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Using linear programming methods, we derive various isoperimetric inequalities in 2 and 4-dimensional Riemannian manifolds whose curvature is bounded from above.

First, we consider the problem of shaping a small planet inside a non-positivily curved surface so as to maximize the gravity feeled by a fixed observer (the Little Prince). This provides a pointwise inequality which, integrated on the boundary of a domain, yields Weil's theorem asserting that the planar Euclidean isoperimetric inequality is satisfied inside all simply connected, non-positively curved surfaces.

Then, generalizing Croke's proof of the dimension 4 version of this result, we obtain similar statements in manifolds satisfying an arbitrary sectional curvature upper bound.

Moreover, the method enables us to state all our results under a relaxed curvature condition.

* benoit.kloeckner@ujf-grenoble.fr; Supported by ANR grant "GMT" JCJC - SIMI 1 - ANR 2011 JS01 011 01
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1. INTRODUCTION

1.1. Weil's isoperimetric conjecture

Isoperimetric inequalities are one of the most ancestral still active subject in geometry. First studied in Euclidean space, it is natural to consider their extension to other Riemannian manifolds: for example it is now well known that in round spheres and real hyperbolic spaces, balls have the least perimeter amoung domains of given volume. One particularly challenging problem is to obtain an isoperimetric inequality that is valid for a whole class of Riemannian manifold; obviously, some geometric constraint must be imposed to such a class. The first result in this spirit has been proved as early as 1926 by Weil: a simply connected surface with boundary whose Gauss curvature is non-positive must satisfy the planar isoperimetric inequality [Wei26].

It has been conjectured, (notably by Aubin [Aub76], see also [BZ88], [Gro81]) that similarly in any dimension n, a domain in any simply-connected manifold of non-positive sectional curvature must satisfy the isoperimetric inequality of $\mathbb{R}^n$. It has been generalized to a negative upper bound κ on the sectional curvature; in this case it is expected that any domain satisfy the optimal isoperimetric inequality of the real hyperbolic space of dimension n and curvature κ.

Up to now, it is known that this conjecture holds in dimension 2 (we refer to [Oss78] for attribution of the general case but note that the proof by Weil [Wei26] when $\kappa = 0$ preceeds the usually cited one by Beckenbach and Radó [BR33]) and 3 [Kle92], and in dimension 4 for $\kappa = 0$ [Cro84].

An interesting feature of Croke's proof is that it applies to all non-positively curved manifold with boundary such that every pair of points is linked by at most one geodesic; this is slightly more general than the usual setting, but more importantly it makes it possible to extend the conjecture to positive curvature bound.

More precisely, it is expected that any compact manifold with boundary whose sectional curvature is bounded above by $\kappa > 0$ and all of whose pairs of points are linked by at most one geodesic satisfies the isoperimetric inequality of the sphere of curvature κ.

Since a simply connected non-positively curved manifold is called a Cartan-Hadamard manifold, the isoperimetric conjecture when $\kappa \leqslant 0$ is sometimes known as the Cartan-Hadamard conjecture. Since Weil's work is the first one in this direction we shall call "Weil's isoperimetric conjecture" both the statement for domains when $\kappa \leqslant 0$, and its adaptation for $\kappa > 0$ described above.

The goal of this paper is to give a method that

- solves the dimension 4 and $\kappa < 0$ case for domains which are small enough in a explicit sense,
- solves the $\kappa > 0$ case in dimension 2 and 4,
- makes Weil's theorem a consequence of a pointwise inequality,
- enable one to use more flexible hypotheses than the usual upper bound on sectional curvature.

Very roughly, the method consists in using the curvature bound (or, rather, a volume comparison property implied by the curvature bound) to derive linear integral inequalities involving the volume of the domain, the volume of its boundary, and a natural measure on the set of geodesic segments, and combining these inequalities in an optimal way using linear programming.

1.2. Notations

Let M be a connected compact Riemannian manifold with boundary of dimension $n = 2$ or 4, and let κ be a real number to be used as a curvature bound. All isoperimetric inequalities below also apply to manifolds with boundary *and corners* by classical smoothing arguments.

We use $|\cdot|$ to denote the size of an object, in particular the Riemannian volume of a manifold.

Let S^n_κ be the simply connected space form of dimension n and curvature κ, and for all V let $B^n_\kappa(V)$ be the metric ball of S^n_κ having volume V (if $\kappa > 0$, we restrict V to be at most the volume $\frac{1}{2}|S^n_\kappa| = \kappa^{-n/2}\omega_n/2$ of an hemisphere, where ω_n denotes the volume of the unit n-sphere S^n_1).

Let A_M be the volume of ∂M, V_M be the volume of M, $K(x,P)$ be the sectional curvature of M at a point $x \in M$ and plane $P \subset T_xM$; by $K \leqslant \kappa$ we mean that this inequality is satisfied uniformly in x and P.

1.3. Candle condition and Root-Ricci curvature

To state our results optimaly, let us describe a few geometric properties whose relations are studied in our previous article [KK12], to which we refer for more details and proofs.

The "candle function" is simply a normalized version of the Jacobian of the exponential map. Let $\gamma = \gamma_u$ be a geodesic curve in M that begins at $p = \gamma(0)$ with initial velocity $u \in UT_pM$. Then the candle function $s(\gamma, r)$ is by definition the Jacobian of the map $u \mapsto \gamma_u(r)$. In other words, it is defined by the equations

$$\mathrm{d}q = s(\gamma_u, r)\,\mathrm{d}u\,\mathrm{d}r \qquad q = \gamma_u(r) = \exp_p(ru),$$

where $\mathrm{d}q$ is Riemannian measure on M, $\mathrm{d}r$ is Lebesgue measure on $\mathbb{R}$, and $\mathrm{d}u$ is Riemannian measure on the sphere UT_pM.

The candle function of a geometry of constant curvature κ is denoted by $s_\kappa(r)$. Croke's Theorem does not really need $K \leqslant 0$, but rather $s(\gamma, r) \geqslant s_0(r)$ for all γ, r. The latter is a consequence of the former according to Günther's inequality [Gün60, BC64], but is also a consequence of a weaker curvature bound to be described below.

Say that a manifold M is Candle(κ) if the inequality

$$s(\gamma, r) \geqslant s_\kappa(r) \tag{1}$$

holds for all γ, r; or LCD(κ), for *logarithmic candle derivative* if the logarithmic condition

$$(\log s(\gamma, r))' \geqslant (\log s_\kappa(r))' \tag{2}$$

(where the derivative is taken with respect to r) holds for all γ, r. The Candle(κ) condition is weaker than LCD(κ), since the former can be deduced from the latter by integration.

If $\kappa > 0$, then these conditions are only meaningful up to the distance $\pi/\sqrt{\kappa}$ between conjugate points in the comparison geometry. We also write Candle(κ, ℓ) and LCD(κ, ℓ) if the same conditions hold up to a distance of $r = \ell$. If L_M is the maximal length of a geodesic in M, then in this vocabulary Croke only uses Candle($0, L_M$) instead of $K \leqslant 0$.

Both to show that the candle conditions are much more general than sectional upper bounds and in order to get equality cases later on, let us introduce the *root-Ricci curvature*. Assume $K \leq \rho$ for some constant $\rho \geqslant 0$. For any unit tangent vector $u \in UT_pM$ with $p \in M$, we define

$$\sqrt{\mathrm{Ric}}(\rho, u) \stackrel{\text{def}}{=} \mathrm{Tr}(\sqrt{\rho - R(\cdot, u, \cdot, u)}).$$

Here $R(u, v, w, x)$ is the Riemann curvature tensor expressed as a tetralinear form, and the square root is the positive square root of a positive semidefinite matrix or operator. This definition should be considered an analogue of Ricci curvature: just as

$$\frac{1}{n-1}\mathrm{Ric}(u, u) = \frac{1}{n-1}\mathrm{Tr}\,R(\cdot, u, \cdot, u)$$

is the arithmetic mean of principal curvatures,

$$\rho - (\frac{1}{n-1}\sqrt{\mathrm{Ric}}(\rho, u))^2 = \rho - \left(\frac{1}{n-1}\mathrm{Tr}(\sqrt{\rho - R(\cdot, u, \cdot, u)})\right)^2 \tag{3}$$

is a non-linear mean of principal curvature, conjugate to the linear mean by the map $f(x) = \sqrt{\rho - x}$.

We refer to [KK12] for a detailled description of various property of root-Ricci curvature; let us simply recall that we say that M *is of* $\sqrt{\mathrm{Ric}}$ *class* (ρ, κ) if $K \leqslant \rho$ and $\sqrt{\mathrm{Ric}}(\rho, u) \geqslant (n-1)\sqrt{\rho - \kappa}$ for all u, which amounts to ask the mean (3) to be $\leqslant \kappa$, and that the following implications hold: for all $\rho' \geqslant \rho \geqslant \kappa$

$$K \leqslant \kappa \implies \sqrt{\mathrm{Ric}}\text{ class }(\rho, \kappa) \implies \sqrt{\mathrm{Ric}}\text{ class}(\rho', \kappa) \implies \mathrm{Ric} \leqslant \kappa(n-1)g$$

All implications are strict when $n > 2$; of course in dimension 2 root-Ricci curvature is equivalent to sectional curvature. Let us give two examples of 4-manifolds of $\sqrt{\mathrm{Ric}}$ class $(0, -1)$ but not satisfying $K \leqslant -1$:

- the complex hyperbolic plane, normalized to have sectional curvature between $-\frac{9}{4}$ and $-\frac{9}{16}$,
- the product of two simply connected surfaces whose Gauss curvature is bounded above by -9.

The main purpose of [KK12] is to prove the following relation between root-Ricci curvature and candle conditions:

$$\sqrt{\mathrm{Ric}} \text{ class } (\rho,\kappa) \implies \mathrm{LCD}(\kappa, \frac{\pi}{2\sqrt{\rho}}).$$

Moreover, if M has $\sqrt{\mathrm{Ric}}$ class (rho,κ) and for all γ there is a positive r that makes (2) or (1) an equality, then M has constant curvature κ.

1.4. Isoperimetric inequalities

We start with a simple application of the linear programming method.

Theorem 1.1 (Little Prince)**.** *Assume M is a compact domain in a convex,*[1] *simply connected, non-positively curved surface. For all $x \in M$ let X_x be the "gravitational field" generated by x, namely the radial vector field centered at x whose divergence is the Dirac measure $-\delta_x$. For any points $p \in \partial M$ and $q \in \partial B_0^2(V_M)$, with inward normal vectors v_p, v_q, we have*

$$\int_M X_x \cdot v_p \, \mathrm{d}x \leqslant \int_{B_0^2(V_M)} X_y \cdot v_q \, \mathrm{d}y$$

In other words, to feel maximal gravity given the density and mass of its planet, the Little Prince should make it a round ball in the Euclidean plane. The nice thing is that while the problem data has a prominent point p, its solution is as symmetric as it could be. As a consequence, when we integrate this inequality over ∂M, we get a relation between A_M and V_M, which happens to be precisely Weil's isoperimetric inequality, see Section 2

After the Little Prince warm-up, we shall show the following results using the linear programming method.

We use L_M to denote the maximal length of a geodesic with endpoints in M; that is, when M is a mere manifold with boundary L_M is the maximal length of a geodesic segment in M, but if it is assumed that M is a domain in a convex ambiant manifold N then L_M is the diameter of M as a subset of N.

Theorem 1.2 (non-negative bound)**.** *Recall that $n = 2$ or 4, and assume $\kappa \geqslant 0$. If M satisfies* $\mathrm{Candle}(\kappa, L_M)$ *(e.g. $K \leqslant \kappa$, or $\sqrt{\mathrm{Ric}}$ class (ρ,κ) with small enough ρ), if every pair of points is linked by at most one geodesic, and if $V_M \leqslant \kappa^{-n/2}\omega_n/2$ then M satisfies the isoperimetric inequality*

$$A_M \geqslant |\partial B_\kappa^n(V_M)|\,. \tag{4}$$

Note that when $\kappa = 0$ the volume upper bound is empty, so that in particular we recover Weil's and Croke's results. The hypothesis on the volume is quite natural in a positive curvature setting; note in particular that limiting the volume to *half* the volume of a sphere is unsurprising considering the hypothesis that M is uniquely geodesic.

The case $n = 2, \kappa > 0$ already appeared with slightly different hypotheses in [MJ00]. The case $n = 4$ is our most satisfactory new result.

When $\kappa < 0$ our method half fails and we need to consider a domain in an ambiant manifold, the stronger LCD condition, and a smallness condition expressed in terms of L_M and the radius $r_M(\kappa)$ of the comparison ball $B_\kappa^n(V_M)$.

[1] By convex, we mean that every pair of points is linked by a geodesic; we do not need the surface to be complete.

Theorem 1.3 (negative bound)**.** *Recall that $n = 2$ or 4 and let κ be negative. Assume M is a compact domain in a convex manifold N which satisfies* $\mathrm{LCD}(\kappa, L_M)$ *(e.g., $K \leqslant \kappa$ or $\sqrt{\mathrm{Ric}}$ class $(0,\kappa)$) and that every pair of points of M is linked by exactly one geodesic in N. If $n = 4$, assume further that M satisfies the smallness condition*

$$\tanh\left(L_M\sqrt{-\kappa}\right)\tanh\left(r_M(\kappa)\sqrt{-\kappa}\right) \leqslant 1/2 \tag{5}$$

Then M satisfies the hyperbolic isoperimetric inequality

$$A_M \geqslant \left|\partial B^n_\kappa(V_M)\right|.$$

Note that $\tanh \leqslant 1$, so that a diameter bound *or* a volume bound is sufficient to get the smallness condition. Of the three restrictions in that case, the smallness condition is the most undesirable, but getting rid of it would need a different approach.

This isoperimetric inequality has been proved in all dimensions for *very small* domains by Morgan and Johnson [MJ00], then by Druet [Dru02] under a very mild scalar curvature condition, but since they use compactness arguments they do not give any uniform (let alone explicit) size condition.

The case of equality in both above results is not completely clear, but we can recover it under a curvature bound.

Theorem 1.4. *Consider M as in Theorem 1.2 or 1.3 but replace the candle condition by $\sqrt{\mathrm{Ric}}$ class (ρ,κ) for any $\rho \geqslant \kappa$ such that $2\sqrt{\rho}L_M \leqslant \pi$; alternatively, if $\kappa > 0$ and no such ρ exists, assume $K \leqslant \kappa$.*

If equality holds in (4)*, then M is isometric to $B^n_\kappa(V_M)$.*

Let us end with a variation when the number of geodesics connecting two points is only bounded.

Theorem 1.5 (relative version)**.** *Consider M as in Theorem 1.2 or 1.3 but:*

- *only assume that every pair of points in M is linked by* at most *m geodesics (in N if $\kappa < 0$),*
- *if $\kappa > 0$, assume $mV_M \leqslant \kappa^{-n/2}\omega_n/2$,*
- *if $\kappa < 0$, assume*

$$\tanh(L_M\sqrt{-\kappa})\tanh(r\sqrt{-\kappa}) \leqslant 1/2$$

where r is the radius of $B^n_\kappa(mV_M)$.

Then M satisfies the following isoperimetric inequality:

$$A_M \geqslant \frac{1}{m}\left|\partial B^n_\kappa(mV_M)\right|$$

We call this result "relative" since it is related to the more classical relative isoperimetric inequality of Choe [Cho06] and Choe-Ritore [CR07], which are Didon version of Weil's isoperimetric conjecture: M lies in some ambiant manifold N outside of a convex body C and the parts of M and ∂M included in C are not counted in V_M and A_M. The geodesic flow then extends by bouncing against the convex, and some pairs of points get linked by two geodesics (one classical, the other one bouncing).

Let us give an example that shows the optimality of Theorem 1.5.

Example 1.1. Let us first construct an orbifold example. Let B_0 be a ball in a space form S^n_κ, A be a totally geodesic 2-codimensional subspace containing the center of B_0, and θ be a rotation of axis A and angle $2\pi/m$ for some positive integer m. Then $R := B_0/\theta$ is an orbifold with a cone singularity along A, and any two points of R are connected by exactly m geodesics (defined as the projections of geodesics of B_0). Moreover R realizes the equality in Theorem 1.5.

One would prefer to have manifold examples. Most simply, consider C a solid cylinder of axis A and very small radius r and let $M := (B \setminus C)/\theta$. Then M is a manifold with boundary and corners, and letting $r \to 0$ we see that Theorem 1.5 is sharp among such manifolds.

1.5. Faber-Krahn inequalities

As usual, isoperimetric inequalities have consequences for the bottom of the spectrum $\lambda(M)$ of the Laplacian with Dirichlet boundary condition. This is a consequence of the Faber-Krahn method, see for example the book of Chavel [Cha84]. In our case, we get that under the assumptions of Theorems 1.2 or 1.3,

$$\lambda(M) \geqslant \lambda(B^n_\kappa(V_M))$$

and under the assumptions of Theorem 1.4, equality of these eigenvalues implies that M is isometric to $B^n_\kappa(V_M)$.

Acknowledgements

B.K. thanks Sylvain Gallot for fruitful conversations about Riemannian geometry.

2. SHAPING THE LITTLE PRINCE PLANET

In this section, we prove Theorem 1.1. Beside the individual interest of having a "pointwise" version of an isoperimetric inequality, the proof already contains all the main ideas for our other results, but in a simple form so that everything appears clearly.

Here $n = 2$ and $\kappa = 0$, so M is a compact smooth domain in a simply connected surface N satisfying Candle(0) and assumed to be convex (every pair of points is linked by a geodesic, which must be unique thanks to the candle condition which prevents conjugate points). In particular, for any $x \in N$ the polar coordinates $y = \exp_x(tu)$ with $u \in UT_xM$, $t > 0$ define a global chart of $N \setminus \{x\}$. To simplify the notations, we shall write $s_x(y) = s_x(tu) = s(\gamma_u, t)$ so that s_x, depending on the context, denotes a function either on N or on T_xN.

We consider a given point $p \in \partial M$. For each $x \in M$, we define

$$X_x = -(2\pi s_x)^{-1} \exp_{x*}\left(\frac{\partial}{\partial t}\right)$$

This defines a radial vector field(singular at x) whose flux through a piece of sphere $\exp_x(tA)$ where $A \subset UT_xN$ is equal to $-|A|/2\pi$, so that $\operatorname{div} X_x = -\delta_x$ in the sense of distributions.

Proof of Theorem 1.1. First, remark that, since we try to bound from above the gravity $\int_M X_x \cdot \nu_p \, \mathrm{d}x$, we can assume that M is starshaped at p and contained in one of the halves of N separated by $T_p\partial M$ (if there is mass above the Little Prince or holes in the planet, we can easily increase gravity by changing the shape of M without changing its volume).

Let $\mathscr{G}_p$ denote the set of maximal unit-speed geodesic segments of M that start at p; it can be identified with the set UT_p^+M of unit, inward tangent vectors at p, which is endowed with its usual measure. Let μ be the corresponding measure on $\mathscr{G}_p$.

Consider the functions α, ℓ defined on $\mathscr{G}_p$ that to a ray associate its starting angle with the inward normal ν_p, and its length. We shall identify μ with its image measure $(\alpha, \ell)_\# \mu$ on $D := [0, \pi/2] \times [0, +\infty)$. Similarly, we identify α, ℓ with the corresponding functions on UT_p^+M, as well as with the canonical coordinates on D.

Using a polar change of coordinates, we have

$$\begin{aligned}\int_M X_x \cdot \nu_p \, \mathrm{d}x &= \frac{1}{2\pi} \int_{\exp_p^{-1}(M)} \cos\alpha(u) \, \mathrm{d}t \, \mathrm{d}u \\ &= \int_D \ell \cos\alpha \, \mathrm{d}\mu(\alpha, \ell)\end{aligned}$$

Similarly, the candle condition yields

$$\begin{aligned}V_M = \int_M \mathrm{d}x &= \int_{\exp_p^{-1}(M)} s_p(tu) \, \mathrm{d}t \, \mathrm{d}u \\ &\geqslant \int_D \frac{1}{2}\ell^2 \, \mathrm{d}\mu(\alpha, \ell)\end{aligned}$$

We have another constraint: we know the α-marginal of μ, namely for all function $f : [0, \pi/2] \to \mathbb{R}$ it holds

$$\int f(\alpha)\,\mathrm{d}\mu(\alpha) = 2\int_0^{\frac{\pi}{2}} f(\alpha)\,\mathrm{d}\alpha$$

We are led to the following linear programming problem: given a number V_M, find a positive measure μ_0 on D satisfying

$$\int \frac{1}{2}\ell^2\,\mathrm{d}\mu_0 \leqslant V_M \tag{6}$$

$$\int f(\alpha)\,\mathrm{d}\mu_0 = 2\int f(\alpha)\,\mathrm{d}\alpha \quad \forall f \tag{7}$$

and maximizing

$$\int \ell\cos\alpha\,\mathrm{d}\mu_0$$

The point is then to use the classical duality principle. Assume that we find μ_0, f and a number $a > 0$ such that:

- there is equality in (6),
- for all (α, ℓ), it holds

$$\frac{a}{2}\ell^2 + f(\alpha) \geqslant \ell\cos\alpha \tag{8}$$

  with equality on $\operatorname{supp}\mu_0$

then we claim that μ_0 achieves the maximum we seek. Indeed, for all μ satisfying the constraints we would have

$$\begin{aligned}\int \ell\cos\alpha\,\mathrm{d}\mu &\leqslant \int \left(\frac{a}{2}\ell^2 + f(\alpha)\right)\mathrm{d}\mu \\ &\leqslant aV_M + 2\int f(\alpha)\,\mathrm{d}\alpha \\ &= \int \left(a\frac{1}{2}\ell^2 + f(\alpha)\right)\mathrm{d}\mu_0 \\ &= \int \ell\cos\alpha\,\mathrm{d}\mu_0\end{aligned}$$

In general, finding such μ_0, f, a could be difficult; but here we know that μ_0 should be the measure μ associated with the domain $B_0^2(V_M)$. It satisfies the constraints by what preceeds, realizes equality in (6), and is concentrated on the set

$$\left\{(\alpha,\ell)\,|\,\cos\alpha = \frac{\ell}{2r}\right\}$$

where r is the radius of $B_0^2(V)$. Moreover, a being fixed we should take f as small as possible, thus we let

$$f(\alpha) = \sup_\ell \left(\ell\cos\alpha - \frac{a}{2}\ell^2\right) = \frac{\cos^2\alpha}{2a}$$

and we see that (8) is realized when $\cos\alpha = a\ell$. Taking $a = 1/2r$, we are done. The maximizaton objective is realized by the planar disk of volume V_M, proving Theorem 1.1. □

Proof of Weil's Theorem. Let M be a domain in a simply connected non-positively curved surface (or more generally, a surface without cut-locus satisfying Candle(0)). Recall that A_M denotes the

length of ∂M, let $B = B_0^2(V_M)$, $q \in \partial B$ and $A_B = |\partial B|$. Integrating the Little Prince inequality over $p \in \partial M$, we get

$$\begin{aligned}
\int_{\partial M}\int_M X_x \cdot \nu_p \,\mathrm{d}x\,\mathrm{d}p &\leqslant A_M \int_B X_y \cdot \nu_q \,\mathrm{d}y \\
\int_M \int_{\partial M} X_x \cdot \nu_p \,\mathrm{d}p\,\mathrm{d}x &\leqslant \frac{A_M}{A_B}\int_B\int_{\partial B} X_y \cdot \nu_q \,\mathrm{d}q\,\mathrm{d}y \\
\int_{M^2} -\operatorname{div} X_x(\xi)\,\mathrm{d}x\,\mathrm{d}\xi &\leqslant \frac{A_M}{A_B}\int_{B^2} -\operatorname{div} X_y(\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\
\int_M \mathrm{d}x &\leqslant \frac{A_M}{A_B}\int_B \mathrm{d}y \\
A_M &\geqslant A_B
\end{aligned}$$

□

This proof is limited to $n = 2$ and $\kappa = 0$ because it needs two quantities to be proportional: the function T defining the Euclidean ball in polar coordinates centered at a boundary point by $\{\cos\alpha = T(\ell)\}$ and the ratio between the derivatives of the contribution ℓ^n/n to the volume and the contribution $\ell = \int_0^\ell |X_{\exp_p(tu)}| s_p(tu)\,\mathrm{d}t$ to the gravitational force of the direction u.

In other cases, one could try to change the expression of the gravitational field to ensure this coincidence, but $|X_{\exp_p(tu)}| s_p(tu)$ needs to be constant to ensure that the divergence is zero outside the singularity. We expect that when $\kappa \neq 0$, this method can be adapted by using a constant divergence vector field to get a similar pointwise inequality implying the isoperimetric inequality, but only in surfaces of constant curvature.

3. THE LINEAR PROGRAMMING PROBLEM BEHIND THE ISOPERIMETRIC INEQUALITY

In this section we shall derive the linear programming problem that lies behind theorems 1.2 and 1.5. The starting point is a set of inequalities that translate the curvature bounds, and are formulated in terms of a measure on geodesic segments we first define. We keep the notations of the introduction all along. From now on we assume that M satisfies Candle(κ).

3.1. A measure on geodesic segments

The cotangent bundle T^*M has a canonical symplectic structure, and since M is Riemannian we can translate it to a symplectic structure on TM. The geodesic flow induces a 1-dimensional foliation of TM, which leaves invariant the unit tangent bundle UTM, a 1-codimensional manifold. By the general principle of symplectic quotients, these data induce a symplectic structure on any hypersurface $U \subset UTM$ transversal to the geodesic flow, with compatibility when the geodesic flow is used to identify two pieces of transversal U_1, U_2. Here M is assumed to be uniquely geodesic, so that the picture becomes global: the set $\mathscr{G}$ of maximal, oriented, unparametrized geodesic segments identifies with the manifold of leaves of the geodesic flow, and therefore inherits a symplectic structure. The corresponding volume μ_M is the measure we are interested in.

If $\mathrm{d}x$ is the Riemannian volume on M and $\mathrm{d}u$ the usual volume on the sphere UT_xM, the Liouville measure is defined as the measure on UTM that writes $\mathrm{d}x\,\mathrm{d}u$ (note that $\mathrm{d}u$ denotes measures on each fiber of UTM, but they are equivariant under parallel transport so that they can all be identified). The measures μ_M and $\mathrm{d}x\,\mathrm{d}u$ are closely related: the latter is obtained from the former by time integration. In other words, the Liouville measure of $A \subset UTM$ is the integral of the function $\gamma \in \mathscr{G} \mapsto |\gamma \cap A|$ (where $|\cdot|$ donotes length) against the measure μ_M.

There is a natural identification between $\mathscr{G}$ and

$$UT^+\partial M = \{(p,v)\,|\,p \in \partial M, v \in UT_pM \text{ inward pointing}\}$$

(a geodesic segment is identified to its starting point and initial velocity). Since $UT^+\partial M$ is transversal to the geodesic flow, the image measure of μ_M under this map is nothing else than the volume of the symplectic structure inherited from the canonical symplectic structure of T^*M. In coordinates, denoting by $\alpha(p,v)$ the angle between u and the inward normal to ∂M at x, this measure is

$\cos\alpha\,\mathrm{d}p\,\mathrm{d}v$ (where $\mathrm{d}p$ is the volume measure on the boundary). Under the above identification, the function α can be defined for a geodesic segment; we define similarly both on $\mathscr{G}$ and $UT^+\partial M$, the functions β and ℓ that give respectively the angle between a geodesic and the outward normal at its final point, and its length.

Santaló's equality states that for all integrable function f,

$$\int_{UTM} f(x,u)\,\mathrm{d}u\,\mathrm{d}x = \int_{UT^+\partial M}\int_0^{\ell(v)} f(\xi^t(p,v))\cos(\alpha(v))\,\mathrm{d}t\,\mathrm{d}v\,\mathrm{d}p$$

where ξ^t is the geodesic flow; this of course follows from the compatibility of the induced symplectic structures on transversals and the relation between μ_M and the Liouville measure. Applying this to the length $\ell(x,u)$ of the geodesic segment $\gamma(x,u)\in\mathscr{G}$ defined by an element of UTM, we get with some abuse of notation

$$\mu_M = \cos\alpha\,\mathrm{d}v\,\mathrm{d}p = \frac{1}{\ell}\,\mathrm{d}u\,\mathrm{d}x$$

Note that compared to the proof of the Little Prince Theorem, we changed the normalization by including the $\cos\alpha$ factor in the measure.

It is a positive measure on $\mathscr{G}$, and we have

$$\int \ell\,\mathrm{d}\mu_M = \omega_{n-1}V_M \tag{9}$$

where as above, ω_{n-1} denotes the total volume of the unit sphere in the n-dimensional Euclidean space.

Moreover, the reversibility of the geodesic flow implies that μ_M is symmetric in (α,β) in the sense that for all function $f:\mathbb{R}\times[0,\pi/2]^2\to\mathbb{R}$, we have

$$\int f(\ell(\gamma),\alpha(\gamma),\beta(\gamma))\,\mathrm{d}\mu_M(\gamma) = \int f(\ell(\gamma),\beta(\gamma),\alpha(\gamma))\,\mathrm{d}\mu_M(\gamma) \tag{10}$$

and the α marginal of μ_M is $A_M\omega_{n-2}\sin^{n-2}(\alpha)\cos(\alpha)\,\mathrm{d}\alpha$ in the sense that for all $f:[0,\pi/2]\to\mathbb{R}$ we have

$$\int f(\alpha(\gamma))\,\mathrm{d}\mu_M(\gamma) = A_M\int_0^{\pi/2} f(\alpha)\delta^n(\alpha)\,\mathrm{d}\alpha \tag{11}$$

where have set $\delta^n(\alpha)=\omega_{n-2}\sin^{n-2}(\alpha)\cos(\alpha)$ for simplicity. Beware that we abusively used α as a mute variable in the second member; we shall proceed similarly in the sequel, using μ_M also to denote its projection on $[0,+\infty)\times[0,\frac{\pi}{2}]^2$ given by the map $\gamma\mapsto(\ell(\gamma),\alpha(\gamma),\beta(\gamma))$, and using the letters ℓ,α,β as coordinates on $[0,+\infty)\times[0,\frac{\pi}{2}]^2$.

3.2. Croke's inequality and its siblings

We shall now introduce three inequalities satisfied by μ_M under the condition Candle(κ), that we will then combine to give informations on A_M and V_M.

Let $s_\kappa:[0,+\infty)\to[0,+\infty)$ (which depends implicitely on n) be the jacobian fonction of S^n_κ, namely

$$s_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin^{n-1}(\sqrt{\kappa}t) & \text{if } \kappa>0 \text{ and } \sqrt{\kappa}t\leqslant\pi \\ t^{n-1} & \text{if } \kappa=0 \\ \frac{1}{\sqrt{\kappa}}\sinh^{n-1}(\sqrt{\kappa}t) & \text{if } \kappa<0 \end{cases}$$

When $\kappa>0$ and $\sqrt{\kappa}t\geqslant\pi$, we set $s_\kappa(t)=0$.

If $f:[0,+\infty)\to\mathbb{R}$ is any function, we denote by $f^\wedge$ its primitive that vanishes at 0.

Lemma 3.1. *If every pair of points in M is linked by at most m geodesic, the following inequalities hold:*

$$\int \frac{s_\kappa(\ell)}{\cos\alpha\cos\beta} d\mu_M \leqslant mA_M^2 \tag{12}$$

$$\int \frac{\hat{s_\kappa}(\ell)}{\cos\alpha} d\mu_M \leqslant mA_M V_M \tag{13}$$

$$\int \hat{\hat{s_\kappa}}(\ell) d\mu_M \leqslant mV_M^2 \tag{14}$$

Moreover, if M is of $\sqrt{\mathrm{Ric}}$ class (ρ,κ) for some ρ such that $2\sqrt{\rho}L_M \leqslant \pi$, or if $K \leqslant \kappa$, then each one of these inequality is an equality if and only if M has constant sectional curvature κ and, respectively:

- *almost every pair of boundary points is connected by exactly m geodesics,*
- *almost every pair of a boundary point and an interior point is connected by exactly m geodesics,*
- *almost every pair of interior points is connected by exactly m geodesics.*

Proof. These three inequalities are proved in a similar way, and when $m = 1$ the first one is Croke's inequality, proved in [Cro84]. Let us give here the proof of the second one, the last one being left to the reader.

Fix a point $p \in \partial M$. Denoting by $\mathrm{d}v$ the Riemannian volume of the hemisphere $UT_p^+\partial M$ we have for $x = \exp_p(tv)$ that $\mathrm{d}x = s_p(tv)\,\mathrm{d}v\,\mathrm{d}t \geqslant s_\kappa(t)\,\mathrm{d}v\,\mathrm{d}t$. Integrating in v and t, and using that each point x is the exponential image of at most m vectors tv, we get

$$mV_M \geqslant \int s_\kappa(t)\,\mathrm{d}v\,\mathrm{d}t = \int \hat{s_\kappa}(\ell(\gamma(p,u)))\,\mathrm{d}v$$

Integrating this inequality in p, we get

$$mA_M V_M \geqslant \int \hat{s_\kappa}(\ell(\gamma(p,v)))\,\mathrm{d}v\,\mathrm{d}p = \int \frac{\hat{s_\kappa}(\ell)}{\cos\alpha}\,\mathrm{d}\mu_M$$

as claimed.

The case of equality, as in Croke's article, follows from the equality case in Günther's Theorem and its refinement. □

We shall mainly use these inequalities when $m = 1$. If M has the property that every pair of points is linked by exactly one geodesic, we say that M is *convex*.

We have now four inequalities and equalities involving μ_M, A_M and V_M, and we would like to combine them in an optimal way. However, to use them efficiently we need a last, more flexible inequality that translates the symmetry and marginal information on μ_M. This is obtained in the most ad hoc way: for all function $f : [0,\pi/2]^2 \to \mathbb{R}^+$ such that

$$f(\alpha,\beta) \leqslant \frac{1}{2}(f(\alpha,\alpha) + f(\beta,\beta)) \tag{15}$$

we obviously have

$$\int f(\alpha,\beta)\,\mathrm{d}\mu_M \leqslant A_M \int_0^{\pi/2} f(\alpha,\alpha)\delta^n(\alpha)\,\mathrm{d}\alpha \tag{16}$$

We denote by $\mathscr{F}$ the set of all non-negative functions f satisfying inequality (15). This set contains in particular all symmetric product functions $(\alpha,\beta) \mapsto g(\alpha)g(\beta)$.

3.3. A linear programming problem

We consider, from now on and until further notice, the case when every pair of point in M is linked by at most one geodesic ($m = 1$).

	A	μ	
a	A_B	$-\frac{s_\kappa(\ell)}{\cos(\alpha)\cos(\beta)}$	$\geqslant 0$
b	V_M	$-\frac{\hat{s_\kappa}(\ell)}{2}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right)$	$\geqslant 0$
c	0	$-\hat{\hat{s_\kappa}}(\ell)$	$\geqslant -V_M^2$
d	0	ℓ	$\geqslant \omega_{n-1}V_M$
$f\in\mathscr{F}$	0	$-f(\alpha,\beta)$	$\geqslant -A_B\int f(\alpha,\alpha)\delta^n(\alpha)\,\mathrm{d}\alpha$
	$\leqslant 1$	$\leqslant 0$	

TABLE I. A linear programming problem

To use linear programming to combine our inequalities, we must first give them a linear form. Let A_B be the surface area of the space form ball that has volume V_M, so that either $A_M > A_B$ (and the desired result holds), or we can replace inequality (12) by its linearized version

$$\int \frac{s_\kappa(\ell)}{\cos\alpha\cos\beta}d\mu_M \leqslant A_B A_M \tag{17}$$

Similarly, either $A_M > A_B$ or the following version of (16) holds for all $f\in\mathscr{F}$:

$$\int f(\alpha,\beta)\,\mathrm{d}\mu_M \leqslant A_B\int_0^{\pi/2} f(\alpha,\alpha)\delta^n(\alpha)\,\mathrm{d}\alpha \tag{18}$$

We can also replace (13) by its symmetric form

$$\int \frac{\hat{s_\kappa}(\ell)}{2}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right)d\mu_M \leqslant A_M V_M \tag{19}$$

Last, we only use the inequality $\int \ell d\mu_M \geqslant \omega_{n-1}V_M$ instead of the whole of (9).

Table I summarizes the following linear programming problem: find a non-negative real A and a non-negative measure μ on $[0,+\infty)\times[0,\pi/2]^2$ satisfying the constraints

$$\begin{aligned}
AA_B - \int \frac{s_\kappa(\ell)}{\cos(\alpha)\cos(\beta)}\,\mathrm{d}\mu &\geqslant 0\\
AV_M - \int \frac{\hat{s_\kappa}(\ell)}{2}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right)\mathrm{d}\mu &\geqslant 0\\
-\int \hat{\hat{s_\kappa}}(\ell)\,\mathrm{d}\mu &\geqslant -V_M^2\\
\int \ell\,\mathrm{d}\mu &\geqslant V_M\\
\forall f\in\mathscr{F} \qquad -\int f\,\mathrm{d}\mu &\geqslant -A_B\int f(\alpha,\alpha)\delta^n(\alpha)\,\mathrm{d}\alpha
\end{aligned}$$

so that A is minimal.

The table is to be read as follows: the first line gives the variables names, each following line expresses a constraints, except the last one that expresses the minimization objective. Since A_M, μ_M satisfy the constraints, we have $A_M \geqslant \inf A$.

The interest of the table is that it can be read in columns to obtain the so-called *dual problem*. Here: find non-negative reals a,b,c,d and a function $f\in\mathscr{F}$ satisfying the constraints

$$\begin{aligned}
aA_B + bV_M &\leqslant 1\\
-a\frac{s_\kappa(\ell)}{\cos(\alpha)\cos(\beta)} - b\frac{\hat{s_\kappa}(\ell)}{2}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right)&\\
-c\hat{\hat{s_\kappa}}(\ell) + d\ell - f(\alpha,\beta) &\leqslant 0 \quad \forall(\ell,\alpha,\beta)
\end{aligned}$$

in order to maximize

$$-cV_M^2 + d\omega_{n-1}V_M - A_B \int f(\alpha,\alpha)\delta^n(\alpha)\,\mathrm{d}\alpha.$$

The relevance of the dual problem shall be recalled below.

3.4. Solving the linear programming problem

In view of the maximization objective in the dual problem, we see that it cannot hurt to take f has small as possible. Given the values of the other dual variables (a,b,c,d), the constraint involving f shows that we should take

$$f(\alpha,\beta) = \sup_\ell \left(-a\frac{s(\ell)}{\cos(\alpha)\cos(\beta)} - b\frac{s\hat{}(\ell)}{2}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right) - cs\hat{}\hat{}(\ell) + d\ell \right) \tag{20}$$

as long as this defines a function in $\mathscr{F}$. From this point will originate some technical difficulties in the following. Note that the Taylor series in ℓ of the supremum argument shows that f is non-negative.

The set where the ℓ derivative of the above supremum argument vanishes will therefore be of primary importance, namely the (ℓ,α,β) satisfying

$$d = a\frac{s'_\kappa(\ell)}{\cos(\alpha)\cos(\beta)} + b\frac{s_\kappa(\ell)}{2}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right) + cs_\kappa\hat{}(\ell) \tag{21}$$

Let us state as a lemma adapted to our case the fundamental result of linear programming, that we shall use several times.

Lemma 3.2 (Duality principle)**.** *Let $B = B^n_\kappa(V_M)$ be the ball of the reference space form that has same volume as M. There is a function T such that μ_B is concentrated on the set $C = \{(\ell,\alpha,\beta)\,|\,\cos\alpha = \cos\beta = T(\ell)\}$. Assume that there are non-negative constants a,b,c,d with at least one of a,b non-zero such that*

1. *equation* (21) *holds on C;*
2. *the* sup *in* (20) *is realized exactly when $(\ell,\alpha,\beta)\in C$;*
3. *the function f defined by* (20) *is in $\mathscr{F}$, and $f(\alpha,\beta) = (f(\alpha,\alpha)+f(\beta,\beta))/2$ only when $\alpha=\beta$.*

Then $A_M \geqslant A_B$, and if M is of $\sqrt{\mathrm{Ric}}$ class (ρ,κ) for some ρ such that $2\sqrt{\rho}L_M \geqslant L_M$ or $K\leqslant\kappa$, then there is equality if and only if M is isometric to B.

For the sake of completeness, let us give the (classical) proof.

Proof. For all A,μ,a,b,c,d,f satisfying the constraints, denoting by $D = -cV_M^2 + d\omega_{n-1}V_M - \int f(\alpha,\alpha)\delta^n(\alpha)\,\mathrm{d}\alpha$ the quantity to be maximized in the dual problem, by using first the primal constraints, then the dual ones we have

$$\begin{aligned} D &\leqslant aAA_B + bAV_M + \int \left(-a\frac{s_\kappa(\ell)}{\cos\alpha\cos\beta} \right. \\ &\qquad \left. -b\frac{s_\kappa\hat{}(\ell)}{2}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right) - cs_\kappa\hat{}\hat{}(\ell) + d\ell - f(\alpha,\beta)\right)\mathrm{d}\mu \\ &\leqslant A. \end{aligned}$$

If a,b,c,d,f satisfy the hypothesis, we can by multiplying a,b,c,d by a suitable constant assume that $aA_B + bV_M = 1$. Then a,b,c,d,f satisfy the dual constraints, with equality in the first one, and equality in the second one precisely for $(\ell,\alpha,\beta)\in C$. For $A = A_B$ and $\mu = \mu_B$, we then have equality in the above inequality.

Since A_M,μ_M satisfy the primal constraints, we have $A_B = D \leqslant A_M$

Assume now that there is equality. Then, since one of a,b is non-zero, one of the three first primal constraints must be an equality. It follows that M must have constant curvature and be convex. Moreover, μ_M must be supported on the set $(\alpha=\beta)$ for the last primal constraint to yield an equality. Similarly, equality in the second dual constraint must occur on the support of μ_M, which must therefore be concentrated on C, so that M must be isometric to B. □

We see that knowing the optimum in advance has a great advantage: we only have to find suitable dual variables, rather that trying to optimize A, μ.

4. CROKE'S AND WEIL'S THEOREMS

In this section we treat the case $\kappa = 0$ in both dimensions 2 and 4 by solving the preceding linear programming problem. From Lemma 3.2, we only have to determine suitable dual variables a, b, c, d, f.

The optimal domain will be shown to be a Euclidean ball B, for which the function T defining the support of μ_B is

$$T(\ell) = \frac{\ell}{2r}$$

where r is the diameter of B. Since here $s_\kappa(t) = t^{n-1}$, equation (21) becomes

$$d = a\frac{(n-1)(2r\cos\alpha)^{n-2}}{\cos^2\alpha} + b\frac{(2r\cos\alpha)^{n-1}}{\cos\alpha} + \frac{c}{n}(2r\cos\alpha)^n \quad \forall\alpha \tag{22}$$

If $n = 4$, the following values for the first four dual variables:

$$d = 12r^2 \quad a = 1 \quad b = c = 0$$

solve this equation. The vanishing of b and c was expected since Croke's theorem does not need the extra two inequalities we introduced.

Then, as noticed before, we shall take

$$\begin{aligned} f(\alpha,\beta) &= \sup_\ell \left(-\frac{\ell^3}{\cos(\alpha)\cos(\beta)} + 12r^2\ell \right) \\ &= 16r^3(\cos\alpha\cos\beta)^{1/2} \end{aligned} \tag{23}$$

Since f is a symmetric product, it does belong to $\mathscr{F}$ and equality $f(\alpha,\beta) = (f(\alpha,\alpha) + f(\beta,\beta))/2$ holds only when $\alpha = \beta$. The duality principle applies and Croke's Theorem is proved.

Note that this proof is extremly close to the proof given by Croke. The only difference is that we did not have to guess how to apply Hölder's inequality and to which function, thanks to the linear programming formulation.

Consider now the case $n = 2$ and $\kappa = 0$. Equation (21) leads to the following choice of dual variables:

$$d = 2r, \quad b = 1, \quad a = c = 0$$

and

$$\begin{aligned} f(\alpha,\beta) &= \sup_\ell \left(-\frac{\ell^2}{4}\left(\frac{1}{\cos\alpha} + \frac{1}{\cos\beta}\right) + 2r\ell \right) \\ &= \frac{4r^2}{\frac{1}{\cos\alpha} + \frac{1}{\cos\beta}} \end{aligned}$$

which is easily seen to belong to $\mathscr{F}$. Moreover $f(\alpha,\beta) = (f(\alpha,\alpha) + f(\beta,\beta))/2$ if and only if $\alpha = \beta$. Weil's Theorem is proved using the Duality principle.

This proof is in fact very close to the one we already gave in Section 2, we simply seeked at once the integrated inequality.

5. POSITIVE BOUND

In this section, we assume $\kappa > 0$ and apply the same method as above to prove Theorem 1.2. Up to a dilation, we can assume $\kappa = 1$; we therefore assume further $V_M < \omega_n/2$ in all this section. The

only additional difficulty is that the expressions of the length of a chord in a ball and of the Jacobian function lead to more intricate values for the dual variables.

More precisely, if $B = B_1^n(V_M)$ has radius r, the function T defining the support C of μ_B is

$$T(\ell) = \frac{\tan\frac{\ell}{2}}{\tan r}$$

5.1. Dimension 4

If $n = 4$, we have

$$s_1(\ell) = \sin^3(\ell), \quad s_1'(\ell) = 3\sin^2(\ell)\cos(\ell), \quad s_1\hat{}(\ell) = \frac{2}{3} - \frac{1}{3}\sin^2(\ell)\cos(\ell) - \frac{2}{3}\cos(\ell)$$

and therefore look for (a,b,c,d) such that

$$\begin{aligned} d = a\frac{3\sin^2(\ell)\cos(\ell)\tan^2(r)}{\tan^2(\ell/2)} + b\frac{\sin^3(\ell)\tan(r)}{\tan(\ell/2)} \\ + c\left(\frac{2}{3} - \frac{1}{3}\sin^2(\ell)\cos(\ell) - \frac{2}{3}\cos(\ell)\right) \qquad \forall \ell \end{aligned}$$

which happens to have a solution (which is the happy coincidence enabling this method to handle this case), namely

$$a = 1, \quad b = 6\tan r, \quad c = 9\tan^2 r, \quad d = 12\tan^2 r$$

Here, no dual variable vanishes, so that we do need the three inequalities (12), (13) and (14).

With these values, by construction equation (21) has a solution at $\cos(\alpha) = \frac{\tan(\ell/2)}{\tan(r)}$. We want to define f by (20) (up to a constant, below we divided by $9\tan r$), but it is no longer obvious that $f \in \mathscr{F}$, nor that $f(\alpha,\beta) = (f(\alpha,\alpha) + f(\beta,\beta))/2$ happens precisely at $\alpha = \beta$. This is proved in the appendix, see Lemma 9.1. Theorem 1.2 in dimension 4 follows from it by duality principle. Note that the lemma is elementary but technical: we need to solve a partial degree 4 polynomial system of equations, which is done using a formal computation software.

5.2. Dimension 2

If $n = 2$, we have for all $\ell < \pi$:

$$s_1(\ell) = \sin\ell, \quad s_1'(\ell) = \cos\ell, \quad s_1\hat{}(\ell) = 1 - \cos\ell, \quad s_1\hat{}\hat{}(\ell) = \ell - \sin\ell$$

and therefore look for (a,b,c,d) such that

$$d = a\frac{\cos\ell\tan^2 r}{\tan^2(\ell/2)} + b\frac{\sin\ell\tan r}{\tan(\ell/2)} + c(1-\cos\ell) \qquad \forall \ell$$

which is solved by

$$a = 0, \quad b = 1, \quad c = \tan r, \quad d = 2\tan r$$

With these values, by construction equation (21) has a solution at $\cos(\alpha) = \frac{\tan(\ell/2)}{\tan(r)}$. Following the usual scheme, we define

$$g(\ell,\alpha,\beta) = -\frac{1-\cos\ell}{2}\left(\frac{1}{\cos\alpha} + \frac{1}{\cos\beta}\right) - \tan r(\ell - \sin\ell) + 2(\tan r)\ell$$

for $\ell < \pi$ and $g(\ell > \pi,\alpha,\beta) = g(\pi,\alpha,\beta)$, then $f(\alpha,\beta) := \sup_\ell g(\ell,\alpha,\beta)$. This supremum is realized precisely for

$$\ell = 2\arctan\left(\frac{2\tan r}{\frac{1}{\cos\alpha} + \frac{1}{\cos\beta}}\right)$$

and we have

$$g(\ell,\alpha,\beta)=\frac{1}{2}\left(g(\ell,\alpha,\alpha)+g(\ell,\beta,\beta)\right)\quad\forall \ell,\alpha,\beta$$

We deduce easily from this two facts that $f\in\mathscr{F}$ and that $f(\alpha,\beta)$ equals $(f(\alpha,\alpha)+f(\beta,\beta))/2$ if and only if $\alpha=\beta$. The last case of Theorem 1.2 follows, once again from the duality principle.

6. NEGATIVE BOUND

Let us turn to Theorem 1.3, and see what happens that perturbs our method.

Here we assume $\kappa=-1$ up to dilating the metric on M, and note that if $B=B^n_{-1}(V_M)$ has radius r, the function T defining the support C of μ_B is

$$T(\ell)=\frac{\tanh\frac{\ell}{2}}{\tanh r}$$

6.1. Dimension 4

If $n=4$, we have

$$s_{-1}(\ell)=\sinh^3(\ell),\quad s'_{-1}(\ell)=3\sinh^2(\ell)\cosh(\ell),$$
$$\hat{s_{-1}}(\ell)=\frac{2}{3}+\frac{1}{3}\sinh^2(\ell)\cosh(\ell)-\frac{2}{3}\cosh(\ell)$$

and therefore look for (a,b,c,d) such that

$$d=a\frac{3\sinh^2(\ell)\cosh(\ell)\tanh^2(r)}{\tanh^2(\ell/2)}+b\frac{\sinh^3(\ell)\tanh(r)}{\tanh(\ell/2)}$$
$$+c\left(\frac{2}{3}+\frac{1}{3}\sinh^2(\ell)\cosh(\ell)-\frac{2}{3}\cosh(\ell)\right)\qquad\forall\ell$$

which happens to have solution

$$a=1,\quad b=-6\tanh r,\quad c=9\tanh^2 r,\quad d=12\tanh^2 r \tag{24}$$

but has no non-negative solution. It is therefore not possible to use the same inequalities and linear programming problem as above. To solve this issue, we need to prove that the linear combination of inequalities (12), (13) and (14) with coefficients a,b,c defined by (24) holds despite the sign of b. Note that it would be of no use to involve d here since the corresponding inequality is in fact an equality.

Proposition 6.1. *Assume that*

- *M lies inside a convex manifold N of the same dimension, that satisfies* $\mathrm{LCD}(-1)$,
- *M is small enough in the sense that* $\tanh(L)\tanh(r)\leqslant 1/2$ *where $L=L_M$ is the maximal length of a geodesic.*

Then it holds

$$\int\left(\frac{s_{-1}(\ell)}{\cos\alpha\cos\beta}-6\tanh(r)\frac{\hat{s_{-1}}(\ell)}{2}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right)+9\tanh^2(r)\hat{\hat{s_{-1}}}(\ell)\right)d\mu_M$$
$$\leqslant A_M^2-6\tanh(r)A_MV_M+9\tanh^2(r)V_M^2 \tag{25}$$

Proof. In fact we prove the inequality involving s_{-1} and $\hat{s_{-1}}$ and add (14) to get the result. It is probably possible to use this term in the proof to increase the bound on L and r, but we do not know how to get rid of it completely.

Let $\gamma : [0,\ell] \to M$ be a maximal geodesic segment of unit speed and denote by α and β the angles of γ with the normals to ∂M at $\gamma(0)$ and $\gamma(\ell)$. Let $j : [0,\ell]^2 \to [0,+\infty)$ be the candle function of M along γ, namely:

$$j(x,y) = s_{\exp_{\gamma(x)}}((y-x)\gamma'(x))$$

(in particular, $j(0,y) = s(\gamma,y)$).

By the exponential candle condition, we have

$$\frac{j_2(x,y)}{j(x,y)} \geqslant \frac{s'(y-x)}{s(y-x)}$$

for all $y > x$, where j_2 denotes the second partial derivative of j. Let $u(x,y) = j(x,y) - s(y-x)$; then u is a non-negative function and

$$u_2(x,y) \geqslant \left(\frac{j(x,y)}{s(y-x)} - 1\right) s'(y-x) = u(x,y)\frac{s'(y-x)}{s(y-x)}$$

Using the expression of s and the monotony of tanh, it comes $u_2(x,y) \geqslant 3u(x,y)/\tanh(L)$ then, by integration

$$u(0,\ell) \geqslant \frac{3}{\tanh(L)} \int_0^\ell u(0,y)\,\mathrm{d}y \quad \text{and} \quad u(0,\ell) \geqslant \frac{3}{\tanh(L)} \int_0^\ell u(x,\ell)\,\mathrm{d}x$$

where the second inequality is derived as the first one, using $-u_1(x,y) \geqslant u(x,y)(s'/s)(y-x)$. Then we combine them into

$$\frac{u(0,\ell)}{\cos\alpha\cos\beta} \geqslant \frac{3}{2\tanh(L)} \left(\frac{\int_0^\ell u(0,y)\,\mathrm{d}y}{\cos\alpha} + \frac{\int_0^\ell u(x,\ell)\,\mathrm{d}x}{\cos\beta}\right)$$

Now we can recall the definition of u and integrate in μ_M (with $j =: j_\gamma$) to get

$$\begin{aligned}\int \left(\frac{j_\gamma(0,\ell)}{\cos\alpha\cos\beta} - \frac{3}{2\tanh(L)} \left(\frac{\int_0^\ell j_\gamma(0,y)\,\mathrm{d}y}{\cos\alpha} + \frac{\int_0^\ell j_\gamma(x,\ell)\,\mathrm{d}y}{\cos\beta}\right)\right) \mathrm{d}\mu_M(\gamma) \\ \geqslant \int \left(\frac{s_{-1}(0,\ell)}{\cos\alpha\cos\beta} - \frac{3}{2\tanh(L)} \hat{s_{-1}}(\ell) \left(\frac{1}{\cos\alpha} + \frac{1}{\cos\beta}\right)\right) \mathrm{d}\mu_M\end{aligned}$$

When M is convex, the left-hand side is equal to $A_M^2 - 3/(\tanh L)A_M V_M$. Then if the smallness condition holds, it is possible to add a positive multiple of (13) to replace the factor $-3/(\tanh L)$ by the wanted $-6\tanh r$ in the second terms.

When M is not convex, we prove that the left-hand side is lesser than $A_M^2 - 3/(\tanh L)A_M V_M$. Given γ starting at $p \in \partial M$, instead of stopping it as soon as it reaches ∂M, extend it in both direction in N. Write $\{t \in \mathbb{R} \,|\, \gamma(t) \in M\}$ as a disjoint union of linearly ordered segments $I_{-k_1} \cup \cdots \cup I_0 \cup \cdots \cup I_{k_2}$ where $I_0 = [0,\ell]$, and write $I_i = [p_i, q_i]$ when $i > 0$, $I_i = [q_i, p_i]$ when $i < 0$. Let $\beta_i \in [0,\pi/2]$ be the angle between γ and the direction normal to the boundary at q_i. Using the LCD(-1) condition and the definition of L as he diameter of M in N, it comes easily for each i

$$j(0,q_i) \geqslant \frac{3}{\tanh L} \int_{I_i} j(0,y)\,\mathrm{d}y$$

Summing on i, and integrating on γ it comes

$$\begin{aligned}&\int \left(\sum_i \frac{j_\gamma(0,q_i)}{\cos\alpha\cos\beta_i} - \frac{3}{\tanh(L)} \frac{\int_{I_i} j_\gamma(0,y)\,\mathrm{d}y}{\cos\alpha}\right) \mathrm{d}\mu_M(\gamma) \\ &\geqslant \int \left(\frac{j_\gamma(0,\ell)}{\cos\alpha\cos\beta} - \frac{3}{\tanh(L)} \frac{\int_0^\ell j_\gamma(0,y)\,\mathrm{d}y}{\cos\alpha}\right) \mathrm{d}\mu_M(\gamma) \\ = \int \left(\frac{j_\gamma(0,\ell)}{\cos\alpha\cos\beta} - \frac{3}{2\tanh(L)} \left(\frac{\int_0^\ell j_\gamma(0,y)\,\mathrm{d}y}{\cos\alpha} + \frac{\int_0^\ell j_\gamma(x,\ell)\,\mathrm{d}y}{\cos\beta}\right)\right) &\mathrm{d}\mu_M(\gamma)\end{aligned}$$

Using Santlò's formula, the left-hand-side is equal to $A_M^2 - 3/(\tanh L)A_M V_M$, and we are done. □

Now we can formulate an alternative linear programming problem involving the *ad hoc* inequality (25), and the last two primal constraints. The duality principle holds as before, and the last missing piece in the proof of Theorem 1.3 in dimension 4 is the technical Lemma 9.2 proved in the appendix, according to which the function f defined by (20) satisfies the properties required by Lemma 3.2. See the Section 8 for a discussion about the smallness condition.

6.2. Dimension 2

Just as when $\kappa > 0$, the 2-dimensional case tastes like the 4-dimensional one but is simpler. We have

$$s_{-1}(\ell) = \sinh \ell, \quad s'_{-1}(\ell) = \cosh \ell, \quad \hat{s_{-1}}(\ell) = \cosh \ell - 1$$

and we look for a, b, c, d such that

$$d = a \frac{\cosh \ell \tanh^2 r}{\tanh^2(\ell/2)} + b \frac{\sinh \ell \tanh r}{\tanh(\ell/2)} + c(\cosh \ell - 1) \quad \forall \ell$$

whose solutions are the multiples of

$$a = 0, \quad b = 1, \quad c = -\tanh r, \quad d = 2 \tanh r$$

The sign of c makes it necessary to adapt the method as in the case $n = 4$.

Lemma 6.2. *If M lies in a convex manifold N with* $\mathrm{LCD}(-1)$*, then it holds*

$$\int \left(\frac{\hat{s_{-1}}(\ell)}{2} \left(\frac{1}{\cos \alpha} + \frac{1}{\cos \beta} \right) - \tanh(r) \hat{\hat{s_{-1}}}(\ell) \right) \mathrm{d}\mu \leqslant A_M V_m - 2\pi \tanh(r) V_M^2$$

Proof. Fix a geodesic γ and define j and u as when $n = 4$. By symmetry of the jacobian we have

$$-j_1(s,t) \geqslant j(s,t) \frac{s'(t-s)}{s(t-s)}$$

from which it follows

$$-u_1(s,t) \geqslant u(s,t) \frac{s'(t-s)}{s(t-s)}$$

Since here $s'/s = 1/\tanh \geqslant 1$, we get $-u_1 \geqslant u$. Integrating for s from 0 to t, then for t from 0 to ℓ, using

$$\frac{1}{2} \left(\frac{1}{\cos \alpha} + \frac{1}{\cos \beta} \right) \geqslant 1 \geqslant \tanh r$$

and finally integrating on γ against the measure μ produces the desired inequality. The case when M is not convex is treated as in dimension 4. □

7. RELATIVE VERSION

Let us turn to the proof of Theorem 1.5. It is completely similar to the previous proofs, using Lemma 3.1 for arbitrary m, as soon as we have primal and dual candidates for optimality that achieves equality in the modified linear programming problem given in table II.

The candidate is given by the orbifold R constructed in example 1.1, starting from a ball B_0 whose volume is mV_M. The singularity is not a problem: the measure μ_R (which equals $\frac{1}{m}\mu_{B_0}$ on $[0, +\infty) \times [0, \pi/2]^2$) and the boundary volume $A_R = |\partial R| = |\partial B_0|/m$ are well-defined and satisfy the needed equalities; by construction using the same values for dual variables as above, we see that (A_R, μ_R) realizes the infimum of our new linear programming problem.

	A	μ	
a	mA_B	$-\dfrac{s_\kappa(\ell)}{\cos(\alpha)\cos(\beta)}$	$\geqslant 0$
b	mV_M	$-\dfrac{\hat{s_\kappa}(\ell)}{2}\left(\dfrac{1}{\cos\alpha}+\dfrac{1}{\cos\beta}\right)$	$\geqslant 0$
c	0	$-\hat{\hat{s_\kappa}}(\ell)$	$\geqslant -\omega_{n-1}mV_M^2$
d	0	ℓ	$\geqslant V_M$
$f\in\mathscr{F}$	0	$-f(\alpha,\beta)$	$\geqslant -A_R\int f(\alpha,\alpha)\delta^n(\alpha)\,\mathrm{d}\alpha$
	$\leqslant 1$	$\leqslant 0$	

TABLE II. The linear programming problem for the strong relative case

8. COMMENTS AND QUESTIONS

It is desirable to get rid of the smallness hypothesis in Theorem 1.3. To achieve that, one coud try to affirmatively answer the following.

Question 8.1. Is it true that the inequality

$$\frac{j(0,\ell)}{\cos\alpha\cos\beta}-3\tanh(r)\left(\frac{\int_0^\ell j(0,y)\,\mathrm{d}y}{\cos\alpha}+\frac{\int_0^\ell j(x,\ell)\,\mathrm{d}x}{\cos\beta}\right)+9\tanh^2(r)\int_0^\ell\int_0^y j(x,y)\,\mathrm{d}x\,\mathrm{d}y\geqslant \frac{s_\kappa(\ell)}{\cos\alpha\cos\beta}-3\tanh(r)\hat{s_\kappa}(\ell)\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right)+9\tanh^2(r)\hat{\hat{s_\kappa}}(\ell)$$

holds whenever $j(x,y)$ is the jacobian of the exponential map $\exp_{\gamma(x)}$ at the vector $(y-x)\gamma'(x)$, where γ is a unit speed geodesic in a n-manifold of sectional curvature bounded above by $\kappa<0$?

It is very tempting to believe in this inequality, but it cannot be consequence of $\mathrm{LCD}(-1)$ since the jacobian of the complex hyperbolic plane, normalized to have sectional curvature between $-9/4$ and $-9/16$, does not satisfy it with $\kappa=-1$ when $\cos\alpha=\cos\beta=1$ and r,ℓ are large enough.

Question 8.2. Does the isoperimetric conjecture hold with the assumption $K\leqslant\kappa$ replaced by $\mathrm{LCD}(\kappa)$ or $\mathrm{Candle}(\kappa)$ in dimension 3, or >4?

In the $\kappa=0$, $n=4$ case Croke's proof needs the candle condition only *between boundary points*, allowing for example for some positively curved parts on the interior of M as long as they are compensated near the boundary.

Question 8.3. Given $n\neq 4$, does the isoperimetric conjecture hold in dimension n for $\kappa=0$, under the weak hypothesis

$$s_p(q)\geqslant s_0(d(p,q)) \tag{26}$$

for all $p,q\in\partial M$?

9. APPENDIX

In this appendix we prove two technical lemmas used in the paper, that are quite intricate but completely elementary. Note that both proof are assisted by a formal computation software, used to solve polynomial systems of degree 4.

9.1. A technical lemma for the positive bound Theorem

Lemma 9.1. *Define a function on* $[0,\pi[\times[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$ *by*

$$g(\ell,\alpha,\beta)=\frac{4}{3}\ell-\frac{s(\ell)}{9\tan^2 r\cos\alpha\cos\beta}-\frac{s\hat{}(\ell)}{3\tan r}\left(\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}\right)-s\hat{}\hat{}(\ell)$$

and let

$$f(\alpha,\beta)=\sup_{\ell} g(\ell,\alpha,\beta).$$

1. *For all fixed* α*, the maximum of* $g(\ell,\alpha,\alpha)$ *is realized at*

$$\ell=2\arctan(\tan r\cos\alpha)$$

   and only there.

2. *For all* α *and* β*, we have*

$$f(\alpha,\beta)\leqslant\frac{1}{2}\left(f(\alpha,\alpha)+f(\beta,\beta)\right)$$

   and equality occurs only for $\alpha=\beta$.

Note that in the proof of Theorem 1.2 we have to enable ℓ to be greater than π; by convention $s(\ell)=0$ for $\ell>\pi$, so that on $[\pi,+\infty)$ the function $s\hat{}$ is constant (with value $s\hat{}(\pi)=4/3$) and $s\hat{}\hat{}(\ell)$ increase linearly. It follows that g extends with $g(\ell,\alpha,\beta)=g(\pi,\alpha,\beta)$, and the above lemma still holds with this extension.

Proof. We first perform the following change of variables: $t:=\tan\frac{\ell}{2}$, $p:=1/(3\tan r\cos\alpha)$ and $q:=1/(3\tan r\cos\beta)$. We now have to study the functions

$$G(t,p,q)=\frac{8}{3}\arctan t-pqS_0(t)-(p+q)S_{-1}(t)-S_{-2}(t)$$

where $S_i(t)=s^{(i)}(2\arctan t)$, and

$$F(p,q)=\sup_t G(t,p,q)$$

Usual trigonometric formulas give

$$\begin{aligned}
S_1(t)&=12\frac{t^2(1-t^2)}{(1+t^2)^3}\\
S_0(t)&=8\frac{t^3}{(1+t^2)^3}\\
S_{-1}(t)&=\frac{2}{3}-\frac{2}{3}\frac{1-t^2}{1+t^2}-\frac{4}{3}\frac{t^2(1-t^2)}{(1+t^2)^3}\\
&=\frac{4t^4(3+t^2)}{3(1+t^2)^3}\\
S_{-2}(t)&=\frac{4}{3}\arctan(t)-\frac{8}{3}\frac{t^3}{(1+t^2)^3}-\frac{4}{3}\frac{t}{1+t^2}
\end{aligned}$$

so that

$$\partial_t G(t,p,q)=\frac{2}{(1+t^2)^4}\left(12pqt^4-8(p+q)t^3+(4-12pq)t^2+\frac{4}{3}\right)$$

The equation $\partial_t G(t,p,p)=0$ is in particular equivalent to

$$12p^2t^4-16pt^3+(4-12p^2)t^2+\frac{4}{3}=0$$

and by the very construction of G, we know that $t = 1/3p$ must be a solution. This enables us to factorize it into

$$4(3pt-1)(pt^3-t^2-pt-\frac{1}{3}) = 0$$

Let $P(t) = pt^3 - t^2 - pt - \frac{1}{3}$. The roots of P' are

$$\frac{1}{3p} \pm \sqrt{\frac{1}{9p^2} + \frac{1}{3p}}$$

Since $P(0) < 0$, P has at exactly one positive zero, which is greater than $1/3p$, which by a sign check is a local maximum of $G(\cdot, p, p)$. It follows that the only other extremum of $G(\cdot, p, p)$ in $[0, +\infty)$ is a local minimum located on the right of the only local maximum, and we are left with proving that $G(1/3p, p, p)$ is greater than or equal to $\lim_{t\to+\infty} G(t, p, p) = \frac{2\pi}{3} - \frac{8}{3}p$. The two expressions have the same limit when p goes to 0, and we have

$$\frac{d}{dp}\left(G\left(\frac{1}{3p}, p, p\right) + \frac{8}{3}p - \frac{2\pi}{3}\right) = \frac{216p^4}{(9p^2+1)^2} \geqslant 0$$

The first point is proved, let us turn to the second one.

We want to prove that the function defined by

$$H(t,p,q) = G\left(\frac{1}{3p}, p, p\right) + G\left(\frac{1}{3q}, q, q\right) - 2G(t,p,q)$$

is non-negative for all $t, p, q \in (0, +\infty)$ and vanishes only when $p = q$ (and $t = 1/3p$ of course). First, let us show that for all sequence $(t_n, p_n, q_n) \in (0, +\infty)^3$ that ultimately gets out of all compact sets, we have

$$\liminf \left(H_n := H(t_n, p_n, q_n)\right) \geqslant 0$$

We will then only have left to study the critical points of H.

Up to taking subsequences, we can assume that $(t_n), (p_n), (q_n)$ have limits in $[0, +\infty]$. If $t_n \to 0$, then

$$\liminf H_n \geqslant \inf_p G\left(\frac{1}{3p}, p, p\right) + \inf_q G\left(\frac{1}{3q}, q, q\right) = 0$$

Otherwise, if one of p_n or q_n goes to $+\infty$, H_n also goes to $+\infty$. If $t_n \to +\infty$, we have

$$\liminf H_n \geqslant \inf_{p,q}\left(G\left(\frac{1}{3p}, p, p\right) + G\left(\frac{1}{3q}, q, q\right) + \frac{8}{3}(p+q) - \frac{4\pi}{3}\right)$$

and the right-hand-side is 0, as proved in the first point. Let us finally assume $p_n \to 0$; then

$$\liminf H_n \geqslant \inf_{t,q}\left(\frac{2\pi}{3} + F\left(\frac{1}{3q}, q, q\right) - 2\left(\frac{8}{3}\arctan t - qS_{-1}(t) - S_{-2}(t)\right)\right)$$

Let $J(t,q)$ be the minimized term in the right-hand-side (this is not a Jacobian!); we have

$$\frac{dJ}{dt} = \frac{16}{3}\frac{6qt^3 - 3t^2 - 1}{(1+t^2)^4}$$

so that, q being fixed, $\inf_t J(t,q) = J(t_0, q)$ where t_0 is the positive real characterized by $q = (3t^2 + 1)/(6t^3)$. it follows that

$$\inf_{t,q} J(t,q) = \inf_t J\left(t, \frac{3t^2+1}{6t^3}\right)$$

the minimized function in the right-hand-side is decreasing and has limit 0 in $+\infty$, so that it is positive, and we are done proving $\liminf H_n \geqslant 0$.

The derivatives of H are given by

$$\frac{\partial H}{\partial t}(t,p,q) = -\frac{16}{3}\frac{9pqt^4 - 6(p+q)t^3 + (3-9pq)t^2 + 1}{(1+t^2)^4}$$
$$\frac{\partial H}{\partial p}(t,p,q) = \frac{8}{3(9p^2+1)^2(1+t^2)^3}\big(81p^4t^6 + 243p^4t^4 + (486qp^4 + 108qp^2 + 6q)t^3 + (-54p^2 - 3)t^2 - 18p^2 - 1\big)$$

and of course the q derivative is symmetric to the p derivative. The critical points of H are therefore given by a system of three polynomial equations in t,p,q, of partial degrees at most 4 in p and q. This system can be solved explicitely, and (using maple !) its only solution in positive coordinates are the points of the curve $(p = q, 3pt = 1)$, where H vanishes. This shows that outside this critical curve H is positive, and concludes the proof. □

9.2. A technical lemma for the negative bound Theorem

Lemma 9.2. *Define a function on* $[0,+\infty) \times [0,\frac{\pi}{2}] \times [0,\frac{\pi}{2}]$ *by*

$$g(\ell,\alpha,\beta) = \frac{4}{3}\ell - \frac{s(\ell)}{9\tanh^2(r)\cos\alpha\cos\beta} + \frac{s\hat{}(\ell)}{3\tanh r}\left(\frac{1}{\cos\alpha} + \frac{1}{\cos\beta}\right) - s\hat{}\hat{}(\ell)$$

and let

$$f(\alpha,\beta) = \sup_{\ell} g(\ell,\alpha,\beta).$$

1. *For all fixed* α*, the maximum of* $g(\ell,\alpha,\alpha)$ *is realized at*

$$\ell = 2\,\mathrm{arctanh}(\tanh r\cos\alpha)$$

and only there.

2. *For all* α *and* β*, we have*

$$f(\alpha,\beta) \leqslant \frac{1}{2}\left(f(\alpha,\alpha) + f(\beta,\beta)\right)$$

and equality occurs only for $\alpha = \beta$.

Proof. As in Lemma 9.1, let us first introduce the variables

$$\begin{aligned} t &= \mathrm{arctanh}(\ell/2) && \in [0,1) \\ p &= 1/(3\tanh(r)\cos\alpha) && \in [1/3,+\infty) \\ q &= 1/(3\tanh(r)\cos\beta) && \in [1/3,+\infty) \end{aligned}$$

We thus consider the functions defined by

$$G(t,p,q) = \frac{8}{3}\mathrm{arctanh}(t) - pqS_0(t) + (p+q)S_{-1}(t) - S_{-2}(t)$$

and

$$F(p,q) = \sup_t G(t,p,q)$$

where $S_i(t) = s^{(i)}(\ell)$, that is

$$S_0(t) = \frac{8t^3}{(1-t^2)^3}$$
$$S_{-1}(t) = \frac{4}{3}\frac{t^4(t^2-3)}{(t^2-1)^3}$$
$$S_{-2}(t) = \frac{4}{3}\mathrm{arctanh}(t) + \frac{8}{9}\frac{t^3}{(1-t^2)^3} - \frac{4}{3}\frac{t}{1-t^2}$$

The sign of $\partial_t G(t,p,p)$ is then the same as the sign of $(1-3pt)P(t)$ where $P(t) = pt^3 - t^2 + pt + 1/3$. If $p > 1/\sqrt{3}$, then P' has no positive root and $P(t)$ is positive for all $t \geqslant 0$. If $p \leqslant 1/\sqrt{3}$, then P attains its only local minimum at the second root of P', which is

$$\frac{1+\sqrt{1-3p^2}}{3p} > 1$$

Since $P(1) \geqslant 0$, we deduce that P is positive on $[0,1]$. In all cases, $G(\cdot,p,p)$ has its only maximum on $[0,1]$ at $t = 1/3p$ as claimed.

Let us turn to the second point. We have to prove that the function defined by

$$H(t,p,q) = G\left(\frac{1}{3p},p,p\right) + G\left(\frac{1}{3q},q,q\right) - 2G(t,p,q)$$

is non-negative for all $t,p,q \in (0,+\infty)$ and vanishes only when $p = q$. It can be checked that if (t_n,p_n,q_n) escapes every compact of the domain $(0,1)\times(1/3,+\infty)^2$, then $\liminf H(t_n,p_n,q_n) \geqslant 0$. Then, using Maple we can solve the system $dH(t,p,q) = 0$ with $t \geqslant 0$ and $p \geqslant 1/3$ to see that only $p = q = 1/3t$ is a solution. Since H vanishes along this curve, it is equal to the set of minimums of H on the domain $(0,1)\times(1/3,+\infty)^2$, and the lemma is proved. □

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