)^3, \\ s'(\ell) &= 3 \cosh(\ell) \sinh(\ell)^2, \\ s^{(-1)}(\ell) &= \frac{\cosh(\ell)^3 - 3 \cosh(\ell) + 2}{3}, \\ s^{(-2)}(\ell) &= \frac{\sinh(\ell)^3 - 6 \sinh(\ell) + 6\ell}{9}. \end{aligned}$$

Equation (48) becomes

$$\frac{3a \tanh(r)^2 \cosh(\ell) \sinh(\ell)^2}{\tanh(\ell/2)^2} + \frac{2b \tanh(r) \sinh(\ell)^3}{\tanh(\ell/2)} + \frac{c(\cosh(\ell)^3 - 3 \cosh(\ell) + 2)}{3} - d = 0.$$

The solution is

$$a = \frac{1}{\tanh(r)^2}, \quad b = -\frac{3}{\tanh(r)}, \quad c = 9, \quad d = 12.$$

We need to check the second case of condition (55):

$$2b + qa = -\frac{6}{\tanh(r)} + \frac{3}{\tanh(L)} \geq 0.$$

This condition is equivalent to the smallness hypothesis (4). We also note that the clean optimality condition (40) does not hold for all  $V$ . We instead check (41), which says that

$$\frac{A_0^2}{\tanh(r)^2} + 12\omega_3 V - 9V^2 > 0. \quad (58)$$

Recall that

$$V = |B_{4,-1}(r)| = \omega_3 s^{(-1)}(r) \\ A_0 = |\partial B_{4,-1}(r)| = \omega_3 s(r).$$

By Theorem 1.15,  $A_0 \geq 3V$ , so the first term is greater than the third term in (58); the middle term is plainly positive.

We also want to confirm that Problem 7.2 is strong enough to exclude all  $A \in [0, A_0]$ , even if (40) does not hold. To this end, we first use  $P(A_0) = 0$  to calculate that

$$\int_0^{\pi/2} f(\alpha) dz(\alpha) = \frac{d\omega_3 V - aA_0^2 - 2bA_0 V - cV^2}{2A_0} \\ = \frac{4\pi^2(\cosh(r) - 1)}{\sinh(r)}.$$

Now let  $A_1 = 3 \tanh(r)V$ . On the one hand,

$$P(A_1) = \left( \frac{A_1}{\tanh(r)} - 3V \right)^2 - 12\omega_3 V + 2A_1 \int_0^{\pi/2} f(\alpha) dz(\alpha) \\ = -\frac{24\pi^2}{\cosh(r)} < 0.$$

On the other hand,  $A \leq A_1$  is excluded by Theorem 1.15, which we will prove using the same Problem 7.2 in Section 8.5.1. Thus we fully establish

Theorem 1.5 as a corollary of Problem 7.2, if we can confirm (43) for the given values of  $a, b, c, d$ .

The cost function (42) is

$$E(\ell, x, y) = \sinh(\ell)^3 xy - (\cosh(\ell)^3 - 3 \cosh(\ell) + 2)(x + y) + \sinh(\ell)^3 - 6 \sinh(\ell) - 6\ell.$$

The potential from (46) and (57) is

$$f(x) = 6 \operatorname{arctanh}\left(\frac{1}{x}\right) + \frac{2x}{x^2 - 1}.$$

The adjusted cost (43) is

$$F(\ell, x, y) = \sinh(\ell)^3 xy - (\cosh(\ell)^3 - 3 \cosh(\ell) + 2)(x + y) + \sinh(\ell)^3 - 6 \sinh(\ell) - 6\ell + 6 \operatorname{arctanh}\left(\frac{1}{x}\right) + \frac{2x}{x^2 - 1} + 6 \operatorname{arctanh}\left(\frac{1}{y}\right) + \frac{2y}{y^2 - 1}. \quad (59)$$

We conclude the proof of Theorem 1.5 with the following lemma. The lemma is also numerically evident but surprisingly tricky (for the authors).

**Lemma 7.4.** *The function  $F(\ell, x, y)$  on  $\mathbb{R}_{\geq 0} \times (1, \infty)^2$  given by (59) is non-negative, and vanishes only when*

$$x = y = \frac{1}{\tanh(\ell/2)}.$$

*Proof.* The proof is analogous to that of Lemma 7.1, but differs in its technical details. Throughout the proof, we will fix  $y$  and minimize  $F(\ell, x, y)$  with respect to  $x$  and  $\ell$ .

To check the non-compact limits of  $x$  and  $\ell$ , we re-express  $F$  as:

$$F(\ell, x, y) = \sinh(\ell)^3(x - 1)(y - 1) + h(\ell)(x + y) - 6 \sinh(\ell) - 6\ell + f(x) + f(y),$$

where

$$\begin{aligned} h(\ell) &= (\sinh(\ell)^3 - \cosh(\ell)^3 + 3 \cosh(\ell) - 2) \\ &= \frac{(3e^\ell + 1)(1 - e^{-\ell})^3}{4} > 0. \end{aligned}$$

We also have

$$f(x) = \frac{1}{x - 1} + \frac{1}{x + 1} + \operatorname{arctanh}\left(\frac{1}{x}\right) > \frac{1}{x - 1}$$

and the elementary relation  $\sinh(\ell) \geq \ell$ . We combine these comparisons to obtain the bound

$$\begin{aligned}\tilde{F}(\ell, x, y) &\stackrel{\text{def}}{=} \sinh(\ell)^3(x-1)(y-1) + \frac{1}{x-1} - 12\sinh(\ell) \\ &< F(\ell, x, y).\end{aligned}$$

The function  $\tilde{F}$  is useful for minimizing with respect to either  $\ell$  or  $x$ , leaving the other variables fixed. It is a bit simpler to use the variables

$$(\tilde{x}, \tilde{y}) \stackrel{\text{def}}{=} (x-1, y-1),$$

which we will need anyway later in the proof. We obtain

$$\begin{aligned}\tilde{F}(\ell, \tilde{x}, \tilde{y}) &= \sinh(\ell)^3\tilde{x}\tilde{y} + \frac{1}{\tilde{x}} - 12\sinh(\ell) \\ \min_{\ell} \tilde{F}(\ell, \tilde{x}, \tilde{y}) &= \frac{-16}{\sqrt{\tilde{x}\tilde{y}}} + \frac{1}{\tilde{x}} \\ \min_{\tilde{x}} \tilde{F}(\ell, \tilde{x}, \tilde{y}) &= 2\sqrt{\sinh(\ell)^3\tilde{y} - 12\sinh(\ell)}.\end{aligned}$$

We obtain these uniform lim infs:

$$\begin{aligned}\liminf_{x \rightarrow \infty} (\inf_{\ell} F(\ell, x, y)) &\geq \lim_{\tilde{x} \rightarrow \infty} (\min_{\ell} \tilde{F}(\ell, \tilde{x}, \tilde{y})) = 0 \\ \liminf_{x \rightarrow 1} (\inf_{\ell} F(\ell, x, y)) &\geq \lim_{\tilde{x} \rightarrow 0} (\min_{\ell} \tilde{F}(\ell, \tilde{x}, \tilde{y})) = \infty \\ \liminf_{\ell \rightarrow \infty} (\inf_x F(\ell, x, y)) &\geq \lim_{\ell \rightarrow \infty} (\min_{\tilde{x}} \tilde{F}(\ell, \tilde{x}, \tilde{y})) = \infty \\ \liminf_{\ell \rightarrow 0} (\inf_x F(\ell, x, y)) &\geq \lim_{\ell \rightarrow 0} (\min_{\tilde{x}} \tilde{F}(\ell, \tilde{x}, \tilde{y})) = 0.\end{aligned}$$

Once we control  $x$ , we can also check the last case more directly by calculating that

$$F(0, x, y) = f(x) + f(y) > 0.$$

Either way, this establishes that we can use the derivative test for each fixed  $y$  to confirm that  $F(\ell, x, y) \geq 0$ .

We use the final change of variables

$$t \stackrel{\text{def}}{=} \tanh\left(\frac{\ell}{2}\right).$$

Sage tells us that

$$\begin{aligned}\frac{\partial F}{\partial \ell}(t, x, y) &= -12 \frac{(t^4 + t^2)xy - 2t^3(x+y) + 3t^2 - 1}{(t^2 - 1)^3}, \\ \frac{\partial F}{\partial x}(t, x, y) &= -4 \frac{(2t^3y - 3t^2 + 1)(x^2 - 1)^2 + x^4(t^2 - 1)^3}{(t^2 - 1)^3(x^2 - 1)^2}.\end{aligned}$$



We again rigorously determine the common zeroes of their numerators by finding their associated prime ideals in the ring  $\mathbb{Q}[x, y, t]$  using Sage. The solution set in this case is characterized by 7 prime ideals. Six of the ideals are inconsistent with the conditions  $0 \leq t < 1$  and  $x, y > 1$ , while the 7th yields the desired locus  $x = y = 1/t$ . We eliminate the 6 unwanted ideals. Two of them contain  $t - 1$ , two contain  $t + 1$ , and one contains  $x$ .

The final unwanted primary ideal is more complicated and includes the left sides of these two equations:

$$\begin{aligned} 3tx^2 + txy - 2x^2y + x + y &= 0 \\ t^3x^2 + t^3xy - t^2x - t^2y + 2t + 2x &= 0. \end{aligned}$$

Since the first equation is linear in  $t$ , we can eliminate it by substitution. The resulting relation between  $x$  and  $y$  has a factor of  $x + y$ , which is excluded easily. The other factor in the numerator is

$$\begin{aligned} C(x, y) \stackrel{\text{def}}{=} 4x^6y^3 - 12x^5y^2 - 8x^4y^3 + 27x^5 + 27x^4y + 17x^3y^2 \\ + 5x^2y^3 - 11x^3 - 11x^2y - 5xy^2 - y^3. \end{aligned}$$

We can confirm that this has no solutions with  $x, y > 1$  by writing

$$C(\tilde{x}, \tilde{y}) = D(\tilde{x}, \tilde{y}) + 9\tilde{x}^2\tilde{y}(\tilde{y} - 1)^2 + 2\tilde{x}\tilde{y}(\tilde{y} - 2)^2,$$

whence it turns out that  $D(\tilde{x}, \tilde{y})$  has non-negative coefficients in  $\tilde{x}$  and  $\tilde{y}$ . Thus  $C(x, y) = 0$  has no solutions with  $x, y > 1$ , which completes the derivative test for  $F(\ell, x, y)$ .  $\square$

## 8. PROOFS OF OTHER RESULTS

**8.1. Uniqueness.** Problems 6.1 and 7.2 both place strong restrictions on  $\mu$  and therefore on  $\Omega$  in the sharp case. First, all of the inequalities in Problem 6.1 become equalities when  $\kappa > 0$ ; all of the inequalities in Problem 7.2 become equalities when  $\kappa < 0$ . In particular, equation (25) becomes an equality, which implies that  $\Omega$  is convex and that the candle comparison is an equality at short distances. That in turn implies that  $\Omega$  satisfies  $\text{Ric} \geq (n - 1)\kappa g$  and that it is the equality case of Bishop's inequality [BC64, §11.10], which implies that it has constant curvature  $K = \kappa$ .

The case  $\kappa = 0$  does not use (25), but it does use (23). This again implies that  $\Omega$  is convex. The stronger assumption that  $\Omega$  is  $\sqrt{\text{Ric}}$  class 0 together with equality in (23) tells us again that  $\Omega$  has constant curvature  $K = 0$ .

Second, sharpness tells us that  $\mu_\Omega$  is concentrated on the locus given by equation (47). In other words, every chord in  $\Omega$  has the same length and incident angles as if  $\Omega$  were a round ball  $B_{n, \kappa}(r)$ . If  $\Omega$  is convex with constant curvature, this implies that  $\Omega$  is isometric to  $B_{n, \kappa}(r)$ .

**8.2. Günther's inequality with reflections.** In this section, we will prove Proposition 5.8.

If  $\gamma(t)$  is a smooth curve in  $M$  with  $t \in [0, r]$ , then it is a constant-speed geodesic if and only if it is a critical point of the energy functional

$$E(\gamma) = \int_0^r \frac{\langle \gamma'(t), \gamma'(t) \rangle}{2} dt$$

assuming Dirichlet boundary conditions (*i.e.*, that we fixed the endpoints of  $\gamma$ ). Let  $\gamma$  be such a geodesic with unit speed, and let  $y(t)$  be a smooth, infinitesimal normal displacement. Then we can define a relative energy

$$E(y) \stackrel{\text{def}}{=} E(\gamma + y) - E(\gamma) + O(\|y\|^3),$$

which is just the second variational derivative of the curve energy, equivalently half of the second variation of the curve length. We can identify the normal bundle to  $\gamma(t)$  with  $\mathbb{R}^{n-1}$  using parallel transport, thus view  $y$  as a function with values  $y(t) \in \mathbb{R}^{n-1}$ . If  $\gamma$  is an ordinary geodesic without reflections, then by a standard calculation,

$$E(y) = \int_0^r [\langle y'(t), y'(t) \rangle - \langle y(t), R(t)y(t) \rangle] dt,$$

where

$$R(t) = R(\cdot, \gamma'(t), \cdot, \gamma'(t))$$

is the Riemann curvature tensor specialized at the unit tangent  $\gamma'$ . This leads to the differential equation

$$y'' = -R(t)y(t),$$

which is satisfied by  $y$  when it is a Jacobi field, *i.e.*, a geodesic displacement of  $\gamma$ .

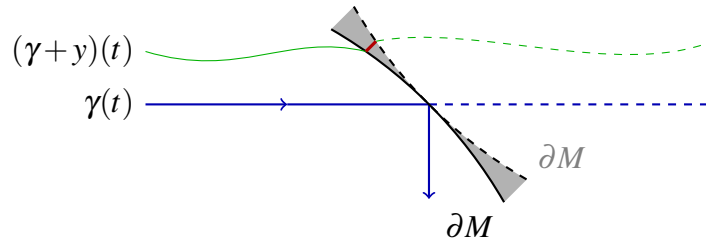


FIGURE 7. Diagram of a vector field  $y$  that displaces a geodesic  $\gamma$  in a curved surface (non-geodesically), and a continuation if  $\gamma$  were straight. The short red segment is the length variation of  $\gamma + y$  due to the reflection.

If  $\gamma$  reflects from  $\partial M$ , then the energy has extra terms. We will derive the energy (60) and a modified Jacobi field equation (61). Although these

equation are not really new [Inn98, §2], we give a geometric argument that we have not seen elsewhere. To understand the extra terms in  $E(y)$  due to the reflections, suppose that  $\gamma$  reflects from  $\partial M$  at a point  $p = \gamma(t)$ , and let  $Q = Q(p)$  be shape operator  $\partial M$  relative to the inward unit normal  $w = w(p)$ , i.e.,

$$Qu = -\nabla_u w.$$

If we give  $\gamma$  a ghost extension as in Figure 7, then the displacement  $\gamma + y$  has a gap when  $\partial M$  is curved. (The figure shows the convex case with a positive gap; the gap can also have negative width.) We first assume the simplest case in which  $\gamma$  is normal to  $\partial M$ . The quadratic form  $\langle \cdot, Q \cdot \rangle / 2$  osculates  $\partial M$ , so that the width of the gap, and thus the negative of the change in length, is  $\langle y, Qy \rangle$ . If the angle of incidence of  $\gamma$  is  $\theta \neq 0$ , then this answer is subject to two corrections. First, the gap is at an angle of  $\theta$  from  $\gamma$ , so the length saved is  $\cos(\theta) \langle \cdot, Q \cdot \rangle$ . Second,  $y$  no longer represents the position that  $\gamma + y$  meets  $T_p \partial M$ , again because the surface is angled.

To derive where  $\gamma + y$  meets  $T_p \partial M$ , we call  $T_p(\partial M)$  the *tangent hyperplane*, the normal  $N_p(\gamma)$  to  $\gamma(t)$  the *coronal hyperplane*, and the 2-dimensional plane spanned by  $w(p)$  and  $\gamma'(t)$  the *sagittal plane*<sup>4</sup>. Let  $P$  be the orthogonal projection from the tangent hyperplane to the coronal hyperplane. If we choose an orthonormal coronal basis  $e_1, \dots, e_{n-1}$  such that  $e_1$  is in the sagittal plane, and a matching tangent basis, then

$$P = \begin{pmatrix} \cos(\theta) & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then the change in length, and therefore the extra energy term, is

$$-\cos(\theta) \langle P^{-1}y, QP^{-1}y \rangle.$$

(This formula still works when  $\theta = 0$  if we take  $P$  to be the identity matrix.) If  $\gamma$  reflects from a sequence of boundary points  $\{p_k\}$  at times  $\{t_k\}$ , then we have the same change in length using angles  $\theta_k$  and symmetric matrices  $P_k$  and  $Q_k$ , and we can abbreviate the result by letting

$$A_k \stackrel{\text{def}}{=} \cos(\theta_k) P_k^{-1} Q_k P_k^{-1}.$$

<sup>4</sup>This terminology is borrowed from human anatomy.

Then energy of the normal field  $y$  is

$$E(y) = \int_0^r [\langle y'(t), y'(t) \rangle - \langle y(t), R(t)y(t) \rangle] dt - \sum_k \langle y(t_k), A_k y(t_k) \rangle. \quad (60)$$

Thus, if  $y$  is a (reflecting) Jacobi field, it satisfies the distributional differential equation

$$y''(t) = -R(t)y(t) - \sum_k A_k y(t_k) \delta_{t_k}(t), \quad (61)$$

where  $\delta_t$  is a Dirac delta measure on  $\mathbb{R}$  concentrated at  $t$ . Note that if  $\partial M$  is concave, then  $Q_k$  is negative semidefinite and therefore so is  $A_k$ .

We now follow a standard proof of Günther's inequality; see Theorem 3.101 in [GHL90]. First, we will need that the energy (60) is positive definite, so that if  $y$  Jacobi field, it is an energy minimum (assuming Dirichlet boundary conditions) and not just a critical point. This is standard in the proof of Günther's inequality without the  $A_k$  terms, with the aid of the length restriction when  $\kappa > 0$ . It is still true with the  $A_k$  terms, since each such term is positive semidefinite.

Second, we consider a matrix solution  $Y$  to (61) with  $Y(0) = 0$  and  $Y(r) = I$ . Then the candle function of  $\gamma$  satisfies

$$j(\gamma, 0, \ell) = \frac{\det Y(\ell)}{\det Y(r)},$$

and the logarithmic derivative at  $r$  is given by

$$\left. \frac{\partial}{\partial t} \right|_{t=r} \log(j(\gamma, 0, t)) = (\det Y)'(r) = \text{Tr}(Y)(r).$$

We generalize the energy (60) to the matrix argument  $Y$ , and we interpret it as a function of  $Y$ ,  $R(t)$ , and each  $A_k$ :

$$E(Y, R, A) = \int_0^r [\langle Y'(t), Y'(t) \rangle - \langle Y(t), R(t)Y(t) \rangle] dt - \sum_k \langle Y(t_k), A_k Y(t_k) \rangle, \quad (62)$$

using the Hilbert-Schmidt inner product

$$\langle X, Y \rangle = \text{Tr}(X^T Y).$$

If  $Y$  is a solution to (61), then integration by parts yields the remarkable equality

$$\text{Tr}(Y)(r) = E(Y, R, A).$$

If we minimize  $E$  with respect to all three arguments  $Y$ ,  $R$ , and  $A$ , then we both solve (61) and minimize the logarithmic derivative of  $\gamma$ . If we fix  $Y$ , then it is immediate from (62) and from the constraints that we should take  $R = \kappa I$  and  $A_k = 0$ , *i.e.*, maximum curvature and flat mirrors.

8.3. **Multiple images.** Lemma 5.7 yields the following model.

**LP Problem 8.1.** *Given  $n$ ,  $\kappa$ ,  $A$ ,  $V$ , and  $m$ , is there a symmetric, positive measure  $\mu(\ell, \alpha, \beta)$  such that*

$$\begin{aligned}\alpha_*(\mu) &= \int_{\ell, \beta} d\mu = A dz(\alpha) \\ \int_{\ell, \alpha, \beta} s_{n, \kappa}(\ell) \sec(\alpha) \sec(\beta) d\mu &\leq mA^2 \\ \int_{\ell, \alpha, \beta} s_{n, \kappa}^{(-1)}(\ell) (\sec(\alpha) + \sec(\beta)) d\mu &\leq 2mAV \\ \int_{\ell, \alpha, \beta} s_{n, \kappa}^{(-2)}(\ell) d\mu &\leq mV^2 \\ \int_{\ell, \alpha, \beta} \ell d\mu &= \omega_{n-1}V?\end{aligned}$$

Theorem 1.8 now follows as a corollary-of-proof<sup>5</sup> of Theorem 1.4. If we apply the transformation

$$\tilde{V} = mV, \quad \tilde{A} = mA, \quad \tilde{\mu} = m\mu,$$

then Problem 8.1 becomes Problem 6.1.

8.4. **Alternative functionals.** In this section we prove Theorem 1.17. The proof is almost the same as the proof of Croke's theorems in Section 7.1.

Given  $L = L(\Omega)$ , we consider the following linear programming problem based on equation 23.

**LP Problem 8.2.** *Given  $n$ ,  $A$ , and  $L$ , is there a symmetric positive measure  $\mu(\ell, \alpha, \beta)$  such that*

$$\begin{aligned}\alpha_*(\mu) &= \int_{\ell, \beta} d\mu = A dz(\alpha) \\ \int_{\ell, \alpha, \beta} \ell^{n-1} \sec(\alpha) \sec(\beta) d\mu &\leq A^2 \\ \int_{\ell, \alpha, \beta} \ell^{n-3} d\mu &= L?\end{aligned}$$

We can apply a version of Proposition 6.5 to establish Theorem 1.17 as a sharp inequality.

Given  $a \geq 0$  and  $d \in \mathbb{R}$ , we consider the energy

$$E(\ell, \alpha, \beta) = a\ell^{n-1} \sec(\alpha) \sec(\beta) - d\ell^{n-3}.$$

<sup>5</sup>Also known as a *porism*.

By design, given a radius  $r > 0$ , there are values of  $a, d > 0$  such that  $E(\ell, \alpha, \alpha)$  is minimized in  $\ell$  when

$$\ell = 2r \cos(\alpha),$$

which thus satisfies (47). We can take

$$a = \frac{n-3}{r^2}, \quad d = 4(n-1).$$

(Note that we need  $n \geq 4$ . If  $n < 3$ , then  $a$  would be negative. If  $n = 3$ , then  $L(\Omega) \propto |\partial\Omega|$  and Theorem 1.17 is vacuous.)

We define the potential

$$f(\alpha) = -\frac{E(2r \cos(\alpha), \alpha, \alpha)}{2} = 2^{n-1}(r \cos(\alpha))^{n-3}.$$

Applying the change of variables (49), the adjusted cost function is

$$F(\ell, x, y) = (n-3)\ell^{n-1}xy - 4(n-1)\ell^{n-3} + 2^{n-1}(x^{3-n} + y^{3-n}).$$

We want to show that  $F \geq 0$ . For any fixed value of  $xy$ ,  $F(\ell, x, y)$  is minimized when  $x = y$ . Then

$$\begin{aligned} F(\ell, x, x) &= (n-3)\ell^{n-1}x^2 - 4(n-1)\ell^{n-3} + 2^n x^{3-n} \\ &= ((n-3)(\ell x)^{n-1} - 4(n-1)(\ell x)^{n-3} + 2^n)x^{3-n}. \end{aligned}$$

The first factor is a polynomial in  $\ell x$  that, by univariate calculus, decreases to 0 at  $\ell x = 2$  and then increases again. This completes the proof of Theorem 1.17.

**8.5. Old wine in new bottles.** In this section, we complete the proof of Theorem 1.16. The rest of this paper has covered all cases except Theorem 1.15, due to Yau, and Theorem 1.14, due to Croke. The arguments given here are equivalent to the original proofs, only restated in linear programming form.

8.5.1. *Yau's linear isoperimetric inequality.* Suppose that  $\Omega$  is  $n$ -dimensional and LCD(-1). Problem 7.2 yields

$$\int_{\ell, \alpha, \beta} (s^{(-1)}(\ell) \sec(\alpha) - (n-1)s^{(-2)}(\ell)) \, d\mu_{\Omega} \leq AV - (n-1)V^2$$

since  $q > n-1$ . The integrand is positive, since it is the second antiderivative of

$$\begin{aligned} s'(\ell) \sec(\alpha) - (n-1)s(\ell) &= \\ &= (n-1) \sinh(\ell)^{n-2} (\cosh(\ell) \sec(\alpha) - \sinh(\ell)) > 0. \end{aligned}$$

Thus the right side is positive, and Theorem 1.15 follows. Stated pedantically in terms of optimal transport, the result follows if we define a cost function

$$E(\ell, \alpha, \beta) = s^{(-1)}(\ell) \sec(\alpha) - (n-1)s^{(-2)}(\ell),$$

and then a vanishing potential  $f(\alpha) = 0$ .

8.5.2. *Croke's curvature-free inequality.* For simplicity, we take  $\rho = 1$ .

Suppose that  $\Omega$  is  $n$ -dimensional with unique geodesics. Lemma 5.6 produces the following simple model that does not use any value of  $\kappa$ , and that could be combined with Problem 6.1.

**LP Problem 8.3.** *Given  $n$ ,  $A$ , and  $V$ , is there a symmetric, positive measure  $\mu(\ell, \alpha, \beta)$  such that*

$$\begin{aligned} \alpha_*(\mu) &= \int_{\ell, \beta} d\mu = A \, dz(\alpha) \\ \int_{\ell, \alpha, \beta} s_{n, (\pi/\ell)^2}^{(-2)}(\ell) \, d\mu &\leq V^2 \\ \int_{\ell, \alpha, \beta} \ell \, d\mu &= \omega_{n-1} V? \end{aligned}$$

In order to analyze this model, we simplify it in two respects. First, we can integrate away  $\alpha$  and  $\beta$ , because none of the integrals explicitly depend on them. We call the resulting measure  $\mu(\ell)$ . Second, we can explicitly evaluate the integrand that arises from Lemma 5.6:

$$s_{n, (\pi/\ell)^2}^{(-2)}(\ell) = \left(\frac{\ell}{\pi}\right)^{n+1} s_{n,1}^{(-2)}(\pi) = \frac{\ell^{n+1} \omega_n}{2\pi^n \omega_{n-1}}.$$

The first equality is just rescaling by  $\ell/\pi$ . The second equality is a tricky but standard integral; the answer can also be inferred from the optimal case of a hemisphere  $Y_{n,1}$ . The simplified model is then as follows.

**LP Problem 8.4.** *Given  $n$ ,  $A$ , and  $V$ , is there a positive measure  $\mu(\ell)$  on  $\mathbb{R}_{\geq 0}$  such that*

$$\begin{aligned} \int_{\ell} d\mu &= \frac{\omega_{n-2}}{n-1} A \\ \int_{\ell} \frac{\ell^{n+1} \omega_n}{2\pi^n \omega_{n-1}} \, d\mu &\leq V^2 \\ \int_{\ell} \ell \, d\mu &= \omega_{n-1} V \end{aligned}$$

As usual, we state the dual of Problem 8.4.

**LP Problem 8.5.** *Given  $n, A,$  and  $V,$  are there constants  $c \geq 0$  and  $f, d \in \mathbb{R}$  such that*

$$\begin{aligned} f + c \frac{\ell^{n+1} \omega_n}{2\pi^n \omega_{n-1}} - d\ell &\geq 0 \\ f \frac{\omega_{n-2}}{n-1} A + cV^2 - d\omega_{n-1}V &< 0? \end{aligned} \quad (63)$$

In the optimal case of  $Y_{n,1}$ , we have  $\ell = \pi$  everywhere and  $V = \omega_n/2$ . We can solve for the constants  $f, c,$  and  $d$  assuming that the left side of equation (63) reaches 0 there and is non-negative for other values of  $\ell$ . We obtain

$$c = 2\omega_{n-1}, \quad d = (n+1)\omega_n, \quad f = n\pi\omega_n.$$

Assuming that  $A$  is feasible rather than infeasible, (63) then gives us the inequality

$$A \geq \frac{(n-1)\omega_n\omega_{n-1}}{2\pi\omega_{n-2}} = \omega_{n-1}.$$

This establishes Theorem 1.14.

*Remark.* It may seem wrong that  $Y_{n,1}$  does not itself have unique geodesics. But it is a limit of manifolds that do, which is good enough. In any case the proof of Theorem 1.14 only really uses that  $\Omega$  has unique geodesics in its interior.

## 9. CLOSING QUESTIONS

Of course, we want Theorem 1.5 without the smallness condition (4). It would suffice to prove a stronger version of Lemma 5.5. At least in the convex case, this would be implied by the following conjecture; the non-convex case might not be much harder.

**Conjecture 9.1.** *Let  $j(r, t) = j_M(\gamma, r, t)$  be the candle function of a geodesic in  $\gamma$  in an  $n$ -manifold  $M$  with curvature  $K \leq -1$ . Then*

$$\left[ j - (n-1) \frac{\partial j}{\partial t} + (n-1) \frac{\partial j}{\partial r} - (n-1)^2 \frac{\partial^2 j}{\partial r \partial t} \right] (r, t)$$

*is minimized when  $M$  has constant curvature  $K = -1$ . Integrating twice,*

$$\begin{aligned} j(0, \ell) - (n-1) \int_0^\ell j(0, t) dt - (n-1) \int_0^\ell j(s, \ell) ds \\ + (n-1)^2 \int_0^\ell \int_0^t j(s, t) ds dt. \end{aligned}$$

*is minimized when  $K = -1$ .*



Note that even second relation in Conjecture 9.1 is not true under the weaker hypothesis  $\text{LCD}(-1)$ . For example, it does not hold when  $\ell$  is large enough if  $M$  is the complex hyperbolic plane  $\mathbb{C}H^2$ , normalized to have sectional curvature between  $-9/4$  and  $-9/16$ .

The following relaxation of Kleiner's theorem is open even though, as explained in Section 1.4.2, it is close to true. The motivation is that the even strongest form holds in dimension  $n = 4$  following the proof of Croke's theorem.

**Question 9.1.** *Suppose that  $\Omega$  is a compact, 3-dimensional manifold with boundary, with unique geodesics, with non-positive curvature, and with fixed volume  $|\Omega|$ . Then is its surface area  $|\partial\Omega|$  minimized when  $\Omega$  is a round, Euclidean ball? What if non-positive curvature is replaced by  $\text{Candle}(0)$ ? What if the  $\text{Candle}(0)$  condition is only required for pairs of boundary points?*

Each part of Question 9.1 could also be asked in dimension  $n \geq 5$  and for other curvature bounds  $\kappa$ .

Finally, the following conjecture would give a more robust proof of Theorem 1.7, with a weaker hypothesis as well when  $\kappa = 0$ .

**Conjecture 9.2.** *Suppose that  $\Omega$  is a convex, compact Riemannian  $n$ -manifold with boundary with unique geodesics. Suppose that for some constants  $\kappa$  and  $r$ , all of the chords in  $\Omega$  satisfy equation (47). Then  $\Omega$  is isometric to  $B_{n,\kappa}(r)$ .*

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